CS/ECE 374 Spring 2017 Lanxiao Bai (lbai5@illinois.edu)
Homework 0 Problem 2

The ***plus one***, w^+, of a string $w \in \{0, 1, 2\}^*$ is obtained from w by replacing each symbol a in w by the symbol corresponding $a + 1 \mod 3$. for example, $0102101^+ = 1210212$. The plus one function is formally defined as follows:

$$w^+ := \begin{cases} \varepsilon & \text{if } w = \varepsilon \\ 1 \cdot x^+ & \text{if } w = 0x \\ 2 \cdot x^+ & \text{if } w = 1x \\ 0 \cdot x^+ & \text{if } w = 2x \end{cases}$$

(b) Prove by induction that $(x \bullet y)^+ = x^+ \bullet y^+$ for all strings $x, y \in \{0, 1, 2\}^*$.

Your proofs must be formal and self-contained, and they must invoke the *formal* definitions of length $|w|$, concatenation $x \bullet y$, and plus one w^+. Do not appeal to intuition!

Solution: (b) Proof:

Apply induction on the length of x.

Base case: When $|x| = 0$, $(x \bullet y)^+ = (\varepsilon \bullet y)^+ = y^+ = \varepsilon \bullet y^+ = \varepsilon^+ \bullet y^+ = x^+ \bullet y^+$

Suppose for all $|x| \le k$, we have $(x \bullet y)^+ = x^+ \bullet y^+$. Then when $|x| = k + 1$, there are following 3 cases.

If $x = 0v$ with $|v| = k$, $(x \bullet y)^+ = (0v \bullet y)^+ = 1 \bullet (v \bullet y)^+ = 1 \bullet v^+ \bullet y^+ = (0v)^+ \bullet y^+ = x^+ \bullet y^+$.

If $x = 1v$ with $|v| = k$, $(x \bullet y)^+ = (1v \bullet y)^+ = 2 \bullet (v \bullet y)^+ = 2 \bullet v^+ \bullet y^+ = (1v)^+ \bullet y^+ = x^+ \bullet y^+$.

If $x = 2v$ with $|v| = k$, $(x \bullet y)^+ = (2v \bullet y)^+ = 0 \bullet (v \bullet y)^+ = 0 \bullet v^+ \bullet y^+ = (2v)^+ \bullet y^+ = x^+ \bullet y^+$.

So as a result, it is proved that $(x \bullet y)^+ = x^+ \bullet y^+$ for all strings $x, y \in \{0, 1, 2\}^*$.

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