1. Note that a automorphism should be able to preserve the structure of the original algebraic structure, which means that if g_λ is an automorphism, then we should have for all $x \leq y \in \mathbb{R}$, there is

$$g_\lambda(x) \leq g_\lambda(y) \Rightarrow \lambda x \leq \lambda y \Rightarrow \lambda(x-y) \leq 0$$

Since $x \leq y$, $x - y \leq 0$, so in order to make g_λ an automorphism, it is required that $\lambda \geq 0$.

2. (a) Since for all $x \in \mathbb{Z}$ we have $-x$ that $g(-x) = x \in \mathbb{Z}$, and for all $x, y \in \mathbb{Z}, (x \neq y) \to g(x) \neq g(y)$ so we see that g is a bijection between $\mathbb{Z}$ and $\mathbb{Z}$.

   We also see that for all $x, y \in \mathbb{Z}$, $g(0) = 0$ and

$$g(x+y) = -(x+y) = (-x) + (-y) = g(x) + g(y)$$

   so g is a homomorphism.

   In conclusion, g is an automorphism of $(\mathbb{Z}, +, 0)$.

   (b) There is no other automorphism.

   Proof:

   In order to preserve the structure, any mapping h and $m \in \mathbb{Z}$ needs to have

$$h(m) = h(\underbrace{1 + 1 + \cdots + 1}_{m}) = \underbrace{h(1) + h(1) + \cdots + h(1)}_{m} = mh(1)$$

   In order to make this mapping surjective, we see that $h(1) = 1$ or $h(1) = -1$, which means that there are no other valid automorphisms for $(\mathbb{Z}, +, 0)$.

3. Similarly, in order to preserve the structure, any mapping h and $m \in \mathbb{N}$ needs to have

$$h(m) = mh(1)$$

In order to make this mapping bijective, it is only possible to let $h(x) = x$. So the identity is the only possible automorphism.

4. **Proof:** We first prove $(\mathcal{P}(X), \oplus, 0)$ is an Abelian group.

- **Closure** For any $x, y \in \mathcal{P}(X)$, there is

$$x \oplus y = x \triangle y \subseteq x \cup y$$

which has $x \oplus y \subseteq x \cup y \subseteq X$, so $x \oplus y \in \mathcal{P}(X)$.

- **Commutivity** By set theory we know for any $x, y \in \mathcal{P}(X)$

$$x \oplus y = x \triangle y = y \triangle x = y \oplus x$$

holds.

- **Associativity** By set theory we know for any $x, y, z \in \mathcal{P}(X)$

$$x \oplus y \oplus z = x \triangle y \triangle z = x \triangle (y \triangle z) = x \oplus (y \oplus z)$$

holds.

- **Additive identity** For any $x \in \mathcal{P}(X)$

$$x \oplus 0 = x \triangle \emptyset = x$$

- **Additive inverse**

$$x \oplus x = x \triangle x = \emptyset = 0$$

Then we want to prove $(\mathcal{P}(X), \otimes, 1)$ is a monoid

- **Closure** For any $x, y \in \mathcal{P}(X)$, there is

$$x \otimes y = x \cap y$$

which has $x \otimes y \subseteq x \subseteq X$ and $x \otimes y \subseteq y \subseteq X$, so $x \otimes y \in \mathcal{P}(X)$

- **Associativity** By set theory we know for any $x, y, z \in \mathcal{P}(X)$

$$x \otimes y \otimes z = x \cap y \cap z = x \cap (y \cap z) = x \otimes (y \otimes z)$$

holds.

- **Multiplicative identity** For any $x \in \mathcal{P}(X)$

$$x \otimes 1 = x \cap X = x$$

In the last, we want to prove the Distributivity. For any $x, y, z \in \mathcal{P}(X)$,

$$\begin{aligned} x \otimes (y \oplus z) &= x \cap (y \triangle z) \\ &= (x \cap y) \triangle (x \cap z) \\ &= (x \otimes y) \oplus (x \otimes z) \end{aligned}$$

$$\begin{aligned} (y \oplus z) \otimes x &= (y \triangle z) \cap x \\ &= (y \cap x) \triangle (z \cap x) \\ &= (y \otimes x) \oplus (z \otimes x) \end{aligned}$$

In conclusion, we proved that $(\mathcal{P}(X), \oplus, \otimes, 0, 1)$ is a ring.