2.1.6 Proof: Suppose there exists a rational s that $s^2 = 6$, then $\exists m, n \in \mathbb{N}$ and $g.c.d.(m, n) = 1$ that $s = m/n$. So $m^2/n^2 = 6$.

Then $m^2 = 6n^2$, so m^2 is divisible by 2, so m is also even. Let $m = 2k, k \in \mathbb{N}$. Then $4k^2 = 6n^2 \Leftrightarrow 2k^2 = 3n^2$, so n^2 is even, as is n.

This corollary implies that $g.c.d.(m, n) = 2$, which contradicts with the assumption.

Thus, there is no rational s that $s^2 = 6$.$\blacksquare$

2.1.8 Solution:

(a) Proof: Since x, y are rational numbers, then $x = m/n, y = p/q, m, n, p, q \in \mathbb{Z}, p$ and q, m and n are relatively prime.

$x + y = (mq + np)/nq$. Since $\mathbb{Z}$ is closed under addition and p and q, m and n are relatively prime, $mq + np$ and n, $mq + np$ and q are relatively prime. So $x + y$ is rational.

$xy = mp/np$. Since $\mathbb{Z}$ is closed under multiplication and p and q, m and n are relatively prime, mp and nq are relatively prime.

So xy is rational.$\blacksquare$

(b) Proof: Suppose $x+y, xy$ is rational, $x+y = m/n, m, n \in \mathbb{Z}, g.c.d(m, n) = 1$, $xy = s/t, s, t \in \mathbb{Z}, g.c.d(s, t) = 1$. Since x is rational, $x = p/q, p, q \in \mathbb{Z}, g.c.d(p, q) = 1$. Then $y = x + y - y = m/n - p/q \in \mathbb{Q}$ or $y = xy/x = s/t/(m/n) = sn/mt \in \mathbb{Q}$, which contradicts with that y is irrational.

Thus, $x + y$ and xy are irrational.

2.1.12 Solution: Let $a = 2, b = 3$, $c = -2, d = -1$, we have $0 < a < b$ and $c < d < 0$ and $ac = -4 < -3 = bd$.

Let $a = 1/2, b = 1$, $c = -1/2, d = -1$, we have $0 < a < b$ and $c < d < 0$ and $bd = -1 < -1/4 = ac$.

2.1.13 Solution: Proof: If $a^2 + b^2 = 0$, since $a^2 \geq 0, b^2 \geq 0$, so if $a \neq 0$ or $b \neq 0$, $a^2 > 0$ or $b^2 > 0$ which means $a^2 + b^2 > 0$.

As a result, $a^2 + b^2 = 0 \Rightarrow a = b = 0$.

If $a = b = 0$, $a^2 = b^2 = 0$, so $a^2 + b^2 = 0$.

Thus, $a^2 + b^2 = 0$ if and only if $a = b = 0$.$\blacksquare$

2.2.5 Proof: Since $a < x < b, a < y < b$, we have that $-b < -x < -a, -b < -y < -a$, so $x - y < b - a$ and $y - x < b - a$.

As a result, $|x - y| < b - a$.$\blacksquare$

Geometrically speaking, this inequality is true since x, y is in the line between a, b, so the length $|x - y| < |b - a| = b - a$.

2.2.6 Solution:

(a)

$$\begin{aligned} &|4x - 5| \leq 13 \\ \Rightarrow &-13 \leq 4x - 5 \leq 13 \\ \Rightarrow &-8 \leq 4x \leq 18 \\ \Rightarrow &-2 \leq x \leq 9/2 \end{aligned}$$

(b)

$$\begin{aligned} &|x^2 - 1| \leq 3 \\ \Rightarrow &-3 \leq x^2 - 1 \leq 3 \\ \Rightarrow &-2 \leq x^2 \leq 4 \\ \Rightarrow &x \leq 2 \end{aligned}$$

2.2.12 Solution:

$$\begin{aligned} &4 < |x + 2| + |x - 1| < 5 \\ \Rightarrow &4 < |x - (-2)| + |x - 1| < 5 \\ \Rightarrow &2|x - 1| < 2, x > 1 \text{ or } 2|x + 2| > 1, x < -2 \\ \Rightarrow &3/2 < x < 2 \text{ or } -3 < x < -5/2 \end{aligned}$$

2.3.6 Proof: Let $m = \sup\{-s : s \in S\}$. Then $\forall n \in \{-s : s \in S\}, n \leq m$, then $-n \geq -m$ for all $n \in \{-s : s \in S\}$. So $-m = -\sup\{-s : s \in S\}$ is a lower bound of S.

Suppose $-m$ is not the infimum of S, then $\exists v$ that $\forall s \in S, v \leq s$ and $v > -m$. Then $-v < m$ and $\forall n \in \{-s : s \in S\}, n \leq -v$, which means m is not the supremum of $\{-s : s \in S\}$ and contradicts with the condition given.

Hence, $\inf S = -\sup\{-s : s \in S\}$.■

2.3.7 Proof: Let the upper bound be m, then $\forall s \in S$, $s \leq m$. Suppose m is not a supremum, then there exists an $v < m$ that $\forall s \in S$, $s \leq v$. Since $m \in S$, $m \leq v$ and contradicts with the assumption.

As a result, this upper bound is the supremum of S.■