Dynamic Systems Part 2

The mathematics behind the "Simulator" computer program

The computer program "Simulator", the use of which is described in a separate manual, enables the simulation of simple dynamic systems and experimentation with them. The code is publicly accessible on GitHub, written in VB.NET, provided with detailed comments and can be extended as required. This requires the free community version of Microsoft Visual Studio, at least version 17.9, which is based on Microsoft Framework 8.0.

This document describes the mathematical principles for the "Simulator". Part 1 deals with the systems listed in the corresponding table of contents. Further systems follow in Part 2. The technical documentation for the "Simulator" can be found in the document "Technical Documentation".

Version 7.0 – 2025/03/01

Contents

Introduction

There are many programs on the Internet that enable the simulation of simple dynamic systems. However, their code is hardly public, and the underlying mathematics is also poorly documented. The "Simulator" enables the iteration of simple real functions, the simulation of mathematical billiards, the investigation of numerical methods for solving ordinary differential equations, the simulation of various coupled pendulums or Newton's universe. The code of the program is written in VB.NET and is publicly available on Github in the repository "HermannBiner/Simulator". To work with it, the community version 2022 of Microsoft Visual Studio is sufficient, which can be downloaded free of charge and easily installed. The installation of Microsoft Framework 8.0 is also a prerequisite for the "Simulator".

The GitHub link is as follows:

https://github.com/HermannBiner/Simulator

The mathematics on which the "Simulator" is based is dealt with at an elementary level in this document. Examples of exercises or suggestions for extending the "Simulator" are intended to encourage students to do their own work. Mathematics at grammar school is more than full of material. Nevertheless, interested pupils may be offered further topics as part of an optional subject or a seminar. The "Simulator" and this document are intended to contribute to this.

The following systems are implemented in version 7 of the program:

- Growth models and iteration of quadratic functions such as logistic growth, tent mapping, iteration on the parabola including related topics such as the Feigenbaum diagram.
- Mathematical billiards with various billiard table shapes: elliptical billiards, stadium billiards, oval billiards. The analogue to the Feigenbaum diagram is the C-diagram.
- The investigation of numerical methods for solving ordinary differential equations. Some simple methods are compared here using the example of the spring pendulum.
- The simulation of coupled pendulums: Double pendulum, oscillating spring pendulum and horizontal shaking pendulum.
- Iterations in the complex plane: Newton iteration and basins of roots of unity. Zeros of third degree polynomials.
- Generation of Mandelbrot- and Julia-sets for the quadratic function and the n-th power function
- Simulation of Newton's universe including our planetary system

The use of the "Simulator" is documented in detail in a manual in German and a manual in English. In addition, technical documentation explains the architecture of the "Simulator". The language in the user interface and in all documentation can be selected between German and English. The code contains detailed comments in English.

The only prerequisite is mathematics, which is either covered at grammar school or which can be made accessible to a secondary school student with little effort. Topics from geometry (conic sections and plane vector geometry), analysis (continuity, differential calculus and ordinary differential equations) and physics (Lagrange formalism, gravitation) are covered.

The individual subject areas are largely independent, so that a selection is possible depending on interest and the time available.

Special thanks go to Prof. em. Dr. Urs Kirchgraber of ETH Zurich, who drew my attention to chaos theory in the nineties. I would also like to thank Prof. Dr. Norbert Hungerbühler of ETH Zurich, who has supported me in many ways and to whom I owe valuable advice.

1. The N-body problem

The N-body problem deals with N bodies that are positioned in three-dimensional space and whose movement is only influenced by the gravitational forces between them. It occurs, for example, when astronomers want to calculate the orbits of stars or planets. Johannes Kepler (1571 - 1630) formulated laws in the years 1599 - 1619 that correctly describe the two-body problem. Isaac Newton (1642 - 1726) derived these laws from his law of gravitation in 1687.

The case N > 2 has later occupied many mathematicians and physicists, without being able to find a closed analytical solution that describes the trajectories of the motion of the N bodies in the general case. Although Quidong Wang found a solution for the general case in 1991 with the help of Taylor series, but these converge so slowly that they are useless in practice. The general case has chaotic properties. Henri Poincaré (1854 - 1912) already discovered this in 1890 when investigating special cases of the three-body problem.

In the following, we will extend the "simulator" with a simulation of the N-body problem. We will only consider the simplified case where all bodies lie in the same plane. You must be aware that the simulation is only a numerical artifact and does not reflect the actual movement of the N-bodies.

An in-depth presentation of the mathematical theory can be found in "Orbital Motion" [1] and "Celestial Mechanics" [2]. Both books report in detail on the N-body problem, but also on general celestial mechanics, while remaining at a largely elementary mathematical level.

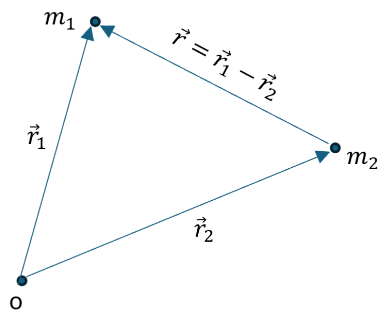
1.1 The Two-Body Problem

Newton's equations

Although the numerical simulation of the two-body problem does not differ significantly from that of the N-body problem, we will treat the two-body problem mathematically here.

We consider two masses m_1, m_2 which are located at the positions $\vec{r}_1(t), \vec{r}_2(t)$ at time t. This should be $\vec{r}_1(0) \neq \vec{r}_2(0)$

The force$\vec{F} = G\frac{m_1 m_2}{|\vec{r}|^2}$ acts between the masses in the direction of the other mass *according to Newton's law of gravitation*, where $\vec{r} = \vec{r}_2 - \vec{r}_1 \neq \vec{0}$ is the difference vector of the positions. $G \approx 6.6743 \cdot 10^{-11}\frac{m^3}{kg \cdot s^2}$ is the gravitational constant.



According to *Newton's laws* and his *law of gravitation*, we have the equations of motion:

$$\begin{cases} m_1\ddot{\vec{r}}_1 = -G\dfrac{m_1 m_2}{|\vec{r}|^2}\cdot\dfrac{\vec{r}}{|\vec{r}|} \\ m_2\ddot{\vec{r}}_2 = G\dfrac{m_1 m_2}{|\vec{r}|^2}\cdot\dfrac{\vec{r}}{|\vec{r}|} \end{cases}$$

Or:

$$(1)\begin{cases} \ddot{\vec{r}}_1 = -G\dfrac{m_2}{|\vec{r}|^2}\cdot\dfrac{\vec{r}}{|\vec{r}|} \\ \ddot{\vec{r}}_2 = G\dfrac{m_1}{|\vec{r}|^2}\cdot\dfrac{\vec{r}}{|\vec{r}|} \end{cases}$$

The conservation laws

The classical conservation laws follow from these equations. We will first check this for the *law of momentum*. If we add the above equations, we obtain:

$$m_1\ddot{\vec{r}}_1 + m_2\ddot{\vec{r}}_2 = \vec{0}$$

And by integrating according to *t*, the total momentum of the system is obtained:

$$m_1\dot{\vec{r}}_1 + m_2\dot{\vec{r}}_2 = \vec{p}$$

Whereby $\vec{p}$ is constant.

If $\vec{R}$ is the coordinate of the center of gravity and $M := m_1 + m_2$, then the total momentum is also

$$M\dot{\vec{R}} = m_1\dot{\vec{r}}_1 + m_2\dot{\vec{r}}_2$$

By integrating to *t* and determining the integration constants using special cases such as $m_1 = 0$, you obtain the position vector for the center of gravity:

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{M}$$

It also follows from $M\dot{\vec{R}} = \vec{p} = constant$ that the center of gravity of the system moves at a constant speed.

Now we will examine the *angular momentum theorem*. The following applies to the total angular momentum $\vec{L}$ of the system:

$$\frac{d}{dt}\vec{L} = \frac{d}{dt}\left(\vec{r}_1 \times m_1\dot{\vec{r}}_1 + \vec{r}_2 \times m_2\dot{\vec{r}}_2\right) = \vec{r}_1 \times m_1\ddot{\vec{r}}_1 + \vec{r}_2 \times m_2\ddot{\vec{r}}_2$$

$$= -\vec{r}_1 \times G\frac{m_1 m_2}{|\vec{r}|^2}\cdot\frac{\vec{r}}{|\vec{r}|} + \vec{r}_2 \times G\frac{m_1 m_2}{|\vec{r}|^2}\cdot\frac{\vec{r}}{|\vec{r}|} = \vec{r} \times G\frac{m_1 m_2}{|\vec{r}|^2}\cdot\frac{\vec{r}}{|\vec{r}|} = \vec{0}$$

The angular momentum $\vec{L}$ of the system is therefore constant.

We also check that the *total energy* of the system is constant. The potential energy is at the distance $|\vec{a}|$ from the center of gravity (if the zero point is defined at infinity):

$$E_{pot} = -\int_{|\vec{a}|}^{\infty} G\frac{m_1 m_2}{|\vec{r}|^2}\cdot\frac{\vec{r}}{|\vec{r}|}d\vec{r} = -G\frac{m_1 m_2}{|\vec{a}|}$$

Therefore

$$E = \frac{1}{2} m_1 \left|\dot{\vec{r}}_1\right|^2 + \frac{1}{2} m_2 \left|\dot{\vec{r}}_2\right|^2 - G \frac{m_1 m_2}{|\vec{r}|}$$

$$\frac{d}{dt} E = m_1 \dot{\vec{r}}_1 \cdot \ddot{\vec{r}}_1 + m_2 \dot{\vec{r}}_2 \cdot \ddot{\vec{r}}_2 + G \frac{m_1 m_2}{|\vec{r}|^3} \vec{r} \cdot \dot{\vec{r}}$$

We set $\gamma = G \frac{m_1 m_2}{|\vec{r}|^3}$. Then after (1): $m_1 \ddot{\vec{r}}_1 = -\gamma \vec{r}$ and $m_2 \ddot{\vec{r}}_2 = \gamma \vec{r}$ and you have:

$$\frac{d}{dt} E = -\dot{\vec{r}}_1 \cdot \gamma \vec{r} + \dot{\vec{r}}_2 \cdot \gamma \vec{r} + \gamma \vec{r} \cdot \dot{\vec{r}} = -\dot{\vec{r}} \cdot \gamma \vec{r} + \gamma \vec{r} \cdot \dot{\vec{r}} = 0$$

□

Choice of reference system

In the following, we will now try to determine the orbits of the masses m_1, m_2. To do this, we first select a suitable coordinate system as a reference system.

It is easy to see that Newton's equations (1) are *independent of the choice of zero point.* If you replace $\vec{r}_{1,2} = \vec{r}_{1,2}{}' + \vec{c}$, whereby the old zero point is pushed into the new one by the translation $\vec{c}$, then $\vec{r}_{1,2}{}'$ fulfill the same equations. Now we choose the centre of gravity of the system, which moves at a constant speed, as the zero point. This gives us an inertial system. Relative to this, the total momentum of the system is zero. Therefore, the following applies:

$$m_1 \vec{r}_1 + m_2 \vec{r}_2 = 0$$

If equations (1) are subtracted from each other, it follows:

$$\ddot{\vec{r}} = \ddot{\vec{r}}_1 - \ddot{\vec{r}}_2 = -GM \frac{\vec{r}}{|\vec{r}|^3}$$

Where $M = m_1 + m_2$ is the total mass. It also follows from the first equation (1) and since the total momentum is zero:

$$\ddot{\vec{r}}_1 = -G \frac{m_2 \vec{r}}{|\vec{r}|^3} = -G \frac{m_2 \vec{r}_1 - m_2 \vec{r}_2}{|\vec{r}|^3} = -G \frac{m_2 \vec{r}_1 + m_1 \vec{r}_1}{|\vec{r}|^3} = -GM \frac{\vec{r}_1}{|\vec{r}|^3}$$

An analogous calculation is carried out for $\vec{r}_2$ and finally the (symmetrical) system of equations relative to the centre of gravity is obtained:

$$(2) \begin{cases} \ddot{\vec{r}}_1 = -GM \dfrac{\vec{r}_1}{|\vec{r}|^3} \\ \ddot{\vec{r}}_2 = -GM \dfrac{\vec{r}_2}{|\vec{r}|^3} \\ \ddot{\vec{r}} = -GM \dfrac{\vec{r}}{|\vec{r}|^3} \end{cases}$$

If we form the cross product with $\vec{r}$, we have:

$$0 = \vec{r} \times \left(\ddot{\vec{r}} + GM \frac{\vec{r}}{|\vec{r}|^3} \right) = \vec{r} \times \ddot{\vec{r}} = \frac{d}{dt} \left(\vec{r} \times \dot{\vec{r}} \right)$$

And integration according to *t* delivers:

$$\vec{r} \times \dot{\vec{r}} = \vec{h}$$

Where $\vec{h}$ is constant and generally $\neq \vec{0}$. It is $\vec{r} \neq \vec{0}$ by assumption and in general $\dot{\vec{r}} \neq \vec{0}$, since the bodies are in each other's gravitational field. If the bodies do not fly towards each other and collide, $\vec{r}$ is also not parallel to $\dot{\vec{r}}$. *The path then runs in a plane* perpendicular to $\vec{r}, \dot{\vec{r}}$ regardless of *t.* We place this plane through the centre of gravity.

Can we place the x- and y-axis of the coordinate system in this plane? To do this, we would have to rotate the coordinate system so that the x- and y-axes fall into the plane and the z-axis is perpendicular to this plane. We refer to this rotation as D. This is bijective and D^{-1} returns the rotated coordinate system to the original one. Let $\vec{r}_{1,2}{}'$ be the positions of the masses in the rotated system. Then the following applies: $\vec{r}_{1,2} = D^{-1}\vec{r}_{1,2}{}'$. D and thus also D^{-1} keep distances invariant, so $|\vec{r}| = |D^{-1}\vec{r}'| = |\vec{r}'|$ applies. Thus, for example, the first equation (1) changes into:

$$D^{-1}\ddot{\vec{r}}_1' = -G\frac{m_2}{|\vec{r}'|^3} \cdot D^{-1}\vec{r}'$$

Since D^{-1} is bijective, it follows:

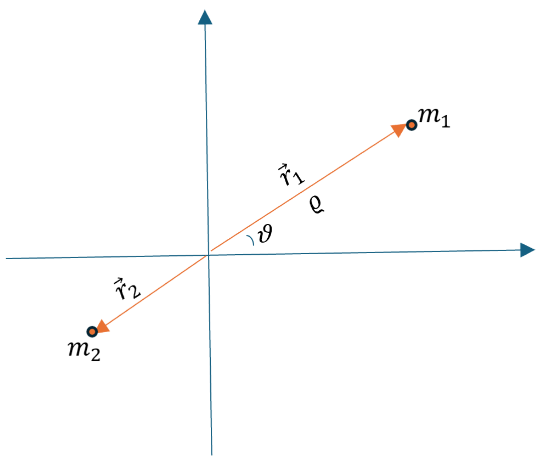
$$\ddot{\vec{r}}_1' = -G\frac{m_2}{|\vec{r}'|^3} \cdot \vec{r}'$$

The same applies to the second equation. This means that the new coordinates fulfil the same equations (*) as the old ones.

The equations of motion are therefore *independent of the rotation of the coordinate system,* and we can place the x and y axes in the plane in which the movement takes place. At the same time, the zero point of the coordinate system lies in the common centre of gravity. We will need this reference point later for the implementation to describe the path of *a* mass relative to the common centre of gravity.

The path of *a* mass relative to the common centre of gravity

The following sketch shows the corresponding situation.



Position of the masses relative to the selected coordinate system

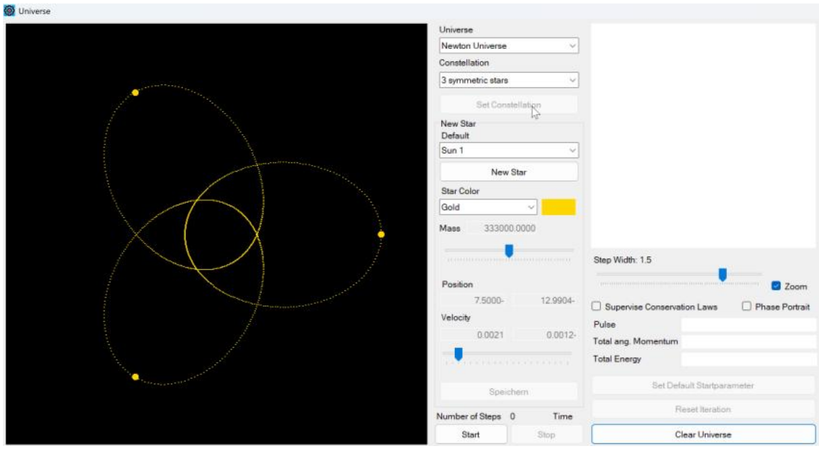
If we look at the first equation (2), we get that the total momentum is zero:

$$\ddot{\vec{r}}_1 = -GM\frac{\vec{r}_1}{|\vec{r}|^3} = -GM\frac{\vec{r}_1}{|\vec{r}_1 - \vec{r}_2|^3} = -GM\frac{\vec{r}_1}{\left|\vec{r}_1 + \frac{m_1}{m_2}\vec{r}_1\right|^3} = -G\frac{Mm_2^3}{M^3} \cdot \frac{\vec{r}_1}{|\vec{r}_1|^3}$$

This gives us an equation in which only the movement of the mass m_1 appears:

$$(3)\ \ddot{\vec{r}}_1 = -G\frac{m_2^3}{M^2} \cdot \frac{\vec{r}_1}{|\vec{r}_1|^3}$$

These are - broken down into components - three second-order differential equations. When solving them, a total of 6 integration constants must be determined.



Three symmetrical stars in the "simulator"

For the calculation of the displayed orbits, it was assumed that the star orbits around the common centre of gravity. The orbit curve was calculated for each star as a two-body problem with respect to the other masses at the centre of gravity.

Polar coordinates

We now introduce polar coordinates:

$$\vec{r}_1 = \varrho\begin{pmatrix} cos\vartheta \\ sin\vartheta \end{pmatrix}, |\vec{r}_1| = \varrho$$

$$\dot{\vec{r}}_1 = \dot{\varrho}\begin{pmatrix} cos\vartheta \\ sin\vartheta \end{pmatrix} + \varrho\dot{\vartheta}\begin{pmatrix} -sin\vartheta \\ cos\vartheta \end{pmatrix}$$

$$\ddot{\vec{r}}_1 = \ddot{\varrho}\begin{pmatrix} cos\vartheta \\ sin\vartheta \end{pmatrix} + \ 2\dot{\varrho}\dot{\vartheta}\begin{pmatrix} -sin\vartheta \\ cos\vartheta \end{pmatrix} + \varrho\ddot{\vartheta}\begin{pmatrix} -sin\vartheta \\ cos\vartheta \end{pmatrix} + \varrho\dot{\vartheta}^2\begin{pmatrix} -cos\vartheta \\ -sin\vartheta \end{pmatrix}$$

Since we assume that there are two differently positioned (point-like) masses, each mass lies outside the common centre of gravity. **Therefore, it holds $\varrho > 0$!**

The aim is now to find two equations for$\varrho(t)$ and $\vartheta(t)$. Eliminate *t* from these equations to obtain an equation for $\varrho(\vartheta)$ and thus the orbital equation we are looking for. To do this, we consider the angular momentum and the energy of the mass m_1. First, we show that this angular momentum is constant

$$\dot{\vec{L}} := \frac{d}{dt}(\vec{r}_1 \times m_1\dot{\vec{r}}_1) = \vec{r}_1 \times m_1\ddot{\vec{r}}_1 = \vec{0}$$

Since according to (3) $\ddot{\vec{r}}_1 \parallel \vec{r}_1$. The angular momentum is always perpendicular to the coordinate plane, and we only consider its magnitude. Then we have in polar coordinates:

$$L \coloneqq |\vec{L}| = |\vec{r}_1 \times m_1\dot{\vec{r}}_1| = m_1\varrho^2\dot{\vartheta} = constant$$

This allows us to eliminate the time derivatives.

First we get:

$$\dot{\vartheta} = \frac{L}{m_1} \cdot \frac{1}{\varrho^2}$$

Furthermore:

$$\dot{\varrho} = \frac{d\varrho}{dt} = \frac{d\varrho}{d\vartheta} \cdot \frac{d\vartheta}{dt} = \varrho' \dot{\vartheta} = \frac{L}{m_1} \cdot \frac{\varrho'}{\varrho^2}$$

Now let's look at equation (3) again:

$$m_1 \ddot{\vec{r}}_1 = -G \frac{m_2^3 m_1}{M^2} \cdot \frac{\vec{r}_1}{|\vec{r}_1|^3}$$

The force $-G \frac{m_2^3 m_1}{M^2} \cdot \frac{\vec{r}_1}{|\vec{r}_1|^3}$ acts on m_1 and the associated potential energy is (if the zero point is chosen at infinity):

$$E_{pot} = -G \frac{m_2^3 m_1}{M^2} \cdot \frac{1}{|\vec{r}_1|}$$

This allows the energy of the mass m_1 to be written as:

$$E = \frac{1}{2} m_1 \left|\dot{\vec{r}}_1\right|^2 - G \frac{m_2^3 m_1}{M^2} \cdot \frac{1}{|\vec{r}_1|}$$

In polar coordinates we have:

$$E = \frac{m_1}{2} \left(\dot{\varrho}^2 + \varrho^2 \dot{\vartheta}^2\right) - G \frac{m_2^3 m_1}{M^2} \cdot \frac{1}{\varrho}$$

Now we eliminate $\dot{\varrho}, \dot{\vartheta}$:

$$E = \frac{m_1}{2} [\left(\frac{L}{m_1}\right)^2 \cdot \frac{{\varrho'}^2}{\varrho^4} + \varrho^2 \left(\frac{L}{m_1}\right)^2 \cdot \frac{1}{\varrho^4}] - G \frac{m_2^3 m_1}{M^2} \cdot \frac{1}{\varrho}$$

And preserved:

$$E = \frac{L^2}{2m_1} \cdot \frac{1}{\varrho^2} \left(\frac{{\varrho'}^2}{\varrho^2} + 1\right) - G \frac{m_2^3 m_1}{M^2} \cdot \frac{1}{\varrho}$$

If the mass m_1 flies directly towards the centre of gravity and towards the other mass m_2, $\vec{r}_1 \parallel \dot{\vec{r}}_1$ and thus $L = 0$. Then we simply have a linear motion that ends with the collision of the masses. We exclude this case. **Therefore, $\boldsymbol{L \neq 0}$** and we divide by L:

$$\frac{2m_1 E}{L^2} = \frac{1}{\varrho^2} \left(\frac{{\varrho'}^2}{\varrho^2} + 1\right) - 2G \frac{m_2^3 m_1^2}{M^2 L^2} \cdot \frac{1}{\varrho}$$

Solution of the differential equation

We set: $p := \frac{M^2 L^2}{G m_2^3 m_1^2}$ and substitute: $\varrho = \frac{1}{u}$, $\varrho' = -\frac{u'}{u^2}$. This gives us

$$\frac{2m_1 E}{L^2} = {u'}^2 + u^2 - \frac{2u}{p}$$

Square complement supplies:

$$\frac{2m_1E}{L^2} = {u'}^2 + (u - \frac{1}{p})^2 - \frac{1}{p^2}$$

We take the approach:

$$u(\vartheta) = Acos(\vartheta - \alpha) + \frac{1}{p} \ \ ,u'(\vartheta) = -Asin(\vartheta - \alpha)$$

And obtain by insertion:

$$\frac{2m_1E}{L^2} = A^2 - \frac{1}{p^2}$$

Then we get:

$$A^2 = \frac{2m_1E}{L^2} + \frac{1}{p^2}$$

We first want to check whether this expression is always defined, i.e. whether the right-hand side is always positive.

E is constant. If we look at the energy at perihelion or aphelion, it is $\dot{\varrho} = 0$ and we obtain the kinetic energy at this point:

$$E_{kin} = \frac{m_1}{2}\varrho^2\dot{\vartheta}^2 = \frac{m_1}{2}\varrho^2\frac{L^2}{m_1^2\varrho^4} = \frac{L^2}{2m_1\varrho^2}$$

And for the total energy:

$$E = \frac{L^2}{2m_1\varrho^2} - G\frac{m_2^3m_1}{M^2}\cdot\frac{1}{\varrho} = \frac{L^2}{2m_1}(\frac{1}{\varrho^2} - \frac{2}{p}\cdot\frac{1}{\varrho})$$

This delivers:

$$\frac{2m_1E}{L^2} + \frac{1}{p^2} = \frac{1}{\varrho^2} - \frac{2}{p}\cdot\frac{1}{\varrho} + \frac{1}{p^2} = \left(\frac{1}{\varrho} - \frac{1}{p}\right)^2 \geq 0$$

□

Now is:

$$A^2 = \frac{2m_1E}{L^2} + \frac{1}{p^2} = \frac{1}{p^2}\left(\frac{2m_1E}{L^2}\cdot p^2 + 1\right) = \frac{1}{p^2}\cdot(\frac{2EL^2M^4}{G^2m_1^3m_2^6} + 1)$$

We define ε by:

$$\varepsilon^2 := A^2p^2 = \frac{2EL^2M^4}{G^2m_1^3m_2^6} + 1 \geq 0$$

It is sufficient to consider the case$\varepsilon \geq 0$, because otherwise you can simply add π to $\vartheta - \alpha$.

You have: $A = {}^{\varepsilon}/_{p}$ and

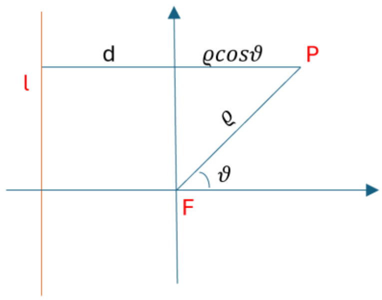
$$u(\vartheta) = \frac{\varepsilon\cos(\vartheta - \alpha) + 1}{p}$$

And thus:

$$(4)\ \ \varrho(\vartheta) = \frac{p}{1 + \varepsilon\cos{(\vartheta - \alpha)}}$$

Discussion of the solution

(4) is the equation of a conic section. If we define this as the location of the points which have a constant distance ratio ε from a fixed point (the focal point F) and a fixed straight line (the guiding line l), we obtain:



The point P and its distance from F or l

$$\frac{\varrho}{d + \varrho cos\vartheta} = \varepsilon$$

$$\varrho = \frac{p}{1 - \varepsilon cos\vartheta}$$

$p = \varepsilon d = \varrho(\frac{\pi}{2})$ is half the cone section at the focal point. For $\alpha = \pi$, equation (4) corresponds to the above equation.

The minimum distance of a planet from the sun or from the focal point is the *perihelion*. If you look at equation (4) again:

$$(4)\ \ \varrho(\vartheta) = \frac{p}{1 + \varepsilon\cos{(\vartheta - \alpha)}}$$

You can then see that the minimum is assumed for $\cos(\vartheta - \alpha) = 1$ or $\vartheta = \alpha$

α is just *the argument of the perihelion*, or the angle between the direction of the perihelion and the x-axis. In our solar system, the x-axis is defined by the intersection of the ecliptic and the Earth's equatorial plane at the vernal equinox

Depending on the value of the numerical eccentricityε , different types of conic section are obtained:

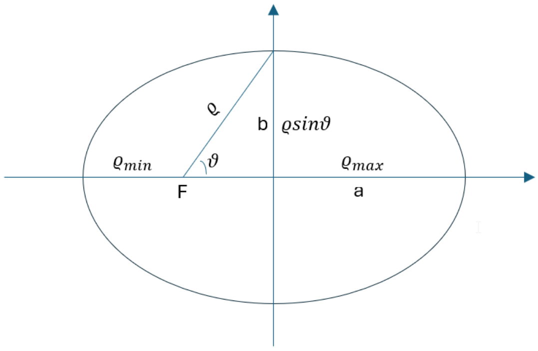
$\varepsilon = 0$: $\varrho = p = constant$. The conic section is a circle.

$0 < \varepsilon < 1$: $\frac{p}{1-\varepsilon} < \varrho < \frac{p}{1+\varepsilon}$. The conic section is an ellipse.

$\varepsilon = 1$: $\varrho \to \infty$ for $\vartheta \to 0$. Otherwise is $\varrho > 0$ and defined for all $\vartheta \in]0,2\pi[$. You have a parabola.

$\varepsilon > 1$: $\varrho > 0$ only for $cos\vartheta < \frac{1}{\varepsilon}$. $\varrho \to \infty$ for $\vartheta \to arccos\frac{1}{\varepsilon}$. One has a hyperbole.

In the case of the ellipse, it is useful to calculate the semi-axes. Let a be the major semi-axis and b the minor semi-axis of the ellipse.



Major and minor semi-axis of the ellipse

The following applies:

$$a = \frac{1}{2}(\varrho_{min} + \varrho_{max}) = \frac{1}{2}\left(\varrho\left(\frac{\pi}{2}\right) + \varrho(0)\right) = \frac{1}{2}\left(\frac{p}{1+\varepsilon} + \frac{p}{1-\varepsilon}\right) = \frac{p}{1-\varepsilon^2}$$

To determine b we are looking for the maximum of $\varrho sin\vartheta$. It is:

$$(\varrho sin\vartheta)' = \frac{pcos\vartheta(1-\varepsilon cos\vartheta) - psin\vartheta \cdot \varepsilon sin\vartheta}{(1-\varepsilon cos\vartheta)^2} = \frac{pcos\vartheta - \varepsilon p}{(1-\varepsilon cos\vartheta)^2} = 0$$

For $cos\vartheta = \varepsilon$. This gives us:

$$b = \varrho sin\vartheta = \frac{p\sqrt{1-\varepsilon^2}}{1-\varepsilon^2} = \frac{p}{\sqrt{1-\varepsilon^2}} = a\sqrt{1-\varepsilon^2}$$

Remark:

We have carried out our calculation with the centre of gravity as the zero point. In the literature, the location of the mass m_2 is sometimes set as the zero point and the movement of m_1 is calculated relative to m_2. This results in slightly different values for the parameters p, ε in the orbit equation.

Finally, we want to make some considerations about the units of the parameters that occurred in the orbit equation in the case of the two-body problem. This was:

$$\varrho(\vartheta) = \frac{p}{1 + \varepsilon\cos(\vartheta - \alpha)}$$

In it was: $p := \frac{M^2L^2}{Gm_1^2m_2^3}$. We write the unit of a size in square brackets, and we use the kilogram - meter - second system. Then$[L] = \frac{kg \cdot m^2}{s}$. This becomes:

$$[p] = [M]^2[L]^2 \cdot [Gm_1^2m_2^3]^{-1} = kg^2 \cdot \frac{kg^2 \cdot m^4}{s^2} \cdot \frac{kg \cdot s^2}{m^3} \cdot \frac{1}{kg^5} = m$$

Therefore, p has the dimension of a length.

Furthermore was:

$$\varepsilon^2 := \frac{2EL^2M^4}{G^2m_1^3m_2^6} + 1$$

It is:

$$\left[\frac{EL^2M^4}{G^2m_1^3m_2^6}\right] = [EL^2M^4]\cdot[G^2m_1^3m_2^6]^{-1} = \frac{kg\cdot m^2}{s^2}\cdot\frac{kg^2\cdot m^4}{s^2}\cdot kg^4\cdot\frac{kg^2\cdot s^4}{m^6}\cdot\frac{1}{kg^9} = 1$$

ε is dimensionless.

Another problem arises during implementation. The speed at which the centrifugal force and gravitational force are balanced in the gravitational field of a mass M is:

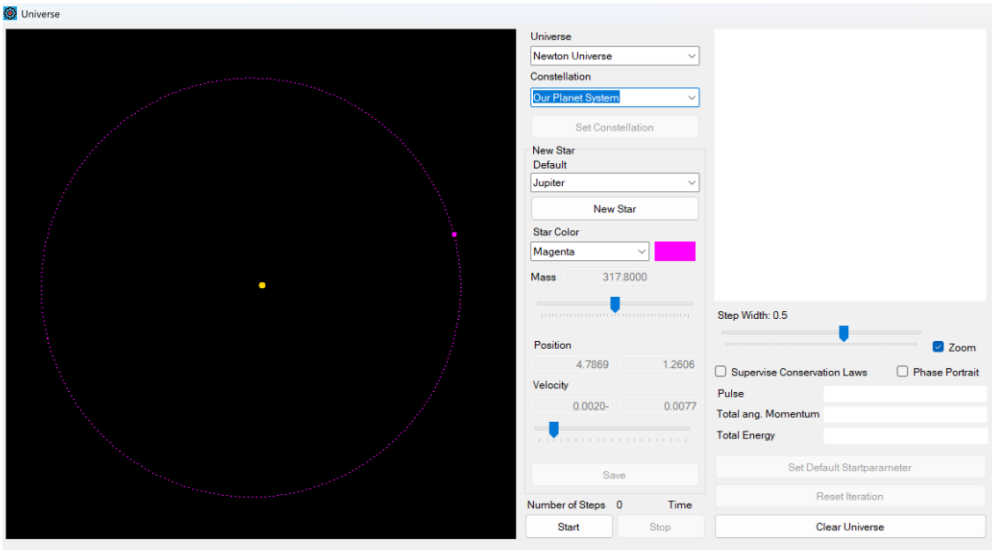
$$v_0 = \sqrt{\frac{GM}{r}}$$

If the moving mass has the distance r to M. In the implementation, when specifying the speed v , we assume that this is the speed at perihelion. However, this is only the case if $v > v_0$. Otherwise, the moving mass will approach the mass M, which means that the starting point of the orbit is actually the aphelion of the orbit. We take this into account in the implementation in that the orbit curve is determined as follows:

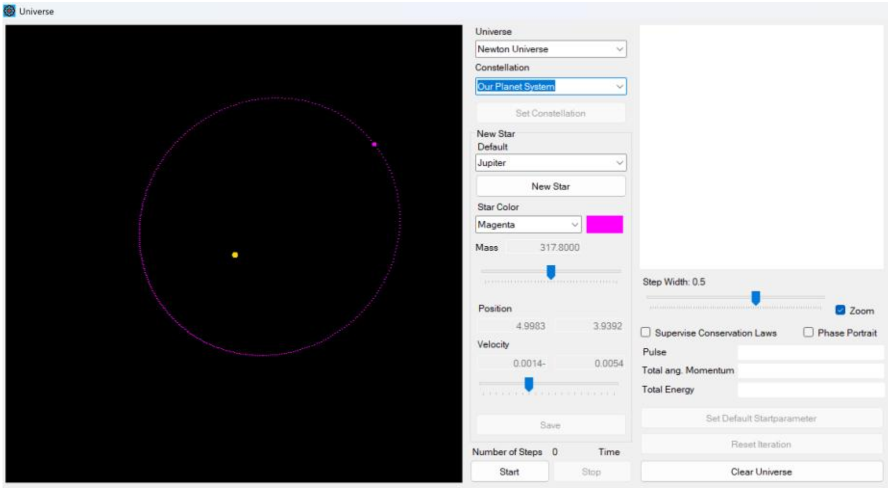
$$\varrho(\vartheta) = \frac{p}{1+\varepsilon\cos(\vartheta-\alpha)}, if\ v \geq v_0$$

$$\varrho(\vartheta) = \frac{p}{1-\varepsilon\cos(\vartheta-\alpha)}\cdot\frac{1-\varepsilon}{1+\varepsilon}, if\ v < v_0$$

It is easy to see that the starting point for $\vartheta = \alpha$ is identical in both cases. However, the second equation provides an elliptical orbit with the starting point at aphelion.



Jupiter, starting from the actual position at perihelion and the actual speed



Here, the user has changed the starting position and speed

1.2 Kepler's laws

Here we can look at Kepler's laws. The orbital curve of the mass m_1, calculated in the previous section, immediately provides Kepler's first law: *the planets move on ellipses with the sun at their focal point*.

An infinitesimal surface element, which is swept by the position vector, is in polar coordinates:

$$dF = \frac{1}{2}\varrho^2\dot{\vartheta}dt = \frac{1}{2}\frac{L}{m_1}dt$$

since the angular momentum L is constant. This means that the area covered during a fixed time Δt is always of the same size. This leads to Kepler's second law: *the line connecting the planet and the sun sweeps over the same area at the same time*.

If T is the orbital period of the mass m_1, then the elliptical surface $F = \pi ab$ is swept during this time. The following therefore applies:

$$\pi ab = \int_0^T \frac{1}{2}\varrho^2\dot{\vartheta}dt = \frac{1}{2}\int_0^T \frac{L}{m_1}dt = \frac{1}{2}T\frac{L}{m_1}$$

So it is:

$$T = 2\pi ab\frac{m_1}{L}$$

The definition of p was: $p := \frac{M^2L^2}{Gm_2^3m_1^2}$. Furthermore, $p = a(1-\varepsilon^2)$ and thus

$$L = \sqrt{Gm_2} \cdot \frac{m_1m_2}{M}\sqrt{a(1-\varepsilon^2)}$$

Furthermore, $b = a\sqrt{1-\varepsilon^2}$. If we use this in the above formula for T, we get

$$T = 2\pi a^2\sqrt{1-\varepsilon^2} \cdot m_1\frac{M}{\sqrt{Gm_2}m_1m_2\sqrt{a(1-\varepsilon^2)}} = 2\pi\sqrt{a}^3\frac{M}{\sqrt{Gm_2^3}}$$

Thus:

$$\frac{T^2}{a^3} = 4\pi^2 \frac{M^2}{Gm_2^3} = constant$$

In the special case of the sun and a planet, we have $m_2 = M_{Sonne}$ and $M = m_1 + m_2 \approx M_{Sonne}$

Then it becomes:

$$\frac{T^2}{a^3} = 4\pi^2 \frac{1}{GM_{Sonne}}$$

In particular, Kepler's third law follows from this: *the squares of the orbital periods are in the same ratio as the cubes of the large semi-major axes*.

1.3 Some data about our solar system

Later, we will check certain calculations for the simulation using our solar system as an example. Individual exercises also refer to this system. Therefore, the necessary data is listed in the following table.

Once again, the gravitational constant: $G \approx 6.6743 \cdot 10^{-11} \frac{m^3}{kg \cdot s^2}$

	Sun	Mercury	Venus	Earth	Mars
Mass in kg	1.9884E+30	3.3010E+23	4.8673E+24	5.9722E+24	6.4200E+23
Mass relative to the earth's mass	332'943	0.055	0.815	1.000	0.107
Mass relative to the mass of the sun	1	1.660E-07	2.448E-06	3.004E-06	3.229E-07
Diameter in km	1'392'000	4'879	12'103	12'735	6'772
Gravitational constant in m/s2	274.00	3.70	8.87	9.80	3.73
Orbital period around the sun in days	-	87.969	224.701	365.256	686.980
Major semi-axis in million km	-	57.909	108.200	149.600	227.990
Large half-axis in AU	-	0.3871	0.7233	1.000	1.524
Eccentricity of the elliptical path	-	0.2056	0.0068	0.0167	0.0934
Speed relative sun in km/h	-	172'332	126'072	107'208	86'868
Speed relative sun in AU/day	-	0.0276	0.0202	0.0172	0.0139
... and in AU/year	-	10.0700	7.3669	6.2646	5.0760
Escape velocity in km/s *)	617.4	4.3	10.4	11.2	5.0
Argument of the perihelion **)	-	1.3519	2.2956	1.7967	5.8650
Perihelion in AU	-	0.3075	0.7184	0.9833	1.3814
Perihelion Speed in km/h	-	212'328	126'936	109'044	95'400
Perihelion velocity in AU/day	-	0.034063	0.020364	0.017494	0.015305
Aphelion in AU	-	0.467	0.728	1.017	1.666

	Jupiter	Saturn	Uranus	Neptune
Mass in kg	1.8980E+27	5.6830E+26	8.6800E+25	1.0024E+26
Mass relative to the earth's mass	317.8	95.2	14.5	16.8
Mass relative to the mass of the sun	9.545E-04	2.858E-04	4.365E-05	5.041E-05
Diameter in km	138'346	114'632	50'532	49'105
Gravitational constant in m/s2	24.79	10.44	8.87	11.15
Orbital period around the sun in days	4'329	10'751	30'664	60'148

Major semi-axis in million km	778.51	1'433.40	2'872.40	4'514.60
Large half-axis in AU	5.204	9.582	19.201	30.178
Eccentricity of the elliptical path	0.0489	0.0542	0.0472	0.0097
Speed relative sun in km/h	47'052	34'884	24'516	19'548
Speed relative sun in AU/day	0.0075	0.0056	0.0039	0.0031
... and in AU/year	2.7393	2.0453	1.4244	1.1322
Escape velocity in km/s *)	60.2	36.1	21.4	23.6
Argument of the perihelion **)	0.2575	1.6132	2.9839	0.7849
Perihelion in AU	4.9501	9.0481	18.3755	29.7667
Perihelion Speed in km/h	49'392	36'648	25'596	19'800
Perihelion velocity in AU/day	0.007924	0.005879	0.004106	0.003176
Aphelion	5.455	10.12	20.11	30.069

*) at the equator

**) Relative to the zero direction, which is defined by the intersection of the ecliptic with the Earth's equatorial plane. The argument is in radian.

The information on the perihelion changes over time. The above data are from the year 2000 and are used as the starting position of a planet when implementing our planetary system. As the numerical approximation becomes imprecise very soon anyway, we refrain from striving for greater accuracy here. Moreover, this would quickly become very time-consuming

1.4 The N-body Problem

The three-body problem is already significantly more complex than the two-body problem. It was already formulated by Newton. There is no closed analytical solution to this problem. Over the last three hundred years, great mathematicians have worked on this problem. Important contributions were made by Pierre Simon Laplace (1749 - 1827) and Henri Poincaré (1854 - 1912). A more recent result from the years 1953 - 1963, which among other things relates to the stability of our planetary system, is the so-called KAM theorem by Andrei Kolmogorov (1903 - 1987), Jürgen Moser (1928 - 1999) and Vladimir Arnold (1937 - 2010).

For n bodies with mass $m_j, j = 1..n$, which are located at the positions $\vec{r}_j, j = 1..n$, Newton's equations of motion are given by

$$m_j \ddot{\vec{r}}_j = \sum_{i \neq j} G \frac{m_j m_i}{\left|\vec{r}_i - \vec{r}_j\right|^3} \cdot (\vec{r}_i - \vec{r}_j)$$

We will use these equations when simulating the movement of these bodies for the numerical methods. These equations are invariant under displacement of the zero point, i.e. when replacing $\vec{r}_j = \vec{r}_j' + \vec{c}$, where $\vec{c}$ is a constant vector. We will choose the common centre of gravity of the masses as the zero point. The coordinate $\vec{R}$ of the center of gravity for any coordinate system is given by

$$M\vec{R} = \sum_{j=1}^{n} m_j \vec{r}_j$$

$$M = \sum_{j=1}^{n} m_j$$

The sum of all forces cancels out, because for all pairs of bodies the force between these bodies is zero in total. It is therefore:

$$\sum_{j=1}^{n} m_j \ddot{\vec{r}}_j = 0$$

The momentum theorem follows from this by integration:

$$\sum_{j=1}^{n} m_j \dot{\vec{r}}_j = \vec{p}$$

Whereby $\vec{p}$ is constant.

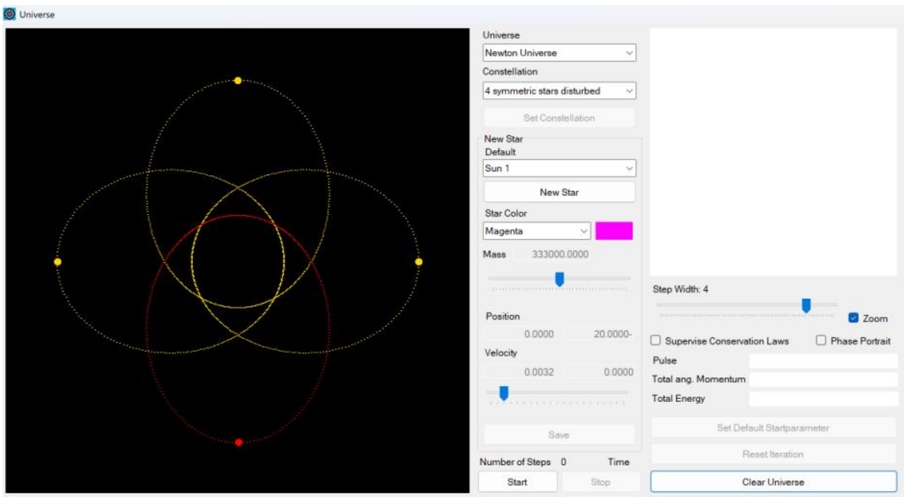
The following applies to the angular momentum $\vec{L}$:

$$\frac{d}{dt}\vec{L} = \frac{d}{dt}\sum_{j=1}^{n} \vec{r}_j \times m_j \dot{\vec{r}}_j = \sum_{j=1}^{n} \vec{r}_j \times m_j \ddot{\vec{r}}_j = \sum_{j=1}^{n} \vec{r}_j \times \{\sum_{i \neq j} G \frac{m_j m_i}{\left|\vec{r}_i - \vec{r}_j\right|^3} \cdot (\vec{r}_i - \vec{r}_j)\}$$

In this somewhat complex sum, the summands always occur in pairs, e.g.

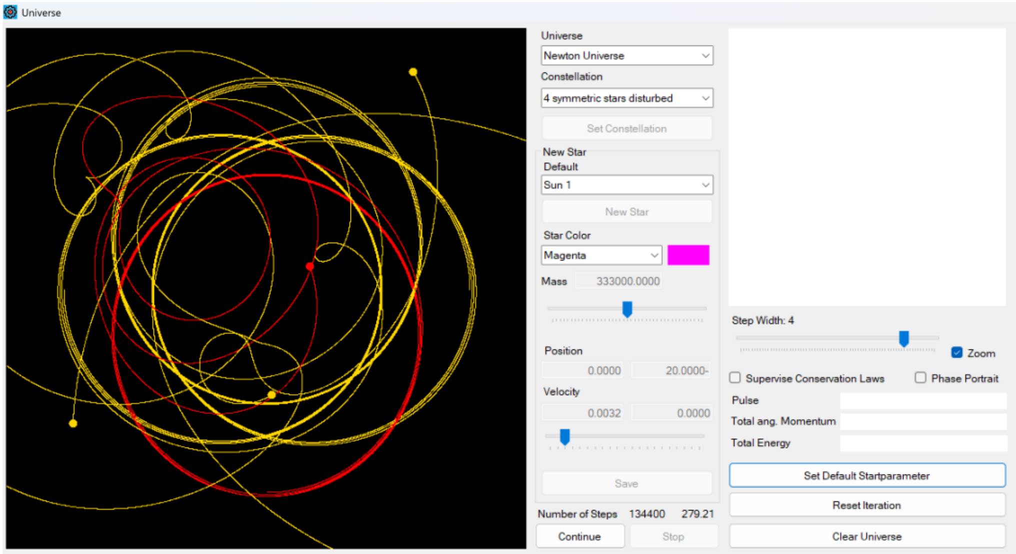
$\vec{r}_1 \times G \frac{m_1 m_2}{|\vec{r}_2 - \vec{r}_1|^3} \cdot (\vec{r}_2 - \vec{r}_1)$ and $\vec{r}_2 \times G \frac{m_1 m_2}{|\vec{r}_1 - \vec{r}_2|^3} \cdot (\vec{r}_1 - \vec{r}_2)$. The sum of such a pair is zero, so in total $\frac{d}{dt}\vec{L} = 0$ and therefore, the angular momentum is constant. For the motion of the N-bodies, the plane perpendicular to $\vec{L}$ is invariant and we can place it in the centre of gravity of the system. It is called the *invariant plane of Laplace*. In the simulation of the N-body problem, we will restrict ourselves to the case where all bodies lie in this plane.

With a similar, but somewhat more complex calculation, it can be shown that the total energy of the system is constant.



4 (almost) symmetrical stars in starting position

The orbits shown were calculated for each star like a two-body problem. The distance of the red star from the centre of gravity is smaller by $5 \cdot 10^{-7}$ ‰ . The system reacts very sensitively to this disturbance and after four orbits it runs "out of control".



The "disturbed" system of the four stars gets out of control after four orbits

1.5 Numerical approximation of the N-body Problem

In the "Simulator", we only consider flat systems, i.e. we calculate with two components.

The acceleration that the body number j experiences is:

$$\ddot{\vec{r}}_j = \sum_{i \neq j} G \frac{m_i}{\left|\vec{r}_i - \vec{r}_j\right|^3} \cdot (\vec{r}_i - \vec{r}_j)$$

We write for the position vector of the body:

$$\vec{r}_j = \begin{pmatrix} x_j \\ y_j \end{pmatrix}$$

We set:

$$R_{ji} := \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$$

And then, wee receive e.g. for the first component of: $\vec{r}_j$

$$\ddot{x}_j = G \sum_{i \neq j} \frac{m_i}{{R_{ji}}^3} \cdot (x_i - x_j)$$

For n bodies, this provides a system of 2n second-order differential equations. With the substitution: $u_{1j} = x_j, v_{1j} = \dot{x}_j$ or $u_{2j} = y_j, v_{2j} = \dot{y}_j$, a system of first-order differential equations is obtained for the acceleration of body number j:

$$\begin{cases} \dot{u}_{1j} = v_{1j} =: f_1(t, u_1, v_1, u_2, v_2) \\ \dot{v}_{1j} = G \sum_{i \neq j} \frac{m_i}{{R_{ji}}^3} \cdot \left(u_{1i} - u_{1j}\right) =: g_1(t, u_1, v_1, u_2, v_2) \\ \dot{u}_{2j} = v_{2j} =: f_2(t, u_1, v_1, u_2, v_2) \\ \dot{v}_{2j} = G \sum_{i \neq j} \frac{m_i}{{R_{ji}}^3} \cdot \left(u_{2i} - u_{2j}\right) =: g_2(t, u_1, v_1, u_2, v_2) \end{cases}$$

With

$$R_{ji} \coloneqq \sqrt{(u_{1i} - u_{1j})^2 + (u_{2i} - u_{2j})^2}$$

As in the section on the double pendulum, we now apply the four-order Runge-Kutta method for a simulation of this system. The result will only be a numerical artifact of a real system. However, in the implementation we will monitor the total momentum, the total angular momentum and the total energy of the system and check that these are (reasonably) constant.

To document the implementation in the "simulator", we prepare it in the following algorithm. In it, $u_{1n}, v_{1n}, u_{2n}, v_{2n}$ are the values of the parameters after the n^{th} iteration step. The (constant) step size is d, which must be chosen sufficiently small. Now we try to keep the representation somewhat compact. We carry out the following steps one after the other:

$$\vec{z}_{1n} \coloneqq (u_{1n}, v_{1n}, u_{2n}, v_{2n})$$

$$\begin{cases} k_{i1} \coloneqq f_i(t_n, \vec{z}_{1n}) \\ h_{i1} \coloneqq g_i(t_n, \vec{z}_{1n}) \end{cases}, i \in \{1,2\}$$

$$\vec{z}_{2n} \coloneqq (u_{1n} + \frac{d}{2}k_{11}, v_{1n} + \frac{d}{2}h_{11}, u_{2n} + \frac{d}{2}k_{21}, v_{2n} + \frac{d}{2}h_{21})$$

$$\begin{cases} k_{i2} \coloneqq f_i(t_n + \frac{d}{2}, \vec{z}_{2n}) \\ h_{i2} \coloneqq g_i(t_n + \frac{d}{2}, \vec{z}_{2n}) \end{cases}, i \in \{1,2\}$$

$$\vec{z}_{3n} \coloneqq (u_{1n} + \frac{d}{2}k_{12}, v_{1n} + \frac{d}{2}h_{12}, u_{2n} + \frac{d}{2}k_{22}, v_{2n} + \frac{d}{2}h_{22})$$

$$\begin{cases} k_{i3} \coloneqq f_i(t_n + \frac{d}{2}, \vec{z}_{3n}) \\ h_{i3} \coloneqq g_i(t_n + \frac{d}{2}, \vec{z}_{3n}) \end{cases}, i \in \{1,2\}$$

$$\vec{z}_{4n} \coloneqq (u_{1n} + k_{13}, v_{1n} + h_{13}, u_{2n} + k_{23}, v_{2n} + h_{23})$$

$$\begin{cases} k_{i4} \coloneqq f_i(t_n + d, \vec{z}_{4n}) \\ h_{i4} \coloneqq g_i(t_n + d, \vec{z}_{4n}) \end{cases}, i \in \{1,2\}$$

$$\begin{cases} t_{n+1} = t_n + d \\ u_{i(n+1)} = u_{in} + \dfrac{d(k_{i1} + 2k_{i2} + 2k_{i3} + k_{i4})}{6} \\ v_{i(n+1)} = v_{in} + \dfrac{d(h_{i1} + 2h_{i2} + 2h_{i3} + h_{i4})}{6} \end{cases}, i \in \{1,2\}$$

These are the equations that are used in the implementation. As the Runge-Kutta method is structurally the same for all universes, it is implemented in the abstract universe class. In contrast, the functions $f_{1,2}, g_{1,2}$ depend on the power law of the universe and are implemented in the specific universe.

After implementation, it was found that the Runge-Kutta method unfortunately delivers very imprecise trajectories, which are only reasonably plausible for very small increments.

There are certain effects in the "simulator", that have not yet been sufficiently clarified, especially when stars collide or come very close to each other.

1.6 Choice of Coordinate System

The zero point of the coordinate system should coincide with the common centre of gravity of all masses. The common centre of gravity is recalculated when a new body is placed. When this placement is complete, all bodies are repositioned so that the centre of gravity coincides with the zero point$(0,0)$ again. If n masses are placed, then the centre of gravity is relative to their centre of gravity:

$$\sum_{i=1}^{n} m_i \vec{r}_i = \vec{0}$$

Now the mass n+1 is added. The common centre of gravity then shifts by a vector $\vec{c}$. The following then applies to the new centre of gravity:

$$\sum_{i=1}^{n+1} m_i (\vec{r}_i + \vec{c}) = \vec{0}$$

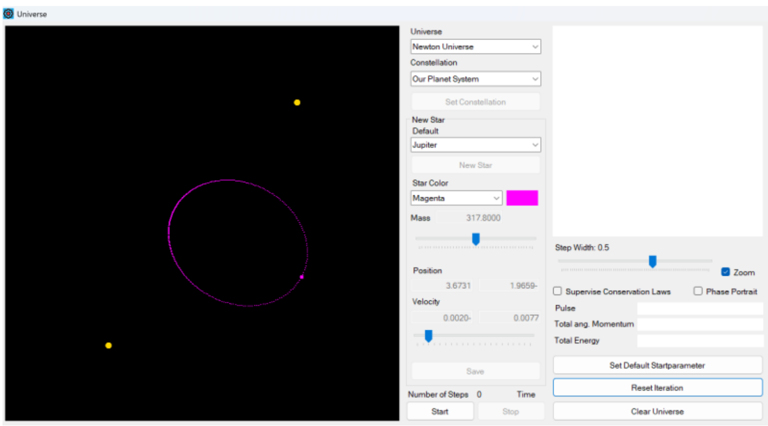
So it is:

$$\vec{c} = -\frac{m_{n+1}\vec{r}_{n+1}}{\sum_{i=1}^{n+1} m_i}$$

In reality, the direction of the x-axis is determined by the intersection of the ecliptic with the Earth's equatorial plane at the vernal equinox. In the simulator, the zero point of the coordinate system is the centre of the display diagram and the x-axis points to the right as usual.

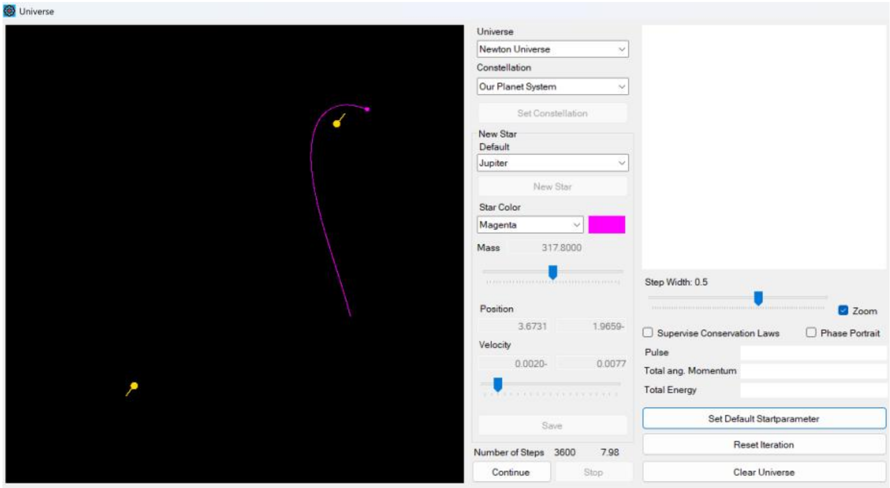
This choice of coordinate system is valid for all universes, as they differ only in the law of force.

In the "Simulator", this causes the entire image to be shifted after a new star has been placed so that the common centre of gravity is at the origin. The coordinates of all previous stars then refer to this new centre of gravity. When a new star is added, its standard coordinates are converted to the new coordinate system during placement.



Two suns and a Jupiter were placed here

In the image above, the coordinates of the suns and Jupiter shown refer to the common centre of gravity. As the suns are at rest, no expected orbit relative to the common shear point is calculated for them. However, this is the case for Jupiter.



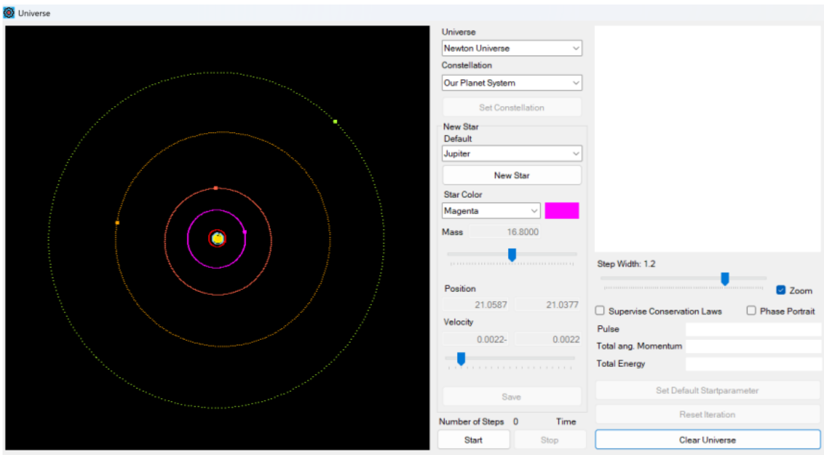
The orbits after the start of the iteration

1.7 Changing the star mass

When placing a new body, its mass can also be adjusted. This can vary between $3.3010 \cdot 10^{23}$ or 0.055 Earth masses (Mercury) and $1.9884 \cdot 10^{30}$ or 332942.6 Earth masses (Sun). The mass can be entered manually or via a shift register. The shift register only adjusts the mass relative to the initial mass by a factor that varies between 1 and 100. This means that Mercury can no longer be reduced in size, but all orders of magnitude between Mercury and 100 times the size of the sun can be represented. It is assumed that a body of the desired minimum size is already selected by default. Changing a mass results in the recalculation of the common centre of gravity.

The size of the mass is indicated in the diagram by the diameter. The number of pixels for the representation is at least 1 and otherwise the exponent of the mass minus 22. 1 pixel stands for the magnitude10^{23} (Mercury, Mars). For the magnitude10^{24} (Earth, Venus) 2 pixels. Jupiter with the magnitude10^{27} is then represented by 5 pixels and the sun by 8 pixels.

The permissible definition range of mass depends on the universe and is defined individually for each universe.



Representation of our planetary system. The inner planets are barely visible.

1.8 Representation of the Universe in the "simulator"

An universe in the "simulator" is defined by the law of motion, for example by Newton's law of gravity. Alternative universes are dealt with in a separate section. A universe contains n bodies, which are generally referred to as "stars" in the "simulator", even if they are planets. However, when it comes to the real planets, this term is also used.

By default, certain stars are set as the default, in particular the components of our planetary system. This is the sun including the associated planets. Such a system, for example consisting of the sun plus a selection of planets, is a *constellation*. Different constellations are available for each universe in the "Simulator". In the example of Newton's universe, this includes our planetary system or the system of inner planets. There are also variants with "disturbances", for example when Jupiter is repositioned so that it comes close to the inner planets. Once you have selected a constellation, you can also place the planets or the sun individually and then move them with the mouse or change their mass and speed.

In the "simulator", the area in which the bodies are placed is a square with a side length that is defined for each universe. The zero point of the coordinate system lies at the centre of this square and is the common centre of gravity of the bodies in the universe.

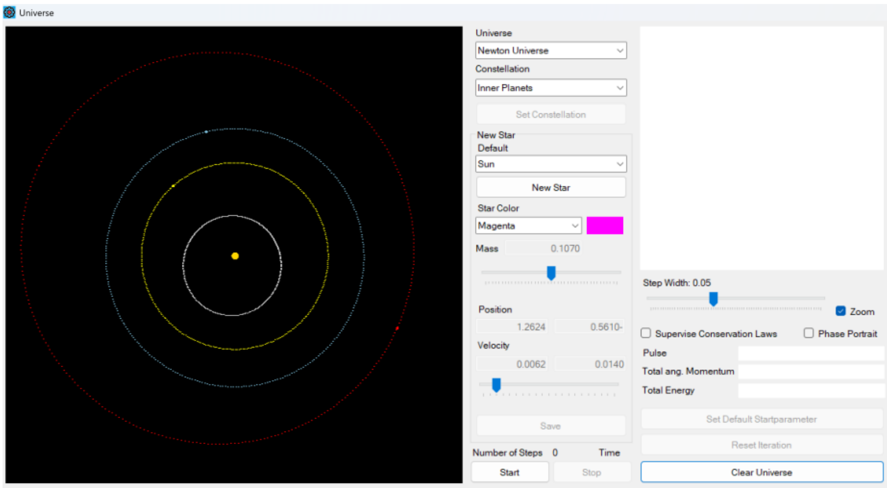
Zoom: If "Zoom" is activated, when new planets or stars are displayed, their position coordinates are converted into mathematical coordinates so that they fit perfectly into the square.

Example

If only the inner planets are considered, then Mars is the outermost planet with a distance of $\sim 1.524\ AU$ from the Sun. However, its aphelion is $1.666\ AU$. The scaling (in the implementation of the "zoom") is then selected so that $1.666\ AU$ just corresponds to the number 30. All planetary orbits are then clearly visible. For example, the perihelion of Mercury is $0.3075\ AU$. This corresponds to the number 5,353 or about 53 pixels.

If the outer planets are also to be shown, then the aphelion of Neptune is $30.069\ AU$. The scaling is selected so that this value corresponds to the number 29 in mathematical coordinates. The perihelion of Mercury is then still 0.297 in mathematical coordinates, which corresponds to about 3 pixels. It is then no longer visible because the mass of the sun is represented by 8 pixels.

After placing a star, the new common centre of gravity is determined and then all stars are placed relative to this centre of gravity. This means that the current start position then refers to this centre of gravity. When placing a new star, its original starting position, which is saved as the star's default, must also be converted to the new coordinate system.



Representation of the inner planets

The movement of the stars in the "simulator" is usually counterclockwise. In the case of planetary motion, it should be noted that the angular momentum vector of a planetary orbit is parallel to the angular momentum vector of the Earth's rotation. If you look at the Earth "from above", i.e. from the North Pole, then the Earth rotates in an anticlockwise direction. This is the usual positive direction of rotation in mathematical angle measurement. The planets in the simulator therefore also rotate in an anticlockwise direction. This direction of rotation is maintained for alternative universes.

1.9 Representation of the orbit sketch when placing a star

As a placement aid for new planets, their orbit is sketched dynamically in the diagram, as if it were a two-body system with the common mass of all existing bodies in the common centre of gravity. The prerequisite for this is that the newly placed planet or star has a velocity of $v > 0$. It is assumed in the calculation that this is the speed at perihelion.

Here we carry out the necessary calculations that the "simulator" must perform. The following is:

M is the sum of all existing masses. m is the mass of the newly added body.

The following applies to the speed v at perihelion ϱ:

$$v = \varrho\dot{\vartheta}$$

This means that the angular momentum of the newly added mass

$$L = m\varrho^2\dot{\vartheta} = m\varrho v$$

For their energy you have:

$$E = \frac{1}{2}mv^2 - G\frac{M^3 m}{(M+m)^2}\cdot\frac{1}{\varrho} \approx \frac{1}{2}mv^2 - G\frac{Mm}{\varrho}$$

$\approx$ applies if $M \gg m$

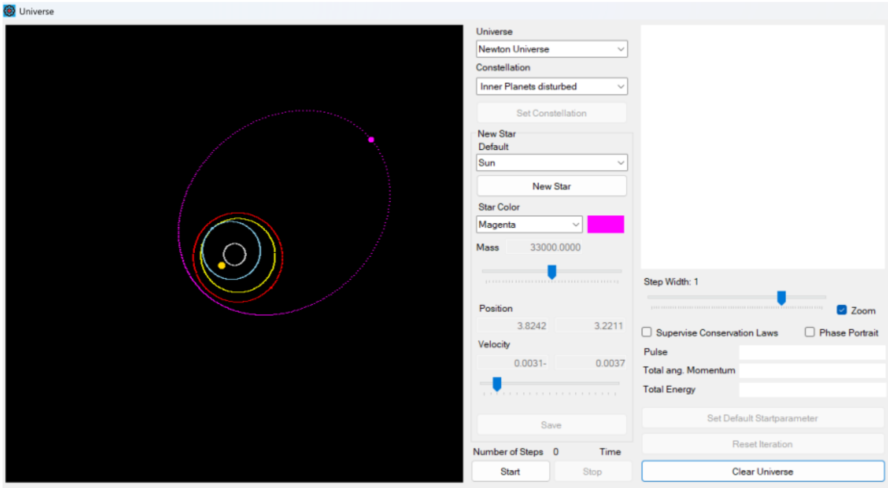
The path parameters are now:

$$p = \frac{(M+m)^2\varrho^2 v^2}{GM^3} \approx \frac{\varrho^2 v^2}{GM}$$

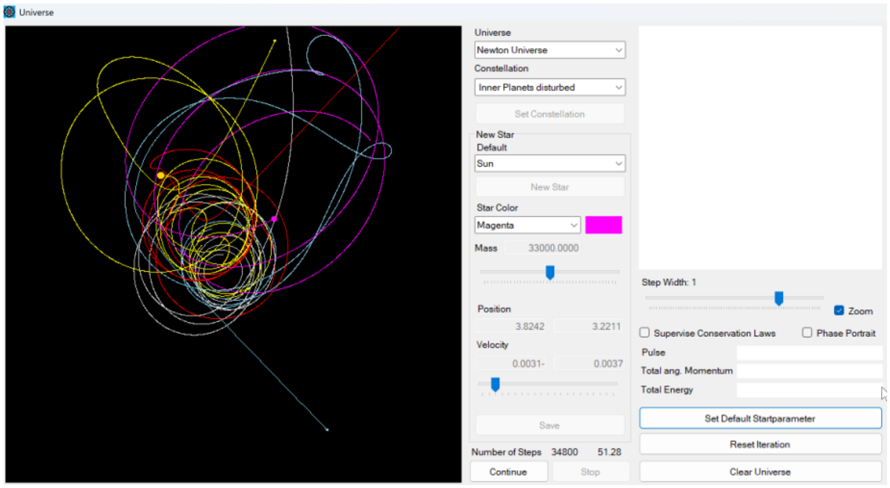
$$\varepsilon^2 - 1 = \frac{2EL^2(M+m)^4}{G^2 m^3 M^6} = \frac{2Em^2\varrho^2 v^2(M+m)^4}{G^2 m^3 M^6} \approx \frac{2E\varrho^2 v^2}{G^2 m M^2}$$

The path can then be drawn considering the argument of the perihelion. This will not be the actual path but serves as a guide.

For alternative universes, orbit calculation is not possible if the two-body problem cannot be solved analytically.



The inner planets together with a disturbed Jupiter. The zero point is the common centre of gravity and not the sun



The disturbed Jupiter has just sent Mars into the afterlife and disturbed the Earth's orbit

1.10 The collision problem

If two bodies come closer to each other than two pixels, which is 0.1 in mathematical units or, depending on the scaling and zoom, a correspondingly large value in astronomical units, then this could be regarded as a plastic collision in the simulation.

This was implemented in a version of the "Simulator": The simulation is not stopped, but we replace the two colliding bodies with a body with the sum of both masses. The velocity vector of the new body is then set so that the momentum remains constant. This does not correspond to a physical reality, but only serves the continuation of the simulation. If $m_{1,2}, \vec{v}_{1,2}$ are the parameters of the bodies before the impact and m, v are the parameters of the new body after the impact, the following applies:

$$m = m_1 + m_2$$

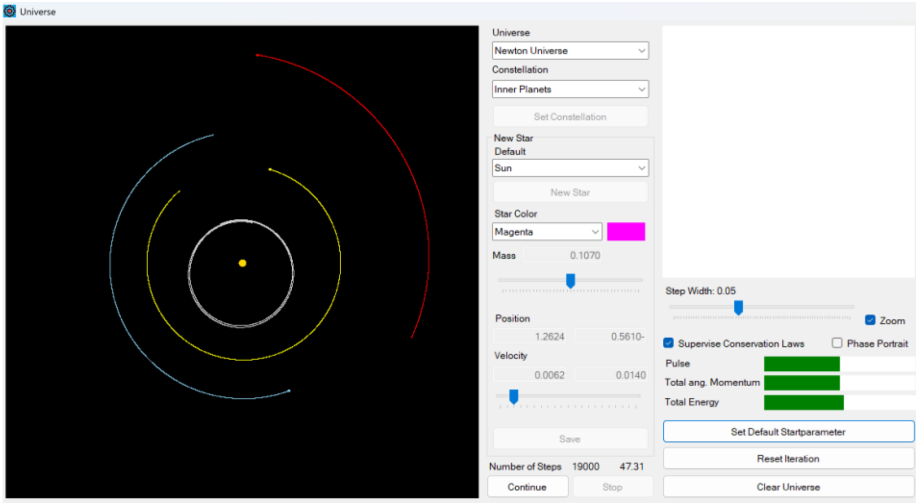
$$\vec{v} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m}$$

As you can easily calculate, the angular momentum of the system remains the same, but not the total energy.

However, after experiments with collisions did not have a satisfactory effect, this implementation was removed again. Now it is ensured that bodies "narrowly miss" each other if they come too close.

1.11 Conservation laws and phase diagram

During the simulation, the total momentum, the total angular momentum and the total energy of the system are continuously calculated and compared with the corresponding values at the start of the simulation. A condition for alternative universes is that these conservation laws also apply there. In the user window, these values are displayed in bars to the right below the phase portrait.



Movement of the inner planets around the sun and monitoring the laws of conservation

However, due to the imprecision of the Runge Kutta-method, deviations can always occur here.

You can also activate the representation of the movement in a phase diagram. In this case, the distance of the star from the common centre of gravity is entered for each star in the horizontal direction and the amount of its velocity relative to this in the vertical direction.

1.12 The Newtonian Universe

The aim is to be able to calculate internally in the "Simulator" with the variable type "decimal". For the Newtonian universe, this means that the spatial distances are managed internally in astronomical units AU . The mass is specified for the user in kg , but is managed internally in the number of earth masses ME. The speed of bodies is specified for the user in km/h . Internally, we keep it in AU/day . The gravitational constant must be adjusted accordingly when calculating the force. The following applies:

$$G = 6.6743 \cdot 10^{-11} \frac{m^3}{kg \cdot s^2}$$

And the simulator uses:

$$G_S = 6.6743 \cdot 10^{-11} \cdot \frac{5.9722 \cdot 10^{24} \cdot (3600 \cdot 24)^2}{149.6^3 \cdot 10^{27}} = 8.88736 \cdot 10^{-10} \frac{AU^3}{ME \cdot day^2}$$

As an example, we can perform a small check by calculating the speed of the earth around the sun from the equilibrium condition centrifugal force = gravitational force, first in the m.kg.s system and then in the AU.ME.day system. The equilibrium condition is

$$m\frac{v^2}{r} = G\frac{Mm}{r^2}$$

Or

$$v^2 = \frac{GM}{r}$$

Where M is the mass of the sun and r is the distance between the earth and the sun.

In the m.kg.s system the equation is:

$$v^2 = 6.6743 \cdot 10^{-11} \frac{1.9884 \cdot 10^{30}}{1.496 \cdot 10^{11}} = 8.8711 \cdot 10^8$$

$$v = 2.9784 \cdot 10^4 \; m/s$$

If we convert this into AU/day , the result is

$$v = 2.9784 \cdot 10^4 \cdot \frac{86400}{1.496 \cdot 10^{11}} = 0.0172 \; AU/day$$

If we calculate in the AU.ME.day system right from the start, we have:

$$v^2 = 8.88736 \cdot 10^{-10} . \frac{332942.6}{1} = 2.959 \cdot 10^{-4}$$

$$v = 0.0172 \; AU/day$$

If the unit of velocity is chosen too high, e.g. AU/year, then the step size of the Runge Kutta method must be chosen very low so that the method leads to a reasonably plausible-looking orbit. This makes the movement of the planets very slow. If, on the other hand,$v \ll |\vec{r}|$, e.g. as here, then the Runge Kutta method is reasonably plausible even with larger step sizes and at the same time the speed of the planets is acceptable. We have not investigated this effect further here. However, the implementation includes a factor τ , which can be used as a factor for changing the unit of velocity. If the speed is adjusted by changing the unit of time: $v' = \tau v$, the gravitational constant in the law of motion must be corrected accordingly: $G' = \tau^2 G$, since time appears in the gravitational constant squared. This factor can be adjusted individually for each universe. Based on experiments, the Newtonian universe, $\tau = 1$

1.13 Alternative Universes

An alternative universe should only differ from Newton's universe in terms of the law of force. Essential properties of Newton's universe should be retained. These include the conservation laws and the independence of the law of motion from the choice of coordinate system.

To this end, we assume that a force law of form also applies in alternative universes between two masses m_1, m_2 and the positions $\vec{r}_1, \vec{r}_2$:

$$\begin{cases} m_1\ddot{\vec{r}}_1 = F(|\vec{r}|) \cdot \frac{\vec{r}}{|\vec{r}|} \\ m_2\ddot{\vec{r}}_2 = -F(|\vec{r}|) \cdot \frac{\vec{r}}{|\vec{r}|} \end{cases}$$

$\vec{r} = \vec{r}_2 - \vec{r}_1$.

Newton's three laws therefore apply:

a) If no force acts on a body, it moves at a constant speed
b) *Force = mass x acceleration*
c) Force = - counterforce

The *momentum theorem* follows from this.

Furthermore, the magnitude of the force depends only on the distance between the masses. This means that the equations of motion are *independent of the choice of zero point.*

As the direction of the force vector is parallel to $\vec{r}$, *the angular momentum theorem* follows and therefore the movement of the masses runs *in a plane*. It also follows that the equations of motion are *invariant to rotations of the coordinate system.*

We also assume that the force can be derived from a potential function. This means that there is a function $\Phi: \mathbb{R}^3 \to \mathbb{R}, \vec{r} \longmapsto \Phi(\vec{r})$ such

$$\vec{F} = -\nabla\Phi \coloneqq \begin{pmatrix} -\dfrac{\partial\Phi(\vec{r})}{\partial x} \\ -\dfrac{\partial\Phi(\vec{r})}{\partial y} \\ -\dfrac{\partial\Phi(\vec{r})}{\partial z} \end{pmatrix}, \vec{r} = (x, y, z)$$

The energy theorem follows from the last condition.

We leave the proof of these statements as an exercise.

The *normalized universe* is implemented in the simulator. Here, the law of force is identical to Newton's universe, but the units are normalized more conveniently. The following applies to the implementation:

The masses are in the range $m_i \in [0.5,2], \forall i$ and their positions in the range $|\vec{r}_i| \leq 2$. The gravitational constant is $G = 1$ and the velocities are in the range $\left|\dot{\vec{r}}_i\right| \leq 1$

The parameters for calculating the orbit are the same as in Newton's universe, with the only difference that the gravitational constant is $G = 1$. For N bodies, we sketch the orbit of a mass as a two-body problem, namely as the movement of the mass relative to the common centre of gravity, with all other masses united at one point. The derivation is therefore the same as for the two-body problem in Newton's universe:

$$L = m_1|\vec{r}_1|\left|\dot{\vec{r}}_1\right|$$

$$E = \frac{1}{2}m_1\left|\dot{\vec{r}}_1\right|^2 - \frac{M^3 m_1}{(m_1 + M)^2} \cdot \frac{1}{|\vec{r}_1|}$$

If $M =$ mass of all other stars.

This reintroduces the parameters:

$$p \coloneqq \frac{(m_1 + M)^2 L^2}{m_1^2 M^3}$$

$$\varepsilon^2 \coloneqq \frac{2EL^2(m_1 + M)^4}{m_1^3 M^6} + 1$$

And obtain the equation for the trajectory in polar coordinates as in the section on the two-body problem:

$$\varrho(\vartheta) = \frac{p}{1 + \varepsilon\cos\,(\vartheta - \alpha)}$$

For universes that do not obey Newton's law of gravity, the trajectory cannot generally be calculated.

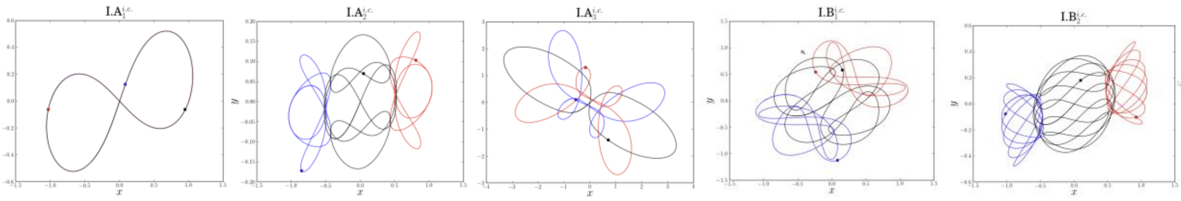
1.14 Periodically stable tracks

An exciting topic in the (normalized) universe is the question of periodically stable orbits, especially in the three-body problem. The first families of such periodic orbits were found by Leonhard Euler (1707 - 1783) and later by Joseph Louis Lagrange (1736 - 1813).

Examples can be found on the Internet under the links:

https://numericaltank.sjtu.edu.cn/three-body/three-body.htm

https://numericaltank.sjtu.edu.cn/three-body/three-body-movies.htm



Examples of periodic stable orbits on the Internet

Further explanations can also be found in the links above. Some of these scenarios are saved as constellations in the "Simulator". In particular, the case of three stars with the same mass $m_1 = m_2 = m_3 = 1$ and the initial positions $\vec{r}_1 = (-1{,}0), \vec{r}_2 = (1{,}0), \vec{r}_3 = (0{,}0)$

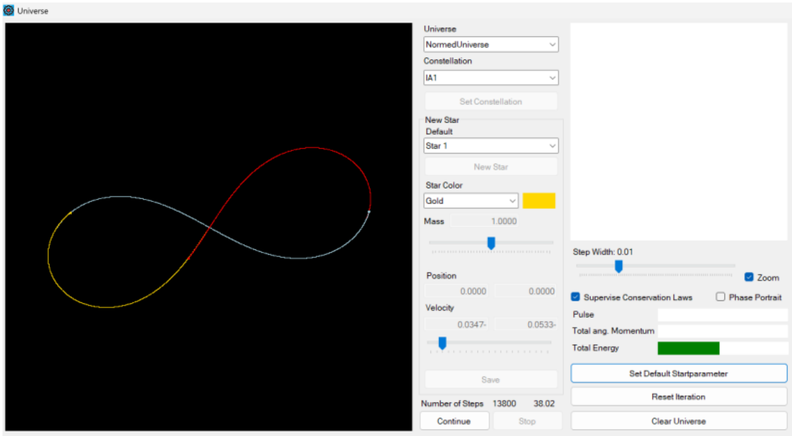
The initial velocities are: $\vec{v}_1 = (c_1, c_2) = \vec{v}_2, \vec{v}_3 = (-2c_1, -2c_2)$ where the values of $c_{1,2}$ can be found in the following table. Note that the total momentum is zero at the start.

Several examples with a low period are implemented, whereby the Runge Kutta method clearly reaches its limits:

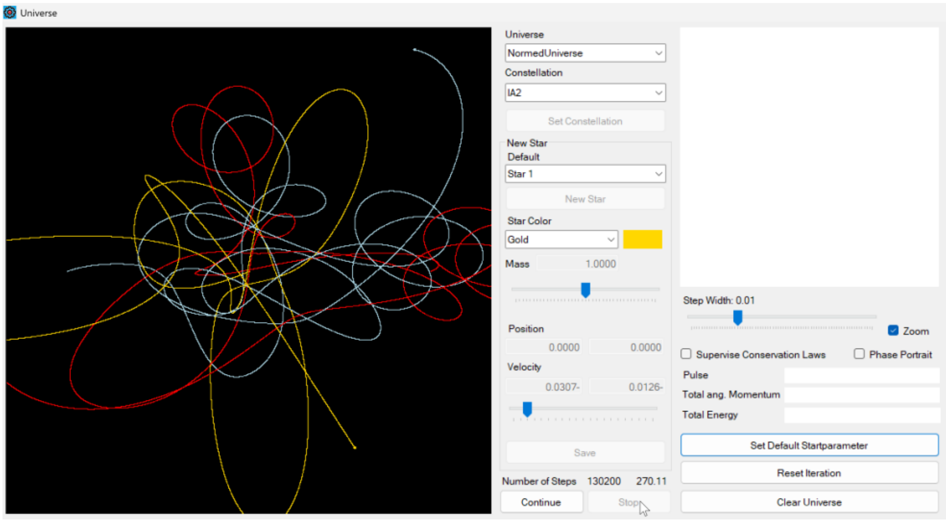
Constellation	C_1	C_2
IA1	0.3471168881	0.5327249454
IA2	0.3068934205	0.1255065670
IA3	0.6150407229	0.5226158545
IB1	0.4644451728	0.3960600146
IB2	0.4059155671	0.2301631260
IIC1	0.2827020949	0.3272089716

Hundreds of thousands of such orbits have been found in recent decades, even with stars of different masses.

In the normalized universe, the factor for the unit of time is set as $.\tau = 0.1$



The periodically stable path IA1



The Runge Kutta method is not precise enough for the representation of path IA2

1.15 Exercise examples

1. Two-body problem: Calculate the path of the mass m_1 relative to the position of the mass m_2 and compare the result with the calculation of the path relative to the centre of gravity.

2. Two body problem: Calculate the path of the mass m_2 relative to the center of mass.

3. From the polar form of the conic section equation, determine its equation in Cartesian coordinates.

4. The gravitational constant is known. The most important data from this system are also given in the chapter "Information on our solar system".

   a) Calculate the orbital period *T* of the earth around the sun from this information.

   b) Calculate the angular momentum of the Earth-Sun system.

   c) Calculate the explicit polar equation of the Earth's orbit around the Sun.

   d) The orbital period of Mars around the sun is 687 Earth days. Use this information to determine the major semi-axis of Mars' orbit.

5. Determine the total energy for a system with N bodies. Show that the derivative of the energy with respect to time is zero because the summands belonging to two bodies cancel to zero in pairs.

6. Investigate symmetrical starting positions in the "simulator". This means that the bodies under consideration all have the same mass and their starting position and starting speed are rotationally symmetrical to the zero point.

7. Determine the equation of Jupiter's orbit around the sun in astronomical units.

8. A body lies on an ellipse with a major axis $a = 2$ and minor axis $b = 1$. At the start, the body is in the position $\vec{r}_0 = \binom{0.5176}{1.9318}$ and $\vartheta_0 = 1.309$. Determine the path equation of the ellipse.

9. Determine the orbital equation of Jupiter in AU. Use it to calculate Jupiter's aphelion and perihelion. Compare the result with data from the internet. Note that the result is only an approximation.

10. If the earth were moving in a circular path around the sun, gravitational force and centrifugal force would just balance each other out. What would then be the speed of the Earth in km/h? Compare the result with the speed given in the table about the solar system.

11. Prove the statements in the section on alternative universes.

12. Search for further examples of stable periodic orbits on the Internet and implement them as corresponding constellations in the normalized universe.

Further reading

[1] Orbital Motion, Archie E. Roy, Taylor & Francis Group, 2005

[2] Celestial Mechanics, John M.A. Danby, Willmann-Bell Inc., 1992