Loop Transformations for Parallelism & Locality

Last week

- Data dependences and loops
- Loop transformations
  - (Parallelization)
  - Scalar expansion
- Value data dependences

Today and Monday

- Loop transformations and transformation frameworks
  - Loop reversal
  - Loop fusion
  - Loop fission
  - Loop interchange
  - Unroll and Jam

Review

Distance vectors

- Concisely represent dependences in loops (*i.e.*, in iteration spaces)
- Dictate what transformations are legal
  - *e.g.,* Permutation and parallelization

Legality

- A dependence vector is **legal** when it is lexicographically nonnegative

Loop-carried dependence

- A dependence $D=(d_1,\ldots d_n)$ is **carried** at loop level i if d_i is the first nonzero element of D

Loop Permutation

Idea

- Swap the order of two loops to increase parallelism, to improve spatial locality, or to enable other transformations
- Also known as **loop interchange**

Example

```
do i = 1,n
    do j = 1,n
        x = A(2,j)
    enddo
enddo
```

This access strides through a row of A

→

```
do j = 1,n
    do i = 1,n
        x = A(2,j)
    enddo
enddo
```

This code is invariant with respect to the inner loop, yielding better locality

Loop Interchange (cont)

Example

```
do i = 1,n
    do j = 1,n
        x = A(i,j)
    enddo
enddo
```

This array has stride n access

→

```
do j = 1,n
    do i = 1,n
        x = A(i,j)
    enddo
enddo
```

This array now has stride 1 access

(Assuming column-major order for Fortran)

Legality of Loop Interchange

Case analysis of the direction vectors

(=,=)

The dependence is loop independent, so it is unaffected by interchange

(=,<)

The dependence is carried by the j loop.
After interchange the dependence will be (<,=), so the dependence will still be carried by the j loop, so the dependence relations do not change.

(<,=)

The dependence is carried by the i loop.
After interchange the dependence will be (=,<), so the dependence will still be carried by the i loop, so the dependence relations do not change.

Legality of Loop Interchange (cont)

Case analysis of the direction vectors (cont.)

(<,<)

The dependence distance is positive in both dimensions.
After interchange it will still be positive in both dimensions, so the dependence relations do not change.

(<,>)

The dependence is carried by the outer loop.
After interchange the dependence will be (>,<), which changes the dependences and results in an illegal direction vector, so interchange is illegal.

(>,*) (=,>)

Such direction vectors are not possible for the original loop.

Loop Interchange Example

Consider the (<,>) case

```
do i = 1,n
   do j = 1,n
      C(i,j) = C(i+1,j-1)
   enddo
enddo
```

$\Rightarrow$

```
do j = 1,n
   do i = 1,n
      C(i,j) = C(i+1,j-1)
   enddo
enddo
```

Before

(1,1) C(1,1) = C(2,0)
(1,2) C(1,2) = C(2,1)
. . .   d = (<,>) δ^a
(2,1) C(2,1) = C(3,0)

After

(1,1) C(1,1) = C(2,0)
(2,1) C(2,1) = C(3,0)
. . .   d = (>,<) δ^f
(1,2) C(1,2) = C(2,1)

Frameworks for Loop Transformations

Unimodular Loop Transformations [Banerjee 90],[Wolf & Lam 91]

- can represent loop permutation, loop reversal, and loop skewing
- unimodular linear mapping (determinant of matrix is + or - 1)
  - T i = i', T is a matrix, i and i' are iteration vectors

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} i'_1 \\ i'_2 \end{bmatrix}$$

- transformation is legal if the transformed dependence vector remain lexicographically positive
- limitations
  - only perfectly nested loops
  - all statements are transformed the same

Legality of Loop Interchange, Reprise

Reduced case analysis of the direction vectors $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} i \\ j \end{bmatrix} = \begin{bmatrix} j \\ i \end{bmatrix}$

(=,=)

The dependence is loop independent, so it is unaffected by interchange

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(=,<)

The dependence is carried by the j loop.

After interchange the dependence will be (<,=), so the dependence will still be carried by the j loop, so the dependence relations do not change.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ < \end{bmatrix} = \begin{bmatrix} < \\ 0 \end{bmatrix}$$

(<,>)

The dependence is carried by the outer loop.

After interchange the dependence will be (>,<), which changes the dependences and results in an illegal direction vector, so interchange is illegal.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} < \\ > \end{bmatrix} = \begin{bmatrix} > \\ < \end{bmatrix}$$

Loop Reversal

Idea

– Change the direction of loop iteration

(*i.e.,* From low-to-high indices to high-to-low indices or vice versa)

Benefits

– Improved cache performance

– Enables other transformations (coming soon)

Example

```
do i = 6,1,-1
   A(i) = B(i) + C(i)
enddo
```

```
do i = 1,6
   A(i) = B(i) + C(i)
enddo
```

Loop Reversal and Distance Vectors

Impact

– Reversal of loop i negates the i^{th} entry of all distance vectors associated with the loop

– What about direction vectors?

When is reversal legal?

– When the loop being reversed does not carry a dependence
(*i.e.*, When the transformed distance vectors remain legal)

Example

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} i \\ j \end{bmatrix} = \begin{bmatrix} i \\ -j \end{bmatrix}$$

```
do i = 1,5
    do j = 1,6
        A(i,j) = A(i-1,j-1)+1
    enddo
enddo
```

Dependence: Flow
Distance Vector: (1,1)
Transformed
Distance Vector: (1,-1) legal

Loop Reversal Example

Legality

– Loop reversal will change the direction of the dependence relation

Is the following legal?

```
do i = 1,6
    A(i) = A(i-1)
enddo
```

Dependence: Flow
Distance Vector: (1)

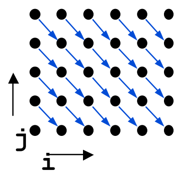
```
do i = 6,1,-1
    A(i) = A(i-1)
enddo
```

Dependence:	Anti	Flow
Distance Vector:	(1)	(−1)

Loop Skewing

Original code

```
do i = 1,6
   do j = 1,5
      A(i,j) = A(i-1,j+1)+1
   enddo
enddo
```

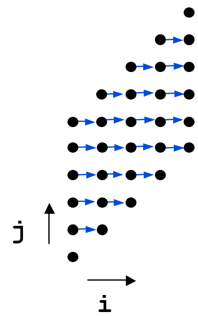


Distance vector: (1, -1)

Can we permute the original loop?

Skewing:

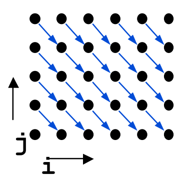
$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} i \\ j \end{bmatrix} = \begin{bmatrix} i \\ i+j \end{bmatrix}$$



Transforming the Dependences and Array Accesses

Original code

```
do i = 1,6
   do j = 1,5
      A(i,j) = A(i-1,j+1)+1
   enddo
enddo
```



Dependence vector:

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

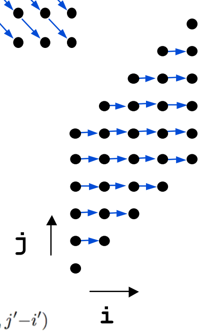
New Array Accesses:

$$A\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} i \\ j \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} i \\ j \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = A(i,j)$$

$$A\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} i' \\ j' \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} i' \\ j' \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = A(i', j'-i')$$

$$A\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} i \\ j \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} i \\ j \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = A(i-1, j+1)$$

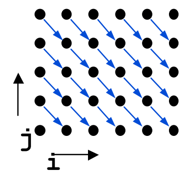
$$A\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} i' \\ j' \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} i' \\ j' \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = A(i'-1, j'-i'+1)$$



Transforming the Loop Bounds

Original code

```
do i = 1,6
     do j = 1,5
          A(i,j) = A(i-1,j+1)+1
     enddo
enddo
```



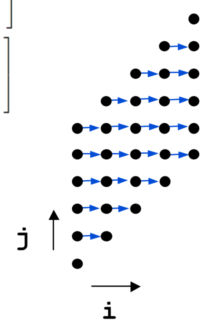
Bounds:

$$\begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} i \\ j \end{bmatrix} \le \begin{bmatrix} -1 \\ 6 \\ -1 \\ 5 \end{bmatrix} \qquad \begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} i' \\ j'-i' \end{bmatrix} \le \begin{bmatrix} -1 \\ 6 \\ -1 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} i' \\ j' \end{bmatrix} \le \begin{bmatrix} -1 \\ 6 \\ -1 \\ 5 \end{bmatrix} \qquad \begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} i' \\ j' \end{bmatrix} \le \begin{bmatrix} -1-i' \\ 6+i' \\ -1-i' \\ 5+i' \end{bmatrix}$$

Transformed code

```
do i' = 1,6
     do j' = 1+i',5+i'
          A(i',j'-i') = A(i'-1,j'-i'+1)+1
     enddo
enddo
```



Loop Fusion

Idea

– Combine multiple loop nests into one

Example

```
do i = 1,n
    A(i) = A(i-1)
enddo
do j = 1,n
    B(j) = A(j)/2
enddo
```

```
do i = 1,n
    A(i) = A(i-1)
    B(i) = A(i)/2
enddo
```

Pros

– May improve data locality
– Reduces loop overhead
– Enables **array contraction** (opposite of scalar expansion)
– May enable better instruction scheduling

Cons

– May hurt data locality
– May hurt icache performance

Legality of Loop Fusion

Basic Conditions

– Both loops must have same structure
  – Same loop depth
  – Same loop bounds
  – Same iteration directions

Can we relax any of these restrictions?

– Dependences must be preserved
  e.g., Flow dependences must not become anti dependences

```
do i = 1,n
    body1
enddo
do i = 1,n
    body2
enddo
```

All cross-loop dependences flow from body1 to body2

```
do i = 1,n
    body1
    body2
enddo
```

Ensure that fusion does not introduce dependences from body2 to body1

Loop Fusion Example

What are the dependences?

```
    do i = 1,n
s1      A(i) = B(i) + 1
    enddo
    do i = 1,n
s2      C(i) = A(i)/2
    enddo
    do i = 1,n
s3      D(i) = 1/C(i+1)
    enddo
```

$s_1 \delta^f s_2$

$s_2 \delta^f s_3$

What are the dependences?

```
    do i = 1,n
s1      A(i) = B(i) + 1
s2      C(i) = A(i)/2
s3      D(i) = 1/C(i+1)
    enddo
```

$s_1 \delta^f s_2$

$s_3 \delta^a s_2$

Fusion changes the dependence between s_2 and s_3, so fusion is illegal

Is there some transformation that will enable fusion of these loops?

Loop Fusion Example (cont)

Loop reversal is legal for the original loops

– Does not change the direction of any dep in the original code

– Will reverse the direction in the fused loop: $s_3\delta^a\ s_2$ will become $s_2\delta^f\ s_3$

```
     do i = n,1
s1      A(i) = B(i) + 1
     enddo
                          s1δf s2
     do i = n,1
s2      C(i) = A(i)/2
     enddo
                          s2δf s3
     do i = n,1
s3      D(i) = 1/C(i+1)
     enddo
```

⇒

```
     do i = n,1,-1
s1      A(i) = B(i) + 1
                   s1δf s2
s2      C(i) = A(i)/2
                   s2δf s3
s3      D(i) = 1/C(i+1)
     enddo
```

After reversal and fusion all original dependences are preserved

Concepts

Using direction and distance vectors

Transformations:

– What is the benefit?

– What do they enable?

– When are they legal?

Unimodular transformation framework

– represents loop permutation, loop reversal, and loop skewing

– provides mathematical framework for ...

  – testing transformation legality,

  – transforming array accesses and loop bounds*,

  – and combining transformations

*** The example did not require Fourier Motzkin elimination.**

Next Time

Lecture

– More loop transformations

– An even cooler transformation framework