Math 461 Probability Final from 2012

(1) (15 pts) A cruise ship has 10 Americans and 12 Brazilians as passengers. For dinner, a group of 8 passengers is selected at random to dine with the captain. Find the probability that at least one American and at least one Brazilian is selected.

(2) (15=5+5+5 pts)

(a) Choose n cards randomly without replacement from a standard deck of 52 cards. What is the probability that the cards contain no aces?

(b) If, instead you choose n cards with replacement (randomly choose a card, note what it is in a list, replace it in the deck, and repeat), then what is the probability that the list of cards contains no aces?

(c) For a certain unfair die, suppose that the probability of rolling the number n is $n/21$, for $n = 1, 2, 3, 4, 5, 6$. Now if you roll the die and then choose n cards from the deck with replacement, what is the probability that the list of chosen cards contains no aces?

(3) (15 points) Compute the probability that a hand of 13 cards (drawn randomly from a standard deck of 52) contains both the ace and the king from at least one suit. (You may leave your answer unsimplified.)

(4) (15=5+10 pts)

(a) What is the definition of conditional probability? And how does this relate to the definition of independence for events?

(b) If a 5-card poker hand contains at least 2 aces, then what is the probability that it is a full house (meaning it has the form xxyyy)?

(5) (15 pts) A blood test is 99% effective in detecting a disease given that the disease is actually present. The test also has a 2% chance of being positive when the disease is not actually present. If 0.5% of the population actually has the disease, then what is the probability that a person actually has the disease given that their test result is positive?

(6) (15=5+5+5 pts) The joint probability mass function of X and Y is given by: $p(1, 1) = \frac{1}{8}$, $p(1, 2) = \frac{1}{4}$, $p(2, 1) = \frac{1}{8}$, $p(2, 2) = \frac{1}{2}$.

(a) What is the conditional mass function of X given $Y = i, i = 1, 2$?

(b) Are X and Y independent? Why or why not?

(c) What is $\mathbb{P}(XY \leq 3)$?

(7) (20pts) A certain baseball player has a batting average p=.2 (meaning that the chance of a hit is .2 in each at-bat). Show that the probability of the player getting fewer than 200 hits in his next 900 at-bats is *approximately* .95.

Hint: $18/12 = 1/5$ and $0.5/12 \simeq 0.04$.

Be sure to briefly describe your assumptions and justify which approximation you apply.

(8) (20=5+5+5+5 pts) Let X be a random variable with cumulative distribution function (cdf) F_X given by

$$F_X(x) = \begin{cases} 0, & x < 0, \\ \frac{x}{4}, & 0 \le x < 1, \\ \frac{1}{2} + \frac{x-1}{4}, & 1 \le x < 2, \\ \frac{11}{12}, & 2 < x < 3, \\ 1, & 3 \le x. \end{cases}$$

(a) Note that I haven't specified the value of F_X at $t = 2$. What must $F_X(2)$ be?

(b) Compute $\mathbb{P}(X < 3)$,

(c) Compute $\mathbb{P}(1 < X < 2)$,

(d) Compute $\mathbb{P}(1 < X \le t | 1 < X < 2)$, for $1 < t < 2$.

(9) (15=5+5+5pts) The joint density of X and Y is given by

$$f(x, y) = \begin{cases} x + y, & 0 < x < 1, 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Are X and Y independent? Why or why not?

(b) What is the density function of X?

(c) What is $\mathbb{P}(X + Y < 1)$?

(10) (15pts) Consider n independent flips of a coin having probability p of landing heads. Say that a changeover occurs whenever an outcome differs from the one preceding it. For instance, if $n = 5$ and the outcome is HHTHT, then there are 3 changeovers.

Hint: Express the number of changeovers as the sum of $n - 1$ Bernoulli random variables.

(11) (20pts) The number of people who enter an elevator on the ground floor is a Poisson random variable with mean 10. If there are N floors above the ground floor and if each person is equally likely to get off

at any one of these N floors, independently of where the others get off, compute the expected number of stops (by conditioning) that the elevator will make before it empties.

(12) (20=5+5+10pts) Let $X_1, \ldots, X_{25}$ be independent Poisson random variables with mean 1.

(a) Use the Markov inequality to get a bound on $\mathbb{P}\left(\sum_{i=1}^{25} X_i > 15\right)$.

(b) The bound that you should get in part (a) is not very informative. Why?

(c) Use the central limit theorem to approximate $\mathbb{P}\left(\sum_{i=1}^{25} X_i > 15\right)$.