Math 415 P13 S13 **Midterm II** April 07, 2014

Name & UI#: ____

- This is a closed-book, closed-notes exam. No electronic aids are allowed.
- Read each question carefully. Unless otherwise stated you need to justify your answer. *Do not use results not proven in class.*
- Answer the questions in the spaces provided on the question sheets. If you need extra room, use the back sides of each page. If you must use extra paper, make sure to write your name on it and attach it to this exam.
- Do not unstaple or detach pages from this exam.

Question	Points	Score
1	15	
2	25	
3	5	
4	10	
5	10	
6	15	
7	10	
8	10	
Total:	100	

1. Consider the matrix $A = \begin{bmatrix} 1 & 4 & 7 & 8 \\ 2 & 0 & 6 & 2 \\ 0 & 1 & 1 & 2 \end{bmatrix}$.

   (a) (8 points) Find a basis for $\mathrm{im}(A)$, the image of A.

   (b) (7 points) Find a basis for $\ker(A)$, the kernel of A.

2. Consider $T : \mathcal{P}_2 \longrightarrow \mathcal{P}_2$ defined by $T(p(x)) = (1+x)p'(x) + p''(x) + 2p(x)$.

   (a) (5 points) Show that T is a linear transformation.

   (b) (6 points) Find the matrix A of T relative to standard basis $\mathfrak{A} = (1, x, x^2)$. Is T an isomorphism?

(c) (5 points) Find the matrix B of T relative to the basis $\mathfrak{B} = (1, 1+x, 2+2x+x^2)$.

(d) (4 points) What is the change of basis matrix S from $\mathfrak{B}$ to $\mathfrak{A}$?

(e) (5 points) Find a formula for A^k for any positive integer k.
(Don't leave the answer as a product of matrices.)

3. (5 points) Let $U^{2\times 2}$ denote the vector space of upper triangular 2×2 matrices. Find all values of a so that

$$\mathfrak{B} = \left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2a \\ 0 & a^2 \end{bmatrix} \right)$$

is a basis for $U^{2\times 2}$.

4. (10 points) Let P be the plane in $\mathbb{R}^3$ given by $x + 2y - 3z = 0$. Consider the linear transformation

$$T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \text{ given by } T(\vec{x}) = \mathrm{proj}_P(\vec{x}),$$

the projection onto P. Construct a basis $\mathfrak{B} = (\vec{v}_1, \vec{v}_2, \vec{v}_3)$ for $\mathbb{R}^3$ such that the matrix B of T relative to $\mathfrak{B}$ is a diagonal matrix. (You need to write the matrix B too).

(Hint: pick two linearly independent vectors $\vec{v}_1$ and $\vec{v}_2$ from the plane and pick $\vec{v}_3$ perpendicular to the plane.)

5. (a) (3 points) Let A be $n\times k$, B be $k\times m$ matrices. Show that $\ker(B) \subseteq \ker(AB)$.

(b) (7 points) Let A be 5×3, B be 3×5 matrices. Prove that AB cannot be invertible.

6. Let V be the subspace of $\mathbb{R}^4$ spanned by the vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 4 \\ 2 \\ -2 \\ 4 \end{bmatrix}$.

(a) (10 points) Find an orthonormal basis for V.

(b) (5 points) Find the projection of $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ on V.

7. (10 points) Find the QR-factorization of the matrix $M = \begin{bmatrix} 1 & 4 & -3 \\ 1 & 2 & 1 \\ -1 & -2 & -3 \\ -1 & 4 & 1 \end{bmatrix}$.

8. Select one (you don't need to justify your answer).

(a) (2 points) We fix a vector $\vec{v}$ in $\mathbb{R}^4$. Let V be the set of all 4×4 matrices A such that $A\vec{v} = \vec{0}$. Then V is a subspace of $\mathbb{R}^{4\times4}$.

(a) True
(b) False

(b) (2 points) If A is a 3×7 matrix then

(a) rank(A) must be greater than 3.
(b) nullity(A) must be 4 or more.
(c) ker(A) can be 2 dimensional.

(c) (2 points) There is a linear isomorphism from $\mathcal{P}_3$ to $U^{2\times2}$.

(a) True
(b) False

(d) (2 points) Let $\vec{v}_1$ and $\vec{v}_2$ be vectors in $\mathbb{R}^n$ such that $\|\vec{v}_1\| = 2$, $\|\vec{v}_2\| = 3$, $\|\vec{v}_1+\vec{v}_2\| = 4$. Then $\cos(\theta)$, the cosine of the angle between $\vec{v}_1$ and $\vec{v}_2$, is

(a) $\frac{1}{3}$ (b) $\frac{1}{4}$ (c) $\frac{1}{5}$ (d) $\frac{1}{6}$

(e) (2 points) Let $C[0,1]$ be the vector space of all continuous functions defined from interval [0,1] to $\mathbb{R}$. Consider $T : C[0,1] \to C[0,1]$ given by

$$T(f(x)) = f(x) - \int_0^1 f(x)dx.$$

(a) T is not a linear transformation, so not an isomorphism.
(b) T is a linear transformation but not 1-to-1 ($\ker(T) \neq \{0\}$), so not an isomorphism.
(c) T is an isomorphism.

9. (10 points) (Bonus problem!) Let $\vec{v}_1, \vec{v}_2, ..., \vec{v}_m$ be vectors in $\mathbb{R}^n$. Prove that

$$\|\vec{v}_1 + \vec{v}_2 + \cdots + \vec{v}_m\| \leq \|\vec{v}_1\| + \|\vec{v}_2\| + \cdots + \|\vec{v}_m\|.$$

State any theorem & inequality you use.