3.5.8 Proof:

Suppose we have a sequence (x_n) that is increasing and bounded by u. Then by Monotone Convergence Theorem, (x_n) converges. Then there is N_1 that when $n \geq N_1$, $|x_n - x_{n-1}| < \varepsilon/(n-m+1)$ for all $\varepsilon > 0$, there is N_2 that when $n \geq N_2$, $|x_{n-1} - x_{n-2}| < \varepsilon/(n-m+1)$ for all $\varepsilon > 0$ and so on.

Then let $n > m$, when we pick $N = \max\{N_1, N_2, \cdots, N_{n-m+1}\}$, then when $n \geq N$, we have

$$\begin{aligned} |x_n - x_m| &= |x_n - x_{n-1} + x_{n-1} - x_{n-2} + x_{n-2} - \cdots + x_{m+1} - x_m| \\ &\leq |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \cdots + |x_{m+1} - x_m| \\ &\leq \sum_{i=1}^{n-m+1} \frac{\varepsilon}{n-m+1} \\ &< \varepsilon \end{aligned}$$

Hence, by definition (x_n) is Cauchy sequence.

■

3.5.9 Proof:

Since $0 < r < 1$, $\lim r^{n+1}/r^n = \lim r = r$, sequence $(x_n) = r^n$ is a Cauchy sequence.

Then let $n > m$ and take N that when $m, n \geq N$ that $r^m - r^n < \varepsilon/(1-r)$, then we have

$$\begin{aligned} |x_n - x_m| &= |x_n - x_{n-1} + x_{n-1} - x_{n-2} + x_{n-2} - \cdots + x_{m+1} - x_m| \\ &\leq |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \cdots + |x_{m+1} - x_m| \\ &< r^{n-1} + r^{n-2} + \cdots + r^m \\ &= \frac{r^m(1 - r^{n-m})}{1-r} \\ &= \frac{r^m - r^n}{1-r} < \varepsilon \end{aligned}$$

when $m, n \geq N$, for all $\varepsilon > 0$.

Hence, by definition, (x_n) is a Cauchy sequence.

■

3.5.10 Proof: Since $x_n = \frac{1}{2}(x_{n-2} + x_{n-1})$, the new term is formed by averaging the previous two terms. So we can see that

$$|x_n - x_{n+1}| = \frac{x_2 - x_1}{2^{n-1}}$$

Thus, if $m > n$,

$$\begin{aligned}
|x_n - x_m| &= |x_n - x_{n-1} + x_{n-1} - x_{n-2} + x_{n-2} - \cdots + x_{m+1} - x_m| \\
&\leq |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \cdots + |x_{m+1} - x_m| \\
&= (x_2 - x_1)(\frac{1}{2^{n-1}} + \frac{1}{2^n} + \cdots + \frac{1}{2^{m-2}}) \\
&< (x_2 - x_1)\frac{1}{2^{n-2}}
\end{aligned}$$

Therefore, given $\varepsilon > 0$, if n is chosen so large that $1/2^n < \varepsilon/4$ and if $m > n$, then it follows that $|x_n - x_m| < \varepsilon$.

Hence, (x_n) is a Cauchy sequence, and as a result, (x_n) converges.

And $\lim(x_n) = \lim(x_2 - x_1)\frac{1}{2^{n-1}} = (x_2 - x_1)\lim\frac{1}{2^{n-1}} = (x_2 - x_1)\frac{5}{3}$(by the result of Example 3.5.6).

■

3.5.11 Proof:

Since $y_n = \frac{1}{3}y_{n-1} + \frac{2}{3}y_{n-2} \Rightarrow y_n - y_{n-1} = -\frac{2}{3}y_{n-1} + \frac{2}{3}y_{n-2} = -\frac{2}{3}(y_{n-1} - y_{n-2})$.

As a result, we see that $|y_{n+1} - y_n| = (-\frac{2}{3})^{n-1}(y_2 - y_1)$.

Thus, if $m > n$,

$$\begin{aligned}
|y_n - y_m| &= |y_n - y_{n-1} + y_{n-1} - y_{n-2} + y_{n-2} - \cdots + y_{m+1} - y_m| \\
&\leq |y_n - y_{n-1}| + |y_{n-1} - y_{n-2}| + \cdots + |y_{m+1} - y_m| \\
&= (y_2 - y_1)((-\frac{2}{3})^n + (-\frac{2}{3})^{n+1} + \cdots + (-\frac{2}{3})^{m-2}) \\
&\leq (y_2 - y_1)((\frac{2}{3})^{n-1} + (\frac{2}{3})^n + (\frac{2}{3})^{n+1} + \cdots + (\frac{2}{3})^{m-2}) \\
&= (y_2 - y_1)(\frac{2}{3})^{n-1}(1 + \frac{2}{3} + (\frac{2}{3})^2 + \cdots + (\frac{2}{3})^{m-n-2}) \\
&< (\frac{2}{3})^{n-2} \qquad (1)
\end{aligned}$$

Then for all $\varepsilon > 0$ if we choose N that $(\frac{2}{3})^{n-2} < \varepsilon$, when $n, m \geq N$ we have $|y_n - y_m| < \varepsilon$.

So (y_n) is Cauchy sequence, and thus converges.

As a result, if we take $b_n = 2n+1$, $lim y_n = \lim y_{b_n} = 1 + (2/3) + (2/3)^3 + \cdots + (2/3)^{2n-1} = 1 - \frac{6}{5}((\frac{2}{3})^{2n} - 1) = \frac{11}{5}$.

■

4.1.2

(a) To make $|\sqrt{x} - 2| < \frac{1}{2}$, we need $-\frac{1}{2} < \sqrt{x} - 2 < \frac{1}{2}$, so that $\frac{3}{2} < \sqrt{x} < \frac{5}{2}$.
Hence, $\frac{9}{4} < x < \frac{25}{4}$, and $-\frac{7}{4} < x - 4 < \frac{9}{4}$ with $-\frac{9}{4} < 4 - x < \frac{7}{4}$.
As a result, $0 < |x - 4| < \frac{9}{4}$ can make $|\sqrt{x} - 2| < \frac{1}{2}$.

(b) To make $|\sqrt{x} - 2| < 10^{-2}$, we need $-10^{-2} < \sqrt{x} - 2 < 10^{-2}$, so that $\frac{199}{100} < \sqrt{x} < \frac{201}{100}$.
Hence, $\frac{39601}{10000} < x < \frac{40401}{10000}$, and $-\frac{399}{10000} < x - 4 < \frac{401}{10000}$ with $-\frac{401}{10000} < 4 - x < \frac{399}{10000}$.
As a result, $0 < |x - 4| < \frac{401}{10000}$ can make $|\sqrt{x} - 2| < 10^{-2}$.

4.1.5 Proof:

$$\begin{aligned}
|g(x) - c^2| &= |(x + c)(x - c)| \\
&= (x + c)|x - c| && \text{since } x, c \geq 0 \\
&\leq 2a(x - c) && \text{since } x, c \leq a \\
&= 2a|x - c|
\end{aligned}$$

Then if for all $\varepsilon > 0$, we pick $\delta = \varepsilon/2a$, then when $0 < |x - c| < \delta$, we have

$$|g(x) - c^2| \leq 2a(x - c) < 2a \cdot \varepsilon/2a = \varepsilon$$

Hence, by definition,

$$\lim_{x \to c} = c^2$$

■

4.1.7 **Proof:** Since $c \in \mathbb{R}$, c is a cluster point, so for all $\delta > 0$ there is a $x \in \mathbb{R}$ that $|x - c| < \delta$. Without losing generality, we can assume $cx \geq 0$ and $|c| > |x|$, then when we pick $\delta < \varepsilon/(4c^2)$

$$\begin{aligned}
|x^3 - c^3| &= |(x - c)(x^2 + cx + c^2)| \\
&= |(x - c)((x - c)^2 + 3cx)| \\
&= ((x - c)^2 + 3cx)|x - c| \\
&\leq (x + c)^2|x - c| \\
&< \delta(x + c)^2 \\
&< 4c^2\delta < \varepsilon
\end{aligned}$$

Hence, by definition, we have

$$\lim_{x \to c} = c^3$$

∎

4.1.9

(a) **Proof:** Since 2 is a cluster point of $\mathbb{R}$, for all $\delta > 0$, there is $x \in \mathbb{R}$ that $|x - 2| < \delta$. So if we pick δ that $|\frac{1}{1-x}| < \varepsilon/\delta$(since it converges), then

$$\begin{aligned}
|\frac{1}{1 - x} - (-1)| &= |\frac{1}{1 - x} + 1| \\
&= |\frac{1 + 1 - x}{1 - x}| \\
&= |\frac{2 - x}{1 - x}| \\
&= |\frac{x - 2}{1 - x}| \\
&= |x - 2||\frac{1}{x - 1}| \\
&< \delta\varepsilon/\delta = \varepsilon
\end{aligned}$$

Hence,

$$\lim_{x \to 2} \frac{1}{1 - x} = -1$$

∎

(b) **Proof:** Since 1 is a cluster point of $\mathbb{R}$, for all $\delta > 0$, there is $x \in \mathbb{R}$ that $|x-1| < \delta$. So if we pick δ that $|\frac{1}{1+x}| < 2\varepsilon/\delta$(since it converges), then

$$\begin{aligned}|\frac{x}{1+x} - \frac{1}{2}| &= |\frac{2x-1-x}{2+2x}| \\ &= |\frac{x-1}{2+2x}| \\ &= \frac{1}{2}|\frac{x-1}{x+1}| \\ &= \frac{|x-1|}{2}|\frac{1}{x+1}| \\ &< \frac{\delta}{2}\frac{2\varepsilon}{\delta} = \varepsilon\end{aligned}$$

Hence,

$$\lim_{x\to 1} \frac{x}{1+x} = 1/2$$

■

(c) **Proof:** Since 0 is a cluster point of $\mathbb{R}$, for all $\delta > 0$, there is $x \in \mathbb{R}$ that $|x-0| = |x| < \delta$. So if we pick δ that $\delta < \varepsilon$, then

$$|\frac{x^2}{|x|} - 0| = |x| < \delta < \varepsilon$$

Hence,

$$\lim_{x\to 1} |x| = 0$$

■

(d) **Proof:** Since 1 is a cluster point of $\mathbb{R}$, for all $\delta > 0$, there is $x \in \mathbb{R}$ that

$|x-1|<\delta$. So if we pick δ that $\delta<\varepsilon/2$, then

$$\begin{aligned}
|\frac{x^2-x+1}{x+1}-\frac{1}{2}| &= |\frac{2x^2-2x+2-x-1}{2x+2}| \\
&= |\frac{2x^2-3x+1}{2x+2}| \\
&= |\frac{(x-1)(2x-1)}{2x+2}| \\
&= |x-1||\frac{2x-1}{2x+2}| \\
&< |x-1||\frac{2x-1}{x}| \\
&< |x-1||\frac{2x}{x}| \\
&< 2\delta = \varepsilon
\end{aligned} \tag{2}$$

Hence,

$$\lim_{x\to 1}\frac{x^2-x+1}{x+1}=\frac{1}{2}$$

■

4.1.12

(a) Let $(x_n)=1/n$, then $f(x_n)=n^2$ then by Divergence Criteria, since (x_n) converges to 0 and $f(x_n)$ diverges, so

$$\lim_{x\to 0}\frac{1}{x^2}(x>0)$$

does not exist.

(b) Let $(x_n)=1/n^2$, then $f(x_n)=n$ then by Divergence Criteria, since (x_n) converges to 0 and $f(x_n)$ diverges, so

$$\lim_{x\to 0}\frac{1}{\sqrt{x}}(x>0)$$

does not exist.

(c) Let $(x_n) = -1/n$ for all $n\mathbb{N}$, then $f(x_n)$ converges to -1, if Let $(x_n) = -1/n$ for all $n\mathbb{N}$ then $f(x_n)$ converges to 1 then by Divergence Criteria, since (x_n) converges to 0 and $f(x_n)$ diverges, so

$$\lim_{x\to 0} x + \operatorname{sgn}(x)$$

does not exist.

(d) Let $(x_n) = 1/n^2$, then by Divergence Criteria, since (x_n) converges to 0 and $f(x_n) = sin(x)$ diverges, so

$$\lim_{x\to 0} \sin(1/x^2)$$

does not exist.