1. Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

   Then

$$AS = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & & 0 \\ & 0 & \ddots & \\ & & \ddots & 1 \\ 1 & & & 0 \end{bmatrix} = \begin{bmatrix} a_{1n} & a_{11} & \cdots & a_{1(n-1)} \\ a_{2n} & a_{21} & \cdots & a_{2(n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{nn} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

   As shown, S is a shift matrix since it can shift each column 1 step to right. And

$$S\text{x} = [x_{n-1}, x_0, \cdots, x_{n-2}]$$

   For the same reason,

$$S^k\text{x} = [x_{n-k}, x_{n-k+1}, \cdots, x_1, x_2, \cdots, x_{n-k-1}]$$

2. No, since all eigenvalues of S, $|\lambda_i| = 1$, power method may not converge.

3. **Proof:**

$$\begin{aligned} & Sx = \lambda x \\ & (S - \lambda I)x = 0 \\ & \begin{bmatrix} -\lambda & 1 & & 0 \\ & -\lambda & \ddots & \\ & & \ddots & 1 \\ 1 & & & -\lambda \end{bmatrix} = 0 \\ & det(\begin{bmatrix} -\lambda & 1 & & 0 \\ & -\lambda & \ddots & \\ & & \ddots & 1 \\ 1 & & & -\lambda \end{bmatrix}) = 0 \end{aligned}$$

So that

$$\begin{bmatrix} 1 & & 0 \\ -\lambda & \ddots & \\ & \ddots & 1 \end{bmatrix} - \lambda \begin{bmatrix} -\lambda & & 0 \\ & \ddots & \\ & \ddots & -\lambda \end{bmatrix} = 0$$

$\Rightarrow 1 \pm \lambda = 0 \Rightarrow |\lambda| = 1$

4. Since $Sx = \lambda x$, $S^2x = \lambda_1\lambda_2 x$ that λ_1, λ_2 is eigenvalues of S.

   Since $\lambda_k = e^{i(2\pi k/n)} = \cos(2\pi k/n) + i\sin(2\pi k/n)$ and $\cos(2\pi\frac{n+k}{n}) = \cos(2\pi\frac{k}{n})$ and $\sin(2\pi\frac{n+k}{n}) = \sin(2\pi\frac{k}{n})$ when $k < n$.

   Thus, there are always n unique eigenvalues. For the same reason, S^4, S^{2^k} and S^n all have n unique eigenvalues.