2.4.2 Solution:

Claim: $\sup S = 1$ and $\inf S = -1$.

Proof: Since $n, m \in \mathbb{N}$, $n \geq 1, m \geq 1 \Rightarrow 0 < 1/n \leq 1$ and $0 < 1/m \leq 1 \Rightarrow -1 \leq -1/m < 0$, thus for all $n, m \in \mathbb{N}$, we have $-1 < 1/n - 1/m < 1$. As a result, 1 is a upper bound of S and -1 is a lower bound of S.

Suppose 1 is not the supremum, then exists s that for all $x \in S$ that $x \leq s < 1$. Let $s = 1 - s'$ where $s' > 0$. Then when $n = 1$, $\forall s' > 0$, we can always find m that $0 < 1/m < s'$, so there exists n, m that make $x > s$, which means s is not an upper bound at all. As a result, $\sup S = 1$.

Similarly, suppose -1 is not the infimum, then exists i that for all $x \in S$ that $-1 < i \leq x$. Let $s = -1 + s' = s' - 1$, where $s' > 0$. When $m = 1$, $\forall s' > 0$ we can always find $0 < 1/n < s'$ that make $x < i$. So i is not a lower bound. As a result, $\inf S = -1$.

In conclusion, $\sup S = 1$ and $\inf S = -1$.$\blacksquare$

2.4.3 Proof: Suppose u is not an upper bound, then there is $x \in S$ that $x > u$, so $x - u > 0$. Hence, there is $n_0 \in \mathbb{N}$ that $0 < 1/n_0 < x - u \Rightarrow x > u + 1/n_0$, which contradicts with the condition given. Thus, u is an upper bound of S, namely $\forall x \in S, x \leq u$.

Suppose u is not the supremum, then there is $x \leq v < u$ for all $x \in S$. Then $\forall n \in \mathbb{N}$, there is $x \in S$ that $v - 1/n < x - 1/n \leq u - 1/n < x$. Since for all $n \in \mathbb{N}$, $u - 1/n$ is not an upper bound, so v can't be an upper bound.

As a result, $u = \sup S$.$\blacksquare$

2.4.9 Solution:

(a) $\sup\{h(x, y) : y \in Y\} = 2x + 1$ and $\inf\{f(x) : x \in X\} = 1$.

(b) $\inf\{h(x, y) : x \in X\} = 2x$ and $\sup\{g(y) : y \in Y\} = 2$.

2.4.10 Solution:

(a) $\sup\{h(x, y) : y \in Y\} = 1$ and $\inf\{f(x) : x \in X\} = 1$.

(b) $\inf\{h(x, y) : x \in X\} = 0$ and $\sup\{g(y) : y \in Y\} = 0$.

2.5.3 Proof: $\forall x \in S$, we have $\inf S \leq x \leq \sup S \in I_S$, so $S \subseteq I_S$.

$\forall x \in I_S$, $\inf S \leq x \leq \sup S$. Since $S \subseteq J$, $J = [a, b]$ in which a is a lower bound of S and b us an upper bound of S. So we have $a \leq \inf S \leq x \leq \sup S \leq b$ by Completeness Axiom for all $x \in J$. As a result, $x \in J$ for all $s \in I_S$.

As a result, $I_S \subseteq J$.■

2.5.7 Proof: Since for all $n \in \mathbb{N}$, $0 \in [0, 1/n]$, so $0 \in \cap_{n=1}^{\infty} I_n$. Suppose there is a nonzero number $\epsilon \in \cap_{n=1}^{\infty} I_n$, so that $\forall n \in \mathbb{N}$, $\epsilon \leq 1/n$. Since for all $t > 0$, there is always n_t that $0 \leq 1/n_t < t$, so there is a $n_\epsilon < \epsilon$ which contradicts with the assumption. Hence, for all $n \in \mathbb{N}$, $n \notin \cap_{n=1}^{\infty} I_n$.

As a result, $\cap_{n=1}^{\infty} I_n = \{0\}$.■

2.5.9 Proof: Suppose there is a real number $t \in \cap_{n=1}^{\infty} K_n$, then for all $n \in \mathbb{N}$, $t \in K_n \Rightarrow t > n$. But by Archimedean Property, there is an $n_t \in \mathbb{N}$ that $n_t > n$, which contradicts with our assumption's corollary.

As a result, for all $x \in \mathbb{R}$, $x \notin \cap_{n=1}^{\infty} K_n$, so we have $\cap_{n=1}^{\infty} K_n = \emptyset$.■