4.2.2

(a)

$$\begin{aligned}\lim_{x\to 2}\sqrt{\frac{2x+1}{x+3}} &= \sqrt{\lim_{x\to 2}\frac{2x+1}{x+3}} && \text{(by Exercise 4.2.15)}\\ &= \sqrt{\frac{\lim_{x\to 2} 2x+1}{\lim_{x\to 2} x+3}} && \text{(by Theorem 4.2.4 (a))}\\ &= \sqrt{\frac{5}{5}} = 1\end{aligned}$$

(b)

$$\lim_{x\to 2}\frac{x^2-4}{x-2} = \lim_{x\to 2} x+2 = 4$$

(c)

$$\lim_{x\to 0}\frac{(x+1)^2-1}{x} = \lim_{x\to 0}\frac{x^2+2x}{x} = \lim_{x\to 0} x+2 = 2$$

(d)

$$\begin{aligned}\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1} &= \lim_{x\to 1}\frac{1}{\sqrt{x}+1}\\ &= \frac{1}{\sqrt{\lim_{x\to 1} x}+1} && \text{(by Exercise 4.2.15)}\\ &= \frac{1}{2}\end{aligned}$$

4.2.4 **Proof:** Let $x_n := 1/n\pi$ for $n \in \mathbb{N}$, then $\lim(x_n) = 0$ and $cos(x_n) = \cos(n\pi)$ does not converge in $\mathbb{R}$, so by Divergence Criteria, $\lim\cos(1/x)$ does not exist.

However, since $-1 \leq \cos(1/x) \leq 1$ for all $x \in \mathbb{R}$, $-|x| \leq x\cos(1/x) \leq |x|$. Since $f(x) = |x|$ and $g(x) = -|x|$ converges to 0 at $x = 0$, by Squeeze Theorem,

$$\lim_{x\to 0} x\cos(\frac{1}{x}) = 0$$

■

4.2.8 Proof:

Base case: when $n = 3$, since $-1 < x < 1$, $0 < x^2 \leq |x| < 1$, so $-x^2 \leq x^3 = |x|x^2 \leq x^2$.

Inductive Hypothesis: Suppose when $n = k$ $(k > 3)$, $-x^2 \leq x^k < x^2$.

Inductive Step: When $n = k + 1$, $|x^{k+1}| = |x^k x| = |x^k||x| \leq x^2|x| \leq |x^3| \leq x^2$ by base case and inductive hypothesis.

So by Mathematics Induction, $-x^2 \leq x^k < x^2$.

Since $\lim_{x\to 0} x^2 = 0 \to \lim_{x\to 0} -x^2 = -\lim_{x\to 0} x^2 = 0$. And by the lemma we just proved and Squeeze Theorem,

$$\lim_{x\to 0} x^n = 0$$

■

4.2.9

(a) **Proof:** Suppose $\lim_{x\to c} f = L_1$ and $\lim_{x\to c}(f + g) = L_2$. So by definition, given any $\varepsilon > 0$, there is a $\delta_1 > 0$ and $\delta_2 > 0$ such that if $0 < |x-c| < \min\{\delta_1, \delta_2\}$ then $|f(x)-L_1| < \varepsilon$ and $|f(x)+g(x)-L_2| < \varepsilon$.

Then we have $|f(x)+g(x)-L_2| \leq |f(x)-L_1|+|g(x)-L_2+L_1|$. Suppose g's limit does exist, even when $0 < |x-c| < \min\{\delta_1, \delta_2\}$, there is x that make $|g(x)-L| > \varepsilon$ for all $L \in \mathbb{R}$, which means $f+g$ does not converges and contradicts with the condition given.

As a result, $g's$ limit exists.

■

(b) No, suppose $f(x) = x$ and $g(x) = 1/x$, then $\lim_{x\to 0} f(x) = 0$ and $lim_{x\to 0}f(x)g(x) = 1$, but $g(x)$ does not converge at 0.

4.2.11

(a) Let $x_n = \frac{1}{\sqrt{n\pi+\pi/2}}$, then (x_n) converges to 0, but $\sin(1/x_n) = \sin(n\pi + \pi/2)$ does not converges, so by Divergence Criteria, $\lim_{x\to 0}\sin(1/x^2)$ does not exist in $\mathbb{R}$.

(b) Since $-1 \leq \sin(1/x) \leq 1$ for all $x \in \mathbb{R}$, $-|x| \leq x\sin(1/x) \leq |x|$. Since $f(x) = |x|$ and $g(x) = -|x|$ converges to 0 at $x = 0$, by Squeeze Theorem,

$$\lim_{x\to 0} x\sin(\frac{1}{x}) = 0$$

■

(c) Let $x_n = \frac{1}{2n\pi+\pi/2}$, and $x'_n = \frac{1}{n\pi+3\pi/2}$, then (x_n) and (x'_n) converges to 0. However, $\lim_{x\to 0} \operatorname{sgn}\sin(1/x_n) = \lim_{x\to 0} \operatorname{sgn}\sin(2n\pi + \pi/2) = 1$ and $\lim_{x\to 0} \operatorname{sgn}\sin(2n\pi + 3\pi/2) = -1$. As a result, the limit does not exist.

(d) Since $-1 \leq \sin(1/x^2) \leq 1$ for all $x \in \mathbb{R}$, $-\sqrt{x} \leq \sqrt{x}\sin(1/x^2) \leq \sqrt{x}$. Since $f(x) = \sqrt{x}$ and $g(x) = -\sqrt{x}$ converges to 0 at $x = 0$, by Squeeze Theorem,

$$\lim_{x\to 0} \sqrt{x}\sin(\frac{1}{x^2}) = 0$$

■

5.1.3 Proof: Since both $f(x)$ and $g(x)$ are continuous on their domain, so naturally $h(x)$ is continuous on $[a, b) \cup (b, c]$. Then we need to prove that $h(x)$ is also continuous when $x = b$.

Also, when $x = b$, $f(x) \leq h(x) \leq g(x)$, and $\lim_{x\to b} f(x) = \lim_{x\to b} g(x) = f(b) = g(b)$. So $\lim_{x\to b} h(x) = f(b) = g(b) = h(b)$. Since, $x = b$ is a cluster point in $\mathbb{R}$, by definition, we see that $h(x)$ is continuous at $x = b$.

Hence, $h(x)$ is continuous on $[a, c]$.

■

5.1.4

(a) $f(x) = [\![x]\!]$ is the ceiling function, which is discontinuous at each integer points $x \in \mathbb{Z}$ and is continuous on the rest points.

(b) Similarly, $g(x) = x[\![x]\!]$, since ceiling function is not continuous, $g(x)$ is discontinuous at each integer points $x \in \mathbb{Z}$ and is continuous on the rest points.

(c) Since $\sin(x) \in [-1, 1]$ for all $x \in \mathbb{R}$, when $x \in (2n\pi, 2n\pi + \pi]$, $[\![\sin(x)]\!] = 1$, but when $x \in (2n\pi + \pi, 2(n + 1)\pi]$, $[\![\sin(x)]\!] = 0$. As a result, when $x = n\pi$, $h(x)$ is discontinuous, and is continuous on the rest points.

(d) For all $x \in [\frac{1}{n}, \frac{1}{n+1})$, $k(x) = [\![\frac{1}{x}]\!] = n + 1$, and as a result, is continuous and is discontinuous otherwise.

5.1.5 Since $\lim_{x\to 2} \frac{x^2+x-6}{x-2} = \lim_{x\to 2} \frac{(x+3)(x-2)}{x-2} = \lim_{x\to 2} x + 3 = 5$. And since $x = 2$ is a cluster point in $\mathbb{R}$, then if we define $f(x) = \lim_{x\to 2} \frac{x^2+x-6}{x-2} = 5$, then $f(x)$ is continuous at this point by definition.

5.1.12 **Proof:** Suppose when $x = a, a \in \mathbb{R} - \mathbb{Q}$, $f(a) \neq 0$, then since $f(x)$ is continuous on $\mathbb{R}$, $f(a) = \lim_{x\to a} f(x) \neq 0$ in a small enough neighborhood V that a is the only rational number in V. And thus, this corollary violates the condition given.

Hence, $f(x) = 0$ for all $x \in \mathbb{R}$.

$\blacksquare$

5.1.13 By density theorem, for all $xy \in \mathbb{R}$ and $x < y$ there is always $r \in \mathbb{Q}$ that $x < r < y$ and there is always $r' \notin \mathbb{Q}$ that $x < r' < y$. As a result, there is always a rational number between two irrational numbers and there is always an irrational numbers between two rational numbers. Hence, $g(x)$ is never continuous in $\mathbb{R} - \{3\}$(the intersection of two branches).

Then we need to discuss the continuity at $x = 3$. Since $3 \in \mathbb{Q}$, $g(3) = 2 \cdot 3 = 6$. And at this point any subsequences will converge to 6 at this point. Then by Convergence Criteria, $\lim_{x\to 3} g(x) = 6$.

As a result, $x = 3$ is the only point where $g(x)$ is continuous.

5.1.15 Let $x_n := \frac{1}{n}$ and $y_n := \frac{1}{n^2}$, obviously, $\lim(x_n) = \lim(y_n) = 0$. However, $f(x_n) = n$ and $f(y_n) = n^2$, since $x_n, y_n \in (0, 1)$, $n > 1$, so $f(x_n) \neq f(y_n)$ for all $n \in \mathbb{N}$.