Let

$$f_1(x) = -\sin(x) - \frac{1}{8}\sin(2x) \text{ on } [0, 2\pi)$$

$$f_2(x) = \frac{x-1}{x+1} \text{ on } [0, 1)$$

1. Provide an exact (symbolic) form for the bound constants for the two functions: C_1 and C_2.
2. Provide values ξ_1 and ξ_2 which yield the maximum C_1 and C_2.

Solutions:

1. By running python code:

```
In [1]: import sympy as sp
In [2]: x = sp.Symbol("x")
In [3]: from math import factorial
In [4]: def C(f, n):
   ...:     return abs(f.diff(x, n + 1)) / factorial(n
        + 1)
In [5]: f1 = -sp.sin(x) - sp.sin(2 * x) / 8
In [6]: f2 = (x - 1) / (x + 1)
In [7]: C1 = C(f1, 2)

In [8]: C2 = C(f2, 2)

In [9]: C1
Out[9]: Abs(cos(x) + cos(2*x))/6

In [10]: C2
Out[10]: Abs(((x - 1)/(x + 1) - 1)/(x + 1)**3)
```

we have:

$$\begin{aligned} C_1 &= \max_{\xi \in [0, 2\pi)} \frac{|f^{(n+1)}(\xi)|}{(n+1)!} \\ &= \max_{\xi \in [0, 2\pi)} \frac{|f^{(2+1)}(\xi)|}{(2+1)!} \\ &= \max_{\xi \in [0, 2\pi)} \frac{|\cos x + \cos 2x|}{6} \end{aligned}$$

$$
\begin{aligned}
C_2 &= \max_{\xi \in [0,2\pi)} \frac{|f^{(n+1)}(\xi)|}{(n+1)!} \\
&= \max_{\xi \in [0,2\pi)} \frac{|f^{(2+1)}(\xi)|}{(2+1)!} \\
&= \max_{\xi \in [0,1)} \left| \frac{1}{(x+1)^3} \left(\frac{x-1}{x+1} - 1 \right) \right| \\
&= -((x-1)/(x+1)-1)/(x+1)^3
\end{aligned}
$$

on $[0, 1)$

2. Solve $C_1' = 0$, we get that $\xi_1 = 0$ to be the critical point, and checked when $\xi_1 = 0$, we got the maximum value by 2-order derivative test.

   So,
   $$C_1 = \max_{\xi \in [0,2\pi)} \frac{|\cos x + \cos 2x|}{6} = \frac{|\cos 0 + \cos 0|}{6} = \frac{1}{3}$$

   Similarly, solve $C_2' = 0$, we get that $\xi_2 = 0$ to be the critical point, and checked when $\xi_2 = 0$, we got the maximum value by 2-order derivative test.

   So,
   $$C_2 = -((0-1)/(0+1)-1)/(0+1)^3 = 2.$$