1.3.1 List the symmetries of an equilateral triangular plate (there are six) and work out the multiplication table for the symmetries.

Solution: There're 6 symmetries in total for an equilalateral triangular plate as listed following:

1. 3 Rotational symmetries: e, r, r^2;
2. 3 Reflectional symmetries: $a, b = ra, c = r^2a$.

And the multiplication table is as following:

	e	r	r^2	a	b	c
e	e	r	r^2	a	b	c
r	r	r^2	e	b	c	a
r^2	r^2	e	r	c	a	b
a	a	b	c	e	r	r^2
b	b	c	a	r^2	e	r
c	c	a	b	r	r^2	e

Table 1: Table of Multiplication for Equilateral Triangle

1.3.3

(a) Solution:

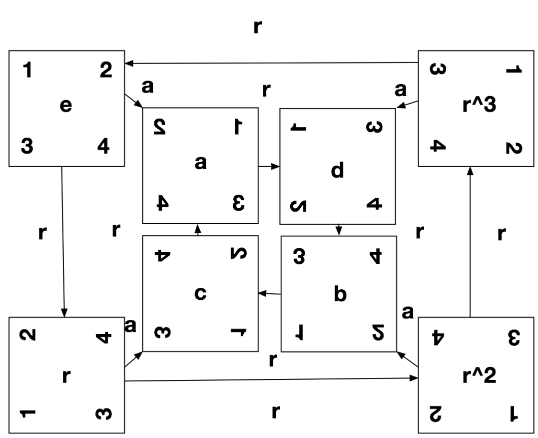

Figure 1: D_4 Symmetry

As the graph above has shown, it's clear that,

$$a = a \tag{1}$$

$$b = ar^2 \tag{2}$$

$$c = ar^3 \tag{3}$$

$$d = ar \tag{4}$$

(b) Solution: According to equation (3), $ar = d$. And as the graph has shown, that $r^{-1}a = d = r^3a$.

As a result, $ar = r^{-1}a = r^3a$ is verified.

(c) Claim: $\forall k \in \mathbb{Z}$, $ar^k = r^{-k}a$.

Proof: Since $ar^k = r^{-k}a \Leftrightarrow r^kar^kr^{-k} = r^kr^{-k}ar^{-k} \Leftrightarrow r^ka = ar^{-k} \Leftrightarrow ar^{-k} = r^ka$, we have $\forall k \in \mathbb{Z}$, $ar^k = r^{-k}a \Leftrightarrow \forall k \in \mathbb{Z}^*$, $ar^k = r^{-k}a$.

When $k = 0$, $ar^0 = ae = ea = r^0a$ is obvious by the definition of e.

Suppose when $k = m \in \mathbb{Z}^*$, $ar^m = r^{-m}a$.

Then when $k = m + 1 \in \mathbb{Z}^*$, $ar^k = ar^{m+1} = ar^mr = (r^{-m}a)r = r^{-1}(r^{-m}a) = (r^{-1}r^{-m})a = r^{-(m+1)}a = r^{-k}a$.

According to the Principle of Mathematical Induction, the claim is proved.

(d) Solution: In the group of $G = e, r, r^2, r^3, a, b, c, d$, any element can be expressed in the form of r^ma^n, $0 \leq m \leq 3$ and $n = 0$ or 1. For arbitary elements $e_1, e_2, ..., e_k \in G$, product

$$P = \prod_{i=0}^{k} e_i = \prod_{i=0}^{k} r^{m_i}a^{n_i} = r^{\sum_{i=1}^{k} m_i}a^{\sum_{i=1}^{k} n_i}$$

And with the rules we verified in question (b) and (c), $r^k = r^{k \mod 4}$ and $a^k = a$(if k is odd) or e(if k is even), so $r^{\sum_{i=1}^{k} m_i}a^{\sum_{i=1}^{k} n_i}$ can be reduced to the form of r^ma^n, $0 \leq m \leq 3$ and $n = 0$ or 1.

As a result, these relations suffice to compute any product.

1.4.2 Solution: In this problem, we need to figure out the coefficiencies in the following function:

$$\begin{cases} x_2 = w_{11} \cdot x_1 + w_{12} \cdot y_1 \\ y_2 = w_{21} \cdot x_1 + w_{22} \cdot y_1 \end{cases}$$

which can be rewrite with matrix:

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

So we can start with those 3 rotational symmetries: as it was demostrated in linear algebra, the matrix that rotate a vector θ radian is

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

. As a result,

- $$e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- $$r = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- $$r^2 = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

And since the question gives the 3 vertices as $(1, 0, 0), (-1/2, \sqrt{3}/2, 0), (-1/2, -\sqrt{3}/2, 0)$, we can give the matrices of the 3 reflectional are:

- $$a = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- $$b = ar = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- $$c = br = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

B.4 Solution: Let $C = A \cap B$, then $(A \cup B) \setminus (A \cap B) = (A \cup B) \setminus C = (A \setminus C) \cap (B \setminus C) = (A \setminus B) \cup (B \setminus A)$

E.1 Claim: $\forall S, T \in Hom_K(K^n, K^m), \alpha \in K, [S+T] = [S] + [T]$ and $[\alpha T] = \alpha[T]$.

Proof: Since $S, T \in Hom_K(K^n, K^m)$, S, T are $m \times n$ matrices. So they are able to be added. $[S+T]\mathbf{x} = (S+T)(\mathbf{x}) = S(\mathbf{x}) + T(\mathbf{x}) = [S]\mathbf{x} + [T]\mathbf{x} = ([S] + [T])\mathbf{x}$, so we can multiply x^{-1} on both sides of the equation and get

$$[S+T] = [S] + [T]$$

And similarly, $[\alpha T]\mathbf{x} = (\alpha T)(\mathbf{x}) = \alpha T(\mathbf{x}) = \alpha[T]\mathbf{x}$. As a result,

$$[\alpha T] = \alpha[T]$$

is proved to be true.