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Given an undirected graph $G = (V, E)$ a *matching* in G is a set of edges $M \subseteq E$ such that no two edges in M share a node. A matching M is *perfect* if $2|M| = |V|$, in other words if every node is incident to some edge of M. PerfectMatching is the following decision problem: does a given graph G have a perfect matching? Describe a polynomial-time reduction from PerfectMatching to SAT. Does this problem that PerfectMatching is a difficult problem?

Solution: First of all we can easily check if $2|M| = |V|$ is true if given an answer in linear time $O(1)$, thus the problem is in NP.

Then we can give a construct to reduce PerfectMatch to SAT:

- For each $v \in V$, encode v as $c(v) \in \{0, 1\}$, then PerfectMatch requires that there is a set of code $c(v_i)$ for all $v_i \in V$ that makes

$$\bigwedge_{v_i \in V} \Big(\bigvee_{v_j \in adj(v_i) - \{v_i\}} (v_i \wedge v_j) \wedge \big(\bigwedge_{v_k \in adj(v_i) - \{v_i, v_j\}} (v_i \wedge \neg v_k) \big) \Big) = 1$$

  to be true, which is essentially the SAT problem.

How we claim that PerfectMatch returns true if SAT returns true and PerfectMatch returns false if SAT returns false.

Proof: "$\Rightarrow$":

When there is a PerfectMatch for $G = (V, E)$, then for any given v_i, there is exactly 1 points that has the same code as it, so

$$\bigvee_{v_j \in adj(v_i) - \{v_i\}} (v_i \wedge v_j) \wedge \big(\bigwedge_{v_k \in adj(v_i) - \{v_i, v_j\}} (v_i \wedge \neg v_k) \big) = 1$$

which gives that

$$\bigwedge_{v_i \in V} \Big(\bigvee_{v_j \in adj(v_i) - \{v_i\}} (v_i \wedge v_j) \wedge \big(\bigwedge_{v_k \in adj(v_i) - \{v_i, v_j\}} (v_i \wedge \neg v_k) \big) \Big) = 1$$

"$\Leftarrow$":

When

$$\bigwedge_{v_i \in V} \Big(\bigvee_{v_j \in adj(v_i) - \{v_i\}} (v_i \wedge v_j) \wedge \big(\bigwedge_{v_k \in adj(v_i) - \{v_i, v_j\}} (v_i \wedge \neg v_k) \big) \Big) = 1$$

for any given v_i, there is exactly 1 points that has the same code with it, so

$$\bigvee_{v_j \in adj(v_i) - \{v_i\}} (v_i \wedge v_j) \wedge \big(\bigwedge_{v_k \in adj(v_i) - \{v_i, v_j\}} (v_i \wedge \neg v_k) \big) = 1$$

which means at least one encoding makes

$$(v_i \wedge v_j) \wedge (\bigwedge_{v_k \in adj(v_i) - \{v_i, v_j\}} (v_i \wedge \neg v_k) = 1$$

so for $G = (V, E)$, then for any given v_i, there is exactly 1 points that has the same code as it.

Hence, we conclude that PerfectMatch returns true if SAT returns true and PerfectMatch returns false if SAT returns false.

Since SAT Problem is a NP-Complete Problem, PerfectMatch is harder than SAT, so PerfectMatch is a difficult problem.

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