1

(a) Solution: Since $1001 \equiv 2 \mod 9, 1001^6 \equiv 1 \mod 9$, $1001^{1001} \equiv 1001^{6\cdot 166+5} \equiv 1001^5 \equiv 5 \mod 9$

(b) Solution: Since $n^3 + 5n = n^3 - n + 6n = n(n-1)(n+1) + 6n$, $n^3+5n \equiv n(n-1)(n+1)+6n \equiv n(n-1)(n+1) \mod 6$, and without losing generality we can assume n is even so $n \equiv 0 \mod 2$ and $n+1 \equiv 0 \mod 3$, so $n(n+1) \equiv 0 \mod 6$.

As a result, $n^3 + 5n \equiv n(n-1)(n+1) + 6n \equiv n(n-1)(n+1) \equiv 0 \mod 6$.■

(c) Solution: $4^{3n+1} + 2^{3n+1} + 1 \equiv 4^{3n} \cdot 4 + 2^{3n} \cdot 2 + 1 \equiv (4^3)^n \cdot 4 + (2^3)^n \cdot 2 + 1 \equiv (2^3) \equiv 4 + 2 + 1 \equiv 7 \equiv 0 \mod 7$.

(d) Solution: $13^{21}+14^{14} = (13^3)^7+(14^2)^7$, then $13^{21}+14^{14} \equiv (13^3)^7+(14^2)^7 \equiv 13^3 + 14^2 \equiv 2393 \mod 7$ by Fermat's Little Theorem.

So it is a composite and $2393|(13^{21} + 14^{14})$.

2 Solution:

Claim: $n \equiv t(n) \mod 11$.

Proof: Let $n = \sum_{i=0}^{m} a_i 10^i$, then $t(n) = \sum_{i=0}^{m}(-1)^i a_i$ and $n - t(n) = \sum_{i=0}^{m} a_i(10^i - (-1)^i)$ and since for each odd $i, 10^i - (-1)^i \equiv 10 - 10 \equiv 0 \mod 11$ and for each even $i, 10^i - (-1)^i \equiv 1 - 1 \mod 11$.

So $\forall i \in [0, m], 11|10^i - (-1)^i$, so $n \equiv t(n) \mod 11$.■

3

(a) Solution:

Claim: Infinitely many bases b for which $347|(347)_b$.

Proof: Since $(347)_b = 3b^2 + 4b + 7 \equiv 0 \mod 347$ when $b = 10$. Since we know that $3 \cdot 10^2 \equiv 3 \cdot 10^2 + 347k \mod 347, 3 \cdot 10 \equiv 3 \cdot 10 + 347k \mod 347, 7 \equiv 7 + 347k \mod 347, k \in \mathbb{Z}$. As a result, we can always find one more $k \in \mathbb{Z}$ that $10^2 + 347k = n^2, 10 + 347k = n$ that $n \in \mathbb{N}$. As a result, infinitely many bases b for which $347|(347)_b$.■

(b) Solution:

Claim: Infinitely many bases b for which $7|(347)_b$.

Proof: Basically let $n = 7k, k \in \mathbb{Z}$, that $(347)_n = 3 \cdot (7k)^2 + 4 \cdot 7k + 7$ must be divisible by 7. So we have a bijection between $\mathbb{Z}$ and all n. As a result, infinitely many bases b for which $7|(347)_b$.■

4

(a) Solution:

Claim: Reflexive, not symmetric transitive, not equivalent.

Proof: Let $x \sim x$, then $\exists k \in \mathbb{Z}$ that $x = kx$, so $x \sim x$ must be true, which proved the reflexivity. Let $s \sim t, t \sim r$, so $\exists k_1, k_2 \in \mathbb{Z}$ that $sk_1 = t, tk_2 = r$, then $sk_1k_2 = r \Rightarrow s \sim r$, which proved the transitivity. And we know that even though $2|4$, 4 does not divide 2, so the relation is not symmetric. As a result, the relation is not equivalent.■

(b) Solution:

Claim: Reflexive, symmetric not transitive, not equivalent.

Proof: Take $x \in \mathbb{R}, |x - x| = 0 \leq 1 \Rightarrow x \sim x \Rightarrow$ reflexive. Take $x, y \in \mathbb{R}, |x - y| = |y - x| \leq 1 \Rightarrow x \sim y \rightarrow y \sim x \Rightarrow$ symmetric. Since while $|3 - 2| \leq 1$ and $|4 - 3| \leq 1, |4 - 2| = 2 > 1$, so this relation is not transitive. As a result, it is not an equivalent relation.■

(c) Solution:

Claim: Reflexive, symmetric transitive, equivalent.

Proof: Take $x \in \mathbb{R}$, $x - x = 0 \in \mathbb{Z} \Rightarrow$ reflexive. Take $x, y \in \mathbb{R}$, let $x \sim y, x - y \in \mathbb{Z}, y - x = -(x - y) \in \mathbb{Z} \Rightarrow$ symmetric. Take $x, y, z \in \mathbb{R}, x \sim y, y \sim z$, so $x - z = (x - y) + (y - z) \in \mathbb{Z}$ by closure $\Rightarrow$ transitive. As a result, this is an equivalent relation.■

(d) Solution:

Claim: Reflexive, symmetric transitive, equivalent.

Proof: Take $x \in \mathbb{R}$, $x = 2^0 x \Rightarrow$ reflexive. Take $x, y \in \mathbb{R}$, let $x \sim y, \exists n \in \mathbb{Z}, x = 2^n y \Rightarrow y = 2^{-n} x \Rightarrow y \sim x \Rightarrow$ symmetric. Take $x, y, z \in \mathbb{R}, x \sim y, y \sim z$, so $\exists k_1, k_2 \in \mathbb{Z}, x = 2^{k_1} y, y = 2^{k_2} z \Rightarrow x = 2^{k_1+k_2} z \Rightarrow x \sim z \Rightarrow$ transitive. As a result, this is an equivalent relation.$\blacksquare$