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MSc Banking & Finance

Structured Products and Derivatives

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Contents

Chapter 1

Introduction

1.1 What are structured products?

There is no unified definition of the notion of a "structured product". In this course, we mean by a structured product a financial instrument which combines different underlyings such as stocks, indices, rates, commodities and so on with a derivative strategy. By combining an underlying with a derivative or by combining different derivatives one tries to generate payoffs and risk profiles which are superior to classical investments. The first structured product in Switzerland was issued by Schweizerischer Bankverein in 1991. It was named "GROI" (Guaranteed Return On Investment) and was a capital protected product.

The advantages of structured products are

+ almost every expectation on the market can be mapped by structured products (adjustment to the risk-return profile of the investor)
+ access to a broad range of investment possibilities
+ fast adaption of issued products to changing markets
+ clearly defined redemption at maturity for all scenarios
+ the fair value of the product can be calculated at any time (at least in principle), i.e., the price is transparent
+ independent of the abilities of a portfolio manager

whereas disadvantages are

- the complexity of the product places high demands on investors and consultants

- often unsatisfactory transparency with respect to the costs, profits are not declared clearly

- default risk of the issuer

1.2 Categories of structured products

There are four main categories of structured products, compare with table 1.1.

Category	Description	Number of tradeable products September 2019	Turn over in Mio. CHF 2018
Leverage	leverage effect in comparison to holding the underlying	20401	6170
Participation	participation in price increase of underlying, possibly > 1, possible conditional capital protection	1657	5203
Yield-enhancement	cheaper than underlying, but with profit cap	12787	3396
Capital protection	unconditional protection against dropping prices of the underlying	556	152

Table 1.1: Structured products can be divided into four categories. The data is from `http://www.six-structured-products.com/de/aktuelles/marktreport`

The "Swiss Structured Product Association" (SSPA) annually publishes the "Swiss Derivative Map", which records all products quoted at "SIX Structured Products"[1], and attaches them to the above four mentioned main groups (and their subcategories), see the next page.

[1] Exchange for structured products, see `http://www.six-structured-products.com/en/latest-news/marktuebersicht`

1.3 Day-count convention

Later on, we will price structured products. To do so, we need an agreement on how to measure time. By convention, we measure time in units of years. But if t_1 and $t_2 > t_1$ denote two dates expressed as day/month/year, it is not clear what $t_2 - t_1$ should be. Different conventions how to count the number of days between t_1 and t_2 and how to express this number as a year fraction exist. In this text, we use the "30/360 European"-rule which is defined as follows. If $t_1 = d_1/m_1/y_1$ and $t_2 = d_2/m_2/y_2$ are two dates, then the year fraction between these dates is defined as

$$\frac{\min\{d_2, 30\} - \min\{d_1, 30\}}{360} + \frac{m_2 - m_1}{12} + y_2 - y_1 \,. \tag{1.1}$$

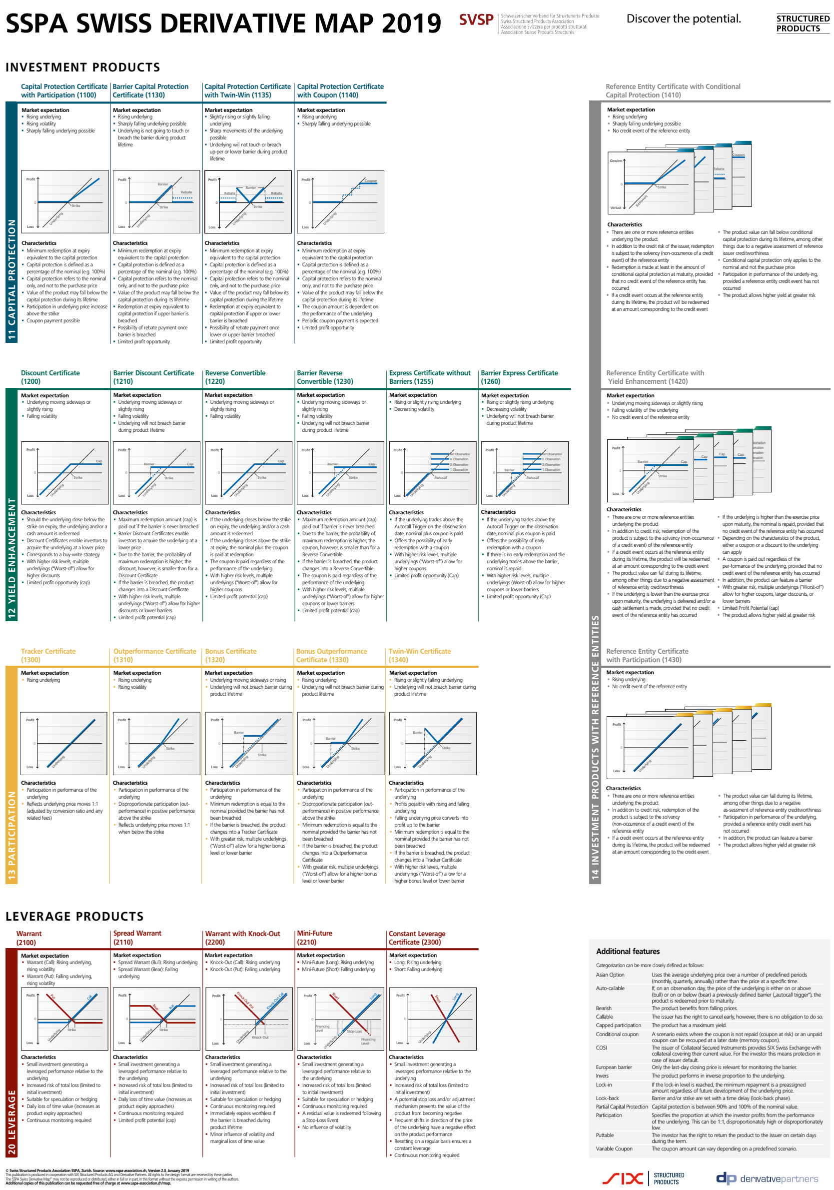

Figure 1.1: The “Swiss Derivative Map”.

For example, according to this definition, the are

$$\frac{\min\{30,30\}-\min\{12,30\}}{360}+\frac{8-1}{12}+2018-2015=3.6\overline{3}$$

years (or 1308 days) between $t_1 = 12/1/2015$ and $t_2 = 30/8/2018$. We implement formula (1.1) in the function `yf.py`[2], see figure 1.2 for an example.

[2] Alternatively, you may download and install the Python package `FinDates`, which also realises the above formula (and much more).

```
In [1]: from yf import yf

In [2]: yf([(12,1,2015),[(30,8,2018)])
Out[2]: array([3.63333333])
```

Figure 1.2: The Python function `yf` realises the “30/360 European” day-count convention.

1.4 Combinations

Structured products are “portfolios” composed of different “ingredients” such as options, stocks, bonds or cash. We call such portfolios “combinations” and assume for simplicity that the involved options are of European style. Combinations can be analysed either graphically or tabularly. We stipulate the following conventions.

- In graphs, we plot the *profit* instead of the payoff of the product.
- The yield of the invested capital as well as transaction costs are neglected.

As an example, we implement at September 11, 2019, the following outperformance product written on U-Blox which expires on December 20, 2019[3]

[3] By $V_c(s,t;X)$ we denote the price or value of a call option with strike X at time t written on a underlying with price $s = S_t$ at time t.

1. buy (long) one share of underlying with price $s = S_0 = 77.45$
2. buy (long) one call option on the underlying with strike $X_1 = S_0$ and price $V_c(s,0;X_1) = 5.50$
3. sell (short) two call options on the underlying each with strike $X_2 = 1.2S_0$ and price $V_c(s,0;X_2) = 1.20$

The initial value V_0 (the price the investor has to pay) of the product is

$$V_0 = S_0 + \underbrace{V_c(S_0,0;X_1) - 2V_c(S_0,0;X_2)}_{=:V_0^O} = 80.55\,.$$

Here, we denote by $V_0^O := V_c(S_0,0;X_1) - 2V_c(S_0,0;X_2) = 3.10$ the price of the option strategy the investor has to pay.

The value

$$V_T = g(S_T)$$

of the product at maturity will solely depend on the price S_T of the underlying at maturity[4]. The payoff function of the product is

[4] We generally denote by g the payoff function of any product.

$$\begin{aligned} g(x) &= x + V_c(x,T;X_1) - 2V_c(x,T;X_2) \\ &= x + \max\{x - X_1, 0\} - 2\max\{x - X_2, 0\} \,. \end{aligned}$$

We determine the profit

$$\begin{aligned} P(S_T) &= g(S_T) - V_0 \\ &= S_T + \max\{S_T - X_1, 0\} - 2\max\{S_T - X_2, 0\} - V_0 \quad (1.2) \end{aligned}$$

in dependence on S_T. To do so, we have to consider three scenarios.

- If $S_T \leq X_1$, all options expire worthless and our profit is

$$P(S_T) \overset{(1.2)}{=} S_T + 0 - 2 \cdot 0 - V_0 = S_T - S_0 - V_0^O$$

- If $X_1 < S_T < X_2$, the calls with strike X_2 expire worthless ($V_c(S_T, T; X_2) = 0$) and the profit is

$$\begin{aligned} P(S_T) &\overset{(1.2)}{=} S_T + (S_T - X_1) - 2 \cdot 0 - V_0 = 2S_T - X_1 - V_0 \\ &= 2S_T - S_0 - (S_0 + V_0^O) = 2(S_T - S_0) - V_0^O \end{aligned}$$

- If $S_T \geq X_2$, all options are in the money and the investor realises the profit

$$\begin{aligned} P(S_T) &\overset{(1.2)}{=} S_T + (S_T - X_1) - 2(S_T - X_2) - V_0 \\ &= 2X_2 - X_1 - V_0 = 2 \cdot 1.2S_0 - S_0 - (S_0 + V_0^O) \\ &= 0.4S_0 - V_0^O =: P_{\max} = 27.88 \end{aligned}$$

From the above considerations it follows that the break even[5] B of the product is

$$B = \frac{V_0^O}{2} + S_0 = 79 \,.$$

Thus, the investor realises a loss whenever $S_T < B$ where the maximal loss is $P(0) = -V_0 = -80.55$. If $S_T > B$, the investor makes a profit which is, however, capped by the maximal profit $P_{\max} = 27.88$. In figure 1.3 we plot the profit as a function of S_T. Besides the graphical representation (figure 1.3) of the combination we may analyse it using a table, compare with the panel in table 1.2. Substituting there the numbers $S_0 = X_1 = 77.45$, $X_2 = 1.2S_0$ and $V_0^O = 3.10$ we obtain the profit function as shown in figure 1.3.

[5] Mathematically, the break even is defined as the root of the profit, i.e., $P(B) = 0$.

Example 1.1. We again consider the above outperformance product.

i) Suppose that at maturity the price of U-Blox is 90. What is the (holding period) return of the product?

ii) At which closing price(s) of the underlying the profit of the strategy "long underlying" is equal to the profit of the outperformance product?

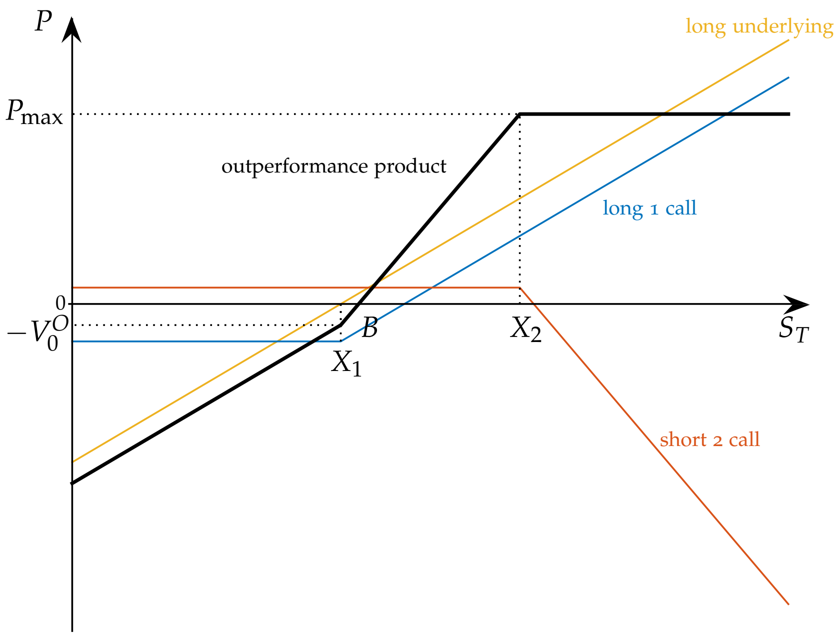

Figure 1.3: The profit and loss of an outperformance product

Portfolio	Value at $t = 0$	Value at maturity $t = T$ $S_T \leq X_1$	$X_1 < S_T < X_2$	$S_T \geq X_2$
Long underlying	S_0	S_T	S_T	S_T
Long 1 call with strike X_1	$V_c(S_0, 0; X_1)$	0	$S_T - X_1$	$S_T - X_1$
Short 2 call with strike X_2	$-2V_c(S_0, 0; X_2)$	0	0	$-2(S_T - X_2)$
Total V_t	$S_0 + V_0^O$	S_T	$2S_T - X_1$	$2X_2 - X_1$
Profit $P = V_T - V_0$		$S_T - S_0 - V_0^O$	$2(S_T - S_0) - V_0^O$	$0.4S_0 - V_0^O$

Table 1.2: Panel of the above outperformance product. Again, V_0^O denotes the price of the option strategy the investor has to pay.

iii) The issuer of the outperformance product would like to offer the product to the same price as the underlying (due to transparency reasons), i.e., the issue price of the product is equal to the market price of the underlying. What consequences result from this?

Solution.

i) Since $X_1 < S_T < X_2$, the return R is by (1.2)

$$R = \frac{P(90)}{V_0} \overset{(1.2)}{=} \frac{2(90 - 77.45) - 3.1}{80.55} \doteq 27.3\% \,.$$

ii) The profit of the strategy "long underlying" is $S_T - S_0$; the profit $P(S_T)$ of the outperformance product is given in (1.2). From figure 1.3 it becomes clear that the equation $P(S_T) = S_T - S_0$ admits two solutions, one in the interval $]X_1, X_2[$, the other one in $[X_2, \infty[$. Whence we calculate, for $X_1 < S_T < X_2$,

$$\begin{aligned} P(S_T) &= S_T - S_0 \\ 2(S_T - S_0) - V_0^O &= S_T - S_0 \end{aligned}$$

from where it follows that $S_T = S_0 + V_0^O = V_0 = 80.55$. Second, for $S_T \geq X_2$ we have

$$\begin{aligned} P(S_T) &= S_T - S_0 \\ 0.4S_0 - V_0^O &= S_T - S_0 \end{aligned}$$

which yields $S_T = 1.4S_0 - V_0^O = 105.33$. Thus, if U-Blox closes in the interval $]80.55, 105.33[$, then the outperformance product realises a higher profit than the investment into the underlying.

iii) The issue price of the product is $V_0 = S_0 + V_c(S_0, 0; X_1) - 2V_c(S_0, 0; X_2)$; it must be equal to S_0. But the equation $V_0 = S_0$ is equivalent to the equation $V_c(S_0, 0; X_1) - 2V_c(S_0, 0; X_2) = 0$, which itself is equivalent to

$$V_c(S_0, 0; X_2) = \frac{1}{2} V_c(S_0, 0; X_1) = 2.75 \, .$$

The price $V_c(S_0, 0; X)$ of a call option depends on the quantities S_0, X, σ, r, T; thus, the issuer can not freely choose the strike X_2; as a consequence, the maximal profit P_{max} is also not freely choosable by the issuer. To find the strike X_2, we need a model/formula for the price $V_c(S_0, 0; X)$ in dependence on the strike X (and, of course, on S_0, σ, r, T), such that we can solve (at least numerically) the equation $V_c(S_0, 0; X) = 2.75$ for X. The problem here is that while S_0, r, T are directly observable, σ is not. Even worse, market data suggests that σ depends on X (and T). Thus, at this stage, we are not able to find the strike X_2. In chapter 3 we will come back to this problem. ◇

1.5 A case study

We consider the hypothetical product written on Roche defined in table 1.3. At the redemption date, the product pays off the difference between the realised maximum and the realised minimum of the underlying (Roche) over a predefined observation period.

Assume that you purchase this product at issuance and that you hold it until maturity. The goal of this case study is to answer the following question:

"How much you are willing to pay at issuance for this product?"

Issuer	Eternity Investments EI
Underlying	Roche ROGN
Initial Fixing IF	CHF 286.10
Initial Fixing Date	October 4, 2019
Final Fixing Date	October 2, 2020
Redemption Date	October 9, 2020
Redemption	The Investor is entitled to receive from the Issuer on the Redemption Date per Product: The Maximum of the Underlying minus the Minimum of the Underlying
Maximum	The realised maximum of the official daily close of the Underlying during the observation period.
Minimum	The realised minimum of the official daily close of the Underlying during the observation period.
Observation period	Each business day from October 4, 2019 to Final Fixing Date.
Calculation Examples	Minimum / Maximum / Redemption: 270 / 340 / 70; 250 / 300 / 50; 220 / 300 / 80

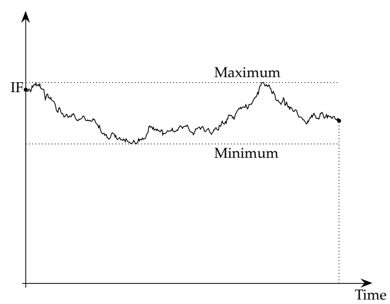

Table 1.3: An example of a structured product.

Here are a few observations. At maturity, the investor is entitled to receive the cash flow

$$g(S_{t_0}, \ldots, S_{t_J}) := \max\{S_{t_0}, \ldots, S_{t_J}\} - \min\{S_{t_0}, \ldots, S_{t_J}\} ,$$

where S_t denotes the price of the underlying at time t (with $S_0 = 286.10$) and $t_0, t_1, \ldots, t_J$ indicate the observation dates. The future time evolution of the underlying (its path) is not known, and so isn't the cash flow the investor receives. For example, the path of the underlying shown in table 1.3 is just one of many possible scenarios, each of these scenarios generates a cash flow at maturity. By the principle of financial equivalence, we need to discount the cash flow generated by a scenario to the date of issuance. This discounted random cash flow might be an estimate for the value/price of the product. To obtain a reasonable price, we should therefore generate as many scenarios for the time evolution of the underlying as possible and calculate for each of these scenarios the discounted cash flow. Then we average over all discounted cash flows.

Hence, we could find the price of the product by the following Monte Carlo simulation.

i) Denote by $t = 0$ the date of issuance of the product and by $t = T$ the date of maturity of the product. In the time interval $[0, T]$ generate n paths $t \mapsto S_t$ of the underlying, i.e., generate

$$(s_0^k, s_1^k, \ldots, s_J^k), \quad k = 1, \ldots, n .$$

Herewith, s_j^k denotes the k-th realisation of the random variable S_{t_j} at observation date t_j.

ii) For each of the generated paths evaluate the payoff function g, that is find the max and the min of the prices s_j^k and take the difference, i.e.,

$$g_k := \max\{s_0^k, \ldots, s_J^k\} - \min\{s_0^k, \ldots, s_J^k\} .$$

This gives the redemption/cash flow for all the generated paths.

iii) Discount the cash flows just found in ii) to the date of issuance,

$$v_k := e^{-rT} g_k .$$

iv) Take the mean of all n discounted cash flows calculated in iii).

$$V_n := \frac{1}{n} \sum_{k=1}^{n} e^{-rT} g_k = e^{-rT} \frac{1}{n} \sum_{k=1}^{n} g_k .$$

This mean is an estimator of the value of the product.

It is easy to imagine that step i) is the hardest one. How do we generate a path of the underlying and what is the connection to the so-called risk-neutral valuation of a derivative? See the next section for a partial answer.

1.6 Why we need models

Step i) in the above described Monte Carlo simulation requires a mathematical model for the stochastic price evolution of the underlying. This fact renders derivatives theory non-trivial. One might ask if it is really necessary to confront one-selfs with such a model. Why not do derivatives pricing without any model just by looking at the historical time evolution of the underlying and to use it to generate future paths? Indeed, we can do the following. We take a time series of daily closing prices $(p_1, p_2, \ldots, p_n, p_{n+1})$ of the underlying and build the daily log-returns $r_j^c := \ln(p_{j+1}/p_j)$, $j = 1, \ldots, n$. Suppose that the time-to-maturity of the derivative we would like to price is $T \leq n$ (business) days and that the underlying actually trades at the price s_0. Now we draw uniformly T log-returns from the vector

$\mathbf{r}^c := (r_1^c, r_2^c, \ldots, r_n^c)$ successively. It is important that the drawing is done with replacement. We obtain a vector $\mathbf{r}^{c,*} := (r_1^{c,*}, r_2^{c,*}, \ldots, r_T^{c,*})$; note that some of the $r_j^{c,*}$'s might be equal (it possible that all of them are equal). Now we build the time series (a path) $(s_1, s_2, \ldots, s_T)$ of a possible price evolution of the underlying up to maturity by setting

$$s_1 = s_0 e^{r_1^{c,*}} \ , \ s_2 = s_1 e^{r_2^{c,*}} \ , \ldots , \ s_T = s_{T-1} e^{r_T^{c,*}} \ ;$$

this recursion is obviously equivalent to

$$s_j = s_0 e^{\sum_{k=1}^{j} r_k^{c,*}} \ .$$

Thus, we have generated one path just by using historical data and without any model! Now we repeat drawing with replacement from $\mathbf{r}^c$ B times, yielding the so-called bootstrap samples

$$\mathbf{r}^{c,*1} \ , \mathbf{r}^{c,*2} \ , \ldots , \ \mathbf{r}^{c,*B} \ .$$

For each of these B vectors we build paths $(s_1^b, s_2^b, \ldots, s_T^b)$ by

$$s_j^b = s_0 e^{\sum_{k=1}^{j} r_k^{c,*b}} \ .$$

To price a derivative with payoff g we then proceed with steps ii)-iv) of the above Monte-Carlo simulation, i.e., we evaluate the payoff $g_b := g(s_1^b, \ldots, s_T^b)$ for each path, discount and finally average, $V_B := 1/B \sum_{b=1}^{B} e^{-rT/360} g_b$ (since T is measured in days, we have to divide time by 360). Before we realise this in Python, we describe how to obtain a bootstrap sample $\mathbf{r}^{c,*}$. The idea is the following. From the set $\{1, 2, \ldots, n\}$ draw randomly one integer and call the drawn integer j_1^*. The first entry $r_1^{c,*}$ of $\mathbf{r}^{c,*}$ is now set to the j_1^*-th entry of the original return vector $\mathbf{r}^c$, i.e., $r_1^{c,*} = r_{j_1^*}^c$. Now draw again from the set $\{1, 2, \ldots, n\}$ to obtain the integer j_2^*. The second entry $r_2^{c,*}$ of $\mathbf{r}^{c,*}$ is then set to the j_2^*-th entry of the original return vector $\mathbf{r}^c$, that is $r_2^{c,*} = r_{j_2^*}^c$. We repeat this until we have drawn j_T^* from the set $\{1, 2, \ldots, n\}$; we finally obtain

$$(r_1^{c,*}, r_2^{c,*}, \ldots, r_T^{c,*}) = (r_{j_1^*}^c, r_{j_2^*}^c, \ldots, r_{j_T^*}^c) \ .$$

In Python, a vector $(j_1^*, j_2^*, \ldots, j_T^*)$ with $j_k^* \in \{1, 2, \ldots, n\}$ can be obtained by `np.random.randint(n, size = (1, T))`. To generate B such vectors, we might set `np.random.randint(n, size = (B, T))`. Note that since Python starts indexing at 0, we actually obtain vectors $(j_0^*, j_1^*, \ldots, j_{T-1}^*)$ with each j_k^* in the set $\{0, 1, \ldots, n-1\}$.

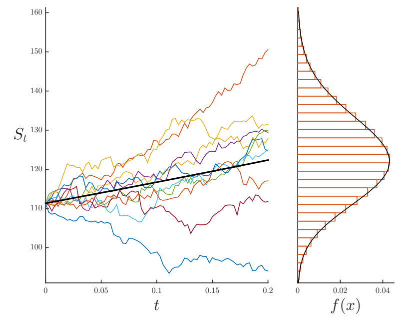

Figure 1.4: Bootstrapping from an empirical distribution. Left. Possible bootstrapped paths $(s_1^b, s_2^b, \ldots, s_T^b)$, i.e. possible future stock price movements (starting in s_0 at time $t = 0$) based on historical prices. The black curve is the average over all paths, that is

$$\bar{s}_j := \frac{1}{B} \sum_{b=1}^{B} s_j^b, \quad j = 0, 1, \ldots, T \ .$$

By the central limit theorem, this average converges, as $B \to \infty$, to the expected stock price

$$\mathbb{E}[S_t] = s_0 e^{(\mu + \sigma^2/2) \frac{360}{T} t} \ ,$$

where $t \in [0, T/360]$ is measured in years and where μ and σ are the mean and standard deviation of the aggregated log-returns $\sum_{k=1}^{T} r_k^{c,*b}$, respectively. For the Nestlé-example below we have $s_0 = 111.34$, $\mu \doteq 0.0915$, $\sigma \doteq 0.0760$ such that at maturity $t = \frac{T}{360} = \frac{72}{360} = 0.2$ years (corresponding to $T = 72$ days) the expected stock price is $\mathbb{E}[S_t] = 122.36$. Right. The empirical distribution of stock prices s_T^b and the corresponding probability density function

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2} x} e^{-\frac{(\ln(x/s_0) - \mu)^2}{2\sigma^2}}$$

of a lognormally distributed random variable.

Example 1.2. On September 3, 2019, the market price of a call option with strike $X = 110$ CHF and with maturity November 15, 2019, on Nestlé (which trades at $s_0 = 111.34$ CHF) is equal to $V^{\mathrm{M}} = 3.72$ CHF (average of bid and ask price). We use the above described bootstrap procedure to price this option without any model.

Solution. We use a time series $(p_1, \ldots, p_{n+1})$ of daily closing prices of Nestlé ranging from September 3, 2018, to September 2, 2019, so $n + 1 = 249$. Using the day-count-convention from section 1.3, we find $T = 72$ (days). To price the call, we only need the possible prices of Nestlé at maturity, hence we just calculate $s_T^b = s_0 e^{\sum_{k=1}^T r_k^{c,*b}}$ (no need to generate the whole path) for $b = 1, \ldots, B$ with $B = 5 \cdot 10^6$ bootstrap samples. The (bootstrap) price of the call is then

$$V_B := e^{-rT/360} \frac{1}{B} \sum_{b=1}^{B} \max\{s_T^b - X, 0\} .$$

where $r = -0.88\%$. We find $V^B \approx 12.72$ CHF, which is completely different from the market price $V^{\mathrm{M}} = 3.72$ CHF. ◇

Note that each sum $\sum_{k=1}^T r_k^{c,*b}$ in the exponent of s_T^b is a realisation of the random variable $\sum_{k=1}^T R_k^c$, where the random variables R_k^c are iid copies of R^c, the (daily) log-return. According to the central limit theorem, this sum is - under the iid assumption - approximatively normal, no matter the (true but unknown) distribution of R^c will be. Thus, the stock price $S_{T/360}$, at maturity will be (approximatively) lognormal, and this is clearly visible in the histogram of the values s_T^b, compare with figure 1.4 and the jupiter notebook `Chapter 1 Introduction.ipynb`. Further note that if we assume that the exponent $\sum_{k=1}^T R_k^c \sim \mathcal{N}(\mu, \sigma^2)$ is normal with mean μ and variance σ^2, then we may calculate the price of the call in example 1.2 analytically. Indeed, the considerations of chapter 3 lead to the "pricing formula"

$$V_c = e^{-rT/360} \big(s_0 e^{\mu + \frac{1}{2}\sigma^2} N(d + \sigma) - X N(d) \big) \tag{1.3}$$

with $d = \frac{\ln \frac{s_0}{X} + \mu}{\sigma}$. For the data in example 1.2 Python finds $\mu \doteq 0.0915$, $\sigma \doteq 0.0760$; evaluating the above formula yields then $V_c \approx 12.70$ CHF, which is of course close to the value V_B obtained from the bootstrap.

So, what went wrong in example 1.2? If the stock price at maturity is indeed lognormal (such that the above "pricing formula" is correct), then we obviously plugged in the wrong values for the parameters μ and/or σ (as we should have obtained the market value $V^{\mathrm{M}} = 3.72$ CHF). To simplify things, assume that the (historical) volatility estimated above is the correct value for σ. Under this assumption, the only parameter which we plugged in wrongly into formula (1.3) is μ. The natural question is then: What value μ must attain such that the price V_c (or equivalently, the price V_B) is equal to the market price $V^{\mathrm{M}} = 3.72$ CHF? To answer this question, we just solve the equation (1.3)

$$e^{-rT/360}\left(s_0 e^{\mu+\frac{1}{2}\sigma^2} N(d+\sigma) - XN(d)\right) = 3.72$$

for μ. Python's `fsolve` finds that the correct value for μ is -0.86%, which is almost equal to the risk free $r = -0.88\%$!

We have seen above that the historical data suggests that the expected stock price of Nestlé will be $\mathbb{E}[S_{T/360}] = s_0 e^{\mu+\sigma^2/2} \doteq 122.36$ CHF on November 15, 2019. However, this "view" of market expectation is - as we have also seen - completely irrelevant to the price of the call: the option price ignores the "expected up-movement" $\mu \approx 9.15\%$ of Nestlé. What is relevant in derivatives pricing is the risk free rate (and of course volatility, but that is a different story...); we will come back to this point in chapter 3.

As the "expected up-movement" of the underlying is implicit in the (bootstrapped) paths $(s_1^b, s_2^b, \ldots, s_T^b)$, $b = 1, \ldots, B$ (compare with figure 1.4), these paths are the wrong ones to price the option! Thus, we need a mathematical model for the time evolution of the underlying which is able to “deliver” the correct paths. We will get to know such models in the forthcoming chapters.

1.7 Problems

Problem 1.1. Consider the following investment strategy

1. Buy (long) of a money market investment (of size K_0)

2. Buy (long) one share of underlying with price S_0

3. Sell (short) one call option with strike $X < S_0$ and price V_c

Suppose that continuously compounded risk free rate is $r > 0$ and the time-to-maturity is T.

i) Draw a profit diagram (i.e., the function $P(S_T)$) of this combination.

ii) Calculate the break even of the combination.

iii) Suppose the parameters of the combination take the following values

- money market investment $K_0 = 200$ CHF (at time $t = 0$)
- price of underlying $S_0 = 184$ CHF (at time $t = 0$)
- strike and value of the call option $X = 178$ CHF and $V_c(S_0, 0; X) = 9.40$ CHF, respectively
- time-to-maturity $T = 9$ months

– continuously compounded risk free rate $r = 2\%$ p.a.

iii$_1$) For which price of the underlying the investment strategy "long underlying" yields a better return than the investment into the combination?

iii$_2$) At maturity, the underlying closes 10% below S_0. Find the (holding period) return of the combination.

Problem 1.2. [Iron condor] An iron condor is a portfolio consisting in four options (written on the same underlying and having the same time-to-maturity) as follows

- long put with strike X_1,
- short put with strike $X_2 > X_1$,
- short call with strike $X_3 > X_2$,
- long call with strike $X_4 > X_3$,

where the differences of the strikes $X_2 - X_1$ and $X_4 - X_3$ are equal and denoted by $h > 0$. Typically, the strikes X_2 and X_3 are chosen such that $s = S_0 = (X_2 + X_3)/2$, whence the differences of the strikes to S_0 are equal, i.e., $X_3 - S_0 = S_0 - X_2$.

i) Argue that if you are buying this strategy, you will realise a positive cash flow at $t = 0$. That is, the initial value of this option strategy $V_0^O := -V_c(s, 0; X_1) + V_c(s, 0; X_2) - V_p(s, 0; X_3) + V_p(s, 0; X_4)$ is negative.

ii) What are the maximal profit and maximal loss of an iron condor. Argue that the negative of V_0^O must be smaller than h.

iii) Draw a profit diagram of an iron condor. The initial value of the iron condor satisfies $0 < -V_0^O < h$ according to part i) and part ii).

iii) Assume that $S_0 - X_2$ and $X_3 - S_0$ are equal to $d > 0$. State the brake even points in terms of S_0, d and V_0^O.

Remark. In spring 2019, several web pages reported on bank clients who suffered losses realised by the so-called "yield-enhancement strategy" YES, which is based on an iron condor. For example, you may read the article "UBS Clients Burned by Iron Condor Strategy" on

`https://www.wealthmanagement.com/alternative-investments/ubs-clients-burned-iron-condor-strategy`.

Problem 1.3. [Butterfly and convexity] A butterfly is a portfolio consisting in three call options (written on the same underlying) with strikes $X_1 < X_2 < X_3$ having the same maturity. To be more precise, for $\alpha > 0$, we are

- long $\alpha \frac{1}{X_2 - X_1}$ call with strike X_1,
- short $\alpha \frac{X_3 - X_1}{(X_2 - X_1)(X_3 - X_2)}$ call with strike X_2,
- long $\alpha \frac{1}{X_3 - X_2}$ call with strike X_3;

compare with figure 1.5.

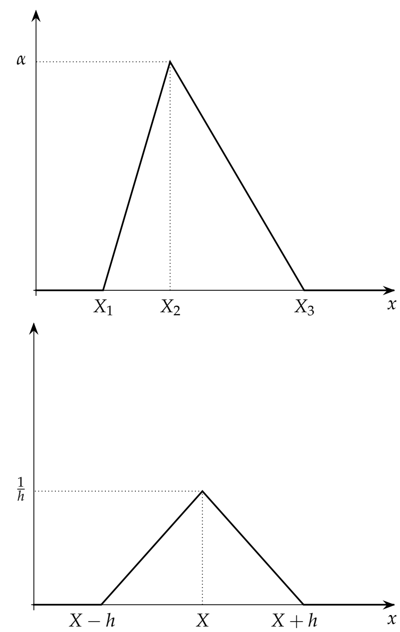

Figure 1.5: Payoff $g_{\mathrm{bf}}(x)$ of a portfolio - called butterfly - consisting in three call options with strikes X_1, X_2 and X_3. Top. We are long $\frac{\alpha}{K_2 - K_1}$ call with strike X_1, short $\alpha \frac{X_3 - X_1}{(X_2 - X_1)(X_3 - X_2)}$ call with strike X_2 and long $\frac{\alpha}{X_3 - X_2}$ call with strike X_3. Bottom. For $h > 0$ we specifically set $X_2 = X$, $X_1 = X - h$, $X_3 = X + h$.

i) Set $X_2 = (X_1 + X_3)/2$ and $\alpha = \frac{2}{X_3 - X_1}$. Show: the area below the payoff becomes 1 and the payoff function g_{bf} of the butterfly is

$$g_{\mathrm{bf}}(x) = \alpha^2 \Big(\max\{x - X_1, 0\} - 2\max\{x - X_2, 0\} + \max\{x - X_3, 0\} \Big) \ .$$

ii) Now set $X := X_2$ as we well $h := X_3 - X_2 = X_2 - X_1$. Show: the payoff function of this particular butterfly is

$$g_{\mathrm{bf}}(x) = \frac{\max\{x - X + h, 0\} - 2\max\{x - X, 0\} + \max\{x - X - h, 0\}}{h^2} \ .$$

iii) The value $V_{\mathrm{bf}}(s, T; X, h)$ of the butterfly at maturity is equal to $g_{\mathrm{bf}}(x) \geq 0$, which is according to part ii) non-negative. Thus, the value $V_{\mathrm{bf}}(s, t; X, h) \geq 0$ of the butterfly for $0 \leq t < T$ is non-negative as well, otherwise there exist arbitrage opportunities. Conclude: the value $V_{\mathrm{bf}}(s, t; X, h)$ is equal to

$$V_{\mathrm{bf}}(s, t; X, h) = \frac{V_c(s, t; X - h) - 2V_c(s, t; X) + V_c(s, t; X + h)}{h^2} \ , \tag{1.4}$$

where $V_c(s, t; X)$ denotes the time t-value of a call option with strike X.

iv) Use the Taylor approximation

$$f(x \pm h) \approx f(x) \pm h f'(x) + \frac{1}{2} h^2 f''(x)$$

to show that the second derivative of a function f can be approximated by

$$f''(x) \approx \frac{f(x - h) - 2f(x) + f(x + h)}{h^2}$$

and conclude from part iii) that

$$V_{\mathrm{bf}}(s, t; X, h) \approx \partial_{XX} V_c(s, t; X) \ .$$

v) In the limit $h \to 0$, there holds

$$\lim_{h \to 0} \frac{f(x - h) - 2f(x) + f(x + h)}{h^2} = f''(x) \ .$$

Use this and the fact that the non-negativity in (1.4) is preserved by passing to the limit to conclude that the second derivative of the price of the call option with respect to the strike (the so-called dual-gamma of the option) is non-negative as well, i.e.,

$$\partial_{XX} V_c(s, t; X) \geq 0 \; ;$$

which means that the call price V_c is convex with respect to strike (note that this is a model-independent result).

Chapter 2

Stock price models

As it should have been become clear by the case study of the previous chapter, a structured product pays off at maturity a cash flow which is not known in advance; the cash flow depends generally on the realised path the underlying has taken. An investor who buys such a product (and holds it until maturity) is entitled to receive this random cash flow. Depending on the realised scenario of the underlying the cash flow may be lower than the product's premium the investor paid at inception, thus generating a loss (a negative return). On the other hand, the performance of the underlying might be such that the structured product yields a profit which is much better than the profit implied by a direct investment into the underlying. To find the price/premium of a structured product an investor is willing to pay she needs a model which map the different scenarios the underlying may take to objective probabilities (probability distribution functions). Using these probabilities, the investor averages over all possible scenarios and discounts - by the principle of financial equivalence - this average to the date of purchase of the product.

We start this chapter by introducing a discrete model (binomial tree) for the time evolution of one underlying. Letting in the binomial tree the number of periods tend to infinity, we end up with the so-called geometric Brownian motion, which is basically the Black-Scholes model. Then we discuss how to simulate in Python a path of such a stochastic process, which we then extend to multiple underlyings. The resulting Python function will be the core to the pricing of structured products via Monte Carlo simulation.

Although it is a mission impossible to do derivatives theory without mathematics, I tried to reduce the mathematics used in this (and the next) chapter to a minimum ...

2.1 Discrete time model: the binomial tree

A binomial tree is a discrete model for the time evolution of the underlying. In a tree with m periods the lifetime of the option is divided in m intervals, each interval has length $\Delta t := \frac{T}{m}$. Starting with today's stock price $s_{0,0}$, we denote the possible prices of the stock at time $t_j = j\Delta t$ (j-th period) by $s_{i,j}$, $i = 0, \ldots, j$, and assume that and the end of the next time period (time $t_{j+1} = (j+1)\Delta t$) the stock can take the two possible values $s_{i,j+1}$ or $s_{i+1,j+1}$ as follows. Either the stock price increases to $s_{i+1,j+1}$ by a factor $u > 1$ with probability p, that is

$$s_{i+1,j+1} = us_{i,j} ,$$

or the stock price decreases by a factor $d < 1$ with probability $q = 1 - p$, i.e.,

$$s_{i,j+1} = ds_{i,j} ,$$

compare with figure 2.1.

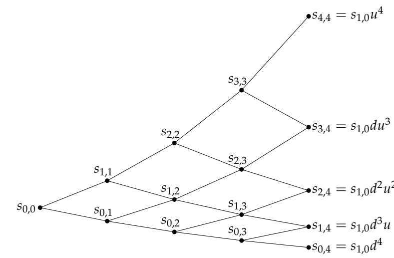

Figure 2.1: A binomial tree having m periods. Starting from the current market price $S_0 =: s_{0,0}$ of the stock it may increase from period j to period $j+1$ by a factor $u > 1$, i.e., $s_{i+1,j+1} = s_{i,j}u$ or it may decrease by a factor $d < 1$, that is $s_{i,j+1} = s_{i,j}d$. The both scenarios will be realised with probability p and probability $q = 1 - p$, respectively. Example with $m = 4$, $u = 1.4$ and $d = 0.8$.

We additionally assume that the stock price movements in the next period are independent of the movements in the previous period. If we consider a tree with m periods, then the parameter $0 < p < 1$ together with the independence assumption defines a probability measure on the set $\Omega := \{0, 1, \ldots, m\}$. We call this measure $\mathbb{P}$. For example,

$$\mathbb{P}[\{i\}] = \binom{m}{i} p^i q^{m-i}$$

is the probability that in a tree with m periods the stock will move upward i times. The expression $\binom{m}{i} := \frac{m!}{i!(m-i)!}$ counts the number of paths in the tree having i up-movements. In figure 2.1 for example, there are $\binom{4}{2} = 6$ paths with 2 up-movements, i.e., there are 6 paths connecting the "nodes" $s_{0,0}$ and $s_{2,4}$ in the tree.

The triple $(\Omega, 2^\Omega, \mathbb{P})$ is called probability space. The set 2^Ω is the so-called power set of Ω; it is the set of all subsets of Ω.

In a tree with m periods there will be $m + 1$ different realisations of the stock price at maturity T

$$s_{i,m} = s_{0,0}u^i d^{m-i}, \quad i = 0, \ldots, m . \tag{2.1}$$

Obviously, the values $s_{i,m}$ are realisations of a discrete random variable which we call S_m. We are interested in the probability distribution of this random variable, i.e., we are interested in the probabilities

$$\mathbb{P}[S_m = s_{i,m}] .$$

By construction, the variable S_m is binomially distributed, hence

$$\mathbb{P}[S_m = s_{i,m}] = \binom{m}{i} p^i q^{m-i}, \quad i = 0, \ldots, m , \tag{2.2}$$

compare with figure 2.2. Note that the probability mass function is not symmetric, even for $p = 0.5$. According to (2.1) and to complete our (discrete) model, we need to specify the values u and d. First of all, u has to be larger than one and d has to be smaller than one. Second, it is reasonable to assume that u (d) should be the larger (the smaller), the more volatile the underlying will be. Third, it is reasonable to assume that volatility scales with square root of time. Thus, if we denote by σ the (annualised) volatility, then the volatility of the underlying within the time period Δt is $\sigma\sqrt{\Delta t}$.

Hence, we work in the following with the values

$$u = 1 + \sigma\sqrt{\Delta t}, \quad d = 1 - \sigma\sqrt{\Delta t}\,. \tag{2.3}$$

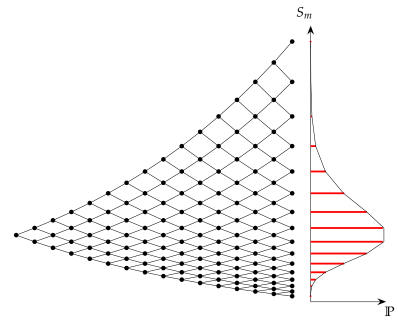

Figure 2.2: Probability mass function (2.2) of the discrete random variable S_m. The chosen parameters are $m = 15$, $u = 1.08$, $d = u^{-1}$ and $p = q = 0.5$.

A natural question in mathematical finance is now: What happens (with the distribution of S_m) if we take more and more periods, i.e., if we let tend m to infinity, or equivalently, $\Delta t \to 0$?

2.2 Continuous time model: the geometric Brownian motion

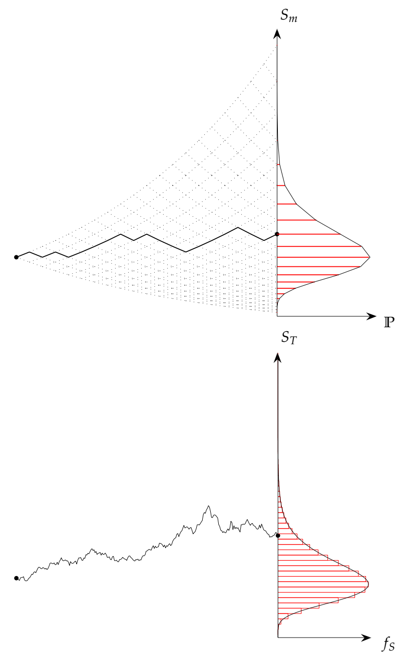

Figure 2.3: Top. One of the 2^m possible paths in a binomial tree having m periods. The random variable S_m is binomially distributed. Bottom. Limit $m \to \infty$ of a binomial tree and one of the uncountable many paths of a geometric Brownian motion. The random variable S_T is log-normally distributed $\ln\mathcal{N}(\mu_S, \sigma_S^2)$ with probability density function $f_S(x)$ in (2.17).

We exemplarily answer the question for the case $p = q = 0.5$ and denote the maturity T by t. By (2.1) and (2.3), we have

$$s_{i,m} = s\left(1 + \sigma\sqrt{\frac{t}{m}}\right)^i \left(1 - \sigma\sqrt{\frac{t}{m}}\right)^{m-i}, \tag{2.4}$$

where we dropped the double index in $s_{0,0}$ for simplicity. Now let be U_m be the number of up-movements of the stock in a tree with m periods and let be $D_m = m - U_m$ the corresponding number of down-movements. Let

$$X_m = U_m - D_m = 2U_m - m$$

be the difference of up- and down-movements in a tree with m periods. Note that X_m is equal to

$$X_m = \sum_{j=1}^{m} Y_j, \quad X_0 = 0$$

where the Y_j's are independent Bernoulli-variables with

$$\mathbb{P}[Y_j = \pm 1] = p = \frac{1}{2}\,.$$

The discrete random variable X_m takes the values

$$-m, \ldots, -1, 0, 1, \ldots, m$$

and is obviously binomially distributed. If we replace the exponents i and $m - i$ in (2.4) by $U_m = \frac{1}{2}(m + X_m)$ and $D_m = \frac{1}{2}(m - X_m)$, respectively, then the random variable S_m taking the values $s_{i,m}$ can be written as

$$S_m = s\left(1+\sigma\sqrt{\frac{t}{m}}\right)^{\frac{1}{2}(m+X_m)}\left(1-\sigma\sqrt{\frac{t}{m}}\right)^{\frac{1}{2}(m-X_m)} .$$

Taking the natural logarithm on both sides yields

$$\begin{aligned} \ln S_m &= \ln s + \frac{1}{2}(m+X_m)\ln\left(1+\sigma\sqrt{\frac{t}{m}}\right) \\ &\quad + \frac{1}{2}(m-X_m)\ln\left(1-\sigma\sqrt{\frac{t}{m}}\right) . \end{aligned} \tag{2.5}$$

Now we use a Taylor expansion of the natural logarithm[1]

$$\ln(1+x) = x - \frac{1}{2}x^2 + \mathcal{O}(x^3), \quad -1 < x \leq 1$$

and apply it to (2.5) with $x = \pm\sigma\sqrt{\frac{t}{m}}$. This yields

$$\begin{aligned} \ln S_m &= \ln s + \frac{1}{2}(m+X_m)\left(\sigma\sqrt{\frac{t}{m}} - \sigma^2\frac{t}{2m} + \mathcal{O}\left(m^{-3/2}\right)\right) \\ &\quad + \frac{1}{2}(m-X_m)\left(-\sigma\sqrt{\frac{t}{m}} - \sigma^2\frac{t}{2m} + \mathcal{O}\left(m^{-3/2}\right)\right) ; \end{aligned}$$

or - by factoring out -

$$\begin{aligned} \ln S_m &= \ln s - \frac{1}{2}\sigma^2 t + \mathcal{O}\left(m^{-1/2}\right) \\ &\quad + \sigma\sqrt{\frac{t}{m}}X_m + \frac{1}{m}X_m\mathcal{O}\left(m^{-1/2}\right) . \end{aligned}$$

Now, as we let tend m to infinity, we have $\mathcal{O}\left(m^{-1/2}\right) \to 0$. Furthermore, it is possible to prove the limits[2]

$$\sqrt{\frac{t}{m}}X_m \xrightarrow{d} W_t, \quad \frac{1}{m}X_m \xrightarrow{d} 0$$

as $m \to \infty$, where W_t is a normally distributed random variable with mean 0 and variance t, compare with the technical note in section 2.6. Thus,

$$\ln S_m \xrightarrow{d} \ln s - \frac{1}{2}\sigma^2 t + \sigma W_t$$

or, equivalently, the random variable S_m tends to a limit which we call S_t, the stock price at maturity t

$$S_m \xrightarrow{d} s e^{-\frac{1}{2}\sigma^2 t + \sigma W_t} =: S_t . \tag{2.6}$$

If we relax the condition $p = q = \frac{1}{2}$ to $p \in [0,1]$ arbitrary, one can show that S_m tends to

$$S_t = s e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t} , \tag{2.7}$$

[1] (Big-oh notation). We write

$$f = \mathcal{O}(g) \ \text{ as } \ x \to x_0$$

provided there exists a constant c such that

$$|f(x)| \leq c|g(x)|$$

for all x sufficiently close to x_0.

[2] Herewith, the symbol $\xrightarrow{d}$ denotes *convergence in distribution*, i.e., the cdf of $\sqrt{\frac{t}{m}}X_m$ converges to the cdf of W_t.

where the precise connection between p and the constant $\mu \in \mathbb{R}$ is not in our interest. The "object" W_t in (2.7) is an example of a so-called stochastic process. You may imagine a stochastic process

$$\{X(t),\, t \geq 0\}$$

as a stochastic "function" which maps a non-negative number t to a random variable $X(t)$ following a certain distribution. Usually, people write X_t instead of $X(t)$. The stochastic process W_t in (2.7) is called a standard Brownian motion[3] and is probably the most important process in (continuous time) finance. For further purposes, we need a proper mathematical definition of W_t.

[3] The Brownian motion is named after the scottish botanist Robert Brown (1773-1858), who watched as pollen made irregular movements in a water drop. A Brownian motion is also called Wiener process, named after the American mathematician Norbert Wiener (1894-1964), who proved the existence of a mathematical model for such a movement.

Definition 2.1. *A standard Brownian motion is a stochastic process W_t with the following properties.*

i) $W_0 = 0$.

ii) W_t *has independent and normally distributed increments, i.e., for any*

$$0 = t_0 < t_1 < t_2 < \ldots < t_{n-1} < t_n\ ,$$

and any $n \in \mathbb{N}^\times$ the n random variables (the increments)

$$X_1 := W_{t_1} - W_{t_0},\ \ X_2 := W_{t_2} - W_{t_1},\ \ \ldots,\ \ X_n := W_{t_n} - W_{t_{n-1}}$$

are independent and there holds

$$X_j \sim \mathcal{N}(0, t_j - t_{j-1})\ .$$

iii) W_t *is continuous in t.*

In particular, W_t is, for a given $t > 0$, a random variable which is normally distributed with mean 0 and standard deviation $\sqrt{t}$ (to see this, take $t_1 = t$ and $t_0 = 0$ in the definition 2.1), i.e.,

$$W_t \sim \mathcal{N}(0, t)\ . \tag{2.8}$$

Before we proceed, we repeat some facts on the above mentioned normal distribution due to its preeminent role in finance and statistics. Recall that a continuous random variable X follows a normal distribution with mean $\mu \in \mathbb{R}$ and standard deviation $\sigma \in \mathbb{R}^+$ if it has the probability density function (pdf)

$$\phi_{\mu,\sigma}(x) := \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}},\ \ x \in \mathbb{R}\ . \tag{2.9}$$

We write $X \sim \mathcal{N}(\mu, \sigma^2)$. Any normally distributed random variable X can be standardised by the affine mapping

$$Z := \frac{X - \mu}{\sigma}$$

to a standard normally distributed random variable with mean 0 and standard deviation 1, i.e., $Z \sim \mathcal{N}(0,1)$. We call the pdf of Z simply ϕ (instead of $\phi_{0,1}$), whence

$$\phi(x) := \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} , \tag{2.10}$$

compare with figure 2.4. The cumulative distribution function (cdf) of Z is called N and - by definition of the cdf of a continuous random variable - given by

$$N(z) := \int_{-\infty}^{z} \phi(x) \mathrm{d}x = \mathbb{P}[Z \leq z] \, . \tag{2.11}$$

As you know from statistics, the function $N(z)$ corresponds to the volume of the area which is embedded by the graph of ϕ, the x-axis (the abscissa) and the vertical axis $x = z$ and is equal to the probability that the random variable Z takes values smaller or equal to z, compare with figure 2.4.

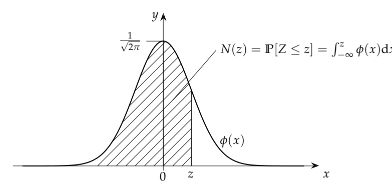

Figure 2.4: Density $\phi(x)$ of a standard normally distributed random variable $Z \sim \mathcal{N}(0,1)$ and volume of area $N(z) = \mathbb{P}[Z \leq z]$ as a probability.

Values of $N(z)$ are usually tabulated for a few values of z in the range $0 \leq z \leq 3.99$ or are calculated by software packages using numerical routines (note that the antiderivative of ϕ (and thus N) is not known in closed form).

The stock price process S_t at time t in (2.7) is a deterministic function of W_t and thus itself a stochastic process. It is called a geometric Brownian motion. Later on, we generalise the model S_t to more realistic ones. For this purpose it is useful (and necessary!) not to state the process S_t itself, but the stochastic differential equation (SDE) it satisfies. Using Itô calculus[4], one can show that S_t in (2.7) solves the SDE

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t, \quad S_0 = s > 0 \, . \tag{2.12}$$

[4] Itô calculus, named after the Japanese mathematician Kiyoshi Itô (1915-2008), is a generalisation of ordinary differential- and integral calculus to stochastic processes.

The above equation states that an infinitesimal change $\mathrm{d}S_t$ at time t of the stock price is composed of two parts. The term $\mu S_t \mathrm{d}t$ describes a deterministic change. In an infinitesimal short time interval $\mathrm{d}t$ this change is proportional to the current stock price S_t (with proportionality μ)[5]. If we invest the "capital" S_t at the annual continuous rate μ, we obtain after the time span $\mathrm{d}t$ the simple interest $\mu S_t \mathrm{d}t$. The term $\sigma S_t \mathrm{d}W_t$ describes a stochastic change, which also proportional to the current stock price, compare with definition 2.1. To understand a little bit better the differential equation (2.12) we now assume for a brief moment that $\sigma = 0$, so there is no randomness in the time evolution of the stock price. By a formal division by $\mathrm{d}t$ we obtain from (2.12) the ordinary differential equation (ODE)

[5] The parameter μ is called "drift". It is *not* the expected return of the stock.

$$\frac{\mathrm{d}S_t}{\mathrm{d}t} := S_t' = \mu S_t, \;\; S_0 = s$$

such that we are looking for a function S_t whose derivative is μ-times the function itself and which "starts" in s (exponential growth). We find that

$$S_t = se^{\mu t}$$

solves this differential equation. If the volatility in (2.12) is not zero, $\sigma > 0$, things become, due to the presence of W_t, much more involved. However, Itô calculus shows that the solution of the SDE (2.12) is indeed given by (2.7), compare also with figure 2.5.

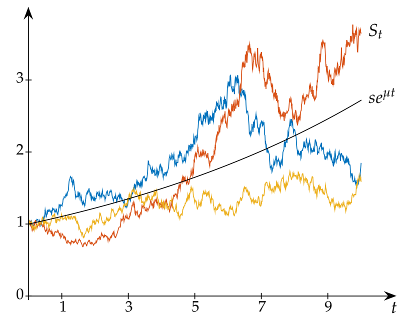

Figure 2.5: Three possible paths $t \mapsto S_t$ of a geometric Brownian motion (with $S_0 = 1$, $\sigma = 0.2$, $\mu = 0.1$).

Now we consider the log-return (the superscript "c" is not an exponent, but indicates that we consider a continuous return)

$$R^c_{t_{k-1},t_k} := \ln \frac{S_{t_k}}{S_{t_{k-1}}}$$

of the stock over the time period $[t_{k-1}, t_k]$ with length $\Delta t = t_k - t_{k-1}$. Using (2.7) we find

$$\begin{aligned}
R^c_{t_{k-1},t_k} &= \ln \frac{S_{t_k}}{S_{t_{k-1}}} \\
&\overset{(2.7)}{=} \ln \frac{se^{(\mu-\frac{\sigma^2}{2})t_k+\sigma W_{t_k}}}{se^{(\mu-\frac{\sigma^2}{2})t_{k-1}+\sigma W_{t_{k-1}}}} \\
&= \ln e^{(\mu-\frac{\sigma^2}{2})(t_k-t_{k-1})+\sigma(W_{t_k}-W_{t_{k-1}})} \\
&= \Big(\mu - \frac{\sigma^2}{2}\Big)\Delta t + \sigma X_k\,. \qquad (2.13)
\end{aligned}$$

By definition 2.1, ii), the random variable X_k is normally distributed with mean 0 and standard deviation $\sqrt{\Delta t}$, such that the random variable $R^c_{t_{k-1},t_k}$ is also normally distributed, with mean $(\mu - \sigma^2/2)\Delta t$ and standard deviation $\sigma\sqrt{\Delta t}$[6], i.e.,

In the model of Black and Scholes, log-returns are normally distributed.

$$R^c_{t_{k-1},t_k} \sim \mathcal{N}\big(\mu_S, \sigma_S^2\big) \qquad (2.14)$$

with

$$\mu_S := (\mu - \frac{1}{2}\sigma^2)\Delta t, \quad \sigma_S^2 := \sigma^2 \Delta t\,. \qquad (2.15)$$

We may use the fact (2.14) to estimate the model parameters μ and σ from a time series. Assume that a time series $(r_1^c, r_2^c, \ldots, r_n^c)$ of log-returns $R^c_{t_{k-1},t_k}$ is available. We calculate the sample mean $\bar{r}$ and the sample standard deviation s_r via

$$\bar{r} := \frac{1}{n}\sum_{k=1}^n r_k^c, \quad s_r := \sqrt{\frac{1}{n-1}\sum_{k=1}^n (r_k^c - \bar{r})^2}\,.$$

If the log-returns are normally distributed, then $\bar{r}$ is an estimator for μ_S and s_r is an estimator for σ_S. Thus, we estimate the annualised volatility σ by

$$\widehat{\sigma} := \frac{s_r}{\sqrt{\Delta t}}$$

[6] We used the fact that for real numbers a and b and any random variable X with mean μ_X and variance σ_X^2 there holds

$$\begin{aligned}
\mathbb{E}[aX+b] &= a\mu_X + b \\
\mathrm{Var}[aX+b] &= a^2\sigma_X^2
\end{aligned}$$

and the annualised drift μ by

$$\widehat{\mu} := \frac{\bar{r}}{\Delta t} + \frac{s_r^2}{2\Delta t} .$$

Typical values for Δt are $\Delta t = \frac{1}{252}$ in the case of daily data, $\Delta t = \frac{1}{50}$ in the case of weekly data or $\Delta t = \frac{1}{12}$ in the case of monthly data.

If the logarithm $\ln(X)$ of a random variable X is normally distributed, this random variable is said to be log-normally distributed. Thus, if the underlying follows a geometric Brownian motion and admits at time $0 \leq t \leq T$ the value $S_t := s$, then the random variable S_T is log-normally distributed with mean μ_S and variance σ_S^2; we write

$$S_T \sim \ln \mathcal{N}(\mu_S, \sigma_S^2) , \tag{2.16}$$

with μ_S, σ_S as in (2.15) and $\Delta t := T - t$. The probability density function of S_T is given by

$$f_S(x) = \frac{1}{\sqrt{2\pi\sigma_S^2}x} e^{-\frac{(\ln(x/s)-\mu_S)^2}{2\sigma_S^2}} , \tag{2.17}$$

compare with figure 2.3. The random variable S_T has mean

$$\mathbb{E}[S_T] = \int_0^\infty x f_S(x) \mathrm{d}x = s e^{\mu(T-t)} \tag{2.18}$$

and variance

$$\mathrm{Var}[S_T] = \int_0^\infty \left(x - \mathbb{E}[S_T]\right)^2 f_S(x) \mathrm{d}x = s^2 e^{2\mu(T-t)} \left(e^{\sigma^2(T-t)} - 1\right) , \tag{2.19}$$

compare with the technical note in section 2.7. Before we proceed, we shortly discuss the meaning of the parameter μ in (2.12). It is often stated that μ is the expected return (p.a.) of the stock. But this is not entirely true. To see this, suppose $T - t$ is one year. Then, by (2.14), the expected *log-return* of the stock is $\mu_S = \mu - \frac{1}{2}\sigma^2 \approx \mu$, which is different from μ (unless the volatility is zero!). On the other hand, we might look at the holding period return of the stock, i.e.,

$$R_{t,T} := \frac{S_T}{S_t} - 1 .$$

The expected *holding period return* is then, since $s = S_t$ and $T - t = 1$

$$\mathbb{E}[R_{t,T}] = \frac{1}{S_t} \mathbb{E}[S_T] - 1 \overset{(2.18)}{=} e^{\mu(T-t)} - 1 = e^{\mu} - 1 \approx \mu$$

which is again different from μ. Thus, μ is in either case only an approximation to the expected return of the stock. Correctly, μ should be called drift.

Example 2.2. Stock A has a drift of 16% p.a. and its annualised volatility is 35%. The stock currently trades at 38 CHF. Consider a European call option on this stock which matures in 8 month and has strike 43 CHF. What is the probability that the option will be exercised at maturity?

Solution. The option will be exercised if $S_T \geq X$; thus we have to calculate the probability $\mathbb{P}[S_T \geq X]$. Since S_T is log-normally distributed, we consider the log-return $R^c_{t,T} = \ln(S_T/S_t) = \ln(S_T/38)$ in (2.14). This random variable is normally distributed, such that we have

$$
\begin{aligned}
\mathbb{P}[S_T \geq X] &= \mathbb{P}[\ln(S_T/38) \geq \ln(X/38)] \\
&= 1 - \mathbb{P}[R^c_{t,T} \leq \ln(43/38)] \\
&= 1 - \mathbb{P}\left[Z \leq \frac{\ln(43/38) - \mu_S}{\sigma_S}\right] \\
&\overset{(2.15)}{=} 1 - \mathbb{P}\left[Z \leq \frac{\ln(43/38) - (\mu - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right] \\
&= 1 - \mathbb{P}\left[Z \leq \frac{\ln(43/38) - (0.16 - 0.5 \cdot 0.35^2) \cdot 2/3}{0.35\sqrt{2/3}}\right] \\
&\doteq 1 - \mathbb{P}[Z \leq 0.202] \\
&= 1 - N(0.202) \doteq 1 - 0.58 = 0.42\,.
\end{aligned}
$$

For an explanation of the distribution function $N(z) := \mathbb{P}[Z \leq z]$ of the standard normal random variable Z see figure 2.4. ◇

2.3 Generating paths of a geometric Brownian motion

Later on, we want to price derivatives by using Monte Carlo simulation as outlined in section 1.5. It is simplest to model the time evolution of the underlying by a geometric Brownian motion (Black-Scholes model), thus we need to simulate paths of S_t in (2.7). We start with the generation of a sample path of a standard Brownian motion W_t, i.e., we plot a graph of the mapping $t \mapsto W_t$. To do so, we choose arbitrary time instances $0 = t_0 < t_1 < \ldots < t_n$. By i) and ii) of definition 2.1, we have for t_k, $k \in \{1, \ldots, n\}$

$$
\begin{aligned}
W_{t_k} &= W_{t_{k-1}} + X_k \\
&= W_{t_{k-2}} + X_{k-1} + X_k \\
&= W_{t_{k-3}} + X_{k-2} + X_{k-1} + X_k \\
&= \vdots \\
&= W_{t_0} + X_1 + X_2 + \ldots + X_k \\
&= \sum_{j=1}^{k} X_j\,.
\end{aligned}
$$

Since by ii) of definition 2.1 each X_j in the sum is normally distributed, $X_j \sim \mathcal{N}(0, t_j - t_{j-1})$, we have

$$W_{t_k} = \sum_{j=1}^{k} \sqrt{t_j - t_{j-1}} Z_j$$

with $Z_j \sim \mathcal{N}(0,1)$ standard normally distributed. If in particular the time points t_j are equidistant, i.e.,

$$t_1 - t_0 = t_2 - t_1 = \ldots = t_n - t_{n-1} =: \Delta t$$

we have

$$W_{t_k} = \sqrt{\Delta t} \sum_{j=1}^{k} Z_j \, . \tag{2.20}$$

Thus, if we want to generate a path (t_k, w_{t_k}), $k = 0, 1, \ldots, n$ of a Brownian motion with $t_k = k\Delta t$, we just have to draw n realisations z_j of the standard normal distribution and then to build the sum (2.20). We can generalise a standard Brownian motion to a Brownian motion. A stochastic process X_t is called a Brownian motion with mean μ and standard deviation σ if the process

$$\frac{X_t - \mu t}{\sigma}$$

is a standard Brownian motion. Thus, a Brownian motion is defined by

$$X_t = \mu t + \sigma W_t$$

where we can add an arbitrary constant $x_0 \in \mathbb{R}$

$$X_t = x_0 + \mu t + \sigma W_t$$

such that the process starts at x_0, i.e., $X_0 = x_0$. Since $W_t \sim \mathcal{N}(0, t)$ by i) and ii) of definition 2.1, the process X_t is - for each t - normally distributed as well with mean $\mathbb{E}[X_t] = x_0 + \mu t$ and variance $\mathrm{Var}[X_t] = \sigma^2 t$, i.e., $X_t \sim \mathcal{N}(x_0 + \mu t, \sigma^2 t)$. The Brownian motion with $\mu = 0$ was used by Louis Bachelier in his doctoral thesis (published 1900 !) to model stock prices and to derive pricing formulas for European put and call options. The main problem of the Brownian motion to model a stock price is the positive probability that the stock price becomes negative, see figure 2.7.

It becomes clear that one can overcome negative stock prices by the geometric Brownian motion (2.7), which we now write as

$$X_t = x_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t} \, .$$

To simulate a path (t_k, x_{t_k}), $k = 1, 2, \ldots, n$ of a geometric Brownian motion with $t_k = k\Delta t$, we just have to draw n realisations z_j of the

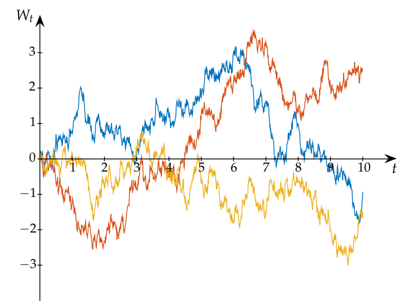

Figure 2.6: Possible paths of a standard Brownian motion with $\Delta t = 0.01$.

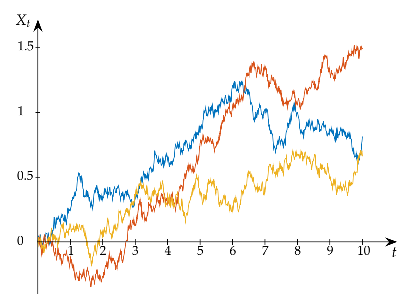

Figure 2.7: Possible paths of a Brownian motion with $x_0 = 0$, $\mu = 0.1$ and $\sigma = 0.2$. W_t is as in figure 2.6.

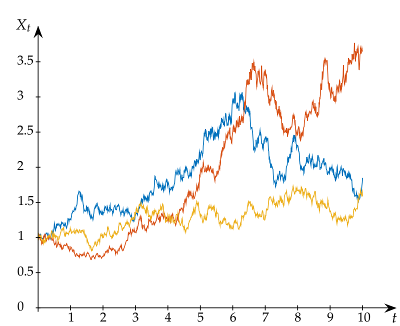

Figure 2.8: Possible paths of a geometric Brownian motion with $x_0 = 1$, $\mu = 0.1$ and $\sigma = 0.2$. W_t is as in figure 2.6.

standard normal distribution, then to build the sum (2.20) $w_{t_k} = \sqrt{\Delta t}\sum_{j=1}^{k} z_j$ and to evaluate

$$x_{t_k} = x_0 e^{(\mu - \frac{1}{2}\sigma^2)t_k + \sigma\sqrt{\Delta t}\sum_{j=1}^{k} z_j} , \tag{2.21}$$

compare with figure 2.8. To realise (2.21), it might be more intuitive to work with a rekursive version, i.e., knowing $x_{t_{k-1}}$, how do we generate x_{t_k}? See problem 2.3 for an answer. The generation of a path of a (geometric) Brownian motion relies heavily on drawing numbers z_j which follow a standard normal distribution. This is not trivial. For example, (standard) Excel does only provide the generation of uniformly distributed random numbers u_j (which is non-trivial as well) such that the natural question is how do we come from u_j to z_j? As it becomes clear from figure 2.9, we just need to invert the cumulative distribution function $N(z)$ of the standard normal distribution, i.e., once we have drawn numbers u_j from the uniform distribution on the interval $[0,1]$, we set

$$z_j = N^{-1}(u_j) ,$$

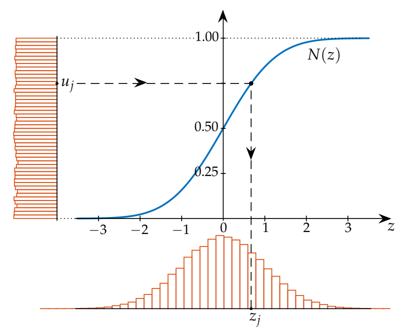

Figure 2.9: We draw 10^5 numbers u_j from the uniform distribution on the interval $[0,1[$, see the histogram on the left, and map them via N^{-1} to numbers z_j. These follow a standard normal distribution, see the histogram at the bottom.

where N^{-1} is the inverse of N. The z_j's obtained in this way follow a standard normal distribution, see problem 2.4. In Python, the function N^{-1} is realised as `norm.ppf`. Note that since N is not known in closed form, so isn't N^{-1} such that one has to invert N numerically, which means that we need to solve an equation of the form: Given $u_j \in]0,1[$, find z_j such that

$$N(z_j) = \int_{-\infty}^{z_j} \phi(x)\mathrm{d}x = u_j .$$

This is not trivial and we will not dwell on the (numerical) realisation of this problem. In Python, we may therefore draw n numbers from a standard normal distribution as follows. First, we draw from the uniform distribution on the interval $[0,1[$ using $u = \texttt{np.random.uniform}(0,1,n)$ and then we set $z = \texttt{norm.ppf}(u)$. However, we may directly generate the numbers using $z = \texttt{randn}(n)$ from `numpy.random`.

As we have to consider later on derivatives and structured products written on multiple underlyings, we need to generalise the concept of the geometric Brownian motion to an arbitrary number d of dimensions (underlyings).

2.4 Multivariate geometric Brownian motion

We consider the d stochastic differential equations

For $d = 1$ these equations reduce to the equation (2.12) with $L_{11} = \sigma$.

$$\mathrm{d}S_t^i = \mu_i S_t^i \mathrm{d}t + S_t^i \sum_{j=1}^{d} L_{ij}\mathrm{d}W_t^j, \quad S_0^i = s_i > 0 . \tag{2.22}$$

Herewith, the processes W_t^j, $j = 1, \ldots, d$, are independent Brownian motions, each of them satisfying the definition 2.1. The solution of the stochastic differential equation (2.22) is given by

$$S_t^i = s_i e^{(\mu_i - \frac{1}{2}\sum_{j=1}^d L_{ij}^2)t + \sum_{j=1}^d L_{ij} W_t^j}$$

such that the log-return $R_{0,t}^{c,i}$ of the i-th underlying in the time interval $[0, t]$ is

$$R_{0,t}^{c,i} := \ln \frac{S_t^i}{s_i} = (\mu_i - \frac{1}{2}\sum_{j=1}^d L_{ij}^2)t + \sum_{j=1}^d L_{ij} W_t^j .$$

In the model (2.22) we need to specify the $d \times d$-matrix $\mathbf{L} = (L_{ij})$. We choose this matrix such that the covariance matrix of the log-returns $R_{0,t}^{c,i}$ is equal to the covariance matrix $\mathbf{\Sigma} t = (\mathbf{\Sigma}_{ij})t$ given by

$$\mathbf{\Sigma}_{ij} t = \sigma_i \sigma_j \rho_{ij} t, \quad 1 \le i, j \le d ,$$

with σ_i the volatility of the i-th underlying and ρ_{ij} the correlation of the returns $R_{0,t}^{c,i}$ and $R_{0,t}^{c,j}$. In problem 2.6 we show that

$$\mathrm{Cov}\left[R_{0,t}^{c,i}, R_{0,t}^{c,j}\right] = \sum_{k=1}^d L_{ik} L_{jk} t = (\mathbf{L}\mathbf{L}^\top)_{ij} t \tag{2.23}$$

such that there must hold

$$\mathbf{L}\mathbf{L}^\top = \mathbf{\Sigma} . \tag{2.24}$$

Given the (positive definite) covariance matrix $\mathbf{\Sigma}$, equation (2.24) for $\mathbf{L}$ has infinite many solutions. However, if we let $\mathbf{L}$ be lower triangular, i.e., the elements L_{ij} above the main diagonal are zero, $L_{ij} = 0$ for $i < j$, then the solution is unique. If $\mathbf{L}$ is lower triangular, then equation (2.24) is called the Cholesky decomposition[7] of $\mathbf{\Sigma}$.

If we define

$$\mathrm{d}\widehat{W}_t^i := \frac{1}{\sigma_i} \sum_{j=1}^d L_{ij} \mathrm{d}W_t^j , \tag{2.25}$$

then we can write the model (2.22) as

$$\mathrm{d}S_t^i = \mu_i S_t^i \mathrm{d}t + \sigma_i S_t^i \mathrm{d}\widehat{W}_t^i, \quad S_0^i = s_i > 0 . \tag{2.26}$$

In contrast to the Brownian motions in equation (2.22), the Brownian motions $\widehat{W}_t^i$ are not independent but have the correlation matrix $\boldsymbol{\rho} = (\rho_{ij})$. The solution of the stochastic differential equation (2.26) is

$$S_t^i = s_i e^{(\mu_i - \frac{1}{2}\sigma_i^2)t + \sigma_i \widehat{W}_t^i} ,$$

compare with the case $d = 1$ in (2.7). In section 2.3 we discussed how to generate a path of a geometric Brownian motion, compare with

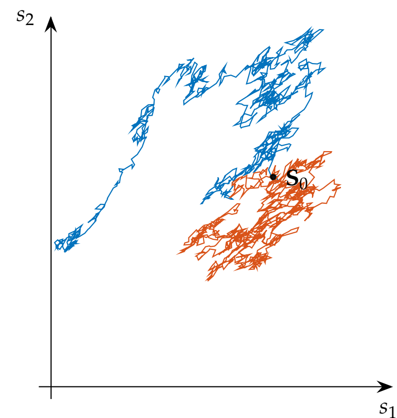

Figure 2.10: Two realisations of a d-dimensional Brownian motion, $d = 2$. Both paths start in $\mathbf{S}_0 = (S_0^1, S_0^2)$ (black point). The parameter values in (2.26) are: $\sigma_1 = 0.18$, $\sigma_2 = 0.15$, $\rho = 0.68$, $\mu_1 = 0.04$, $\mu_2 = 0.02$, $T = 2$.

[7] Named after the French mathematician André Louis Cholesky (1875–1918). The $n \times n$-matrix $\mathbf{L}$ in the factorisation $\mathbf{A} = \mathbf{L}\mathbf{L}^\top$ is unique if the symmetric $n \times n$-matrix $\mathbf{A}$ is positive definite. A symmetric matrix $\mathbf{A}$ is called positive definite if $\mathbf{x}^\top \mathbf{A}\mathbf{x} > 0$ for all vectors $\mathbf{0} \neq \mathbf{x} \in \mathbb{R}^n$. Covariance matrices are positiv definite.

(2.21). Obviously, we need to extend this to $d > 1$ dimensions. In the technical note 2.8 we focus on the generation of paths of the vector process

$$\mathbf{S}_t := (S_t^1, S_t^2, \dots, S_t^d)$$

and call the resulting Python function `gbm_d.py`.

2.5 Problems

Problem 2.1. Stock A currently trades at 760 CHF. Its drift is 14% p.a., its volatility is 20% p.a.

i) In which interval the stock price will be after 6 months with probability 95%?

ii) What is the expected stock price in 6 months?

Problem 2.2. In the time period 2001-2014 an index realised a drift of -14% and a volatility of 38%. The value of the index at the end of 2014 ($t = 0$) was $s = S_0 = 3027$.

i) For a continuous random variable X (with strictly increasing distribution function) the so-called Value at Risk $\mathrm{VaR}_\alpha(X)$ at the confidence level $1 - \alpha$ is defined as the quantile

$$\mathbb{P}[X \leq \mathrm{VaR}_\alpha(X)] = \alpha \; .$$

Find the Value at Risk of the index at the end of October 2015 ($t = T$), i.e., the random variable X in this example is the final value S_T of the index, $X = S_T$. Take $\alpha = 0.05$.

ii)* For a continuous random variable X (with strictly increasing distribution function) the so-called Expected Shortfall $\mathrm{ES}_\alpha(X)$ at confidence niveau $1 - \alpha$ is defined as the conditional expectation

$$\mathrm{ES}_\alpha(X) = \mathbb{E}\big[X | X \leq \mathrm{VaR}_\alpha(X)\big] \; .$$

Find the Expected Shortfall of the index at the end of October 2015. Take $\alpha = 0.05$.

Hint. Show that there holds

$$\mathrm{ES}_\alpha(S_T) = \frac{1}{\alpha}\mathbb{E}[S_T] N\big(z_\alpha - \sigma\sqrt{T-t}\big)$$

Herein, we denote by z_α the quantile satisfying $N(z_\alpha) = \alpha$.

Problem 2.3. Show that (2.21) is equivalent to the recursion $x_{t_0} = x_0$,

$$x_{t_k} = x_{t_{k-1}} e^{(\mu - \frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t} z_k}, \quad k = 1, 2 \dots, n \; .$$

Problem 2.4. Let U be a continuous random variable uniformly distributed on the interval $[0,1]$ and let $N^{-1} : [0,1] \to \mathbb{R}$ be the inverse of the cumulative distribution function (cdf) of the standard normal distribution.
Show: the random variable Z defined by $Z = N^{-1}(U)$ is standard normally distributed, i.e., $Z \sim \mathcal{N}(0,1)$.

Hint. Use the following fact. If X is a random variable with probability density function (pdf) $f_X(x)$ and if one sets $Y = g(X)$ for some strictly increasing function g with inverse h, i.e.,

$$y = g(x) \Leftrightarrow x = h(y) \, ,$$

then the pdf of Y is

$$f_Y(y) = f_X(h(y))h'(y)$$

where h' is the derivative of h.

Problem 2.5. Show that the expectation of the stock price S_m defined in (2.2) is given by

$$\mathbb{E}[S_m] = s_{0,0}(up + dq)^m. \tag{2.27}$$

Hint. Use the binomial formula $(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$, $n \in \mathbb{N}$, $a, b \in \mathbb{R}$.

Problem 2.6. Show equation (2.23).
Hint. To show this, you might use the following properties of the covariance. Let a, b, c and d deterministic, and let U, V, X and Y be random variables. Then there holds

$$\begin{aligned} \mathrm{Cov}[aU + bV, cX + dY] &= ac\mathrm{Cov}[U,X] + ad\mathrm{Cov}[U,Y] \\ &\quad + bc\mathrm{Cov}[V,X] + bd\mathrm{Cov}[V,Y] \end{aligned}$$

as well as $\mathrm{Cov}[X,1] = \mathrm{Cov}[1,X] = 0$.

Problem 2.7. Let the covariance matrix $\mathbf{\Sigma} = (\mathbf{\Sigma}_{ij}) = (\rho_{ij}\sigma_i\sigma_j)$ be given. Find the matrix $\mathbf{L}$ in the Cholesky factorisation.

Hint. Consider the equation $\mathbf{L}\mathbf{L}^\top = \mathbf{\Sigma}$ with $\mathbf{L}$ a $d \times d$ lower triangular matrix.

2.6 Technical note. The limit $\lim_{m\to\infty} \sqrt{t/m}X_m$

In this section we show the limits (in distribution)

$$\sqrt{\frac{t}{m}}X_m \xrightarrow{d} W_t, \quad \frac{1}{m}X_m \xrightarrow{d} 0 \, ,$$

with W_t a standard Brownian motion and X_m the discrete random variable

$$X_m = \sum_{j=1}^{m} Y_j, \quad X_0 = 0, \quad \mathbb{P}[Y_j = \pm 1] = \frac{1}{2}$$

counting the difference of up- and down-movements in a binomial tree with m periods. The number $m \in \mathbb{N}^\times$ defines an equidistant partition of the interval $[0,t]$ as follows

$$0 = t_0 < t_1 < \ldots < t_m = t, \quad t_j = j\Delta t, \quad \Delta t := \frac{t}{m}$$

For $\tau = t_j$, $j = 0, 1, \ldots, m$, we now define a scaled version of X_m via

$$W_\tau^{(m)} := \sqrt{\frac{t}{m}} \sum_{j=1}^{\frac{\tau}{\Delta t}} Y_j \, .$$

For values $\tau \in]t_j, t_{j+1}[$ we define $W_\tau^{(m)}$ via linear interpolation, i.e.,

$$W_\tau^{(m)} = \frac{W_{t_{j+1}}^{(m)} - W_{t_j}^{(m)}}{t_{j+1} - t_j} (\tau - t_j) + W_{t_j}^{(m)}, \quad t \in]t_j, t_{j+1}[\, .$$

By construction, $W_\tau^{(m)}$ is - for each $\tau \in]0,t]$ - a continuous random variable; we are interested in its distribution in the limit $m \to \infty$ (or $\Delta t \to 0$). A useful tool in calculation limits of distributions is the so-called moment generating function. For a random variable X the moment generating function $\varphi_X(u)$ is defined as

$$\varphi_X(u) := \mathbb{E}[e^{uX}], \ \ u \in \mathbb{R} \, .$$

We calculate the moment generation function of $W_t^{(m)}$. We have

$$\begin{aligned}
\varphi_{W_t^{(m)}}(u) \quad &= \quad \mathbb{E}[e^{uW_t^{(m)}}] = \mathbb{E}\left[e^{u\sqrt{\frac{t}{m}}\sum_{j=1}^m Y_j}\right] \\
&= \quad \mathbb{E}\left[e^{u\sqrt{\frac{t}{m}}Y_1} \cdot \ldots \cdot e^{u\sqrt{\frac{t}{m}}Y_m}\right] \\
&\overset{Y_j \text{ indep.}}{=} \quad \mathbb{E}\left[e^{u\sqrt{\frac{t}{m}}Y_1}\right] \cdot \ldots \cdot \mathbb{E}\left[e^{u\sqrt{\frac{t}{m}}Y_m}\right] \\
&\overset{\text{def. } Y_j}{=} \quad \frac{1}{2}\left(e^{u\sqrt{\frac{t}{m}}} + e^{-u\sqrt{\frac{t}{m}}}\right) \cdot \ldots \cdot \frac{1}{2}\left(e^{u\sqrt{\frac{t}{m}}} + e^{-u\sqrt{\frac{t}{m}}}\right) \\
&= \quad \left(\frac{e^{u\sqrt{\frac{t}{m}}} + e^{-u\sqrt{\frac{t}{m}}}}{2}\right)^m .
\end{aligned}$$

In the function $\ln \varphi_{W_t^{(m)}}(u)$ we now let tend m to infinity (it is simpler to consider the logarithm of the moment generating function). Setting for simplicity

$$x := \frac{1}{\sqrt{m}}$$

we therefore consider the limit

$$\lim_{m\to\infty} \ln \varphi_{W_t^{(m)}}(u) = \lim_{x\to 0^+} \frac{\ln\left(\frac{e^{xu\sqrt{t}} + e^{-xu\sqrt{t}}}{2}\right)}{x^2} \, .$$

In this fraction both numerator as well as enumerator tend to 0 as x tends to zero, whence we have a limit of the form $\frac{0}{0}$. In such a case, the rule of Bernoulli-L'Hôpital[8] tells us that we are allowed to take first the derivative (with respect to x) of numerator and enumerator separately and then taking the limit. Thus

$$\begin{aligned}
\lim_{m\to\infty} \ln \varphi_{W_t^{(m)}}(u) &= \lim_{x\to 0^+} \frac{\frac{2}{e^{xu\sqrt{t}}+e^{-xu\sqrt{t}}} \cdot \left(\frac{1}{2}u\sqrt{t}e^{xu\sqrt{t}} - \frac{1}{2}u\sqrt{t}e^{-xu\sqrt{t}}\right)}{2x} \\
&= \lim_{x\to 0^+} \frac{u\sqrt{t}\left(e^{xu\sqrt{t}} - e^{-xu\sqrt{t}}\right)}{2x\left(e^{xu\sqrt{t}} + e^{-xu\sqrt{t}}\right)} \\
&= \frac{u\sqrt{t}}{2} \underbrace{\lim_{x\to 0^+} \frac{1}{e^{xu\sqrt{t}} + e^{-xu\sqrt{t}}}}_{=\frac{1}{2}} \cdot \lim_{x\to 0^+} \frac{e^{xu\sqrt{t}} - e^{-xu\sqrt{t}}}{x} \\
&= \frac{u\sqrt{t}}{4} \lim_{x\to 0^+} \frac{u\sqrt{t}(e^{xu\sqrt{t}} + e^{-xu\sqrt{t}})}{1} \\
&= \frac{1}{2}u^2 t\,,
\end{aligned}$$

and it follows that

$$\lim_{m\to\infty} \mathbb{E}\left[e^{uW_t^{(m)}}\right] = \lim_{m\to\infty} \varphi_{W_t^{(m)}}(u) = e^{\lim_{m\to\infty} \ln \varphi_{W_t^m}(u)} = e^{\frac{1}{2}u^2 t}\,.$$

If we can show that $e^{\frac{1}{2}u^2t}$ is the moment generating function of the random variable W_t we are (almost) done. Indeed, using the pdf $\phi_{0,\sqrt{t}}(x)$ in (2.9) of the normally distributed random variable $W_t \sim \mathcal{N}(0,t)$ we find

$$\varphi_{W_t}(u) = \mathbb{E}\left[e^{uW_t}\right] = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{ux} e^{-\frac{x^2}{2t}}\,\mathrm{d}x = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{-\frac{x^2-2utx}{2t}}\,\mathrm{d}x\,.$$

We rewrite the exponent in the integrand as (quadratic completion)

$$-\frac{x^2-2utx}{2t} = -\frac{(x-ut)^2 - u^2t^2}{2t} = -\frac{(x-ut)^2}{2t} + \frac{u^2t}{2}\,,$$

hence

$$\begin{aligned}
\varphi_{W_t}(u) &= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{-\frac{x^2-2utx}{2t}}\,\mathrm{d}x = e^{\frac{1}{2}u^2t} \cdot \underbrace{\frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-ut)^2}{2t}}\,\mathrm{d}x}_{=1} \\
&= e^{\frac{1}{2}u^2t}\,.
\end{aligned}$$

Thus there holds

$$\lim_{m\to\infty} \varphi_{W_t^{(m)}}(u) = \varphi_{W_t}(u)\,;$$

Lévy's continuity theorem[9] now implies the convergence we have looked for

$$W_t^{(m)} \xrightarrow{d} W_t\,.$$

Herewith, the symbol $\xrightarrow{d}$ denotes *convergence in distribution*, i.e., the sequence of the cdfs of $W_t^{(m)}$ converge to the cdf of W_t. This convergence holds at any point x at which the cdf of W_t is continuous (hence for all $x \in \mathbb{R}$).

[8] The rule is named after the Swiss mathematician Johann Bernoulli (1667-1748) and the French mathematician Guillaume de L'Hôpital (1661-1704). It states that, provided the involved functions f and g meet some conditions, there holds

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}\,.$$

[9] Named after the French mathematician Paul Lévy, (1886–1971).

2.7 Technical note. Calculating the central moments of S_T

For $k \in \mathbb{N}$, the k-th central moment μ_k of a random variable X is defined as the expectation

$$\mu_k := \mathbb{E}\left[(X - \mathbb{E}[X])^k\right] .$$

From statistics we know that $\mu_1 = 0$ and $\mu_2 = \mathrm{Var}[X]$. The third central moment is used to define the skewness, the fourth central moment appears in the definition of the kurtosis of X. We now consider the stock price S_T at time T (starting in $s = S_t$ at time t) with pdf as in (2.17). We want to calculate the moments $\mathbb{E}[S_T^k]$, $k = 1, 2, \ldots$. To do so we rewrite

$$S_T = e^{\ln(\frac{S_T}{s}) + \ln(s)} = s e^{R^c_{t,T}}$$

with $R^c_{t,T}$ the log-return as in (2.14). Since the log-return is normal with mean μ_S and volatility σ_S, we have (in distribution) $R^c_{t,T} \stackrel{d}{=} \mu_S + \sigma_S Z$, where Z is standard normal with pdf ϕ as in (2.10). Thus there holds

$$\begin{aligned}
\mathbb{E}[S_T^k] &= \mathbb{E}[s^k e^{kR^c_{t,T}}] = s^k \mathbb{E}[e^{kR^c_{t,T}}] = s^k \mathbb{E}[e^{k\mu_S + k\sigma_S Z}] = s^k e^{k\mu_S} \mathbb{E}[e^{k\sigma_S Z}] \\
&= s^k e^{k\mu_S} \int_{-\infty}^{\infty} e^{k\sigma_S z} \phi(z) \mathrm{d}z = s^k e^{k\mu_S} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{k\sigma_S z - \frac{1}{2}z^2} \mathrm{d}z .
\end{aligned}$$

We complete squares in the exponent of the integrand

$$-\frac{1}{2}z^2 + k\sigma_S z = -\frac{1}{2}(z - k\sigma_S)^2 + \frac{1}{2}k^2\sigma_S^2$$

and obtain

$$\begin{aligned}
\mathbb{E}[S_T^k] &= s^k e^{k\mu_S + \frac{1}{2}k^2\sigma_S^2} \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z - k\sigma_S)^2} \mathrm{d}z}_{=1} = s^k e^{k\mu_S + \frac{1}{2}k^2\sigma_S^2} \\
&= s^k e^{k(\mu - \frac{1}{2}\sigma^2(1-k))(T-t)} . \qquad (2.28)
\end{aligned}$$

For $k = 1$ we therefore have $\mathbb{E}[S_T] = s e^{\mu(T-t)}$, for the variance we find

$$\begin{aligned}
\mu_2 = \mathrm{Var}[S_T] &= \mathbb{E}\left[(S_T - \mathbb{E}[S_T])^2\right] = \mathbb{E}[S_T^2] - \mathbb{E}[S_T]^2 \\
&\stackrel{(2.28)}{=} s^2 e^{2(\mu + \frac{1}{2}\sigma^2)(T-t)} - s^2 e^{2\mu(T-t)} \\
&= s^2 e^{2\mu(T-t)} \left(e^{\sigma^2(T-t)} - 1\right) .
\end{aligned}$$

These are the formulas (2.18) and (2.19), respectively. The skewness of S_T is defined by $\mathrm{Skew} = \frac{\mu_3}{\mu_2^{3/2}}$ and follows by (2.28) and some algebraic manipulations

$$\begin{aligned}
\mathrm{Skew} &= \frac{\mu_3}{\mu_2^{3/2}} = \frac{1}{\mu_2^{3/2}} \left(\mathbb{E}[S_T^3] - 3\mathbb{E}[S_T^2]\mathbb{E}[S_T] + 2\mathbb{E}[S_T]^3\right) \\
&\stackrel{(2.28)}{=} \left(e^{\sigma^2(T-t)} + 2\right) \sqrt{e^{\sigma^2(T-t)} - 1} > 0 ,
\end{aligned}$$

which means that the stock price S_T is right-skewed, compare with figure 2.3. Similarly, the kurtosis Kurt $= \frac{\mu_4}{\mu_2^2}$ becomes - again with algebraic manipulations -

$$\begin{aligned}
\text{Kurt} &= \frac{\mu_4}{\mu_2^2} = \frac{1}{\mu_2^2}\left(\mathbb{E}[S_T^4] - 4\mathbb{E}[S_T^3]\mathbb{E}[S_T] + 6\mathbb{E}[S_T^2]\mathbb{E}[S_T]^2 - 3\mathbb{E}[S_T]^4\right) \\
&\overset{(2.28)}{=} e^{4\sigma^2(T-t)} + 2e^{3\sigma^2(T-t)} + 3e^{2\sigma^2(T-t)} - 3\,.
\end{aligned}$$

2.8 Technical note. Simulating a d-dimensional GBM

In this section we discuss the generation of one path of a d-dimensional geometric Brownian motion $\mathbf{S}_t = (S_t^1,\ S_t^2, \ldots, S_t^d)$; each component is given by

$$S_t^i = S_0^i e^{(\mu_i - \frac{1}{2}\sigma_i^2)t + \sum_{j=1}^d L_{ij} W_t^j}\,, \qquad (2.29)$$

compare with (2.26). At time points

$$0 = t_0 < t_1 < t_2 < \ldots < t_n$$

we want to evaluate/simulate the process S_t^i. To do so, we write for arbitrary $k \in \{1, \ldots, n\}$

$$\begin{aligned}
S_{t_k}^i &= S_0^i \frac{S_{t_k}^i}{S_{t_{k-1}}^i} \frac{S_{t_{k-1}}^i}{S_{t_{k-2}}^i} \cdot \ldots \cdot \frac{S_{t_2}^i}{S_{t_1}^i} \frac{S_{t_1}^i}{S_0^i} = S_0^i \prod_{\ell=1}^k \frac{S_{t_\ell}^i}{S_{t_{\ell-1}}^i} \\
&\overset{(2.29)}{=} S_0^i \prod_{\ell=1}^k e^{(\mu_i - \frac{1}{2}\sigma_i^2)(t_\ell - t_{\ell-1}) + \sum_{j=1}^d L_{ij}(W_{t_\ell}^j - W_{t_{\ell-1}}^j)} \\
&= S_0^i e^{\sum_{\ell=1}^k (\mu_i - \frac{1}{2}\sigma_i^2)(t_\ell - t_{\ell-1}) + \sum_{j=1}^d L_{ij}(W_{t_\ell}^j - W_{t_{\ell-1}}^j)} \\
&= S_0^i e^{(\mu_i - \frac{1}{2}\sigma_i^2)t_k + \sum_{\ell=1}^k \sum_{j=1}^d L_{ij}(W_{t_\ell}^j - W_{t_{\ell-1}}^j)}\,.
\end{aligned}$$

We define $X_{j\ell} := W_{t_\ell}^j - W_{t_{\ell-1}}^j$. According to definition 2.1, the random variable $X_{j\ell} \sim \mathcal{N}(0, t_\ell - t_{\ell-1})$ is normal with mean 0 and standard deviation $\sqrt{t_\ell - t_{\ell-1}}$. Whence it suffices to consider the random vectors

$$\mathbf{Z}_\ell = \begin{pmatrix} Z_{1\ell} \\ Z_{2\ell} \\ \vdots \\ Z_{d\ell} \end{pmatrix} \in \mathbb{R}^d,\ \ell = 1, \ldots, k$$

with $Z_{j\ell} \sim \mathcal{N}(0,1)$ standard normal and to write

$$S_{t_k}^i = S_0^i e^{(\mu_i - \frac{1}{2}\sigma_i^2)t_k + \sum_{\ell=1}^k \sqrt{t_\ell - t_{\ell-1}} \sum_{j=1}^d L_{ij} Z_{j\ell}}\,.$$

The sum $\sum_{j=1}^d L_{ij} Z_{j\ell}$ is the product of the i-th row of $\mathbf{L}$ (which we denote by $\mathbf{L}_i$) and the column vector $\mathbf{Z}_\ell$, thus

$$S_{t_k}^i = S_0^i e^{(\mu_i - \frac{1}{2}\sigma_i^2)t_k + \mathbf{L}_i \sum_{\ell=1}^k \sqrt{t_\ell - t_{\ell-1}}\, \mathbf{Z}_\ell}\,.$$

We obtain a realisation $s_{i,k}$ of the random variable $S^i_{t_k}$ by drawing in total n random vectors $\mathbf{z}_\ell$ of length d (in Python via `randn`) and setting

$$s_{i,k} = s_i e^{(\mu_i - \frac{1}{2}\sigma_i^2)t_k + \mathbf{L}_i \sum_{\ell=1}^{k} \sqrt{\Delta t_\ell}\mathbf{z}_\ell}, \;\; i = 1, \dots, d, \;\; k = 1, \dots, n\,.$$

Herein, we set $\Delta t_\ell := t_\ell - t_{\ell-1}$ for notational simplicity. According to this formula, the generation of a path of $\mathbf{S}_t$ requires a loop "over time t" (w.r.t. k) and a loop "over the dimension d" (w.r.t. i). We want to avoid these time consuming loops and vectorise w.r.t. k first. To do so, we collect the n column vectors $\mathbf{z}_\ell$ in the matrix $\mathbf{Z} \in \mathbb{R}^{d\times n}$; in Python, we generate $\mathbf{Z}$ directly via $\mathbf{Z} = \mathtt{randn}(d, n)$. Next, we define the row vector $\Delta\mathbf{t} := (\Delta t_1, \Delta t_2, \dots, \Delta t_n)$, from which we stack d copies into the matrix $\Delta\mathbf{T} \in \mathbb{R}^{d\times n}$ (each of the d rows of $\Delta\mathbf{T}$ is equal to $\Delta\mathbf{t}$). In Python, this can be achieved by $\Delta\mathbf{T} = \mathtt{repeat}(\Delta\mathbf{t}, d, \mathtt{axis} = 0)$. The n sums

$$\sum_{\ell=1}^{1} \sqrt{\Delta t_\ell}\mathbf{z}_\ell,\; \sum_{\ell=1}^{2} \sqrt{\Delta t_\ell}\mathbf{z}_\ell,\; \dots\;, \sum_{\ell=1}^{n} \sqrt{\Delta t_\ell}\mathbf{z}_\ell$$

can now be realised in Python via $\mathbf{X} = \mathtt{cumsum}(\sqrt{\Delta\mathbf{T}} \odot \mathbf{Z}), \mathtt{axis} = 1)$. Here, the k-th row of $\mathbf{X} \in \mathbb{R}^{d\times n}$ equals to the sum $\sum_{\ell=1}^{k} \sqrt{\Delta t_\ell}\mathbf{z}_\ell$. The n products $\mathbf{L}_i \sum_{\ell=1}^{k} \sqrt{\Delta t_\ell}\mathbf{z}_\ell$ are now given by $\mathbf{L}_i\mathbf{X}$; these are row vectors of length n. If we define the column vectors of length n

Two remarks on the notation. Let $\mathbf{A} \in \mathbb{R}^{n\times m}$ be a matrix, and let be f some univariate function. Then by $f(\mathbf{A}) \in \mathbb{R}^{n\times m}$ we denote the matrix whose ij-element is given by $f(A_{ij})$, i.e., f is applied element-wise. Similarly, for a matrix $\mathbf{B}$ having the same dimension as $\mathbf{A}$ we define $\mathbf{A} \odot \mathbf{B} \in \mathbb{R}^{n\times m}$ to be the matrix whos ij-element is given by $A_{ij}B_{ij}$.

$$\mathbf{s}_i := \begin{pmatrix} s_{1,i} \\ s_{2,i} \\ \vdots \\ s_{n,i} \end{pmatrix}, \;\; \mathbf{t} := \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_n \end{pmatrix}$$

then a realisation of S^i_t at the time points t_k, $k = 1, \dots, n$, is given by

$$\mathbf{s}_i = S^i_0 e^{(\mu_i - \frac{1}{2}\sigma_i^2)\mathbf{t} + (\mathbf{L}_i\mathbf{X})^\top}, \;\; i = 1, \dots, d\;\;. \tag{2.30}$$

The exponent in (2.30) is a column vector $\mathbf{x}$ of length n; according to the above side remark, the object $e^{\mathbf{x}}$ has to be understood component-wise.

The implementation of (2.30) still requires a `for`-loop w.r.t. the dimension d such that we apply a second vectorisation. We realise the d column vectors $(\mathbf{L}_i\mathbf{X})^\top$ at once by $(\mathbf{L}\mathbf{X})^\top \in \mathbb{R}^{n\times d}$. The d column vectors $(\mu_i - \frac{1}{2}\sigma_i^2)\mathbf{t}$ can be realised at once by defining the row vector of length d

$$\mathbf{a} := \left(\mu_1 - \sigma_1^2/2, \mu_2 - \sigma_2^2/2, \dots, \mu_d - \sigma_d^2/2\right)$$

and then stack n copies of it into the matrix $\mathbf{A} \in \mathbb{R}^{n\times d}$ (each of the n rows of $\mathbf{A}$ is equal to $\mathbf{a}$). In Python, this can be done by $\mathbf{A} = \mathtt{repeat}(\mathbf{a}, n, \mathtt{axis} = 0)$. Similarly, we create the matrix $\mathbf{T} \in \mathbb{R}^{n\times d}$

of which each column corresponds to the vector $\mathbf{t}$, in Python $\mathbf{T} = \texttt{repeat}(\mathbf{t}, d, \texttt{axis} = 1)$. The d column vectors $(\mu_i - \frac{1}{2}\sigma_i^2)\mathbf{t}$ can now be realised by the element-wise product $\mathbf{A} \odot \mathbf{T}$. Finally, let

$$\mathbf{s}_0 := \left(S_0^1, S_0^2, \ldots, S_0^d\right)$$

be the row vector of the initial values and let $\mathbf{S}_0 \in \mathbb{R}^{n \times d}$ be the matrix whose n rows are all equal to $\mathbf{s}_0$. If $\mathbf{s}_i$ in (2.30) is the i-th column of the matrix $\mathbf{S} \in \mathbb{R}^{n \times d}$, then

$$\mathbf{S} = \mathbf{S}_0 \odot e^{\mathbf{A} \odot \mathbf{T} + (\mathbf{L}\mathbf{X})^\top} \tag{2.31}$$

is an exact simulation of a d-dimensional geometric Brownian motion at the time points t_k, $k = 1, \ldots, n$. The Python function `gbm_d.py` realises (2.31); the function returns the matrix

$$\begin{pmatrix} \mathbf{s}_0 \\ \mathbf{S} \end{pmatrix} \in \mathbb{R}^{(n+1) \times d}$$

as well as the column vector $\mathbf{t}_0 = (0, t_1, t_2, \ldots, t_n)^\top \in \mathbb{R}^{n+1}$. It can be used via $\texttt{plot}(\mathbf{t}_0, \mathbf{S})$ to visualise the time evolution of each S_t^i.

Chapter 3

European option pricing

In the previous chapter we introduced the geometric Brownian motion as mathematical model for the stochastic price evolution of a stock. At this stage, this model can not be used for pricing, since it is stated with respect to the "wrong probability measure". As it turns out and as we have already heuristically seen in section 1.6, the drift μ in the model (which is approximatively the expected return of the underlying) is completely irrelevant to the price of a derivative. To understand this a little bit better, we need to go back to binomial trees and to consider the pricing of European options (the simplest case!) in such a model. We combine then the findings of this section with the geometric Brownian motion of the previous chapter. This leads us to the famous formula of Black and Scholes (to price a call or a put). We then must have a look on the shortcomings of the Black-Scholes model and discuss two possible extensions of this model. Both these extensions are used in practice and address the utmost importance of modelling volatility. Since a large part of structured products involve options on multiple stocks, we have to extend the presented material to (European) options written on multiple underlyings.

3.1 The principles of option pricing in a nut shell

The binomial tree model set up in the previous chapter contains the three parameters u, d and p. It turns out that in order to price options in a "meaning full" way the parameter p can not be chosen arbitrarily. To see this, we consider the following situation in a tree with just one period.

- A stock with current price $s_{0,0} = 100$ CHF. Either, the price of the stock will increase to $s_{1,1} = 105$ CHF tomorrow with probability $p = 0.6$ or it will decrease to $s_{0,1} = 95$ CHF with probability

$q = 1 - p = 0.4.$

- A call option with strike $X = 100$ CHF on this stock. The option expires tomorrow.
- The continuously compounded risk free rate is $r = 0$.

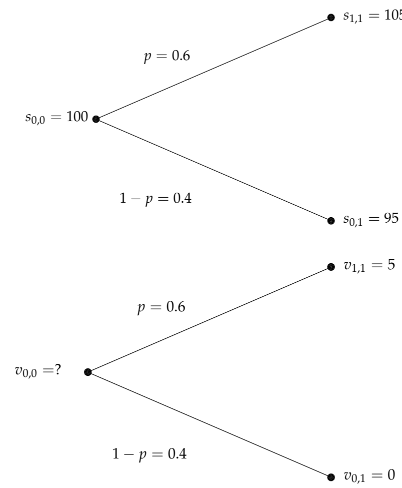

Figure 3.1: Value of stock and option at maturity.

The random variable S_1 takes two possible values, $s_{1,1} = 105$ and $s_{0,1} = 95$. Now, there are two possible scenarios respectively cash flows generated by our option tomorrow.

u) The stock increases, such that the option pays off

$$v_{1,1} = \max\{s_{1,1} - X, 0\} = \max\{105 - 100, 0\} = 5 \text{ CHF}$$

d) The stock decreases and the cash flow from the option is

$$v_{0,1} = \max\{s_{0,1} - X, 0\} = \max\{95 - 100, 0\} = 0 \text{ CHF}$$

compare with figure 3.1. We observe that the cash flows $v_{1,1}$ and $v_{0,1}$ are realisations of a random variable V_1, which itself is a function of the random variable S_1, i.e., $V_1 = \max\{S_1 - 100, 0\}$. The fundamental question is: What is the price/value $v_{0,0}$ of the option today? The answer will *not* be the expectation

$$v_{0,0} = pv_{1,1} + qv_{0,1} = 0.6 \cdot 5 + 0.4 \cdot 0 = 3 \text{ CHF} .$$

To see this, we construct a portfolio consisting in a short position in the call option and a long position in δ shares of the stock, where δ is yet undefined. Again, the are two scenarios (at maturity) for the (random) value Π of this portfolio.

u) If the stock price increases, the value will be

$$\pi_{1,1} = \delta s_{1,1} - v_{1,1} = \delta \cdot 105 - 5 .$$

d) If, however, the stock price decreases, Π takes the value

$$\pi_{0,1} = \delta s_{0,1} - v_{0,1} = \delta \cdot 95 - 0 .$$

We can eliminate the randomness in Π by choosing δ accordingly. We set δ such that, no matter the stock price will go up or down, the value of the portfolio is the same in both scenarios, i.e., $\pi_{1,1} = \pi_{0,1}$. Thus, δ solves the equation

$$\begin{aligned} \pi_{1,1} &= \pi_{0,1} \\ \delta s_{1,1} - v_{1,1} &= \delta s_{0,1} - v_{0,1} \end{aligned}$$

with solution

$$\delta = \frac{v_{1,1} - v_{0,1}}{s_{1,1} - s_{0,1}} = \frac{5}{10} = \frac{1}{2} . \tag{3.1}$$

Choosing δ in this way, the value of the portfolio tomorrow becomes independent of the stock movement and is given by

$$\pi_{1,1} = \pi_{0,1} = \frac{1}{2} \cdot 105 - 5 = \frac{1}{2} \cdot 95 - 0 = 47.5 \text{ CHF} .$$

The portfolio is *riskless*, since it generates the cash flow 47.5 for sure. Now, if the value of this riskless portfolio tomorrow is 47.5, and the continuously compounded risk free rate is r by assumption, the value $\pi_{0,0}$ of the portfolio today must be $e^{-r\Delta t}47.5 = 47.5$ (with $\Delta t = 1/252$ in this particular case). Otherwise, there would exist arbitrage opportunities.[1] The value of the portfolio today is by construction also given by $\pi_{0,0} = \delta s_{0,0} - v_{0,0}$, thus we can solve for the unknown option price $v_{0,0}$

$$v_{0,0} = \delta s_{0,0} - \pi_{0,0} = \delta s_{0,0} - e^{-r\Delta t}\pi_{1,1}$$

from where it follows that

$$v_{0,0} = \frac{1}{2} \cdot 100 - e^{-0 \cdot \Delta t} \cdot 47.5 = 2.5 \text{ CHF} .$$

We are going to formalise the above steps. Symbolically, we have done the following (with $\pi = \pi_{1,1} = \pi_{0,1}$)

$$\begin{aligned}
v_{0,0} &= \delta s_{0,0} - e^{-r\Delta t}\pi \\
&= \delta s_{0,0} - e^{-r\Delta t}(\delta s_{1,1} - v_{1,1}) \\
&= \delta(s_{0,0} - e^{-r\Delta t}s_{1,1}) + e^{-r\Delta t}v_{1,1} \\
&\overset{(3.1)}{=} \frac{v_{1,1} - v_{0,1}}{s_{1,1} - s_{0,1}}(s_{0,0} - e^{-r\Delta t}s_{1,1}) + e^{-r\Delta t}v_{1,1} \\
&= \frac{v_{1,1} - v_{0,1}}{us_{0,0} - ds_{0,0}}(s_{0,0} - e^{-r\Delta t}us_{0,0}) + e^{-r\Delta t}v_{1,1} \\
&= \frac{v_{1,1} - v_{0,1}}{u - d}(1 - e^{-r\Delta t}u) + e^{-r\Delta t}v_{1,1}
\end{aligned}$$

$$\begin{aligned}
v_{0,0} &= \left(\frac{1 - e^{-r\Delta t}u}{u - d} + e^{-r\Delta t}\right)v_{1,1} + \frac{e^{-r\Delta t}u - 1}{u - d}v_{0,1} \\
&= \frac{1 - e^{-r\Delta t}u + e^{-r\Delta t}u - e^{-r\Delta t}d}{u - d}v_{1,1} + \frac{e^{-r\Delta t}u - 1}{u - d}v_{0,1} \\
&= e^{-r\Delta t}\underbrace{\frac{e^{r\Delta t} - d}{u - d}}_{=p^*}v_{1,1} + e^{-r\Delta t}\underbrace{\frac{u - e^{r\Delta t}}{u - d}}_{=q^*}v_{0,1} \\
&= e^{-r\Delta t}\left(p^* v_{1,1} + q^* v_{0,1}\right) . \qquad (3.2)
\end{aligned}$$

Herewith, we have defined the probabilities

$$p^* := \frac{e^{r\Delta t} - d}{u - d}, \quad q^* := \frac{u - e^{r\Delta t}}{u - d} \qquad (3.3)$$

[1] We dispense with the formal definition of arbitrage and content ourselves with the following description. Arbitrage means that the value of the portfolio is not negative and there is a positive probability that the return on the portfolio is positive. The risk of losing money is zero.

which sum up to one, i.e., $p^* + q^* = 1$. Note that the parameters u and d have to satisfy

$$0 < d < e^{r\Delta t} < u \ ,$$

otherwise p^* and q^* are not probabilities (they would become negative). If the above conditions are satisfied, then p^* defines a new probability measure $\mathbb{Q}$ on $(\Omega, 2^\Omega)$. Under the measure $\mathbb{Q}$ the random variable S_m (the stock price at maturity) is still binomially distributed

$$\mathbb{Q}[S_m = s_{i,m}] = \binom{m}{i} {p^*}^i {q^*}^{m-i}, \quad i = 0, \ldots, m \ . \tag{3.4}$$

Since $\mathbb{Q}$ is constructed by risk-neutrality, this measure is also called a risk-neutral probability measure or martingale measure. A martingale is a stochastic process for which the best prediction (the expectation) of the martingale's future value is its present value. The discounted stock price (the random variable)

$$\widehat{S}_m := e^{-rT} S_m$$

is a martingale since it satisfies by problem 3.1 the definition

$$\mathbb{E}^{\mathbb{Q}}[\widehat{S}_m] = \widehat{S}_0 \ , \tag{3.5}$$

i.e., "the expected future value is equal to the present value", where $\widehat{S}_0 = e^{-r \cdot 0} S_0 = s_{0,0}$. (The "random variable" S_0 admits only one value with probability one, namely the stock price today $S_0 = s_{0,0}$.)

According to equation (3.2), the option price $v_{0,0}$ *is* a (discounted) expectation, however not with respect to the probability measure $\mathbb{P}$ (which has become entirely irrelevant) but with respect to the probability measure $\mathbb{Q}$, i.e.,

$$v_{0,0} = e^{-r\Delta t} \mathbb{E}^{\mathbb{Q}}[V_1] \ .$$

Since the random variable $V_1 = \max\{S_1 - 100, 0\} =: g(S_1)$ (the possible cash flows at maturity) is a function of the random variable S_1 (the possible stock prices at maturity), we can write

$$v_{0,0} = e^{-r\Delta t} \mathbb{E}^{\mathbb{Q}}\left[g(S_1)\right] \ . \tag{3.6}$$

We generalise the above construction to binomial trees having an arbitrary number m of periods. To do so, we consider the "node" (i, j) in the tree (corresponding to time $t_j = j\Delta t$) and its two neighbours $(i + 1, j + 1)$ and $(i, j + 1)$ on the next time level (corresponding to time $t_{j+1} = (j + 1)\Delta t$). We assume that the option prices $v_{i+1,j+1}$ and $v_{i,j+1}$ are known. We want to find the option price $v_{i,j}$, compare with figure 3.2.

At time t_j, we construct a portfolio consisting in a long position of a certain number δ of shares of the stock and a short position of a call

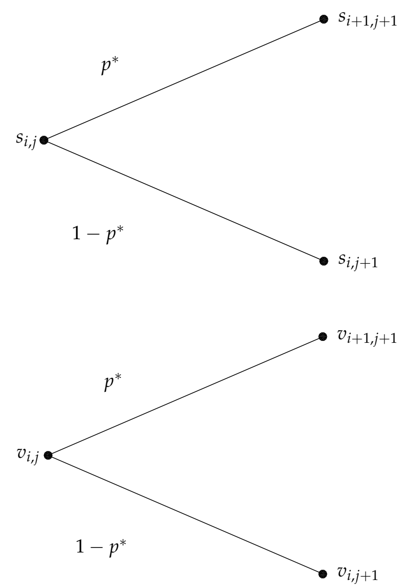

Figure 3.2: Stock and option prices somewhere in the tree.

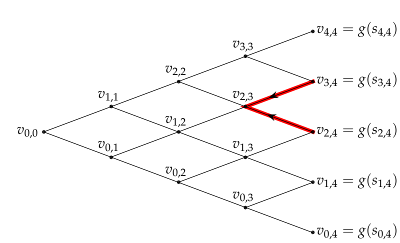

Figure 3.3: Starting with the option prices $v_{i,m} = g(s_{i,m})$ at maturity ($t = T = m\Delta t$) we are able to calculate recursively the option price $v_{0,0}$ today ($t = 0$) by going "backward" through the tree. Each value $v_{i,j}$ is calculable from the options prices $v_{i+1,j+1}$ and $v_{i,j+1}$. Schematic representation with $m = 4$ periods.

option on the stock. Repeating verbatim the arguments that lead to pricing formula (3.2), we end with

$$v_{i,j} = e^{-r\Delta t}(p^* v_{i+1,j+1} + q^* v_{i,j+1}), \quad j = m-1, \ldots, 0, \quad i = 0, 1, \ldots, j \tag{3.7}$$

where the probabilities p^* and q^* are as in (3.3). If we want to price an option with payoff function g using a binomial tree, we thus first calculate the option prices $v_{i,m}$ at maturity, i.e.,

$$v_{i,m} = g(s_{i,m}), \quad i = 0, \ldots, m$$

and then go recursively backward through the tree via (3.7) until we have found the option price $v_{0,0}$ today, compare with figure 3.3. Solving the recursion (3.7) we may calculate the option price $v_{0,0}$ directly. Indeed, one can show that the recursion (3.7) is equivalent to

$$v_{0,0} = e^{-rT} \sum_{i=0}^{m} \binom{m}{i} {p^*}^i {q^*}^{m-i} v_{i,m} \, . \tag{3.8}$$

In this weighted sum the option prices $v_{i,m}$ are realisations

$$v_{i,m} = g(s_{i,m})$$

of the random variable $V_m = g(S_m)$, and the product $\binom{m}{i}{p^*}^i {q^*}^{m-i}$ is the risk-neutral probability that the stock price S_m at maturity admits the value $s_{i,m}$, compare with (3.4). Thus, the option price $v_{0,0}$ can also be written as

$$v_{0,0} = e^{-rT} \sum_{i=0}^{m} \mathbb{Q}[S_m = s_{i,m}] g(s_{i,m}) \, ,$$

such that the value $v_{0,0}$ is a discounted conditional expectation (with respect to the risk-neutral probability measure) of random cash flows $V_m = g(S_m)$ in the future, i.e.,

$$v_{0,0} = e^{-rT} \mathbb{E}^{\mathbb{Q}} \left[g(S_m) \mid s_{0,0} = s \right] . \tag{3.9}$$

Note that equation (3.9) leads to a "discrete" version of the Black-Scholes formula. Indeed, for a call option with $g(x) = \max\{x - X, 0\}$ for example we find the above expectation to be

$$v_{0,0} = s B_{m,\widehat{p}}(i^*) - X e^{-rT} B_{m,p^*}(i^*) \, , \tag{3.10}$$

where $s = s_{0,0}$ is the current stock price and $B_{n,p}(x)$ denotes the complementary distribution function of the binomial distribution[2] with parameters n and p, i.e., for $x \in \mathbb{N}$,

$$\begin{aligned} B_{n,p}(x) &:= \sum_{i=x}^{n} \binom{n}{i} p^i (1-p)^{n-i} = 1 - \sum_{i=0}^{x-1} \binom{n}{i} p^i (1-p)^{n-i} \\ &= 1 - F_{n,p}(x-1) \, ; \end{aligned}$$

[2] The cumulative distribution function $F_{n,p}(x)$ can be realised in Python via `binom.cdf(x, n, p)`.

compare with problem 3.2. As we have seen in section 2.2, the discrete random variable S_m converges (in distribution) to the continuous random variable S_T in (2.7) if one let the number of periods m tend to infinity (where we have to replace μ by r, compare also with figure 2.3) and the price of the option converges to

$$V(s,0) = e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[g(S_T) \mid S_0 = s\right] . \tag{3.11}$$

Note carefully that the measure $\mathbb{Q}$ in (3.11) is different from the measure $\mathbb{Q}$ in (3.9). Whereas the former is a discrete measure with corresponding probability mass function (3.4), the latter corresponds to the (cumulative) distribution function of the log-normal distribution with density (2.17), where $\mu = r$. Since S_T is a continuous random variable, the expectation in (3.11) is not a sum any more but the integral

$$V(s,0) = e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[g(S_T) \mid S_0 = s\right] = e^{-rT}\int_0^\infty g(x) f_S(x)\mathrm{d}x \, ,$$

where $f_S(x)$ is the probability density function in (2.17) of the random variable S_T. The insight that the value of a derivative corresponds to a discounted expectation with respect to a martingale measure (i.e., a measure such that the discounted stock price is a martingale) as in (3.11) is one of the backbones of modern finance.

3.2 The formula of Black and Scholes

We proceed to the pricing of options in the model of the Black and Scholes. The model is based on the following assumptions.

- The price S of the stock follows a geometric Brownian motion (2.7). In particular, the volatility σ is constant.
- There exists a (continuously compounded) risk free rate r and there are no arbitrage opportunities. This means that the drift μ in (2.15) has to be replaced by the risk free rate, $\mu = r$.[3]

If these assumptions are met, the price or the value $V = V(s,t)$ of a European option[4] with payoff function g is given by

$$V(s,t) = \mathbb{E}^{\mathbb{Q}}[e^{-r(T-t)}g(S_T) \mid S_t = s] \, , \tag{3.12}$$

with S_T as in (2.7), where we have to replace μ by $r - q$. Thus, for example, a European call option with strike X has the price

$$V_c(s,t) = e^{-r(T-t)}\int_0^\infty \max\{x - X, 0\} f_S(x)\mathrm{d}x \, ,$$

where $f_S(x)$ denotes the density of S_T in (2.17), with mean and variance given by

$$\mu_S = (r - q - \frac{1}{2}\sigma^2)(T-t), \quad \sigma_S = \sigma\sqrt{T-t} \, . \tag{3.13}$$

[3] If the stock pays a (continuous) dividend q, we have $\mu = r - q$.

[4] In dependence on the price of the underlying $S_t = s$ and time $0 \le t < T$.

Evaluating this integral leads to the formula of Black and Scholes[5],

$$V_c(s,t) = V(s,t;X,T,\sigma,r,q,1)$$

(compare with the technical note in section 3.7), where V is given by

$$V(s,t;X,T,\sigma,r,q,\omega) := \omega\big(se^{-q(T-t)}N(\omega d_1) - Xe^{-r(T-t)}N(\omega d_2)\big)\ . \tag{3.14}$$

Herewith, the auxiliary variables d_1 and d_2 depend on $s, T-t, X, \sigma, r$ as well as on q and are given by

$$\begin{aligned} d_1 &:= \frac{1}{\sigma\sqrt{T-t}}\left(\ln\frac{s}{X} + \left(r - q + \frac{\sigma^2}{2}\right)(T-t)\right) \\ d_2 &:= d_1 - \sigma\sqrt{T-t}\ . \end{aligned}$$

For a (European) put option we correspondingly find

$$V_p(s,t) = V(s,t;X,T,\sigma,r,q,-1)\ .$$

Exemplarily, we plot in figure 3.4 the graph of the function

$$[80,160] \ni s \mapsto V_c(s,0)$$

for the parameter values $T = 1$, $X = 120$, $\sigma = 0.1$, $r = 0.02$ and $q = 0$.

[5] Named after the financial economists Fischer Sheffey Black (USA, 1938–1995) and Myron Samuel Scholes (Canada, 1941–).

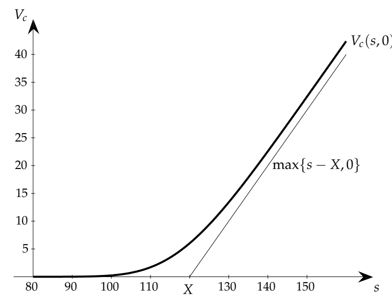

Figure 3.4: Value $V_c(s,0)$ of a European call option as function in the price s of stock (parameter values are $T = 1$, $X = 120$, $\sigma = 0.1$, $r = 0.02$, $q = 0$).

Example 3.1. On June 4, 2019, we find the theoretical (Black-Scholes) price of the following warrant (see figure 3.5) and compare it with the issue price, which is 0.38 CHF. The price of the underlying (ABB) is $s = 18.38$ CHF. The (implied) volatility is 23.83%, and as a proxy for the risk free rate we take the 6-months Swiss franc LIBOR which is -0.657%.

Solution. Since during the life time of the warrant there are no dividend payments we have $q = 0$. For $r - q < 0$ it is known that the early exercise of an American option is not optimal, such that in this particular case the value of the American put is equal to its European counterpart. Thus, we may use formula (3.14) with $\omega = -1$ to price the warrant. We first find the auxiliary variables d_1 and d_2. According to the day-count convention (1.1) the time-to-maturity is $T - t = 196/360$ (June 4, 2019 until December 20, 2019) and $r - q = \ln(1 - 0.00657) \doteq -0.00659$, hence

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{T-t}}\left(\ln\frac{s}{X} + \left(r - q + \frac{\sigma^2}{2}\right)(T-t)\right) \\ &= \frac{1}{0.2383\sqrt{196/360}}\left(\ln\frac{18.38}{18} + \left(-0.00659 + \frac{0.2383^2}{2}\right)\frac{196}{360}\right) \\ &\doteq 0.186325 \\ d_2 &= d_1 - \sigma\sqrt{T-t} \doteq 0.186325 - 0.2383\sqrt{196/360} \doteq 0.010492\ . \end{aligned}$$

Thus we have $N(-d_1) \doteq 0.42609$ as well as $N(-d_2) \doteq 0.49581$, and we find for the price of the put

$$
\begin{aligned}
V_p(s,0) &\overset{(3.14)}{=} Xe^{-r(T-t)}N(-d_2) - se^{-q(T-t)}N(-d_1) \\
&\doteq 18 \cdot e^{0.00659 \cdot \frac{196}{360}} \cdot 0.49581 - 18.38 \cdot 0.42609 \\
&\doteq 1.1251 \, .
\end{aligned}
$$

We need to divide V_p by the ratio 3 and obtain the theoretical warrant price of $V_p/3 \doteq 0.375$ CHF; this corresponds to the stated issue price. ◇

PUT WARRANTS ON ABB LTD / "ABBPJB"

(the "Products")

SSPA SWISS DERIVATIVE MAP© / EUSIPA DERIVATIVE MAP© WARRANT (2100)

EXERCISE PRICE/STRIKE CHF 18.00 – AMERICAN STYLE – PHYSICAL SETTLEMENT – CHF

This document is for information purposes only.

A Product does not constitute a collective investment scheme within the meaning of the Swiss Federal Act on Collective Investment Schemes. Therefore, it is not subject to authorisation by the Swiss Financial Market Supervisory Authority FINMA ("FINMA") and potential investors do not benefit from the specific investor protection provided under the CISA and are exposed to the credit risk of the Issuer.

I. Product Description

Terms

Swiss Security Number (Valor)	48233913
ISIN	CH0482339139
Symbol	ABBPJB
Issue Size	10,000,000 Products (CHF 3,800,000) (may be increased/decreased at any time)
Issue Currency	CHF
Exercise Currency	CHF
Issue Price	CHF 0.38 (per Product)
Exercise Style	American
Initial Volatility	23.83%
Initial Delta	-42.77%
Initial Gearing	16.22x
Initial Premium	8.23%

Launch Date: 04 June 2019, being the date on which the Exercise Price/Strike and the Initial Level are fixed.

Issue Date/Payment Date: 05 June 2019, being the date on which the Products are issued and the Issue Price is paid.

Exercise Period: From 04 June 2019 to 20 December 2019, being the period during which the Products may be exercised.

Exercise Date: with respect to any Product, the Exchange Business Day during the Exercise Period on which such Product is exercised.

Exercise Lot: 3 Products and multiples thereof, being the smallest number of Products required in order to exercise the Products.

Expiration Date: 20 December 2019, being the day on which the Products will expire at the Expiration Time. If the Expiration Date is not an Exchange Business Day, the immediately following Exchange Business Day shall be deemed to be the Expiration Date.

Expiration Time: Scheduled Closing Time

Settlement Date: 3 Business Days after the Exercise Date excluding the Exercise Date, being the date on which any cash payment or delivery of Underlying(s) due under the relevant Products upon exercise shall be made.

Last Trading Date: 20 December 2019, until 12:00 CET, being the last date on which the Products may be traded.

Underlying

ABB Ltd (ABBN SW <EQUITY>; SIX Swiss Exchange)

Initial Level	CHF 18.38
Exercise Price/Strike	CHF 18.00 (97.93%)[1)]
Ratio	3:1, i.e. 3 Products to 1 Share
Currency	CHF

Valuation Time	Scheduled Closing Time
ISIN	CH0012221716
Valor	1222171

[1)] in % of the Underlying's Initial Level

Figure 3.5: Part of term sheet for a put warrant on ABB.

The formula (3.14) of Black and Scholes is a so-called closed-form solution to the (call) option pricing problem and is only valid under the assumption that the price of the underlying follows a geometric Brownian motion (or equivalently, that the log-returns of the underlying are normal, compare with (2.14)). Naturally, one has to extend the pricing problem in two directions

E_1 (Extension 1.) Can we still derive closed-form solutions to price other derivatives than European? An American option for example gives its holder the right to exercise not only at maturity, but at any time during the life time $]0, T]$ of the option. The payoff of an Asian fixed strike option on the other hand depends not just on the value S_T of the underlying at maturity, but on the (discrete) average $A_T = \sum_{k=1}^{n} S_{t_k}$ over predetermined dates $t_k \in [0, T]$. For both examples (American, discrete fixed strike Asian) there are no closed form solutions such that we have to rely on numerical methods. Barrier options become worthless (or worth-full), if the underlying hits the barrier(s) during the life time of the option. For such options there exist formulas to find their price, but some of them are quit involved (this is in particular true for double barrier options), compare with chapter 4. At this stage, we could name many other types of options/derivatives for which there are no pricing formulas or for which the pricing formulas are too complicated to evaluate them directly (see example 3.2 below).

E_2 (Extension 2.) By now it is well accepted and demonstrated in many empirical studies that the Black-Scholes model (2.7) is inappropriate to map real stock price/return data. In particular, the model of Black and Scholes assumes normally distributed log-returns (2.13), which is not observable in reality. Furthermore, the volatility $\sigma > 0$ is not constant (over time) as proposed by the model. Even worse, it seems that volatility itself is stochastic. In section 3.3 we will come back to the short comings of the Black-Scholes model. Due to this mis-modelling people suggested alternative models for the time evolution of stock prices S_t in the sequel. Some of these models are mathematically very demanding and for a larger part of them there are no closed form solutions to the pricing problem, even not for plain vanillas. Again we have to dwell on numerical methods.

Of course, we can (and have to) combine the above extensions. For example, how can we price an American option in a so-called local volatility model?

Example 3.2 (to E_1). We look at a pricing problem for which there exists a pricing formula, but the formula is almost impossible to evaluate. The following description is taken from Heynen and Kat[6]. "A floating strike lookback option grants its holder the right to buy at the lowest price (when it is a call) or to sell at the highest price (when it is a put) recorded during the lookback period, i.e., the period during which the underlying price is monitored." If the lookback period consists in a discrete set of dates t_j with $0 = t_0 < t_1 < \ldots < t_J \leq T$ the option is said to be discretely monitored. If we denote by $m_{t_0}^{t_J}$ the minimum

$$m_{t_0}^{t_J} = \min\{S_{t_0}, S_{t_1}, \ldots, S_{t_J}\} ,$$

[6] R. Heynen and H. Kat. Lookback options with discrete and partial monitoring of the underlying price. *Applied Mathematical Finance*, 2:273–284, 1995

then the payoff of a call is $\max\{S_T - m_{t_0}^{t_J}, 0\}$ and the value of a discretely monitored floating strike lookback call option is

$$V_{c,\text{fls}}(s,m,t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\big[\max\{S_T - m_{t_0}^{t_J}, 0\} \mid S_t = s,\ m_{t_0}^{t} = m\big] .$$

Similarly, the value of the corresponding put is

$$V_{p,\text{fls}}(s,m,t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\big[\max\{m_{t_0}^{t_J} - S_T, 0\} \mid S_t = s,\ m_{t_0}^{t} = m\big] ;$$

here, $m_{t_0}^{t_J}$ denotes the maximum of the prices.

It turns out that the product defined in table 1.3 is a portfolio of discretely monitored floating strike lookback options. More precisely, the value $V(s,0)$ of the product at inception is

$$V(s,0) = V_{p,\text{fls}}(s,s,0) + V_{c,\text{fls}}(s,s,0)$$

such that we need to calculate both the above expectations. If we take the Black-Scholes model for S_t, then these expectations can be calculated analytically. For example, for a call there holds the formula

$$\begin{aligned} V_{c,\text{fls}}(s,s,0) &= s\Big(1 - N(\mathbf{a};\boldsymbol{\rho}_1, J+1) - \sum_{j=1}^{J} e^{r(t_j - T)} f_j\Big) \\ &\qquad -e^{-rT} s\big(N(\mathbf{b};\boldsymbol{\rho}_2, J) - N(\mathbf{c};\boldsymbol{\rho}_3, J+1)\big) , \end{aligned}$$

where $N(\mathbf{x};\boldsymbol{\rho},d)$ denotes the cdf of the d-dimensional standard normal distribution with $d \times d$-correlation matrix $\boldsymbol{\rho}$ evaluated at the point $\mathbf{x} = (x_1, \dots, x_d)$, i.e.,[7]

$$N(\mathbf{x};\boldsymbol{\rho},d) = c_d \int_{-\infty}^{x_d} \cdots \int_{-\infty}^{x_1} e^{-\frac{1}{2}\mathbf{y}^\top \boldsymbol{\rho}^{-1}\mathbf{y}} \mathrm{d}\mathbf{y} .$$

[7] The correlation matrices $\boldsymbol{\rho}_1$, $\boldsymbol{\rho}_2$ and $\boldsymbol{\rho}_3$ depend on the t_j and/or T. The vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ (at $t = 0$) depend on r, σ and/or T, t_j. The f_j's in the sum involve terms of the form $N(\cdot;\boldsymbol{\rho}_{3,j}, j)$, $N(\cdot;\boldsymbol{\rho}_{4,j}, J-j)$ as well as $N(\cdot;\boldsymbol{\rho}_{5,j}, J+1-j)$. The precise dependencies are not of our interest.

Typically, the number J of observation dates is "large"; for the product in table 1.3 we have $J = 252$ (equal to the number of business days per year). According to the above formula, we thus need to evaluate a 253-dimensional integral numerically, since the cdf is not known in closed form. Evaluating high-dimensional integrals to a high precision within an acceptable computation time is very hard; the above pricing formula is of no practical relevance and we need other approximation methods.

The function `min_max.py` finds the Black-Scholes price V of the product given in table 1.3 via Monte Carlo simulation using n paths. Each path is generated according to (2.21) with $\mu = r - q$. We take the model parameters $\sigma = 0.0753$, $r = -0.0063$ and $q = 0.0310$ and $n = 10^6$. Python then finds $V \approx V_n \doteq 33.722$. $\diamond$

3.3 Shortcomings of the model of Black and Scholes

We are coming back to the model of Black and Scholes and to its in section 3.2, E_2) mentioned mismodelling of the underlying, respectively. As already said, in the model of Back and Scholes the realised

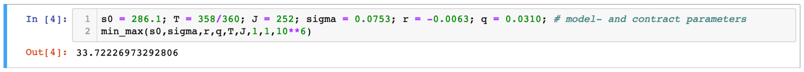

Figure 3.6: The Python function `min_max.py` finds the price of the product in table 1.3.

log-returns (2.13) are normally distributed. We may test this assumption using real data. To do so, in figure 3.7 we exemplarily have a look on the DAX over the time period 26/11/1990 to 4/7/2012 and plot a histogram of the daily log-returns $\ln(p_{t+\Delta t}/p_t)$ for $\Delta t = 1/252$ (p_t is the daily closing price of the DAX at time t). In the same graph, we also plot the probability density function (2.9)

$$\phi_{\mu,\sigma}(x) := \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

of the normal distribution $R^c_{t,t+\Delta t} \sim \mathcal{N}(\mu, \sigma^2)$ with $\mu = \bar{r} \doteq 0.000276$ and $\sigma = s_r \doteq 0.014737$ given by the (usual) point estimators for (daily) mean and (daily) standard deviation. We observe that the theoretical distribution (the Black-Scholes model) does not fit the empirical distribution well. In particular, the empirical distribution has heavier tails than the theoretical one which means that the model underestimates the probability of having large absolute returns (large deviations from the mean).

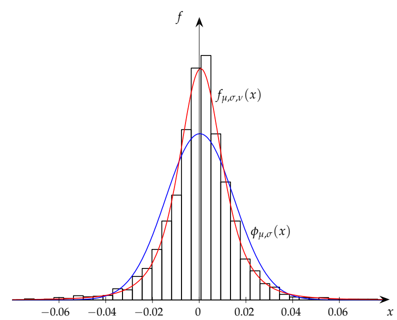

Figure 3.7: Histogram of DAX daily log-returns and estimated theoretical distributions.

In this regard, the student t distribution does a much better job. Besides the normal distribution, we estimate and plot in figure 3.7 also the probability density function $f_{\mu,\sigma,\nu}$ of the student t distribution given by

$$f_{\mu,\sigma,\nu}(x) = c_\nu \frac{1}{\sqrt{\nu\pi}\sigma} \left(1 + \frac{1}{\nu}\left(\frac{x-\mu}{\sigma}\right)^2\right)^{-(\nu+1)/2}. \tag{3.15}$$

Herewith, c_ν is a constant which depends on the number ν of degrees of freedom. This distribution becomes more heavy-tailed the smaller ν is, and it holds

$$\lim_{\nu\to\infty} f_{\mu,\sigma,\nu}(x) = \phi_{\mu,\sigma}(x) .$$

The above time-series of the DAX log-returns has the estimated parameter values $\mu \doteq 0.00065$, $\sigma \doteq 0.00983$ and $\nu \doteq 3.2907$; see the technical note 3.8 on how to get these values using maximum likelihood estimation. Since the decay of the distributions is not clearly visible in figure 3.7, we may not consider the densities $f(x)$ themselves, but their logarithm $\ln(f(x))$, compare with figure 3.8.

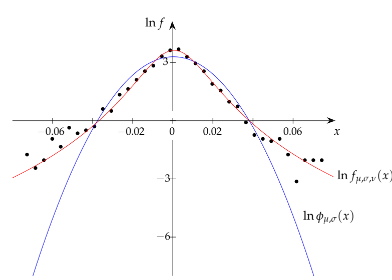

Figure 3.8: Daily log-returns of the DAX (•) and estimated theoretical distributions.

Would be the model of Black and Scholes correct, the (graph of the) log-returns would follow a parabola, since

$$\ln \phi_{\mu,\sigma}(x) = -\frac{1}{2}\ln(2\pi\sigma^2) - \frac{(x-\mu)^2}{2\sigma^2}$$

is obviously a quadratic function. This is, however, not observable as we can clearly see in figure 3.8. Not surprisingly, the student t distribution is able to follow the empirical distribution much better.

Now we consider volatility. We may check the assumption of constant volatility (with respect to strike and to time-to-maturity) as follows. We look at market prices

$$V_1^{\mathrm{M}}, \ V_2^{\mathrm{M}}, \ \ldots, \ V_\nu^{\mathrm{M}}$$

(the superscript "M" stands for "market") of ν traded options on the same underlying with different strikes

$$X_1, \ X_2, \ \ldots, \ X_\nu$$

but with the same time-to-maturity $T - t$. Since up to the volatility all the model parameters are known we may ask for the volatility σ_ℓ which, inserted into the formula (3.14) of Black and Scholes, yields the market value V_ℓ^{M} of the ℓ-th option in the data set. In other words, for each option in the data set we are looking for σ_ℓ such that

$$V(s, t; X_\ell, T, \sigma_\ell, r, q, \omega) = V_\ell^{\mathrm{M}} ,$$

which is equivalent to the root finding problem

$$V(s, t; X_\ell, T, \sigma_\ell, r, q, \omega) - V_\ell^{\mathrm{M}} = 0 . \tag{3.16}$$

The in this way calculated volatility is called *implied volatility*, symbolically σ^{i}. As an example, on September 11, 2019, we consider $\nu = 14$ call options on U-Blox[8] which all expire on December 20, 2019 (thus the time-to-maturity is $T - t = 99/360$), compare with table 3.1.

[8] The market prices are from EUREX, http://www.eurexchange.com.

X_ℓ	V_ℓ^{M}	X_ℓ	V_ℓ^{M}	X_ℓ	V_ℓ^{M}
48	29.57	64	14.73	84	2.93
50	27.62	68	11.51	88	1.97
52	25.67	72	8.66	92	1.33
56	21.88	76	6.23	96	0.90
60	18.21	80	4.31		

Table 3.1: Market price V_ℓ^{M} on September 11, 2019 ($t = 0$) of call options on U-Blox which expire on December 20, 2019 ($t = T$). The closing price of U-Blox at $t = 0$ is $s = S_0 = 77.45$, and the risk free is $r = -0.81\%$. Since U-Blox pays no dividends in the time period $[0, T]$, we have $q = 0$.

For example we have $X_9 = 76$ and $V_9^{\mathrm{M}} = 6.23$ such that we are looking for the implied volatility σ_9^{i} with the property that

$$V(77.45, 0; 76, 99/360, \sigma_9^{\mathrm{i}}, -0.0081, 0, 1) - 6.23 = 0 .$$

Mathematically, calculating the implied volatility in (3.16) is equivalent to the finding of the root of the function

$$f(\sigma) := V(s, t; X_\ell, T, \sigma, r, q, \omega) - V_\ell^{\mathrm{M}} . \tag{3.17}$$

Note that the function f indeed has only one root, since the map $\sigma \mapsto f(\sigma)$ is strictly increasing. Although there exists exactly one solution to the equation $f(\sigma) = 0$ it is impossible to find it in closed form and we have to rely on a numerical procedure. Here, we may use Python to find the implied volatility (to do so, we write the function `implied_vola.py`, which uses the so-called Newton method to find the root of equation (3.17)).

Example 3.3. Calculate the implied volatility of the call option with strike $X_9 = 76$ and market price $V_9^{\mathrm{M}} = 6.23$, compare with table 3.1.

Solution. Since there holds $s = 77.45$, $T - t = 99/360$, $r = -0.0081$ and $q = 0$ we have

```
In [2]: help(implied_vola)
        s = 77.45; # the price of the underlying
        T = yf((11,9,2019),[(20,12,2019)])[0]; r = -0.0081; # time-to-maturity, cc risk-free
        implied_vola(6.23,s,76,T,r,0,1,0.5)

Help on function implied_vola in module implied_vola:

implied_vola(VM, s, X, T, r, q, omega, init)
    Gives the implied volatility of a call (omega = 1) or put (omega = -1)
    option with market price VM, strike X and maturity T.
    The price of the underlying is s; r and q are cc risk free and cc
    dividend yield, respectively.

Out[2]: 0.3468833779533675
```

Figure 3.9: We use Python to find the implied volatility.

The function `implied_vola.py` finds the implied volatility to be $\sigma_9^{\mathrm{i}} \doteq 0.3469$. ◇

We now repeat example 3.3 for all options listed in table 3.1. Would be the model of Black and Scholes correct, the implied volatility would be the same for all considered options, i.e.,

$$\sigma_1^{\mathrm{i}} = \sigma_2^{\mathrm{i}} = \ldots = \sigma_\nu^{\mathrm{i}} .$$

To graphically test this, we just plot the implied volatility σ_ℓ^{i} against the corresponding strike X_ℓ, compare with figure 3.10.

We observe that the corresponding graph is not a horizontal line as the model of Black and Scholes suggests, but a (usually) convex curve, which is called *volatility smile* by practitioners. Note that between the points $(X_\ell, \sigma_\ell^{\mathrm{i}})$ we use some interpolation procedure: for a not tradable option with strike $X \in]X_\ell, X_{\ell+1}[$ we stipulate an implied volatility by some function f, $X \mapsto \sigma^{\mathrm{i}} = f(X)$. In particular, f defines indirectly a price for options that are not available on the market through the Black-Scholes formula

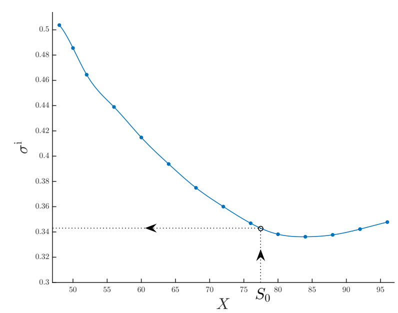

Figure 3.10: The implied volatility σ^{i} (as for the data in table 3.1) is not constant. Would the model of Black and Scholes be correct, the volatility smile would be a horizontal line. The curve is the graph of a so-called cubic spline $X \mapsto \sigma^{\mathrm{i}} = f(X)$. The function f stipulates implied volatilities for options that are not available on the market. For example, what is the price of an at-the-money call ($X = S_0$) on U-Blox on September 11, 2019, where $s = S_0 = 77.45$? Since $\sigma^{\mathrm{i}} = f(S_0) \doteq 0.3430$, the Black-Scholes formula yields $V(S_0, 0; S_0, T, \sigma^{\mathrm{i}}, r, 0, 1) \doteq 5.47$.

$$X \mapsto V(s, t; X, T, \sigma^{\mathrm{i}}, r, q, \omega) = V(s, t; X, T, f(X), r, q, \omega) .$$

Here, we have to be careful, since by problem 1.3 we know that the map $X \mapsto V$ is convex or, equivalently, the so-called dual-gamma of the option is non-negative, that is

$$\partial_{XX} V(s, t; X, T, f(X), r, q, \omega) = \frac{e^{-r(T-t)}\phi(d_2)}{X f(X)\sqrt{T-t}}\big(1 + \chi(X)\big) \geq 0 . \quad (3.18)$$

The function χ in (3.18) is given by

$$\begin{aligned} \chi(X) := {} & 2\sqrt{T-t}\,X d_1 f'(X) \\ & + (T-t) d_1 d_2 X^2 f'(X)^2 \\ & + (T-t) X^2 f(X) f''(X) \end{aligned}$$

If the interpolation function f implies a violation of the convexity, there are arbitrage opportunities (so-called butterfly arbitrage) and the "model" f is not a good model, compare with figure 3.11. Note that the expression (3.18) for the dual-gamma $\partial_{XX}V$ implies two facts. First, the dual-gamma is the same for call and put options (as the formula does not depend on $\omega = \pm 1$). Second, if volatility does not depend on X, $f(X) =: \sigma$ for all X, then $\chi \equiv 0$ and the dual-gamma becomes $\partial_{XX}V(s,t;X,T,\sigma,r,q,\omega) = \frac{e^{-r(T-t)}\phi(d_2)}{X\sigma\sqrt{T-t}}$.

Now we generalise the concept of a volatility smile. To do so, we assume that there is available a set of a total of ν put and/or call options. We assume that there are n different times-to-maturity $T_1 < T_2 < \ldots < T_n$; for each maturity T_i there are ν_i options (there holds $\sum_{i=1}^{n} \nu_i = \nu$). We numerate the options as follows

j	strike	maturity	market price	implied vola
1	$X_{1,1}$	$T_{1,1}$	$V^{\mathrm{M}}_{1,1}$	$\sigma^{\mathrm{i,M}}_{1,1}$
$\vdots$				
ν_1	X_{1,ν_1}	T_{1,ν_1}	V^{M}_{1,ν_1}	$\sigma^{\mathrm{i,M}}_{1,\nu_1}$
ν_1+1	$X_{2,1}$	$T_{2,1}$	$V^{\mathrm{M}}_{2,1}$	$\sigma^{\mathrm{i,M}}_{2,1}$
$\vdots$				
$\nu_1+\nu_2$	X_{2,ν_2}	T_{2,ν_2}	V^{M}_{2,ν_2}	$\sigma^{\mathrm{i,M}}_{2,\nu_2}$
$\nu_1+\nu_2+1$	$X_{3,1}$	$T_{3,1}$	$V^{\mathrm{M}}_{3,1}$	$\sigma^{\mathrm{i,m}}_{3,1}$
$\vdots$				
$\nu_1+\nu_2+\nu_3$	X_{3,ν_3}	T_{3,ν_3}	V^{M}_{3,ν_3}	$\sigma^{\mathrm{i,M}}_{3,\nu_3}$
$\vdots$				
$\sum_{i=1}^{n-1}\nu_i+1$	$X_{n,1}$	$T_{n,1}$	$V^{\mathrm{M}}_{n,1}$	$\sigma^{\mathrm{i,M}}_{n,1}$
$\vdots$				
$\sum_{i=1}^{n}\nu_i$	X_{n,ν_n}	T_{n,ν_n}	V^{M}_{n,ν_n}	$\sigma^{\mathrm{i,M}}_{n,\nu_n}$

Table 3.2: For further purposes, we need a proper numbering of the options in a given data set.

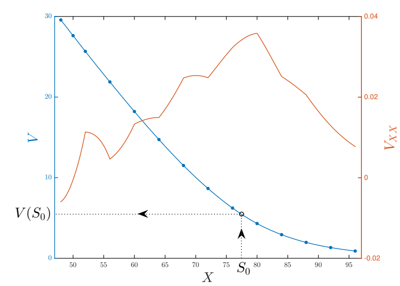

Figure 3.11: The graph of the function $X \mapsto V(s,0;X,T,f(X),r,0,1)$ for the implied volatility "model" $\sigma^{\mathrm{i}} = f(X)$ for the option data as in table 3.1 and f being the cubic spline as in figure 3.10. The "model" f yields the price $V(S_0) = 5.47$ for an at-money-call, but in this particular case f leads to the violation of the butterfly-arbitrage condition $\partial_{XX}V \geq 0$.

Thus, the j-th option with

$$j = \sum_{i=1}^{k-1} \nu_i + \ell, \quad k = 1,\ldots,n, \quad \ell = 1,\ldots,\nu_k$$

in the data set has strike $X_{k,\ell}$, market price $V^{\mathrm{M}}_{k,\ell}$, (market) implied volatility $\sigma^{\mathrm{i,M}}_{k,\ell}$ and time-to-maturity $T_{k,\ell}$. Note that

$$T_{k,1} = T_{k,2} = \cdots = T_{k,\nu_k} = T_k, \quad k = 1,\ldots,n.$$

Now, if we calculate the volatility smile for two (different) times-to-maturity, we obtain generally different smiles. Thus the implied volatility is not only a function of the strike $X_{k,\cdot}$, but also of the time-to-maturity $T_{k,\cdot}$, i.e., $\sigma^{\mathrm{i,M}} = f(X_{k,\cdot}, T_{k,\cdot})$, $k = 1,\ldots,n$. In figure 3.10, we plot this function for a fixed T_k. If we plot the graph of the function $(X_{k,\ell}, T_{k,\ell}) \mapsto \sigma^{\mathrm{i,M}}_{k,\ell}$, we obtain the so-called (implied) volatility surface, see figure 3.12 for an example. Note that only very few points

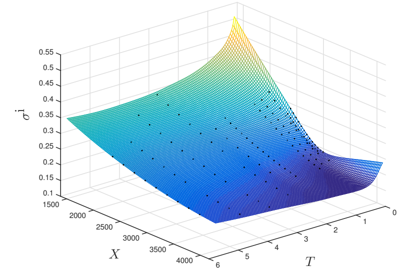

Figure 3.12: Implied volatilities $\sigma^{\mathrm{i,M}}_{k,\ell}$ of EURO STOXX 50 options $(\cdot)$ and the corresponding volatility surface.

of this surface are known (market data). Between these points, we interpolate or we assume some parametric form, see section 3.4 for an example. The shape of f for all strikes X and all times-to-maturity T is not known.

3.4 Alternatives to the model of Black and Scholes

As long as we are concerned with the pricing of European put or call options there is nothing wrong with the Black-Scholes formula. We just take for the volatility σ the implied volatility $\sigma^{\mathrm{i}}(X, T)$ depending on the strike X and on the time-to-maturity T of the option to be priced. As we have seen, the market only quotes implied volatilities for options which are available on the market. However, we need to price options with any strike or maturity. For example, how do we price an option with strike $1.12 S_0$ and maturity $T = 5$ years if the listed strikes are $0.9 S_0$, $0.98 S_0$, $1.05 S_0$, $1.2 S_0$ and the longest listed maturity is two years? A naive answer would be that we obtain the implied volatility $\sigma^{\mathrm{i}}(1.12 S_0, 5)$ by some interpolation and/or extrapolation applied to quoted implied volatilities. This is, however, not a good advice, since these inter- and/or extrapolations are generally not free from arbitrage, see figure 3.11 for an example. While it is possible to eliminate arbitrage implied by interpolation, extrapolation is more subtle. Thus, we typically have to resort on implied volatility models; these models are linked to the notion of local volatility. This is a model which extends the Black-Scholes model and which is used to price some standard exotic options such as digitals, barrier, Asian or lookback options.

Local volatility

In the local volatility model the volatility is a deterministic function of time t and level of the underlying s, i.e.,

$$\sigma = \sigma(s, t)$$

for some bivariate, deterministic function $\sigma : \mathbb{R}^+ \times [0, T] \to \mathbb{R}^+$. We replace in the stochastic differential equation (2.12) the constant σ by $\sigma(s, t)$ and obtain the so-called local volatility model, that is,

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma(S_t, t) S_t \mathrm{d}W_t, \quad S_0 = s > 0 \,. \tag{3.19}$$

In the technical note of section 3.7 we already encountered a special form of this model. There, the volatility σ only depends on t, but not on s. Now we could - for a fixed strike X - choose the function $\sigma(t)$ such that the term structure of implied volatility, i.e., the function $T \mapsto \sigma^{\mathrm{i}}(T)$, is exactly mapped by the model. This means that the model $\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma(t) S_t \mathrm{d}W_t$ would be able to reproduce market prices of European options, however only for a fixed strike. For other values of the strike such a model would yield prices which would be different from market prices. Thus, the natural question is: does there exist a bivariate function $(s,t) \mapsto \sigma(s,t)$ such that the prices $V_c(s,0;X,T)$ of European (call) options implied by the model (3.19) are equal to the market prices, i.e., such that for every strike X and every time-to-maturity T the equation

$$V_c(s,0;X,T) = V^{\mathrm{M}}(s,0;X,T)$$

becomes true? The answer to this question was given by Bruno Dupire[9] and is "yes". Once the question of the existence of such a function is answered we ask for the construction of it. Dupire shows: if there are available market prices $V_c(X,T)$ of (call) options (we skip the dependence on s) for a continuum of strikes X and times-to-maturity T, then the function σ is given by

[9] B. Dupire. Pricing with a Smile. *Risk*, 7(1):18–20, 1994

$$\sigma^2(X,T) = 2\frac{\partial_T V_c(X,T) + (r-q)X\partial_X V_c(X,T) + qV_c(X,T)}{X^2 \partial_{XX} V_c(X,T)} \,. \qquad (3.20)$$

In practice, however, we only have available a finite set of market prices which means that the function $V_c(X,T)$ and its partial derivates are not known. Thus, we need interpolation as well as extrapolation tools such that

- the partial derivatives $\partial_T V_c$, $\partial_X V_c$ and $\partial_{XX} V_c$ appearing in (3.20) exist and such that
- there exist no arbitrage opportunities.

Here, we need to make precise the notion of "arbitrage". We say that the volatility surface is free of arbitrage, iff

(i) there is no calendar-spread arbitrage,

(ii) for each fixed T, there is no butterfly arbitrage.

We shortly discuss calendar-spread arbitrage, for the butterfly arbitrage see problem 1.3 according to which the denominator in the Dupire equation (3.20) and in the transformed Dupire equation (3.22), respectively, must be non-negative.

By calendar-spread we mean the price difference of two options written on the same underlying having the same strike, but different time-to-maturity. Since we can write the numerator of (3.20) as

$$\begin{aligned} &\partial_T V_c(X,T) + (r-q)X\partial_X V_c(X,T) + qV_c(X,T) \\ &\quad = e^{-qT}\partial_T\big(e^{qT} V_c(Xe^{(r-q)T}, T)\big) \,, \end{aligned}$$

we observe that the numerator becomes non-negative if

$$\partial_T(e^{qT}V_c(Xe^{(r-q)T},T)) \geq 0$$

holds. Thus, the function in the bracket must be increasing in T, which means that for $T_1 \leq T_2$ the condition

$$e^{qT_1}V_c(Xe^{(r-q)T_1},T_1) \leq e^{qT_2}V_c(Xe^{(r-q)T_2},T_2)$$

must be satisfied. Otherwise, one can construct an arbitrage opportunity, see for example Bergomi[10].

[10] L. Bergomi. *Stochastic Volatility Modeling*. Financial Mathematics Series. Chapman&Hall/CRC, 2016

How do we construct the local volatility function σ in (3.20) when there are only a finite number of market prices $V_c(X_j, T_j)$ available? Several approaches to construct σ are discussed in the scientific literature, we briefly describe the approach of Gatheral and Jaquier[11]. The authors do not directly consider the equation/definition (3.20), but a transformed version of it. To be more precise, in (3.20) we do not consider option prices V but so called total implied variances w, defined as

[11] J. Gatheral and A. Jacquier. Arbitrage-free SVI volatility surfaces. *Quantitative Finance*, 14(1):59–71, 2014

$$w(x,T) := \big(\sigma^{\mathrm{i}}(x,T)\big)^2 T \ .$$

Herein, $\sigma^{\mathrm{i}}(x,T)$ denotes the implied volatility of a European call or put option with strike[12]

[12] F denotes the forward price of the underlying, S_0 is the price of the underlying at the date of calibration.

$$X = F_T e^x = S_0 e^{(r-q)T} e^x \qquad (3.21)$$

and time-to-maturity T. Thus, we need to re-express equation (3.20) in terms of w. It is possible to show (for simplicity we skip the argument (x,T) in $w(x,T)$)

$$\sigma^2(F_T e^x, T) = \frac{\partial_T w}{\big(\frac{x}{2w}\partial_x w - 1\big)^2 + \frac{1}{2}\partial_{xx} w - \frac{1}{4}\big(\frac{1}{4} + \frac{1}{w}\big)(\partial_x w)^2} \ . \qquad (3.22)$$

The next step in the construction of the local volatility function $\sigma(x,T)$ in (3.22) is the specification of w. For a fixed time-to-maturity T, Gatheral assumes a parametric form for the total implied variance. To be more precise, he assumes that - for a vector of parameters $\boldsymbol{\eta} := (a,b,\rho,m,s)$ - the function $x \mapsto w(x)$ takes the form

$$w(x;\boldsymbol{\eta}) := a + b\Big(\rho(x-m) + \sqrt{(x-m)^2 + s^2}\Big) \ . \qquad (3.23)$$

Herein, $a, m \in \mathbb{R}$, $b \geq 0$, $-1 \leq \rho \leq 1$, and $s > 0$ are parameters to be found.[13] We call the graph of the function $x \mapsto w(x;\boldsymbol{\eta})$ a slice. Since w is a total variance, it can not be negative, $w(x;\boldsymbol{\eta}) \geq 0$ for all $x \in \mathbb{R}$, thus the parameters have to satisfy the additional constraint

[13] Note that the function w still depends on T, but for fixed T the function only depends on x and we suppress the dependence on T. For another time-to-maturity, the parameters $\boldsymbol{\eta}$ will admit different values.

$$a + bs\sqrt{1-\rho^2} \geq 0 \ , \qquad (3.24)$$

compare with problem 3.5. Further constraints on the parameters $\boldsymbol{\eta}$ are implied by non-arbitrage conditions, i.e., we ask for conditions the parameters $\boldsymbol{\eta}$ have to satisfy such that the function w generates a volatility surface which is free from arbitrage. We consider briefly the question of calendar-spread arbitrage. Recall that the function w does not admit calendar-spread arbitrage iff it is increasing in T. Thus, for arbitrary $T_1 \neq T_2$, their corresponding slices $x \mapsto w(x; \boldsymbol{\eta}_1)$ and $x \mapsto w(x; \boldsymbol{\eta}_2)$ should not intersect. This is equivalent to the requirement that the equation (in x)

$$w(x; \boldsymbol{\eta}_1) = w(x; \boldsymbol{\eta}_2)$$

has no real solutions. Now suppose that for two maturities $T_{k-1} < T_k$ we a have found the functions $w(x; \boldsymbol{\eta}_{k-1})$ and $w(x; \boldsymbol{\eta}_k)$ such that their corresponding slices do not intersect. For times-to-maturity $T_{k-1} \leq T \leq T_k$ between T_{k-1} and T_k we have to construct a bivariate function $w(x, T)$ such that there holds $w(x, T_{k-1}) = w(x; \boldsymbol{\eta}_{k-1})$ and $w(x, T_k) = w(x; \boldsymbol{\eta}_{k-1})$ and furthermore such that its derivative with respect to T is increasing. Obviously, the function

$$w(x, T) = \frac{T_k - T}{T_k - T_{k-1}} w(x; \boldsymbol{\eta}_{k-1}) + \frac{T - T_{k-1}}{T_k - T_{k-1}} w(x; \boldsymbol{\eta}_k), \quad T \in [T_{k-1}, T_k] \tag{3.25}$$

does the job. If we repeat this procedure for all of the times-to-maturity $T_1 < T_2 < \ldots < T_n$ of the options in the data set the function $w(x, T)$ is free from calendar-arbitrage and can be taken as an input to calculate the local volatility $\sigma(X, T)$ according to (3.22).

We switch to the butterfly-arbitrage for the model (3.23). There is no butterfly-arbitrage iff the denominator of (3.22) is non-negative, i.e.,

$$\begin{aligned} \psi(x; \boldsymbol{\eta}) := \Big(\frac{x}{2w(x; \boldsymbol{\eta})} \partial_x w(x; \boldsymbol{\eta}) - 1 \Big)^2 + \frac{1}{2} \partial_{xx} w(x; \boldsymbol{\eta}) \\ - \frac{1}{4} \Big(\frac{1}{4} + \frac{1}{w(x; \boldsymbol{\eta})} \Big) \big(\partial_x w(x; \boldsymbol{\eta}) \big)^2 \geq 0 \end{aligned} \tag{3.26}$$

for all $x \in \mathbb{R}$. The condition $\psi(x; \boldsymbol{\eta}) \geq 0$ is implicit, as it does not directly provide conditions (like the inequality (3.24) for example) which need to be satisfied by parameters $\boldsymbol{\eta}$. Since the inequality $\psi(x; \boldsymbol{\eta}) \geq 0$ is rather cumbersome and the function w is non-linear in a unfavourable way it turns out that it is almost impossible to find conditions satisfied by the parameters a, b, ρ, m, s such that the inequality holds. Due to this inability Gatheral and Jaquier suggest a sub-class of the model (3.23) as follows

$$w(x, T) := \frac{1}{2} \theta_T \Big(1 + \rho \varphi(\theta_T) x + \sqrt{(\varphi(\theta_T) x + \rho)^2 + 1 - \rho^2} \Big) . \tag{3.27}$$

Herein,

$$\theta_T := \big(\sigma^{\mathrm{i}}(0, T) \big)^2 T$$

is the at-the-money (ATM) implied total variance of a European option with time-to-maturity T. Furthermore, $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is some function with the property that the limit $\lim_{T\to 0} \theta_T \varphi(\theta_T)$ exists. Note that the model (3.27) is just a re-parameterisation of the model (3.23). This "new" parameterisation still depends on T (via θ_T) and incorporates only the two "parameters" ρ and φ. Gatheral and Jaquier state conditions on these parameters such that the volatility surface is free from arbitrage. Furthermore, they discuss a two-stage calibration procedure how to obtain - starting by the model (3.27) - volatility slices (3.23) for each time-to-maturity T_j such that the time-interpolated slices (3.25) generate a volatility surface which is free from arbitrage. Instead of going into technical details here, we present a numerical example.

Example 3.4. We consider in table 3.3 the (market) implied volatilities $\sigma^{\mathrm{i,M}}$ of $\nu = 139$ call options written on EURO STOXX 50 as of 01.03.2010 (the data set is from Andreasen and Huge[14]). There are $n = 11$ times-to-maturity, the shortest one being $T_1 = 0.101$ years, the longest one $T_{11} = 5.774$ years. The strikes $x_{k,\ell} = X_{k,\ell}/S_0$ are relative to level $S_0 = 2772.70$ of the EURO STOXX 50 at 01.03.2010.

[14] J. Andreasen and B Huge. Volatility interpolation. *Risk Magazine*, (March):76–79, 2011

Besides the market implied volatilities, we also give in table 3.3 the implied volatilities implied by the model (3.25). We observe that some of the implied volatilities generated by the model agree up do two decimal places with their market counterparts. The largest absolut difference between market- and model implied volatility is 0.51, which is realised by the option with strike $X_{k,\ell} = 0.7697 S_0$ and time-to-maturity $T_k = 0.274$. The average relative percent error, i.e.,

$$\mathrm{ARPE} = \frac{1}{\nu} \sum_{k=1}^{n} \sum_{\ell=1}^{\nu_k} \frac{|\sigma^{\mathrm{i,model}}_{k,\ell} - \sigma^{\mathrm{i,M}}_{k,\ell}|}{\sigma^{\mathrm{i,M}}_{k,\ell}} \cdot 100\% \,, \tag{3.28}$$

is $\mathrm{ARPE} \doteq 0.22\%$, thus the quality of the fit is very high. Here, the implied volatilities $\sigma^{\mathrm{i,model}}_{k,\ell}$ implied by the model are given by

$$\sigma^{\mathrm{i,model}}_{k,\ell} = \sqrt{\frac{w(x_{k,\ell}, T_{k,\ell})}{T_{k,\ell}}}$$

with the implied total variance $w(x, T)$ as in (3.25).

In figure 3.12, we plot the market implied volatilities together with the volatility surface

$$(X, T) \mapsto \sqrt{w\big(\ln(X/S_0) - (r-q)T, T\big)/T}$$

with the function $w(x, T)$ as in (3.25) and with the argument x as in (3.21) , constructed using the $n = 11$ functions $w(x; \widehat{\boldsymbol{\eta}}_k)$, $k = 1, \ldots, 11$. In figure 3.14, we plot the corresponding local volatility $\sigma(X, T)$. $\diamond$

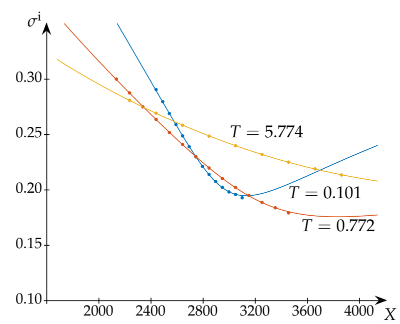

Figure 3.13: Implied market volatilities $(\cdot)$ $\sigma^{\mathrm{i,M}}_{k,\ell}$ of EURO STOXX 50 options as of 01.03.2010 for three different times-to-maturity $T_1 = 0.101$, $T_5 = 0.772$ and $T_{11} = 5.774$, compare with table 3.3, and their corresponding volatility smiles implied by the local volatility model (3.23).

x/T	0.101	0.197	0.274	0.523	0.772	1.769	2.267	2.784	3.781	4.778	5.774
51.31								33.66 (33.69)	32.91 (33.05)		
58.64								31.78 (31.85)	31.29 (31.36)		
65.97								30.19 (30.15)	29.76 (29.82)	29.75 (29.85)	
73.30								28.63 (28.58)	28.48 (28.39)	28.48 (28.47)	
76.97			32.62 (32.11)	30.79 (30.68)	30.01 (29.96)	28.43 (28.38)					
80.63			30.58 (30.47)	29.36 (29.31)	28.76 (28.74)	27.53 (27.49)	27.13 (27.11)	27.11 (27.11)	27.11 (27.07)	27.22 (27.20)	28.09 (28.08)
84.30			28.87 (28.84)	27.98 (27.96)	27.50 (27.54)	26.66 (26.64)					
87.96	29.06 (28.97)	27.64 (27.51)	27.17 (27.23)	26.63 (26.64)	26.37 (26.36)	25.75 (25.81)	25.55 (25.55)	25.80 (25.74)	25.85 (25.85)	26.11 (26.03)	26.93 (26.92)
89.79	27.97 (27.94)	26.72 (26.64)									
91.63	26.90 (26.91)	25.78 (25.78)	25.57 (25.65)	25.31 (25.34)	25.19 (25.21)	24.97 (25.00)					
93.46	25.90 (25.89)	24.89 (24.94)									
95.29	24.88 (24.90)	24.05 (24.11)	24.07 (24.12)	24.04 (24.08)	24.11 (24.09)	24.18 (24.23)	24.10 (24.11)	24.48 (24.47)	24.69 (24.72)	25.01 (24.95)	25.84 (25.85)
97.12	23.90 (23.93)	23.29 (23.32)									
98.96	23.00 (23.00)	22.53 (22.56)	22.69 (22.68)	22.84 (22.87)	22.99 (23.01)	23.47 (23.49)					
100.79	22.13 (22.15)	21.84 (21.85)									
102.62	21.40 (21.38)	21.23 (21.21)	21.42 (21.40)	21.73 (21.74)	21.98 (22.00)	22.83 (22.78)	22.75 (22.78)	23.22 (23.30)	23.84 (23.68)	23.92 (23.97)	24.86 (24.88)
104.45	20.76 (20.73)	20.69 (20.65)									
106.29	20.24 (20.21)	20.25 (20.17)	20.39 (20.34)	20.74 (20.71)	21.04 (21.05)	22.13 (22.11)					
108.12	19.82 (19.84)	19.84 (19.80)									
109.95	19.59 (19.61)	19.44 (19.52)	19.62 (19.57)	19.88 (19.82)	20.22 (20.20)	21.51 (21.49)	21.61 (21.59)	22.19 (22.23)	22.69 (22.74)	23.05 (23.08)	23.99 (23.99)
111.78	19.29 (19.50)	19.20 (19.34)									
113.62			19.02 (19.11)	19.14 (19.10)	19.50 (19.46)	20.91 (20.91)					
117.28			18.85 (18.91)	18.54 (18.57)	18.88 (18.86)	20.39 (20.38)	20.58 (20.57)	21.22 (21.28)	21.86 (21.86)	22.23 (22.29)	23.21 (23.20)
120.95			18.67 (18.92)	18.11 (18.20)	18.39 (18.38)	19.90 (19.90)					
124.61			18.71 (19.06)	17.85 (17.99)	17.93 (18.04)	19.45 (19.49)		20.54 (20.45)	21.03 (21.16)	21.64 (21.61)	22.51 (22.50)
131.94								19.88 (19.76)	20.54 (20.52)	21.05 (21.02)	21.90 (21.89)
139.27								19.30 (19.20)	20.02 (19.99)	20.54 (20.53)	21.35 (21.37)
146.60								18.49 (18.76)	19.64 (19.56)	20.12 (20.13)	

Table 3.3: Implied volatilities of EURO STOXX 50 options as of 01.03.2010. The values in brackets are implied volatilities according to the model (3.23).

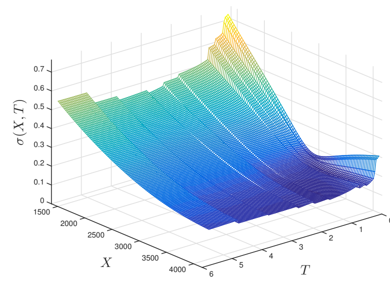

Figure 3.14: Local volatility $\sigma(X, T)$ constructed using $w(x, T)$ and the Dupire equation (3.22).

Before we proceed to stochastic volatility models, we are going back to question on how to price European calls or puts which are not available on the market. We might apply two equivalent approaches. The first one is to use the local volatility model (3.19). Once we have found the function $\sigma(s, t)$, we can either solve the corresponding pricing equation[15] or we numerically integrate the SDE (3.19) by some appropriate Monte Carlo scheme. The second approach is much simpler. Since we have the function/model $w(x, T)$ available, we may calculate the implied volatility

$$\sigma^{\mathrm{i}} = \sqrt{\frac{w(x,T)}{T}}, \; x \overset{(3.21)}{=} \ln\left(\frac{X}{S_0}\right) - (r-q)T \,,$$

implied by $w(x, T)$ and then plug in this value for σ into the formula (3.14) of Black and Scholes. We already sketched the procedure to calibrate the model $w(x, T)$ to market data. Here, we take a slightly different approach to find w by assuming that we have available mar-

[15] An example of a pricing equation in a local voatiltiy model can be found in the next chapter.

ket quotes of options with time-to-maturity equal to some fixed T and that we need to price European puts or calls having exactly this time-to-maturity T, such that there is no need to model the whole map $(x, T) \to w(x, T)$; we just need to find the slice $w(x; \boldsymbol{\eta})$ in (3.23) for this given T. So, assume that we know the implied volatility $\sigma_\ell^{\mathrm{i,M}}$ of (European) call and/or put options with strike X_ℓ and maturity T, $\ell = 1, \ldots, \nu$. The parameter vector $\boldsymbol{\eta} := (a, b, \rho, m, s)$ is now chosen such that the sum of squared differences between total implied variances $w(x_\ell; \boldsymbol{\eta})$ implied by the model and the quoted total implied variances $(\sigma_\ell^{\mathrm{i,M}})^2 T$ becomes minimal. If we denote by $\mathcal{S}$ the set

$$\mathcal{S} := \left\{ \boldsymbol{\eta} \in \mathbb{R}^5 \mid b \geq 0,\ s > 0,\ |\rho| \leq 1,\ a + bs\sqrt{1 - \rho^2} \geq 0 \right\} \subset \mathbb{R}^5 ,$$

then the parameter vector $\boldsymbol{\eta}$ is thus solution of the constrained non-linear regression problem

$$\widehat{\boldsymbol{\eta}} := \underset{\boldsymbol{\eta} \in \mathcal{S}}{\operatorname{argmin}} f(\boldsymbol{\eta}) ,$$

where the objective function $f(\boldsymbol{\eta})$ is given by the sum of squared residuals

$$f(\boldsymbol{\eta}) := \sum_{\ell=1}^{\nu} \left[w(x_\ell; \boldsymbol{\eta}) - (\sigma_\ell^{\mathrm{i,M}})^2 T \right]^2 .$$

However, the solution $\widehat{\boldsymbol{\eta}}$ does not necessarily fulfil the butterfly-arbitrage condition $\psi(x; \widehat{\boldsymbol{\eta}}) \geq 0$ in (3.26), such that we have to incorporate this inequality into the calibration of the model too. If we denote by $\mathcal{S}_b$ the "butterfly set"

$$\mathcal{S}_b := \{ \boldsymbol{\eta} \in \mathbb{R}^5 \mid \psi(x; \boldsymbol{\eta}) \geq 0,\ \forall x \in \mathbb{R} \} ,$$

then the parameter vector is solution of the extended constrained non-linear regression problem

$$\widehat{\boldsymbol{\eta}} := \underset{\boldsymbol{\eta} \in \mathcal{S} \cap \mathcal{S}_b}{\operatorname{argmin}} f(\boldsymbol{\eta}) .$$

Since we do not know the set $\mathcal{S}_b$ explicitly, we can not apply any of the optimisation algorithms provided by Python, and we are enforced to reformulate the above minimisation problem. The idea is to add to the objective function f a penalty term whenever the constraint $\psi(x; \boldsymbol{\eta}) \geq 0$ is violated, i.e., the new objective function to be minimised is $f(\boldsymbol{\eta}) + \lambda 1_{\{\exists x \in \mathbb{R} | \psi(x;\boldsymbol{\eta}) < 0\}}$. Here λ is a large positive number and $1_{\{\exists x \in \mathbb{R} | \psi(x;\boldsymbol{\eta}) < 0\}}$ returns 1 if there is at least one x for which $\psi(x; \boldsymbol{\eta}) < 0$. How do we check this condition in Python? One possibility is to find the global minimum of ψ (with respect to x) and then to check wether it is positive. To find the global minimum we need to solve the equation $\partial_x \psi(x; \boldsymbol{\eta}) = 0$ for x numerically. If this equation has multiple solutions, Pythons `fsolve` does not necessarily find the one which corresponds to the global minimum such that we refuse this idea. A second possibility is to calculate the roots of ψ, since these split $\mathbb{R}$ into intervals on which $\psi > 0$ (good) or $\psi < 0$ (bad). Again, `fsolve` might not be able to find a solution of the equation $\psi(x;\boldsymbol{\eta}) = 0$, and we skip this possibility too. Instead solving any equation, we define the discrete subset

$$\mathcal{G} := \{x_j \in \mathbb{R} \mid j = 0, 1, \ldots, N\}$$

where, for some given $N \in \mathbb{N}^\times$, $x_0 \in \mathbb{R}$ and $x_0 < x_N \in \mathbb{R}$ the x_j's are defined as

$$x_j := x_0 + \frac{x_N - x_0}{N} j \,,$$

and then evaluate ψ on $\mathcal{G}$, i.e., we set $\psi_j := \psi(x_j;\boldsymbol{\eta})$ for $x_j \in \mathcal{G}$. For each ψ_j we then check wether it is non-negative by $1_{\{\psi_j < 0\}}$. If $\psi_j \geq 0$, then this expression returns 0 (condition $\psi(x_j;\boldsymbol{\eta}) \geq 0$ is satisfied), otherwise 1 (condition $\psi(x_j;\boldsymbol{\eta}) \geq 0$ is violated). Finally, we add all the zeros and ones, multiply the resulting sum by λ and then add this product to the sum of squared residuals. Whence, we try to solve the problem

$$\widehat{\boldsymbol{\eta}} := \underset{\boldsymbol{\eta} \in \mathcal{S}}{\operatorname{argmin}} f(\boldsymbol{\eta}) + \lambda \sum_{j=0}^{N} 1_{\{\psi(x_j;\boldsymbol{\eta}) < 0\}} \,. \tag{3.29}$$

The penalty is the larger the more points x_j violate the butterfly arbitrage condition; if all x_j satisfy the condition, the penalty is zero, which is reasonable. Please note that if the condition is satisfied by all the x_j's, this thus not necessarily imply that it holds for all $x \in \mathbb{R}$, compare with figure 3.15.

Example 3.5. We are coming back to example/problem 1.1, iii), which we repeat here. On September 11, 2019 ($t = 0$) find the strike X of a European call option on U-Blox which expires on December 20, 2019 ($t = T$) such that $V_c(s, 0; X) = 2.75$ (taking into account that (implied) volatility depends on X, $\sigma^{\mathrm{i}} = f(X)$).

Solution. There holds $s = S_0 = 77.45$, $r = -0.81\%$ and $q = 0$ (since there are no dividend payments made by U-Blox in the period $[0, T]$). The market quotes (as of September 11, 2019) can be found in table 3.1. We solve the minimisation problem (3.29); the parameters which define the set $\mathcal{G}$ are $x_0 = -5$, $x_N = 5$ and $N = 10^4$. We obtain

$$\widehat{\boldsymbol{\eta}} \doteq (-0.098, 0.195, 0.131, 0.191, 0.670) \,,$$

which leads via $\sigma^{\mathrm{i}} = \sqrt{w(x;\widehat{\boldsymbol{\eta}})/T}$ to the volatility smile shown in figure 3.16. In particular, for an at-the-money call option the calibrated model (3.23) yields $\sigma^{\mathrm{i}} \doteq 0.3447$, this results in a call price

$$V_c(S_0, 0; S_0) = V(S_0, 0; S_0, T, \sigma^{\mathrm{i}}, r, 0, 1) \doteq 5.50 \,.$$

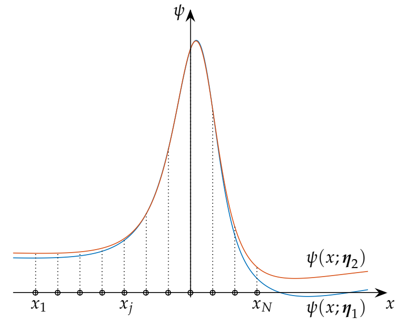

Figure 3.15: Butterfly-arbitrage of slices $w(x;\boldsymbol{\eta}_i)$, $i = 1, 2$. The slice corresponding to the parameter vector $\boldsymbol{\eta}_1 = (-0.195, 0.293, 0.346, 0.406, 0.823)$ does not satisfy the butterfly-arbitrage condition, the slice with parameters $\boldsymbol{\eta}_2 = (-0.098, 0.195, 0.131, 0.191, 0.670)$ does. On the set $\mathcal{G}$ indicated in the figure both slices fulfil the condition and the optimisation algorithm is not able to detect that $\boldsymbol{\eta}_1$ does in fact not satisfy the condition. Note that although the two parameter vectors are different, they lead to (almost) equal implied volatilities $\sigma^{\mathrm{i}} = \sqrt{w(x;\boldsymbol{\eta})/T}$. For example, for an at-the-money option $X = S_0$ with maturity $T = 0.2$ and $r = q = 0$ (whence $x = 0$), we obtain $\sigma^{\mathrm{i}} = \sqrt{w(0;\boldsymbol{\eta}_1)/0.2} \overset{(3.23)}{=} 0.40453$ and $\sigma^{\mathrm{i}} = \sqrt{w(0;\boldsymbol{\eta}_2)/0.2} \overset{(3.23)}{=} 0.40452$.

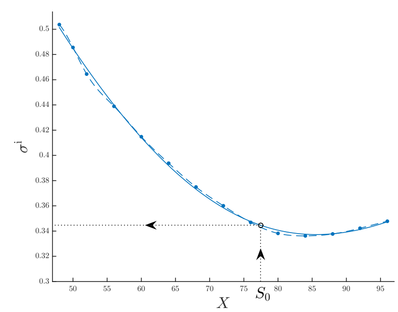

Figure 3.16: The solution of the optimisation problem (3.29) yields the shown volatility smile. The dashed line is the implied volatility according to the cubic spline interpolation f as in figure 3.10.

Recall that the cubic spline interpolation gives the $V_c(S_0, 0; S_0) \doteq 5.47$ (see figure 3.10), but this value is not free from butterfly-arbitrage. In the formula of Black and Scholes (3.14) we then use the expression

$$\sigma^{\mathrm{i}}(X) \overset{(3.21),(3.23)}{=} \sqrt{w\big(\ln(X/S_0) - (r-q)T; \widehat{\boldsymbol{\eta}}\big)/T}$$

for σ; we find $X \doteq 84.70$ as the (unique) solution of the equation $V_c(s, 0; X) = 2.75$, compare with the jupiter notebook of this chapter. $\diamond$

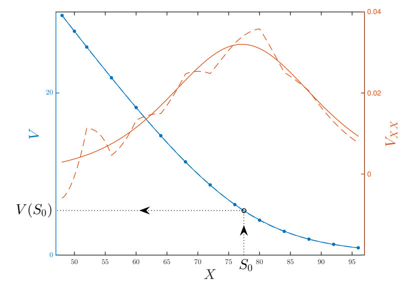

Figure 3.17: The solution of the optimisation problem (3.29) yields the graph of the functions $X \mapsto V(S_0, 0; X, T, \sigma^{\mathrm{i}}(X), r, 0, 1)$ and $X \mapsto \partial_{XX} V(S_0, 0; X, T, \sigma^{\mathrm{i}}(X), r, 0, 1)$. The dashed lines are the corresponding graphs if we "model" implied volatility by a cubic spline, compare with figure 3.11. Note that both "models", i.e., cubic spline interpolation and regression yields almost the same prices (there are no differences visible), the second approach, however, is free from (butterfly) arbitrage and hence favourable.

The model of local volatility (3.19) is used to price some standard exotic options like digitals, barrier, Asian or lookback options. However, to price other exotics like forward start or cliquet options a local volatility model is not appropriate. Here, it is market practice to use so-called stochastic volatility models.

Stochastic volatility

In such a model, the constant volatility σ in the model of Black and Scholes (2.12) is replaced by a stochastic process σ_t, i.e., one considers the stochastic differential equation

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma_t S_t \mathrm{d}W_t, \quad S_0 = s > 0\,. \tag{3.30}$$

There exists a vast number of ways to specify σ_t, here we pick the model of Heston[16] since this one has become the benchmark stochastic volatility model. Heston sets $\sigma_t = \sqrt{V_t}$ and models the variance V_t by a CIR process. The CIR process is named after Cox, Ingersoll and Ross[17]; they use this process to model interest rates. A CIR process is defined as the solution of the stochastic differential equation

$$\mathrm{d}V_t = \kappa(m - V_t)\mathrm{d}t + \delta\sqrt{V_t}\mathrm{d}\widetilde{W}_t, \quad V_0 = v > 0\,, \tag{3.31}$$

with $\kappa, m, \delta > 0$. The Brownian motion $\widetilde{W}_t$ driving the variance and the Brownian motion W_t in (3.30) driving the stock price S_t are correlated $\rho \in [-1, 1]$, compare also with figure 3.18. We remark that stochastic volatility models are related to the famous discrete time GARCH model (Generalised Autoregressive Conditional Heteroscedasticity). Indeed, if one takes in the GARCH model limits to continuous time, we end up with the model $\mathrm{d}V_t = \kappa(m - V_t)\mathrm{d}t + \delta V_t \mathrm{d}\widetilde{W}_t$, where the parameters κ, m, δ, ρ depend on the GARCH parameters.

To price derivatives within the model of Heston, we need to calibrate it to market data. To be more precise, we have to find the vector

$$\boldsymbol{\eta} := (\kappa, m, \delta, \rho, v) \tag{3.32}$$

[16] S. Heston. A closed-form Solution for Options with Stochastic Volatility, with Applications to Bond and Currency Options. *The Review of Financial Studies*, 6:327–343, 1993

[17] J.C. Cox, J.E. Ingersoll, and S.A. Ross. A theory of the term structure of interest rates. *Econometrica*, 53(2):385–407, 1985

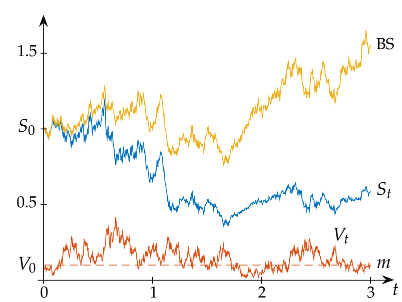

Figure 3.18: A path (—) of the stock price process S_t in the model of Heston and its corresponding variance process V_t (—). The same path (—) in the model of Black and Scholes (BS) with (constant) volatility $\sigma = \sqrt{V_0}$ is displayed for comparison. Parameter values are $S_0 = 1$, $V_0 = 0.3^2$, $\mu = 0.5$, $\kappa = 5$, $m = 0.1$, $\delta = 0.8$ and $\rho = -0.6$.

of model parameters by solving a constrained non-linear least squares problem of the form

$$\widehat{\boldsymbol{\eta}} = \operatorname*{argmin}_{\boldsymbol{\eta}\in\mathcal{S}} \sum_{k=1}^{n} \sum_{\ell=1}^{\nu_k} \left[V^{\mathrm{H}}(X_{k,\ell}, T_{k,\ell}; \boldsymbol{\eta}) - V^{\mathrm{M}}_{k,\ell} \right]^2 \tag{3.33}$$

with $V^{\mathrm{M}}_{k,\ell}$ the market value of a (European) option with strike $X_{k,\ell}$ and time-to-maturity $T_{k,\ell}$ and $V^{\mathrm{H}}(X_{k,\ell}, T_{k,\ell}; \boldsymbol{\eta})$ its corresponding theoretical value according to the Heston model (using the parameter vector $\boldsymbol{\eta}$). Furthermore, in (3.33) $\mathcal{S}$ is the set

$$\mathcal{S} = \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times [-1;1] \times \mathbb{R}^+ \subset \mathbb{R}^5 \, .$$

Example 3.6. On march 1, 2010, we calibrate the Heston model to call options on the EURO STOXX 50, compare with table 3.3. Running (3.33) we find

$$\widehat{\boldsymbol{\eta}} \doteq (0.2115, 0.1615, 0.3771, -0.7128, 0.0536) \, .$$

The average absolute relative error of the implied volatilities (3.28) is ARPE $= 1.84\%$, which is much larger than the ARPE of the local volatility model. Indeed, we observe in figure 3.19 that the fit of the Heston model for short maturities is of rather poor quality. ◇

We may visualise the quality of the calibration in example 3.6 in different ways. Here, we repeat figure 3.13, i.e., we plot the volatility smile implied by the Heston model for three different times-to-maturity. To obtain the volatility smiles implied by the Heston model, we do not solve the problem (3.16), but the problem

$$V(s, 0; X, T_k, \sigma^{\mathrm{i}}, r, q, \omega) - V^{\mathrm{H}}(X, T_k; \widehat{\boldsymbol{\eta}}) = 0$$

for σ^{i}, where $V^{\mathrm{H}}(X, T_k; \widehat{\boldsymbol{\eta}})$ is the price of the option with strike X, time-to-maturity T_k and parameters $\widehat{\boldsymbol{\eta}}$ in the Heston model.

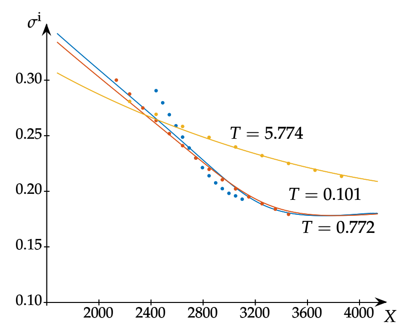

Figure 3.19: Implied market volatilities $(\cdot)$ $\sigma^{\mathrm{i,M}}_{k,\ell}$ of EURO STOXX 50 options as of 01.03.2010 for three different times-to-maturity $T_1 = 0.101$, $T_5 = 0.772$ and $T_{11} = 5.774$, compare with table 3.3, and their corresponding volatility smiles implied by the Heston model (3.30).

One may improve the quality of the fit by generalising the Heston model. First, we may introduce a second factor (stochastic process) to model the instantaneous variance V_t. For example, we may instead of (3.30) consider the two-factor Heston model

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sqrt{V_t^1} S_t \mathrm{d}W_t^1 + \sqrt{V_t^2} S_t \mathrm{d}W_t^2, \quad S_0 = s > 0$$

with uncorrelated (standard) Brownian motions W_t^1 and W_t^2 and where both the variances V_t^1 and V_t^2 follow a CIR process as in (3.31); the Brownian motions $\widetilde{W}_t^1$ and $\widetilde{W}_t^2$ driving these CIR processes are uncorrelated as well, but there holds $\mathrm{Corr}[\mathrm{d}W_t^i, \mathrm{d}\widetilde{W}_t^i] = \rho_i \mathrm{d}t$. Furthermore, we may add jumps to the above model. To be more precise, Bates adds to the above model of S_t a so-called compound Poisson[18]

[18] Name after the french mathematician S. D. Poisson (1781–1840). J in J_t stands for "jump".

process J_t, i.e.,

$$\mathrm{d}S_t = \mu(V_t^1, V_t^2)S_t\mathrm{d}t + \sqrt{V_t^1}S_t\mathrm{d}W_t^1 + \sqrt{V_t^2}S_t\mathrm{d}W_t^2 + S_t\mathrm{d}J_t, \quad S_0 = s > 0\,. \tag{3.34}$$

Here, the function $\mu(v_1, v_2)$ models the drift of S_t. By adding jumps the paths of the underlying become discontinuous in time t (in contrast to the paths in figure 2.5 for example), a property which is observable in reality. The model of Bates[19] consists in 15 parameters: in addition to the 2×5 parameters

$$(\kappa_1, \kappa_2, m_1, m_2, \delta_1, \delta_2, \rho_1, \rho_2, v_1, v_2)$$

of the two-factor Heston model there are five parameters

$$(\lambda_0, \lambda_1, \lambda_2, \mu_J, \sigma_J)$$

of the process modelling the jumps. Here, we have the variance-dependent intensity $\lambda_t = \lambda_0 + \lambda_1 V_t^1 + \lambda_2 V_t^2$ which measures the (expected) number of jumps occurring within a time unit, and the parameters μ_J, σ_J of the log-normally distributed jumps sizes.

[19] D.S. Bates. Post-'87 crash fears in the S&P 500 futures option market. *Journal of Econometrics*, 94(1–2):181–238, 2000

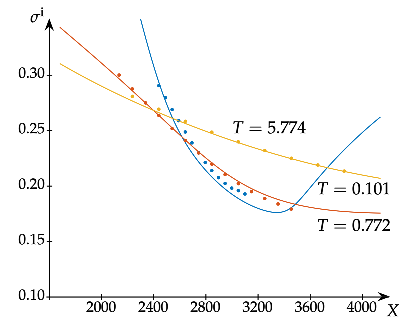

Figure 3.20: Implied market volatilities $(\cdot)$ $\sigma^{\mathrm{i,m}}_{k,\ell}$ of EURO STOXX 50 options as of 01.03.2010 for three different times-to-maturity $T_1 = 0.101$, $T_5 = 0.772$ and $T_{11} = 5.774$, compare with table 3.3, and their corresponding volatility smiles implied by the Bates two-factor model (3.34).

Calibrating the two-factor Bates model leads to a similar least-squares problem as in (3.33), the set $\mathcal{S}$ however is here a subset of $\mathbb{R}^{15}$, which makes the calibration of the model quite involved. Repeating example 3.6 for the Bates model yields an average relative percentage error of ARPE $= 0.89\%$. In particular, adding a second factor and jumps to the Heston model leads to a much better volatility smile for short maturities, compare with figure 3.20.

We compare the calibration of the local volatility model (3.23), the Heston model (3.30) and the Bates model (3.34) with the calibration of the Black-Scholes model[20]. To do so, we solve the minimisation problem (compare with (3.33))

$$\widehat{\sigma} = \underset{\sigma \in \mathbb{R}^+}{\operatorname{argmin}} \sum_{k=1}^{n} \sum_{\ell=1}^{\nu_k} \left[V^{\mathrm{BS}}(X_{k,\ell}, T_{k,\ell}; \sigma) - V^{\mathrm{M}}_{k,\ell} \right]^2$$

with $V^{\mathrm{BS}}(X_{k,\ell}, T_{k,\ell}; \sigma)$ being the Black-Scholes price (using the volatility σ) of the option with strike $X_{k,\ell}$ and time-to-maturity $T_{k,\ell}$. We find $\widehat{\sigma} \doteq 0.2342$ with ARPE $= 13.12\%$, which is huge. Even worse, the volatility smile is - for all times-to-maturity T_k - the same and independent of the strike $X_{k,\cdot}$. That is, the implied volatility $\sigma^{\mathrm{i}}_{k,\ell}$ implied by the Black-Scholes model is the same across all strikes and all times-to-maturity and is equal - by construction - to $\widehat{\sigma}$. The corresponding volatility surface is flat.

[20] In practice, we do not calibrate the Black-Scholes model to market data in this way. However, it is not entirely wrong to treat the Black-Scholes model as any other model, i.e., find all the model parameters such that the sum of squared residuals between market and model prices becomes minimal. The only one parameter to be found in the Black-Scholes model is σ.

We close this section with a remark. No matter which model (local volatility, stochastic volatility etc) we decide to use to price a derivative, the general principle of derivatives pricing still applies. To be more precise, in all these models a (European) option price is given as the discounted expectation with respect to a martingale measure, that is

$$V(s,t) = \mathbb{E}^{\mathbb{Q}}[e^{-r(T-t)}g(S_T) \mid S_t = s] \ ,$$

as in (3.12) in case of a local volatility model and

$$V(s,\sigma,t) = \mathbb{E}^{\mathbb{Q}}[e^{-r(T-t)}g(S_T) \mid S_t = s,\ \sigma_t = \sigma] \ ,$$

in a single-factor stochastic volatility model. The challenge is then to calculate or at least to approximate these expectations.

3.5 Options written on multiple stocks

Almost for all of the structured products categorised in the "Swiss Derivative Map" there exist variants of these products written on multiple underlyings, such as "Multi Barrier Reverse Convertibles" or "Multi Bonus Certificates". The general principle of derivatives pricing sketched in section 3.1 can be extended to the multivariate case. Indeed, if we denote by

$$\mathbf{S}_t := (S_t^1, S_t^2, \ldots, S_t^d) \tag{3.35}$$

the vector price process of d stocks, the value V of a European option with payoff function g and maturity T is with $\mathbf{s} := (s_1, s_2, \ldots, s_d)$ defined as

$$V(\mathbf{s},t) = \mathbb{E}^{\mathbb{Q}}[e^{-r(T-t)}g(\mathbf{S}_T) \mid \mathbf{S}_t = \mathbf{s}] \ .$$

To calculate/approximate this expectation we need a model for the joint price dynamics of $\mathbf{S}_t$. The first such model we already introduced in section 2.4. It is the multivariate Black-Scholes model (2.26)

$$\mathrm{d}S_t^i = \mu_i S_t^i \mathrm{d}t + \sigma_i S_t^i \mathrm{d}\widehat{W}_t^i, \quad S_0^i = s_i > 0 \ ,$$

or equivalently,

$$S_t^i = s_i e^{(\mu_i - \frac{1}{2}\sigma_i^2)t + \sigma_i \widehat{W}_t^i} \ . \tag{3.36}$$

Under the measure $\mathbb{Q}$ we have $\mu_i = r - q_i$ where $q_i \geq 0$ is the continuously compounded dividend yield of the i-th underlying. Recall that the dependence structure of the underlyings in this model is given by the covariance matrix $\mathbf{\Sigma}$ of the log-returns $R_{0,t}^{c,i} := \ln(S_t^i/s_i)$ of the involved underlyings. For later purposes, it is more convenient not to specify the covariance matrix $\mathbf{\Sigma}$ directly, but to express it as a product of volatilities σ_i and correlations ρ_{ij}. Indeed, if we define $\boldsymbol{\sigma} := \mathrm{diag}(\sigma_1, \ldots, \sigma_d)$ to be the $d \times d$ diagonal matrix with the (main) diagonal filled with the volatilities, and if $\boldsymbol{\rho} = (\rho_{ij})$ is the correlation matrix of the log-returns of the underlyings, then we have the factorisation

$$\mathbf{\Sigma} = \boldsymbol{\sigma}\boldsymbol{\rho}\boldsymbol{\sigma} \ .$$

To use the model (3.36) for pricing, we need to specify the volatilities and the correlations. To be (partly) consistent with the case $d = 1$, we choose for the volatilities the implied volatilities σ^{i}_j, whence we set

$$\boldsymbol{\Sigma} = \boldsymbol{\sigma}^{\mathrm{i}}\boldsymbol{\rho}\boldsymbol{\sigma}^{\mathrm{i}} \ , \quad \boldsymbol{\sigma}^{\mathrm{i}} = \mathrm{diag}(\sigma^{\mathrm{i}}_1, \ldots, \sigma^{\mathrm{i}}_d) \ .$$

It remains to specify/model the correlation, which is non-trivial. For $d \times d$ "up" and "down" correlation matrices $\boldsymbol{\rho}_u$ and $\boldsymbol{\rho}_d$ and $\lambda \in [0,1]$ it is market practice to consider the convex combination

$$\boldsymbol{\rho} = (1-\lambda)\boldsymbol{\rho}_d + \lambda\boldsymbol{\rho}_u \qquad (3.37)$$

such that we will work with the model

$$\boldsymbol{\Sigma} = \boldsymbol{\sigma}^{\mathrm{i}}\big((1-\lambda)\boldsymbol{\rho}_d + \lambda\boldsymbol{\rho}_u\big)\boldsymbol{\sigma}^{\mathrm{i}} \ . \qquad (3.38)$$

A possible specification is $\boldsymbol{\rho}_d = \boldsymbol{\rho}^{\mathrm{h}} = (\rho^{\mathrm{h}}_{k\ell})$ and $\boldsymbol{\rho}_u = \mathbf{1}$, where $\rho^{\mathrm{h}}_{k\ell}$ denotes the historical correlation between underlying k and ℓ (this information is extracted from the corresponding time series), and where $\mathbf{1}$ is the full correlation matrix, i.e., a $d \times d$-matrix for which all its elements are equal to 1. As λ is an additional model parameter, the natural question is then how to calibrate it to market data. We might skip the solution to this problem by setting $\lambda = 0$, such that $\boldsymbol{\rho} = \boldsymbol{\rho}^{\mathrm{h}}$. For simplicity, we will mostly use this model specification. Another possible specification of the model (3.38) is to set $\boldsymbol{\rho}_d = \mathbf{I}$, $\boldsymbol{\rho}_u = \mathbf{1}$ and $\lambda = \rho^{\mathrm{i}}$, where $\mathbf{I}$ is the $d \times d$-identity matrix, and where ρ^{i} is the so-called implied correlation, compare with the side remark. Within this specification, all the correlations $\rho_{k\ell}$, $\ell \neq k$, are then equal to ρ^{i}.

The d-dimensional model of Black and Scholes (3.36) is - by the same reasons as for the Black-Scholes model (2.7) - too crude. In practice, banks use a generalisation of the local volatility model (3.19), i.e.,

$$\mathrm{d}S^i_t = \mu_i S^i_t \mathrm{d}t + \sigma_i(S^i_t, t) S^i_t \mathrm{d}\widehat{W}^i_t, \quad S^i_0 = s_i > 0 \ .$$

Thus, the constant volatility σ_i in (2.26) is replaced by the local volatility function $\sigma_i(s_i, t)$ of the i-th underlying and in (3.38), we replace the matrix $\boldsymbol{\sigma}^{\mathrm{i}}$ by the matrix

$$\boldsymbol{\sigma}^{\mathrm{loc}}(\mathbf{s}, t) := \mathrm{diag}\big(\sigma_1(s_1, t), \ldots, \sigma_d(s_d, t)\big) \ .$$

This model is called local volatility with constant correlation (LVCC) model. However, assuming that correlation is constant over time is as wrong as assuming volatility is constant over time. Whence, it is natural and necessary to develop models that allow correlation depend on time; this dependence can either be deterministic or stochastic. A local volatility model with local correlation

Implied correlation. We might define implied correlation $\rho^{\mathrm{i}}_{k\ell}$ between underlyings k and ℓ as follows. Suppose there is an option on a basket with value $B_t = \omega_k S^k_t + \omega_\ell S^\ell_t$ containing these two underlyings only. The market states the price of the option in terms of implied volatility, this is the basket implied volatility σ^{i}_B. From portfolio theory, we know that the volatility of a basket built with two constituents k and ℓ is given by $\sigma^2_B = \omega^2_k\sigma^2_k + \omega^2_\ell\sigma^2_\ell + 2\omega_k\omega_\ell\sigma_k\sigma_\ell\rho_{k\ell}$. Since we know the weights ω_k, ω_ℓ as well as the implied volatilities of the basket constituents, we can solve for $\rho_{k\ell}$ to get

$$\rho^{\mathrm{i}}_{k\ell} = \frac{{\sigma^{\mathrm{i}}_B}^2 - \omega^2_k{\sigma^{\mathrm{i}}_k}^2 - \omega^2_\ell{\sigma^{\mathrm{i}}_\ell}^2}{2\omega_k\omega_\ell\sigma^{\mathrm{i}}_k\sigma^{\mathrm{i}}_\ell} \ .$$

In practice, we typically do not have available the quotes σ^{i}_B, because basket options written on the underlyings k and ℓ only do not exist. For example, there is no European basket option on Novartis and Roche only, but these two stocks are SMI members, and the SMI implied volatility is quoted. Thus, it might be better to assume that $\rho := \rho_{k\ell}$ is the same for all pairs (k, ℓ), $k \neq \ell$, of underlyings in the basket. Under this assumption, we obtain

$$\rho^{\mathrm{i}} = \frac{{\sigma^{\mathrm{i}}_B}^2 - \boldsymbol{\omega}^\top\boldsymbol{\sigma}^{\mathrm{i}}\boldsymbol{\sigma}^{\mathrm{i}}\boldsymbol{\omega}}{\boldsymbol{\omega}^\top\boldsymbol{\sigma}^{\mathrm{i}}(\mathbf{1}-\mathbf{I})\boldsymbol{\sigma}^{\mathrm{i}}\boldsymbol{\omega}} \ ,$$

where $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_d)^\top$ is the (column) vector of basket weights. We rewrite the above expression for ρ^{i}. To do so, let $\mathbf{e} := (1, \ldots, 1)^\top$ be the column vector of length d with each entry equal to one. Since the basket weights sum up to one, we can write $1 = \mathbf{e}^\top\boldsymbol{\omega}$ as well as $1 = 1^\top 1 = \boldsymbol{\omega}^\top\mathbf{e}\mathbf{e}^\top\boldsymbol{\omega} = \boldsymbol{\omega}^\top\mathbf{1}\boldsymbol{\omega}$. Using this identity, we wright the expression for the implied correlation more distinctly as

$$\rho^{\mathrm{i}} = \frac{\boldsymbol{\omega}^\top(\sigma^{\mathrm{i}}_B\mathbf{1}\sigma^{\mathrm{i}}_B - \boldsymbol{\sigma}^{\mathrm{i}}\boldsymbol{\sigma}^{\mathrm{i}})\boldsymbol{\omega}}{\boldsymbol{\omega}^\top(\boldsymbol{\sigma}^{\mathrm{i}}\mathbf{1}\boldsymbol{\sigma}^{\mathrm{i}} - \boldsymbol{\sigma}^{\mathrm{i}}\boldsymbol{\sigma}^{\mathrm{i}})\boldsymbol{\omega}} \ .$$

If we calculate ρ^{i} for different strikes X and plot then the graph of the mapping $X \mapsto \rho^{\mathrm{i}}$, we obtain the so-called implied correlation smile, compare for example with Bossu's book .

S. Bossu. *Advanced Equity Derivatives*. Wiley Finance Series. Wiley, 2014

(LVLC) is an extension of the LVCC by assuming that the correlation $\rho_{k\ell} = \rho_{k\ell}(S_t^k, S_t^\ell, t)$ is a deterministic function of the price levels S_t^k, S_t^ℓ of the underlyings k and ℓ (and of time t). Once we have specified all the $d(d-1)/2$ local correlation functions, we obtain a local correlation matrix $\boldsymbol{\rho}^{\mathrm{loc}}(\mathbf{s}, t) := \big(\rho_{k\ell}(s_k, s_\ell, t)\big)$. As it might be too hard to find all the functions $\rho_{k\ell}(S_t^k, S_t^\ell, t)$ through calibration to market quotes, it is simpler to assume that the local correlation matrix is of the form (3.37), where now λ is not a number anymore, but a function $\mathbb{R}^d \times \mathbb{R} \ni (\mathbf{s}, t) \mapsto \lambda(\mathbf{s}, t) \in [0, 1]$. The local covariance matrix is thus

$$\begin{aligned} \boldsymbol{\Sigma}^{\mathrm{loc}}(\mathbf{s}, t) &= \boldsymbol{\sigma}^{\mathrm{loc}}(\mathbf{s}, t)\boldsymbol{\rho}^{\mathrm{loc}}(\mathbf{s}, t)\boldsymbol{\sigma}^{\mathrm{loc}}(\mathbf{s}, t) \qquad (3.39)\\ &= \boldsymbol{\sigma}^{\mathrm{loc}}(\mathbf{s}, t)\big((1-\lambda(\mathbf{s}, t))\boldsymbol{\rho}_d + \lambda(\mathbf{s}, t)\boldsymbol{\rho}_u\big)\boldsymbol{\sigma}^{\mathrm{loc}}(\mathbf{s}, t)\ . \end{aligned}$$

How can we extract the function $\lambda(\mathbf{s}, t)$ from market data? Define the (abstract weight) vector $\mathbf{w} = (w_1, \ldots, w_d)^\top$, where the i-th component is the product of the weight ω_i and the price S_t^i of the i-th underlying in a basket (for example the SMI), i.e., $w_i := \omega_i S_t^i$. Then, the value B_t of the corresponding basket can be written as $B_t = \sum_{i=1}^d \omega_i S_t^i = \sum_{i=1}^d w_i = \mathbf{e}^\top \mathbf{w}$, where $\mathbf{e} := (1, \ldots, 1)^\top$ is the column vector of length d with each entry equal to one. Suppose the local volatility function $\sigma_B(B_t, t)$ of the basket is known. Langnau [21] shows that under certain arbitrage conditions there holds

$$\sigma_B^2(B_t, t)B_t^2 = \mathbf{w}^\top \boldsymbol{\Sigma}^{\mathrm{loc}}(\mathbf{S}_t, t)\mathbf{w}\ .$$

If we plug in (3.39) into the above equation and solve for λ, we find

$$\lambda(\mathbf{S}_t, t) = \frac{\sigma_B^2(B_t, t)B_t^2 - \mathbf{w}^\top \boldsymbol{\sigma}^{\mathrm{loc}}\boldsymbol{\rho}_d\boldsymbol{\sigma}^{\mathrm{loc}}\mathbf{w}}{\mathbf{w}^\top \boldsymbol{\sigma}^{\mathrm{loc}}(\boldsymbol{\rho}_u - \boldsymbol{\rho}_d)\boldsymbol{\sigma}^{\mathrm{loc}}\mathbf{w}} \qquad (3.40)$$

where for notational simplicity we omit the dependence of the local volatility (diagonal) matrix $\boldsymbol{\sigma}^{\mathrm{loc}} = \boldsymbol{\sigma}^{\mathrm{loc}}(\mathbf{S}_t, t)$ on the values S_t^i and t, compare with problem 3.7.[22] In practice, we might again specify $\boldsymbol{\rho}_d = \mathbf{I}$ and $\boldsymbol{\rho}_u = \mathbf{1}$. Using the identity $B_t^2 = \mathbf{w}^\top \mathbf{e}\mathbf{e}^\top \mathbf{w} = \mathbf{w}^\top \mathbf{1}\mathbf{w}$, we find the formula

$$\lambda(\mathbf{S}_t, t) = \frac{\mathbf{w}^\top(\sigma_B \mathbf{1}\sigma_B - \boldsymbol{\sigma}^{\mathrm{loc}}\boldsymbol{\sigma}^{\mathrm{loc}})\mathbf{w}}{\mathbf{w}^\top(\boldsymbol{\sigma}^{\mathrm{loc}}\mathbf{1}\boldsymbol{\sigma}^{\mathrm{loc}} - \boldsymbol{\sigma}^{\mathrm{loc}}\boldsymbol{\sigma}^{\mathrm{loc}})\mathbf{w}}$$

which becomes similar to the formula for the implied correlation.

We will use the model (2.26) together with (3.38) to price structured products on multiple stocks, such as multi bonus certificates, multi barrier reverse convertibles or multi capital protection products, respectively.

[21] A. Langnau. A Dynamic Model for Correlation. *Risk Magazine*, (April):74–78, 2010

[22] Note that also $\mathbf{w}$ depends on $\mathbf{S}_t$, $\mathbf{w} = \boldsymbol{\omega} \odot \mathbf{S}_t$; $\odot$ denotes the elementwise multiplication of matrices having the same dimension.

3.6 Problems

Problem 3.1. Show that $\mathbb{E}^{\mathbb{Q}}[S_m] = s_{0,0}e^{rT}$, i.e., the expected future value of the stock with respect to the probability measure $\mathbb{Q}$ is just equal to its present value $s_{0,0}$ continuously compounded with rate equal to the risk free.
Hint. Use equation (2.27) and the definition of p^*, q^* in (3.3).

Problem 3.2. We consider a European call option with strike X and time-to-maturity T within a binomial tree having m periods. In this problem we show the "discrete" Black and Scholes formula (3.10). According to (3.8) the price $v_{0,0}$ of the call is given by

$$v_{0,0} = e^{-rT} \sum_{i=0}^{m} \binom{m}{i} p^{*i} q^{*m-i} \max\{s_{i,m} - X, 0\} .$$

i) Find the unique index $i^* \in \mathbb{N}$ with the property that

$$\max\{s_{i,m} - X, 0\} = \begin{cases} 0 & \text{if} \quad i < i^* \\ s_{i,M} - X & \text{if} \quad i \geq i^* \end{cases} .$$

ii) By part i), the price of the call can be written as

$$v_{0,0} = e^{-rT} \sum_{i=i^*}^{m} \binom{m}{i} p^{*i} q^{*m-i} (s_{i,m} - X) .$$

Show that this sum can be explicitly evaluated as

$$v_{0,0} = sB_{m,\widehat{p}}(i^*) - Xe^{-rT}B_{m,p^*}(i^*) , \tag{3.41}$$

where $\widehat{p} = p^* u e^{-r\Delta t}$. Here, $B_{n,p}(x)$ denotes the complementary distribution function of the binomial distribution with parameters n and p, that is

$$B_{n,p}(x) := 1 - \sum_{i=0}^{x-1} \binom{n}{i} p^i (1-p)^{n-i} = 1 - F_{n,p}(x-1) .$$

iii) Implement formula (3.41) in Python. Now take the values $s = 105$, $X = 100$, $T = 0.5$, $\sigma = 0.22$ and $r = 0.001$. How may periods (the m) you have to take such that the binomial tree price $v_{0,0}$ differs not more than 1 basis point from the Black-Scholes price (which is $V \doteq 9.19383$)?

Problem 3.3. Implement the Black-Scholes formula (3.14) in Python and call the corresponding function `call_but_bs.py`.

Problem 3.4. [Dual-gamma*] Show that the dual-gamma $V_{XX}(s, t; X, T, f(X), r, q, \omega)$ of a European put and call option with strike-dependent volatility $\sigma = f(X)$ is given by expression (3.18). Proceed as follows:

i) Show, by using the definition of ϕ and $d_{1,2}$, that there holds

$$\phi(d_1) = e^{-(r-q)(T-t)} \frac{X}{s} \phi(d_2) \; .$$

ii) Show, by using the definition of $d_{1,2}$, that the partial derivative of d_2 with respect to X is given by

$$\partial_X d_2 = -\frac{1}{X f(X) \sqrt{T-t}} - \frac{f'(X)}{f(X)} d_1 \; .$$

iii) Use part i) to show that the dual-delta of a European put or call option is given by

$$\begin{aligned} &\partial_X V(s, t; X, T, f(X), r, q, \omega) \\ &= X e^{-r(T-t)} \sqrt{T-t} \phi(d_2) f'(X) - \omega e^{-r(T-t)} N(\omega d_2) \end{aligned}$$

iv) Finally, utilise parts ii) and iii) to show (3.18).

Problem 3.5. Prove inequality (3.24) by setting $y := x - m$ and calculating the (global) minima $y^\star$ of $w(y) = a + b\Big(\rho y + \sqrt{y^2 + s^2}\Big)$. Then show that the minimal total implied variance is $w(y^\star) = a + bs\sqrt{1-\rho^2}$.

Problem 3.6. Show: if we model the total implied variance as in (3.27), then the parameters a, b, m and s depend on ρ, φ and $\theta(T)$ in the following way

$$\begin{aligned} a &= \frac{1}{2}\theta(T)(1-\rho^2) \\ b &= \frac{1}{2}\theta(T)\varphi(\theta(T)) \\ m &= -\frac{\rho}{\varphi(\theta(T))} \\ s &= \frac{\sqrt{1-\rho^2}}{\varphi(\theta(T))} \; . \end{aligned}$$

Problem 3.7. Show equation (3.40) by plugging the expression (3.39) into $\sigma_B^2(B_t, t) B_t^2 = \mathbf{w}^\top \boldsymbol{\Sigma}^{\mathrm{loc}}(\mathbf{S}_t, t)\mathbf{w}$ and then solving for λ.

3.7 Technical note. The (generalised) Black-Scholes formula

In this section we derive the Black-Scholes formula stated in (3.14) for $\omega = 1$. To do so, we first consider the log-return in the time-interval $[t, T]$ (compare with (2.14))

$$R^c_{t,T} = \ln \frac{S_T}{S_t} \sim \mathcal{N}(\mu_S, \sigma_S^2)$$

and the problem of calculating the undiscounted expectation

$$\widehat{V}(s,t) := \mathbb{E}^{\mathbb{Q}}\left[g(S_T) \mid S_t = s\right] .$$

According to the definition of the expectation of a continuous random variable and since $S_T = S_t e^{R^c_{t,T}}$ as well as $S_t = s$ we have to evaluate the integral

$$\widehat{V}(s,t) = \mathbb{E}^{\mathbb{Q}}\left[g(se^{R^c_{t,T}})\right] = \int_{\mathbb{R}} g(se^x)\phi_{\mu_S,\sigma_S}(x)\mathrm{d}x ,$$

with ϕ_{μ_S,σ_S} the pdf of a normally distributed random variable as in (2.9). We standardise $z := (x - \mu_S)/\sigma_S$ and obtain

$$\widehat{V}(s,t) = \int_{\mathbb{R}} g(se^{\sigma_S z + \mu_S})\phi(z)\mathrm{d}z ,$$

where ϕ is the pdf of a standard normally distributed random variable as in (2.10). The payoff $g(x) = \max\{x - X, 0\}$ of a call option becomes zero if the argument x is smaller than the strike X, i.e.

$$se^{\sigma_S z + \mu_S} < X \Leftrightarrow z < \frac{\ln \frac{X}{s} - \mu_S}{\sigma_S} = -\frac{\ln \frac{s}{X} + \mu_S}{\sigma_S} .$$

For simplicity, we define the auxiliary variable d as

$$d = \frac{\ln \frac{s}{X} + \mu_S}{\sigma_S} ,$$

such that it is enough to evaluate the integral

$$\widehat{V}_c(s,t) = \int_{-d}^{\infty} \left(se^{\sigma_S z + \mu_S} - X\right)\phi(z)\mathrm{d}z .$$

We separate this integral in two parts, namely

$$\widehat{V}_c(s,t) = se^{\mu_S} \underbrace{\int_{-d}^{\infty} e^{\sigma_S z}\phi(z)\mathrm{d}z}_{=:\mathcal{I}} - X \int_{-d}^{\infty} \phi(z)\mathrm{d}z .$$

Now, the second integral is due to (2.11) just

$$\int_{-d}^{\infty} \phi(z)\mathrm{d}z = 1 - N(-d) = N(d)$$

and it remains to find the first integral

$$\mathcal{I} := \int_{-d}^{\infty} e^{\sigma_S z}\phi(z)\mathrm{d}z \overset{(2.10)}{=} \frac{1}{\sqrt{2\pi}} \int_{-d}^{\infty} e^{\sigma_S z - \frac{1}{2}z^2}\mathrm{d}z .$$

To do so, we complete squares in the exponent of the integrand, that is

$$-\frac{1}{2}z^2 + \sigma_S z = -\frac{1}{2}(z - \sigma_S)^2 + \frac{1}{2}\sigma_S^2 .$$

Hence, the first integral becomes

$$\begin{aligned}
\mathcal{I} &= e^{\frac{1}{2}\sigma_S^2} \frac{1}{\sqrt{2\pi}} \int_{-d}^{\infty} e^{-\frac{1}{2}(z-\sigma_S)^2} \mathrm{d}z \\
&\overset{u=z-\sigma_S}{=} e^{\frac{1}{2}\sigma_S^2} \frac{1}{\sqrt{2\pi}} \int_{-d-\sigma_S}^{\infty} e^{-\frac{1}{2}u^2} \mathrm{d}u \\
&= e^{\frac{1}{2}\sigma_S^2} \big(1 - N(-d-\sigma_S)\big) = e^{\frac{1}{2}\sigma_S^2} N(d+\sigma_S) \; .
\end{aligned}$$

Thus, the undiscounted value $\widehat{V}_c(s,t)$ of a call option is

$$\widehat{V}_c(s,t) = s e^{\mu_S + \frac{1}{2}\sigma_S^2} N(d+\sigma_S) - XN(d) \tag{3.42}$$

with

$$d = \frac{\ln \frac{s}{X} + \mu_S}{\sigma_S} \; . \tag{3.43}$$

As an application of formula (3.42) we now consider the case where the stock price S_t follows a geometric Brownian motion as in (2.12), but with time-dependent parameters μ and σ. In particular we consider

$$\mathrm{d}S_t = \big(r(t) - q(t)\big) S_t \mathrm{d}t + \sigma(t) S_t \mathrm{d}W_t, \quad S_0 = s > 0 \; . \tag{3.44}$$

Here, r, q and σ are the continuous risk free rate, the continuous dividend yield and the volatility, respectively. These functions are deterministic; the model (3.44) is called the generalised Black-Scholes model. One can show that in the model (3.44) the log-return $R^c_{t,T} = \ln \frac{S_T}{S_t} \sim \mathcal{N}(\mu_S, \sigma_S^2)$ is normally distributed, with mean and variance given by

$$\mu_S = \int_t^T \Big(r(\tau) - q(\tau) - \frac{1}{2}\sigma^2(\tau) \Big) \mathrm{d}\tau, \quad \sigma_S^2 = \int_t^T \sigma^2(\tau) \mathrm{d}\tau \; .$$

To simplify the notation, we introduce the following quantities, averaged over the time interval $[t, T]$

$$\begin{aligned}
\overline{\sigma}^2 &:= \frac{1}{T-t} \int_t^T \sigma^2(\tau) \mathrm{d}\tau \\
\overline{r} &:= \frac{1}{T-t} \int_t^T r(\tau) \mathrm{d}\tau \\
\overline{q} &:= \frac{1}{T-t} \int_t^T q(\tau) \mathrm{d}\tau \; .
\end{aligned}$$

Using these we can write μ_S and σ_S as

$$\mu_S = \big(\overline{r} - \overline{q} - \frac{1}{2}\overline{\sigma}^2\big)(T-t), \quad \sigma_S^2 = \overline{\sigma}^2 (T-t) \; . \tag{3.45}$$

In case of a time-dependent risk free $r(t)$ future cash flows have to be discounted with $e^{-\int_t^T r(\tau)\mathrm{d}\tau} = e^{-\overline{r}(T-t)}$. Hence, the value of a call option is

$$\begin{aligned} V_c(s,t) &= e^{-\bar{r}(T-t)}\widehat{V}_c(s,t) \\ &\overset{(3.42)}{=} se^{-\bar{r}(T-t)+\mu_S+\frac{1}{2}\sigma_S^2}N(d+\sigma_S) - Xe^{-\bar{r}(T-t)}N(d) \\ &\overset{(3.45)}{=} se^{-\bar{q}(T-t)}N(d_1) - Xe^{-\bar{r}(T-t)}N(d_2) \end{aligned}$$

with

$$\begin{aligned} d_1 &:= d+\sigma_S \overset{(3.43)}{=} \frac{\ln\frac{s}{X}+\mu_S+\sigma_S^2}{\sigma_S} \\ &\overset{(3.45)}{=} \frac{\ln\frac{s}{X}+(\bar{r}-\bar{q}+\frac{1}{2}\bar{\sigma}^2)(T-t)}{\sqrt{\bar{\sigma}^2(T-t)}} \\ d_2 &:= d = d_1 - \sigma_S \overset{(3.45)}{=} d_1 - \sqrt{\bar{\sigma}^2(T-t)}\,. \end{aligned}$$

If in particular the parameters r, q and σ are constant (Black-Scholes model), we have $\bar{r}=r$, $\bar{q}=q$ as well as $\bar{\sigma}^2=\sigma^2$ and thus

$$V_c(s,t) = se^{-q(T-t)}N(d_1) - Xe^{-r(T-t)}N(d_2)$$

with

$$d_1 = \frac{\ln\frac{s}{X}+(r-q+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = d_1 - \sigma\sqrt{T-t}\,.$$

This is the formula of Black and Scholes (3.14) for $\omega = 1$.

3.8 Technical note. Maximum likelihood estimation

Maximum likelihood is a common approach to fit parametric distributions to data. Suppose $\mathbf{x} = (x_1, x_2, \ldots, x_n)^\top$ is random sample of a (univariate) random variable X. We assume that the density $f(x;\boldsymbol{\eta})$ of X depends on m parameters $\boldsymbol{\eta} = (\eta_1, \eta_2, \ldots, \eta_m)$. For example, the normal distribution with density (2.9) has two parameters, $\boldsymbol{\eta} = (\mu, \sigma)$, and the student t distribution with density (3.15) is described by three parameters $\boldsymbol{\eta} = (\mu, \sigma, \nu)$. We select the parameters $\boldsymbol{\eta}$ in such a way that, according to the chosen distribution, the realisation $\mathbf{x}$ is most likely to appear. Since $\mathbf{x}$ is a random sample, it can be viewed as a realisation of the random vector $\mathbf{X} = (X_1, X_2, \ldots, X_n)$, where the X_i's are iid copies of X. The random vector $\mathbf{X}$ has then density $\prod_k f(x_k;\boldsymbol{\eta})$. For a given and fixed $\mathbf{x}$, this is a function on the m variables $(\eta_1, \eta_2, \ldots, \eta_m)$; it is called the likelihood function L. Thus, L is given by

$$L(\mathbf{x};\boldsymbol{\eta}) = \prod_{k=1}^{n} f(x_k;\boldsymbol{\eta})\,. \tag{3.46}$$

We are looking for the parameter vector $\boldsymbol{\eta}$ that maximises L. The necessary first order condition for an extremum is that all partial derivatives $\partial_{\eta_i} L$ have to be zero. We assume that the corresponding (in general non-linear) system of m equations

$$\begin{cases} \partial_{\eta_1} L(\mathbf{x};\boldsymbol{\eta}) & = & 0 \\ \partial_{\eta_2} L(\mathbf{x};\boldsymbol{\eta}) & = & 0 \\ & \vdots & \\ \partial_{\eta_m} L(\mathbf{x};\boldsymbol{\eta}) & = & 0 \end{cases} \tag{3.47}$$

admits a solution $\widehat{\boldsymbol{\eta}}$. Since is unpleasant to take derivatives of a n-fold product $\partial_{\eta_i} \prod_{k=1}^{n} f(x_k;\boldsymbol{\eta})$, it is more convenient to consider the (natural) logarithm of L; the corresponding function is called the log-likelihood function

$$\ell(\mathbf{x};\boldsymbol{\eta}) := \ln L(\mathbf{x};\boldsymbol{\eta}) \stackrel{(3.46)}{=} \sum_{k=1}^{n} \ln f(x_k;\boldsymbol{\eta}) \ .$$

Since $x \mapsto \ln x$ is strictly increasing, the solution $\widehat{\boldsymbol{\eta}}$ of (3.47) is also the solution of

$$\begin{cases} \partial_{\eta_1} \ell(\mathbf{x};\boldsymbol{\eta}) & = & 0 \\ \partial_{\eta_2} \ell(\mathbf{x};\boldsymbol{\eta}) & = & 0 \\ & \vdots & \\ \partial_{\eta_m} \ell(\mathbf{x};\boldsymbol{\eta}) & = & 0 \end{cases} \tag{3.48}$$

and vice-versa. How does (3.48) translate to the case when X follows a student t distribution with density f as in (3.15), that is

$$f(x_k;\boldsymbol{\eta}) = c_\nu \frac{1}{\sqrt{\nu\pi}\sigma} \left(1 + \frac{1}{\nu}(\frac{x_k - \mu}{\sigma})^2\right)^{-(\nu+1)/2} ;$$

and $\boldsymbol{\eta} = (\mu, \sigma, \nu)$? Well, if we denote by $z_k := \frac{x_k - \mu}{\sigma}$ and if we remark that the logarithm of f is in this case just[23]

$$\ln f(x_k;\mu,\sigma,\nu) = -\frac{1}{2}\ln\pi + \ln c_\nu - \frac{1}{2}\ln\nu - \ln\sigma - \frac{\nu+1}{2}\ln\left(1 + \frac{1}{\nu}z_k^2\right) ,$$

then it is easy to see that the log-likelihood function is

$$\ell(\mathbf{x};\mu,\sigma,\nu) = -\frac{n}{2}\ln\pi + n\ln c_\nu - \frac{n}{2}\ln\nu - n\ln\sigma - \frac{\nu+1}{2}\sum_{k=1}^{n}\ln g_k(\mu,\sigma,\nu) \ .$$

Here, we have set

$$g_k(\mu,\sigma,\nu) := 1 + \frac{1}{\nu}z_k^2 = 1 + \frac{1}{\nu}(\frac{x_k-\mu}{\sigma})^2 \tag{3.49}$$

to simplify the notation. The system (3.48) in this particular case becomes

$$\begin{aligned} \partial_\mu \ell(\mathbf{x};\mu,\sigma,\nu) &= -\frac{\nu+1}{2}\sum_{k=1}^{n}\frac{\partial_\mu g_k}{g_k} = 0 \\ \partial_\sigma \ell(\mathbf{x};\mu,\sigma,\nu) &= -\frac{n}{\sigma} - \frac{\nu+1}{2}\sum_{k=1}^{n}\frac{\partial_\sigma g_k}{g_k} = 0 \\ \partial_\nu \ell(\mathbf{x};\mu,\sigma,\nu) &= n\frac{\partial_\nu c_\nu}{c_\nu} - \frac{n}{2\nu} - \frac{1}{2}\sum_{k=1}^{n}\ln g_k - \frac{\nu+1}{2}\sum_{k=1}^{n}\frac{\partial_\nu g_k}{g_k} = 0 \end{aligned}$$

[23] Carefully note that z_k depends on the unknown parameters μ and σ, such that we should better write $z_k(\mu,\sigma) = \frac{x_k-\mu}{\sigma}$.

Calculating the partial derivatives $\partial_\nu c_\nu$, $\partial_\mu g_k$, $\partial_\sigma g_k$ and $\partial_\nu g_k$ and doing some algebraic manipulations, we finally obtain the following non-linear system to find the parameter vector $\boldsymbol{\eta} = (\mu, \sigma, \nu)$ corresponding to the student t distribution with density (3.15)

$$\begin{cases} \displaystyle\sum_{k=1}^{n} \frac{z_k}{g_k} &= 0 \\ \displaystyle\frac{1}{n}\sum_{k=1}^{n} \frac{z_k^2}{g_k} - \frac{\nu}{\nu+1} &= 0 \\ \displaystyle\psi\Big(\frac{\nu+1}{2}\Big) - \psi\Big(\frac{\nu}{2}\Big) - \frac{1}{n}\sum_{k=1}^{n} \ln g_k &= 0 \end{cases}$$

with

$$z_k = \frac{x_k - \mu}{\sigma}, \quad g_k = 1 + \frac{1}{\nu} z_k^2 \,.$$

The above system can be solved numerically in Python using `fsolve`, compare with the corresponding jupiter notebook to this chapter. Alternatively, we might use `dist.fit` from SciPy directly. Here, `.fit` realises MLE for more than 80 different distributions.

The function $\psi(x)$ appearing in this system is the so-called Digamma- or Psi-function, denoted by $\psi(x)$, and is implemented in many software packages under the name "psi". It is defined as $\psi(x) := \partial_x \ln \Gamma(x)$, where the Gamma function $\Gamma(x)$ appears in the normalisation constant c_ν as $c_\nu := \Gamma(\frac{\nu+1}{2})/\Gamma(\frac{\nu}{2})$. The Gamma function is - under certain conditions - the only one extension of the factorial $x!$ to real (and even complex) numbers.

Chapter 4

Barrier options

Many structured products have barrier options as components. Thus it makes sense to discuss these in a separate chapter. We consider single barrier options only (since we do not need the more complicated double barrier options) and give pricing formulas in the Black-Scholes model.

4.1 Functionality of barrier options

A barrier option is an example of a so-called path-dependent option, since its payoff depends on the realised path of the underlying. Whereas the payoff of a European option is only a function of the price of the underlying at maturity S_T and hence it is irrelevant "which path" from S_0 to S_T the underlying has taken, a barrier options becomes either worthless of worth full if - during the lifetime of the option - the underlying has touched or breached a *barrier* $B > 0$. Before we proceed, we need a proper mathematical definition of such "barrier events". Consider a (open) set $G \in \mathbb{R}^d$ and let $\mathbf{X}_t = (X_t^1, \ldots, X_t^d)$ be some vector-valued stochastic process with continuous paths (each component X_t^j is a stochastic process). You may image $\mathbf{X}_t$ to model the time-evolution of d stocks. Assuming that $\mathbf{X}$ starts in $\mathbf{X}_0 = (X_0^1, \ldots, X_0^d) \in G$ at time $t = 0$, define the random time

$$\tau_G := \min\{t \geq 0 \mid \mathbf{X}_t \notin G\} \ . \tag{4.1}$$

This is the first exit time of $\mathbf{X}$ from the domain G, compare with figure 4.1.

For $T > 0$ define the (discrete) random variable

$$I := 1_{\{T<\tau_G\}} := \begin{cases} 1 & \text{if } T < \tau_G \\ 0 & \text{else} \end{cases} \ . \tag{4.2}$$

Thus, $I = 1$ means that $\mathbf{X}$ did not leave G before T. For barrier options with barrier $B > 0$ written on one underlying we have $d = 1$

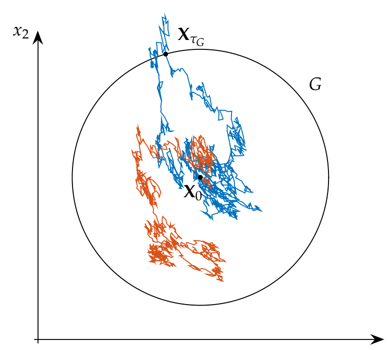

Figure 4.1: First exit time τ_G. In this example, the domain G is a disk in $\mathbb{R}^2$. Within a certain time period $[0, T]$, one path (starting in $\mathbf{X}_0$) leaves G, another path does not. Compare also with figure 5.4.

and the domain is either $G =]B, \infty[$ or $G =]0, B[$. The random variable I is completely characterised if we calculate the probability $\mathbb{P}[I = 1] = \mathbb{P}[T < \tau_G]$ that the process $\mathbf{X}$ does not exit G before T, see section 4.3. According to the two values the random variable I can take we distinguish two main types of barrier options.

- An out-option only pays off if the underlying does *not* reach the barrier; $I = 1$. Otherwise the option becomes worthless.

- An in-option only pays off if the underlying reaches the barrier; $I = 0$. Otherwise the option becomes worthless.

Out- or in options respectively can be further grouped in up-or down-options.

- If $S_0 < B$, the option is an up-option (the domain G is $G =]0, B[$).

- If $S_0 > B$, the option is a down-option (the domain G is $G =]B, \infty[$).

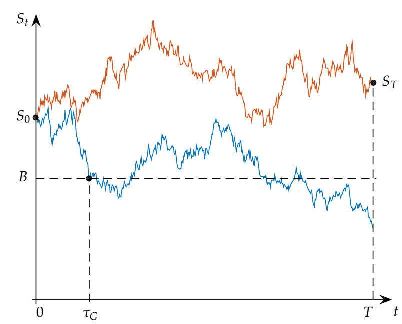

Figure 4.2: A down-and-out option. One path (—) of the underlying S_t breaches the barrier $B < S_0$ from above (and whence the options becomes worthless), another path (—) does not. In the first case, the first exit time τ_G of S_t from the domain $G =]B, \infty[$ is smaller than T, in the second case it is larger than T.

To completely characterise a barrier option, we additionally need to specify the payoff function g at maturity. For example, a down-and-out put pays off at maturity $g(S_T) = \max\{X - S_T, 0\}$ if the underlying did not hit the barrier from above, compare also with figures 4.2 and 4.3.

Thus, there are $2^3 = 8$ (classical) types of single barrier options: an in- or out-option may be an up- or down-option, each of these four possibilities can be further specified as put or call options. In particular, we consider the payoff function of a down-and-out put. Now, if the barrier was not hit during the lifetime of the option, we have $I = 1$ and the option pays off the amount of its European counterpart, i.e., $g(S_T) = \max\{X - S_T, 0\}$. If, however, there was a barrier event, i.e., $I = 0$, the option becomes worthless no matter the further development of the underlying and thus $g(S_T) = 0$. Using the variable I the payoff function of an down-and-out put (still the barrier has to satisfy $B < S_0$) can thus written as

$$g(S_T) = I \max\{X - S_T, 0\} , \tag{4.3}$$

compare with figure 4.3.

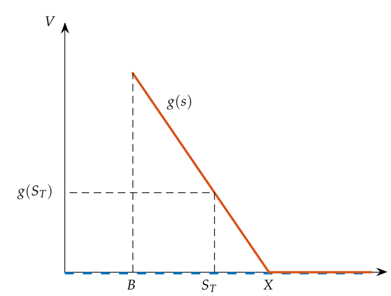

Figure 4.3: The payoff function of a down-and-out put is $I \max\{X - S_T, 0\}$ with I defined as in (4.2), where $G =]B, \infty[$. At maturity T, such an option pays off $\max\{X - S_T, 0\}$ but only if the underlying did not hit the barrier (from above), i.e. if $I = 1$. The event "barrier never hit" is coloured red (—), whereas the event "barrier hit" is marked by (− −).

Example 4.1. We denote by $V_{p,\text{do}}$ and $V_{p,\text{di}}$ the value (price) of a down-and-out put and a down-and-in put, respectively.

i) What is the price of a down-and-out put if the barrier B is larger than the strike X?

ii) Give - for $B < X$ - the payoff function for a down-and-in put.

iii) Which relation exists between the prices $V_{p,\text{di}}$ and $V_{p,\text{do}}$ under the assumption that the considered options are written on the same underlying and have the same strike X, the same barrier B and the same maturity T?

Solution.

i) If there is a barrier event, the out-option is worthless by definition. If there is no barrier event, i.e., $S_t > B$ for all $t \in [0, T]$, in particular $S_T > B > X$, the option pays off $\max\{X - S_T, 0\} = 0$, compare with figure 4.3. Thus, for $B > X$, the price of down-and-out put is $V_{p,\text{do}} = 0$.

ii) A down-and-in put option pays off $\max\{X - S_T, 0\}$, but only if the underlying hits the barrier (from above). Otherwise, the option is worthless. In terms of the variable I we can thus write the payoff of a down-and-in put as

$$g(S_T) = I^c \max\{X - S_T, 0\}$$

with $I^c = 1 - I$ the complement of I.

iii) Suppose we are long a down-and-in and a down-and-out put option written on the same underlying and both having the same parameters. The payoff or the value at maturity $t = T$ of this portfolio is

$$\underbrace{I^c \max\{X - S_T, 0\}}_{V_{p,\text{di}}} + \underbrace{I \max\{X - S_T, 0\}}_{V_{p,\text{do}}}$$
$$\overset{(I^c = 1 - I)}{=} \max\{X - S_T, 0\} = V_p$$

which is the payoff of a European put. The equation

$$V_{p,\text{di}} + V_{p,\text{do}} = V_p$$

must also hold for $t < T$, otherwise there would exist arbitrage opportunities. ◇

4.2 Pricing single barrier options

By the general principles of derivatives pricing discussed in chapter 3 the price of a down-and-out option with payoff function g is - under the assumption of time-continuous barrier monitoring - given by the expectation

$$V(s, t) = e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}} \left[g(S_T) 1_{\{T < \tau_G\}} \mid S_t = s \right] .$$

To evaluate this expectation we need to know explicitly the (risk neutral) joint probability density function $f_{\mathbb{Q}}$ of the two random variables (S_T, τ_G) given that $S_t = s$. Knowing $f_{\mathbb{Q}}$ we then would need to calculate the double integral

$$\mathbb{E}^{\mathbb{Q}}\left[g(S_T)1_{\{T<\tau_G\}} \mid S_t = s\right] = \int_0^\infty \int_0^\infty g(x)1_{\{T<y\}} f_{\mathbb{Q}}(x,y)\mathrm{d}x\mathrm{d}y \ .$$

It is easily imaginable that calculating analytically this double integral is cumbersome and - for arbitrary processes S_t - not possible, even if the joint density is known. For example, we are not able to calculate the expectation in a local volatility model (3.19), such that we need some numerical approximation, for example a Monte Carlo procedure. However, Monte Carlo pricing of continuously monitored barrier options is not straight forward, and it is computationally better to solve (numerically) the corresponding pricing partial differential equation (PDE) satisfied by the price $V = V(s,t)$ of the option[1],

$$\begin{cases} \partial_t V + \frac{1}{2}\sigma^2(s,t)s^2\partial_{ss}V + (r-q)s\partial_s V - rV &= 0 \\ V(B,t) &= 0 \\ V(s,T) &= g(s) \end{cases}$$

in the domain $(s,t) \in G \times [0,T[$. If we assume that the underlying follows a geometric Brownian motion (2.7), i.e., $\sigma(s,t) \equiv \sigma$ and that dividend payments are made continuously (with rate q) then the above expectation (or, equivalently, the corresponding PDE) can be calculated/solved analytically, i.e., there are pricing formulas available. For example, for a down-and-out put (where $g(s) = \max\{X - s, 0\}$) with strike X and barrier B one finds

$$\begin{aligned} V_{p,\mathrm{do}}(s,t) &= V_c(s,t;X,X) - V_c(s,t;X,B) \\ &\quad -\left(\frac{s}{B}\right)^\alpha \left(V_c(s,t;XY,XY) - V_c(s,t;XY,BY)\right) , \end{aligned} \tag{4.4}$$

whereas for a down-and-out call with $g(s) = \max\{s - X, 0\}$ we have

$$V_{c,\mathrm{do}}(s,t) = V_c(s,t;X,X) - \left(\frac{s}{B}\right)^\alpha V_c(s,t;XY,XY) \ . \tag{4.5}$$

Once the out-options are priced, the value of the corresponding in-options can be found by applying example 4.1 iii). In particular, we have

$$V_{p,\mathrm{di}}(s,t) = V_p(s,t) - V_{p,\mathrm{do}}(s,t) \tag{4.6}$$

$$V_{c,\mathrm{di}}(s,t) = V_c(s,t) - V_{c,\mathrm{do}}(s,t) \ . \tag{4.7}$$

In formulas (4.4)–(4.5) we use the following notation. We denote by (compare with (3.14)[2])

$$V_c(s,t;X,Z) := se^{-q(T-t)}N(d_1) - Xe^{-r(T-t)}N(d_2) \ ,$$

with

$$d_1 := \frac{1}{\sigma\sqrt{T-t}}\left(\ln\frac{s}{Z} + \left(r - q + \frac{\sigma^2}{2}\right)(T-t)\right), \quad d_2 := d_1 - \sigma\sqrt{T-t} \ ,$$

[1] Here, we use the deep connection between probability theory and differential equations. This connection is the so-called Feynman-Kac theorem and states that conditional expectations (which derivative prices are) involving stochastic processes can be equivalently written as solutions of partial differential equations. The theorem is named after the famous American physicist Richard Feynman (1918–1988), Nobel price winner in physics, and the Polish mathematician Mark Kac (1914–1984).

[2] Note that V_c also depends on T, σ, r, q, but for notational simplicity we omit this dependence. Using this notation, the value of a European call with strike X becomes $V_c(s,t;X,X)$.

and let Y and α, respectively, be given by

$$\begin{aligned} Y &:= \left(\frac{s}{B}\right)^2 \\ \alpha &:= -1 - \frac{2(r-q)}{\sigma^2}. \end{aligned}$$

Note carefully that the formulas (4.4)-(4.7) are only valid if $B \le X$.

Example 4.2. On September 26, 2018, we consider a down-and-out put on Nestlé. The underlying has the following values: $s = S_0 = 80.80$ CHF, $\sigma = 14.74\%$ and $q = 3.30\%$ (σ and q are from Bloomberg). The parameters of the down-and-out put are: strike $X = 84.84$ CHF, barrier $B = 55.752$ CHF, maturity September 27, 2021. The continuously compounded risk free is $r = -0.329\%$.
Find the value $V_{p,\text{do}}$ of the option.

Solution. We use the Python function `barrier_bs.py` from problem 4.2 to calculate the value of the option. The time-to-maturity is $T - t \approx 3.003$ years.

```
In [9]:  T = yf((26,9,2018),[(27,9,2021)])
         V = barrier_bs(80.8,55.752,84.84,T[0],0.1474,-0.00329,0.033)
         T[0],V[0]
Out[9]:  (3.0027777777777778, 6.4554971129640695)
```

Figure 4.4: We use a self-written Python function to calculate the value of the down-and-out put.

We find $V_{p,\text{do}}(s,0) \doteq 6.4555$ CHF. ◇

How does the down-and-out put relate to the ordinary put? In figure 4.5 we plot the graph of the functions $s \mapsto V_{p,\text{do}}(s,0)$ and $s \mapsto V_p(s,0)$. Herein, we use the same parameters as in the example 4.2, except that we do not take $T = 3.003$, but $T = 0.5$. We observe that the down-and-out put is (much) cheaper than the ordinary put, in particular if the underlying is close to the barrier. Looking at the definition of the payoff of a down-and-out put, this behaviour becomes intuitively clear.

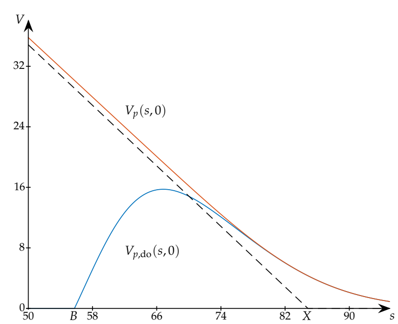

Figure 4.5: A down-and-out put is cheaper than its corresponding European counterpart.

In problem 4.5 we consider a specific example of a down-and-out call option, see equation (4.5) for the corresponding pricing formula. In practice, these options often have the property that $B = X$, compare with the term sheet of the so-called "Turbo Call" of UBS in figure 4.6. A consequence of this property is that such an option becomes worthless whenever the underlying hits the strike.

Turbo Call Warrant
Linked to Roche
Issued by UBS AG, Zurich

Leverage

Cash settled
SVSP/EUSIPA Product Type: Knock-Out Warrants (2200)
Valor: 32225467

Final Termsheet

This Product does not represent a participation in any of the collective investment schemes pursuant to Art. 7 ff of the Swiss Federal Act on Collective Investment Schemes (CISA) and thus does not require an authorisation of the Swiss Financial Market Supervisory Authority (FINMA). Therefore, Investors in this Product are not eligible for the specific investor protection under the CISA. Moreover, Investors in this Product bear the issuer risk.

This document (Final Termsheet) constitutes the Simplified Prospectus for the Product described herein; it can be obtained free of charge from UBS AG, P.O. Box, CH-8098 Zurich (Switzerland), via telephone (+41-(0)44-239 47 03), fax (+41-(0)44-239 69 14) or via e-mail (swiss-prospectus@ubs.com). The relevant version of this document is stated in English; any translations are for convenience only. For further information please refer to paragraph «Product Documentation» under section 4 of this document.

1. Description of the Product

Information on Underlying

Underlying(s)	Reference Level	Strike	Knock-Out Barrier	Conversion Ratio
Roche Holding AG Bloomberg: ROG VX / Valor: 1203204	CHF 243.50	CHF 230.00	CHF 230.00	50:1

Product Details

Security Numbers	Valor: 32225467 / ISIN: CH0322254670
Issue Size	up to 5,000,000 Units (with reopening clause)
Issue Price	CHF 0.30 (unit quotation)
Redemption Currency	CHF
Type of Product	Down and Out Call Warrant
Option Style	European
Exercise at Expiry	Automatically

Dates

Launch Date	02 May 2016
Fixing Date	29 April 2016
Initial Payment Date (Issue Date)	10 May 2016
Last Trading Day/Time	16 September 2016, 17:15 CET
Expiration Date ("Expiry")	**16 September 2016** (subject to Market Disruption Event provisions)

Figure 4.6: Part of the term sheet of a knock-out call warrant on Roche.

Barrier options are difficult to hedge dynamically since their delta $\partial_s V$ can become very large if the underlying is close to the barrier, see problem 4.6. Moreover, the function $s \mapsto \partial_s V$ may have a non-intuitive behaviour, which is in strong contrast to the delta of plain vanillas. For example, the delta of a European put is

$$\partial_s V_p(s,t) = -e^{-q(T-t)} N(-d_1) \, ,$$

which is a strictly increasing function (between -1 and 0), i.e., the larger the value s of the underlying, the closer the delta is to 0. The delta of a down-and-out put, however, shows a completely different behaviour if s is close to B, compare with figure 4.7.

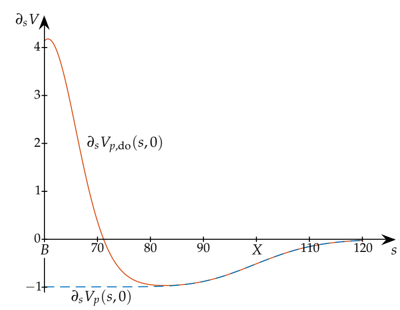

Figure 4.7: The delta of a down-and-out put (—) can be large, whereas the delta of the corresponding European put (− −) is between -1 and 0. Parameters are $X = 100$, $B = 60$, $T = 0.25$, $\sigma = 0.18$, $r = 0$, $q = 0.03$

4.3 Exit probability

We have seen that a down-and-out option becomes worthless if the underlying hits the barrier. We now ask for the probability of such an event. In case of a down-and-out option the underlying starts above the barrier ($B < S_0 < \infty$) such that we are looking for the probability that the underlying S_t "leaves" the set $G =]B, \infty[$ before some $T > 0$, i.e., we want to calculate

$$p = \mathbb{P}[\tau_G < T] ,$$

where τ_G is defined in (4.1). The probability p is called exit probability; note that p can also be written as

$$p = \mathbb{E}\left[1_{\{\tau_G < T\}}\right] ;$$

thus calculating exit probabilities is closely related to calculating prices of binary barrier options (the former is with respect to the physical measure $\mathbb{P}$, whereas the later is with respect to a martingale measure $\mathbb{Q}$.) If the underlying S_t follows a geometric Brownian motion (2.7) with drift μ and volatility σ, then this probability can be computed analytically. Indeed, if the underlying starts at time $t = 0$ in $s = S_0$, then p is a function of s and $0 < t \le T$ and there holds

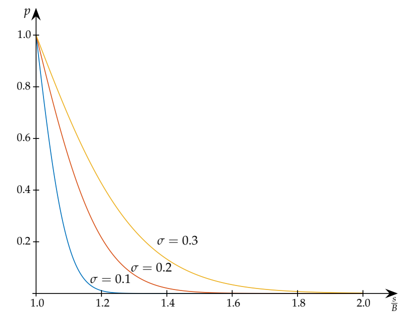

Figure 4.8: Exit probability p of a geometric Brownian motion from the domain $G =]B, \infty[$. The probability p that the stock S_t starting in $S_0 = s$ exits the domain $]B, \infty[$ before T is large if $S_0 = s$ is close to the barrier B (if $\frac{B}{s}$ is closed to 1 and/or if the volatility σ is "large". The chosen values in the figure are: $\mu = 0$, $T = 0.5$.

$$p(s,t) = \left(\frac{B}{s}\right)^{\frac{2\mu}{\sigma^2}-1} N(d_+) + N(d_-) \tag{4.8}$$

with

$$d_\pm := \frac{\ln\frac{B}{s} \pm (\mu - \sigma^2/2)t}{\sigma\sqrt{t}} .$$

Formula (4.8) shows that if at the beginning ($t = 0$) the stock is already "close" to the barrier, then the probability that the stock breaches the barrier is "high" (the fraction $\frac{s}{B}$ is in this case close to one); this is intuitively clear. The formula also states that the exit probability becomes higher the more volatile the stock is. This is also intuitive, compare with figure 4.8.

Example 4.3. On September 26, 2018, we calculate the probability that Nestlé from example 4.2 hits the barrier in the next 3 years. To do so, we may estimate μ and σ using historical data. A time series of daily closing prices of Nestlé ranging from September 25, 2015 to September 25, 2018, yields $\mu = 5.3\%$ and $\sigma = 14.8\%$

Solution. We have $t = 3$, $s = 80.80$ and $B = 55.752$. Hence $d_\pm$ are given by

$$d_\pm = \frac{\ln(55.752/80.8) \pm (0.053 - 0.148^2/2)\cdot 3}{0.148\cdot\sqrt{3}}$$

and the probability is therefore

$$\begin{aligned} p &\overset{(4.8)}{=} \left(\frac{B}{s}\right)^{\frac{2\mu}{\sigma^2}-1} N(d_+) + N(d_-) \\ &\doteq \left(\frac{55.752}{80.8}\right)^{3.839299} \cdot N(-0.955435) + N(-1.39615) \doteq 0.06704 \end{aligned}$$

The probability of a barrier event is thus 6.70%. ◇

4.4 Problems

Problem 4.1. Let $B < X$. How does the price $V_{p,\mathrm{do}}$ of a down-and-out put option change if - ceteris paribus -

i) the barrier B is increased

ii) the strike X is decreased?

Problem 4.2. Implement formulas (4.4)–(4.7) in Python and call the resulting file `barrier_bs.py`.

Problem 4.3. Consider the down-and-out put from example 4.2 and suppose that at maturity (of the option) the underlying (Nestlé) closes at 75 CHF. Also suppose there was no barrier event.
Compare the holding period return of the investment strategies "long European put" and "long down-and-out put", both implemented on September 26, 2018. (Needless to say that the parameters of the European put are the same as for the barrier option.)

Problem 4.4. Consider the graph of the function $s \mapsto V_{p,\mathrm{do}}(s, 0)$ in figure 4.5.
We increase the volatility by 100/3%. How does the corresponding graph change?

Problem 4.5. On June 14, 2016 ($t = 0$), we consider the knock-out call warrant in figure 4.6. The underlying takes the values $s = S_0 =$ 240.60 CHF and $q = 0$. The model parameters are $\sigma = 0.264$ (implied volatility of the corresponding European call warrant with market price $V_c^{\mathrm{M}} = 0.365$ CHF) and $r = \ln(1 - 0.0076)$ (-0.76% is the 3-months CHF Libor).

i) Calculate the model price of the knock-out call warrant (for simplicity, take the given volatility from the European counterpart).

ii) Plot the graph of the functions $s \mapsto V_{c,\mathrm{do}}(s, 0)$, $s \mapsto V_c(s, 0)$ and $s \mapsto g(s)$ in the range $s \in [X - 40, X + 60]$.

iii) On June 14, 2016, you invest 50′000 CHF in

- iii$_1$) the underlying (Roche),
- iii$_2$) the call warrant,
- iii$_3$) the knock-out call warrant,

and you hold the position until September 16, 2016 ($t = T$). For all three investments calculate the holding period return as of September 16, 2016, in dependence on the closing price of the underlying $S_T \in [X, X + 30]$ and plot the graph of the corresponding functions. Additionally, find the break even of these investments. Assume that there was no barrier event.

Problem 4.6. [Delta of a down-and-out put*] Show that the delta $\partial_s V_{p,\mathrm{do}}(s,t)$ of a down-and-out put in the Black-Scholes market model is given by

$$\begin{aligned}
\partial_s V_{p,\mathrm{do}}(s,t) &= e^{-q(T-t)}\left\{ N\big(d_1(X)\big) - N\big(d_1(B)\big) + \frac{\phi(d_1(B))}{\sigma\sqrt{T-t}}\Big(\frac{X}{B}-1\Big)\right\} \\
&\quad -\Big(\frac{s}{B}\Big)^{\alpha}\left\{ e^{-q(T-t)}\left[N\big(d_1(XY)\big) - N\big(d_1(BY)\big) - \frac{\phi(d_1(BY))}{\sigma\sqrt{T-t}}\Big(\frac{X}{B}-1\Big)\right]\right. \\
&\qquad\qquad \left. -\frac{2X}{B^2} s e^{-r(T-t)}\left[N\big(d_2(XY)\big) - N\big(d_2(BY)\big)\right]\right\} \\
&\quad -\frac{\alpha}{B}\Big(\frac{s}{B}\Big)^{\alpha-1}\Big(V_c(s,t;XY,XY) - V_c(s,t;XY,BY)\Big)
\end{aligned}$$

with $d_{1,2}$ as in (4.4)–(4.5), i.e.,

$$\begin{aligned}
d_1(Z) &:= \frac{1}{\sigma\sqrt{T-t}}\Big(\ln\frac{s}{Z} + \big(r-q+\frac{\sigma^2}{2}\big)(T-t)\Big) \\
d_2(Z) &:= d_1(Z) - \sigma\sqrt{T-t}.
\end{aligned}$$

Hint. First find the partial derivative $\partial_s V_c(s,t;x(s),z(s))$, with $V_c(s,t;X,Z)$ as in (4.4)–(4.5) and x, z given arbitrary functions in s.

Chapter 5

Bonus certificates

Bonus certificates belong to the group of participation products, compare with the "Swiss Derivative Map" (category 1320), figure 1.1. Typical underlyings are stocks, indices and commodities, sometimes currencies. The main part of bonus certificates are written on a single underlying, but there exist also multi-bonus certificates, i.e., certificates written on several underlyings. The lifetime of such products is mostly larger than one year.

On "SIX Structured Products" there are about 370 bonus certificates available (October 2019).

5.1 Payoff and profit

As an example, we consider the bonus certificate on Nestlé defined in figure 5.1.
This product has the following parameters. The value of the underlying at issuance is $S_0 = 80.80$ CHF, strike and barrier are $X = 84.84$ CHF and $B = 55.752$ CHF, respectively. The lifetime is $T - t = 3$ years and the continuously compounded risk free rate is $r = -0.329\%$.

It becomes clear by the description of the "Redemption" in the term sheet that we have to distinguish between a barrier event and a no barrier event. We find the payoff function g. If we again denote

	payoff (at maturity T)		
	$S_T \leq B$	$B < S_T \leq X$	$S_T > X$
B never hit	-	X	S_T
B hit	S_T	S_T	S_T

Table 5.1: Payoff panel of a bonus certificate.

by I the discrete random variable as in (4.2) with domain $G =]B, \infty[$, then the function

$$g(S_T) = S_T + I \max\{X - S_T, 0\} \tag{5.1}$$

I. Product Description

Terms

Swiss Security Number (Valor)	43070457
ISIN	CH0430704574
Symbol	SAABJB
Issue Size	up to 247,525 Products (CHF 20,000,000) (may be increased/decreased at any time)
Issue Currency	CHF
Issue Price	CHF 80.80 (per Product; including the Distribution Fee)
Denomination	CHF 80.80
Strike	CHF 84.84 (105%)[2)]

Conditional Protection	105%

Initial Fixing Date: 26 September 2018, being the date on which the Initial Level and the Strike and the Barrier and the Ratio are fixed.

Issue Date/Payment Date: 03 October 2018, being the date on which the Products are issued and the Issue Price is paid.

Final Fixing Date: 27 September 2021, being the date on which the Final Level will be fixed.

Last Trading Date: 27 September 2021, until the official close on the SIX Swiss Exchange, being the last date on which the Products may be traded.

Final Redemption Date: 04 October 2021, being the date on which each Product will be redeemed at the Final Redemption Amount.

Underlying

NESTLE SA-REG (NESN SE <EQUITY>; SIX Swiss Exchange)

Initial Level	CHF 80.80[1)]
Strike	CHF 84.84 (105%)[2)]
Barrier	CHF 55.752 (69%)[2)]
Ratio	1.0000

Currency	CHF
Valuation Time	Scheduled Closing Time
ISIN	CH0038863350
Valor	3886335

Redemption

Final Redemption	Unless previously redeemed, repurchased or cancelled, the Issuer shall redeem each Product on the Final Redemption Date by payment of a cash amount or delivery of a number of Underlyings equal to the Final Redemption Amount to the Holder thereof.
Final Redemption Amount	(i) if **no** Barrier Event has occurred, a cash amount equal to the product of the ratio between the Denomination and the Initial Level and the greater of (x) the Strike and (y) the Final Level, as calculated by the Calculation Agent in accordance with the following formula: $\frac{\text{Denomination}}{\text{Initial Level}} \times \max[\text{Strike};\text{Final Level}]$ (ii) if a Barrier Event has occurred, and (a) the Final Level is **at or above** the Strike, a cash amount equal to the product of the ratio between the Denomination and the Initial Level and the Final Level, calculated by the Calculation Agent in accordance with the following formula: $\frac{\text{Denomination}}{\text{Initial Level}} \times \text{Final Level}$ (b) the Final Level is **below** the Strike, the number of Underlyings specified in the Ratio. In case of a physical settlement according to scenario (ii) (b), the number of Underlyings to be delivered will be rounded down to the nearest integral number of Underlyings. In addition, the holder will receive a cash amount in lieu for any fractional amount.
Settlement Type	Physical settlement or cash settlement
Level	the Share Price
Final Level	the Level at the Valuation Time on the Final Fixing Date, as determined by the Calculation Agent
Barrier Event	If the Level at any time (observed continuously) on any Barrier Observation Date is **at or below** the Barrier.
Barrier Observation Dates	each Exchange Business Day during the Barrier Observation Period(s), being the dates on which the Level is observed for purposes of determining whether a Barrier Event has occurred.
Barrier Observation Period	from and including Initial Fixing Date to and including the Final Fixing Date

Figure 5.1: Part of the term sheet of a bonus certificate.

is a "translation" of the panel 5.1 to a formula. You may compare this with the "Final Redemption Amount" in the term sheet in figure 5.1, which is obviously the formula (5.1) for the case $I = 1$ ($\frac{\text{Denomination}}{\text{Initial Level}} = 1$; make sure you understand the equivalence of $S_T + \max\{X - S_T, 0\}$ and $\max\{X, S_T\}$, and why we use the former). In figure 5.2 we plot the profit at maturity of a bonus certificate in dependence on the closing price S_T of the underlying, i.e., we plot the function $P(S_T) = g(S_T) - V_{\text{bc}}(s, 0)$. Here, as usual, V denotes the value of the product, and the subscript "bc" is the abbreviation for "bonus certificate". Since the issue price $V_{\text{bc}}(s, 0)$ of the certificate is equal to the price S_0 of the underlying at issuance, the profit is

$$\begin{aligned} P(S_T) &= g(S_T) - V_{\text{bc}}(s,0) = g(S_T) - S_0 \\ &\overset{(5.1)}{=} S_T - S_0 + I \max\{X - S_T, 0\} \,. \end{aligned}$$

If there was no barrier ($I = 1$) event during the life time of the product, then the investor realises a "bonus" of size $X - S_0 = 1.05S_0 - S_0 = 0.05S_0 = 4.04$ CHF (note that $\frac{X-S_0}{S_0} = 5\%$) for $B < S_T < X$. In this range, the profit of the bonus certificate is (much) larger than the profit resulting form a direct investment into the underlying, but only if there is no barrier event. Thus, we will purchase a bonus certificate if we expect the underlying to move sideways or to increase slightly, compare with the "Swiss Derivative Map" in figure 1.1, category 1320.

If these expectations are met, the bonus certificate realises a higher profit than the direct investment into the underlying. A disadvantage of a bonus certificate is that the holder will not profit from possible dividend payments of the underlying during the life time of the certificate. We may use this fact to estimate the dividend yield q. The investor is indifferent between the investment into the certificate and the investment into the underlying if both investments realise the same return (we do not take into account the risk aversion of the investor). If there is no barrier event (and if $S_T \leq X$), then the bonus certificate realises the holding period return $\frac{X-S_0}{S_0}$ (in the time period $[0, T]$), whereas the return of underlying due to dividend payments is $\frac{S_T - S_0}{S_0} = \frac{S_0 e^{qT} - S_0}{S_0}$. Thus, the continuous dividend yield satisfies the equation $S_0 e^{qT} = X$ with solution

$$q = \frac{1}{T} \ln(X/S_0) .$$

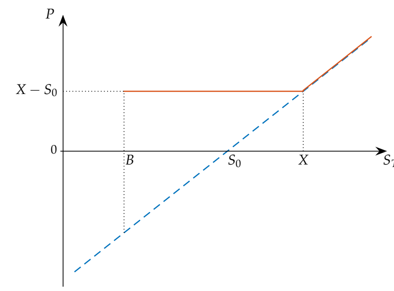

Figure 5.2: Profit of a bonus certificate. The event "barrier never hit" is coloured red (—), whereas the event "barrier hit" is coloured blue (− −).

5.2 Replication and pricing

By replication we mean the combination of financial instruments in such a way that - at maturity and for all possible scenarios of the underlying - the payoff of the replication is equal to the payoff of the structured product. We now replicate the payoff (5.1)

$$g(S_T) = S_T + I \max\{X - S_T, 0\}$$

of a bonus certificate. The term S_T corresponds to the price of the underlying; we may generate this cash flow by looking at a so-called zero strike call. As its name suggests, a zero strike call is a call option with strike $X = 0$. Indeed, such an option has the payoff (the subscript "zs" is an abbreviation for "zero strike")

$$V_{c,\mathrm{zs}}(s, T) = \max\{S_T - X, 0\} = \max\{S_T - 0, 0\} = S_T ,$$

where the last equality follows by the non-negativity of S_T. The term $I \max\{X - S_T, 0\}$ corresponds to the payoff of a down-and-out put option with strike X (and barrier B), compare with (4.3). Thus, at maturity, we may decompose a bonus certificate into a sum of a zero strike call and a down-and-out put. However, the price of the replication has to be equal to the price V_{bc} of the certificate not only at maturity $t = T$, but for every instant of time $0 \le t \le T$, i.e., there holds

$$V_{\mathrm{bc}}(s,t) = V_{c,\mathrm{zs}}(s,t) + V_{p,\mathrm{do}}(s,t) \; .$$

If this equation would be wrong there would exist arbitrage opportunities. To find the value V_{bc}, we use the Black-Scholes model with continuous dividend yields. While in this setting the price $V_{p,\mathrm{do}}$ of the down-and-out put is given by formula (4.4), the value $V_{c,\mathrm{zs}}$ of the zero strike call follows by the Black-Scholes formula (3.14) together with a limit argument. If we let tend the strike X to 0, the auxiliary variables d_1 and d_2 in the formula of Black and Scholes tend to ∞. As a consequence, the values $N(d_1)$ and $N(d_2)$ tend to 1, whence we find

Remember: by s we denote the price of the underlying at time t, i.e., $s = S_t$.

$$\begin{aligned} V_{c,\mathrm{zs}}(s,t) &= \lim_{X\to 0} V(s,t;X,T,\sigma,r,q,1) \\ &\overset{(3.14)}{=} \lim_{X\to 0} \left(se^{-q(T-t)}N(d_1) - Xe^{-r(T-t)}N(d_2)\right) \\ &= se^{-q(T-t)} \; . \end{aligned}$$

Thus, the price of a bonus certificate written on a single underlying is given by

$$V_{\mathrm{bc}}(s,t) = se^{-q(T-t)} + V_{p,\mathrm{do}}(s,t) \; . \tag{5.2}$$

Example 5.1. We check the issue price $V_{\mathrm{bc}}^{\mathrm{M}} = 80.80$ CHF of the bonus certificate in figure 5.1.

Solution. The value of the down-and-out put is given in example 4.2 (compare the parameter values of the certificate with the given values in the example). The value is $V_{p,\mathrm{do}}(s,0) \doteq 6.4555$ CHF (with $s = 80.80$) and it remains to calculate the value of the zero strike call. Since

$$V_{c,\mathrm{zs}}(s,0) = se^{-qT} = 80.80 \cdot e^{-0.033\cdot 3.003} \doteq 73.1768 \text{ CHF} \; ,$$

the Black-Scholes price of the bonus certificate is

$$V_{\mathrm{bc}}(s,0) \overset{(5.2)}{=} se^{-qT} + V_{p,\mathrm{do}}(s,0) \doteq 79.6323 \text{ CHF} \; .$$

Thus, the issue price is about 1.5% (agio) above the model price. ◇

5.3 Bonus certificates on multiple underlyings

We consider a bonus certificate on $d = 2$ underlyings (compare with figure 5.3) and denote by B_t the value of the basket at time t. This value is defined via the performance (the holding period return) R_t

$$B_t = B_0(1 + R_t)$$

of the basket over the time period $[0, t]$. The basket return R_t is set to the average performance of the d basket constituents over the same time period, i.e.,

$$R_t := \frac{1}{d} \sum_{i=1}^{d} \frac{S_t^i - S_0^i}{S_0^i} \ .$$

Herewith, we denote by S_t^i the price of the i-th underlying at time t. Using this definition, the basket value can be written as

$$B_t = B_0 \left(1 + \frac{1}{d} \sum_{i=1}^{d} \frac{S_t^i - S_0^i}{S_0^i} \right) = \frac{B_0}{d} \sum_{i=1}^{d} \frac{S_t^i}{S_0^i} = \sum_{i=1}^{d} \omega_i S_t^i$$

where the weights ω_i are defined by

$$\omega_i := \frac{B_0}{d S_0^i}, \ \ i = 1, \dots, d \ .$$

The payoff of the certificate in figure 5.3 is analogous to the payoff of

Basket Bonus Certificate

Final Terms, 3 May 2011

Roche / Novartis

10 May 2011 until 10 May 2013

An investment in a Basket Bonus Certificate corresponds to a direct investment in the underlying Basket, whereby a negative performance only comes into account after at least one Barrier has been breached. If none of the Barriers has been touched the Basket Bonus Certificate will be redeemed at the Bonus Level or, if higher, at the Final Basket Value. If at least one Barrier has been touched the minimum redemption of the Bonus Level will no longer apply and the Basket Bonus Certificate will turn into a Tracker Certificate on the equally weighted Basket (see Redemption Mode).

Your market expectation: sideways to positive

This structured product does not constitute a participation in a collective investment scheme within the meaning of the Swiss Federal Act on Collective Investment Schemes (CISA) and is therefore not subject to authorization and supervision by the Swiss Financial Market Supervisory Authority (FINMA).

Issuer	Clariden Leu Ltd., Nassau Branch, Bahamas
Lead Manager	Clariden Leu Ltd., Zurich
Rating	Aa2 (Moody's)
Basket	

Underlying	Bloomberg	Initial Fixing Price	Barrier	Exchange
Roche Holding Ltd dividend-right certificate	ROG VX	CHF 143.40	CHF 107.5500	SIX Swiss Exchange
Novartis Inc. registered share	NOVN VX	CHF 51.35	CHF 38.5125	SIX Swiss Exchange

Initial Basket Value	CHF 1,000 (both Underlyings are equally weighted)
Swiss Sec. Number / ISIN	12 474 150 / CH0124741502 (WKN: CLA0VM)
Ticker	VRMSF
Product Category	Complex product. Complex products require specific knowledge regarding the product and its associated risks. Therefore, it is recommended that the investor obtains adequate information regarding the risks associated with the specific product before making an investment decision. For more information on the product's risks, please see page 2.
Product Type	Participation Product (category 1320), according to the Swiss Derivative Map of the Swiss Structured Products Association (www.svsp-verband.ch)
Issue Price	100% (CHF 1,000)
Issue Size	CHF 10,000,000 (10,000 Basket Bonus Certificates)
Denomination	CHF 1,000 (Notional Amount) = 1 Basket Bonus Certificate
Ratio	1 Basket Bonus Certificate corresponds to 1 Basket
Initial Fixing Price	100% of the official closing price of the respective Underlying on the relevant Exchange on the Initial Fixing Date
Barrier	**75%** of the respective Initial Fixing Price
Initial Fixing Date	3 May 2011
Payment Date	10 May 2011
Bonus Level	**CHF 1,140**, corresponds to **114%** of the Initial Basket Value
Last Trading Date	3 May 2013 (until 12:00 p.m. CET)
Final Fixing Date	3 May 2013
Redemption Date	10 May 2013
Barrier Period	**4 May 2011 until 3 May 2013 (continuous monitoring)**
Final Fixing Price	100% of the official closing price of the respective Underlying on the relevant Exchange on the Final Fixing Date
Final Basket Value	$Basket_0 \times \left(100\% + \frac{1}{n} \sum_{i=1}^{n} \frac{S_i(T) - S_i(0)}{S_i(0)} \right)$

whereas:

Basket_0	Initial Basket Value
n	Number of components in the Basket
$S_i(0)$	Initial Fixing Price of the Underlying i
$S_i(T)$	Final Fixing Price of the Underlying i

Redemption Mode

a) If the Underlyings have never been traded at or below their Barriers during the Barrier Period, each Basket Bonus Certificate will be redeemed at the Bonus Level (CHF 1,140), or, if higher, at the Final Basket Value.

b) If at least one Underlying has been traded at or below its Barrier during the Barrier Period, the minimum redemption of the Bonus Level will not apply. Instead, the Basket Bonus Certificate will turn into a Tracker Certificate on the equally weighted Basket. At maturity, each Basket Bonus Certificate will be redeemed at the Final Basket Value.

Figure 5.3: Part of the term sheet of a basket bonus certificate.

a bonus certificate on a single underlying (5.1) i.e.,

$$g(\mathbf{S}_T) = B_T + I \max\{X - B_T, 0\} \; . \tag{5.3}$$

Herein, $\mathbf{S}_T$ is as in (3.35), and $I = 1_{\{T<\tau_G\}}$ is the random variable defined in (4.2) with

$$G =]B_1, \infty[\times \cdots \times]B_d, \infty[\; .$$

Thus, $I = 0$ indicates that at least one underlying hits its barrier before T, compare with the "Redemption Mode" given in the term sheet. Using the same arguments which led to the pricing formula for a bonus certificate written on a single underlying, we conclude that the value of a multi bonus certificate with payoff function (5.3) is given by

$$V_{\mathrm{bc},d}(\mathbf{s}, t) = \sum_{i=1}^{d} \omega_i s_i e^{-q_i(T-t)} + V_{p,\mathrm{do},d}(\mathbf{s}, t) \; .$$

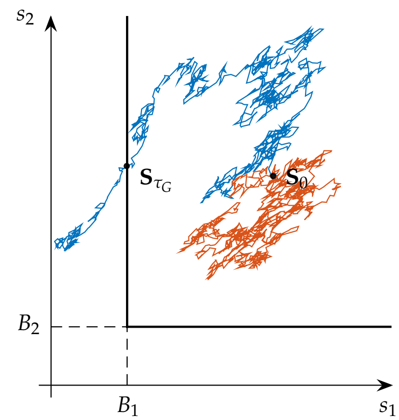

Figure 5.4: Two possible paths $t \mapsto (S_t^1, S_t^2)$ of $d = 2$ underlyings. Both paths start in $\mathbf{S}_0 = (S_0^1, S_0^2)$ (black point). One path (—) leaves the domain $G =]B_1, \infty[\times]B_2, \infty[$, so $I = 0$, the other one (—) does not, hence $I = 1$.

Furthermore, we denote by $V_{p,\mathrm{do},d}(\mathbf{s}, t)$ the value of a down-and-out put on d underlyings.[1] By the general principles of derivatives pricing discussed in chapter 3 the price of a down-and-out put option is - under the assumption of time-continuous barrier monitoring - given by the expectation

$$V_{p,\mathrm{do},d}(\mathbf{s}, t) = e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}} \left[g(\mathbf{S}_T) 1_{\{T<\tau_G\}} \mid \mathbf{S}_t = \mathbf{s} \right] \; . \tag{5.4}$$

Herewith, g is the function

$$g(\mathbf{s}) = \max \left\{ X - \sum_{i=1}^{d} \omega_i s_i, 0 \right\}$$

(compare with the payoff (5.3)) and τ_G as in (4.1) denotes the first exit time of the process $\mathbf{S}_t$ from the domain G, compare with figure 5.4 for the case $d = 2$. To find the expectation (5.4), we need a model for the time evolution of the underlying S_t^i. Even for the simple Black-Scholes model (2.26)

$$S_t^i = s_i e^{(\mu_i - \frac{1}{2}\sigma_i^2)t + \sigma_i \widehat{W}_t^i}$$

the expectation (5.4) seems not to have an analytical expression (except for the case $d = 1$, for which there holds formula (4.4)) such that we have to rely on approximative methods, for example Monte Carlo simulation or solving partial differential equations numerically. Recall that if we decide to approximate the expectation by a Monte Carlo simulation, we need the Cholesky decomposition $\mathbf{L}\mathbf{L}^\top$ of the covariance matrix $\mathbf{\Sigma}$. In problem 2.7 we discuss on how Python might calculate, for a given $\mathbf{\Sigma}$, the matrix $\mathbf{L}$.

[1] Obviously, there are many different types of down-and-out barrier options on multiple underlyings. Here, we mean by a down-and-out option an option which becomes worthless if at least one barrier is hit from above, $B_i < S_0^i$.

Example 5.2. We check the issue price $V^{\mathrm{M}}_{\mathrm{bc},d}(\mathbf{s},t) = 1000$ CHF of the basket bonus certificate in figure 5.3.

Solution. We use the Python function `gbm_d.py` to generate paths of the 2-dimensional geometric Brownian motion. Let $S_0^1 = 143.4$, $S_0^2 = 51.35$; the implied volatilities are $\boldsymbol{\sigma}^{\mathrm{i}} = \mathrm{diag}(\sigma_1^{\mathrm{i}}, \sigma_2^{\mathrm{i}}) = \mathrm{diag}(0.181, 0.170)$, the correlation matrix $\boldsymbol{\rho}$ in (3.37) is (where $\lambda = 0$)

$$\boldsymbol{\rho} = \boldsymbol{\rho}^{\mathrm{h}} = \begin{pmatrix} 1 & \rho \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0.416 \\ & 1 \end{pmatrix} .$$

The discrete dividends are $D_1 = 6.60$ and $D_2 = 2.34$, resulting in the continuously compounded dividend yields $q_1 = \ln(1 + D_1/S_0^1) = 4.50\%$ and $q_2 = \ln(1 + D_2/S_0^2) = 4.46\%$, respectively. The continuously compounded risk free is $r = \ln(1 + 0.0087)$, where 0.87% is the "internal rate of return" given on the term sheet (not shown in figure 5.3). A Monte Carlo simulation based on $n = 10^6$ paths and time step $\Delta t = 1/(8 \cdot 252)$ (monitoring the barrier every hour at each business day) now gives $V^{n,\Delta t}_{\mathrm{bc},2}(\mathbf{s}, 0) \approx 960.66$ CHF, see figure 5.5, such that the agio of the issuer is about 3.9%. The superscript $n, \Delta t$ indicates that the price depends on the number n of generated paths and the time step $\Delta t = t_\ell - t_{\ell-1}$ used for the monitoring of a possible barrier event. ◇

```
In [4]: # input parameters (Monte Carlo simulation)
        n = 10**6; dt = 1/(8*252); v = np.zeros(n)

In [5]: # Monte Carlo simulation
        for j in range(0,n):
            D = gbm_d(s0,mu,Sigma,[],T,dt)
            S = D[0][1:] # simulated stock prices, excluding s0
            # the minimium stock prices
            Smin = np.amin(S,axis=0); Smin = Smin.reshape(2,1)
            ST = S[-1,:]; ST = ST.reshape(2,1)
            BT = np.sum(omega*ST) # final basket value

            if np.any(Smin-B<=0): # barrier hit?
                v[j] = BT
            else:
                v[j] = BT+np.maximum(X-BT,0)

        V = np.exp(-r*T)*np.mean(v)

In [6]: V

Out[6]: 960.6582374899954
```

Figure 5.5: Monte Carlo simulation in Python to find the issue price of a basket bonus certificate (the input of the model and contract parameters is not shown).

The payoff of a multi bonus certificate is not unique. For example, a payoff different to (5.3) is

$$g(\mathbf{S}_T) = I \max\{X, B_T\} + (1 - I) B_0 \min_i \frac{S_T^i}{S_0^i} \tag{5.5}$$

with $B_t = \sum_{i=1}^d \omega_i S_t^i$ again the value of basket at time t and I as in (4.2) with

$$G =]B_1, \infty[\times \cdots \times]B_d, \infty[\, ,$$

compare with figure 5.6.

Bonus Certificate on Nestlé, Novartis, Roche

Bonus Level 142.00% | Continuous Multi Barrier Observation

Final Fixing Date 22/10/2018; issued in CHF; listed on SIX Swiss Exchange AG

ISIN CH0281549854 | Swiss Security Number 28154985 | SIX Symbol LTQJRC

Investors should read the section "Significant Risks" below as well as the section "Risk Factors" of the relevant Programme.
This Product is a derivative instrument. It does not qualify as unit of a collective investment scheme pursuant to art. 7 et seqq. of the Swiss Federal Act on Collective Investment Schemes (CISA) and is therefore neither registered nor supervised by the Swiss Financial Market Supervisory Authority FINMA. Investors do not benefit from the specific investor protection provided under the CISA.
In addition, investors are subject to the credit risk of the Issuer and Guarantor, if any.
This document is not a prospectus within the meaning of article 1156 of the Swiss Code of Obligations (CO).

I. PRODUCT DESCRIPTION

Market expectation of the Investor
Underlyings trade sideways to slightly higher or lower.
Underlyings will not touch or fall below the Barrier Levels.

Product description
Unless a Barrier Event has occurred, this Product offers the Investor on the Redemption Date the maximum of the Denomination multiplied with the Bonus Level (in %) or the Denomination multiplied with the Final Basket Level divided by the Initial Basket Level. If a Barrier Event has occurred, the redemption of the Product will depend on the value of the Underlying with the Worst Performance, as described in section "Redemption".

UNDERLYING

Underlying	Related Exchange	Bloomberg Ticker	Initial Fixing Level (100%)*	Barrier Level (75.00%)*	Conversion Ratio	Weight
NESTLE SA-REG	SIX Swiss Exchange AG	NESN VX	CHF 74.80	CHF 56.10	13.3690	4.456328
NOVARTIS AG-REG	SIX Swiss Exchange AG	NOVN VX	CHF 89.45	CHF 67.09	11.1794	3.726477
ROCHE HOLDING AG-GENUSSSCHEIN	SIX Swiss Exchange AG	ROG VX	CHF 260.70	CHF 195.53	3.8358	1.278609

PRODUCT DETAILS

Swiss Security Number	**28154985**
ISIN	**CH0281549854**
SIX Symbol	**LTQJRC**
Issue Price	100.00%
Issue Size	CHF 10'000'000 (can be increased at any time)
Denomination	CHF 1'000
Initial Basket Level	CHF 1'000
Settlement Currency	CHF
Bonus Level	142.00% (CHF 1'420.00)
Bondfloor at issuance	100.00% (implied Yield p.a.: 0.00%)

DATES

Subscription End Date	22/10/2015 14.00 CET
Initial Fixing Date	22/10/2015
Issue Date	29/10/2015
First Exchange Trading Date	29/10/2015
Last Trading Day/Time	22/10/2018 / Exchange market close
Final Fixing Date	22/10/2018 (subject to Market Disruption Event provisions)
Redemption Date	29/10/2018 (subject to Settlement Disruption Event provisions)

REDEMPTION

The Investor is entitled to receive from the Issuer on the Redemption Date per Product:

Scenario 1

If a Barrier Event has NOT occurred and

a. If the Final Basket Level is at or below the Initial Basket Level multiplied with the Bonus Level (in %), the Investor will receive a Cash Settlement in the Settlement Currency according to the following formula:
Denomination × Bonus Level (in %)

b. If the Final Basket Level is above the Initial Basket Level multiplied with the Bonus Level (in %), the Investor will receive a Cash Settlement in the Settlement Currency according to the following formula:
Denomination × Final Basket Level / Initial Basket Level

Scenario 2

If a Barrier Event HAS occurred, the Investor will receive a round number (i.e. Conversion Ratio) of the Underlying with the Worst Performance per Product. Any potential fractional Conversion Ratio entitlements (Fraction of Underlyings) will be paid in cash, based on the Final Fixing Level. Fractions of Underlyings will not be cumulated.

Initial Fixing Level	Official close of the respective Underlying on the Initial Fixing Date on the Related Exchange, as determined by the Calculation Agent.
Final Fixing Level	Official close of the respective Underlying on the Final Fixing Date on the Related Exchange, as determined by the Calculation Agent.
Final Basket Level	Sum of the products of the Final Fixing Levels of the Underlyings and the respective Weights.
Worst Performance	For each Underlying the performance is calculated by dividing its Final Fixing Level by the respective Initial Fixing Level. The Worst Performance corresponds to the lowest of all so calculated values, as determined by the Calculation Agent.
Barrier Event	A Barrier Event shall be deemed to occur if at any time on any Exchange Business Day during the Barrier Observation Period the level of at least one of the Underlyings' prices has been traded at or below the respective Barrier Level, as reasonably determined by the Calculation Agent.
Barrier Observation Period	22/10/2015 - 22/10/2018

Figure 5.6: Part of the term sheet of a multi bonus certificate.

Note that the weights ω_i are indeed given by $\omega_i = \frac{B_0}{dS_0^i}$ and that the "Conversion Ratio" of the i-th underlying is given by $d\omega_i$. Make sure you understand the equivalence of the above payoff (5.5) and the "Redemption" given in the term sheet.

To close this chapter, we return to example 5.2 and discuss some typical features of a Monte Carlo simulation. Because the simulation draws paths of the d-dimensional geometric Brownian motion randomly, the output (the price $V^{n,\Delta t}_{\mathrm{bc},d}(\mathbf{s},0)$, called V in the above Python code) is random and differs from the true model price $V_{\mathrm{bc},d}(\mathbf{s},0)$. This difference (the Monte Carlo approximation error) is composed of two types of error. The first error stems from the fact that the number n of generated paths used to estimate the price is finite. The more paths we generate, the smaller the approximation error: to calculate the price exactly, we would need to draw infinitely many paths. In case of continuous barrier monitoring, there is a second source of approximation error (even if we could realise infinitely many paths, $n = \infty$), since we can not generate a path over a continuum of time points (we can not set $\Delta t = 0$ on the computer).

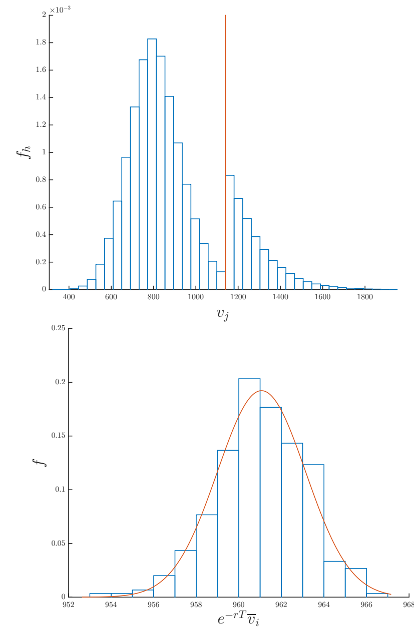

Figure 5.7: Top. Approximate probability distribution function of the random variable V_T. The vertical line (—) located at X indicates the probability $\mathbb{Q}[V_T = X] \approx 0.317$. Bottom. The CLT states that the random variable $e^{-rT}\overline{V}$ is approximately normal.

In figure 5.7, we plot the approximate risk-neutral distribution (the histogram) of the cash flow (5.3)

$$V_T := B_T + \max\{X - B_T, 0\}1_{\{T<\tau_G\}}, \quad B_T = \sum \omega_i S^i_T$$

of the product at maturity based on $n = 10^6$ generated paths (scenarios) of $\mathbf{S}_t$, each on an equidistant time grid with time step $\Delta t = 1/(8 \cdot 252)$. This gives the n realisations $\{v_1, v_2, \ldots, v_n\}$; their discounted average

$$e^{-rT}\overline{v} = e^{-rT}\frac{1}{n}\sum_{j=1}^{n} v_j \tag{5.6}$$

is an approximation to the price $V_{\mathrm{bc},2}(\mathbf{s},0) = \mathbb{E}^{\mathbb{Q}}[e^{-rT}V_T]$ of the product. We observe that the distribution of V_T is of mixed type, i.e., the random variable V_T is both discrete and continuous. For example, the (risk-neutral) probability $\mathbb{Q}[V_T = X]$ is not zero as it would be for a continuous random variable, but is estimated by the proportion

$$p := \frac{1}{n}\sum_{j=1}^{n} 1_{\{v_j = X\}}\ ;$$

for the simulation displayed in figure 5.5 we have $\mathbb{Q}[V_T = X] \approx p = 0.317$. Note that the estimator $e^{-rT}\overline{V}$ is biased[2], $\mathbb{E}[e^{-rT}\overline{V}] \neq V_{\mathrm{bc},2}(\mathbf{s},0)$, since Δt is not zero. The bias gets smaller the smaller Δt becomes, and in the limit $\Delta t \to 0$ the estimator is unbiased. If the product would incorporate a discretely monitored barrier option, we could simulate exactly on the monitoring dates and the corresponding estimator would be unbiased. Since we draw n times from a certain distribution (the distribution of V_T) and then take the average, the central limit theorem (CLT) applies: the distribution of $e^{-rT}\overline{V}$ is, for n large enough, close to the normal distribution. This is clearly visible in figure 5.7, where we plot a histogram of $\{e^{-rT}\overline{v}_1, e^{-rT}\overline{v}_2, \ldots, e^{-rT}\overline{v}_N\}$ for $N = 300$, each average $\overline{v}_i$ is as in (5.6), with $n = 10^4$. Notice the variability in the averages (the prices) in figure 5.7; the standard deviation is around 2 or around 20 basis

[2] By $\overline{V}$ we mean the average $\overline{V} = \frac{1}{n}\sum_{j=1}^{n} V^j_T$, each V^j_T is an independent copy of V_T.

points of the true price. The standard deviation can be reduced by increasing n, at the cost of computation time. The randomness in the outcome is a major drawback of Monte Carlo simulations, and great efforts have been made to construct Monte Carlo methods which reduce the variance. For a overview of variance reduction techniques, the interested reader may have a look on the beautiful monograph of Paul Glasserman[3], where one also find remedies for the bias of the Monte Carlo estimator described above for continuously monitored barrier options.

[3] P. Glasserman. *Monte Carlo Methods in Financial Engineering*, volume 53 of *Applications of Mathematics*. Springer, 2003

5.4 Problems

Problem 5.1. We consider the UBS bonus certificate "PERLES Plus" on Swisscom (Valor 42853339). On September 3, 2018 (17:15 pm), the market price of the certificate is $V_{\mathrm{bc}}^{\mathrm{M}} = 920.50/925.50$ CHF (bid/ask), and the underlying takes the values $s = 435.4$, $\sigma = 0.103$. The yearly dividend payment is 22 and the continuously compounded risk free is $r = \ln(1 - 0.00481)$. The remaining data (the term sheet) can be found on the webpage of the issuer. Note that the payoff (5.1) has to be multiplied by the conversion ratio $N/s_0 = 1000/466.90 = 2.1418$ in this case and that there was no barrier event so far.

i) Compare the market price of the "PERLES Plus" with its Black-Scholes price.

ii) Suppose that at maturity the closing price of Swisscom is 460 CHF. Calculate the holding period return of an investor who buys on September 3, 2018,

   ii$_1$) the underlying,

   ii$_2$) the bonus certificate.

Problem 5.2. The bonus of the "PERLES Plus" in problem 5.1 is 15.2%. What consequences would have resulted if UBS would have set the bonus to 30%? Use the risk free $r = \ln(1 - 0.00457)$.

Hint. To answer this question, you will need a solver.

Suppose that the issue price of the bonus certificate is 102% of its Black-Scholes price and that the agio of 2% is charged on the down-and-out put.

Problem 5.3. Plot the value $V_{\mathrm{bc}}(s,t)$ of the bonus certificate written on Nestlé in figure 5.1 as a function of time-to-maturity, i.e., plot a graph of the function $T - t \mapsto V_{\mathrm{bc}}(s,t)$ for

i) the data as of the issuance of the certificate, compare with example 5.1,

ii) the data as in part i), but with $s = X$.

Problem 5.4. Consider the multi bonus certificate described in figure 5.6. Adapt the jupiter notebook `Chapter 5 Bonus Certificates.ipynb` such that it can find the issue price of the product in a Black-Scholes market model. Use the parameters in the term sheet and the implied volatilities (the numbering is Nestlé $\to 1$, Novartis $\to 2$, Roche $\to 3$) $\boldsymbol{\sigma}^{\mathrm{i}} = \mathrm{diag}(\sigma_1^{\mathrm{i}}, \sigma_2^{\mathrm{i}}, \sigma_3^{\mathrm{i}}) = \mathrm{diag}(0.115, 0.146, 0.145)$ as well as the correlation matrix

$$\boldsymbol{\rho} = \boldsymbol{\rho}^{\mathrm{h}} = \begin{pmatrix} 1 & \rho_{12} & \rho_{13} \\ & 1 & \rho_{23} \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0.726 & 0.634 \\ & 1 & 0.742 \\ & & 1 \end{pmatrix}$$

in (3.37). The continuously compounded dividend yields and the continuously compounded risk free are $q_1 = 0.034$, $q_2 = 0.038$, $q_3 = 0.0373$ and $r = \ln(1 - 0.00713)$, respectively.

Chapter 6

Barrier reverse convertibles

Barrier reverse convertibles, abbreviated by BRC, belong to the group of yield enhancement products (category 1230), compare with the "Swiss Derivative Map" in figure 1.1. At "SIX Structured Products" there are about 11700 BRCs tradable (October 2019). There exist also multi barrier reverse convertibles (MBRC), which are written on multiple underlyings (usually $d = 3, 4, 5$). Nowadays, 5/6 of all barrier reverse convertibles are MBRCs.

6.1 Payoff and profit

A BRC pays its holder a guaranteed coupon C at m predefined dates T_i, $i = 1, \ldots, m$, in the future, irrespective of the performance of the underlying. Since these cash flows occur before maturity, we need to compound them to obtain the payoff of the BRC. The final value C_i of the coupon payment made at T_i is[1]

$$C_i = e^{r(T-T_i)} C \ .$$

If there are m coupon payments the guaranteed payoff implied by the coupon of the BRC is

$$C^m := \sum_{i=1}^{m} C_i = C \sum_{i=1}^{m} e^{r(T-T_i)} \ .$$

Additionally to this guaranteed payoff there is a second, stochastic cash flow which does not only depend on the performance of the underlying but also on whether there was a barrier event or not.

In table 6.1 we exemplarily obtain the payoff panel of a BRC, compare with figure 6.1. Therewith, we denote by N the nominal ("Denomination"), by n the conversion ratio and by $X(= S_0)$ the strike of the BRC.

It turns out that the corresponding payoff function g is given by

$$g(S_T) = C^m + N - nI^c \max\{X - S_T, 0\} \ , \tag{6.1}$$

[1] Here, T is the final fixing date which typically does not correspond to the redemption date. Usually, the last coupon payment date is after T, $T_m > T$. The other coupon payment dates satisfy $T_i < T$, $i = 1, \ldots, m-1$.

I. Product Description

Terms

Swiss Security Number (Valor)	45628173
ISIN	CH0456281739
Symbol	SABXJB
Issue Size	up to CHF 20,000,000 (may be increased/decreased at any time)
Issue Currency	CHF
Issue Price	100.00% (per Product; including the Distribution Fee)
Denomination	CHF 1,000.00
Maximum Yield	5.50%
Coupon	2.75% p.a.

Initial Fixing Date: 26 February 2019, being the date on which the Initial Level and the Strike and the Barrier and the Ratio are fixed.

Issue Date/Payment Date: 05 March 2019, being the date on which the Products are issued and the Issue Price is paid.

Final Fixing Date: 26 February 2021, being the date on which the Final Level will be fixed.

Last Trading Date: 26 February 2021, until the official close on the SIX Swiss Exchange, being the last date on which the Products may be traded.

Final Redemption Date: 05 March 2021, being the date on which each Product will be redeemed at the Final Redemption Amount.

Underlying

Nestlé Ltd (NESN SW <EQUITY>; SIX Swiss Exchange)

Initial Level	CHF 91.26[1)]
Strike	CHF 91.26 (100%)[2)]
Break-even	CHF 86.2407
Barrier	CHF 68.445 (75%)[2)]
Ratio	10.9577

Currency	CHF
Valuation Time	Scheduled Closing Time
ISIN	CH0038863350
Valor	3886335

[1)] as of 26 February 2019 17:30 CET
[2)] in % of the Underlying's Initial Level

Coupon

Coupon Amount	2.75% p.a. of the Denomination.
For Swiss tax purposes only, the Coupon is split into two components:	
Interest Amount	0.00% p.a. of the Denomination.
Premium Amount	2.75% p.a. of the Denomination.
Coupon Payment Dates	Being the date(s) on which the Issuer shall pay the Interest Amount and Premium Amount per Product to the Holders, unless previously redeemed, repurchased or cancelled.

Coupon Period	From and including the Start Date	To and excluding the End Date	Coupon Payment Date
1st	05 March 2019	05 September 2019	05 September 2019
2nd	05 September 2019	05 March 2020	05 March 2020
3rd	05 March 2020	05 September 2020	07 September 2020
4th	05 September 2020	05 March 2021	05 March 2021

Business Day Convention / Day Count Fraction	modified following, unadjusted / 30/360

Redemption

Final Redemption	Unless previously redeemed, repurchased or cancelled, the Issuer shall redeem each Product on the Final Redemption Date by payment of a cash amount or delivery of a number of Underlyings equal to the Final Redemption Amount to the Holder thereof.
Final Redemption Amount	(i) if **no** Barrier Event has occurred, a cash amount equal to **100%** of the Denomination; or (ii) if a Barrier Event has occurred, and (a) the Final Level is **at or above** the Strike, a cash amount equal to **100%** of the Denomination; or (b) the Final Level is **below** the Strike, the number of Underlyings specified in the Ratio. In case of a physical settlement according to scenario (ii) (b), the number of Underlyings to be delivered will be rounded down to the nearest integral number of Underlyings. In addition, the holder will receive a cash amount in lieu for any fractional amount.
Settlement Type	Physical settlement or cash settlement
Level	the Share Price
Final Level	the Level at the Valuation Time on the Final Fixing Date, as determined by the Calculation Agent
Barrier Event	If the Level at any time (observed continuously) on any Barrier Observation Date is **at or below** the Barrier.
Barrier Observation Dates	each Exchange Business Day during the Barrier Observation Period(s), being the dates on which the Level is observed for purposes of determining whether a Barrier Event has occurred.
Barrier Observation Period	from and including Initial Fixing Date to and including the Final Fixing Date

Figure 6.1: Part of the term sheet of a BRC on Nestlé.

	payoff (at maturity T)		
	$S_T \leq B$	$B < S_T < X$	$S_T \geq X$
B never hit	-	$C^m + N$	$C^m + N$
B hit	$C^m + nS_T$	$C^m + nS_T$	$C^m + N$

Table 6.1: Payoff panel of a BRC.

where by I^c we denote the discrete random variable $I^c = 1 - I$ with I defined in (4.2) (for $G =]B, \infty[$). By remarking that the conversion ratio is defined by

$$n := \frac{N}{X} ,$$

we convince ourselves that the function (6.1) indeed reproduces the panel 6.1. We plot the profit $P(S_T)$ of a BRC at maturity. Since the "Issue Price" is equal to N, the profit $P(S_T) = g(S_T) - N$ is therefore

$$P(S_T) = C^m - nI^c \max\{X - S_T, 0\} .$$

We observe that the profit of a BRC is capped at the final value of the sum of the coupon payments.

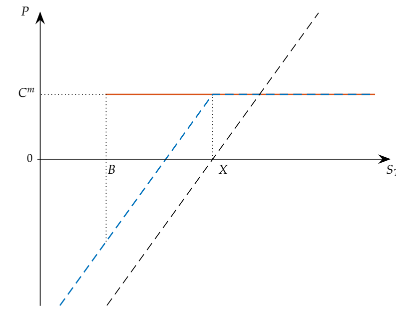

Figure 6.2: Profit diagram of a BRC. The event "barrier never hit" ($I^c = 0$) is marked by (—), whereas the event "barrier hit" ($I^c = 1$) is marked by (−−). We compare this with the profit of the investment into n shares of the underlying (−−), where $X = S_0$.

6.2 Replication and pricing

We replicate the payoff (6.1)

$$g(S_T) = C^m + N - nI^c \max\{X - S_T, 0\} .$$

The cash flow $C^m + N$ can be generated by a riskless investment into the money market with continuously compounding interest rate r. The present value of $C^m + N$ at time $t \le T$ is thus

$$e^{-r(T-t)}(C^m + N) .$$

The cash flow $I^c \max\{X - S_T, 0\}$ corresponds to the payoff of a down-and-in put option. If the barrier $B < S_0$ is hit during the life time of the BRC, then the down-and-in put pays off $\max\{X - S_T, 0\}$, which is the payoff of a European put. If, however, the underlying does not reach the barrier, the down-and-in option becomes worthless. The value of the down-and-in put is by (4.6) given as

$$V_{p,\mathrm{di}}(s,t) = V_p(s,t) - V_{p,\mathrm{do}}(s,t) ;$$

we realised this formula already in the Python function `barrier_bs.py`, compare with problem 4.2. Hence, the term $-nI^c \max\{X - S_T, 0\}$ corresponds to a short position in n down-and-in puts such that for $0 \le t \le T$ the value $V_{\mathrm{brc}}(s,t)$ of the BRC must be

$$V_{\mathrm{brc}}(s,t) = e^{-r(T-t)}(C^m + N) - nV_{p,\mathrm{di}}(s,t) .$$

Using (4.6) once more we have

$$V_{\mathrm{brc}}(s,t) = e^{-r(T-t)}(C^m + N) - n\big(V_p(s,t) - V_{p,\mathrm{do}}(s,t)\big) \tag{6.2}$$

with V_p being the value of the European put and $V_{p,\mathrm{do}}$ the value of the down-and-out put; in the Black and Scholes market model, this values are given by (3.14) and (4.4), respectively.

Example 6.1. We check the market price $V_{\text{brc}}^{\text{M}} = 1041/1046$ (bid/ask) of the BRC on the Nestlé (compare with figure 6.1) as of August 26, 2019 ($t = 0$). We take the following data: $s = 107.96$ (value Nestlé), $\sigma = 19.34\%$, $q = \ln(1 + 0.0168)$, $r = -0.96\%$, (values for σ, r and q are from Bloomberg) and the maturity $t = T$ corresponds to 26/2/2021. The remaining data is taken from the term sheet. In particular, the coupon payment dates T_i are $T_1 = 5/9/2019$, $T_2 = 5/3/2020$, $T_3 = 7/9/2020$ and $T_4 = 5/3/2021$. We use the function `yf.py` to calculate the time spans $T - T_i$ in the evaluation of the final value C^m of the coupon payments. The value $V_{p,\text{di}}$ follows from the Python function `barrier_bs.py`.

```
In [2]:
# time-to-maturity, time-to-coupon payments
T = yf((26,8,2019),[(26,2,2021)])[0]
Ti = yf((26,8,2019),[(5,9,2019),(5,3,2020),(7,9,2020),(5,3,2021)])

# the value of the down-and-in put
s = 107.96; B = 68.445; X = 91.26; r = -0.0096; q = np.log(1+0.0168)
Vpdi = barrier_bs(s,B,X,T,0.1934,r,q)[1]

# final value of the coupon payments
Cm = 13.75*np.sum(np.exp(r*(T-Ti)))

# the value of the BRC
V = np.exp(-r*T)*(Cm+1000)-1000/X*Vpdi
```

```
In [3]:
Vpdi, V
Out[3]: (2.183441388762827, 1045.9909411537612)
```

Figure 6.3: We use Python to find the value $V_{p,\text{di}}$ of the down-and-in put.

We find $V_{\text{brc}}(s, 0) \doteq 1045.99$, thus the Black-Scholes price is equal to the ask price. ◇

We are now interested in the time evolution $t \mapsto V_{\text{brc}}(s, t)$ of the value of the BRC in example 6.1. To understand it, we simulate the (value of the) underlying S_t for $0 < t \leq 0.5$ as a geometric Brownian motion, where we take $t = 0$ to be the date 26/2/2019 of initial fixing (such that $S_0 = X = 91.26$). The model parameters of the geometric Brownian motion are $\mu = 0\%$ (we expect a sideway moving underlying), $\sigma = 16.5\%$; for the risk neutral pricing we take $q = \ln(1 + 0.018)$ and $r = \ln(1 - 0.0073)$. A (simulated) path of the underlying is then given by the points (t_j, s_j) with $t_j = j\Delta t$ and $s_j = S_{t_j}$; we take $\Delta t = 1/720$, see the technical note 2.8 on how to generate paths of a geometric Brownian motion. For each t_j we calculate the value $v_j = V_{\text{brc}}(s_j, t_j)$ of the BRC in dependence on s_j and t_j; we implicitly assume that the model parameters σ, μ and r, q do not change over time. Then we plot the normalised values $(t_j, s_j/X)$ and $(t_j, v_j/V_{\text{brc}}(S_0, 0))$ in figure 6.4, where we normalise the underlying by $S_0 = X$ (the value of the underlying at initial fixing) and the BRC by $V_{\text{brc}}(S_0, 0) \approx 0.985N$.

We observe that if the underlying has a negative performance then the BRC performs better than the underlying, compare with the bottom graph of figure 6.4. If, however, the performance of the underlying is positive, then the performance of the BRC is below the performance of the underlying. We do not invest into a BRC when we expect the underlying to increase.

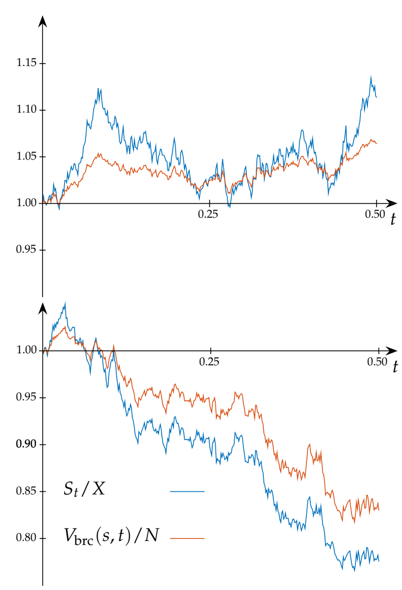

Figure 6.4: Two possible time evolutions of the underlying (Nestlé) and the BRC from figure 6.1. Here, $t = 0$ corresponds to the date of initial fixing, i.e., 26/2/2019. The prices are normalised.

6.3 Multi barrier reverse convertibles

Multi barrier reverse convertibles (MBRC) are barrier reverse convertibles written on multiple stocks; most of them are on $d = 3, 4, 5$ underlyings. We have seen that the payoff of a BRC is given by (since $n = N/X$)

$$g(S_T) = C^m + N - NI^c \max\left\{\frac{X - S_T}{X}, 0\right\},$$

compare with (6.1). Since usually there holds $S_0 = X$, the expression $(X - S_T)/X$ is just the negative performance (the negative of the holding period return) of the underlying. For a MBRC the payoff is similar; now the maximum of the negative performance of all involved underlyings is considered, that is

$$\begin{aligned} g(\mathbf{S}_T) &= C^m + N - NI^c \max\left\{\max_i \frac{X_i - S_T^i}{X_i}, 0\right\} \\ &= C^m + N - NI^c \max\left\{1 - \min_i \frac{S_T^i}{X_i}, 0\right\}, \end{aligned} \tag{6.3}$$

where $\mathbf{S}_t = (S_t^1, \ldots, S_t^d)$ is the vector of the stock prices as in (3.35) and X_i is the strike price of the i-th underlying, compare with the term sheet in figure 6.5. From the considerations made in the previous section it follows that the value V_{mbrc} of a MBRC is given by

$$V_{\text{mbrc}}(\mathbf{s}, t) = e^{-r(T-t)}(C^m + N) - NV_{p,\text{di},d}(\mathbf{s}, t), \tag{6.4}$$

where we denote by $V_{p,\text{di},d}(\mathbf{s}, t)$ the time t-value of a down-and-in put on d stocks with prices $\mathbf{s} = (s_1, \ldots, s_d)$. To calculate this value we model the stock prices $\mathbf{S}_t$ as a d-dimensional geometric Brownian motion (2.22). As in the case of a basket bonus certificate in chapter 5 the value $V_{p,\text{di},d}(\mathbf{s}, t)$ can not be found analytically and we have to rely on numerical approximations.

Symbol/ Swiss Security Code/ISIN	**Z90K0Z/** 47 321 711/CH0473217112
Notional Amount/Denomination/ Trading Units	Up to CHF 3'000'000, with the right to increase / denomination of CHF 1'000 per Structured Product / CHF 1'000 or multiples thereof
Issue Price	100.00% of the Denomination
Currency	CHF
Underlying	**Credit Suisse Group AG registered share**/CH0012138530/SIX Swiss Exchange/Bloomberg: CSGN SE **UBS Group Inc registered share**/CH0244767585/SIX Swiss Exchange/Bloomberg: UBSG SE
Cap Level (100%) **Knock-in Level (65%)** **Ratio**	see table below

Underlying	Cap Level	Knock-in Level	Ratio
CS Group rs	11.4700	7.4555	87.183958
UBS Group rs	11.6550	7.5758	85.800086

Coupon	**12.50% (6.2500% p.a.)**, interest payment 0.0000% p.a., premium payment 6.2500% p.a.
Coupon Payment Date(s)	18 June 2020 18 June 2021 The Coupon will be paid out on a pro rata basis on the Coupon Payment Dates.
Coupon Calculation Method	30/360 (German), modified following
Initial Fixing Date	13 June 2019
Settlement Date	18 June 2019
Last Trading Date	11 June 2021
Final Fixing Date	11 June 2021
Redemption Date/ Date of Delivery	18 June 2021
Initial Fixing Level	Closing prices of Underlyings on Related Exchanges, on 13 June 2019 CS Group rs / CHF 11.4700 UBS Group rs / CHF 11.6550
Final Fixing Level	Closing prices of Underlyings on Related Exchanges on 11 June 2021
Redemption Method	If the price of none of the Underlyings has traded at or below the Knock-in Level between the Initial Fixing Date and the Final Fixing Date, redemption will be 100% of the Denomination independent of the Final Fixing Levels of the Underlyings. If the price of one or more of the Underlyings has traded at or below the Knock-in Level between the Initial Fixing Date and the Final Fixing Date, ▪ redemption will be 100% of Denomination if the Final Fixing Levels of all Underlyings trade at or higher than the Cap Level or ▪ the investor will receive a physical delivery of the Underlying with the worst relative performance (between Initial Fixing Date and Final Fixing Date). The number of Underyings per Denomination is defined according to Ratio (Fractions will be paid in cash, no cumulation). The Coupon(s) will be paid out on the defined Coupon Date(s) independent of the performance of the Underlyings.

Figure 6.5: Part of the term sheet of a MBRC on two underlyings.

Example 6.2. On June 13, 2019, ($t = 0$) we check the issue price 1000 CHF of the MBRC with 6.25% p.a. coupon and 65% barrier on CS and UBS given in figure 6.5.

We take the values $s_1 = X_1 = 11.47$ CHF (CS), $s_2 = X_2 = 11.655$ (UBS), $B_1 = 7.4555 = 0.65s_1$, $B_2 = 7.5758 = 0.65s_2$, $r = -0.735\%$. The continuous dividend yields are $q_1 = \ln(1.03547)$ (CS) and $q_2 = \ln(1.07783)$ (UBS). The implied volatilities are $\boldsymbol{\sigma}^{\mathrm{i}} = \operatorname{diag}(\sigma_1^{\mathrm{i}}, \sigma_2^{\mathrm{i}}) = \operatorname{diag}(0.226, 0.204)$, the correlation matrix $\boldsymbol{\rho}$ in (3.37) is

$$\boldsymbol{\rho} = \boldsymbol{\rho}^{\mathrm{h}} = \begin{pmatrix} 1 & \rho \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0.856 \\ & 1 \end{pmatrix} .$$

The remaining data is from the term sheet given in figure 6.5; time spans are calculated using `yf.py`. We find $V_{\mathrm{mbrc}}(s_1, s_2, 0) \approx 986.12$, compare with figure 6.7. ⋄

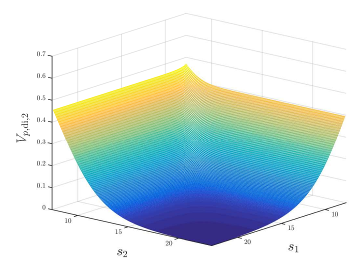

Figure 6.6: The value of the down-and-in put is $V_{p,\mathrm{di},2}(11.47, 11.655, 0) \approx 0.15505$ (this value has to be multiplied by $N = 1000$). The graph of the function $\mathbf{s} \mapsto V_{p,\mathrm{di},2}(\mathbf{s}, 0)$ is obtained by a PDE solver. It is very time consuming to obtain such graphs (how do derivative prices depend on the value of the underlying?) with Monte Carlo.

In figure 6.6 we plot the value of the down-and-in put from example 6.2 i.e., the graph of the function $\mathbf{s} \mapsto V_{p,\mathrm{di},2}(\mathbf{s}, 0)$ for stock prices $\mathbf{s} = (s_1, s_2)$ in the range $(s_1, s_2) \in]B_1, 2X_1[\times]B_2, 2X_2[$.

```
In [11]:
# input parameters (model, contract)
s0 = [11.47,11.655]; q = [np.log(1.03547),np.log(1.07783)]; r = -0.00735
sigma1 = 0.226; sigma2 = 0.204; rho = 0.856;
Sigma = [[sigma1**2,rho*sigma1*sigma2],[rho*sigma1*sigma2,sigma2**2]]
B = [7.4555,7.5758]; X = [11.47,11.655]; N = 1000
T = yf((13,6,2019),[(11,6,2021)]); Ti = yf((13,6,2019),[(18,6,2020),(18,6,2021)])
mu = [r-x for x in q]; q = np.reshape(np.asarray(q),[2,1])
B = np.reshape(np.asarray(B),[2,1]); X = np.reshape(np.asarray(X),[2,1])

# input parameters (Monte Carlo simulation)
n = 10**5; dt = 1/(16*252); v = np.zeros(n)
```

```
In [12]:
# Monte Carlo simulation
for j in range(0,n):
    D = gbm_d(s0,mu,Sigma,[],T,dt)
    S = D[0][1:] # simulated stock prices, excluding s0
    # the minimium stock prices
    Smin = np.amin(S,axis=0); Smin = Smin.reshape(len(s0),1)
    ST = S[-1,:]; ST = ST.reshape(len(s0),1)

    if np.any(Smin-B<=0): # barrier hit?
        v[j] = np.maximum(1-np.min(ST/X),0)

Vpdi = np.exp(-r*T)*np.mean(v)
```

```
In [13]:
# final value of the coupon payments
Cm = 0.0625*N*np.sum(np.exp(r*(T-Ti)))
# the value of the BRC
V = np.exp(-r*T)*(Cm+N)-N*Vpdi; display((V[0],Vpdi[0]))

(986.1205518892768, 0.15504609330611954)
```

Figure 6.7: We estimate the price of the MBRC in example 6.2 to $V_{\text{mbrc}}(s_1, s_2, 0) \approx 986.12$. In particular, $V_{p,\text{di},2} \approx 0.15505$ (this value has to be multiplied by $N = 1000$).

6.4 Problems

Problem 6.1. We consider the investment strategies A and B defined as

- A: investment of N units of currency into a BRC,
- B: investment of N units of currency into n shares of the corresponding underlying.

For simplicity, we set $r = 0$. At which final fixing level S_T strategy B realises a higher return than strategy A? First answer the question generally when there are m coupon payments, then apply your findings to the BRC in figure 6.1.

Problem 6.2. Consider the BRC in figure 6.1.

i) What is the theoretical value $V_{\text{brc}}(s, 0)$ of the BRC at the initial fixing date? Conclude the agio of the issuer.
Take the following values: $\sigma = 16.5\%$, $q = \ln(1 + 0.018)$ and $r = \ln(1 - 0.0073)$.

ii) Data as in part i). We are interested in the dependence of the (theoretical) value of the BRC on the ratio $\alpha := \frac{B}{X}$ (with $X = 91.26$ fixed; in the term sheet we have $\alpha = 0.75$). For $\alpha \in [0.6, 0.9]$ plot the graph of the function $\alpha \mapsto V_{\text{brc}}(s, 0)$ and comment.

Problem 6.3. Consider the MBRC in figure 6.8. For simplicity, take $r = 0$.

i) For the following scenarios find the cash flow at the repayment date.

   i_1) No barrier-event and the final fixings: Nestlé 72.30 CHF, Novartis 83.40 CHF, Roche 272.55 CHF, Lindt 4804.30 CHF i$_2$) Barrier-event and the final fixings: Nestlé 72.30 CHF, Novartis 83.40 CHF, Roche 272.55 CHF, Lindt 4904.30 CHF

i$_3$) Barrier-event and the final fixings: Nestlé 72.30 CHF, Novartis 83.40 CHF, Roche 252.55 CHF, Lindt 4804.30 CHF

ii) For all three scenarios in part i) find the holding period return of an investment into the MBRC at issuance.

Figure 6.8: Part of the term sheet of a MBRC issued by Vontobel.

VONTOBEL

Termsheet (Final Terms) Vontobel Investment Banking

MULTI DEFENDER VONTI +41(0)58 283 78 88 or www.derinet.ch

SSPA DESIGNATION: BARRIER REVERSE CONVERTIBLE (1230)

5.00% p.a. Multi Defender VONTI on Nestlé/Novartis/Roche/Lindt & Sprüngli

PRODUCT DESCRIPTION

Multi Defender VONTI (Multi Barrier Reverse Convertible) refer to several underlyings and are characterized by a guaranteed coupon, several barriers as well as by a – albeit only conditional – redemption at the nominal value. The redemption at the end of the term is determined on the basis of the performance and final fixing of the respective underlyings: A redemption at the nominal value is guaranteed as long as the underlyings have not touched their barriers during relevant barrier monitoring. If one of the underlyings has touched its barrier but all underlyings are again above the respective strike prices at final fixing, the nominal price is also repaid. If, however, one of the underlyings has touched its barrier during barrier monitoring and at least one of the underlyings closes below its strike price at final fixing, the predetermined number of the underlying with the poorest performance is delivered or a cash compensation is paid (for details see "Redemption/delivery").

In Switzerland, these financial instruments are considered structured products. They are not collective investment schemes within the meaning of the Swiss Federal Act on Collective Investment Schemes (CISA), and are therefore not subject to the regulations of the CISA or the supervision of the Swiss Financial Market Supervisory Authority FINMA.

Product Information

Issuer	Vontobel Financial Products Ltd., DIFC Dubai			
Keep-Well Agreement	With Bank Vontobel AG, Zurich (Standard & Poor's A; Moody's A2; see its complete wording in the Issuance Programme)			
Guarantor	Vontobel Holding AG, Zurich (Standard & Poor's A-; Moody's A3)			
Lead Manager	Bank Vontobel AG, Zurich			
Paying, exercise and calculation agent	Bank Vontobel AG, Zurich			
SSPA product type	Barrier Reverse Convertible (1230), see also www.svsp-verband.ch			
Underlyings	Nestlé SA Novartis AG Roche Holding AG Lindt & Sprüngli AG (further details on the underlyings see below)			
Issue price	100%			
Notional	CHF 5000.00			
Strikes / Barriers	Underlying	Strike	Barrier (in %)	Number of underlyings
	Nestlé SA	CHF 69.70 (100%)	CHF 46.70 (67%)	71.73601
	Novartis AG	CHF 79.25 (100%)	CHF 53.10 (67%)	63.09148
	Roche Holding AG	CHF 263.20 (100%)	CHF 176.34 (67%)	18.99696
	Lindt & Sprüngli AG	CHF 4821.00 (100%)	CHF 3230.07 (67%)	1.03713
Barrier monitoring	August 18, 2014 until August 17, 2018, continuous monitoring			
Coupon	5.00% p.a. - yearly payment, 30/360 (total duration: 1440 days)			
Coupon payment	Coupon	Date	Interest component	Premium component
	5.00%	August 27, 2015	0.13%	4.87%
	5.00%	August 29, 2016	0.13%	4.87%
	5.00%	August 28, 2017	0.13%	4.87%
	5.00%	August 27, 2018	0.13%	4.87%
Initial fixing	August 18, 2014, closing prices			
Payment date	August 27, 2014			
Last trading day	August 17, 2018 (12:00 CET)			
Final fixing	August 17, 2018, closing prices			
Repayment date	August 27, 2018			
Reference currency	CHF; issue, trading and redemption will follow in the reference currency			
Swiss Security Number / ISIN / Telekurs Symbol	2417 2805 / CH0241728051 / VONLJG			

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Problem 6.4. Again, consider the MBRC in figure 6.8. In this problem, we study the connection between the coupon size c (in % of the notational N) and the relative barrier b (relative to the initial fixing S_0^i). In the product defined in figure 6.8, there holds $c = 5\%$ and $b = 67\%$. Use Python to find a model $c = f(b)$, i.e., given $40\% < b < 95\%$, find c such that the issue price of the MBRC is $V_{\mathrm{mbrc}}(\mathbf{s}, 0) = 0.985N$.

Use the contract parameters given in figure 6.8, and the model parameters

$$\boldsymbol{\sigma}^{\mathrm{i}} = \mathrm{diag}(\sigma_1^{\mathrm{i}}, \sigma_2^{\mathrm{i}}, \sigma_3^{\mathrm{i}}, \sigma_4^{\mathrm{i}}) = \mathrm{diag}(0.104, 0.113, 0.116, 0.159) \ ,$$

and

$$\boldsymbol{\rho} = \boldsymbol{\rho}^{\mathrm{h}} = \begin{pmatrix} 1 & \rho_{12} & \rho_{13} & \rho_{14} \\ & 1 & \rho_{23} & \rho_{24} \\ & & 1 & \rho_{34} \\ & & & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0.579 & 0.515 & 0.358 \\ & 1 & 0.651 & 0.331 \\ & & 1 & 0.321 \\ & & & 1 \end{pmatrix}$$

as well as $q_1 = 0.0382$, $q_2 = 0.0407$, $q_3 = 0.0392$, $q_4 = 0.0172$, $r = \ln(1 + 0.0016)$. We use the numbering Nestlé $\to$ 1, Novartis $\to$ 2, Roche $\to$ 3, Lindt $\to$ 4.

Chapter 7

Capital protection

According to the "Swiss Derivative Map" (see figure 1.1) capital protection products are applied if one expects the underlying to rise but there is also a substantial probability for a sharp drop in the underlying. At maturity, the holder of such a product receives a certain percentage of the denomination (the capital protection) independent of the performance of the underlying (however, some of the products permit only a conditioned capital protection). Moreover, if the underlying closes above the strike of the product, the investor participates in the (positive) performance of the underlying. Typical underlyings are stocks, indices and interest rates (Libor), rarely currencies and commodities. The lifetime of capital protection products is usually (much) larger than one year, and there exist products with a (return) cap, barriers or coupon payments.

On "SIX Structured Products" there are about 590 capital protection products available (as of October 2019).

7.1 Payoff of an uncapped capital protection product

We use the following notation. We denote by N the denomination, by k the capital protection and by z the participation. In term sheets, k and z are usually given in percents of the denomination. The holder of a capital protection product anticipates via z in the performance

$$R_{\mathrm{u},0,T} := \frac{S_T - X}{S_0} \tag{7.1}$$

of the underlying over the time period $[0, T]$ [1]. The capital protection product defined in figure 7.1 for example takes the values $N = 1000$ CHF, $k = 1.01$ and $z = 1$. The strike price is equal to the value of the underlying at initial fixing, i.e., $X = S_0$, such that $\gamma = 1$.

[1] The strike X can be written as $X = \gamma S_0$ for some $\gamma \in \mathbb{R}^+$. If in particular $\gamma = 1$ (the typical case), then the expression $\frac{S_T - X}{S_0}$ is indeed a holding period return. The subscript "u" is an abbreviation of "underlying".

According to the redemption described in this term sheet, the payoff panel of a capital protection product without cap is as follows,

UNDERLYING			
Underlying	**Index Sponsor**	**Bloomberg Ticker**	**Initial Fixing Level (100%)***
Swiss Market Index®	SIX Swiss Exchange AG	SMI	CHF 8532.09

PRODUCT DETAILS	
Swiss Security Number	**24204717**
ISIN	**CH0242047170**
SIX Symbol	**LTQEJZ**
Issue Price	100.00%
Issue Size	CHF 10'000'000 (can be increased at any time)
Denomination	CHF 1'000
Settlement Currency	CHF
Capital Protection	101.00%
Participation	100.00%
Bondfloor at issuance	96.43% (implied Yield p.a.: 0.58%)

DATES	
Subscription Start Date	20/10/2014
Subscription End Date	24/10/2014 14.00 CET
Initial Fixing Date	24/10/2014
Issue Date	31/10/2014
First Exchange Trading Date	31/10/2014
Last Trading Day/Time	24/10/2022 / Exchange market close
Final Fixing Date	24/10/2022 (subject to Market Disruption Event provisions)
Redemption Date	31/10/2022 (subject to Settlement Disruption Event provisions)

REDEMPTION	
The Investor is entitled to receive from the Issuer on the Redemption Date per Product:	
Scenario 1	If the Final Fixing Level is at or below the Initial Fixing Level, the Investor will receive a Cash Settlement in the Settlement Currency according to the following formula: Denomination × Capital Protection
Scenario 2	If the Final Fixing Level is above the Initial Fixing Level, the Investor will receive a Cash Settlement in the Settlement Currency according to the following formula: Denomination × (Capital Protection + Participation × (Final Fixing Level - Initial Fixing Level) / Initial Fixing Level)
Initial Fixing Level	Official close of the Underlying on the Initial Fixing Date as calculated and published by the Index Sponsor and as determined by the Calculation Agent.
Final Fixing Level	Official close of the Underlying on the Final Fixing Date as calculated and published by the Index Sponsor and as determined by the Calculation Agent.

Figure 7.1: Part of the term sheet of a capital protection certificate on the SMI.

compare with table 7.1.

Table 7.1: Payoff panel of a capital protection product without cap.

	payoff (at maturity T)	
	$S_T \leq X$	$S_T > X$
in % of N	k	$k + zR_{\mathrm{u},0,T}$
in CCY	kN	$(k + zR_{\mathrm{u},0,T})N$

Here, $R_{\mathrm{u},0,T}$ is as in (7.1). Translating the payoff panel to a mathematical function yields the payoff

$$g(S_T) = kN + \frac{zN}{S_0}\max\{S_T - X, 0\} \ . \tag{7.2}$$

The graph of the function $s \mapsto g(s)/N$ is plotted in figure 7.2.

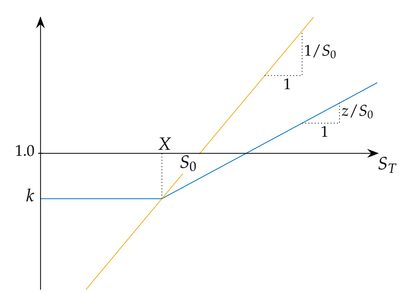

Figure 7.2: Payoff (—), normalised by N, of an uncapped capital protection product (with $k < 1$, $z < 1$, $X < S_0$). We compare this with the payoff of the investment into the underlying (—), normalised by S_0.

Example 7.1. Express the return $R_{\mathrm{u},0,T}$ of the underlying in terms of the return $R_{\mathrm{cp},0,T}$ of the capital protection product (at maturity).

Solution. Since the issue price of the product is N, we need to solve the equation $(g(S_T) - N)/N = R_{\mathrm{cp},0,T}$ for S_T. First assume that $S_T > X$. Then the equation to be solved reads

$$k + \frac{z}{S_0}(S_T - X) = 1 + R_{\mathrm{cp},0,T}$$

The solution is $S_T = \frac{S_0}{z}(1 + R_{\mathrm{cp},0,T} - k) + X$, in terms of the return

$R_{\mathrm{u},0,T}$ of the underlying this becomes

$$\begin{aligned} S_T &= \frac{S_0}{z}(1 + R_{\mathrm{cp},0,T} - k) + X \\ \frac{S_T}{S_0} - 1 &= \frac{\frac{S_0}{z}(1 + R_{\mathrm{cp},0,T} - k) + X}{S_0} - 1 \\ R_{\mathrm{u},0,T} &= \frac{1}{z}R_{\mathrm{cp},0,T} + \frac{1-k}{z} + \frac{X}{S_0} - 1 \end{aligned}$$

Now we assume that $S_T \leq X$. Then the return of the capital protection product is - irrespective of the return of the underlying - $R_{\mathrm{cp},0,T} = k - 1$. The return of the underlying is in this case at most $R_{\mathrm{u},0,T} = \frac{S_T}{S_0} - 1 \leq \frac{X}{S_0} - 1$. In the particular case $X = S_0$ we can therefore state the following. If the return of the capital protection product is $k - 1$, then the return of the underlying is non-positive and can take any value between -1 and 0. If the return of the capital protection product is $R_{\mathrm{cp},0,T} > k - 1$, then the return of the underlying is

$$R_{\mathrm{u},0,T} = \frac{1}{z}R_{\mathrm{cp},0,T} + \frac{1-k}{z} \,.$$

For $k = 1.01$ and $z = 1$ as it is the case for the product in figure 7.1 we simply have $R_{\mathrm{u},0,T} = R_{\mathrm{cp},0,T} - 0.01$ such that the return of the underlying is 1% lower than the return of the capital protection product. Note carefully that if $k < 1$ and $z < 1$ (as is often the case) then return of the capital protection product is (much) smaller that the return of the underlying. ◇

7.2 Replication and pricing

We replicate the payoff (7.2), which is

$$g(S_T) = kN + \frac{zN}{S_0}\max\{S_T - X, 0\} \,.$$

The cash flow kN - which is independent of the performance of the underlying - can be generated by a riskless investment in the money market with continuously compounding interest rate r. The present value of this cash flow at time $t \leq T$ is thus

$$e^{-r(T-t)}kN \,.$$

The term $\frac{zN}{S_0}\max\{S_T - X, 0\}$ corresponds to the payoff of

$$n := \frac{zN}{S_0} \tag{7.3}$$

call options with strike X. Thus, the value V_{cp} of the uncapped capital protection product at time $0 \leq t \leq T$ is

$$V_{\mathrm{cp}}(s,t) = e^{-r(T-t)}kN + \frac{zN}{S_0}V_c(s,t) \,. \tag{7.4}$$

Example 7.2. Consider a capital protection product with the following parameters. Denomination $N = 2000$ CHF, price (of the product) at issuance 2060 CHF (which is 103% of N), capital protection $k = 0.95$, strike $X = S_0$, final fixing in 3 years. Market parameters are $S_0 = 640$ CHF, $\sigma = 23.7045\%$, $q = 1.1\%$, $r = \ln 1.02$.

What is the participation z of this product?

Solution. We need solve the equation (7.4) for z. We find

$$z = \underbrace{\frac{V_{\mathrm{cp}}(s,0) - e^{-rT}kN}{V_c(s,0)}}_{=n} \frac{S_0}{N}$$

the (first) fraction shows how many call options we need to go long to replicate the product. The Black-Scholes formula (3.14) gives the call price $V_c(s,0) \doteq 107.678$ CHF. Thus n is given by

$$n = \frac{V_{\mathrm{cp}}(s,0) - e^{-rT}kN}{V_c(s,0)} \doteq \frac{2060 - 1.02^{-3} \cdot 0.95 \cdot 2000}{107.678} \doteq 2.5$$

and hence

$$z = n\frac{S_0}{N} = 2.5 \cdot \frac{640}{2000} = 0.8 \; .$$

◇

7.3 Capped capital protection

The capital protection product considered so far may generate an arbitrary high profit/return, compare with the payoff in figure 7.2.

Besides the uncapped capital protection products we find on the market also products with a return cap. This means that above a certain final fixing level X_2 of the underlying the return of the product is capped to a fixed value c. The payoff $g(S_T)/N$ of such a product is given in figure 7.3. If we denote by

$$R_{\mathrm{u},0,T} := \frac{S_T - X_1}{S_0} \; ,$$

then the payoff panel of a capped capital protection product is given as in table 7.2 (you might compare it with table 7.1)

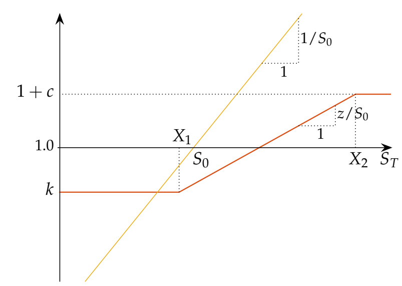

Figure 7.3: Payoff (—), normalised by N, of a capital protection product with cap c (and with $k < 1$, $z < 1$, $X_1 < S_0 < X_2$). We compare this with the payoff of the investment into the underlying (—), normalised by S_0.

	payoff (at maturity T)		
	$S_T \leq X_1$	$X_1 < S_T < X_2$	$X_2 \leq S_T$
in % of N	k	$k + zR_{\mathrm{u},0,T}$	$1+c$
in CCY	kN	$(k + zR_{\mathrm{u},0,T})N$	$(1+c)N$

Table 7.2: Payoff panel of a capital protection product without cap.

We remark that for given values of the strikes X_1, X_2, the participation z and the capital protection k the return cap c can not be chosen arbitrarily. The slope of the function $g(S_T)/N$ in the range $X_1 < S_T < X_2$ is

$$\frac{1+c-k}{X_2 - X_1} .$$

But it is also defined via the participation z as

$$\frac{z}{S_0} .$$

Hence, the return cap c is (the unique) solution of the equation

$$\frac{1+c-k}{X_2 - X_1} = \frac{z}{S_0} ;$$

this solution is

$$c = k - 1 + \frac{z}{S_0}(X_2 - X_1) . \tag{7.5}$$

More generally we need to know four of the quantities X_1, X_2, c, k, z to find the fifth.

We now turn to the replication of a capped capital protection product. To do so, we add to the already existing uncapped capital protection product (with strike X_1) a second (yet unknown) component or product p in such a way that we obtain the payoff given in figure 7.3. In terms of payoffs this reads as[2]

$$\begin{aligned} g_{\text{ccp}}(S_T) &= g_{\text{cp}}(S_T) + g_{\text{p}}(S_T) \\ &\overset{(7.2)}{=} kN + \frac{zN}{S_0}\max\{S_T - X_1, 0\} + g_{\text{p}}(S_T) . \end{aligned}$$

For final fixing levels $S_T \leq X_2$ there must hold

$$g_{\text{p}}(S_T) = 0$$

whereas for the range $S_T > X_2$ there holds $g_{\text{ccp}}(S_T) = (1+c)N$ and thus

$$g_{\text{ccp}}(S_T) = kN + \frac{zN}{S_0}(S_T - X_1) + g_{\text{p}}(S_T) = (1+c)N .$$

It follows that

$$g_{\text{p}}(S_T) = (1+c)N - kN - \frac{zN}{S_0}(S_T - X_1)$$

whenever $S_T > X_2$. If we rewrite this using the relationship (7.5) we end with

$$\begin{aligned} g_{\text{p}}(S_T) &= (1+c)N - kN - \frac{zN}{S_0}(S_T - X_1) \\ &\overset{(7.5)}{=} kN + \frac{zN}{S_0}(X_2 - X_1) - kN - \frac{zN}{S_0}(S_T - X_1) \\ &= -\frac{zN}{S_0}(S_T - X_2) . \end{aligned}$$

[2] The subscript "ccp" stands for "capped capital protection", whereas "cp" stands for (uncapped) "capital protection".

Thus the product p we are looking for has the payoff

$$g_{\mathrm{p}}(S_T) = -\frac{zN}{S_0}\begin{cases} 0 & \text{if } S_T \leq X_2 \\ S_T - X_2 & \text{if } S_T > X_2 \end{cases} ,$$

which obviously corresponds to a short position in $n = \frac{zN}{S_0}$ call options with strike X_2. Thus the price $V_{\mathrm{ccp}}(s,t)$ of a capped capital protection product is given by

$$V_{\mathrm{ccp}}(s,t) = V_{\mathrm{cp}}(s,t) - \frac{zN}{S_0}V_c(s,t;X_2) ,$$

where by $V_c(s,t;X)$ we denote the price of a European call option with strike X. With (7.4) we finally have

$$V_{\mathrm{ccp}}(s,t) = e^{-r(T-t)}kN + \frac{zN}{S_0}\big(V_c(s,t;X_1) - V_c(s,t;X_2)\big) . \tag{7.6}$$

Example 7.3. The product from example 7.2 is being capped at strike $X_2 = 1.25S_0$. The other parameters do not change.

Find the participation and the cap of the product.[3]

Solution. To obtain the participation we solve equation (7.6) for z

$$z = \underbrace{\frac{V_{\mathrm{ccp}}(s,0) - e^{-rT}kN}{V_c(s,0;S_0) - V_c(s,0;1.25S_0)}}_{=n}\frac{S_0}{N} .$$

Herein, we have to calculate the call price $V_c(s,0;1.25S_0)$; using the volatility $\sigma_2 = 0.21$ we find $V_c(s,0;1.25S_0) \doteq 45.802$ CHF. Hence we obtain

$$n = \frac{2060 - 1.02^{-3}\cdot 0.95 \cdot 2000}{107.836 - 45.802} \doteq 4.3458 ,$$

such that the participation z becomes

$$z = n\frac{S_0}{N} \doteq 4.3458 \cdot \frac{640}{2000} \doteq 1.3907 .$$

From this we find the cap c to be

$$c \overset{(7.5)}{=} k - 1 + \frac{z}{S_0}(1.25S_0 - S_0) = -0.05 + 0.25z \doteq 0.2977 .$$

The capped capital protection product has a much higher participation than the uncapped product (for which $z = 0.8$). In contrast, the return of the capped product can not be higher than 29.77%. ◇

[3] We take a smaller volatility $\sigma_2 = 21\% < 23.7045\%$ to price the call with strike X_2. This is due to the volatility smile (see for example figure 3.10) after which OTM options do (usually) have a smaller volatility than ATM options.

7.4 Capital protection products on multiple underlyings

We exemplarily consider the capital protection product written on multiple underlyings described in figure 7.4. To understand the payoff of such a product, we again have a look on the payoff of a capped capital protection product written on one underlying, where we denote the strikes by X and Y, respectively

$$g(S_T) = kN + \frac{zN}{S_0}\left(\max\{S_T - X, 0\} - \max\{S_T - Y, 0\}\right) .$$

Using the cap c in (7.5) we may rewrite this function as

$$g(S_T) = kN + N \max\left\{ \min\left\{1 + c - k, z\frac{S_T - X}{S_0}\right\}, 0\right\} , \tag{7.7}$$

compare with problem 7.4. If we now assume that $X = \gamma S_0$ we are able to generalise the payoff (7.7) to $d > 1$ underlyings. Indeed, if we replace the performance S_T/S_0 by the lowest performance $\min_i S_T^i/S_0^i$ of the involved underlyings and if the relative strikes $\gamma = \frac{X_i}{S_0^i}$ are all equal, we have

$$kN + N \max\left\{ \min\left\{1 + c - k, z\left(\min_i \frac{S_T^i}{S_0^i} - \gamma\right)\right\}, 0\right\} .$$

The price $V_{\mathrm{ccp},d}(\mathbf{s}, t)$ of a capital protection product with the above payoff is for $t \leq T$ given by

$$V_{\mathrm{ccp},d}(\mathbf{s}, t) = e^{-r(T-t)}kN + NV_{\mathrm{cwo},d}(\mathbf{s}, t) .$$

Herein, we denote by $V_{\mathrm{cwo},d}(\mathbf{s}, t)$ the value of a capped "worst-of"-option on d underlyings with payoff

$$g(\mathbf{s}) := \max\left\{ \min\left\{1 + c - k, z\left(\min_i \frac{s_i}{S_0^i} - \gamma\right)\right\}, 0\right\} ; \tag{7.8}$$

by the general principles of derivatives pricing discussed in chapter 3 the price of this option is given by the expectation

$$V_{\mathrm{cwo},d}(\mathbf{s}, t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}g(\mathbf{S}_T) \mid \mathbf{S}_t = \mathbf{s}\right] .$$

To find this expectation we again model the time evolution of the stock prices $\mathbf{S}_t$ as a d-dimensional geometric Brownian motion (2.22); for this model it seems that the above expectation can not be calculated analytically and one has to rely on numerical methods.

In example 7.4 we consider the case $d = 5$.

Example 7.4. We consider the product written on Nestlé $\to 1$, Novartis $\to 2$, Roche $\to 3$, Swiss Re $\to 4$ and Zurich $\to 5$ described in figure 7.4 and check the issue price $V_{\mathrm{cp},5} = 1000$ CHF.

The initial stock prices $\mathbf{S}_0 = (S_0^1, \ldots, S_0^5)$ are from the term sheet; to estimate the dividend yields $q_i = \ln(1 + D_i/S_0^i)$ we take the dividend payments made in 2019, thus $(D_1, D_2, D_3, D_4, D_5) = (2.45, 2.89, 8.7, 5.2, 19.33)$. The risk free is taken from the term sheet

UNDERLYING

Underlying	Related Exchange	Bloomberg Ticker	Initial Fixing Level (100%)*	Strike Level (80.00%)*
NESTLE SA-REG	SIX Swiss Exchange AG	NESN SW	CHF 100.56	CHF 80.45
NOVARTIS AG-REG	SIX Swiss Exchange AG	NOVN SW	CHF 89.21	CHF 71.37
ROCHE HOLDING AG-GENUSSSCHEIN	SIX Swiss Exchange AG	ROG SW	CHF 273.55	CHF 218.84
SWISS RE AG	SIX Swiss Exchange AG	SREN SW	CHF 98.48	CHF 78.78
ZURICH INSURANCE GROUP AG-REG	SIX Swiss Exchange AG	ZURN SW	CHF 339.40	CHF 271.52

PRODUCT DETAILS

Swiss Security Number	**47533836**
ISIN	**CH0475338361**
SIX Symbol	**JTQLTQ**
Issue Price	100.00% (plus Initial Load)
Initial Load	Up to 3.00% of the Denomination
Issue Size	CHF 10'000'000 (can be increased at any time)
Denomination	CHF 1'000
Settlement Currency	CHF
Protection Level	80.00%
Participation	459.00%
Bondfloor at issuance	79.52% (implied Yield p.a.: 0.06%)

DATES

Subscription End Date	27/06/2019 14:00 CEST
Initial Fixing Date	27/06/2019
Issue Date	04/07/2019
First Exchange Trading Date	04/07/2019
Last Trading Day/Time	27/06/2029 / Exchange market close
Final Fixing Date	27/06/2029 (subject to Market Disruption Event provisions)
Redemption Date	04/07/2029 (subject to Settlement Disruption Event provisions)

REDEMPTION

The Investor is entitled to receive from the Issuer on the Redemption Date per Product:

Scenario 1	If the Final Fixing Level of the Underlying with the Worst Performance is at or below the respective Strike Level, the Investor will receive a Cash Settlement in the Settlement Currency according to the following formula: Denomination × Protection Level
Scenario 2	If the Final Fixing Level of the Underlying with the Worst Performance is above the respective Strike Level, the Investor will receive a Cash Settlement in the Settlement Currency according to the following formula: Denomination × (Protection Level + Participation × (Worst Performance - Strike Level of the Underlying with the Worst Performance / Initial Fixing Level of the Underlying with the Worst Performance))
Initial Fixing Level	Official close of the respective Underlying on the Initial Fixing Date on the Related Exchange, as determined by the Calculation Agent.
Final Fixing Level	Official close of the respective Underlying on the Final Fixing Date on the Related Exchange, as determined by the Calculation Agent.
Worst Performance	For each Underlying the performance is calculated by dividing its Final Fixing Level by the respective Initial Fixing Level. The Worst Performance corresponds to the lowest of all so calculated values, as determined by the Calculation Agent.

Figure 7.4: Part of the term sheet of a capital protection product on $d = 5$ underlyings.

("Bondfloor at issuance") as $79.52e^{rT} = 80 \Rightarrow r = \frac{1}{T}\ln(80/79.52)$. We take the implied volatilities

$$\boldsymbol{\sigma}^{\mathrm{i}} = \operatorname{diag}(0.1703, 0.1817, 0.1765, 0.1901, 0.1788)\ .$$

To estimate the correlation matrix, we do not take the historical correlations as they yield a wrong price. Instead, we apply the model (3.37) with $\boldsymbol{\rho}_d = \mathbf{I}$, $\boldsymbol{\rho}_u = \mathbf{1}$ and $\lambda = \rho^{\mathrm{i}}$, where ρ^{i} is the implied correlation of a (hypothetical) basket with the considered underlyings only. From chapter 3 we know that the implied correlation is given by

$$\rho^{\mathrm{i}} = \frac{\boldsymbol{\omega}^\top(\sigma_B^{\mathrm{i}}\mathbf{1}\sigma_B^{\mathrm{i}} - \boldsymbol{\sigma}^{\mathrm{i}}\boldsymbol{\sigma}^{\mathrm{i}})\boldsymbol{\omega}}{\boldsymbol{\omega}^\top(\boldsymbol{\sigma}^{\mathrm{i}}\mathbf{1}\boldsymbol{\sigma}^{\mathrm{i}} - \boldsymbol{\sigma}^{\mathrm{i}}\boldsymbol{\sigma}^{\mathrm{i}})\boldsymbol{\omega}}\ .$$

However, as options on a basket containing the considered underlyings only do not exist, we can neither calculate σ_B^{i} nor $\boldsymbol{\omega}$. But since the underlyings are members of the SMI and we have available all the corresponding data (the implied volatility of options on the SMI; the weights of the underlyings under consideration in the SMI) with respect to the SMI, we estimate/calculate ρ^{i} to be $\rho^{\mathrm{i}} \doteq 0.878$ (we skip details). The remaining data to define the payoff function in (7.8) is again from the term sheet (figure 7.4). In particular $c = \infty$ (i.e., there is no return cap), $k = 0.8$, $z = 4.59$ (!) as well as $\gamma = 0.8$.

```
In [2]:
T = yf((27,6,2019),[(27,6,2029)])
s0 = [100.56,89.21,273.55,98.48,339.4]; D = [2.45,2.89,8.7,5.2,19.33]; r = np.log(80/79.52)/T
q = np.log(1+np.reshape(np.asarray(D),[5,1])/np.reshape(np.asarray(s0),[5,1]))
sigma = np.diag([0.1703,0.1817,0.1765,0.1901,0.1788]); lam = 0.878;
rho = (1-lam)*np.eye(5)+lam*np.ones((5,5)); Sigma = sigma.dot(rho).dot(sigma)
N = 1000; c = np.inf; k = 0.8; z = 4.59; gamma = 0.8;
mu = r-q;

In [3]:
# input parameters (Monte Carlo simulation)
n = 10**5; v = np.zeros(n)

In [4]:
# Monte Carlo simulation
for j in range(0,n):
    S = gbm_d(s0,mu,Sigma,[],T,T)[0]
    v[j] = np.maximum(np.minimum(1+c-k,z*(np.min(S[-1,:]/S[0,:])-gamma)),0) # payoff worst-of-option

Vcwo = np.exp(-r*T)*np.mean(v)
V = N*(np.exp(-r*T)*k+Vcwo); display((V[0],Vcwo[0]))

(983.3629326117828, 0.18816293261178288)
```

Figure 7.5: We estimate the issue price of the product in example 7.4 to $V_{\mathrm{cp},d}(\mathbf{s},0) \approx 983.36$. In particular, $V_{\mathrm{cwo},5}(\mathbf{s},0) \approx 0.18816$ (this value has to be multiplied by $N = 1000$).

For the given data we find $V_{\mathrm{ccp},d}(\mathbf{s},0) \approx 983.36$; this is about 1.7% below the issue price. ◇

7.5 Capital protection product with Asian option component

An Asian option is a further example of a path-dependent option since its payoff does not only depend on the final fixing level S_T of the underlying, but also on values S_t before maturity, $t < T$. To be more precise the payoff of an Asian option is a function of an average A_T. We distinguish between the continuous (arithmetic) average

$$A_T^c := \frac{1}{T}\int_0^T S_\tau \mathrm{d}\tau \tag{7.9}$$

and the discrete (arithmetic) average

$$A_T^d := \frac{1}{J}\sum_{j=1}^{J} S_{t_j}\ , \tag{7.10}$$

with the $J \geq 1$ predefined observation dates

$$0 < t_1 < t_2 < \ldots < t_J \leq T\ .$$

Only the discrete average is realised in concrete derivates.[4] Besides the arithmetic average one also finds in the literature Asian options based on a geometric average. For example, the discrete geometric average over n observation dates is defined as $G_T^d := \sqrt[J]{S_{t_1}\cdot\ldots\cdot S_{t_J}}$; we will not consider such options since they are of more theoretical interest.[5]

Of course one has to distinguish furthermore between Asian put and call options. But since there exists also a put-call parity for Asian options one may consider the call only. Generally, the payoff function of an Asian call option can be written as[6]

[4] Often, the initial fixing level S_0 is included in the average, i.e., (with $t_0 = 0$)

$$A_T^d := \frac{1}{J+1}\sum_{j=0}^{J} S_{t_j}\ .$$

Additionally, usually there holds $t_J = T$.

[5] Interestingly, there are closed form solutions for Asian options relying on the geometric average, but not for options relying on the arithmetic average. This is due to the fact that in case of S_t being a geometric Brownian motion the distribution of G_T^d is known, whereas the distribution of A_T^d is not.

[6] For simplicity, we denote the average by A_T and do not distinguish between continuous and discrete average.

$$g(A_T, S_T) = \max\{A_T - X_1 S_T - X_2, 0\} \ . \tag{7.11}$$

If $X_1 = 0$ (the usual case), the option is a “fixed strike” Asian call option; if $X_2 = 0$ we have a “floating strike” Asian call.

As an example, we consider the capital protection product in figure 7.6.

ÖSTERREICH-INVESTMENT MIT KAPITALSCHUTZ

Auf den Punkt gebracht:

Mit dem Garantie-Zertifikat Wiener Walzer 3 von Raiffeisen Centrobank investieren Anleger in den ATX® und partizipieren am Laufzeitende (Mai 2017) zu 105 % an der positiven durchschnittlichen Wertentwicklung des Index – vor negativen Kursentwicklungen sind Anleger durch 100 % Kapitalgarantie zum Laufzeitende geschützt.

KEY FACTS

Emittent	Raiffeisen Centrobank AG
Garantiegeber*	Raiffeisen Centrobank AG
ISIN / WKN	AT0000A0PCG1 / RCE10B
Nominale	EUR 1.000,-
Ausgabepreis	100 % zzgl. 3 % Ausgabeaufschlag
Zeichnungsfrist	18.04. - 13.05.2011
Anfänglicher Bewertungstag	16.05.2011
Zeichnungs-Valuta	17.05.2011
Finaler Bewertungstag	12.05.2017
Fälligkeitstag	17.05.2017
Kapitalgarantie	100 % zum Laufzeitende
Partizipationsfaktor	105 %
Bewertung	jährlich
Bewertungstage	14.05.2012, 13.05.2013, 12.05.2014,12.05.2015, 12.05.2016, 12.05.2017
Rückzahlung	Zusätzlich zum Nominale von EUR 1.000,- werden 105 % der positiven durchschnittlichen Wertentwicklung des ATX® ausbezahlt (Tilgung am Fälligkeitstag durch Raiffeisen Centrobank*).
Börsennotiz	Wien, Stuttgart, Frankfurt
Kursinfo	RCB01 / www.rcb.at

* Raiffeisen Centrobank AG ist eine 100 %ige Tochter der Raiffeisen Bank International AG – Rating der RBI: www.rbinternational.com/ir/ratings

Der erfreuliche Start des ATX® in das Jahr 2011 war vom Überschreiten der 3.000er-Marke gekennzeichnet. Seither hat der Index – geprägt von den politischen Unruhen im arabischen Raum sowie dem Erdbeben in Japan und dessen Folgen – etwas nachgegeben, aber mittlerweile auf rund 2.900 Punkte aufgeholt. Die Vorzeichen für den österreichischen Aktienmarkt deuten auf eine weiterhin positive Entwicklung des ATX® hin: Die Prognose für ein robustes Wirtschaftswachstum und die soliden Rahmenbedingungen sollten ein Plus für den Finanzplatz Österreich sein.

Für Anleger, die überproportional an der positiven durchschnittlichen Wertentwicklung der 20 im ATX® enthaltenen „Blue Chips“ partizipieren wollen und gleichzeitig die Absicherung gegen Kurseinbrüche der österreichischen Aktien-Benchmark schätzen, stellt das **Garantie-Zertifikat Wiener Walzer 3** ein **sicherheitsorientiertes Anlageinstrument** dar. Die Laufzeit des Zertifikats beträgt sechs Jahre.

Funktionsweise

Am Anfänglichen Bewertungstag wird der **Startwert** des ATX® festgehalten (Schlusskurs). Einmal jährlich – an den jeweiligen jährlichen Bewertungstagen (2012-2017) – wird der Schlusskurs des Index mit dem Startwert verglichen und die **Indexentwicklung** berechnet (prozentuelle Entwicklung des Index vom Startwert bis zum Schlusskurs am jeweiligen Bewertungstag). Am Ende der Laufzeit wird der **arithmetische Durchschnitt der Wertentwicklung des ATX®** ermittelt:

BEISPIEL zur Feststellung der durchschnittlichen Wertentwicklung des Index

	INDEX-STAND	WERTENTWICKLUNG DES INDEX
Anfänglicher Bewertungstag	Startwert = 2.900	–
Bewertungstag 2012	3.306	+14 %
Bewertungstag 2013	2.668	-8 %
Bewertungstag 2014	3.625	+25 %
Bewertungstag 2015	3.625	+25 %
Bewertungstag 2016	3.074	+6 %
Bewertungstag 2017	4.060	+40 %
		positive durchschnittliche Wertentwicklung = +17 %

Die durchschnittliche Wertentwicklung des ATX® bildet die Grundlage für den Abrechnungspreis: Am Fälligkeitstag wird die positive durchschnittliche **Wertentwicklung des Index zu 105 % zusätzlich zur Kapitalgarantie von 100 % ausbezahlt**. Durch die Durchschnittskursberechnung können gegebenenfalls Kursrückgänge an anderen Bewertungstagen abgefedert werden.

Figure 7.6: Part of the term sheet of a capital protection product with an Asian option component.

According to the term sheet the product pays off at maturity

$$kN + zN \max\{\overline{R}_{\mathrm{u},0,T}, 0\}$$

where we again denote by k the capital protection, by z the participation and by $\overline{R}_{\mathrm{u},0,T}$ a discrete arithmetic average return of the underlying at maturity. This average is calculated from returns $R_{\mathrm{u},0,t_j}$ defined at certain observation dates $t_j \in [0, T]$, i.e.,

$$\overline{R}_{\mathrm{u},0,T} = \frac{1}{J}\sum_{j=1}^{J} R_{\mathrm{u},0,t_j} = \frac{1}{J}\sum_{j=1}^{J}\frac{S_{t_j} - S_0}{S_0} = \frac{1}{S_0}\left(\frac{1}{J}\sum_{j=1}^{J} S_{t_j} - S_0\right) .$$

Thus the product pays off at maturity

$$kN + \frac{zN}{S_0} \max\{A_T - S_0, 0\}$$

with A_T the (discrete, arithmetic) average as in (7.10). According to (7.11) the expression $\max\{A_T - S_0, 0\}$ is the payoff of a "fixed strike" Asian call with strike $X = S_0$. The time t-value of this capital protection product is thus given by

$$V_{\mathrm{cp}}(s,t) = e^{-r(T-t)} kN + \frac{zN}{S_0} V_{\mathrm{fs},c}(s,t)$$

(compare with (7.4)), where we denote by $V_{\mathrm{fs},c}(s,t)$ the value of a "fixed strike" Asian call.

```
In [2]: # input parameters (model, contract)
        s0 = [2792.62]; q = [0.0205]; r = np.log(1+0.0308)
        Sigma = [[0.222**2]]
        X = s0; N = 1000; k = 1; z = 1.05
        T = yf((16,5,2011),[(12,5,2017)])[0]
        ti = yf((16,5,2011),[(14,5,2012),(13,5,2013),(12,5,2014),(12,5,2015),(12,5,2016),(12,5,2017)]).tolist()
        mu = [r-x for x in q]; q = np.asarray(q)

In [8]: # input parameters (Monte Carlo simulation)
        n = 10**6; v = np.zeros(n)

In [9]: # Monte Carlo simulation
        for j in range(0,n):
            D = gbm_d(s0,mu,Sigma,ti,T,T)
            S = D[0][1:] # simulated stock prices, excluding s0
            AT = np.mean(S) # average price at maturity

            v[j] = np.maximum(AT-X,0) # payoff

        Vfsc = np.exp(-r*T)*np.mean(v)
        Vcp = N*(np.exp(-r*T)*k+z/np.asarray(s0)*Vfsc); display((Vcp[0],Vfsc))

        (974.8370513731028, 374.913304769027)
```

Figure 7.7: We estimate the issue price of the product in figure 7.6 to $V_{\mathrm{cp}}(s,0) \approx 974.84$; in particular $V_{\mathrm{fs},c}(s,0) \approx 374.91$. Notice the rather large agio; according to the term sheet, the investor has to pay Eur 1030.

Even for the simple model of Black and Scholes (geometric Brownian motion) there is no formula to calculate $V_{\mathrm{fs},c}(s,t)$ and (once more) one needs numerical methods. For the parameters given in the term sheet and the market values $\sigma = 22.2\%$, $r = \ln(1+0.0308)$, $q = \ln(1+0.0207)$ (values from Bloomberg) and $s = S_0 = 2792.62$ (ATX at 16/5/2011) a Monte Carlo simulation based on 10^6 paths yields the price $V_{\mathrm{fs},c}(s,0) \approx 374.91$, see figure 7.7. Observe that this value is (much) lower than the price $V_c(s,0) \doteq 587.40$ of the corresponding European call, compare also with figure 7.8. Hence, by considering the average price of the underlying at maturity rather than its final value gives the issuer the possibility to offer a higher capital protection and/or participation rate to the client.

We remark that the value $V_{\mathrm{fs},c}(s,0)$ of an Asian call decreases with increasing number J of observation dates t_j. However, the option price can not become arbitrarily small but is bounded from below by the value $V_{\mathrm{fs},c,\infty}(s,0)$. Here, we denote by $V_{\mathrm{fs},c,\infty}(s,0)$ the value of a "fixed strike" Asian call option which uses "infinitely many" observation points, i.e., if the average A_T is as in (7.9). For the capital protection product in figure 7.6 there holds $V_{\mathrm{fs},c,\infty}(s,0) \doteq 331.345$.

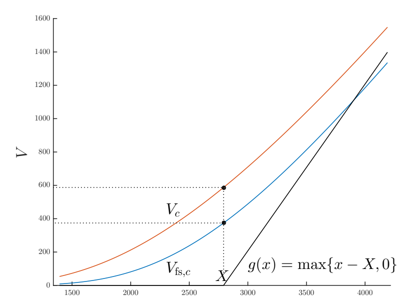

Figure 7.8: An Asian (fixed strike) call option is (much) cheaper than its European counterpart.

7.6 A return comparison

We are coming back to the capital protection products from sections 7.1 - 7.3 written on one underlying and consider the following investment. At time $0 \leq t < T$, we purchase the product at the price $V(s,t)$ (with $s = S_t$ the time t-price of the underlying) and hold it until maturity. For $t = 0$ the purchase price $V(s,0)$ is equal to the denomination N; we may express the purchase price as a multiple of N, i.e.,

$$V(s,t) = v_t N$$

for some $v_t \in \mathbb{R}^+$; note that $v_0 = 1$. Usually, the market states the value v_t multiplied by 100%. We ask which of the capital protection products - uncapped or capped - offers the better return over the holding period $[t,T]$. Clearly, for both products the return depends on the final fixing level S_T of the underlying and is given by

$$R_{t,T} := \frac{g(S_T)}{V(s,t)} - 1 \tag{7.12}$$

with the function

$$g = g_{\text{cp}}(x) = kN + \frac{zN}{S_0}\max\{x - X, 0\}$$

in case of the uncapped product and

$$g = g_{\text{ccp}}(x) = kN + \frac{zN}{S_0}\big(\max\{x - X_1, 0\} - \max\{x - X_2, 0\}\big)$$

for the product with cap, compare with figure 7.9.

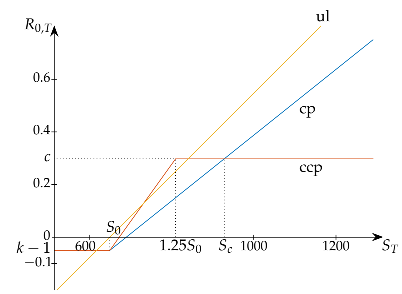

Figure 7.9: Return profile $R_{0,T} = g(S_T)/N - 1$ of the (fictitious) capital protection products of example 7.2 and 7.3, respectively.

As S_T is a random variable, so is $R_{t,T}$; to understand the return possibilities of a structured product we look at the distribution (either the probability density function (pdf) or the cumulative distribution function (cdf)) of this random variable. Obtaining the distribution of $R_{t,T}$ for arbitrary structured products is in general only possible through Monte Carlo simulation. However, since for the capital protection products under consideration the variable $R_{t,T}$ only depends on S_T and since the function g in (7.12) is piecewise affine linear, we are able to find the pdf f_R of $R_{t,T}$ in closed form. Before we do so, we introduce the quantities[7]

$$\alpha := \frac{k}{v_t} - 1, \quad \beta := \frac{c+1}{v_t} - 1 \tag{7.13}$$

for notational simplicity.

Consider first the uncapped product. If $S_T \leq X$, the payoff is $g(S_T) = kN$ and the return becomes $R_{\text{cp},t,T} = \frac{kN}{v_t N} - 1 = \alpha$. The probability that $R_{\text{cp},t,T} = \alpha$ is equal to $\mathbb{P}[S_T \leq X]$. If $S_T > X$, then $R_{\text{cp},t,T} > \alpha$, in particular

$$R_{\text{cp},t,T} = \frac{z}{v_t S_0}(S_T - X) + k - 1\,.$$

[7] The value α corresponds to the minimal return of a capital protection product (uncapped and capped), whereas β corresponds to the maximal return of a capped capital protection product.

Denote by f_S the pdf of the final fixing level S_T of the underlying (starting in $s = S_t$). Then, by problem 7.5, the density $f_{R_{\mathrm{cp}}}$ of the return of an uncapped capital product is

$$f_{R_{\mathrm{cp}}}(x) = \begin{cases} \mathbb{P}[S_T \leq X] & \text{if} \quad x = \alpha \\ \frac{v_t S_0}{z} f_S\big((v_t x + v_t - k)\frac{S_0}{z} + X\big) & \text{if} \quad \alpha < x \end{cases} . \tag{7.14}$$

Note that this distribution is an example of a so-called mixed distribution, compare with figure 7.10. A mixed distribution is a distribution which is both discrete and continuous, see also figure 5.7 for an another example.

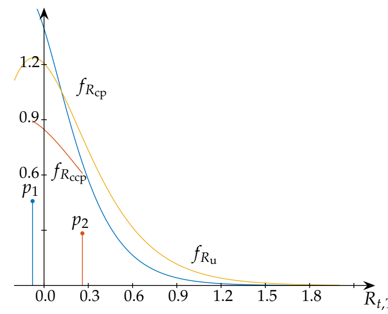

Figure 7.10: Probability density function f of the random variables $R_{\mathrm{cp},t,T}$ (—), the uncapped capital protection product) and $R_{\mathrm{ccp},t,T}$ (—), the capped product), respectively. We compare these with the pdf of the return of the direct investment into the underlying (—).

Now we turn to the capped product. Here, we additionally need to consider $S_T \geq X_2$. In this case the payoff is

$$g_{\mathrm{ccp}}(S_T) = kN + \frac{zN}{S_0}(X_2 - X_1) \overset{(7.5)}{=} (c+1)N ,$$

and thus $R_{\mathrm{ccp},t,T} = \frac{c+1}{v_t} - 1 = \beta$. The probability that $R_{\mathrm{ccp},t,T} = \beta$ is equal to $\mathbb{P}[S_T \geq X_2]$. Thus, the density $f_{R_{\mathrm{ccp}}}$ of the return of a capped capital product is

$$f_{R_{\mathrm{ccp}}}(x) = \begin{cases} \mathbb{P}[S_T \leq X_1] & \text{if} \quad x = \alpha \\ \frac{v_t S_0}{z} f_S\big((v_t x + v_t - k)\frac{S_0}{z} + X_1\big) & \text{if} \quad \alpha < x < \beta \\ \mathbb{P}[S_T \geq X_2] & \text{if} \quad x = \beta \end{cases} . \tag{7.15}$$

As it is simpler to model the log-returns (of the underlying) than the stock prices themselves, it is more convenient to use in (7.14) and (7.15) the pdf f_{R^c} of the log-returns $R^c_{t,T}$ rather than the pdf f_S of S_T, i.e., we make use of the relationship $S_T = S_t e^{R^c_{t,T}}$. Applying this, we may use other distributions than normal, for example we might take the student t distribution with pdf as in (3.15) to model the log-returns. An application of the hint in problem 7.5 reveals that

$$f_S(u) = \frac{1}{u} f_{R^c}\Big(\ln \frac{u}{S_t}\Big) . \tag{7.16}$$

This means that once the density f_{R^c} of the log-returns of the underlying is known, we can explicitly calculate the density of the holding period return of the capital protection products by first applying (7.16) and then using (7.14) or (7.15), depending on which type of product we are considering.

In figure 7.10 we plot the pdfs defined in equations (7.14)-(7.15) for the products in figure 7.9 in the case that the log-returns of the underlying $R^c_{t,T} \sim \mathcal{N}(\mu_S, \sigma_S^2)$ are normal with pdf given in (2.9)[8]. We compare these with the density $f_{R_{\mathrm{u}}}$ of the return $R_{\mathrm{u},t,T} = \frac{S_T}{S_t} - 1$ of a direct investment into the underlying. Furthermore, by p_1 and p_2 we denote the probabilities $\mathbb{P}[S_T \leq X]$ and $\mathbb{P}[S_T \geq X_2]$, respectively. For

[8] The parameter values are as in examples 7.2 and 7.3, respectively, and $t = 0$, $T = 3$. In particular, $S_0 = 640$, $v_0 = 1.03$, $X = X_1 = S_0$, $X_2 = 1.25S_0$, $k = 0.95$, and $z = 0.8$ for the uncapped product, $z = 1.3907$, $c = -0.05 + 0.25z$ for the capped product. Furthermore, the parameters of the geometric Brownian motion modelling S_T are $\mu = \ln(1.03)$ and $\sigma = 0.19$.

the given model they take the values $p_1 \doteq 0.4582$ and $p_2 \doteq 0.2833$, compare with problem 7.5.

If the pdf f_{R^c} of the log-returns is known we are able to calculate the return distributions (7.14)-(7.15) explicitly and hence have access to any measure of interest (expected value, quantiles (VaR), moments etc.) of these distributions without simulation. In particular, the expected return μ_{cp} and the variance $\sigma^2_{R_{\text{cp}}}$ of the uncapped product becomes

$$\begin{aligned} \mu_{\text{cp}} &= \mathbb{E}[R_{\text{cp},t,T}] = \int_\alpha^\infty x f_{R_{\text{cp}}}(x)\mathrm{d}x + p_1\alpha \\ \sigma^2_{R_{\text{cp}}} &= \mathbb{E}[(R_{\text{cp},t,T} - \mu_{\text{cp}})^2] \\ &= \int_\alpha^\infty x^2 f_{R_{\text{cp}}}(x)\mathrm{d}x + \alpha^2 p_1 - \mu^2_{\text{cp}} \,. \end{aligned} \tag{7.17}$$

Likewise, we have for the capped product

$$\mu_{\text{ccp}} = \mathbb{E}[R_{\text{ccp},t,T}] = \int_\alpha^\beta x f_{R_{\text{ccp}}}(x)\mathrm{d}x + p_1\alpha + p_2\beta \tag{7.18}$$

$$\begin{aligned} \sigma^2_{R_{\text{ccp}}} &= \mathbb{E}[(R_{\text{ccp},t,T} - \mu_{\text{ccp}})^2] \\ &= \int_\alpha^\beta x^2 f_{R_{\text{ccp}}}(x)\mathrm{d}x + \alpha^2 p_1 + \beta^2 p_2 - \mu^2_{\text{ccp}} \,. \end{aligned} \tag{7.19}$$

In case that the log-returns of the underlying are normal, all the above integrals can be calculated analytically. However, for more realistic distributions, these integrals can only be evaluated numerically. In Python this can be accomplished using `quad`[9]. We use the self-written Python function `hpr_cp` to calculate the expected return and the volatility (formulas (7.17)-(7.19)) of (capped or uncapped) capital protection products. As an example, consider the uncapped capital protection product from example 7.2. Using `hpr_cp` we then obtain $\mathbb{E}[R_{\text{cp},t,T}] \doteq 0.0684$ and $\sigma_{R_{\text{cp}}} \doteq 0.2213$, compare with figure 7.11.

[9] We will not dwell on numerical quadrature but remark that `quad` is a rather general "blackbox" to evaluate integrals $\int_a^b f(x)\mathrm{d}x$ for $-\infty \leq a < b \leq \infty$ numerically.

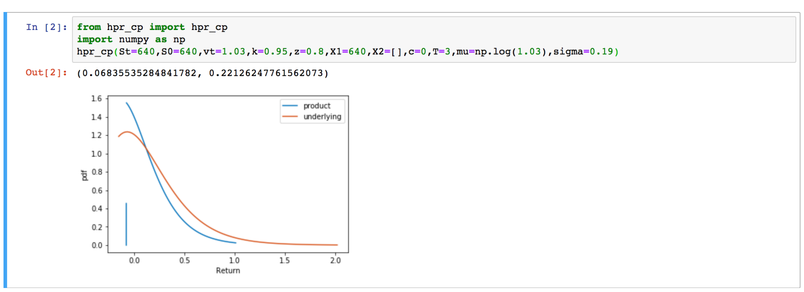

Figure 7.11: A numerical quadrature gives expected return and volatility of a capital protection product (here uncapped).

In table 7.3 we compress the distributions of $R_{\text{cp},t,T}$, $R_{\text{ccp},t,T}$ and $R_{\text{u},t,T}$ of examples 7.2 and 7.3. For the calculation of the median return compare with problem 7.5. We observe that whereas all three investments generate comparable returns, the capped capital protection product is (by far) the least risky investment. Note that the

investment	$\mathbb{E}[R_{t,T}]$	σ_R	median$_R$
underlying	9.27	36.96	3.51
uncapped capital protection	6.84	22.13	−5.04
capped capital protection	5.88	15.00	−3.02

Table 7.3: Risk and return of capital protection products compared to a direct investment into the underlying. All numbers are in %.

values of the return of the direct investment follow from

$$\begin{aligned}\mathbb{E}[R_{\mathrm{u},t,T}] &= e^{\mu(T-t)} - 1\\ \sigma^2_{R_\mathrm{u}} &= e^{2\mu(T-t)}(e^{\sigma^2(T-t)} - 1)\\ \mathrm{median}_{R_\mathrm{u}} &= e^{(\mu-\sigma^2/2)(T-t)} - 1\,.\end{aligned}$$

7.7 Problems

Problem 7.1. The issuer of the capital protection product in example 7.2 changes the parameters.

i) She increases the price at issuance to 105% but offers instead the participation of 112%. Find the corresponding capital protection.

ii) She increases the life time of the product to 4 years; for this time period the risk free is $r = \ln(1.0325)$. The remaining market parameters do not change. Additionally, the capital protection is decreased to 90% and the participation is increased to 100%. Find the price of the product at issuance.

Problem 7.2. Consider the capital protection product in figure 7.1. Market data as of August 11, 2016: value of SMI $s = 8296.14$, volatility $\sigma = 17.59\%$ (interpolated from the implied volatility surface, source Bloomberg), dividend yield $q = \ln(1 + 0.0572)$ (5.72% is the estimated discrete dividend yield, source Bloomberg) and risk free $r = \ln(1 - 0.00564)$ (-0.564% is the linearly interpolated value from the 6- and 7-year CHF swap rates).

i) Calculate the theoretical value of the product and compare with the market price $V^{\mathrm{M}}_{\mathrm{cp}} = 102.61\%$.

ii) On August 11, 2016, investor A invests into the SMI, investor B invests the same amount of money into the capital protection product (to the market price). For which final fixing level S_T the return difference between these investments is

ii$_1$) +5% in favour of investor A

ii$_2$) +5% in favour of investor B?

iii) Part i) shows that the assumption of a constant risk free is too simple. We therefore model the risk free as deterministic function

$r(t)$ of time. Certainly, the most simpel specification for $r(t)$ is a polynomial of degree one, i.e., we assume that the risk free is given by $r(t) = r_0(1 - \beta t)$ with $r_0 = \ln(1 - 0.00564)$ as in part i) and $\beta > 0$ to be found by calibration. If the interest rate is not constant (but still deterministic), then the pricing formula (7.4) changes to

$$V_{\text{cp}}(s,t) = e^{-\int_t^T r(s)\mathrm{d}s} kN + \frac{zN}{S_0} V_c(s,t) \ .$$

Additionally, in the Black-Scholes formula for calculating V_c the constant r has to be replaced by the average

$$\bar{r} := \frac{1}{T-t} \int_t^T r(s)\mathrm{d}s \ .$$

Find β such that the theoretical value is equal to the market price, i.e., $V_{\text{cp}}(s,0) = V_{\text{cp}}^{\text{M}}$. What is the risk free $r(T)$ at maturity according to this model?

Problem 7.3. An institutional client of the issuer of the product in example 7.3 expects an only moderate increase of the underlying in the next 3 years and thus wants to realise a maximal participation. In return the client is happy with a cap of 20%.

Find the strike X_2 and the participation of this product. (Again, take $\sigma_2 = 21\%$.)

Hint. To solve this problem you will need a solver.

Problem 7.4. Let $Y > X > 0$, $z, S_0 > 0$ and $c := k - 1 + \frac{z}{S_0}(Y - X)$. Show: for $s > 0$ the functions

$$f_1(s) := \frac{z}{S_0}\big(\max\{s - X, 0\} - \max\{s - Y, 0\}\big)$$

and

$$f_2(s) := \max\Big\{\min\Big\{1 + c - k, \frac{z}{S_0}(s - X)\Big\}, 0\Big\}$$

are equivalent.

Problem 7.5. i) Show the values for the probabilities p_1 and p_2 in figure 7.10.

Hint. These values have been already calculated numerically in the Python function `hpr_cp`. In this problem, you derive a closed form expression for these probabilities under the assumption that the log-return $R_{t,T}^c$ of the underlying is normal.

ii) Show equations (7.14) and (7.15).

Hint. Use the following fact. Let Y be a (continuous) random variable with pdf f_Y, and let be g a strictly increasing function with inverse $h := g^{-1}$. Then the pdf of the random variable $U := g(Y)$ is given by $f_U(u) = f_Y(h(u))h'(u)$, where h' is the derivative of h.

iii) We consider the cdf $F_R(x) = \mathbb{P}[R_{t,T} \le x]$ of a capital protection product. According to the definition, we have for the uncapped product

$$F_{R_{\mathrm{cp}}}(x) = \begin{cases} p_1 & \text{if} \quad x = \alpha \\ p_1 + \int_\alpha^x f_{R_{\mathrm{cp}}}(y)\mathrm{d}y & \text{if} \quad \alpha < x \end{cases}$$

whereas for the capped product there holds

$$F_{R_{\mathrm{ccp}}}(x) = \begin{cases} p_1 & \text{if} \quad x = \alpha \\ p_1 + \int_\alpha^x f_{R_{\mathrm{ccp}}}(y)\mathrm{d}y & \text{if} \quad \alpha < x < \beta \\ 1 & \text{if} \quad x = \beta \end{cases}$$

For both products use `quad` to generate the graph of the function $x \mapsto F_R(x)$.

iv) Using part iii), find for both products the median return as given in table 7.3.

Basiswerte

Basiswert	Index Sponsor	Bloomberg Ticker	Anfangslevel (100%)*	Ausübungspreis (90.00%)*
Swiss Market Index®	SIX Swiss Exchange AG	SMI	CHF 8359.51	CHF 7523.56

Produktdetails

Valorennummer	**22829495**
ISIN	**CH0228294952**
SIX Symbol	**NPABGY**
Ausgabepreis	100.00%
Emissionsvolumen	CHF 5'000'000 (mit Aufstockungsmöglichkeit)
Denomination	CHF 1'000
Auszahlungswährung	CHF
Kapitalschutz	90.00%
Partizipation	100.00%
Cap Level	126.00%
Bondfloor bei Ausgabe	87.70% (implizierter Zins p.a.: 0.52%)

Daten

Fixierung	11.03.2014
Liberierung	25.03.2014
Erster Börsenhandelstag	25.03.2014
Letzte/r Handelstag/-zeit	11.03.2019 / Börsenschluss
Verfall	11.03.2019 (vorbehältlich Anpassung bei Marktstörungen)
Rückzahlungsdatum	25.03.2019 (vorbehältlich Anpassung bei Abwicklungsstörungen)

Rückzahlung

Der Anleger erhält am Rückzahlungsdatum von der Emittentin pro Produkt:

Rückzahlungsszenario 1	Sofern der Endlevel auf oder unter dem Ausübungspreis liegt, erhält der Anleger eine Barauszahlung in der Auszahlungswährung, entsprechend folgender Formel: Denomination × Kapitalschutz
Rückzahlungsszenario 2	Sofern der Endlevel über dem Ausübungspreis jedoch auf oder unter dem Anfangslevel multipliziert mit dem Cap Level (in %) liegt, erhält der Anleger eine Barauszahlung in der Auszahlungswährung, entsprechend folgender Formel: Denomination × (Kapitalschutz + Partizipation × (Endlevel - Ausübungspreis) / Anfangslevel)
Rückzahlungsszenario 3	Sofern der Endlevel über dem Anfangslevel multipliziert mit dem Cap Level (in %) liegt, erhält der Anleger eine Barauszahlung in der Auszahlungswährung, entsprechend folgender Formel: Denomination × (Kapitalschutz + Partizipation × (Cap Level (in %) - Ausübungspreis (in %)))
Anfangslevel	Der offizielle Schlusskurs des Basiswertes bei Fixierung, welcher vom Index Sponsor berechnet und publiziert wird, festgelegt durch die Berechnungsstelle.
Endlevel	Der offizielle Schlusskurs des Basiswertes bei Verfall, welcher vom Index Sponsor berechnet und publiziert wird, festgelegt durch die Berechnungsstelle.

Figure 7.12: Part of the term sheet of a capital protection product with cap.

Problem 7.6. We consider the capped capital protection product defined in figure 7.12. Data as of April 25, 2017: level SMI 8761.14, market price of the product 103.87/104.62 in % of the denomination (bid/ask), time-to-maturity $T - t = \frac{676}{360}$.

i) Suppose that at maturity the SMI closes at 8600. For an investment at initial fixing, what is the return difference between the capital protection product and the SMI?

ii) On April 25, 2017, find the theoretical value of the product.

Hint. Use $\sigma_1 = 16.681\%$, $\sigma_2 = 12.250\%$ (interpolated volatilities from the volatility surface, source Bloomberg), $q = \ln(1 + 0.026)$ (2.60% is the estimated discrete dividend yield, source Bloomberg) and $r = \ln(1 - 0.00627)$ (-0.627% is the linearly extrapolated value from the 2- and 3-year CHF swap rates).

iii) On April 25, 2017, investor A purchases the capped capital protection product to the mid-price of bid/ask price and holds it until maturity. Investor B invests the same amount of money into the SMI and also holds the position until maturity of the product. Find the expected return of both investors.

Hint. A time series analysis of the log-returns of the SMI in the period 24/4/2015 to 24/4/2017 yields $\mu = -1.925\%$ and $\sigma = 16.43\%$.

iv) Find the volatility σ_R of the return of the investments in part iii).

Hint. Compare with (7.19) and with the Python function `hpr_cp`.

Chapter 8

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