1

(a) **Claim:** If $d|a$ and $d|b$ then $d|ax+by$ for all $x, y \in \mathbb{Z}$.

Proof: $d|a \Rightarrow \exists k_1 \in \mathbb{Z}, a = k_1 d, d|b \Rightarrow \exists k_2 \in \mathbb{Z}, b = k_2 d, ax + by = k_1 x d + k_2 y d = (k_1 x + k_2 y)d$ with $k_1 x + k_2 y$ for all $x, y \in \mathbb{Z}$, so $d|ax + by$ for all $x, y \in \mathbb{Z}$.■

(b) **Claim:**If $a|b$ and $c|d$, then $ac|bd$.

Proof: $a|b \Rightarrow \exists k_1 \in \mathbb{Z}, b = k_1 a, c|d \Rightarrow \exists k_2 \in \mathbb{Z}, c = k_2 d$, then $bd = k_1 a k_2 d = k_1 k_2 a d$ with $k_1 k_2 \in \mathbb{Z}$ by closure. So if $a|b$ and $c|d$, then $ac|bd$.■

(c) **Claim:** If $a|b$ and $c|d$, then $(a+c)|(b+d)$.

Proof: $a|b \Rightarrow \exists k_1 \in \mathbb{Z}, b = k_1 a, c|d \Rightarrow \exists k_2 \in \mathbb{Z}, c = k_2 d$, then $b + d = (k_1 + k_2)d$ with $k_1 + k_2 \in \mathbb{Z}$.

2

(a) **Claim:** If $a \equiv b \mod m$ and $c \equiv d \mod m$, then $ac \equiv bd \mod m$.

Proof: $a \equiv b \mod m \Rightarrow \exists k_1 \in \mathbb{Z}, (a - b) = k_1 m, c \equiv d \mod m \Rightarrow \exists k_2 \in \mathbb{Z}, (c - d) = k_2 m, ac - bd = (b + k_1 m)(d + k_2 m) - bd = bd + k_1 m d + k_2 m b + k_1 k_2 m^2 - bd = k_1 m d + k_2 m b + k_1 k_2 m^2 = m(k_1 d + k_2 b + k_1 k_2 m)$.

Thus, $ac \equiv bd \mod m$.■

(b) **Claim:** If $a \equiv b \mod m$, then for any $k \in \mathbb{N}$, $a^k \equiv b^k \mod m$.

Proof: Base case, let $c = a, d = b$, we have statement to be true when $k = 2$.

Suppose, the statement is true when $k = n$, $a^n \equiv b^n \mod m$. Then when $k = n + 1$, let $a = b + tm$, $a^{n+1} - b^{n+1} = (b + tm)^{n+1} - b^{n+1} = \sum_{i=0}^{n+1} \binom{n+1}{i} b^{n+1-i} y^i - b^{n+1}$, which is a multiple of m, so $m|(a^{n+1} - b^{n+1}) \Rightarrow a^{n+1} \equiv b^{n+1} \mod m$.

As a result, if $a \equiv b \mod m$ and $c \equiv d \mod m$, then $ac \equiv bd \mod m$.■

3

(a) Since $347 \equiv 7 \mod 10 \Rightarrow 347^4 \equiv 1 \mod 10 \Rightarrow 347^{100} \equiv 1 \mod 10 \Rightarrow 347^{101} \equiv 7 \mod 10$, so the last decimal digit is 7.

(b) According to Fermat's Little Theorem, $347^{101} \equiv 347 \mod 101 = 44$ since 101 is a prime.

(c) According to Fermat's Little Theorem, $2^{13} \equiv 2 \mod 13$, then $2^{100} \equiv 2^{12 \cdot 8+4} \equiv (2^12)^8 \cdot 2^4 = 3 \mod 13$.

(d)

(i) Since $9 = 11_8$, $11^2 = 121, 11^3 = 1331 \cdots$, so $9^{1000} = (11_8)^{1000}$ ends with 1.

(ii) Since $10 = 12_8$, $12^2 = 144, 12^3 = 1750 \cdots$, 10^1000 ends with 0.

(iii) Since $11 = 13_8$, then $13 = 13$, $13^2 = 171 \cdots$, so 11^{1000} ends with 1.