CS/ECE 374 Spring 2017
Homework 10 Problem 3

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Consider the following problem. You are managing a communication network, modeled by a directed graph $G = (V, E)$. There are c *users* who are interested in making use of this network. User i (for each $i = 1, 2, \ldots, c$) issues a *request* to reserve a specific path P_i in G on which to transmit data.

You are interested in accepting as many of these path requests as possible, subject to the following restriction: if you accept both P_i and P_j, then P_i and P_j can not share any modes.

Thus the *Path Selection Problem* asks: Given a directed graph $G = (V, E)$, a set of requests $P_1, \ldots, P_c$-each of which must be a path in G- and a number k, is it possible to select at least k of the paths so that no two of the selected paths share any nodes?

Prove that the Path Selection is NP-Complete.

Solution: Reduce Independent Set to Path Selection as follows.
Let (G, k) be an instance of the Independent Set problem. Let $G = (V, E)$ with $|V| = n$ and $|E| = m$. The reduction creates a new directed graph $G = (V, E)$ and n paths P_1, P_2,..., P_n such that G has an independent set of size k if and only if there are k paths in P_1, P_2,..., P_n that are node-disjoint in G. The graph G has m vertices, one corresponding to each edge of G.
We let a_e denote a vertex in G where e is an edge in E. We make G a complete directed graph which means that there is a directed edge between every pair of vertices (a_e, a_e). It remains to define the paths. The paths correspond to vertices in G. For each vertex $i \in V$ (of G), there is a path P_i. Let e_{j_1}, e_{j_2},..., e_{j_n} be the edges incident to i in G (in some arbitrary order). Then the path P_i is $a_{e_{j_1}} \to a_{e_{j_2}} \to \ldots \to a_{e_{j_n}}$; note that this is a valid path since G is a complete directed graph. It can be seen that G and the paths $P_1, \ldots, P_n$ can be constructed in polynomial time from G. One can show that $S = i_1, i_2, \ldots, i_i$ is an independent set in G if and only if the path $P_1, \ldots, P_i$ are node-disjoint. The reason is that if (i, j) is an edge in E then P_i and P_j both contain the vertex a_e in G where $e = (i, j)$. And conversely if P_i and P_j contain a node a_e in G then $e = (i, j)$ in G.

The solution is taken from CS 473 HOMEWORK 10 solutions in Spring 2013. ■