5.1.4

Claim: Let G be any group and H any subgroup. Then G acts on the set G/H of left cosets of H in G by left multiplication. Then action is transitive.

Proof: $\forall aH \in G/H, \forall g \in G, g \cdot G/H = gaH = (ga)H$, since $ga \in G$ by closure, $gaH \in G/H$. $\forall bH \in G/H, g = ba^{-1} \in G$ so that $gaH = bH$, so we proved that the number of orbit $\mathcal{O}(x)$ is 1.

As a result, G acts on the set G/H of left cosets of H in G by left multiplication is transitive.■

5.1.6

Claim: Let G act on X, and let $x \in X$. We have $\mathrm{Stab}(x) \subseteq G$ and if x and y are in the same orbit, then the subgroups $\mathrm{Stab}(x)$ and $\mathrm{Stab}(y)$ are conjugate subgroups.

Proof: Take $s \in \mathrm{Stab}(x) = \{g \in G : g \cdot x = x\}$, obviously $s \in G$, so $\mathrm{Stab}(x) \subseteq G$.

If x and y are in the same orbit, then $\exists g \in G$ that $y = g \cdot x = gxg^{-1}$. Then for $\mathrm{Stab}(x) = \{g \in G : g \cdot x = x\} = \{g \in G : gxg^{-1} = x\}$ and $\mathrm{Stab}(y) = \{g \in G : g \cdot y = y\} = \{g \in G : gyg^{-1} = y\}$. $\forall g' \in G, g'\mathrm{Stab}(x)g'^{-1} = \{g \in G : g'g \cdot xg^{-1}g'^{-1} = x\} = \{g \in G : g'gxg^{-1}g'^{-1} = x\} = \{g \in G : g'gxg^{-1}g'^{-1} = x\} = \{g \in G : g \cdot y = y\} = \mathrm{Stab}(y)$.■

5.2.2

Claim: 2 red beads, 2 green beads and 2 violet heads can make 11 necklaces.

Proof: Since the symmetries consist 6 rotational symmetries and 6 reflectional symmetries. We choose $G = D_6$, so $|G| = 12$. And since

$$|X| = \frac{6!}{2!2!2!} = 90,$$

so $|\mathrm{Fix}(e)| = 90, |\mathrm{Fix}(r^3)| = 6, |\mathrm{Fix}(a)| = |\mathrm{b}| = |\mathrm{Fix}(c)| = |\mathrm{Fix}(d)| = |\mathrm{Fix}(e)| = |\mathrm{Fix}(f)| = 6$.

Thus, by Burnside's Lemma,

$$n = \frac{1}{12}(90 + 7 \cdot 6) = 11. \blacksquare$$

5.3.7

Claim: $\mathrm{Aut}(\mathbb{Z}_2 \times \mathbb{Z}_2) \cong S_3$.

Proof 1: Since $\mathbb{Z}_2 \times \mathbb{Z}_2$ can be grouped into a vector of $2D$ vector, which is basically a 2×2 matrix, in which each elements can only be either 1 or 0. So, obviously, the only way to transform a 2×2 matrix to a 2×2 matrix is to multiply a 2×2 matrix, which means $\mathrm{Aut}(\mathbb{Z}_2 \times \mathbb{Z}_2) \subseteq \mathrm{GL}(2, \mathbb{R})$. Of them, only 6 matrices are invertible including:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

Those matrices can be easily checked to be a group by doing the multiplication.

Since there're only 2 kinds of groups of order 6 up to isomorphism S_3 and $\mathbb{Z}_6$. Since we can check that

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

So we know $\mathrm{Aut}(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is not commutative, which means it isn't isomorphic to $\mathbb{Z}_6$, and thus we know that $\mathrm{Aut}(\mathbb{Z}_2 \times \mathbb{Z}_2) \cong S_3$.$\blacksquare$

Proof 2: Since in any group mapping, we always send identity to identity in order to preserve the structure. So what really matters is that to which we send $\{a, b, c\}$ to from the group of $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{e, a, b, c\}$. Then for each homomorphism there exists a unique permutation in S_3 to fulfill this requirement.

Then $\forall g \in S_3$, by permutation composition, g acts on $\{a, b, c\}$ sends bijectively send a, b, c to $\{a, b, c\}$ so we can see such permutation determines an automorphism by definition.

As a result, $\mathrm{Aut}(\mathbb{Z}_2 \times \mathbb{Z}_2) \cong S_3$.$\blacksquare$