3.1.14

Claim: $\mathbb{Z}_4 \times \mathbb{Z}_4$ is not isomorphic $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof: For the group $\mathbb{Z}_4 \times \mathbb{Z}_4$, the elements of order 4 are as following: $(1,1),(3,3),(1,3),(3,1)$, which means 4 in total. And for $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, the elements of order 4 are $(1,1,0),(1,0,1),(1,1,1),(3,1,0),(3,0,1),(3,1,1)$, which means 6 in total.

Obviously, we see that the 2 groups does not have the same structure, so there is no isomorphism between them.■

3.1.15 .

Claim: $K_1 \trianglelefteq G_1, K_2 \trianglelefteq G_2$, $K_1 \times K_2 \trianglelefteq G_1 \times G_2$ and

$$(G_1 \times G_2)/(K_1 \times K_2) \cong G_1/K_1 \times G_2/K_2.$$

Proof: $K_1 \trianglelefteq G_1 \Rightarrow \forall g_1 \in G_1, g_1 K_1 g_1^{-1} = K_1, K_2 \trianglelefteq G_2 \Rightarrow \forall g_2 \in G_2, g_2 K_2 g_2^{-1} = K_2$. Then take $k = (k_1,k_2) \in K_1 \times K_2, g = (g_1,g_2) \in G_1 \times G_2$, $gkg^{-1} = (g_1,g_2)(k_1,k_2)(g_1^{-1},g_2^{-1}) = (g_1k_1g_1^{-1}, g_2k_2g_2^{-1}) \in K_1 \times K_2$, so $K_1 \times K_2 \trianglelefteq G_1 \times G_2$.

Since $K_1 \times K_2 \trianglelefteq G_1 \times G_2$, we proved that $\pi : G_1 \times G_2 \to (G_1 \times G_2)/(K_1 \times K_2)$ is a quotient map. We can construct a map $\varphi : G_1 \times G_2 \to G_1/K_1 \times G_2/K_2$ by sending (g_1,g_2) to (g_1K_1, g_2K_2), namely $\varphi((g_1,g_2)) = (g_1K_1, g_2K_2)$.

Since when $g_1 = g_3, g_2 = g_4 \Rightarrow \varphi((g_1,g_2)) = (g_1K_1, g_2K_2) = (g_3K_1, g_4K_2) = \varphi((g_3,g_4))$, so the map is well-defined.

Take $(g_{11},g_{12}),(g_{21},g_{22}) \in G_1 \times G_2$, then $\varphi((g_{11},g_{12})(g_{21},g_{22})) = \varphi((g_{11}g_{21}, g_{12}g_{22})) = (g_{11}g_{21}K_1, g_{12}g_{22}K_2) = (g_{11}K_1, g_{12}K_2)(g_{21}K_1, g_{22}K_2) = \varphi((g_{11},g_{12}))\varphi((g_{21},g_{22}))$.

So we can conclude that the map φ we just defined is a group homomorphism.

Take an arbitrary $(g_1K_1, g_2K_2) \in G_1/K_1 \times G_2/K_2$, we have $(g_1,g_2) \in G_1 \times G_2$, that $\varphi((g_1,g_2)) = (g_1K_1, g_2K_2)$ so that the homomorphism φ is surjective.

Finally, we proved that $(G_1 \times G_2)/(K_1 \times K_2) \cong G_1/K_1 \times G_2/K_2$ by the First Isomorphism Theorem.■

3.5.3

Claim: Let G be an abelian group and let $x_1, \cdots, x_n$ be distinct nonzero elements of G. If and only if the set $B = \{x_1, \cdots, x_n\}$ is a basis of G, then for each i, the map $r \mapsto rx_i$ is injective, and

$$G = \mathbb{Z}x_1 \times \mathbb{Z}x_2 \times \cdots \times \mathbb{Z}x_n.$$

Proof: Prove sufficiency first.

Take $B = \{x_1, \cdots, x_n\}$ to be the basis of G, $x_1, \cdots, x_n$ are linearly independent, namely, let $r_1, r_2, \cdots \in \mathbb{Z}$ if

$$\sum r_i x_i = 0,$$

then $r_i = 0$ for all i.

Then we take $r_1 x_i = r_2 x_i$ for all i, then $r_1 x_i - r_2 x_i = 0 \Rightarrow (r_1 - r_2)x_i = 0 \Rightarrow r_1 = r_2$ for all i.

As a result, we proved that the map $r \mapsto rx_i$ is injective.

And take $g \in G$, since B is a basis $\exists r_1, r_2, \cdots, r_n$ to make $r_1x_1 + r_2x_2 + \cdots + r_nx_n = g \in \mathbb{Z}x_1 \times \mathbb{Z}x_2 \times \cdots \times \mathbb{Z}x_n$. So $G \subseteq \mathbb{Z}x_1 \times \mathbb{Z}x_2 \times \cdots \times \mathbb{Z}x_n$.

Then we take $(r_1x_1, r_2x_2, \cdots, r_nx_n) \in \mathbb{Z}x_1 \times \mathbb{Z}x_2 \times \cdots \times \mathbb{Z}x_n$. Obviously, correspondent $(r_1, r_2, \cdots, r_n) \in G$. Thus, $\mathbb{Z}x_1 \times \mathbb{Z}x_2 \times \cdots \times \mathbb{Z}x_n \subseteq G$.

As a result, $G = \mathbb{Z}x_1 \times \mathbb{Z}x_2 \times \cdots \times \mathbb{Z}x_n$.

Then we can prove necessity.

Assume that the map $r \mapsto rx_i$ is injective and $G = \mathbb{Z}x_1 \times \mathbb{Z}x_2 \times \cdots \times \mathbb{Z}x_n$, then $\forall i, r_1x_i = r_2x_i \to r_1 = r_2 \Rightarrow \sum_i r_{1i}x_i = \sum_i r_{2i}x_i \Rightarrow \sum_i (r_{1i} - r_{2i})x_i = 0 \to r_{1i} = r_{2i} \Rightarrow r_{1i} - r_{2i} = 0$.

Thus, we proved that B is a basis of G.

And finally, if and only if the set $B = \{x_1, \cdots, x_n\}$ is a basis of G, then for each i, the map $r \mapsto rx_i$ is injective, and

$$G = \mathbb{Z}x_1 \times \mathbb{Z}x_2 \times \cdots \times \mathbb{Z}x_n.$$

is proved.■

3.6.10

Claim: There are 15 abelian groups of order 128 up to isomorphism.

Proof: According to Fundamental Theorem of Finitely Generated Abelian Groups, any group of order 128 is one of the following groups up to isomorphism:

- $\mathbb{Z}_{128}$
- $\mathbb{Z}_{64} \times \mathbb{Z}_2$
- $\mathbb{Z}_{32} \times \mathbb{Z}_4$
- $\mathbb{Z}_{32} \times \mathbb{Z}_2 \times \mathbb{Z}_2$
- $\mathbb{Z}_{16} \times \mathbb{Z}_8$
- $\mathbb{Z}_{16} \times \mathbb{Z}_4 \times \mathbb{Z}_2$
- $\mathbb{Z}_{16} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
- $\mathbb{Z}_8 \times \mathbb{Z}_8 \times \mathbb{Z}_2$
- $\mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_4$
- $\mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
- $\mathbb{Z}_8 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
- $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_2$
- $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
- $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
- $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

There are 15 kinds of groups in total.$\blacksquare$