Math 414 - Homework 1

due Jan 24

1. We define a binary relation on $\mathbb{Z}$ by declaring

$$m \equiv n \pmod 2 \quad \text{if and only if} \quad m - n \quad \text{is even.}$$

(a): Show that $m \equiv n \pmod 2$ is an equivalence relation.

(b): Show that if $m \equiv m' \pmod 2$ and $n \equiv n' \pmod 2$ then

$$m + n \equiv m' + n' \pmod 2 \quad \text{and} \quad mn \equiv m'n' \pmod 2.$$

Let U be a set. Given an arbitrary subset A of U we let $\chi_A : U \to \{0, 1\}$ be the function given by

$$\chi_A(x) = \begin{cases} 0 & x \notin A \\ 1 & x \in A \end{cases}$$

(c): Show that if A, B are subsets of U then

$$\chi_{A\cap B}(x) \equiv \chi_A(x)\chi_B(x) \pmod 2 \quad \text{for all } x \in U$$

and

$$\chi_{A\triangle B}(x) \equiv \chi_A(x) + \chi_B(x) \pmod 2 \quad \text{for all } x \in U.$$

Recall that $A\triangle B$ is the *symmetric difference* of A and B, i.e. is equal to $(A \cup B) \setminus (A \cap B)$.

2. Let A, B, C be sets and $f : A \to B, g : B \to C$ be functions. Prove or give a counterexample to each of the following. Here $g \circ f$ is the composition of f and g.

(1) If f, g are both injective then $g \circ f$ is injective.
(2) If f, g are both surjective then $g \circ f$ is surjective.
(3) If $g \circ f$ is injective then f is injective.
(4) If $g \circ f$ is injective then g is injective.
(5) If $g \circ f$ is surjective then f is surjective.
(6) If $g \circ f$ is surjective then g is surjective.

3. (a): Describe an explicit bijection between $\mathbb{N}$ and $\mathbb{Z}$.

(b): Show that the union of two countable sets is countable.

4. Suppose that A is a finite set with $n \geq 1$ elements. Show that A^k has n^k elements for all integers $k \geq 1$ by applying induction to k.

5. Let $\mathcal{P}(A)$ be the power set of a set A. Let A, B, C, D be sets. Prove or give a counterexample to each of the following.

(1) $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.
(2) $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$.