1. (a) Let $p(x) = ax^3 + bx^2 + cx + d$, then

$$p(0) = 0 \Rightarrow d = 0$$

$$p(1) = 1 \Rightarrow a + b + c + d = 1 \Rightarrow a + b + c = 1$$
$$p'(0) = 0 \Rightarrow c = 0 \Rightarrow a + b = 1$$

Thus, we have that $p(x) = ax^3 + (1 - a)x^2$. For a more specific answer, we can let $a = 1$, so that $p(x) = x^3$.

(b) Let $r(x) = ax^2 + bx + c$, then

$$r(0) = c = 0$$

$$r(2) = 4a + 2b + c = 0 \Rightarrow 2a + b = 0$$
$$r'(1) = 2a + b = 0$$
$$\int_1^3 r''(x)dx = r'(3) - r'(1) = 2a(3 - 1) = 1 \Rightarrow 4a = 1 \Rightarrow a = 1/4$$

So

$$b = -2a = -1/2$$

As a result,

$$r(x) = \frac{1}{4}x^2 - \frac{1}{2}x$$

(c) Let $q(x) = ax + b$, then

$$q(0) = b = 1$$

$$q'(0) = a = 1$$

$$\int_{-1}^1 q(x)dx = 1 \Rightarrow \frac{a}{2}(1^2 - (-1)^2) + b(1 - (-1)) = 2b = 1 \Rightarrow b = \frac{1}{2}$$

Since b can't be 1 and 1/2 at the same time, so such $q(x)$ does not exist.

2. Let

$$f(x) = \sum_{i=1}^{n+1} p_i x^{n-1}$$

$$g(x) = \sum_{i=1}^{n+1} q_i x^{n-1}$$

$$h(x) = \sum_{i=1}^{n+1} m_i x^{n-1}$$

$$V_a = \begin{bmatrix} 1 & a_1 & \cdots & a_1^n \\ 1 & a_2 & \cdots & a_2^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & a_{n+1} & \cdots & a_{n+1}^n \end{bmatrix}$$

$$V_b = \begin{bmatrix} 1 & b_1 & \cdots & b_1^n \\ 1 & b_2 & \cdots & b_2^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & b_{n+1} & \cdots & b_{n+1}^n \end{bmatrix}$$

then we have

$$V_1 \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_{n+1} \end{bmatrix} = \begin{bmatrix} f(a_1) \\ f(a_2) \\ \vdots \\ f(a_{n+1}) \end{bmatrix}$$

$$V_2 \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_{n+1} \end{bmatrix} = \begin{bmatrix} g(b_1) \\ g(b_2) \\ \vdots \\ g(b_{n+1}) \end{bmatrix}$$

Hence,

$$\begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_{n+1} \end{bmatrix} = V_2^{-1} \begin{bmatrix} g(b_1) \\ g(b_2) \\ \vdots \\ g(b_{n+1}) \end{bmatrix}$$

Then

$$V_1 V_2^{-1} \begin{bmatrix} g(b_1) \\ g(b_2) \\ \vdots \\ g(b_{n+1}) \end{bmatrix} = \begin{bmatrix} g(a_1) \\ g(a_2) \\ \vdots \\ g(a_{n+1}) \end{bmatrix}$$

Since $f(x) = g(x) \cdot h(x)$, we have

$$\begin{bmatrix} f(a_1) \\ f(a_2) \\ \vdots \\ f(a_{n+1}) \end{bmatrix} = \left(V_1 V_2^{-1} \begin{bmatrix} g(b_1) \\ g(b_2) \\ \vdots \\ g(b_{n+1}) \end{bmatrix} \right)^T \begin{bmatrix} h(a_1) & 0 & \cdots & 0 \\ 0 & h(a_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & h(a_{n+1}) \end{bmatrix}$$

So

$$\begin{bmatrix} h(a_1) & 0 & \cdots & 0 \\ 0 & h(a_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & h(a_{n+1}) \end{bmatrix} = \left(\left(V_1 V_2^{-1} \begin{bmatrix} g(b_1) \\ g(b_2) \\ \vdots \\ g(b_{n+1}) \end{bmatrix} \right)^T \right)^{-1} \begin{bmatrix} f(a_1) \\ f(a_2) \\ \vdots \\ f(a_{n+1}) \end{bmatrix}$$

And

$$\begin{bmatrix} h(a_1) \\ h(a_2) \\ \vdots \\ h(a_{n+1}) \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} \left(\left(V_1 V_2^{-1} \begin{bmatrix} g(b_1) \\ g(b_2) \\ \vdots \\ g(b_{n+1}) \end{bmatrix} \right)^T \right)^{-1} \begin{bmatrix} f(a_1) \\ f(a_2) \\ \vdots \\ f(a_{n+1}) \end{bmatrix}$$

As a result, the coefficients are

$$\begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_{n+1} \end{bmatrix} = V_1 \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} \left(\left(V_1 V_2^{-1} \begin{bmatrix} g(b_1) \\ g(b_2) \\ \vdots \\ g(b_{n+1}) \end{bmatrix} \right)^T \right)^{-1} \begin{bmatrix} f(a_1) \\ f(a_2) \\ \vdots \\ f(a_{n+1}) \end{bmatrix}$$