1.1.7 Solution:

(a) Claim: $A_1 \cap A_2 = \{6k : k \in \mathbb{N}\}$.

Proof: Since $\forall n \in \mathbb{N}, A_n = \{(n+1)k : k \in \mathbb{N}\}$, we have $A_1 = \{2k : k \in \mathbb{N}\}$ and $A_2 = \{3k : k \in \mathbb{N}\}$. So $\forall x \in A_1 \cap A_2, \exists k_1, k_2$ that $x = 2k_1 = 3k_2$, so $2|x \Rightarrow 2|3k_2 \Rightarrow 2|k_2 \Rightarrow \exists k_3$ that $k_2 = 2k_3$, so $x = 6k_3, k \in \mathbb{N}$, which means $A_1 \cap A_2 \subseteq \{6k : k \in \mathbb{N}\}$.

On the other hand, $\forall x \in \{6k : k \in \mathbb{N}\}, \exists k_0$ that $x = 6k_0$, so naturally, we have $k_1 = 3k_0, k_2 = 2k_1$ that $x = 2k_1 = 3k_2$, so $\{6k : k \in \mathbb{N}\} \subseteq A_1 \cap A_2$.

As a result, $A_1 \cap A_2 = \{6k : k \in \mathbb{N}\}$.$\blacksquare$

(b) $\bigcup\{A_n : n \in \mathbb{N}\} = \mathbb{N} - \{1\}$, $\bigcap\{A_n : n \in \mathbb{N}\} = \emptyset$.

1.1.10 Solution:

(a) Since $E := \{x \in \mathbb{R} : 1 \le x \le 2\}$, $x^2 \in [1, 4]$, so $f(E) = \{x \in \mathbb{R} : 1/4 \le x \le 1\}$.

(b) Since $G := \{x \in \mathbb{R} : 1 \le x \le 4\}, f^{-1}(G) = \{x \in \mathbb{R} : 1/2 \le x \le 1\}$

1.1.14 Solution: Proof: Prove $f(E \cup F) = f(E) \cup f(F)$ first.

$\forall x \in E \cup F$, $x \in E$ or $x \in F$, so $f(x) \in f(E)$ or $f(x) \in f(F)$. That is saying that $f(x) \in f(E) \cup f(F)$, so $f(E \cup F) = f(E) \cup f(F)$.

$\forall f(x) \in f(E) \cup f(F)$, then $\exists x \in E$ or $x \in F$, which means $x \in E \cup F$, so $f(x) \in f(E \cup F)$. Thus, $f(E) \cup f(F) \subseteq f(E \cup F)$.

As a result, $f(E \cup F) = f(E) \cup f(F)$.

Then $\forall x \in E \cap F$, $x \in E$ and $x \in F$, so $f(x) \in f(E)$ and $f(x) \in f(F)$ which means $f(x) \in f(E) \cap f(F)$.

As a result, $f(E \cap F) \subseteq f(E) \cap f(F)$.$\blacksquare$

1.1.16 Solution: Proof: $\forall x_1, x_2 \in \mathbb{R}$, if $f(x_1) \neq f(x_2)$, $x_1/\sqrt{x_1^2+1} \neq x_2/\sqrt{x_2^2+1} \Rightarrow x_1\sqrt{x_2^2+1} \neq x_2\sqrt{x_1^2+1} \Rightarrow x_1^2x_2^2 + x_1^2 \neq x_1^2x_2^2 + x_2^2 \Rightarrow x_1^2 \neq x_2^2 \Rightarrow x_1 \neq x_2$, which proves that f is injective.

$\forall y \in B$, $y = x/\sqrt{x^2+1} \Rightarrow y^2 = x^2/(x^2+1) \Rightarrow x^2(y^2-1) = -y^2 \Rightarrow x^2 = y^2/(1-y^2) \Rightarrow x = y/\sqrt{1-y^2}$ exists since $y \in B$ and $y^2 < 1$ as a result.

So f is a bijection.$\blacksquare$

1.2.2 Solution: Proof:

Base case: when $n = 1$, $1^3 = 1 = [\frac{1}{2}1(1+1)]^2$.

Suppose when $n = k$, we have

$$1^3 + 2^3 + \cdot + k^3 = [\frac{1}{2}k(k+1)]^2$$

Then when $n = k + 1$,

$$\begin{aligned}
1^3 + 2^3 + \cdot + k^3 + (k+1)^3 &= [\frac{1}{2}k(k+1)]^2 + (k+1)^3 \\
&= \frac{1}{4}k^4 + \frac{1}{2}k^3 + \frac{1}{4}k^2 + k^3 + 3k^2 + 3k + 1 \\
&= \frac{1}{4}(k^4 + 6k^3 + 26k^2 + 3k + 1) \\
&= \frac{1}{4}(k+1)^2(k+2)^2 \\
&= [\frac{1}{2}(k+2)(k+1)]^2
\end{aligned}$$

So by mathematical induction, we proved that $1^3 + 2^3 + \cdot + n^3 = [\frac{1}{2}n(n+1)]^2$, $\forall n \in \mathbb{N}$.■

1.2.6 Solution: Proof:

Base case: when $n = 1$, $n^3 + 5n = 6$ which is divisible by 6.

Suppose that when $n = k$, we have $k^3 + 5k$ is divisible by 6. Then when $n = k+1$, $(k+1)^3+5(k+1) = k^3+3k^2+5k+3k+6 = (k^3+5k)+(3k^2+3k+6) = 6(k_0+1)+3k(k+1)$. Since one of k and $k+1$ have to be even, so $3k(k+1)$ is divisible by 6. Thus $(k+1)^3+5(k+1) = 6(k_0+1)+3k(k+1)$ is divisible by 6.

As a result, $n^3 + 5n$ is divisible by 6 for all $n \in \mathbb{N}$.■

1.2.13 Solution: Proof:

Base case: when $n = 1$, $1 < 2^1 = 2$.

Suppose when $n = k$, we have $k < 2^k$. Then when $n = k + 1$, $k + 1 < 2^k + 1 < 2^k + 2^1 < 2^k + 2^k < 2^{k+1}$.

As a result, $n < 2^n$ for all $n \in \mathbb{N}$.

1.3.2 Solution:

(b) Proof: Since $|A| = m$, $|C| = 1$ and $C \subset A$, $|A \cap C| = |C| = 1$, $|A/C| = |A| - |A \cap C| = m - 1$.■

(c) Proof: Suppose C/B is a finite set, either $|C/B| = 0$ or $n \in \mathbb{N}$ that $|C/B| = n$. Without generality, we can suppose $(B/C) = m(0 \leq m \leq |B|)$. Then $|C| = |B| - |B/C| + |C/B| = |B| - m + n$, which means C is finite. That is not possible according to the condition given.

Thus, C/B is infinite.■

1.3.4 Solution: Claim: Let $f : \mathbb{N} \to (13, \infty)$, $f(x) = x + 13$, it is a bijection.

Proof: $\forall x_1, x_2 \in \mathbb{N}$, $f(x_1) \neq f(x_2) \Rightarrow x_1 + 13 \neq x_2 + 13 \Rightarrow x_1 \neq x_2$, which means f is injective.

And $\forall y \in (13, \infty)$, $\exists x = y - 13 \in \mathbb{N}$ that $y = x + 13$. So f is surjective.

As a result, f is a bijection.■

1.3.5 Solution:

$$f(x) = \begin{cases} x/2 \text{ if } x \text{ is even} \\ -(x+1)/2 \text{ if } x \text{ is odd} \end{cases}$$