6.8 Solution:

(a)

$$\begin{aligned}224 &= 126 + 98\\ 126 &= 98 + 28\\ 98 &= 28 \cdot 3 + 14\\ 28 &= 14 \cdot 2\end{aligned}$$

Thus, $gcd(224, 126) = 14$ and $14 = 98 - 28 \cdot 3 = 4 \cdot 98 - 3 \cdot 126 = 4 \cdot 98 - 3 \cdot 126 = 4 \cdot 224 - 7 \cdot 126$.

In conclusion, $gcd(224, 126) = 14 = 4 \cdot 224 - 7 \cdot 126$.

(b)

$$\begin{aligned}299 &= 221 + 78\\ 221 &= 2 \cdot 78 + 65\\ 78 &= 65 + 13\\ 65 &= 5 \cdot 13\end{aligned}$$

Thus, $gcd(299, 221) = 13$ and $13 = 78 - 65 = 78 - (221 - 2 \cdot 78) = 3 \cdot 78 - 221 = 3 \cdot (299 - 221) - 221 = 3 \cdot 299 - 4 \cdot 221$.

In conclusion, $gcd(299, 221) = 13 = 3 \cdot 299 - 4 \cdot 221$.

6.18 Solution:

If $gcd(a, b) = 1$, then $gcd(a^2, b^2) = 1$ and $gcd(a, 2b) = 1$(if a is odd) or 2(if a is even).

6.28 Solution:

Claim: If $gcd(a, b) = 1$ and $a|n, b|n$ then $ab|n$.

Proof: Since $a|n$, $\exists k \in \mathbb{Z}, n = ak_1$. And since $gcd(a, b) = 1$ and $b|n$, $\exists k_2 \in \mathbb{Z}$, $k_1 = k_2 b$. Thus, $n = abk_2$ and as a result, $ab|n$.$\blacksquare$

6.29 Solution:

Claim: $lcm(a, b)gcd(a, b) = ab$.

Proof: Let $a = k_1 gcd(a,b), b = k_2 gcd(a,b)$, obviously $gcd(k_1, k_2) = 1$, so $ab = k_1 gcd(a,b) k_2 gcd(a,b) = k_1 k_2 gcd(a,b)^2 = lcm(a,b) gcd(a,b)$.■

6.48 Solution:

Claim: Given $a, b, c \in \mathbb{Z}$, let $d = gcd(a,b)$ and $d|c$, that the set of integer solutions to $ax + by = c$ is nonempty.

Proof: According to Euclidean Algorithm, d is a linear combination of a, b, namely $\exists x, y \in \mathbb{Z}$ that $xa + yb = d$, then $akx + bky = dk = c$ must have integer solutions where $x' = kx, y' = ky$.

And if x_0, y_0 is a pair of solution, $x = x_0 + bt/d, y = y_0 - at/d$ for $t \in \mathbb{Z}$.■