Thursday, September 10 $**$ *Functions of several variables; Limits.*

1. For each of the following functions $f\colon \mathbb{R}^2 \to \mathbb{R}$, draw a sketch of the graph together with pictures of some level sets.

   (a) $f(x, y) = xy$

   (b) $f(\mathbf{x}) = |\mathbf{x}|$. Please note here that $\mathbf{x}$ is a vector. In coordinates, this function is $f(x, y) = \sqrt{x^2 + y^2}$.

   For (a), the result is one of the many quadric surfaces. What is the name for this type? Is the graph in (b) also a quadric surface?

2. Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x, y) = \frac{2x^3 y}{x^6 + y^2} \quad \text{for } (x, y) \neq \mathbf{0}$$

   In this problem, you'll consider $\lim_{(x,y)\to\mathbf{0}} f(x, y)$.

   (a) Look at the values of f on the x- and y-axes. What do these values show the limit $\lim_{(x,y)\to\mathbf{0}} f(x, y)$ must be **if it exists**?

   (b) Show that along each line in $\mathbb{R}^2$ through the origin, the limit of f exists and is 0.

   (c) Despite this, show that the limit $\lim_{(x,y)\to\mathbf{0}} f(x, y)$ does not exist by finding a curve over which f takes on the constant value 1.

3. Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x, y) = \frac{xy^2}{\sqrt{x^2 + y^2}} \quad \text{for } (x, y) \neq \mathbf{0}$$

   In this problem, you'll show $\lim_{\mathbf{h}\to\mathbf{0}} f(\mathbf{h}) = 0$.

   (a) For $\epsilon = 1/2$, find some $\delta > 0$ so that when $0 < |\mathbf{h}| < \delta$ we have $|f(\mathbf{h})| < \epsilon$. Hint: As with the example in class, the key is to relate $|x|$ and $|y|$ with $|\mathbf{h}|$.

   (b) Repeat with $\epsilon = 1/10$.

   (c) Now show that $\lim_{\mathbf{h}\to\mathbf{0}} f(\mathbf{h}) = 0$. That is, given an arbitrary $\epsilon > 0$, find a $\delta > 0$ so that that when $0 < |\mathbf{h}| < \delta$ we have $|f(\mathbf{h})| < \epsilon$.

   (d) Explain why the limit laws that you learned in class on Wednesday aren't enough to compute this particular limit.