CS/ECE 374 Spring 2017 Lanxiao Bai (lbai5@illinois.edu)
Homework 1 Problem 3 Renheng Ruan (rruan2@illinois.edu)

Let $L_1, L_2$, and $L_3$ be regular languages over $\Sigma$ accepted by DFAs $M_1 = (Q_1, \Sigma, \delta_1, s_1, A_1)$, $M_2 = (Q_2, \Sigma, \delta_2, s_1, A_2)$, and $M_3 = (Q_3, \Sigma, \delta_3, s_3, A_3)$, respectively.

1. Describe a DFA $M = (Q, \Sigma, \delta, s, A)$ in terms of $M_1, M_2$, and $M_3$ that accepts $L = \{w \mid w \text{ is in exactly two of } \{L_1, L_2, L_3\}\}$. Formally specify the components $Q, \delta, s$, and $A$ for $M$ in terms of the components of $M_1, M_2$, and $M_3$.
2. Prove by induction that your construction is correct.

---

**Solution:** 1. For $M = (Q, \Sigma, \delta, s, A)$ that accepts exactly two of $\{L_1, L_2, L_3\}$,

We require that:

- $Q = Q_1 \times Q_2 \times Q_3$
- $\Sigma$ to be the same as $M_1, M_2, M_3$
- $\delta : Q \times \Sigma \to Q$, that $\delta((q_1, q_2, q_3), a) = (\delta_1(q_1, a)), \delta_2(q_2, a), \delta_3(q_3, a)), a \in \Sigma, q_1 \in Q_1, q_2 \in Q_2, q_3 \in Q_3$
- $s = (s_1, s_2, s_3)$
- $A = (A_1 \times A_2 \times (Q_3 - A_3)) \cup (A_1 \times (Q_2 - A_2) \times A_3) \cup ((Q_1 - A_1) \times A_2 \times A_3)$

2. **Proof:** Apply induction on the length of $w$.

Base case: When $|w| = 0$, namely, $w = \varepsilon$, $\delta(s, w) = \delta(s, \varepsilon) = (\delta_1(s_1, \varepsilon), \delta_2(s_2, \varepsilon), \delta_3(s_3, \varepsilon)) = (s_1, s_2, s_3) = s$.

If $w \in L$, without losing generality we can assume $w \in L_1, L_2$ only, so $\delta_1(s_1, w) = s_1 \in A_1, \delta_2(s_2, w) = s_2 \in A_2, \delta_3(s_3, w) = s_3 \notin A_3$. Thus, $s \in A$, which means $M$ accepts $w$.

If $M$ accepts $w$, then $s = (s_1, s_2, s_3) \in A$, which means that $s_1 \in A_1, s_2 \in A_2, s_3 \notin A_3$ or $s_1 \in A_1, s_2 \notin A_2, s_3 \in A_3$ or $s_1 \notin A_1, s_2 \in A_2, s_3 \in A_3$. Then exactly 2 of $M_1, M_2, M_3$ accepts $w$, so $w$ in exactly 2 of $L_1, L_2, L_3$. Hence, $w \in L$.

Suppose for all $|w| \le k$, if $w \in L$, then $M$ accepts $w$. And if $M$ accepts $w$, then $w \in L$.

Then when $|w| = k + 1$, let $w = w_0 \bullet a, a \in \Sigma$, $\delta(s, w) = \delta(\delta(s, a), w_0)$. By induction hypothesis, we know that $\delta(s, a)$ corrected judged if $a$ should be accepted. And let $\delta(s, a) = q$, $\delta(q, w_0)$, since $|w_0| = k$, again, by induction hypothesis, we know that $\delta(q, w_0)$ corrected judged if $w_0$ should be accepted by $M$.

If $M$ accepts $w$, then $\delta(s, w) \in A \Rightarrow (\delta_1(s_1, w), \delta_2(s_2, w), \delta_3(s_3, w)) \in A$. That means that $\delta_1(s_1, w) \in A_1, \delta_2(s_2, w) \in A_2, \delta_3(s_3, w) \notin A_3$ or $\delta_1(s_1, w) \in A_1, \delta_2(s_2, w) \notin A_2, \delta_3(s_3, w) \in A_3$ or $\delta_1(s_1, w) \notin A_1, \delta_2(s_2, w) \in A_2, \delta_3(s_3, w) \in A_3$, so exactly 2 of $M_1, M_2, M_3$ accepts $w$, so $w$ in exactly 2 of $L_1, L_2, L_3$. Hence $w \in L$.

In conclusion, the automata $M$ described above is correct.

■