5.2.1

(a) Since $x^2 \geq 0 \Rightarrow x^2 + 1 \geq 1 > 0$ for all $x \in \mathbb{R}$. And since $x^2 + 2x + 1$ is continuous on $\mathbb{R}$ by Example 5.2.3, so $f(x)$ is continuous on $\mathbb{R}$ by Theorem 5.2.2(b).

(b) Since x is continuous on $\mathbb{R}^*$, $\sqrt{x}$ is continuous on $\mathbb{R}^*$ by Theorem 5.2.5(b). So $x + \sqrt{x}$ is continuous on $\mathbb{R}^*$ by Theorem 5.2.2(a), so $g(x)$ is continuous on $x \geq 0$ by Theorem 5.2.5(b).

(c) Since $x \neq 0$ is continuous, $\sin x$ is continuous, so $|\sin x|$ is continuous by Theorem 5.2.4 and $\sqrt{1 + |\sin x|}$ is continuous by Theorem 5.2.2 and Theorem 5.2.5. As a result, by Theorem 5.2.2(b) $h(x)$ is continuous when $x \neq 0$.

(d) Since x continuous on $\mathbb{R}$, $1 + x^2$ is continuous by Theorem 5.2.2(b), and since $\cos x$ is continuous on $\mathbb{R}$, so $k(x)$ is also continuous on $\mathbb{R}$ by Theorem 5.2.7.

5.2.3

(a) Let

$$f(x) = \begin{cases} x & \text{if } x \neq 0 \\ 1 & \text{otherwise} \end{cases}$$

and

$$g(x) = \begin{cases} x & \text{if } x \neq 0 \\ -1 & \text{otherwise} \end{cases}$$

Then f, g are discontinuous at $x = 0$, but $f + g = 2x$ and is continuous on $\mathbb{R}$.

(b) Let

$$f(x) = \begin{cases} x & \text{if } x \neq 1 \\ 2 & \text{otherwise} \end{cases}$$

and

$$g(x) = \begin{cases} x & \text{if } x \neq 1 \\ 1/2 & \text{otherwise} \end{cases}$$

Then f, g are discontinuous at $x = 1$, but $fg = x^2$ and is continuous on $\mathbb{R}$.

5.2.4 When $x \in \mathbb{Z}$, $[\![x]\!] = x + 1$, so $x - [\![x]\!] = -1$, and when $x \notin \mathbb{Z}$, $x - [\![x]\!]$ is the decimal part of x. As a result, $f(x)$ is discontinuous when $x \in \mathbb{Z}$ and continuous otherwise.

5.2.5 By substitution, we see that $g \circ f(0) = g(1) = 0$ and from the definition of f and g, we see that

$$g \circ f(x) = \begin{cases} 2 & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Thus, $\lim_{x \to 0} g \circ f(x) = 2 \neq 0 = g \circ f(0)$.

5.2.9 **Proof:** Suppose that there is a $c \in \mathbb{R}$ that $h(c) \neq 0$. We notice that for any $m \in \mathbb{Z}$, $(m/2^n)$ converges to 0, so we can always find a δ that there is $x \in (c - \delta, c + \delta)$, so that

$$|h(x) - 0| = |h(x)| > \varepsilon$$

So we have $f(x)$ does not converge at c and thus is discontinuous, which contradicts with our condition given.

Hence, for all $x \in \mathbb{R}$, $h(x) = 0$.

■

5.3.1 **Proof:** Since I is a closed and bounded interval and f is continuous, then there is $x^* \in I$ that $f(x^*) \leq f(x)$ for all $f(x)$ by Maximum-Minimum Theorem.

Since for all $x \in I$, $f(x^*) > 0$, then by Corollary 2.4.5, there is $n \in \mathbb{N}$ that $\alpha = 1/n$ have $0 < \alpha < f(x^*) \leq f(x)$ for all $x \in I$.

■

5.3.3 **Proof:** If for all $x \in I$, $f(x) \geq 0$, then we have b_n that $f(b_n) = \frac{1}{2} f(b_{n-1})$, by Archimedean Property, we have $\inf\{f(b_n)\} = 0$. Since I is closed, bounded interval and f is continuous, we have that there is $c \in I$ that $f(c) = 0$.

Similarly, if for all $x \in I$, $f(x) \leq 0$, we have that there is $c \in I$ that $f(c) = 0$.

If there is $a, b \in I$ that $f(a) \leq 0, f(b) \geq 0$, then by Location of Roots Theorem, there is a number $c \in (a, b) \subseteq I$ that $f(c) = 0$.

In conclusion, there is a $c \in I$ that $f(c) = 0$.

■

5.3.4 Proof: Let polynomial of odd degrees be

$$f(x) = c_1x + c_2x^3 + \cdots + c_nx^{2n-1}$$

Since for all $n \in \mathbb{N}$, $x = 0 \Rightarrow x^n = 0 \Rightarrow cx^n = 0$ for any $c \in \mathbb{R}$. As a result, $f(0) = 0$.

Hence, for all polynomials of odd degree, there is always a real root.

■

5.3.13 Proof: Since f is bounded then there is M that $|f(x)| \leq M$ for all $x \in \mathbb{R}$, then by Completeness Axiom, there is supremum and infimum of $f(x)$ in $\mathbb{R}$. Since $f(x)$ converges to 0 when at ∞ and $-\infty$.

Then for any bounded closed interval $[a_n, b_n] \subseteq \mathbb{R}$, f has both maximum $f(x_n^*)$ and minimum $f(x_{*n})$. Then for $\mathbb{R}$, we know that $\sup\{f(x)\} = \max\{x_1^*, x_2^*, \cdots, x_n^*, 0\}$ and $\min\{x_{*1}, x_{*2}, \cdots, x_{*n}, 0\}$.

Hence, either maximum or minimum can be attained.

■

And $y = x^3$ does not have any maximum or minimum.

5.3.14 Proof: Suppose there is not a δ-neignborhood of x_0 that make $f(x) < \beta$, then either $f(x)$ diverges, or it converges to α that $\alpha \geq \beta$.

If $f(x)$ diverges, then f is discontinuous at $x = x_0$, if $\lim_{x\to x_0} f(x) = \alpha \geq \beta$, then $\lim_{x\to x_0} f(x) \neq f(x_0)$, f is still discontinuous, which contradicts with the condition given.

Hence, there must be a δ-neignborhood of x_0 that make $f(x) < \beta$.

■

5.4.2 Proof: Since $f(x) = 1/x^2$, then for all $\varepsilon > 0$ and $x, u \in [1, \infty)$, then when $|x - u| < \delta = \frac{\varepsilon x^2u^2}{x+u}$, we have

$$|f(x) - f(u)| = \frac{|u^2 - x^2|}{x^2u^2} = \frac{|u - x|(x + u)}{x^2u^2} < \frac{\delta(x + u)}{x^2u^2} = \varepsilon$$

Hence, by definition, f is uniformly continuous on $[1, \infty)$.

However, when in $(0, \infty)$, let $(x_n) = \frac{1}{\sqrt{n}}$ and $(u_n) = \frac{1}{\sqrt{n+1}}$, then $\lim(x_n - u_n) = 0$, but $|f(x_n) - f(u_n)| = 1$ for all $n \in \mathbb{N}$.

Hence, by Nonuniform Continuity Criteria, f is not uniformly continuous on $(0, \infty)$.

■

5.4.3

(a) **Proof:** Suppose $\varepsilon = 1$, then whenever $|x - u| < \delta$,

$$|x^2 - u^2| = |x^2 - (x + \delta)^2| = |2x\delta + \delta^2| \geq 1$$

can be reached if we pick $x > \frac{1}{2\delta}$.

Hence, by Nonuniform Continuity Criteria, f is not uniformly continuous on $[0, \infty)$.

■

(b) **Proof:** Let $(x_n) = \frac{1}{n}$ and $(u_n) = \frac{1}{n+\pi}$, then

$$\lim(x_n - u_n) = \lim \frac{\pi}{n(n + \pi)} = 0$$

but

$$|f(x_n) - f(u_n)| = 2|\sin n|$$

For $\varepsilon > 0$, there is always n that $2|\sin n| \geq \varepsilon$.

Hence, by Nonuniform Continuity Criteria, f is not uniformly continuous on $(0, \infty)$.

5.4.6 **Proof:** Since f, g is uniformly continuous, then both f, g are continuous on A and whenever $x, u \in A$ has that $|x-u| < \delta$, there is $|f(x)-f(u)| < \varepsilon$ and $|g(x) - g(u)| < \varepsilon$ for all $\varepsilon > 0$ be definition.

Thus, there is a $M > 0$ that $|f(x)| \leq M$ and $|g(x)| \leq M$. Then if we pick δ that $|f(x) - f(u)| < \varepsilon/2M$ and $|g(x) - g(u)| < \varepsilon/2M$.

$$\begin{aligned}
|f(x)g(x) - f(u)g(u)| &= |f(x)g(x) - f(x)g(u) + f(x)g(u) - f(u)g(u)| \\
&\leq |f(x)(g(x) - g(u))| + |(f(x) - f(u))g(u)| \\
&= |f(x)||g(x) - g(u)| + |f(x) - f(u)||g(u)| \\
&\leq M(|g(x) - g(u)| + |f(x) - f(u)|) < \varepsilon
\end{aligned}$$

Hence, by definition, fg is uniformly continuous on A.

■

5.4.7 Proof: For $f(x) = x$, since $|f(u) - f(x)| = |u - x|$, so f us a Lipschitz function, so f is uniformly continuous on $\mathbb{R}$ by Theorem 5.4.5.

For $g(x) = \sin x$, $|g(u) - g(x)| = |2\sin(x/2 - u/2)\cos(x/2 + u/2)| \leq |2\sin(x/2 - u/2)|$. If we pick $\delta = 2\arcsin(\varepsilon)$, then there is $|g(u) - g(x)| \leq \varepsilon$. Thus, by definition, g is uniformly continuous on $\mathbb{R}$.

Then for $fg(x) = x\sin x$, suppose it is uniformly continuous, then for $\varepsilon > 0$, if $|x - u| < \delta$, $|f(x) - f(u)| < \varepsilon$, but if we let $p = x + 2n\pi, q = u + 2n\pi$, then when $|p - q| < \delta$, but $|f(p) - f(u)| = |(x\sin(x) - y\sin(y)) + 2n\pi(\sin(x) - \sin(y))| > \varepsilon$ when n is sufficiently large.

Hence, fg is not uniformly continuous on $\mathbb{R}$.

$\blacksquare$