Rice's Theorem. *Let $\mathcal{L}$ be any set of languages that satisfies the following conditions:*

- *There is a Turing machine Y such that $\textsc{Accept}(Y) \in \mathcal{L}$.*
- *There is a Turing machine N such that $\textsc{Accept}(N) \notin \mathcal{L}$.*

The language $\textsc{AcceptIn}(\mathcal{L}) := \big\{ \langle M \rangle \bigm| \textsc{Accept}(M) \in \mathcal{L} \big\}$ is undecidable.

Prove that the following languages are undecidable *using Rice's Theorem*:

1. $\textsc{AcceptRegular} := \big\{ \langle M \rangle \bigm| \textsc{Accept}(M) \text{ is regular} \big\}$
2. $\textsc{AcceptIllini} := \big\{ \langle M \rangle \bigm| M \text{ accepts the string } \texttt{ILLINI} \big\}$
3. $\textsc{AcceptPalindrome} := \big\{ \langle M \rangle \bigm| M \text{ accepts at least one palindrome} \big\}$
4. $\textsc{AcceptThree} := \big\{ \langle M \rangle \bigm| M \text{ accepts exactly three strings} \big\}$
5. $\textsc{AcceptUndecidable} := \big\{ \langle M \rangle \bigm| \textsc{Accept}(M) \text{ is undecidable } \big\}$

Solution: Undecidability proofs for all of these languages appear in the undecidability lecture notes. ■

To think about later. Which of the following are undecidable? How would you prove that?

1. $\text{ACCEPT}\{\{\varepsilon\}\} := \left\{\langle M\rangle \;\middle|\; M \text{ accepts only the string } \varepsilon\text{; that is, } \text{ACCEPT}(M) = \{\varepsilon\}\right\}$

   Solution: Undecidable by Rice's theorem. Let $\mathcal{L} = \{\{\varepsilon\}\}$ — the set containing one language, which contains one string, which is empty. Let $M_{\{\varepsilon\}}$ be a Turing machine with the transitions

   $$\delta(\text{start}, \square) = (\text{accept}, \cdot, \cdot)$$
   $$\delta(\text{start}, a) = (\text{reject}, \cdot, \cdot) \qquad \text{for all } a \in \Sigma.$$

   Clearly $\text{ACCEPT}(M_{\{\varepsilon\}}) = \{\varepsilon\} \in \mathcal{L}$. On the other hand, let M_{REJECT} be the Turing machine that always rejects its input; clearly $\text{ACCEPT}(M_{\text{REJECT}}) = \varnothing \notin \mathcal{L}$. ■

2. $\text{ACCEPT}\{\varnothing\} := \left\{\langle M\rangle \;\middle|\; M \text{ does not accept any strings; that is, } \text{ACCEPT}(M) = \varnothing\right\}$

   Solution: Undecidable by Rice's theorem. Let $\mathcal{L} = \{\varnothing\}$ — the set containing one language, which contains no strings. We immediately have $\text{ACCEPT}(M_{\text{REJECT}}) = \varnothing \in \mathcal{L}$ but $\text{ACCEPT}(M_{\{\varepsilon\}}) = \{\varepsilon\} \notin \mathcal{L}$. ■

3. $\text{ACCEPT}\varnothing := \left\{\langle M\rangle \;\middle|\; \text{ACCEPT}(M) \text{ is not an acceptable language}\right\}$

   Solution: Trivially decidable. For any Turing machine M the language $\text{ACCEPT}(M)$ is acceptable by definition. Thus, $\text{ACCEPT}\varnothing = \varnothing$ is correctly decided by the machine M_{REJECT}. ■

4. $\text{ACCEPT=REJECT} := \left\{\langle M\rangle \;\middle|\; \text{ACCEPT}(M) = \text{REJECT}(M)\right\}$

   Solution: Undecidable by definition-chasing. $\text{ACCEPT}(M) = \text{REJECT}(M)$ if and only if M diverges on every input string. Thus, $\text{ACCEPT=REJECT} = \text{NEVERHALT}$, which is proved undecidable in the notes. ■

5. $\text{ACCEPT}{\neq}\text{REJECT} := \left\{\langle M\rangle \;\middle|\; \text{ACCEPT}(M) \neq \text{REJECT}(M)\right\}$

   Solution: Undecidable by closure properties. $\text{ACCEPT}(M) \neq \text{REJECT}(M)$ if and only if M halts on at least input string. Thus, $\text{NeverHalt} = \text{TMENCODINGS} \setminus \text{ACCEPT}{\neq}\text{REJECT}$, where TMENCODINGS is the language of all Turing machine encodings. TMENCODINGS is decidable, but NEVERHALT is not. Thus, Corollary 3(d) in the undecidability notes implies that $\text{ACCEPT}{\neq}\text{REJECT}$ is undecidable. ■

6. $\text{ACCEPT}{\cup}\text{REJECT} := \left\{\langle M\rangle \;\middle|\; \text{ACCEPT}(M) \cup \text{REJECT}(M) = \Sigma^*\right\}$

   Solution: Undecidable by definition-chasing. $\text{ACCEPT}(M) \cup \text{REJECT}(M) = \Sigma^*$ if and only if M halts on every input string. Thus, $\text{ACCEPT}{\cup}\text{REJECT} = \text{NEVERDIVERGE}$, which is proved undecidable in the notes. ■