13.8

Claim: If S is bounded set of real numbers, and $\sup(S), \inf(S) \in S$ then S is a closed interval.

Proof: Since $S \subseteq \mathbb{R}$ is bounded, $\sup(S), \inf(S)$ exist by Completeness Axiom. Thus $\forall n \geq \sup(S), \forall s \in S, n \geq s$. Similarly, $\forall n \leq \sup(S), \forall s \in S, n \leq s$. Since $\sup(S), \inf(S) \in S$, the equality holds, so we have $\forall s \in S, \inf(S) \leq s \leq \sup(S)$.

So we can conclude that S is a closed interval.$\blacksquare$

13.11 Suppose $\langle a \rangle, \langle b \rangle$ converge,

(a) **Claim:** $\lim a_n < \lim b_n \to \exists N \in \mathbb{N}, (n \geq N \to a_n < b_n)$.

Proof: Suppose $\exists \varepsilon \forall n \in \mathbb{N}, b_n \leq a_n \Rightarrow b_n - a_n \leq 0 \Rightarrow \lim b_n - a_n \leq 0 \Rightarrow \lim b_n - \lim a_n \leq 0$.

This implies that $\lim a_n \geq \lim b_n$ which is not true.

As a result, we can conclude that $\lim a_n < \lim b_n \to \exists N \in \mathbb{N}, (n \geq N \to a_n < b_n)$.$\blacksquare$

(b) **Claim:** It is not true that $\lim a_n \leq \lim b_n \to \exists N \in \mathbb{N}, (n \geq N \to a_n \leq b_n)$.

Proof: If $a_n = 2/n, b_n = 1/n$, we have $\lim a_n = \lim b_n = 0$ but $\forall n \in \mathbb{N}$, $a_n > b_n$.$\blacksquare$

13.22 **Solution:**

(a) $x^2 < 5x \Rightarrow x^2 - 5x < 0 \Rightarrow x(x-5) < 0 \Rightarrow 0 < x < 5$, so we know S is bounded and $\sup(S) = 5, \inf(S) = 0$.

(b) $2x^2 < x^3 + x \Rightarrow x^3 - 2x^2 + x > 0 \Rightarrow x(x-1)^2 > 0 \Rightarrow x > 1$ so S is only lower bounded and $\inf(S) = 0$.

(c) $4x^2 > x^3 + x \Rightarrow x^3 - 4x^2 + x < 0 \Rightarrow x(x^2 - 4x + 1) < 0 \Rightarrow x((x-2)^2 - 3) < 0 \Rightarrow 0 < x < 2 - \sqrt{3}$, so S is bounded and $\sup(S) = 0, \inf(S) = 2 - \sqrt{3}$.

13.25

Claim: $\lim \sqrt{1+n^{-1}} = 1$.

Proof: Since $x > 0$, $1+n^{-1} > 1 \Rightarrow \sqrt{1+n^{-1}} < 1+n^{-1} \Rightarrow \sqrt{1+n^{-1}} - 1 < 1+n^{-1} - 1 \Rightarrow \sqrt{1+n^{-1}} - 1 < n^{-1}$.

Since $1+n^{-1} > 1$, $|\sqrt{1+n^{-1}} - 1| = \sqrt{1+n^{-1}} - 1 < 1+n^{-1} - 1 = n^{-1}$. Then $n \geq N \to n^{-1} \leq N^{-1} < \varepsilon$.

Thus, we can conclude that $\lim \sqrt{1+n^{-1}} = 1$.■

13.29

Claim: Let $x_n = (1+n)/(1+2n)$, $\lim_{n\to\infty} = 1/2$.

Proof: $x_n = (1+n)/(1+2n) = (1/2+1/2+n)/(1+2n) = (1/2)/(1+2n) + 1/2 = 1/(2+4n) + 1/2$.

Take $n_1, n_2 \in \mathbb{N}$ and $n_1 < n_2$, then $[(1/2)/(1+2n_2)]/[(1/2)/(1+2n_1)] = (1+2n_1)/(1+2n_2)$. Let $n_2 - n_1 = 1 > 0$, $[(1/2)/(1+2n_2)]/[(1/2)/(1+2n_1)] = 1 - 2/(1+2n_2) > 1$.

So we know (x_n) is increasing.

$x_n = (1+n)/(1+2n) = (1/2)/(1+2n) + 1/2 = 1/(2+4n) + 1/2$. Since $n > 0$, $2+4n > 2 \Rightarrow 0 < 1/(2+4n) < 1/2 \Rightarrow 1/2 < x_n < 1$. So we proved that (x_n) is bounded with $\sup(x_n) = 1$ and $\inf(x_n) = 1/2$.

Thus we can conclude that $\lim x_n$ exists by Monotone Convergence Theorem.

And since $|x_n - 1/2| = 1/(2+4n) < 1/n \leq 1/N < \varepsilon$ when $n \geq N \in \mathbb{N}$, we proved that $\lim x_n = 1/2$.■

13.37 **Solution:** Because not every numbers construct with Cantor's technique represent a rational number.