7.30 Solution: Since $E[X^2] = E[Y^2] = Var(X) + E[X]^2 = \sigma^2 + \mu^2$

$$\begin{aligned} E[(X-Y)^2] &= E[X^2 - 2XY + Y^2] \\ &= E[X^2] - 2E[XY] + E[Y^2] \\ &= E[X^2] - 2E[X][Y] + E[Y^2] \\ &= 2\sigma^2 \end{aligned}$$

7.31 Solution:

$$\begin{aligned} Var(\sum_{i=0}^{10} X) &= \sum_{i=0}^{10} Var(X) \\ &= \sum_{i=0}^{10} (E[X^2] - E[X]^2) \\ &= \sum_{i=0}^{10} \frac{35}{12} = \frac{175}{6} \end{aligned} \tag{1}$$

7.33 Solution:

(a) Since $Var(X) = E[X^2] - E[X]^2 \Rightarrow 5 = E[X^2] - 1 \Rightarrow E[X^2] = 6$

$$\begin{aligned} E[(2+X)^2] &= E[X^2] + 4E[X] + 4 \\ &= 6 + 4 + 4 = 14 \end{aligned} \tag{2}$$

(b)

$$Var(4+3X) = 9Var(X) = 45$$

7.38 Solution: Since $f(x,y) = 2e^{-2x}/x, 0 \le x < \infty, 0 \le y \le x$

$$\begin{aligned} E[XY] &= \int_0^\infty \int_0^x 2ye^{-2x} dydx = \frac{1}{4} \\ E[X] &= \int_0^\infty 2e^{-2x} dydx = \frac{1}{2} \\ E[Y] &= \int_0^\infty \int_0^x \frac{2y}{x} e^{-2x} dydx = \frac{1}{4} \\ Cov(X,Y) &= E[XY] - E[X]E[Y] \\ &= \frac{1}{4} - \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8} \end{aligned} \tag{3}$$

7.39 Solution:

$$\begin{aligned}
&E[Y_n] = E[X_n + X_{n+1} + X_{n+2}] = \sum_i E[X_{n+i}] = 3\mu \\
&Cov(Y_n, Y_n) = Var(Y_n) = \sum_i Var(X_{n+i}) = 3\sigma^2 \\
&Cov(Y_n, Y_{n+1}) = Cov(X_n + X_{n+1} + X_{n+2}, X_{n+1} + X_{n+2}, X_{n+3}) \\
&\qquad = Cov(X_{n+1} + X_{n+2}, X_{n+1} + X_{n+2}) \\
&\qquad = Var(X_{n+1} + X_{n+2}) = Var(X_{n+1}) + Var(X_{n+2}) = 2\sigma^2 \\
&Cov(Y_n, Y_{n+2}) = Cov(X_n + X_{n+1} + X_{n+2}, X_{n+2} + X_{n+3}, X_{n+4}) \\
&\qquad = Cov(X_{n+2}, X_{n+2}) = Var(X_{n+2}) = \sigma^2
\end{aligned} \tag{4}$$

For $j \geq 3$, $Cov(Y_n, Y_{n+j}) = 0$.

7.41 Solution:

$$E[X] = 20 \cdot 30/100 = 6$$

$$Var(X) = \frac{20(100-20)}{100-1}\frac{3}{10}\frac{7}{10} = \frac{112}{33}$$

7.42 Solution:

(a)

$$E[\sum_i X_i] = \sum_i E[X_i = 1] = 10 \cdot \frac{2 \cdot 10 \cdot 10}{20 \cdot 19} = \frac{100}{19}$$

$$Var(X_i) = E[X_i^2] - E[X_i]^2 = \frac{10}{19}$$

$$Cov(X_i, X_j) = E[X_i X_j] - E[X_i]E[X_j] = \frac{10}{19} \cdot \frac{9}{17} - \frac{100}{361} = \frac{90}{6137}$$

$$Var(\sum_i X_i) = \sum_{i=1}^{10} Var(X_i) + 2\sum\sum Cov(X_i, X_j) = \frac{900}{361} + 10 \cdot 9 \cdot \frac{10}{6137} = \frac{16200}{6137}$$

(b)

$$E[\sum_i Y_i] = \sum_i E[Y_i] = 10 \cdot \frac{1}{19} = \frac{10}{19}$$

$$Var(Y_i) = E[Y_i^2] - E[Y_i]^2 = \frac{18}{361}$$

$$Cov(Y_i, Y_j) = E[Y_iY_j] - E[Y_i]E[Y_j] = \frac{8\binom{10}{2}\cdot 16!}{20!} - \frac{1}{361} = \frac{2}{6137} = \frac{10}{6137}$$

$$Var(\sum_i Y_i) = \sum_{i=1}^{10} Var(Y_i) + 2\sum\sum Cov(Y_i, Y_j) = \frac{10\cdot 18}{361} + 90\cdot\frac{2}{6137} = \frac{3240}{6137}$$

7.50 Solution: Since $f(x,y) = \frac{e^{-x/y}e^{-y}}{y}, 0 < x < \infty, 0 < y < \infty$, then

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f(x,y)}{f_Y(y)} \\ &= \frac{\frac{e^{-x/y}e^{-y}}{y}}{\int_0^\infty \frac{e^{-x/y}e^{-y}}{y}dx} \\ &= \frac{\frac{e^{-x/y}e^{-y}}{y}}{e^{-y}} \\ &= \frac{e^{-x/y}}{y}, x > 0 \end{aligned}$$

So $E_{X|Y}[X^2|Y=y] = \int_0^\infty \frac{x^2e^{-x/y}}{y}dx = 2y^2$.

7.51 Solution: Since $f(x,y) = \frac{e^{-y}}{y}, 0 < x < y, 0 < y < \infty$, then

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f(x,y)}{f_Y(y)} \\ &= \frac{\frac{e^{-y}}{y}}{\int_0^y \frac{e^{-y}}{y}dx} \\ &= \frac{1}{y}, 0 < x < y \end{aligned}$$

So $E_{X|Y}[X^3|Y=y] = \int_0^y \frac{x^3}{y}dx = \frac{y^3}{4}$.

7.56 Solution: Let $Y_i = 1$ if elevator stops at i-th floor, so $E[Y_i|X=k] = 1 - (\frac{N-1}{N})^k$, so

$$E[\sum_i Y] = \sum_i E[Y_i] = N(1 - (\frac{N-1}{N})^k)$$

As a result,

$$\begin{aligned} E[Y] &= E[E[Y|X]] \\ &= E[N(1-(\frac{N-1}{N})^k)] \\ &= N - NE[(\frac{N-1}{N})^k)] \\ &= N - N\sum_{i=0}^{\infty}(\frac{N-1}{N})^k \frac{10k}{k!}e^{-10} \\ &= N(1-e^{-\frac{10}{N}}). \end{aligned}$$

7.57 Solution:

$$E[\sum_i Y_i] = E[E[\sum_i Y_i|X]] = 2.5 \cdot 5 = 12.5$$