UIUC Math 415 M13 **Final Exam** May 10, 2013

Name & UI#: ______________________________

- This is a closed-book, closed-notes exam. No electronic aids are allowed.
- Read each question carefully. Unless otherwise stated you need to justify your answer. *Do not use results not proven in class.*
- Answer the questions in the spaces provided on the question sheets. If you need extra room, use the back sides of each page. If you must use extra paper, make sure to write your name on it and attach it to this exam.
- Do not unstaple or detach pages from this exam.

Question	Points	Score
1	15	
2	10	
3	7	
4	8	
5	10	
6	10	
7	15	
8	10	
9	15	
Total:	100	

1. Let A be an $n \times n$ matrix.

(a) (4 points) Show that $A + A^T$ is a symmetric matrix.

$$(A+A^T)^T = A^T + A \Rightarrow A^T + A \text{ is symmetric.}$$

(b) (4 points) Show that $A - A^T$ is a skew-symmetric matrix.

$$(A-A^T)^T = A^T - A = -(A-A^T)$$

$$\Rightarrow A - A^T \text{ is skew-symmetric.}$$

(c) (7 points) Prove that A can be written as sum of a symmetric and a skew-symmetric matrix.

$$A = \underbrace{\frac{1}{2}(A+A^T)}_{\text{sym.}} + \underbrace{\frac{1}{2}(A-A^T)}_{\text{skew-sym.}}$$

2. (a) (7 points) Let $\mathcal{P}_3$ be the linear space of polynomials with degree less than or equal 3 and let D be the differential operator on $\mathcal{P}_3$, that is,

$$D : \mathcal{P}_3 \longrightarrow \mathcal{P}_3 \quad \text{given by} \quad D(p(x)) = p'(x).$$

Find the matrix A of D with respect to standard basis $\mathfrak{B} = (1, x, x^2, x^3)$.

$D(1) = 0$

$D(x) = 1$

$D(x^2) = 2x$

$D(x^3) = 3x^2$

$$\Rightarrow \quad B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) (3 points) Consider the basis $\mathfrak{B} = (1, 1+x, 1+x+x^2)$ for $\mathcal{P}_2$. Find the $\mathfrak{B}$-coordinate of $f(x) = 1 - x - x^2$.

$$f(x) = 1 - x - x^2 = \alpha(1) + \beta(1+x) + \theta(1+x+x^2)$$

$\Rightarrow \theta = -1;\ \beta = 0;\ \alpha = 2. \quad \Rightarrow$

$$[f(x)]_{B} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$$

3. (7 points) Find a 2×2 matrix A such that $A^k \neq I_2$ for $k = 1, 2, 3, 4, 5$ but $A^6 = I_2$.

How about rotation by $60°$? $A = \begin{bmatrix} \cos(\pi/3) & -\sin(\pi/3) \\ \sin(\pi/3) & \cos(\pi/3) \end{bmatrix}$

For $k = 1, 2, 3, 4, 5$. $A^k \vec{x} \neq \vec{x}$ as rotation by $k\frac{\pi}{3}$ yields a distinct vector.

but

$$A^6 \vec{x} = \vec{x} \text{ for every } \vec{x} \Rightarrow A^6 = I.$$

4. (8 points) Let A be an $n \times m$ matrix. Prove that rank(A)=rank(A^TA). State any theorem you use.

(Hint: In class we showed that ker(A) =ker(A^TA).)

A is $n \times m$ matrix A^TA is $m \times m$ matrix.

$\text{rank}(A) + \text{nullity}(A) = m$ (Fundamental thm. of linear alg.)

$\text{rank}(A^TA) + \text{nullity}(A^TA) = m$

Since $\text{nullity}(A) = \text{nullity}(A^TA)$ we get

$$\text{rank}(A) = \text{rank}(A^TA).$$

5. (10 points) Consider the subspace V of $\mathbb{R}^4$ given by $V = \text{span}(\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\1\\-1\\0\end{bmatrix})$. Find a basis for $V^\perp$, the orthogonal complement of V.

We want to find all the vectors $\vec{x} = \begin{bmatrix}a\\b\\c\\d\end{bmatrix}$ such that

$$\vec{x} \cdot \begin{bmatrix}1\\1\\1\\1\end{bmatrix} = 0 \quad \& \quad \vec{x} \cdot \begin{bmatrix}2\\1\\-1\\0\end{bmatrix} = 0.$$

Thus; $\quad a+b+c+d = 0$

$\quad\quad\quad 2a+b-c = 0.$

$$\left[\begin{array}{cccc:c}1 & 1 & 1 & 1 & 0\\ 2 & 1 & -1 & 0 & 0\end{array}\right] \overset{-2}{\curvearrowright}$$

$$\rightarrow \left[\begin{array}{cccc:c}1 & 1 & 1 & 1 & 0\\ 0 & -1 & -3 & -2 & 0\end{array}\right] \begin{matrix}\curvearrowleft{\scriptstyle \times 1}\\ {\scriptstyle \times(-1)}\end{matrix} \longrightarrow \begin{array}{c} \begin{array}{cccc:c} & & s & t & \end{array} \\ \left[\begin{array}{cccc:c}1 & 0 & -2 & -1 & 0\\ 0 & 1 & 3 & 2 & 0\end{array}\right]\end{array}$$

$$\vec{x} = \begin{bmatrix}a\\b\\c\\d\end{bmatrix} = \begin{bmatrix}2s+t\\-3s-2t\\s\\t\end{bmatrix} = s\begin{bmatrix}2\\-3\\1\\0\end{bmatrix} + t\begin{bmatrix}1\\-2\\0\\1\end{bmatrix}$$

$$V^\perp = \text{span}\left(\begin{bmatrix}2\\-3\\1\\0\end{bmatrix}, \begin{bmatrix}1\\-2\\0\\1\end{bmatrix}\right)$$

6. Consider $A = \begin{bmatrix} 9 & 1 & 0 \\ 5 & 0 & 1 \\ 0 & 11 & 4 \end{bmatrix}$.

(a) (7 points) Find adj(A), the adjoint of A, by using the definition of adjoint.

$$\operatorname{adj}(A) = \begin{bmatrix} +\det(A_{11}) & -\det(A_{21}) & +\det(A_{31}) \\ -\det(A_{12}) & +\det(A_{22}) & -\det(A_{32}) \\ +\det(A_{13}) & -\det(A_{23}) & +\det(A_{33}) \end{bmatrix}$$

$$= \begin{bmatrix} -11 & -4 & 1 \\ -20 & 36 & -9 \\ 55 & -99 & -5 \end{bmatrix}$$

(b) (3 points) Find the inverse of A.

$$\det(A) = \det \begin{bmatrix} 9 & 1 & 0 \\ 5 & 0 & 1 \\ 0 & 11 & 4 \end{bmatrix} = 9 \times (0-11) - 1 \times (20-0) + 0 \times (—)$$

$$= -99 - 20 = -119$$

$$\Rightarrow A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) = \frac{1}{-119} \begin{bmatrix} -11 & -4 & 1 \\ -20 & 36 & -9 \\ 55 & -99 & -5 \end{bmatrix}$$

7. (15 points) Orthogonally diagonalize the matrix $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

(*Hint:* $\lambda = 2$ is an eigenvalue.)

$$f_A(\lambda) = \det \begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix} = -(\lambda+1)^2(\lambda-2).$$

e-values are $\lambda_1 = -1$ with alg. mul. 2 & $\lambda_2 = 2$ with alg. mult. 1.

$$E_{-1} = \ker \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \operatorname{span}\left(\begin{bmatrix} -1 \\ +1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right) \overset{G.S.P}{=\!=} \operatorname{span}\left(\begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} +\frac{1}{\sqrt{6}} \\ +\frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \end{bmatrix} \right)$$

$$E_2 = \ker \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} = \operatorname{span}\left(\begin{bmatrix} +1 \\ +1 \\ +1 \end{bmatrix} \right) \overset{G.S.P}{=\!=} \operatorname{span}\left(\begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \right)$$

Put $S = \begin{bmatrix} -\frac{1}{\sqrt{2}} & +\frac{1}{\sqrt{6}} & 1/\sqrt{3} \\ \frac{1}{\sqrt{2}} & +\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$; $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

then; $AS = SD$ & S is orthogonal matrix.

8. (a) (7 points) Find the singular values of $A = \begin{bmatrix} 2 & 2 \\ 0 & 2 \\ 2 & 0 \end{bmatrix}$

$$A^TA = \begin{bmatrix} 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 0 & 2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ 4 & 8 \end{bmatrix}$$

$$f_{A^TA} = \lambda^2 - 16\lambda + 48 = (\lambda - 4)(\lambda - 12)$$

$\Rightarrow$ singular values of A are:

$$\sigma_1 = \sqrt{12} = 2\sqrt{3} \quad \& \quad \sigma_2 = \sqrt{4} = 2$$

(b) (3 points) What is the Σ matrix in the singular value decomposition of A above?

[Recall: Σ matrix in SVD of A has the same size that A has, & "diagonal" entries of Σ are singular values of A].

$$\Sigma = \begin{bmatrix} 2\sqrt{3} & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$$

9. Select one (you don't need to justify your answer).

(a) (2 points) Let $\vec{v}_1$ and $\vec{v}_2$ be vectors in $\mathbb{R}^n$ such that $\vec{v}_1 \cdot \vec{v}_1 = 9$, $\vec{v}_1 \cdot \vec{v}_2 = 10$, $\vec{v}_2 \cdot \vec{v}_2 = 25$. Then $\|\vec{v}_1 + 2\vec{v}_2\|$ equals

(a) $\sqrt{151}$ (b) $\sqrt{149}$ (c) $\sqrt{117}$ (d) $\sqrt{159}$ (e) 59

$\|v_1 + 2v_2\|^2 = (v_1+2v_2)\cdot(v_1+2v_2) = v_1 \cdot v_1 + 4 v_1 \cdot v_2 + 4 v_2 \cdot v_2 = 9 + 40 + 100$

(b) (2 points) There are vectors $\vec{v}_1$ and $\vec{v}_2$ in $\mathbb{R}^n$ such that $\vec{v}_1 \cdot \vec{v}_1 = 4$, $\vec{v}_1 \cdot \vec{v}_2 = -7$, $\vec{v}_2 \cdot \vec{v}_2 = 9$.

Cauchy-Schwarz

(a) True (b) False

$|v_1 \cdot v_2| \leq \|v_1\| \, \|v_2\|$

$\rightarrow \operatorname{im}(A) = \operatorname{span}\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} \right)$

(c) (4 points) Consider $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 6 & 0 \\ 1 & 2 & 3 & 4 \end{bmatrix}$. For which vectors $\vec{b}$ below the system $A\vec{x} = \vec{b}$ is consistent? (Select two).

$A\vec{x} = \vec{b}$ is consistent $\Leftrightarrow$ $\vec{b} \in \operatorname{im}(A)$

(a) $\begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}$ (b) $\begin{bmatrix} 8 \\ 2 \\ 7 \end{bmatrix}$ (c) $\begin{bmatrix} 1 \\ 2 \\ 8 \end{bmatrix}$ (d) $\begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}$ (e) $\begin{bmatrix} 8 \\ 8 \\ 1 \end{bmatrix}$

(d) (3 points) The matrix $A = \begin{bmatrix} p & 2 \\ 2 & p \end{bmatrix}$ is positive semi-definite when p is in

(a) $[2, \infty)$ (b) $(-\infty, 2]$ (c) $[-2, 0]$ (d) $[-2, 2]$ (e) $[-\sqrt{2}, \sqrt{2}]$.

(e) (2 points) Let W be the subspace of $\mathbb{R}^{2\times 2}$ including all the matrices that commutes with $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$. Then the dimension of W is

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & a+b \end{bmatrix}$

$\rightarrow c = 0$
$\rightarrow a + b = d$
$\rightarrow c + d = d$

(f) (2 points) Let A be a 4×4 matrix such that $f_A(\lambda) = (1-\lambda)(2-\lambda)^3$. Then

(a) $\det(A) = -7$, $\operatorname{tr}(A) = 8$ (b) $\det(A) = 8$, $\operatorname{tr}(A) = 7$ (c) $\det(A) = 8$, $\operatorname{tr}(A) = -7$ (d) $\det(A) = 7$, $\operatorname{tr}(A) = -8$

~~a~~ 1, 2, 2, 2