3.1.5

(a) **Proof:** Given $\varepsilon > 0$, we note that if $n \in \mathbb{N}$, then

$$\frac{n}{n^2+1} < \frac{n}{n^2} = \frac{1}{n}$$

Then if we choose N that $1/N < \varepsilon$, then when $n \geq N$, we have that

$$\left|\frac{n}{n^2+1} - 0\right| = \frac{n}{n^2+1} < \frac{n}{n^2} = \frac{1}{n} \leq \frac{1}{N} < \varepsilon$$

Then by definition, we have

$$\lim\left(\frac{n}{n^2+1}\right) = 0$$

■

(b) **Proof:** Given $\varepsilon > 0$, we note that if $n \in \mathbb{N}$, then

$$\frac{1}{n+1} < \frac{1}{n}$$

Then if we choose N that $2/N < \varepsilon$, then when $n \leq N$, we have that

$$\left|\frac{2n}{n+1} - 2\right| = \left|\frac{-2}{n+1}\right| = \frac{2}{n+1} < \frac{2}{n} \leq \frac{2}{N} < \varepsilon$$

Then by definition, we have

$$\lim\left(\frac{2n}{n+1}\right) = 2$$

■

(c) **Proof:** Given $\varepsilon > 0$, we note that if $n \in \mathbb{N}$, then

$$\frac{1}{4n+10} < \frac{1}{4n}$$

Then if we choose N that $13/4N < \varepsilon$, then when $n \leq N$, we have that

$$\left|\frac{3n+1}{2n+5} - \frac{3}{2}\right| = \left|\frac{-13}{4n+10}\right| = \frac{13}{4n+10} < \frac{13}{4n} \leq \frac{13}{4N} < \varepsilon$$

Then by definition, we have

$$\lim\left(\frac{3n+1}{2n+5}\right) = \frac{3}{2}$$

■

(d) **Proof:** Given $\varepsilon > 0$, we note that if $n \in \mathbb{N}$, then

$$\frac{1}{4n^2+6} < \frac{1}{4n^2} \leq \frac{1}{4n}$$

Then if we choose N that $2/N < \varepsilon$, then when $n \leq N$, we have that

$$\left|\frac{n^2-1}{2n^2+3} - \frac{1}{2}\right| = \left|\frac{-5}{4n^2+6}\right| = \frac{5}{4n^2+6} < \frac{5}{4n^2} \leq \frac{5}{4n} \leq \frac{5}{4N} < \varepsilon$$

Then by definition, we have

$$\lim\left(\frac{n^2-1}{2n^2+3}\right) = \frac{1}{2}$$

■

3.1.9 **Proof:** Since $\lim(x_n) = 0$, then for all $\varepsilon > 0$ there is N that when $n \leq N$, we have $|x_n - 0| = |x_n| = x_n < \varepsilon$.

Then we have $|x_n| < \varepsilon^2$ for $n \leq N \in \mathbb{N}$. As a result, $\sqrt{x_n} < \varepsilon \Rightarrow |\sqrt{x_n} - 0| < \varepsilon$.

As a result, $\lim(\sqrt{x_n}) = 0$.

■

3.1.12 **Proof:** Given $\varepsilon > 0$, we note that if $n \in \mathbb{N}$, then

$$\frac{1}{\sqrt{n^2+1}+n} \leq \frac{1}{\sqrt{n^2+1}} \leq \frac{1}{n^2+1}$$

Then if we choose N that $1/N < \epsilon$, then when $n \leq N$, we have

$$\begin{aligned}
\left|\sqrt{n^2+1} - n\right| &= \left|(\sqrt{n^2+1} - n)\frac{(\sqrt{n^2+1}+n)}{(\sqrt{n^2+1}+n)}\right| \\
&= \left|\frac{1}{\sqrt{n^2+1}+n}\right| \\
&= \frac{1}{\sqrt{n^2+1}+n} \\
&\leq \frac{1}{\sqrt{n^2+1}} \\
&\leq \frac{1}{n^2+1} < \frac{1}{n^2} \leq \frac{1}{n} \leq \frac{1}{N} < \varepsilon
\end{aligned}$$

As a result, by definition

$$\lim \sqrt{n^2+1} - n = 0$$

■

3.1.17 Proof: Given $\varepsilon > 0$, we note that if $n \in \mathbb{N}$, then

$$\frac{1}{2^{n-1}} \leq \frac{1}{n-1} < \frac{1}{n}$$

Then if we choose N that $1/N < \varepsilon$, then when $n \geq N$, we have that

$$\begin{aligned}
\left|\frac{2^n}{n!} - 0\right| &= \frac{2^n}{n!} \\
&\leq 2(\frac{2}{3})^{n-2} && \text{(by hint)} \\
&< 2(\frac{1}{2})^{n-2} && (\text{since } \frac{2}{3} > \frac{1}{2}) \\
&= (\frac{1}{2})^{n-1} \\
&= \frac{1}{2^{n-1}} \\
&\leq \frac{1}{n-1} < \frac{1}{n} \leq \frac{1}{N} < \varepsilon
\end{aligned}$$

Then by definition, we have

$$\lim \frac{2^n}{n!} = 0$$

■

3.2.6 Solution:

(a)

$$\lim\left((2+1/n)^2\right) = \lim(4+4/n+1/n^2) = \lim(4)+\lim(4/n)+\lim(1/n^2) = 4+0+0 = 4$$

(b)

$$\lim\left(\frac{(-1)^n}{n+2}\right) = \lim\left((-1)^n\right) \cdot \lim\left(\frac{1}{n+2}\right) = \lim\left((-1)^n\right) \cdot 0 = 0$$

(c)

$$\lim\left(\frac{\sqrt{n}-1}{\sqrt{n}+1}\right) = \lim\left(\frac{(\sqrt{n}-1)(\sqrt{n}-1)}{(\sqrt{n}+1)(\sqrt{n}-1)}\right) = \lim\left(\frac{n-2\sqrt{n}+1}{n-1}\right) = \lim\left(1-\frac{2}{\sqrt{n}+1}\right)$$
$$= \lim(1) - \lim\left(\frac{2}{\sqrt{n}+1}\right) = 1-0=1$$

(d)

$$\lim\left(\frac{n+1}{n\sqrt{n}}\right) = \lim\left(\frac{n}{n\sqrt{n}}+\frac{1}{n\sqrt{n}}\right) = \lim\left(\frac{1}{n\sqrt{n}}\right)+\lim\left(\frac{1}{n^{3/2}}\right) = 0+0=0$$

3.2.10 Solution:

(a)

$$\lim(\sqrt{4n^2+n}-2n) = \lim\left((\sqrt{4n^2+n}-2n)\frac{\sqrt{4n^2+n}+2n}{\sqrt{4n^2+n}+2n}\right) = \lim\left(\frac{1}{\sqrt{4+\frac{1}{n}}+2}\right)$$
$$= \frac{1}{\lim\left(\sqrt{4+\frac{1}{n}}+2\right)} = \frac{1}{4}$$

(b)

$$\lim\left(\sqrt{n^2+5n}-n\right) = \lim\left((\sqrt{n^2+5n}-n)\frac{\sqrt{n^2+5n}+n}{\sqrt{n^2+5n}+n}\right)$$
$$= \lim\left(\frac{5n}{\sqrt{n^2+5n}+n}\right)$$
$$= \frac{5}{\lim\left(\sqrt{1+\frac{5}{n}}+1\right)} = \frac{5}{2}$$

3.2.11a Solution:

$$\lim((3\sqrt{n})^{1/2n}) = \lim(3^{1/2n}(\sqrt{n})^{1/2n}) = \lim(3^{1/2n})\lim(n^{1/4n}) = \lim(n^{1/4n}) = \lim(n^n)^{1/4} = \infty$$

3.2.16 Proof:

(a) Since
$$L = \lim(x_{n+1}/x_n) = \lim(a^{n+1}/a^n) = a < 1,$$
(a^n) converges and $\lim(a^n) = 0$.

■

(b) Since

$$L = \lim(x_{n+1}/x_n) = \lim((b^{n+1}/2^{n+1})/(b^n/2^n)) = \lim(b/2) = b/2 > 0.5$$

Then if $b/2 < 1 \Rightarrow b < 2$, then $(b^n/2^n)$ converges and $\lim(b^n/2^n) = 0$.

■

(c) Since

$$L = \lim(x_{n+1}/x_n) = \lim(((n+1)/b^{n+1})/(n/b^n)) = \lim(n/b) = \infty$$

does not converge.

So (n/b^n)'s convergence is not guaranteed.

■

(d) Since

$$L = \lim(x_{n+1}/x_n) = \lim((2^{3(n+1)}/3^{2(n+1)})/((2^{3n}/3^{2n})) = \lim(2^3/3^2) = 8/9 < 1$$

does not converge.

$(2^{3n}/3^{2n})$ converges and $\lim(2^{3n}/3^{2n}) = 0$.

■

3.2.17 Solution:

(a) When $(x_n) = 1$, $\lim(x_{n+1}/x_n) = 1$ and (x_n) converges to 1.

(b) When $(x_n) = n$, $\lim(x_{n+1}/x_n) = \lim(n+1)/n = 1$ and (x_n) diverges.

3.2.18 Proof: Suppose X is bounded by u, then by Completeness Axiom there is a $u' = \sup X$. Then there is a $x \in X$ that $x > u' - 1 \Rightarrow x + 1 > u'$. However, since $\lim(x_{n+1}/x_n) > 1$, for n that $x_n + 1 > u'$, $x_{n+1} > x_n + 1 > u'$. Hence, u' is not the supremum, and as a result, X is not bounded.

■