3.1 Solution: Let a function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x(x < 100), f(x) = x + 1(x \geq 100)$ and $P(n) = n$, then $P(n)$ is true for $n = 1 \cdots 99$ but is false when $n = 100$.

3.5 Solution:

Claim: It is true that $\forall n \in \mathbb{N}, \sum_{k=1}^{n}(2k + 1) = n^2 + 2n$.

Proof: We can proof it directly with Gauss Summation.

$$\sum_{k=1}^{n}(2k+1) = 2\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1$$
$$= 2 \cdot \frac{n(n+1)}{2} + n$$
$$= n^2 + 2n \blacksquare$$

Or, we can proof it by induction.
Base case: if $n = 1$, $2k + 1 = 3 = 1^2 + 2$
Suppose that when $n = m$, the claim above is true.
Then when $n = m + 1$,

$$\sum_{k=1}^{m+1}(2k+1) = \sum_{k=1}^{m}(2k+1) + 2(m+1) + 1$$
$$= m^2 + 2m + 2(m+1) + 1$$
$$= (m+1)^2 + 2(m+1)$$

Thus, we can conclude that It is true that $\forall n \in \mathbb{N}, \sum_{k=1}^{n}(2k+1) = n^2+2n \blacksquare$.

3.11 Solution:

Claim: Set of n elements has 2^n subsets.

Proof: Base case, when $n = 0$, there is only $1 = 2^0$ subset which is $\emptyset$.

Suppose when $n = k$, set of k elements has 2^k subsets, then when $n = k+1$, we can add the $(k+1)_{th}$ elements in each subset of 2^k to get a new subset. Thus the number of subsets of the set consists of $k + 1$ elements is $2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$.

So we can conclude that it is true that set of n elements has 2^n subsets.$\blacksquare$

3.15 Solution:

Claim: $\forall n \in \mathbb{N}$,

$$\sum_{i=1}^{n}(-1)^i i^2 = (-1)^n \frac{n(n+1)}{2}$$

Proof: Base case: When $n = 1$, $-1 \cdot (-1)^2 = -1 = -1 \cdot \frac{1 \cdot 2}{2}$
Suppose this claim is true when $n = k$. Then when $n = k + 1$,

$$\begin{aligned}
\sum_{i=1}^{k+1}(-1)^i i^2 &= \sum_{i=1}^{k}(-1)^i i^2 + (-1)^{k+1}(k+1)^2 \\
&= (-1)^k \frac{k(k+1)}{2} + (-1)^{k+1}(k+1)^2 \\
&= (-1)^k (k+1)[\frac{k}{2} + (-1)(k+1)] \\
&= (-1)^k (k+1)(-1 - \frac{k}{2}) \\
&= (-1)^{k+1} \frac{(k+1)(k+2)}{2}
\end{aligned}$$

Thus, we can conclude that it is true that $\forall n \in \mathbb{N}$,

$$\sum_{i=1}^{n}(-1)^i i^2 = (-1)^n \frac{n(n+1)}{2} \blacksquare$$

3.23 Solution: This proof is flawed because the validity of induction step that gives the $P(k+1)$ relies on the trueness of $P(k)$ and $P(k-1)$. However, $P(k-1)$ is not verified to be correct, so it risks the validity of the whole proof.

3.28 Solution:

Claim: We can expand the series, that

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = 1 - 1/2 + 1/2 - 1/3 + \cdots + 1/n - 1/(n+1) = 1 - 1/(n+1)$$

Proof: Base case: when $n = 1$, $1/(1 \cdot 2) = 1/2 = 1 - 1/(1+1)$.

Suppose that when $n = k$, the claim above is true. Then when $n = k + 1$,

$$\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \sum_{i=1}^{k} \frac{1}{i(i+1)} + \frac{1}{(k+1)(k+2)} = 1 - 1/(k+1) + \frac{1}{(k+1)(k+2)}$$
$$= \frac{k(k+2)+1}{(k+1)(k+2)}$$
$$= \frac{(k+1)^2}{(k+1)(k+2)} = 1 - 1/(k+2)$$

Thus, we can conclude that $\forall n \in \mathbb{N}$,

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = 1 - 1/(n+1) \ \blacksquare$$

3.33 Solution: In closed interval $[1, n]$, there're $n - 1$ integer points we can choose to be the start point of sub-intervals. And for $1 \le i \le n - 1$, there are $n - i$ integer points can be chosen to be the right endpoint of the sub-intervals. As a result, the number of all sub-intervals is

$$n = \sum_{i=1}^{n-1}(n-i) = (n-1) + (n-2) + \cdots + 2 + 1 = \sum_{i=1}^{n-1} n = \frac{n(n-1)}{2}$$