6.2.5 Proof: Let $f(x) = x^{1/n} - (x-1)^{1/n}, (n \geq 2)$, then solving $f'(x) \leq 0 \Rightarrow x \geq 1$, so $f(x)$ is decreasing when in $[1, \infty)$. Thus since $a > b > 0 \Rightarrow a/b > 1$, so $f(a/b) - f(1) < 0 \Rightarrow (a/b)^{1/n} - (a/b-1)^{1/n} < 1 \Rightarrow b^{1/n}((a/b)^{1/n} - (a/b-1)^{1/n}) < b^{1/n} \Rightarrow a^{1/n} - (a-b)^{1/n} < b^{1/n} \Rightarrow a^{1/n} - b^{1/n} < (a-b)^{1/n}$

■

6.2.6 Proof: Let $f(x) = \sin x$, without losing generality we suppose $x \leq y$, so by Mean Value Theorem, there is $\sin x - \sin y = f'(c)(x-y)$ for some $c \in [x, y]$, then $|\sin x - \sin y| = |f'(c)||x-y|$.

Since $|f'(x)| = |\cos x| \leq 1$ for all $x \in \mathbb{R}$, so $|\sin x - \sin y| = |f'(c)||x-y| \leq |x-y|$ is proved.

■

6.2.11 Since $[0, 1]$ is closed bounded interval, we can just pick a function that is continuous on $[0, 1]$ to guarantee its uniform continuity, and

$$f(x) = \begin{cases} x^2 \sin(1/x^2) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

whose derivative

$$f'(x) = 2x \sin(1/x^2) - \frac{2}{x} \cos(1/x^2)$$

is not bounded.

6.2.13 Proof: Since $f'(x) > 0$ on I, there is

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} > 0$$

for all $c \in I$, so $(f(x) - f(c))(x - c) > 0$. As a result, when $x \neq c$, $f(x) \neq f(c)$. For any $x > c$, there is $f(x) > f(c)$, for any $x < c$ there is $f(x) < f(c)$.

Thus, by definition, $f(x)$ is strictly increasing on I.

■

7.1.2

1.

$$\begin{aligned} S(f;\dot{\mathcal{P}}) &= \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) \\ &= 0 + 1^2(2-1) + 2^2(4-2) = 1 + 8 = 9 \end{aligned}$$

2.

$$\begin{aligned} S(f;\dot{\mathcal{P}}) &= \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) \\ &= 1^2 + 2^2(2-1) + 4^2(4-2) = 1 + 4 + 32 = 37 \end{aligned}$$

3.

$$\begin{aligned} S(f;\dot{\mathcal{P}}) &= \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) \\ &= 0 + 2^2(3-2) + 3^2(4-3) = 4 + 9 = 13 \end{aligned}$$

4.

$$\begin{aligned} S(f;\dot{\mathcal{P}}) &= \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) \\ &= 2^2(2-0) + 3^2(3-2) + 4^2(4-3) = 8 + 9 + 16 = 33 \end{aligned}$$

7.1.8 Proof: Since

$$S(f;\dot{\mathcal{P}}) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$$

We have that

$$\begin{aligned} |\int_a^b f| &\le \sum_{i=1}^{n} |f(\max\{[x_{i-1}, x_i]\})(x_i - x_{i-1})| \\ &\le |f(\max\{[x_1, x_n]\})|(x_i - x_{i-1}) \\ &= |f(\max\{[x_{i-1}, x_i]\})|(b-a) \end{aligned}$$

Since $M \geq |f(x)|$ for all $x \in [a, b]$,

$$|\int_a^b f| \leq M(b-a)$$

$\blacksquare$