6.2.17

Claim: Let $I_1, I_2, \cdots, I_n$ and $J_1, J_2, \cdots, J_n$ be R's ideals, then

1. For any $i_1, i_2, \cdots, i_m \in [1, n], \bigcap_{k=1}^{m} I = I_{i_k}$ is also an ideal.
2. $(\mathcal{S}) = \bigcap_{i=1}^{n} J_i$, that $\mathcal{S} \subseteq J_i \subseteq R$.

Proof 1: Let $x, y \in I$, and specifically $\forall k \in [1, m], x, y \in I_{i_k}$. Since I_i is an ideal, we know $x - y \in I_i$, so $x - y \in I$.

Let $x \in I$, so $x \in I_{i_1}, I_{i_2}, \cdots, I_{i_m}$, we have that $\forall r \in R, rx \in I_{i_1}, I_{i_2}, \cdots, I_{i_m}$ because they are all ideals. As a result, $\forall r \in R, rx \in I$.

In conclusion, For any $i_1, i_2, \cdots, i_m \in [1, n], \bigcap_{k=1}^{m} I = I_{i_k}$ is also an ideal.■

Proof 2: Let $J = \bigcap_{i=1}^{n} J_i$, $\forall s \in (\mathcal{S}), \forall i \in [1, n], s \in J_i$ since $\mathcal{S} \subseteq J_i$, so $s \in J$. As a result, $(\mathcal{S}) \subseteq J$.

On the other hand, let $J_m = (\mathcal{S})$, $J = \bigcap_{i=1}^{n} J_i$, so $|J| \leq |J_m|$ and for any $j \in J$, $j \in J_m = (\mathcal{S})$. As a result, $J \subseteq (\mathcal{S})$.

In conclusion, $(\mathcal{S}) = \bigcap_{i=1}^{n} J_i$, that $\mathcal{S} \subseteq J_i \subseteq R$.■

6.3.7

(a)

Claim: $n\mathbb{Z}$ is maximal ideal in $\mathbb{Z} \Leftrightarrow |n|$ is prime.

Proof: Suppose $M = n\mathbb{Z}$, then if whenever an ideal $M \subseteq I \subseteq R$ then $I = M = n\mathbb{Z}$ or $I = \mathbb{Z}$. So there are only 2 ideals contain $M = n\mathbb{Z}$, then there are 2 correspondent ideals of $\mathbb{Z}/n\mathbb{Z}$ which are $\mathbb{Z}/n\mathbb{Z}$ and $\{0\}$ by Fourth Ring Isomorphism Theorem.

Since 1 is in $\mathbb{Z}/n\mathbb{Z}$ and it is commutative, $\mathbb{Z}/n\mathbb{Z}$ a field, so $|n|$ must be prime.

Suppose $|n|$ is prime, we known that every non-zero element in $\mathbb{Z}/n\mathbb{Z}$ is invertible, so $\mathbb{Z}/n\mathbb{Z}$ is a field. As a result, $\mathbb{Z}/n\mathbb{Z}$ has only $\mathbb{Z}/n\mathbb{Z}$ and $\{0\}$ as ideals. By Fourth Ring Isomorphic Theorem, correspondently, only $\mathbb{Z}$ and $n\mathbb{Z}$ contain $n\mathbb{Z}$. So $n\mathbb{Z}$ is the maximal ideal.

In conclusion, $n\mathbb{Z}$ is maximal ideal in $\mathbb{Z} \Leftrightarrow |n|$ is prime.■

(b)

Claim: $(f) = fK[x]$ is a maximal ideal in $K[x] \Leftrightarrow f$ is irreducible.

Proof: Suppose $M = (f) = fK[x]$ is a maximal ideal in $K[x]$, then if whenever an ideal $M \subseteq I \subseteq R$, then $fK[x] \subseteq I \subseteq K[x]$. Then only $K[x]$ and $fK[x]$ contains ideal $fK[x]$. By The Fourth Isomorphism Theorem, we know that there're only 2 ideals in $K[x]/fK[x]$ which are $\{0\}$ and $K[x]/fK[x]$.

As a result, $K[x]/fK[x]$ is a field so every nonzero element is invertible, so f is irreducible.

Suppose f is irreducible, so every nonzero element in $K[x]/fK[x]$ is invertible and is a field as a result, thus $fK[x]$ is a maximal ideal in $K[x]$.■

(c)

Claim: $\mathbb{Z}_n$ is a field if and only if $|n|$ is prime, $K[x]/(f)$ is a field if and only if f is irreducible.

Proof: Since we know that $n\mathbb{Z}$ is maximal ideal in $\mathbb{Z}$, $(f) = fK[x]$ is a maximal ideal in $K[x] \Leftrightarrow f$ is irreducible and if M is a proper ideal in a commutative ring R with 1, then R/M is a field if and only if M is maximal.

$\mathbb{Z}_n$ is a field if and only if $|n|$ is prime, $K[x]/(f)$ is a field if and only if f is irreducible is proved to be true by the transitivity of equivalence.■

6.3.8

Claim: If J is an ideal of the ring R, that $J[x]$ is an ideal in $R[x]$ and $R[x]/J[x] \cong (R/J)[x]$.

Proof: Let $x = \sum_{i=0}^{n} a_i x^i, y = \sum_{i=0}^{n} b_i x^i, \forall i \in [0, n], a_i, b_i \in J \in J[x]$, $\forall r \in R$, so $rx - y = r\sum_{i=0}^{n} a_i x^i - \sum_{i=0}^{n} b_i x^i = \sum_{i=0}^{n}(ra_i - b_i)x^i$. Since J is an ideal of R, $ra_i - b_i \in J$, so $rx - y \in J[x]$. As a result, if J is an ideal of the ring R, that $J[x]$ is an ideal in $R[x]$.

Since quotient map defines a partition of R, mapping $\varphi : R \to R/J$ is surjective. As a result, $\varphi' : R[x] \to (R/J)[x]$ is also a surjective map. Let $\varphi'(x) = x + r$, then $\varphi'(x + y) = (x + y + r) = (x + r) + (y + r) = \varphi'(x) + \varphi'(y)$, $\varphi'(xy) = xy + r = (x + r)(y + r) = \varphi'(x)\varphi'(y)$, φ' is a surjective homomorphism.

As a result, by the First Ring Isomorphism Theorem, $R[x]/J[x] \cong (R/J)[x]$.■

6.4.14

Claim: Let J be an ideal of ring R that is commutative with identity, J is prime if and only if R/J has no zero divisors.

Proof: Suppose J is prime. Since R is commutative with identity, R/J is commutative ring with identity. So if $a + J, b + J \in R/J$, and $ab + J = 0$ then for some $a + J = 0$ or $b + J = 0$, which violates our assumption, so R/J has no zero divisors.

Suppose R/J has no zero divisors, then for any $ab \in J = 0$, and $ab + J = (a + J)(b + J) \in J = 0$, then $a + J = 0$ or $b + J = 0$, which means $a \in J$ or $b \in J$.

As a result, let J be an ideal of ring R that is commutative with identity, J is prime if and only if R/J has no zero divisors.■