Part1 The graph with fill rank adjacent matrix:

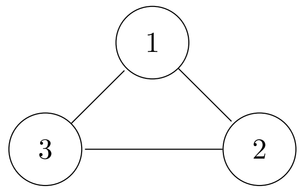

Figure 1: 3 node graph with full-rank adjacent matrix

And its adjacent matrix is

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

The graph with rank-deficient adjacent matrix: And its adjacent matrix

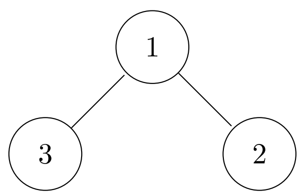

Figure 2: 3 node graph with rank-deficient adjacent matrix

is

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Part2 Define $G = (V, E)$ that $V = \{v_m : m \in \{0, 1, \cdots, n\}\}$ and $E = \{(v_0, v_m) : m \in \{0, 1, \cdots, n\}\}$.

Part3 For an arbitrary graph G, let A be G's adjacent matrix so $L = D - A$. We notice that for all natural number $m \in [1, n]$,

$$\sum_{i=1}^{n} l_{mi} = 0$$

then if we scale it with constant c,

$$\sum_{i=1}^{n} cl_{mi} = c0 = 0$$

still holds, which means

$$[l_{m1}, l_{m2}, \cdots, l_{mn}][c, c, \cdots, c]^T = 0$$

and as a result,

$$\mathbf{L}[c, c, \cdots, c]^T = \mathbf{0}$$

Hence, for all $c \in \mathbb{R}$, $\mathbf{v} = c[1, 1, \cdots, 1]$, $\mathbf{Lv} = \mathbf{0}$.

This means that the nullity $\text{Null}(\mathbf{L}) \geq 1$, so by Rank-nullity theorem $\text{Rank}(\mathbf{L}) \leq n - 1$, which means $\mathbf{L}$ is rank-deficient.

Part4 To prove that $\text{Rank}(\mathbf{L}) = n - 1$, we want to rule out the possibility that $\text{Rank}(\mathbf{L}) = n - 1$, which means we want to see that its impossible that $\text{Null}(\mathbf{L}) > 1$.

Suppose $\text{Null}(\mathbf{L}) > 1$, then $\mathbf{w}$ is another solution to equation $\mathbf{Lw} = \mathbf{0}$ that $w \notin \{c[1, 1, \cdots, 1] : c \in \mathbb{R}\}$, and $\text{Rank}([\mathbf{v}, \mathbf{w}]) = 2$, then $\mathbf{w} \neq c\mathbf{v}$ for all $c \in \mathbf{R}$.

Let an arbitrary vector $\mathbf{z} = \mathbf{w} - \alpha\mathbf{v}$, that the largest in $\mathbf{w}$ is annihilated. Then consider the structure of $\mathbf{L}$, there is always a diagonal element $\mathbf{L}_{ii}$ that $\mathbf{w}_i = 0$, so

$$\sum_{i=1}^{n} l_{mi} \leq 0$$

since elements in $-\mathbf{A}$ is nonpositive, the the equality holds only when $\mathbf{z} = \mathbf{0}$. But this is impossible since $\text{Rank}([\mathbf{v}, \mathbf{w}]) = 2$.

Hence, we conclude that $\text{Null}(\mathbf{L}) = 1$, and, by Rank-Nullity Theorem, $\text{Rank}(\mathbf{L}) = n - 1$.