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Cycles of Functions on Finite Types

We assume some notion of finite types which allows a proof of the pigeonhole principle:

Proposition 1. *If X is a finite type with $N : \mathbb{N}$ elements, then every for every list $x_0, \ldots, x_N$ of $N+1$ elements from X, there are $i \neq j$ with $x_i = x_j$.*

Using this, we can show that by iterating functions on finite types, we must at some point enter a cycle, and we can even give some more details on the cycle length and when they occur.

Definition 1. *We define iteration for functions $f : X \to X$ and a given starting value $x : X$ by:*

$$f^0 x := x \qquad f^{(n+1)} x := f(f^n x).$$

Lemma 1. *Let X be a type with $N > 0$ elements. Then for every $f : X \to X$ and $x : X$, there are $k < N$ and $1 < c \leq N$ such that $f^{c+k} x = f^k x$, meaning $f^k x$ is a cycle for f with length at most length c.*

Proof. Consider the list of $N+1$ values $f^0 x, f^1 x, \ldots, f^N x$ of type X. Since X has N elements, the pigeonhole principle tells us that at least two elements of the above list must coincide. Let this be $f^n x$ and $f^m x$ where $n < m \leq N$ without loss of generality. We then have

$$f^n x = f^m x = f^{(m-n)+n} x$$

and therefore $k := n$ and $c := m - n > 1$ as desired. $\square$

Note that both k and c depend on the specific f and x we choose at the start. The final result we want to show will state that there is a kind of *global cycle length*, which holds for all functions and for all starting values, and only depends on the number of elements of the finite type.

Fact 1. *Given $n, m : \mathbb{N}$ we have $f^{(n+m)} x = f^n (f^m x)$.*

Lemma 2. *Given a function $f : X \to X$ and $k, m : \mathbb{N}$ with $f^{k+m} x = f^m x$, if $m \leq n$ then for all $d : \mathbb{N}$ we have $f^{k \cdot d + n} x = f^n x$.*

Proof. We proceed by induction on d. The case $d = 0$ is trivial, and we additionally show the statement for $d = 1$. By assumption we have $n-m \geq 0$ and therefore:

$$\begin{aligned} f^{k+n}x &= f^{(n-m)+(k+m)}x = f^{(n-m)}(f^{(k+m)}x) \\ &= f^{(n-m)}(f^{m}x) = f^{(n-m)+m}x = f^{n}x \end{aligned}$$

showing the claim. For the inductive step, we have:

$$f^{k\cdot(d+1)+n}x = f^{k+(k\cdot d+n)}x = f^{k}(f^{k\cdot d+n}x) = f^{k}(f^{n}x) = f^{k+n}x = f^{n}x$$

where we made use of the induction hypothesis and the $d = 1$ case. □

Theorem 1. *Let X be finite with $N > 0$ elements, and $a : \mathbb{N}$ be divisible by all numbers $1, \ldots, N$. Then for every function $f : X \to X$ and $x : X$ we have $f^{a+(N-1)}x = f^{(N-1)}x$.*

Proof. Let $f : X \to X$ and $x : X$ be given. Then by Lemma 1 there are $k, c : \mathbb{N}$ with $1 < c \leq N$ and $f^{c+k}x = f^{k}x$. By assumption, a is divisible by c, so there is $d : \mathbb{N}$ with $a = c \cdot d$. Since $k \leq N - 1$ we can make use Lemma 2 to conclude that:

$$f^{a+(N-1)}x = f^{c\cdot d+(N-1)}x = f^{(N-1)}x$$

□

One number which satisfies the property of a in Theorem 1 is $N!$. The smallest number with this property is the least common multiple of the numbers $1, \ldots, N$. The theorem can be read as saying that not matter the starting value x of function $f : X \to X$; after $(N - 1)$ applications of f to x, we get to a cycle and it has at most length a.