3.3.2 Lemma 1: (x_n) is bounded by 1.

Proof: Base case: When $n = 1$, $1 < x_1$ is given.

Inductive hypothesis: Suppose when $n = k$, $1 < x_k$ is true.

Inductive step: Then when $n = k + 1$, $x_{k+1} = 2 - 1/x_k$. By inductive hypothesis, we know that $x_k > 1 \Rightarrow 1/x_k < 1 \Rightarrow -1/x_k > -1$. As a result, $x_{k+1} = 2 - 1/x_k < 2 - 1 = 1$.

Hence, by mathematical induction, we have $x_n > 1$ for all $n \in \mathbb{N}$. As a result, (x_n) is bounded by 1.

■

Lemma 2: (x_n) is monotone, moreover, it is decreasing.

Proof: Base case: When $n = 1$, $x_2 = 2 - 1/x_1$, so $x2 - x1 = (2 - 1/x_1 - x_1) = 1 - (1/x_1 + x_1) = 2 - ((1 + x_1^2)/x_1)$. By AGM inequality, we know that $1 + x_1^2 \geq 2\sqrt{x_1^2 \cdot 1} = 2x_1 \Rightarrow (1 + x_1^2)/2x_1 \geq 2$. Since $x_1 > 1$, $(1 + x_1^2)/2x_1 > 2$. As a result, $x_2 - x_1 \leq 0 \Rightarrow x_1 > x_2$.

Inductive hypothesis: Suppose that when $n = k$, $x_{k+1} > x_k$.

Inductive step: Then when $n = k + 1$, $x_{k+2} = 2 - 1/x_{k+1}$. Then $x_{k+2} - x_{k+1} = 2 - (1 + 1/x_{k+1})$. By AGM inequality, we have $1 + x_{n+1}^2 \geq 2\sqrt{x_{k+1}} \Rightarrow (1 + x_{n+1}^2)/\sqrt{x_{k+1}} \geq 2$. Since $1 < x_n$ for all $n \in \mathbb{N}$. We have that $(1 + x_{n+1}^2)/\sqrt{x_{k+1}} > 2$. As a result, $x_{k+2} - x_{k+1} < 0$, so $x_{k+1} > x_{k+2}$.

So we conclude that $x_{n+1} < x_n$ for all $n \in \mathbb{N}$, which means (x_n) is decreasing.

■

Lemma 3: $\inf(x_n) = 1$.

Proof: First of all, since (x_n) is bounded by 1 below, then by Completeness Axiom, $\inf(x_n)$ exists.

Suppose $\inf(x_n) \neq 1$, then suppose $u = \inf(x_n)$. By definition, we know $u > 1$ and for all $n \in \mathbb{N}$, $u < x_n$. Then for all $\varepsilon > 0$, there is a $N \in \mathbb{N}$, that when $n \geq N$, $x_n < u + \varepsilon \Rightarrow x_n - \varepsilon < u$.

Then take $0 < \varepsilon < x_n - 1$, and $\varepsilon = x_n - x_k$ for some $k > n$. We have $x_n + \varepsilon = x_k \in (x_n)$, so $x_k < 1$. But by assumption, we know that $x_k = x_n - \varepsilon < u$, which contradicts with our corollary of assumption.

Hence, $\inf(x_n) = 1$.

■

Claim: $\lim_{x\to\infty}(x_n) = 1$.

Proof: By lemma 1, 2, 3 and Monotone Convergence Theorem, $\lim_{x\to\infty}(x_n) = \inf(x_n) = 1$.

■

3.3.3 Lemma 1: (x_n) is bounded by 2.

Proof: Base case: When $n = 1, x_1 \geq 2$ is given.

Inductive Hypothesis: When $n = k$, suppose $x_k \geq 2$.

Inductive step: Then when $n = k+1$, $x_{k+1} = \sqrt{x_k - 1}$. Since $x_k \geq 2$ by hypothesis, $x_k - 1 \geq 1 \Rightarrow \sqrt{x_k - 1} \geq 1 \Rightarrow x_{k+1} \geq 1 + \sqrt{x_k - 1} \geq 2$.

Hence, by mathematical induction, we have $x_n \geq 2$ for all $n \in \mathbb{N}$. As a result, (x_n) is bounded by 2.

■

Lemma 2: (x_n) is decreasing.

Proof: Base case: When $n = 1$, by definition, $x_2 = 1 + \sqrt{x_1 + 1}$, so $x_1 - x_2 = x_1 - (1 + \sqrt{x_1 + 1}) = (x_1 - 1) - \sqrt{x_1 - 1}$. Since $x_1 \geq 2 \Rightarrow x_1 - 1 \geq 1 \Rightarrow x_1 - 1 \geq \sqrt{x_1 - 1} \Rightarrow x_1 - x_2 = (x_1 - 1) - \sqrt{x_1 - 1} \leq 0 \Rightarrow x_1 > x_2$.

Inductive Hypothesis: When $x = k$, $x_{k+1} < x_k$.

Inductive step: When $x = k+1$, $x_{k+2} = 1 + \sqrt{x_{k+1} - 1}$, so $x_{k+1} - x_{k+2} = (x_{k+1} - 1) - \sqrt{x_{k+1} - 1}$. Since $x_{k+1} \geq 2$ by lemma 1, $x_{k+1} - 1 \geq 1 \Rightarrow x_{k+1} - 1 \geq \sqrt{x_{k+1} - 1} \Rightarrow x_{k+1} - 1 - \sqrt{x_{k+1} - 1} \geq 0 \Rightarrow x_{k+1} \geq x_{k+2}$.

Hence, by mathematical induction, we have $x_{n+1} > x_n$ for all $n \in \mathbb{N}$.

■

Lemma 3: $\inf(x_n) = 2$.

Proof: First of all, since (x_n) is bounded by 2 below, then by Completeness Axiom, $\inf(x_n)$ exists.

Suppose $\inf(x_n) \neq 2$, suppose $u = \inf(x_n)$, so by definition, $u > 2$, for all $n \in \mathbb{N}, u \leq x_n$. Then for all $\varepsilon > 0$, there is a $N \in \mathbb{N}$ that when $n \geq N, x_n < u + \varepsilon \Rightarrow x_n - \varepsilon < u$.

However, we can choose ε that $x_k = x_n - \varepsilon \geq 2$ so that $x_k \in (x_n)$ and $x_k < u$ which contradicts with our assumption.

As a result, $\inf(x_n) = 2$.

■

Claim: $\lim_{x\to\infty}(x_n) = 2$. By lemma 1, 2, 3 and Monotone Convergence Theorem, $\lim_{x\to\infty}(x_n) = \inf(x_n) = 2$.

■

3.3.8 **Proof:** Since $a_n \leq b_n$ for all $n \in \mathbb{N}$, we have $a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n \leq b_n \leq b_{n-1} \leq \cdots \leq b_2 \leq b_1$. Hence, for all $n \in \mathbb{N}$, b_n is an upper bound of (a_n). Since $(a_n), (b_n)$ are bounded, $\lim a_n, \lim b_n$ exist.

Suppose $\lim(b_n)$ is not an upper bound of (a_n), then there is a N_1 that when $n \geq N$, $a_n \geq \lim(b_n)$. However, by Monotone Convergence Theorem, $\lim(b_n) = \inf(b_n)$, , there is N_2 that when $n \geq N_2$, $b_n \leq \lim(b_n) + \varepsilon$ for all $\varepsilon \geq 0$.

Then let $n \geq \max\{N_1, N_2\}$, we have $a_n \leq b_n$ which contradicts with the condition given.

Hence, $\lim(b_n)$ is an upper bound of (a_n). Then by definition of supremum, $\sup(a_n) = \lim(a_n) \leq \lim(b_n)$.

■

3.4.2 Since $0 < c < 1$, note that if $z_n = c^{1/n}$, then $0 < z_n < 1$ and $z_{n+1} > z_n$, so by Monotone Convergence Theorem, $z = \lim z_n$ exists. As a result, $z = \lim z_{2n}$.

Since $z_{2n} = c^{1/2n} = (c^{1/n})^{1/2} = z^{1/2}$, we have that $z = \lim(z_{2n}) = z^{1/2} = z$. Therefore we conclude that $z = 1$ since $0 < z_n < 1$ and z_n is increasing.

Hence, if $0 < c < 1$, $\lim c^{1/n} = 1$.

■

3.4.4

(a) **Proof:** Let $x_n = 1 - (-1)^n + 1/n$.

If we take $b_n = 2n+1$, $(x_{b_n}) = 2+1/n$ and $\lim(x_{b_n}) = \lim 2 + \lim 1/n = 2$.

However, if we take $b_n = 2n$, subsequence $(x_{b_n}) = 1/n$ which converges to 0.

Hence, we conclude that (x_n) is divergent.

■

(b) **Proof:** Let $x_n = \sin n\pi/4$, x_n.

If we take $b_n = 8n$, then $(x_{b_n}) = \sin 2n\pi$ which converges to 0.

If we take $b_n = 4n$, then subsequence $(x_{b_n}) = \sin n\pi$ does not converge.

Hence, we conclude that (x_n) is divergent.

■

3.4.9 Proof: Suppose $\lim X \neq 0$, let $b_n = n$, the subsequence X of X does not converges to 0. As a result, all its subsequences do not converge to 0, which violates the condition given.

Hence, $\lim X = 0$.

■

3.4.11 Proof:

Since $\lim((-1)^n x_n)$ exists, take $b_n = 2n$, then $\lim x_{b_n}$, and if we take $b'_n = 2n+1$, then $\lim -x_{b'_n} = -\lim x_{b_n}$ exists and $\lim x_{b_n} = -\lim x_{b'_n} = \lim((-1)^n x_n)$, so $\lim x_{b'_n} = -\lim((-1)^n x_n) = \lim((-1)^(n+1) x_n) = \lim x_{b_n} = z$.

Then by definition, for all $\varepsilon > 0$, there is N_1 that when $n \geq N_1$, $|x_{2n} - z| < \varepsilon$ and there is N_2 that when $n \geq N_2$, $|x_{2n+1} - z| < \varepsilon$. Hence, when $n \geq \max\{N_1, N_2\}$, $|x_n - z| < \varepsilon$.

As a result, (x_n) converges.

3.4.14 Proof: Suppose there is no increasing subsequence in (x_n), then there is a $n_0 \in \mathbb{N}$ that $x_{n_0} \geq x_n$ for all $n \in \mathbb{N}$. By definition, x_{n_0} is an upper bound of (x_n). Suppose it is not the supremum, then there is $v \in (x_n)$ that $v < x_{n_0}$ which contradicts with our corollary. As a result $s = x_{n_0} \in (x_n)$.

Hence, there is at least an increasing subsequence in (x_n). And since (x_n) is bounded, this subsequence is also bounded. By Monotone Convergence Theorem, this increasing subsequence converges to s.

■

3.5.5 Proof: Since $x_n = \sqrt{n}$, $\lim |x_{n+1} - x_n| = \lim(\sqrt{n+1} - \sqrt{n}) = \lim(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})/(\sqrt{n+1} + \sqrt{n}) = \lim 1/(\sqrt{n+1} + \sqrt{n}) = 0$.

However, by Archimedean Property, (x_n) is unbounded, so that (x_n) does not converge, so it is not a Cauchy sequence.

■