**2.7.4**

**Claim:** Suppose $G$ is a finite group. Let $N$ be a normal subgroup of $G$ and $A$ an arbitrary subgroup.

$$|AN| = \frac{|A||N|}{|A \cap N|}.$$

**Proof:** We know that $AN/N \cong A/(A \cap N)$, so that $|AN/N| = |A/(A \cap N)|$. So we have $|AN/N| = |AN|/|N|$ and $|A/(A \cap N)| = |A|/|A \cap N|$ by Lagrange's Theorem.

As a result,

$$|AN|/|N| = |A|/|A \cap N| \Rightarrow |AN| = \frac{|A||N|}{|A \cap N|} \blacksquare$$

**2.7.6**

**(a) Claim:** Aut$(G)$ is a group.

**Proof:** Since Aut is an isomorphism, $|G| = |\text{Aut}(G)|$, so as $G$ is not empty, Aut$(G)$ is not empty. Associativity is guaranteed since $G$ is a group. And again, since Aut$(G) = G$, identity and inverses exist in Aut$(G)$.

Thus, we can conclude that Aut$(G)$ is also a group.$\blacksquare$

**(b) Claim:** If map $c_g : G \to G$ defined by $c_g(x) = gxg^{-1}$ is an element of Aut$(G)$, $c : g \mapsto c_g$ is a homomorphism from $G$ to Aut$(G)$.

**Proof:** Take 2 arbitrary $g_1, g_2 \in G$, $c(g_1)c(g_2) = gg_1g^{-1}gg_2g^{-1} = gg_1g_2g^{-1} = c(g_1g_2)$.

So $c : g \mapsto c_g$ is a homomorphism from $G$ to Aut$(G)$.$\blacksquare$

**(c) Claim:** $\text{Ker}(c) = Z(G)$.

**Proof:** $\text{Ker}(c) = \{g \in G, c = id_G\} = \{g \in G, \forall x \in G, gxg^{-1} = x\} = \{g \in G, \forall x \in G, gx = xg\} = Z(G)\blacksquare$

**(d) Claim:** $\text{Int}(G) \cong G/Z(G)$.

**Proof:** According to First Isomorphism Theorem, $G/\text{Ker}(c) \cong \text{Int}(G)$. So we have $\text{Int}(G) \cong G/Z(G)$ by the conclusion of last question that $\text{Ker}(c) = Z(G)$.$\blacksquare$

**2.7.7**

**Claim:** $D_4/C_4 \cong C_2$.

**Proof:** We can establish map $\varphi : D_4 \to C_2$ by maps all element in $C_4$ to $e_{C_2}$ and the rest to $e_{C_2}^{-1}$ so we know it is surjective. Take $d_1, d_2 \in D_4$, $\varphi(d_1 d_2) = \varphi(d_1)(d_2)$ can be easily verified by listing multiplication table. So $\varphi$ is a surjective homomorphism. Since $C_4$ is a normal group containing identity $e$, $C_4$ is the kernel of quotient map.

As a result, we can conclude that $D_4/C_4 \cong C_2$ by the First Isomorphism Theorem.■

**2.7.11**

**Claim:** If $G/Z(G)$ is cyclic, then $G$ is abelian.

**Proof:** Take $G/Z(G) = \langle zG \rangle$. Then for each coset $xZ$, there exists some $i \in \mathrm{Z}$ that $xZ = (gZ)^i = g^i Z$.

Then take $x, y \in G$ and denote $x = g^i z, y = g^j z_0$. So we have $xy = g^i z g^j z_0 = g^{i+j} z z_0 = g^j g^i z z_0 = g^j z_0 g^i z = yx$.

As a result, we proved that $G$ is abelian.■