2.1.5

Claim: The group form by $\mathbb{Z}_4$ under addition C_4 and the group form by the symmetries of rectangle under the composition of symmetries K_4 are not isomorphic.

Proof: The multiplication tables of each group are as the following:

+	[0]	[1]	[2]	[3]
[0]	[0]	[1]	[2]	[3]
[1]	[1]	[2]	[3]	[0]
[2]	[2]	[3]	[0]	[1]
[3]	[3]	[0]	[1]	[2]

Table 1: Multiplication Table of C_4

$\cdot$	e	r_1	r_2	r_3
e	e	r_1	r_2	r_3
r_1	r_1	e	r_3	r_2
r_2	r_2	r_3	e	r_1
r_3	r_3	r_2	r_1	e

Table 2: Multiplication Table of K_4

And their Cayley graphs can show the differences more clearly.

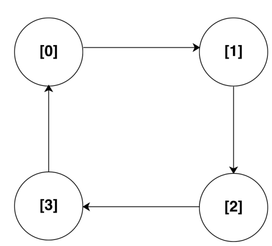


Figure 1: Cayley Graph of C_4

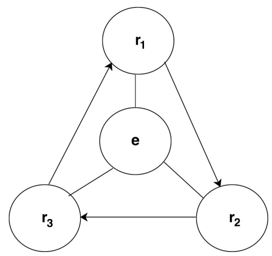


Figure 2: Cayley Graph of K_4

So we can conclude that The group form by $\mathbb{Z}_4$ under addition C_4 and the group form by the symmetries of rectangle under the composition of symmetries K_4 are not isomorphic.$\blacksquare$

2.1.15

Claim: The following mathematical statements are equivalent for a group G:

1. G is abelian
2. $\forall a, b \in G, (ab)^{-1} = a^{-1}b^{-1}$
3. $\forall a, b \in G, aba^{-1}b^{-1} = e$
4. $\forall a, b \in G, (ab)^2 = a^2b^2$
5. $\forall a, b \in G, n \in \mathbb{N}, (ab)^n = a^nb^n$

Proof: To prove all those statements are equivalent, we can prove that $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5 \Rightarrow 1$. So we can prove them separately.

$1 \Rightarrow 2$: Since G is an abelian group, so $\forall a, b \in G, ab = ba$.
Then since $(ab)^{-1} = b^{-1}a^{-1}$ and $b^{-1}a^{-1} = a^{-1}b^{-1}$, so $(ab)^{-1} = a^{-1}b^{-1}$ is proved.

$2 \Rightarrow 3$: $aba^{-1}b^{-1} = ab(a^{-1}b^{-1}) = (ab)(ab)^{-1} = e$.

$3 \Rightarrow 4$: $(ab)^2 = (ab)(ab)$. Since $aba^{-1}b^{-1} = e$, $aba^{-1}b^{-1}ba = ba \Leftrightarrow ab = ba$, so $(ab)(ab) = (ab)(ba) = a(bb)a = ab^2a = aab^2 = a^2b^2$.

$4 \Rightarrow 5$: Since $(ab)^2 = a^2b^2$, $(ab)^2(ab)^{-1} = a^2b^2(ab)^{-1} \Leftrightarrow ab = a^2b^2b^{-1}a^{-1} = a^2ba^{-1} \Leftrightarrow a^{-1}aba = a^{-1}a^2ba^{-1}a \Leftrightarrow ba = ab$.

Then we can use mathematical induction to prove this statement.

Base case: when $n = 1$, $ab = ab$ is obvious.

Suppose that when $n = k$, we have $(ab)^k = a^kb^k$, then if $n = k+1$, $(ab)^{k+1} = (ab)^k \cdot (ab) = a^kb^k(ab) = a^kb^k(ba) = a^k(b^kb)a = a^kb^{k+1}a = a^kab^{k+1} = a^{k+1}b^{k+1}$.

So we can conclude that $\forall a, b \in G, n \in \mathbb{N}, (ab)^n = a^nb^n$.

$5 \Rightarrow 1$: Since $\forall a, b \in G, n \in \mathbb{N}, (ab)^n = a^nb^n$, let $n = 2$, then $(ab)^2 = a^2b^2$. So $(ab)^2 = a^2b^2$, $(ab)^2(ab)^{-1} = a^2b^2(ab)^{-1} \Leftrightarrow ab = a^2b^2b^{-1}a^{-1} = a^2ba^{-1} \Leftrightarrow a^{-1}aba = a^{-1}a^2ba^{-1}a \Leftrightarrow ba = ab$.

Thus, we can finally conclude that the mathematical statements above are equivalent for a group G.■

2.2.2 Solution:

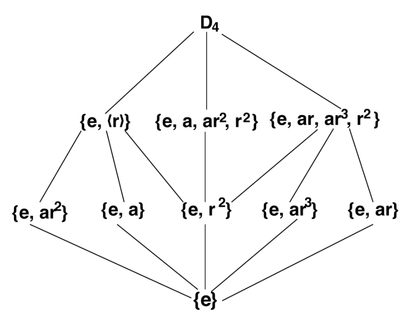


Figure 3: Subgroup Lattice of D_4

2.2.7

Claim: If $H_1, H_2, \cdots, H_n$ are subgroups of G, then $\bigcap_\alpha H_\alpha$ is s subgroup of G.

Proof: We can use induction to prove that $\forall n \in \mathbb{N}$, we have $\bigcap_\alpha H_\alpha$ is s subgroup of G.

Base case, when $n = 1$, if H_1 is a subgroup of G, $\bigcap_{i=1}^{1} H_i = H_1$ is a subgroup of G is obviously true.

Suppose that when $n = k$, $\bigcap_{i=1}^{k} H_i$. Then if $n = k + 1$, $\bigcap_{i=1}^{k+1} H_i = (\bigcap_{i=1}^{k} H_i) \bigcap H_{k+1}$ while $\bigcap_{i=1}^{k} H_i$ and H_{k+1} are both subgroups of G.

If $\bigcap_{i=1}^{k} H_i \subseteq H_{k+1}$ or $H_{k+1} \subseteq \bigcap_{i=1}^{k} H_i$, then $\bigcap_{i=1}^{k+1} H_i = H_{k+1}$ or $\bigcap_{i=1}^{k} H_i$. According to base case, $\bigcap_{i=1}^{k+1} H_i = H_{k+1}$ is a subgroup of G.

If $\bigcap_{i=1}^{k} H_i \nsubseteq H_{k+1}$ or $H_{k+1} \nsubseteq \bigcap_{i=1}^{k} H_i$ and $\bigcap_{i=1}^{k+1} H_i \neq \emptyset$, since for all i, $e \in H_i$, $e \in \bigcap_{i=1}^{k+1} H_i$. Take $g \in \bigcap_{i=1}^{k+1} H_i$, then for all i, $g \in H_i$. Since each H_i is a subgroup, $g^{-1} \in H_i$ for each i, thus $g^{-1} \in \bigcap_{i=1}^{k+1} H_i$. Take $g, h \in H$. Then $g, h \in H_i$ for every i, so $gh \in H_i$ for every i. Thus $gh \in H$.

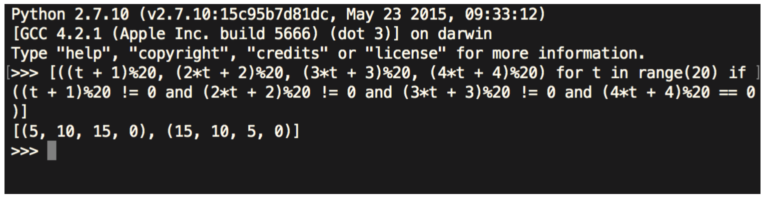
So we can conclude that If $H_1, H_2, \cdots, H_n$ are subgroups of G, then $\bigcap_\alpha H_\alpha$ is s subgroup of G.■

2.2.24

Claim: If there's at least 3 elements of order 4, the group of order 20 cannot be cyclic. If there's exactly 2 element of order 4, the group of order 20 can be cyclic.

Proof: Suppose the group G of order 20 has at least 3 elements of order 4 is cyclic. That is to say that $G \cong C_{20} \cong \mathbb{Z}_{20}$. Since there're only 20 elements in $\mathbb{Z}_{20}$, we can check all elements if they are element of order 4.

By definition, if $[i] \in \mathbb{Z}_{20}$ is an order 4 element, it is expected that $(4 \cdot x) \mod 20 = 0$ and for $i = 1, 2, 3$, $(i \cdot x) \mod 20 \neq 0$. Since it's tiring to calculate all 20 elements, so I wrote some code to do this for me as the following with Python 2.7.10(next page).



```
Python 2.7.10 (v2.7.10:15c95b7d81dc, May 23 2015, 09:33:12)
[GCC 4.2.1 (Apple Inc. build 5666) (dot 3)] on darwin
Type "help", "copyright", "credits" or "license" for more information.
>>> [((t + 1)%20, (2*t + 2)%20, (3*t + 3)%20, (4*t + 4)%20) for t in range(20) if
((t + 1)%20 != 0 and (2*t + 2)%20 != 0 and (3*t + 3)%20 != 0 and (4*t + 4)%20 == 0
)]
[(5, 10, 15, 0), (15, 10, 5, 0)]
>>>
```

Figure 4: Python Code

As the code shows, there's only 2 elements of order 4 in C_{20}.

As a result, we proved that, if there's at least 3 elements of order 4, the group of order 20 cannot be cyclic. If there's exactly 2 element of order 4, the group of order 20 can be cyclic.■