2.5.7 Claim: For a subgroup N of a group G, the following are equivalent:

1. N is normal
2. $\forall a \in G, \exists b \in G, aN = Nb$.
3. $\forall a \in G, aN = Na$.

Proof: To prove all three statements are equivalent to each other, we can prove that $1 \Rightarrow 2, 2 \Rightarrow 3$ and $3 \Rightarrow 1$.

$1 \Rightarrow 2$:

Since N is normal, we know $\forall g \in G, gNg^{-1} = N$ by definition.

Then, we have $\forall g \in G, gN = Ng$, which means $b = a$.

$2 \Rightarrow 3$:

Since $\forall a \in G, \exists b \in G, aN = Nb$, and aN is a coset of N, we know it's either $a \in aN = bN = Nb$ or $aN \cap bN = \emptyset$. Since $aN = Nb$, $b \in Nb = aN$, so the second possibility is ruled out. Thus, we have $bN = Nb$ for all $b \in G$ which is equivalent to $aN = Na$ for all $a \in G$.

$3 \Rightarrow 1$:

Since $\forall a \in G, aN = Na$, we can get $aNa^{-1} = N$ by multiply a^{-1} on both sides, which means N is normal.

As a result, we can conclude that all three statements are equivalent.■

2.5.13

(a) Claim: The center of group G is a normal subgroup of G.

Proof: Since $\text{Center}(G) = \{x | \forall g \in G, gx = xg\}$, we have $\text{Center}(G) = x | gxg^{-1} = x$ which is exactly the definition of normal subgroup.■

(b) Based on the multiplication table of S_3:

Element	$()$	$(1,2)$	$(2,3)$	$(1,3)$	$(1,2,3)$	$(1,3,2)$
$()$	$()$	$(1,2)$	$(2,3)$	$(1,3)$	$(1,2,3)$	$(1,3,2)$
$(1,2)$	$(1,2)$	$()$	$(1,2,3)$	$(1,3,2)$	$(2,3)$	$(1,3)$
$(2,3)$	$(2,3)$	$(1,3,2)$	$()$	$(1,2,3)$	$(1,3)$	$(1,2)$
$(1,3)$	$(1,3)$	$(1,2,3)$	$(1,3,2)$	$()$	$(1,2)$	$(2,3)$
$(1,2,3)$	$(1,2,3)$	$(1,3)$	$(1,2)$	$(2,3)$	$(1,3,2)$	$()$
$(1,3,2)$	$(1,3,2)$	$(2,3)$	$(1,3)$	$(1,2)$	$()$	$(1,2,3)$

Figure 1: S_3

$$\text{Center}(S_3) = \{e\}$$

2.6.1

Claim: Relation defined on X by $x_1 \sim x_2$ if $f(x_1) = f(x_2)$ is an equivalence relation. And the associated partition of X is the partition into $f^{-1}(y)$ for $y \in Y$.

Proof: We can prove the relation is equivalent relation first by proving its reflexivity, symmetry and transitivity.

Reflexivity: $\forall x \in X$, it's obvious that $x = x$ and $f(x) = f(x)$, so $x \sim x$.

Symmetry: Take 2 arbitrary $x, y \in X$, then if $x \sim y$, $x = y \Rightarrow f(x) = f(y) = f(y) = f(x) \Rightarrow y \sim x$. So we have $x \sim y \Leftrightarrow y \sim x$.

Transitivity: Take $x, y, z \in X$. If $x \sim y$ and $y \sim z$, then $x = y \Rightarrow f(x) = f(y)$ and $y = z \Rightarrow f(y) = f(z)$. So $x = y = z \Rightarrow f(x) = f(y) = f(z)$, which means $x \sim z$.

So we proved this relation is an equivalent relation.

Then we can prove the associated partition of X is the partition into fiber.

Since f is a surjective, $\forall y \in Y, \exists x \in X$ that $f^{-1}(y) = x$. And as we have an equivalent relation on X, each $x \in X$ is and only is in one equivalent class is guaranteed by the correspondent partition of X. So each subsets of f^{-1} are disjoint. And since we know f is a injection by the definition of the relation, f and f^{-1} are bijections. Thus, $X = f^{-1}(Y)$, so the partition into fibers is exactly the partition of X.■

2.7.9

Claim: The commutator subgroup C of group G is normal. And quotient group G/C is abelian. If $H \trianglelefteq G$ and G/H is abelian, then $C \subseteq H$.

Proof: Let $a, b \in G$ and $x = gag^{-1}, y = gbg^{-1}, x^{-1} = ga^{-1}g^{-1}, y^{-1} = gb^{-1}g^{-1}$. Then take $c = xyx^{-1}y^{-1} \in C$, $gcg^{-1} = aba^{-1}b^{-1} \in C \Rightarrow gCg^{-1} = C$.

Thus, C is a normal group.

Take $a, b \in C$, so $aC, bC, Ca, Cb \in G/C$. Since C is normal as we've proved, $aCbC = abC = Cab$. To prove G/H is abelian, we want to prove that $Cab = Cba$ for all $a, b \in G$. Since we know Cab and Cba are both right cosets of C, so it's either they are equal, or they are disjoint. Let $b = x = y = e$, we have $a \in Cab$ and $a \in Cba$, so they can't be disjoint.

As a result, $Cab = Cba$, and G/C is abelian by definition.

Since H is normal and G/H is abelian, take arbitrary $c \in C$ as implied in the first part and second part of the proof, $gcg^{-1} = aba^{-1}b^{-1} \in C, Cab = Cba \Rightarrow c \in H$.

Thus, $C \subseteq H$ by definition.■