6.1.2 Proof: Suppose $f(x) = x^{1/3}$ is differentiable at $x = 0$, then given $\varepsilon > 0$, there is $\delta > 0$ that when $0 < |x - 0| = |x| < \delta$, then

$$\left|\frac{f(x) - f(0)}{x - 0}\right| < \varepsilon$$

so

$$\left|\frac{x^{1/3}}{x}\right| = x^{-2/3} < \varepsilon$$

However, when $\varepsilon = 1$, for all $0 < x < 1$, whenever $0 < |x| < \delta$, there is $|x^{-2/3}| > 1 = \varepsilon$, which leads to a contradiction.

Hence, $f(x) = x^{1/3}$ is not differentiable at $x = 0$.

$\blacksquare$

6.1.4 Proof: Given that $\varepsilon > 0$, if $\delta = \varepsilon$, then when $0 < |x - 0| < \delta = \varepsilon$, we have

$$\begin{aligned}\left|\frac{f(x) - f(0)}{x - 0}\right| &= \left|\frac{f(x)}{x}\right| \\ &= \left|\frac{f(x)}{x}\right| \\ &= \begin{cases} |x| & \text{if } x \text{ is rational} \\ 0 & \text{otherwise} \end{cases} \\ &< \delta = \varepsilon \end{aligned}$$

Hence, by definition f is differentiable at $x = 0$ and $f'(0) = 0$.

$\blacksquare$

6.1.7 Proof:

"⇒": Since $g(x)$ is differentiable, then by definition, for all $\varepsilon > 0$, if we pick $\delta > 0$ that when $0 < |x - c| < \delta$, then

$$\begin{aligned}\left|\frac{g(x)-g(c)}{x-c} - L\right| &= \left|\frac{|f(x)|-|f(c)|}{x-c} - L\right| \\ &= \left|\frac{|f(x)|}{x-c} - L\right| \le \left|\frac{|f(x)|}{x-c}\right| + |L| \\ &= \left|\frac{f(x)}{x-c}\right| + |L| \\ &< \varepsilon + |L|\end{aligned}$$

Then

$$\left|\frac{f(x)-f(c)}{x-c} - 0\right| < \varepsilon$$

Hence, by definition, $g(x) = |f(x)|$ is differentiable at $x = c \Rightarrow f(x)$ is differentiable and $f'(c) = 0$.

"⇐": Since $f'(c) = 0$, then for all $\varepsilon > 0$ if we pick δ that when $x \in (c - \delta, c + \delta)$, there is

$$\begin{aligned}&\left|\frac{f(x)-f(c)}{x-c} - 0\right| = \left|\frac{f(x)}{x-c}\right| < \varepsilon \\ &\Rightarrow |f(x)| < \varepsilon|x-c| < \varepsilon\delta - |L|\end{aligned}$$

As a result, for any $L \geq 0$,

$$\begin{aligned}\left|\frac{g(x)-g(c)}{x-c} - L\right| &= \left|\frac{|f(x)|}{x-c} - L\right| \\ &\le \frac{|f(x)|}{|x-c|} + |L| \\ &< \varepsilon\end{aligned}$$

Hence by definition, $f(x)$ is differentiable at c and $f'(c) = 0 \Rightarrow g(x) = |f(x)|$ is differentiable at $x = c$.

6.2.3

(a) Since $f(x) = x + 1/x$, $f'(x) = 1 - 1/x^2$, so that let $f'(x) = 0 \Rightarrow x = 1$ or $x = -1$, which are local extremums. Since $f(0.5) = 2.5 > 2 = f(1)$ and $f(2) = 2.5 > 2 = f(1)$, $f(-0.5) = -2.5 < -2 = f(-1)$ and $f(-2) = -2.5 < -2 = f(-1)$. We know that relative maximum is $(-1, -2)$ and relative minimum is $(1, 2)$ and the increasing intervals are $(-\infty, -1] \cup [1, \infty)$ and the decreasing intervals are $(-1, 0) \cup (0, 1)$.

(b) Since $g(x) = x/(x^2 + 1$, $g'(x) = (1 - x^2)/(x^2 + 1)^2$, so that let $f'(x) = 0 \Rightarrow x = 1$ or $x = -1$, which are relative extremums. Since $g(0.5) = 0.4 < 0.5 = g(1)$ and $g(2) = 0.4 < 0.5 = g(1)$, $g(-0.5) = -0.5 < -0.4 = g(-1)$ and $g(-2) = -0.5 < -0.4 = g(-1)$. We know that relative maximum is $(1, 0.5)$ and relative minimum is $(-1, -0.5)$ and the decreasing intervals are $(-\infty, -1] \cup [1, \infty)$ and the increasing intervals are $(-1, 1)$.

(c) Since $h(x) = x^3 - 3x - 4$, $h'(x) = 3x^2 - 3$, so that let $h'(x) = 0 \Rightarrow x = 1$ or $x = -1$, which are relative extremums. Since $h(0) = -4 > -6 = h(1)$ and $h(2) = -2 > -6 = h(1)$, $h(0) = -4 < -2 = h(-1)$ and $h(-2) = -6 < -2 = h(-1)$. We know that relative maximum is $(-1, -2)$ and relative minimum is $(1, -6)$ and the increasing intervals are $(-\infty, -1] \cup [1, \infty)$ and the decreasing intervals are $(-1, 1)$.

(d) Since $k(x) = 2x + 1/x^2$, $g'(x) = 2 - 2/x^3$, so that let $k'(x) = 0 \Rightarrow x = 1$, which is relative extremums. Since $f(0.5) = 5 > 3 = f(1)$ and $f(2) = 4.25 > 3 = f(1)$, $f(-0.5) = -0.75 > -1 = f(-1)$. We know that relative minimum is $(1, 3)$ and the increasing intervals are $(-\infty, 0) \cup [1, \infty)$ and the decreasing intervals are $(0, 1)$.