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Homework 0 Problem 3

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ be two fixed vectors in the real plane. Recursively define a set $L_n \subseteq \mathbb{R}^2$ as follows.

- $L_0 = \{\mathbf{u}, \mathbf{v}, \mathbf{0}\}$. ($\mathbf{0}$ denotes the zero vector $(0,0)$ in $\mathbb{R}^2$.)
- For integer $n > 0$, $L_n = \{\mathbf{x} - \mathbf{y} \mid \mathbf{x}, \mathbf{y} \in L_{n-1}\}$.

Let $L = \bigcup_{n=0}^{\infty} L_n$. Also, let $D = \{a\mathbf{u} + b\mathbf{v} \mid a, b \in \mathbb{Z}\}$ be the set of vectors obtained as integer linear combinations of $\mathbf{u}$ and $\mathbf{v}$.

1. Prove that $D \subseteq L$, by giving, for each $a, b \in \mathbb{Z}$, an explicit value of n such that $a\mathbf{u} + b\mathbf{v} \in L_n$. (You don't need to minimize the value of n; but you must argue why $a\mathbf{u} + b\mathbf{v} \in L_n$ for your choice of n.)
2. Use mathematical induction to prove that for all integers $n \geq 0$, $L_n \subseteq D$, and hence $L \subseteq D$.

Solution: 1. Proof: First, we can prove two lemmas.

Lemma 1 $L_{n-1} \subseteq L_n$.

Proof: $\forall \mathbf{l} \in L_{n-1}$, let $\mathbf{x} = \mathbf{y}$, we have $\mathbf{0} \in L_n$ for all n. So let $\mathbf{y} = \mathbf{0}, \mathbf{x} = \mathbf{l}$, we have $\mathbf{l} \in L_n$.

As a result, $L_{n-1} \subseteq L_n$.$\blacksquare$

Lemma 2 $\forall n \in \mathbb{N}, \forall k \leq n+1 \in \mathbb{N}$, if $\mathbf{a} \in L_0$, $k\mathbf{a} \in L_n$ and $\forall k < n \in \mathbb{N}, -k\mathbf{a} \in L_n$.

Proof: Base case: when $n = 0$, $\mathbf{u} - \mathbf{0} = \mathbf{u} \in L_1$, $\mathbf{v} - \mathbf{0} = \mathbf{v} \in L_1$, $\mathbf{0} - \mathbf{0} = \mathbf{0} \in L_1$, $\mathbf{0} - \mathbf{u} = -\mathbf{u} \in L_1$, $\mathbf{0} - \mathbf{v} = -\mathbf{v} \in L_1$ by definition.

Suppose for all $n \leq m$, we have $\forall k \leq m+1 \in \mathbb{N}$, if $\mathbf{a} \in L_0$, $k\mathbf{a} \in L_m$ and $\forall k < m \in \mathbb{N}, -k\mathbf{a} \in L_m$. Then when $n = m+1$. By Lemma 1, $\forall k \leq m+1 \in \mathbb{N}$, if $\mathbf{a} \in L_0$, $k\mathbf{a} \in L_{m+1}$ and $\forall k < m \in \mathbb{N}, -k\mathbf{a} \in L_{m+1}$.

Then $m\mathbf{u} - (-\mathbf{u}) = (m+1)\mathbf{u} \in L_{m+1}$, $m\mathbf{v} - (-\mathbf{v}) = (m+1)\mathbf{v} \in L_{m+1}$, $\mathbf{0} - \mathbf{0} = \mathbf{0} \in L_{m+1}$, $-(m-1)\mathbf{u} - \mathbf{u} = -\mathbf{u} \in L_{m+1}$, $-(m-1)\mathbf{y} - \mathbf{v} = -\mathbf{v} \in L_{m+1}$ by definition.

As a result, by strong mathematical induction, $\forall n \in \mathbb{N}, \forall k \leq n+1 \in \mathbb{N}$, if $\mathbf{a} \in L_0$, $k\mathbf{a} \in L_n$ and $\forall k < n \in \mathbb{N}, -k\mathbf{a} \in L_n$.$\blacksquare$

Hence, by Lemma 2, $\forall a, b \in \mathbb{N}, a\mathbf{u}, b\mathbf{v} \in L_{\max\{a-1,b-1\}}, -a\mathbf{u}, -b\mathbf{v} \in L_{\max\{a,b\}}$, as a result, if $n = \max\{a, b\} + 1$, we have $a\mathbf{u} + b\mathbf{v} \in L_n \subseteq L$.

As a result, $D \subseteq L$.

2. Proof: Base case: when $n = 0$, $L_n = L_0 = \{\mathbf{u}, \mathbf{v}, \mathbf{0}\} \subseteq D$.

   Suppose for all $n \leq k \in \mathbb{N}$, we have $L_n = L_k \subseteq D$. Then when $n = k+1$, $L_{k+1} = \{\mathbf{x} - \mathbf{y} | \mathbf{x}, \mathbf{y} \in L_k\}$. Since $\mathbf{Z}$ is closed under addition, $L_{k+1} \subseteq D$.

   As a result, by strong induction, for all $n > 0$, $L_n \subseteq D$. And $L = \bigcup_{n=0}^{\infty} L_n \subseteq D$.

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