(1) Suppose Σ satisfies $(\neg\alpha)$, then Σ does not satisfy α, so $\alpha \notin \Delta$. Since Δ is a finitely satisfiable set, $(\neg\alpha) \in \Delta$.

Suppose $(\neg\alpha) \in \Delta$, then $\alpha \notin \Delta$, so Σ does not satisfy α. Thus, Σ satisfies $(\neg\alpha)$.

In conclusion, Σ satisfies $(\neg\alpha)$ iff $(\neg\alpha) \in \Delta$.

$\blacksquare$

(2) Suppose Σ satisfies $(\alpha \wedge \beta)$ then Σ satisfies α and Σ satisfies β. So $\alpha \in \Delta$ and $\beta \in \Delta$, which by definition means $(\alpha \wedge \beta) \in \Delta$.

Suppose $(\alpha \wedge \beta) \in \Delta$, then $\alpha \in \Delta$ and $\beta \in \Delta$, so Σ satisfies α and Σ satisfies β, which by definition means Σ satisfies $(\alpha \wedge \beta)$.

In conclusion, Σ satisfies $(\alpha \wedge \beta)$ iff $(\alpha \wedge \beta) \in \Delta$.

$\blacksquare$

(3) Suppose Σ satisfies $(\alpha \vee \beta)$ then Σ satisfies α or Σ satisfies β. So $\alpha \in \Delta$ or $\beta \in \Delta$, which by definition means $(\alpha \vee \beta) \in \Delta$.

Suppose $(\alpha \vee \beta) \in \Delta$, then $\alpha \in \Delta$ or $\beta \in \Delta$, so Σ satisfies α or Σ satisfies β, which by definition means Σ satisfies $(\alpha \vee \beta)$.

In conclusion, Σ satisfies $(\alpha \vee \beta)$ iff $(\alpha \vee \beta) \in \Delta$.

$\blacksquare$

(4) Suppose Σ satisfies $(\alpha \to \beta)$, then Σ satisfies $(\neg\alpha)$ or Σ satisfies β. So by what we has proved above, $(\neg\alpha) \in \Delta$ or $\beta \in \Delta$. So by definition, $(\alpha \to \beta) \in \Delta$.

Suppose $(\alpha \to \beta) \in \Delta$, then $(\neg\alpha) \in \Delta$ or $\beta \in \Delta$, so by what we has proved above, Σ satisfies $(\neg\alpha)$ or Σ satisfies β. Thus, Σ satisfies $(\alpha \to \beta)$.

In conclusion, Σ satisfies $(\alpha \to \beta)$ iff $(\alpha \to \beta) \in \Delta$.

(5) Suppose Σ satisfies $(\alpha \leftrightarrow \beta)$ then Σ satisfies $(\alpha \to \beta)$ and $(\beta \to \alpha)$, so $(\alpha \to \beta) \in \Delta$ and $(\beta \to \alpha) \in \Delta$. Thus, $(\alpha \leftrightarrow \beta) \in \Delta$.

Suppose $(\alpha \leftrightarrow \beta) \in \Delta$, then $(\alpha \to \beta) \in \Delta$ and $(\beta \to \alpha) \in \Delta$, so Σ satisfies $(\alpha \to \beta)$ and $(\beta \to \alpha)$. Thus, Σ satisfies $(\alpha \leftrightarrow \beta)$.

In conclusion, Σ satisfies $(\alpha \leftrightarrow \beta)$ iff $(\alpha \leftrightarrow \beta) \in \Delta$.

$\blacksquare$

(7) Base case: Suppose φ is a sentence symbol, φ is a tautology iff any Σ satisfy φ iff $\varphi \in \Delta$.

Suppose wff α and β are tautology iff $\alpha \in \Delta$, $beta \in \Delta$.

Then by the proposition we proved above $(\neg\alpha)$ is tautology iff any Σ satisfies $(\neg\alpha)$ iff $(\neg\alpha) \in \Delta$. $(\alpha \circ \beta)$ is tautology iff any Σ satisfies $(\alpha \circ \beta)$ iff $(\alpha \circ \beta) \in \Delta$ for any binary connective $\circ$.

Thus, by the Principle of Induction, we conclude that for any wff φ, φ is a tautology iff $\varphi \in \Delta$.

■