1. (a) Suppose we have $a, b, c \in \mathbb{Z}$.
Since $a - a = 0$ which is even, we have $a \equiv a \pmod 2$, so reflexivity is proved.
Suppose $a \equiv b \pmod 2$, then we have $k \in \mathbb{Z}$ that $a - b = 2k$ so that $b - a = -2k$ and $-2k \in \mathbb{Z}$. So we have $b \equiv a \pmod 2$ and thus the symmetry is proved.
Suppose $a \equiv b \pmod 2$ and $b \equiv c \pmod 2$, so we have $k_1, k_2 \in \mathbb{Z}$ that $a - b = 2k_1$, $b - c = 2k_2$. So $a - c = (a - b) - (b - c) = 2(k_1 - k_2)$ that $k_1 - k_2 \in \mathbb{Z}$. Thus, $a \equiv c \pmod 2$. So transitivity is proved.
In conclusion, this relation is equivalent. $\blacksquare$

(b) Since we have $m \equiv n \pmod 2$ and $m' \equiv n' \pmod 2$, so let $k_1, k_2 \in \mathbb{Z}$ that $m - n = 2k_1$ and $m' - n' = 2k_2$.
As a result,

$$\begin{aligned}(m + n) - (m' + n') &= (m - m') + (n - n') \\ &= 2(n - n') + 2(k_1 - k_2) \\ &= 2(n - n' + k_1 - k_2)\end{aligned}$$

Since $n, n', k_1, k_2 \in \mathbb{Z}$, $n - n' + k_1 - k_2 \in \mathbb{Z}$. So we have $m + n \equiv m' + n' \pmod 2$.
Similarly,

$$\begin{aligned}mm' - nn' &= (n + 2k_1)(n' + 2k_2) - nn' \\ &= nn' + 2k_1 n' + 2k_2 n + 4k_1 k_2 - nn' \\ &= 2k_1 n' + 2k_2 n + 4k_1 k_2\end{aligned}$$

So we proved that $mm' \equiv nn' \pmod 2$. $\blacksquare$

(c) For all $x \in U$, if $x \in A$ and $x \in B$ then $x \in A \cap B$ so that

$$\chi_{A\cap B}(x) = 1 = \chi_A(x)\chi_B(x)$$

if $x \notin A$ or $x \notin B$, then

$$\chi_{A\cap B}(x) = 0 = \chi_A(x)\chi_B(x)$$

so we have

$$\chi_{A\cap B}(x) \equiv \chi_A(x)\chi_B(x) (mod\ 2)$$

Suppose $x \in A\Delta B$, then $x \in A$ xor $x \in B$ so

$$\chi_{A\Delta B}(x) = 1 = 1 + 0 = \chi_A(x) + \chi_B(x)$$

Suppose $x \notin A\Delta B$, then $x \notin B$ and $x \notin A$, then so

$$\chi_{A\Delta B}(x) = 0 = 0 + 0 = \chi_A(x) + \chi_B(x)$$

or $x \notin A\Delta B$, then $x \in B$ and $x \in A$, then so

$$\chi_{A\Delta B}(x) = 0$$

$$\chi_A(x) + \chi_B(x) = 1 + 1 = 2$$

So we have

$$\chi_{A\Delta B}(x) \equiv \chi_A(x) + \chi_B(x)(mod\ 2)$$

■

2. (1). Since f, g injective, for all $x_1, x_2 \in A$, $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$ so $g(f(x_1)) \neq g(f(x_2))$, so we conclude that $g \circ f$ is injectives.

   (2). Since f, g surjective for all $z \in C$ there is $y \in B$ that $g(y) = z$ and for all $y \in B$ there is $x \in A$ that $f(x) = y$. So for all $z \in C$ there is $x \in A$ that $g(f(x)) = z$. We conclude that $g \circ f$ is surjective.

   (3). Suppose f is not injective then there exist $x_1, x_2 \in A$ that $f(x_1) = f(x_2)$ so $g(f(x_1)) = g(f(x_2))$ which means $g \circ f$ is not injective. So we conclude that f must be injective.

   (4). Suppose $g \circ f(x) = x$, $f(x) = \sqrt{(}x)$ and $g(x) = x^2$, we see that $g \circ f$ is injective but not g. So we conclude that g is not necessarily injective.

   (5). Let $A = \{0\}, B = \{0, 1\}$ and $f(0) = 0, g(x) = 0$, so $g \circ f$ is surjective, but f is not surjective.

   (6). If g is not surjective, then there is $z \in C$ that no $y \in B$ that $g(y) = z$. Since for all $x \in A$, $f(x) \in B$. So there is no $x \in A$ that $g(f(x)) = z$. So we conclude that g must be surjective.

3. (a)

$$f(x) = \begin{cases} x/2 & \text{if } x \text{ is even} \\ -\frac{x+1}{2} & \text{otherwise} \end{cases}$$

(b) Suppose we have countable sets A, B, then we can construct a bijection between A and odd numbers and between B and even numbers. Then we see that there is a bijection between the union of A and B to the union of odd numbers and between B and even numbers, which is $\mathbb{Z}$. So we conclude that the union of two countable sets is countable.

4. Base case: When $k = 1$, $A^k = A$, which has $1^k = 1$ element.

   Suppose when $k = m$, $|A^k| = n^m$.

   Then when $k = m+1$, $|A^{m+1}| = |A \times A^m| = |A||A^m| = n \cdot n^m = n^{m+1}$.

   In conclusion, $|A^k| = n^k$ for all $k \geq 1$. ■

5. (1).

$$
\begin{aligned}
& V \in \mathcal{P}(A) \cap \mathcal{P}(B) \\
\Leftrightarrow\ & \mathcal{P}(A) \vee V \in \mathcal{P}(B) \\
\Leftrightarrow\ & V \subseteq A \vee V \subseteq B \\
\Leftrightarrow\ & V \subseteq A \cap \subseteq B \\
\Leftrightarrow\ & V \in \mathcal{P}(A \cap B)
\end{aligned}
$$

   So $\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$.

   (2). Let $A = \{1, 2\}$, $B = \{2, 3\}$, so $A \cap B = \{1, 2, 3\}$.

   (3). Note that $A \cap B = \{1, 2, 3\} \not\subseteq A$ and $A \cap B = \{1, 2, 3\} \not\subseteq B$.
   So $A \cap B \notin \mathcal{P}(A) \wedge A \cap B \notin \mathcal{P}(B)$.
   Hence, $A \cap B \notin \mathcal{P}(A) \cup \mathcal{P}(B)$.
   So $\mathcal{P}(A \cup B) \not\subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$.
   We conclude that this is not necessarily correct.