7.1.10 Proof: Suppose $\varepsilon = 1$, and for any $\delta > 0$, if we choose tags in each partition to be irrational in $[0, 1]$, each for any $L \in \mathbb{R}$

$$S(f; \dot{\mathcal{P}}) = \delta \sum_{i=0}^{1/\delta} \frac{1}{x_i}$$

which does not converge since it's a p-series with $p < 1$. Hence, $g \notin \mathcal{R}[0, 1]$.

However, if we choose tags in each equal partition to be rational in $[0, 1]$, and order them by the number of subintervals of the partitions, then

$$||\dot{\mathcal{P}}_n|| = \frac{1}{n}$$

converges to 0.

And

$$|\lim S(f; \dot{\mathcal{P}}) - 0| = |\lim 0||\dot{\mathcal{P}}|| - 0| = 0 < \varepsilon$$

for all $\varepsilon > 0$.

Hence, by definition,

$$\lim S(f; \dot{\mathcal{P}}) = 0$$

■

7.2.2 Proof: Let $\dot{\mathcal{Q}}_n$ be a partition of $[0, 1]$ whose tags are all irrational, so

$$S(h; \dot{\mathcal{Q}}_n) = 0$$

for all n.

And if for $\dot{\mathcal{P}}$, we take all the right endpoints to be tags, then $S(h; \dot{\mathcal{P}}_n)$ we have

$$S(f; \dot{\mathcal{P}}_n) \geq 1 + 1 = 2$$

Hence, if $\varepsilon = 2$, then for any partition with $||\dot{\mathcal{P}}_n|| < \eta$ and $||\dot{\mathcal{Q}}_n|| < \eta$ there is always

$$|\dot{\mathcal{P}}_n - \dot{\mathcal{Q}}_n| \geq 2 = \varepsilon$$

So by Cauchy Criterion, h is not integrable on $[0, 1]$.

■

7.2.4 **Proof:** Since for any $\varepsilon > 0$, if $x \geq \varepsilon/2$, there is $|\omega - \alpha| = |2x| = 2x \geq \varepsilon$.

Hence, this does not satisfy the requirement of Squeeze Theorem.

■

7.2.6 **Claim:** ψ is not necessarily a step function.

Proof: Let

$$\psi(x) = \begin{cases} 1 & x \text{ is rational} \\ 0 & \text{otherwise} \end{cases}$$

then ψ only takes 2 values. However, since ψ is not in $\mathcal{R}[a, b]$ as shown in Example 7.2.2(b), by Theorem 7.2.5, ψ is not a step function.

■

7.2.13 **Example:** $f(x) = 1/x$ is integrable in $[c, 1]$ for any $c \in [0, 1]$ but not on $[0, 1]$.

7.2.15 **Proof:** Let $\mathcal{P} = \{I_i\}_{i=1}^n$ be a partition that $||\mathcal{P}|| < \delta$, that all discontinuous points are on the endpoints of subintervals, then we have $u_i \in I_i$ to be the minimum of I_i and $v_i \in I_i$ to be the maximum of I_i by Maximum-minimum Theorem.

Then we let $\alpha(x) = f(u_i)$ and $\omega(x) = f(v_i)$ when $x \in [x_{i-1}, x_i)$ for $i = 0, 1, \cdots, n-1$ and $\alpha(x) = f(u_n)$ and $\omega(x) = f(v_n)$ when $x \in [x_{n-1}, x_n]$.

Then by definition we have

$$\alpha(x) \leq f(x) \leq \omega(x)$$

Since f in each subintervals in $\mathcal{P}$ is continuous, and is uniformly continuous as a result, we have that for any $\varepsilon > 0$, there is $\delta > 0$ that when $u, v \in [a, b]$ and $0 < |u - v| < \delta$, there is

$$|f(c) - f(x)| < \varepsilon/(b - a)$$

So

$$\begin{aligned} \int_a^b (\omega(x) - \alpha(x)) &= \sum_{i=0}^{n} (f(v_i) - f(u_i))(x_{i+1} - x_i) \\ &< \sum_{i=0}^{n} \frac{\varepsilon}{b-a}(x_i - x_{i-1}) = \varepsilon \end{aligned}$$

Hence, by Squeeze Theorem, $f \in \mathcal{R}[a, b]$.

■

7.3.3

$$\begin{aligned}\int_{-2}^{3} g(x)dx &= \int_{-2}^{-1} g(x)dx + \int_{-1}^{1} g(x)dx + \int_{1}^{3} g(x)dx \\ &= (\frac{x^2}{2})|_{-2}^{-1} + (-\frac{x^2}{2})|_{-1}^{1} + (\frac{x^2}{2})|_{1}^{3} \\ &= \frac{1-4}{2} + 0 + \frac{9-1}{2} = \frac{5}{2}\end{aligned}$$

7.3.12

$$F(x) = \int_0^x f = \begin{cases} \int_0^x xdx = \frac{x^2}{2} & 0 \leq x < 1 \\ \int_0^x 1dx = x & 1 \leq x < 2 \\ \int_0^x xdx = \frac{x^2}{2} & 2 \leq x < 3 \end{cases}$$

With sketch

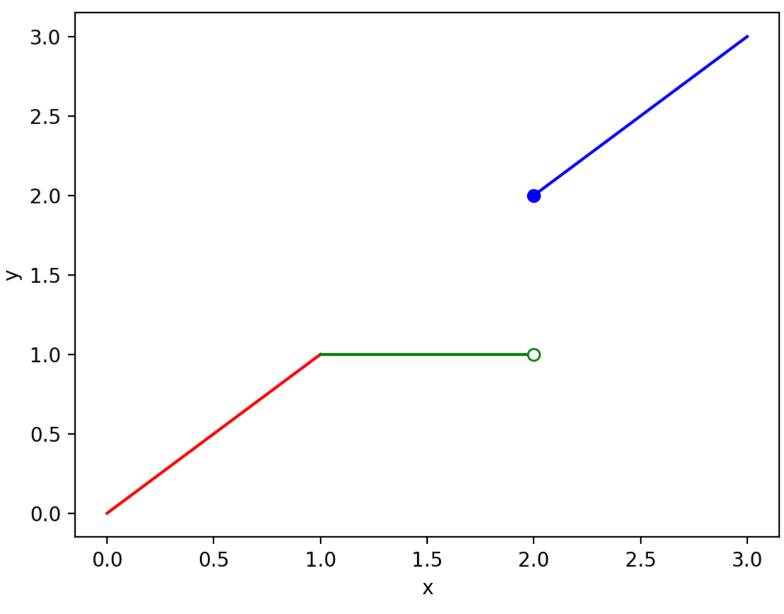

Figure 1: f(x)

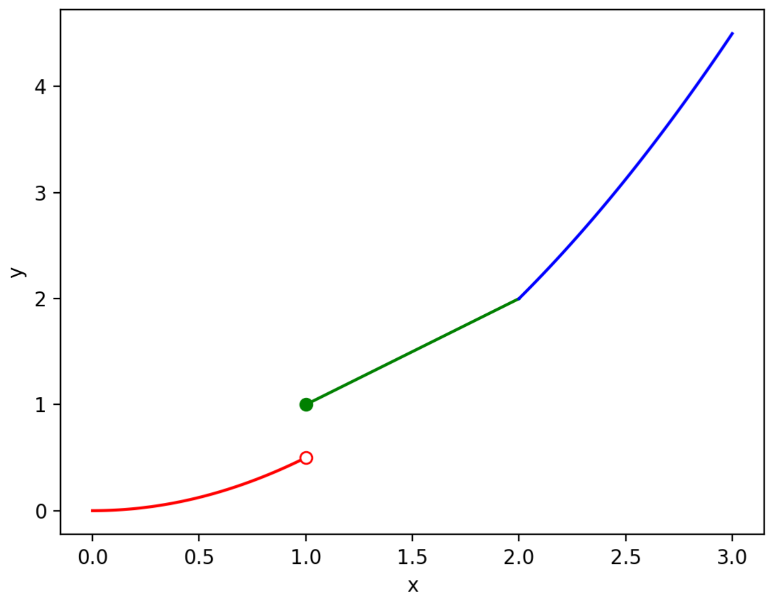

Figure 2: F(x)

So $F(x)$ is differentiable in $[0,1)\cup(1,2)\cup(2,3])$, and

$$F'(x) = \begin{cases} x & 0 \leq x < 1 \\ 1 & 1 < x < 2 \\ x & 2 < x \leq 3 \end{cases}$$