**Math 415 P13 S13** **Midterm III** May 05, 2014

Name , UI# & Section: ______________________________

- This is a closed-book, closed-notes exam. No electronic aids are allowed.
- Read each question carefully. Unless otherwise stated you need to justify your answer. *Do not use results not proven in class.*
- Answer the questions in the spaces provided on the question sheets. If you need extra room, use the back sides of each page. If you must use extra paper, make sure to write your name on it and attach it to this exam.
- Do not unstaple or detach pages from this exam.

| Question | Points | Score |
|---|---|---|
| 1 | 15 | |
| 2 | 15 | |
| 3 | 15 | |
| 4 | 15 | |
| 5 | 25 | |
| 6 | 15 | |
| Total: | 100 | |

15

1. Let $A$ be an $n \times n$ matrix.

(a) (4 points) Show that $A + A^T$ is a symmetric matrix.

$$(A + A^T)^T = A^T + (A^T)^T$$

$$= A^T + A = A + A^T$$

$\therefore A + A^T$ is symmetric

(b) (4 points) Show that $A - A^T$ is a skew-symmetric matrix.

$$(A - A^T)^T = A^T - (A^T)^T$$

$$= A^T - A = -(A - A^T)$$

$\therefore$ $A - A^T$ is skew symmetric

(c) (7 points) Prove that $A$ can be written as sum of a symmetric and a skew-symmetric matrix.

$$(A + A^T) + (A - A^T) = 2A$$

$$\therefore A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$$

where $\frac{1}{2}(A + A^T)$ is symmetric

and $\frac{1}{2}(A - A^T)$ is skew symmetric

15

2. (15 points) Solve the following linear system by using Cramer's rule:

$$\begin{cases} 4x + 2y + 6z = 0 \\ 2x + 6y + 8z = 2 \\ 6x + 8y + 18z = 0 \end{cases}$$

(Hint: you can use the fact that the determinant of the coefficient matrix is 80.)

$$\det \begin{bmatrix} 0 & 2 & 6 \\ 2 & 6 & 8 \\ 0 & 8 & 18 \end{bmatrix} = -2 \times (2\times 18 - 6\times 8) = 24$$

$$\det \begin{bmatrix} 4 & 0 & 6 \\ 2 & 2 & 8 \\ 6 & 0 & 18 \end{bmatrix} = 2\times (4\times 18 - 6\times 6) = 72$$

$$\det \begin{bmatrix} 4 & 2 & 0 \\ 2 & 6 & 2 \\ 6 & 8 & 0 \end{bmatrix} = -2\times (4\times 8 - 2\times 6) = -40$$

$$\therefore \quad x = \frac{24}{80} = \frac{3}{10}$$

$$y = \frac{72}{80} = \frac{9}{10}$$

$$z = \frac{-40}{80} = -\frac{1}{2}$$

15

3. (15 points) Find the quadratic polynomial $f(x) = a + bx + cx^2$ that best fits the data points $(-1,-1)$, $(0,0)$, $(1,1)$, $(2,0)$.

(Hint: solve the previous problem first!)

$$\underbrace{\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\vec{b}}$$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 6 \\ 2 & 6 & 8 \\ 6 & 8 & 18 \end{bmatrix}$$

$$A^T \vec{b} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$\because$ $A^T A \vec{x}^* = A^T \vec{b}$ → normal equation, which we solved in previous problem.

$\therefore$ $\vec{x}^* = \begin{bmatrix} 0.3 \\ 0.9 \\ -0.5 \end{bmatrix}$

$f(x) = 0.3 + 0.9x - 0.5x^2$

15

4. (a) (8 points) Calculate the determinant of the matrix $A = \begin{bmatrix} 7 & 2 & 0 & 0 \\ 3 & 8 & 5 & 4 \\ 2 & 7 & 4 & 1 \\ 5 & 1 & 0 & 0 \end{bmatrix}$.

$$\det A = 7 \times \det \begin{bmatrix} 8 & 5 & 4 \\ 7 & 4 & 1 \\ 1 & 0 & 0 \end{bmatrix} - 2 \times \det \begin{bmatrix} 3 & 5 & 4 \\ 2 & 4 & 1 \\ 5 & 0 & 0 \end{bmatrix}$$

$$= 7 \times [1 \times (5 \times 1 - 4 \times 4)]$$

$$- 2 \times [5 \times (5 \times 1 - 4 \times 4)]$$

$$= 7 \times (-11) - 2 \times (-55)$$

$$= 33$$

(b) (7 points) Let $A$ and $B$ are similar matrices ($AS = SB$ or $A = SBS^{-1}$). Show that $A$ and $B$ have the same characteristic polynomial, that is, $f_A(\lambda) = f_B(\lambda)$.

$$f_A(\lambda) = \det(A - \lambda I)$$

$$= \det(SBS^{-1} - \lambda I)$$

$$= \det(SBS^{-1} - S\lambda I S^{-1})$$

$$= \det(S(B - \lambda I)S^{-1})$$

$$= \det(S) \cdot \det(B - \lambda I) \cdot \det(S^{-1})$$

$$= 1 \times \det(B - \lambda I)$$

$$= f_B(\lambda)$$

15

5. Consider the matrix $A = \begin{bmatrix} 8 & 0 & 7 \\ 7 & 1 & 7 \\ 0 & 0 & 1 \end{bmatrix}$

(a) (7 points) Find the characteristic polynomial of $A$ and determine the eigenvalues and their algebraic multiplicities.

$$f_A(\lambda) = \det \begin{bmatrix} 8-\lambda & 0 & 7 \\ 7 & 1-\lambda & 7 \\ 0 & 0 & 1-\lambda \end{bmatrix}$$

$$= (1-\lambda)(8-\lambda)(1-\lambda) = 0$$

$\lambda_1 = 8$ with algebraic multiplicity 1

$\lambda_2 = 1$ with algebraic multiplicity 2.

(b) (8 points) Find the eigenspaces associated with each eigenvalue and determine the geometric multiplicities.

$$E_{\lambda_1} = \ker \begin{bmatrix} 0 & 0 & 7 \\ 7 & -7 & 7 \\ 0 & 0 & -7 \end{bmatrix} = \text{span}\left( \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right)$$

geometric multiplicity = 1

or: $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$$E_{\lambda_2} = \ker \begin{bmatrix} 7 & 0 & 7 \\ 7 & 0 & 7 \\ 0 & 0 & 0 \end{bmatrix} = \text{span}\left( \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right)$$

geometric multiplicity = 2

10

(c) (5 points) Is A diagonalizable? If so, find an invertible matrix $S$ and a diagonal matrix $D$ such that $AS = SD$.

Yes

$$S = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$D = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(d) (5 points) Find a matrix $C$ such that $C^3 = A$.
(Note: don't leave the answer as product of matrices.)

$$C = S D_0 S^{-1}$$

$$C^3 = S D_0^3 S^{-1} = A$$

$$\therefore \quad D_0 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right]_{x(-1)} \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right] \Rightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right] \qquad \therefore \quad S^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

12

6. Select one (you don't need to justify your answer).

(a) (3 points) If $A$ is skew-symmetric matrix then $A^2$

$A^T = -A$

$(A^2)^T = (A^T)^2 = (-A)^2 = A^2$

(b)

(a) must be skew-symmetric too.
(b) must be a symmetric matrix.
(c) may or may not be symmetric.

$(A^2)^T = A^T A^T = (-A)(-A) = A^2$

(b) (3 points) (Select 3) If $A$ and $B$ are orthogonal matrices then

(b, e, f)

(a) $A^2 - B^2$, (b) $A^3B^{-6}$, (c) $A + B$, (d) $A^2 + B^2$, (e) $A^{-1}$, (f) $A^TB^2$.

must be orthogonal matrices too.

(c) (3 points) $T : \mathbb{R}^3 \to \mathbb{R}^3$ is an orthogonal transformation with $T(\begin{bmatrix}1\\1\\2\end{bmatrix}) = \begin{bmatrix}2\\1\\1\end{bmatrix}$. Then

(c) $T(\begin{bmatrix}1\\1\\-1\end{bmatrix})$ can be

$V_1 \perp V_2 \Rightarrow T(V_1) \perp T(V_2)$

Also: $\|T(\vec{x})\| = \|\vec{x}\|$.

(a) $\begin{bmatrix}-2\\2\\2\end{bmatrix}$, (b) $\begin{bmatrix}-1\\-1\\1\end{bmatrix}$, (c) $\begin{bmatrix}1\\-1\\-1\end{bmatrix}$, (d) $\begin{bmatrix}-2\\-2\\-2\end{bmatrix}$,

(d) (3 points) If $A$ is a $5 \times 5$ matrix with $\det(A) = 3$, then the determinant of adj$(A)$, the adjoint of $A$, must be

(c)

$A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$

$A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$

(a) 9
(b) 27
(c) 81

$\det(\text{adj}(A)) = \det(3A^{-1}) = 3^5 \times \frac{1}{3} = 81$

$\Rightarrow \text{adj}(A) = \det(A) A^{-1} = 3A^{-1}$

$\det(\text{adj}(A)) = 3^5 \det(A^{-1})$
$= 3^5 \cdot \frac{1}{3} = 81$

(e) (3 points) Let $V$ be the set of all $3\times3$ matrices $A$ such that $\text{tr}(A) = 0$ (the trace of $A$). Then $V$ is a subspace of $\mathbb{R}^{3\times3}$.

(b).

$\text{tr}(0) = 0$; thus $0 \in V$.

(a) True
(b) False

if $A, B$ are in $V$ then $\text{tr}(A+B) = \text{tr}(A) + \text{tr}(B) = 0+0$
$\Rightarrow A+B$ is in $V$.

Similarly if $A \in V$ then $\alpha A \in V$.

10

7. (Bonus problem, 5+5 points) A square matrix $P$ is called a *projection matrix* if $P^T = P$ and $P^2 = P$. The following are examples of projection matrices:

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

Note that in each case the matrix is symmetric ($P^T = P$) and when we square the matrix we obtain the same matrix ($P^2 = P$). Prove the following:

(a) If $P$ is a projection matrix then $2P - I$ is symmetric and orthogonal.

$$(2P-I)^T = 2P^T - I^T = 2P - I$$

$\therefore$ symmetric

$$(2P-I)^T(2P-I) = (2P-I)(2P-I)$$

$$= 4P^2 - 4P + I^2$$

$$= 4P - 4P + I = I$$

$\therefore$ orthogonal

(b) If $Q$ is symmetric and orthogonal then $\frac{1}{2}(Q+I)$ is a projection matrix.

$$Q^T = Q, \quad Q^TQ = I$$

$$\therefore \quad \tfrac{1}{2}(Q+I)^T = \tfrac{1}{2}(Q^T + I^T) = \tfrac{1}{2}(Q+I)$$

$$[\tfrac{1}{2}(Q+I)]^2 = \cancel{\tfrac{1}{4}(Q^2+2Q+I)}$$

$$= \tfrac{1}{4}(Q^T+I)(Q+I)$$

$$= \tfrac{1}{4}(Q^TQ + IQ + Q^TI + I^2)$$

$$= \tfrac{1}{4}(I + Q + Q^T + I)$$

$$= \tfrac{1}{4}(2Q+2I) = \tfrac{1}{2}(Q+I)$$