

☞ Homework 10 Question 2 ☞

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- Question 2

BALLOON is a problem that given a graph  $G$  and integer  $k$ , decide whether there is a subgraph in  $G$  which is a balloon of  $k$ . balloon of  $k$  is a graph with a directed path of length  $k + 1$  and the endpoint of the directed path is a part of directed cycle of length  $k$ .

To show BALLOON is NP-complete, we need to show BALLOON is in NP and is NP-hard.

To show BALLOON is in NP, given graph  $G$  and integer  $k$ , we assume there is a certificate that is  $\{v_1, v_2, \dots, v_{2k}\} \subset V(G)$ , where we check  $\{v_1, v_2, \dots, v_k, v_{k+1}\}$  has a directed path from  $v_1$  to  $v_{k+1}$  which is the "tail" of balloon, and check  $\{v_{k+1}, v_{k+2}, \dots, v_{2k}\}$  has a directed cycle. As checking directed path and directed cycle with given vertices need only linear time, BALLOON is in NP.

To show that BALLOON is NP-hard, we try to reduce from directed Hamiltonian Cycle. Given a graph  $G$ , we build graph  $G'$  by first constructing a directed path  $P$  with  $|V(G)|$  vertices which those vertices not in  $V(G)$  and adding a directed edge from the end of  $P$  to an arbitrary vertex in  $V(G)$ . Then, finding whether  $G$  has a directed Hamiltonian Cycle is equivalent to finding whether  $G'$  has a balloon of  $|V(G)|$ .

To be clear, if  $G$  has directed Hamiltonian Cycle, it has directed cycle of length  $|V(G)|$ . As that cycle is directed Hamiltonian Cycle, the directed path (the "tail") must linked to one of the its vertices. The graph  $G'$  has a balloon of  $|V(G)|$ .

If  $G$  has no directed Hamiltonian Cycle, it has no directed cycle of length  $|V(G)|$ . As cycle of length  $|V(G)|$  do not exist in  $G$ , the graph  $G'$  has no balloon of  $|V(G)|$ .

As we proved that directed Hamiltonian Cycle  $<_p$  BALLOON, and we know that SAT is NP-Complete and SAT  $<_p$  3-SAT  $<_p$  directed Hamiltonian Cycle, we get SAT  $<_p$  3-SAT  $<_p$  directed Hamiltonian Cycle  $<_p$  BALLOON and thus BALLOON is NP-hard.

As BALLOON is in NP and is NP-hard, BALLOON is NP-complete.