

1. For eigenvalues of $A^T A$, $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_n$, there is $\lambda_i = \sigma_i^2$, in which σ_i is the singular value of A .
2. Let

$$A = \mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

then

$$A^T = [a_1 \quad a_2 \quad \cdots \quad a_n]$$

and

$$A^T A = \sum_{i=1}^n a_i^2 = \lambda$$

So

$$\sigma = \sqrt{\lambda} = \sqrt{\sum_{i=1}^n a_i^2}$$

and

$$\Sigma = \begin{bmatrix} \sigma \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

and

$$V = 1$$

Hence,

$$U = \begin{bmatrix} \frac{a_1}{\sigma_1} & \frac{a_2}{\sigma_2} & \cdots & \frac{a_n}{\sigma_n} \end{bmatrix}$$

And $A = U\Sigma V^T$.

3. Since $B = \mathbf{b} = [b_1, b_2, \cdots, b_n]$, we have

$$B^T B = \begin{bmatrix} b_1^2 & b_1 b_2 & \cdots & b_1 b_n \\ b_2 b_1 & b_2^2 & \cdots & b_2 b_n \\ \vdots & \vdots & \ddots & \vdots \\ b_n b_1 & b_n b_2 & \cdots & b_n^2 \end{bmatrix}$$

So $\sigma = \sqrt{\lambda} = \sqrt{\sum_{i=1}^n a_i^2}$, and

$$\Sigma = \begin{bmatrix} \sigma & 0 & \cdots & 0 \end{bmatrix}$$

$$U = 1$$

and

$$V = B^{-1}\sigma$$

And $B = U\Sigma V^T$

4. Since $A^+ = V\Sigma^+U^T$, $x = A^+b = V\Sigma^+U^Tb$, then $(Ax - b) \cdot a_i = (AV\Sigma^+U^Tb - b) \cdot a_i = (AV\Sigma^+U^T - I)b \cdot a_i = (U\Sigma V^T V\Sigma^+U^T - I)b \cdot a_i = (U\Sigma\Sigma^+U^T - I)b \cdot a_i = 0$

This means that a_i and $Ax - b$ is always orthogonal.