14.5 Solution: When $b_n = 0, a_n = -2^n$, $\forall n \in \mathbb{N}, b_n > a_n, b_n \to 0$ but a_n does not converge.

14.12 The statement is false, we can construct $(a_n) = 1/\sqrt{n}, (b_n) = 1/\sqrt{n}$, $(a_n) \to 0, (b_n) \to 0$, but $\sum a_n b_n = \sum 1/n$ does not converge by the divergence of harmonic series.

14.13

Claim: If (a_n) converges, then any subsequence of (a_n) converges to the limit of (a_n).

Proof: Let $(a_n) \to L$, then $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ when $n > N$, $|a_n - L| < \varepsilon$. Take $K = N$, then when $k > K$, $n_k > n_K = n_N \geq N$, then $|a_{n_k} - L| < \varepsilon \Rightarrow a_{n_k} \to L.$ ■

14.30

Claim: If $x_1 = 1, x_{n+1} = 1/(x_1 + x_2 + \cdots + x_n)$, then (x_n) converges and $(x_n) \to 0$.

Proof: We know that $x_{n+1}/x_n = (1/(x_1+x_2+x_3+\cdots+x_n))/(1/(x_1+x_2+x_3+\cdots+x_{n-1})) = (x_1+x_2+x_3+\cdots+x_{n-1})/(x_1+x_2+x_3+\cdots+x_n) = 1 - x_n x_{n+1}$.

Since $x_n > 0$ for all x, $x_{n+1}/x_n < 1$, and therefore (x_n) converges by Ratio test.

Since $x_{n+1}/x_n = 1 - x_n x_{n+1} \Rightarrow x_{n+1} = \frac{x_n}{x_n^2+1}, n \geq 2$, then we have $x_{n+1} < \frac{x_n}{x_n^2} = 1/x_n < 0$ for all n so (x_n) is bounded by 0.

And we know that $x_{n+1}/x_n < 1$ for all n so $(x_n) \to 0$ by Monotone Convergence Theorem. ■

14.44 Solution:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{n\to\infty} (1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{n}-\frac{1}{n+1}) = \lim_{n\to\infty}(1-\frac{1}{n+1}) = 1$$

$$
\begin{aligned}
&\frac{1}{n(n+1)} < \frac{1}{n^2} < \frac{1}{n(n-1)} \\
\Rightarrow &\sum_{n=1}^{\infty} \frac{1}{n(n+1)} < \sum_{n=1}^{\infty} \frac{1}{n^2} < \sum_{n=1}^{\infty} \frac{1}{n(n-1)} \\
\Rightarrow &1 - \frac{1}{n+1} < \sum_{n=1}^{\infty} \frac{1}{n^2} < 1 - \frac{1}{n}
\end{aligned}
$$