1. Let $G = (V, E)$ be an undirected graph. Unless we say otherwise, a graph has no loops or parallel edges.

   - Prove that if $|V| \geq 2$ there are two distinct nodes u and v such that degree of u is equal to degree of v. Recall that the degree of a node x is the number of edges incident to x.

     Solution: Since there are no loops, each vertex can be adjacent to at most $|V| - 1$ vertices. Therefore, degree of a vertex is one of the $|V|$ integers in the range $[0, |V| - 1]$. Suppose towards contradiction that all vertices have distinct degrees. Then for each integer $i = 0, 1, 2, \ldots, |V| - 1$ there is a vertex of degree i. In particular, there is a vertex v of degree $|V| - 1$ and there is a vertex u of degree 0. Vertex v has to be adjacent to every other vertex of graph and vertex u cannot be adjacent to any other vertex. Note that u and v are distinct vertices, because they have different degrees (as $|V| \geq 2$). So they are either adjacent or non-adjacent. By choice of u they are non-adjacent and by choice of v they are adjacent. This is a contradiction. Hence, our assumption was false and there are two vertices of the same degree in the graph. ∎

     Rubric: 6 points for correctly arguing that there are two vertices of the same degree. -1 for minor errors. -1 for not using the fact that $|V| \geq 2$.

   - Prove that if G has at least one edge then there is a path between two distinct nodes u and v such that degree of u is equal to degree of v.

     Solution: Let H be a connected component of G with at least one edge. This subgraph has at least two vertices. By result of part (a), there are two vertices u and v of the same degree in H. Since all neighbors of u and v are in H, u and v have the same degree in G as well. Also, since H is connected, by definition, there is a path between u and v. Thus, u and v satisfy the requirements of the problem. ∎

   Rubric: 4 points for arguing that two vertices of the same degree are in the same component. -1 for minor errors. -1 for assuming the graph was connected.

2. The ***plus one***, w^+, of a string $w \in \{\mathtt{0}, \mathtt{1}, \mathtt{2}\}^*$ is obtained from w by replacing each symbol a in w by the symbol corresponding $a + 1 \mod 3$. for example, $\mathtt{0102101}^+ = \mathtt{1210212}$. The plus one function is formally defined as follows:

$$w^+ := \begin{cases} \varepsilon & \text{if } w = \varepsilon \\ \mathtt{1} \cdot x^+ & \text{if } w = \mathtt{0}x \\ \mathtt{2} \cdot x^+ & \text{if } w = \mathtt{1}x \\ \mathtt{0} \cdot x^+ & \text{if } w = \mathtt{2}x \end{cases}$$

(a) **Not to submit:** Prove by induction that $|w| = |w^+|$ for every string w.

Solution: We will prove by induction on $|w|$.

- **Induction Hypothesis:** For all $n \geq 0$, for any string w of length n $|w| = |w^+|$.
- **Base Case:** Let w be an arbitrary string of length 0. $w = \varepsilon$, since there is only one such string. Then $w^+ = \varepsilon$ by definition, so $w = w^+$, which trivially implies $|w| = |w^+|$.
- **Inductive Step:** Let w be an arbitrary string of length $n > 0$. Assume inductive hypothesis holds for all strings x of length $< n$. There are three cases to consider:
  - If $w = \mathtt{0}x$ for some string x, then

$$\begin{aligned} |w^+| &= |\mathtt{1}x^+| && \text{by definition of } w^+ \\ &= 1 + |x^+| && \text{by definition of } |\,| \\ &= 1 + |x| && \text{by the inductive hypothesis} \\ &= |\mathtt{0}x| && \text{by definition of } |\,| \\ &= |w|. \end{aligned}$$

  - Similarly, if $w = \mathtt{1}x$ for some string x, then

$$\begin{aligned} |w^+| &= |\mathtt{2}x^+| && \text{by definition of } w^+ \\ &= 1 + |x^+| && \text{by definition of } |\,| \\ &= 1 + |x| && \text{by the inductive hypothesis} \\ &= |\mathtt{1}x| && \text{by definition of } |\,| \\ &= |w| \end{aligned}$$

  - Finally, if $w = \mathtt{2}x$ for some string x, then

$$\begin{aligned} |w^+| &= |\mathtt{0}x^+| && \text{by definition of } w^+ \\ &= 1 + |x^+| && \text{by definition of } |\,| \\ &= 1 + |x| && \text{by the inductive hypothesis} \\ &= |\mathtt{2}x| && \text{by definition of } |\,| \\ &= |w| \end{aligned}$$

  In all cases, we conclude that $|w| = |w^+|$.

■

Rubric: 0 points: Not to be submitted

(b) Prove by induction that $(x \bullet y)^+ = x^+ \bullet y^+$ for all strings $x, y \in \{\texttt{0}, \texttt{1}, \texttt{2}\}^*$.

Solution: We will prove by induction on $|x|$.

- **Induction Hypothesis:** For all $n \geq 0$, for any string x of length n, for all strings y, $(x \bullet y)^+ = x^+ \bullet y^+$.
- **Base Case:** Let x be an arbitrary string of length 0. $x = \varepsilon$, since there is only one such string. Then

$$\begin{aligned}
(x \bullet y)^+ &= y^+ && \text{by definition of } \bullet \\
&= \varepsilon \bullet y^+ && \text{by definition of } \bullet \\
&= x^+ \bullet y^+ && \text{by definition of } x^+
\end{aligned}$$

- **Inductive Step:** Let x be an arbitrary string of length $n > 0$. Assume inductive hypothesis holds for all strings z of length $< n$. There are three cases to consider:
  - If $x = \texttt{0}z$ for some string z, then

$$\begin{aligned}
(x \bullet y)^+ &= (\texttt{0} \cdot (z \bullet y))^+ && \text{by definition of } \bullet \\
&= \texttt{1} \cdot (z \bullet y)^+ && \text{by definition of } w^+ \\
&= \texttt{1} \cdot (z^+ \bullet y^+) && \text{by the induction hypothesis} \\
&= (\texttt{1} \cdot z^+) \bullet y^+ && \text{by definition of } \bullet \\
&= x^+ \bullet y^+ && \text{by definition of } x^+
\end{aligned}$$

  - Similarly, if $x = \texttt{1}z$ for some string z, then

$$\begin{aligned}
(x \bullet y)^+ &= (\texttt{1} \cdot (z \bullet y))^+ && \text{by definition of } \bullet \\
&= \texttt{2} \cdot (z \bullet y)^+ && \text{by definition of } w^+ \\
&= \texttt{2} \cdot (z^+ \bullet y^+) && \text{by the induction hypothesis} \\
&= (\texttt{2} \cdot z^+) \bullet y^+ && \text{by definition of } \bullet \\
&= x^+ \bullet y^+ && \text{by definition of } x^+
\end{aligned}$$

  - Finally, if $x = \texttt{2}z$ for some string z, then

$$\begin{aligned}
(x \bullet y)^+ &= (\texttt{2} \cdot (z \bullet y))^+ && \text{by definition of } \bullet \\
&= \texttt{0} \cdot (z \bullet y)^+ && \text{by definition of } w^+ \\
&= \texttt{0} \cdot (z^+ \bullet y^+) && \text{by the induction hypothesis} \\
&= (\texttt{0} \cdot z^+) \bullet y^+ && \text{by definition of } \bullet \\
&= x^+ \bullet y^+ && \text{by definition of } x^+
\end{aligned}$$

  In all cases, we conclude that $(x \bullet y)^+ = x^+ \bullet y^+$.

■

Rubric: 10 points: standard induction rubric

3. Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ be two fixed vectors in the real plane. Recursively define a set $L_n \subseteq \mathbb{R}^2$ as follows.

   - $L_0 = \{\mathbf{u}, \mathbf{v}, \mathbf{0}\}$. ($\mathbf{0}$ denotes the zero vector $(0,0)$ in $\mathbb{R}^2$.)
   - For integer $n > 0$, $L_n = \{\mathbf{x} - \mathbf{y} \mid \mathbf{x}, \mathbf{y} \in L_{n-1}\}$.

   Let $L = \bigcup_{n=0}^{\infty} L_n$. Also, let $D = \{a\mathbf{u} + b\mathbf{v} \mid a, b \in \mathbb{Z}\}$ be the set of vectors obtained as integer linear combinations of $\mathbf{u}$ and $\mathbf{v}$.

   (a) Prove that $D \subseteq L$, by giving, for each $a, b \in \mathbb{Z}$, an explicit value of n such that $a\mathbf{u} + b\mathbf{v} \in L_n$. (You don't need to minimize the value of n; but you must argue why $a\mathbf{u} + b\mathbf{v} \in L_n$ for your choice of n.)

   Solution: We divide the proof into three stages:
   Claim 1. For all $i > 0$, $\{u, v, 0, -v-u\} \subseteq L_i$.
   Claim 2. For any $a \in \mathbb{Z}$, $au \in L_n$, where $n = |a|$.
   Claim 3. For any $a, b \in \mathbb{Z}$, $au + bv \in L_n$, where $n = |a| + |b|$.
   Proof of Claim 1. Claim 1 can be proven using an easy induction. Alternatively, the following observations are sufficient.

   - For $i > 0$, L_i is non-empty. (This really needs induction, but could be taken as evident.)
   - For all $i \geq 0$, $0 \in L_i$. (This is true for $i = 0$ by definition. For $i > 0$, this follows from the fact that L_{i-1} is non-empty, so $\exists x \in L_{i-1}$ such that $x - x = 0 \in L_i$.)
   - For all $i > 0$, $L_{i-1} \subseteq L_i$. (This follows from the fact that $0 \in L_{i-1}$ and the definition of L_i.)
   - Hence, for all $i > 0$, $L_1 \subseteq L_i$. (This again could be taken as evident, though formally it needs induction.)
   - $\{u, v, 0, -u, -v\} \subseteq L_1$. (By applying the definition of L_n for $n = 1$.)

   Hence, for $i > 0$, $\{u, v, 0, -v, -u\} \subseteq L_1 \subseteq L_i$.
   Proof of Claim 2. By induction on $n = |a|$.

   - **Base case:** $|a| = 0, 1$ (verified by hand for $a = 0, -1, 1$).
   - **Iduction:** $|a| > 1$. Assume inductively that for any $c \in \mathbb{Z}$ such that $|c| = n'$ and $n' < n$, we have $cu \in L_{n'}$. Because $n = |a|$, we have two possibilities:
   -
     - If $n = a$, then $au = (n-1)u - (-u)$.
     - If $n = -a$, then $au = -(n-1) - u$.

     By the inductively hypothesis and Claim 1, in both cases, L_{n-1} contains both the terms on the RHS, and hence the LHS is in L_n.

   Therefore, for any $a \in \mathbb{Z}$, $au \in L_n$, where $n = |a|$.
   Proof of Claim 3. By induction on $m = |b|$.

   - **Base case:** When $|b| = 0$ the claim follows from Claim 2. There is a special case $|b| = 1$, $|a| = 0$, which should also be verified separately ($v, -v \in L_1$), as the induction step below can't handle it (it's okay not to take off points if this is overlooked).
   - **Induction:** $|b| \geq 1$. Assume inductively that for any $a, c \in \mathbb{Z}$ such that $|c| = m'$ and $m' < m$, we have $au + cv \in L_{|a|+m'}$. Because $m = |b|$, we have two possibilities:
     - If $m = b$, then $au + bv = [au + (m-1)v] - (-v)$.
     - If $m = -b$, then $au + bv = [au + (1-m)v] - v$.

By the induction hypothesis and Claim 1, in both cases, $L_{|a|+m-1}$ contains both the terms on the RHS, and hence the LHS is in $L_{|a|+m}$. Note that to apply Claim 1, we need $|a|+m-1>0$, and hence the case $|a|=0, |b|=1$ should be handled separately.

Therefore, for any $a, b \in \mathbb{Z}$, $au+bv \in L_n$, where $|a|+|b|$.

Proof of Claim 3 (alternative using $n = \max(|a|,|b|)+1$). Using Claim 2 from above and performing a similar proof induction on $|b|$, we can conclude that $au \in L_{|a|}$ and $-bv \in L_{|b|}$.

We showed that $L_{i-1} \subseteq L_i$ for all $i > 0$. This implies $L_m \subseteq L_n$ when $0 \leq m \leq n$, which can be easily proven by induction. Because $|a| \leq \max(|a|,|b|)$ and $|b| \leq \max(|a|,|b|)$, au and $-bv$ are in $L_{\max(|a|,|b|)}$. Thus, when $n = \max(|a|,|b|)+1$, $au+bv = au-(-bv) \in L_n$ by definition of L_n.

Proof of Claim 3 (alternative using $n = |a|+|b|+1$). The reasoning is similar to the previous alternative, bu let $n = |a|+|b|+1$.

These are not the only possible definitions of n and formulation of the claims. Any n that works is acceptable. ■

(b) Use mathematical induction to prove that for all integers $n \geq 0$, $L_n \subseteq D$, and hence $L \subseteq D$.

Solution: Let n be an arbitrary non-negative integer. Assume inductively that $L_m \subseteq D$ for all non-negative integers m such that $m < n$. There are two cases to consider.

- **Base Case:** If $n = 0$, then $L_0 = \{u, v, 0\} = \{1u+0v, 0u+1v, 0u+0v\}$. Therefore $L_0 \subseteq D$.
- **Induction:** If $n > 0$, then let z be an arbitrary element in L_n. By definition of L_n, there are $x, y \in L_{n-1}$ such that $z = x - y$. By the inductive hypothesis, x and y are in D. Therefore, they can be written as $x = a_1u + b_1v$ and $y = a_2u + b_2v$, respectively, where $a_1, a_2, b_1, b_2 \in \mathbb{Z}$. Thus,

$$\begin{aligned} z &= x - y \\ &= (a_1u + b_1v) - (a_2u + b_2v) \\ &= (a_1 - a_2)u + (b_1 - b_2)v \end{aligned}$$

  where $a_1 - a_2$ and $b_1 - b_2$ are in $\mathbb{Z}$. Therefore, $L_n \subseteq D$.

We conclude that $L_n \subseteq D$ for all integers $n \geq 0$, and hence $L = \bigcup_{n=0}^{\infty} L_n \subseteq D$. ■

Rubric: On a scale of 10 points, we would allocate

- 5 points for part (a)
  - 1 point for demonstrating Claim 1
  - 2 points for demonstrating Claim 2
  - 2 points for demonstrating Claim 3
- 5 points for proof by induction of part (b)

Grading notes for part (a):

- Any valid n is acceptable.
- Claims 2 and 3 need induction, but it's okay to give induction only for Claim 2 and say that it works similarly for Claim 3 (though it is a little more involved for the latter).
- If induction is glossed over for Claim 3. the essential point is that au and $-bv$ are shown to exist in L_n, therefore, $au + bv$ is in L_{n+1} by the definition of L_{n+1}.

Rubric (induction): For problems worth 10 points:

+ 1 for explicitly considering an *arbitrary* object
+ 2 for a valid **strong** induction hypothesis
  - **Deadly Sin!** Automatic zero for stating a weak induction hypothesis, unless the rest of the proof is *perfect*.
+ 2 for explicit exhaustive case analysis
  - No credit here if the case analysis omits an infinite number of objects. (For example: all odd-length palindromes.)
  - −1 if the case analysis omits an finite number of objects. (For example: the empty string.)
  - −1 for making the reader infer the case conditions. Spell them out!
  - No penalty if cases overlap (for example:
+ 1 for cases that do not invoke the inductive hypothesis ("base cases")
  - No credit here if one or more "base cases" are missing.
+ 2 for correctly applying the *stated* inductive hypothesis
  - No credit here for applying a *different* inductive hypothesis, even if that different inductive hypothesis would be valid.
+ 2 for other details in cases that invoke the inductive hypothesis ("inductive cases")
  - No credit here if one or more "inductive cases" are missing.