Lattice-Theoretic Framework for Data-Flow Analysis

Last time

- Generalizing data-flow analysis
- Introduced lattices

Today

- Introduce lattice-theoretic frameworks for data-flow analysis

Context

Goals

- Provide a single formal model that describes all data-flow analyses
- Formalize the notions of "safe," "conservative," and "optimistic"
- Place bounds on time complexity of data-flow analysis
- Correctness proof for IDFA

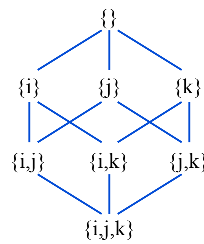
Approach

- Define **domain** of program properties (flow values) computed by data-flow analysis, and organize the domain of elements as a **lattice**
- Define flow functions and a merge function over this domain using lattice operations
- Exploit lattice theory in achieving goals

Data-Flow Analysis via Lattices

Relationship

– Elements of the lattice (V) represent flow values (in[] and out[] sets)
  – *e.g.*, Sets of live variables for liveness
– ⊤ represents "best-case" information (initial flow value)
  – *e.g.*, Empty set
– ⊥ represents "worst-case" information
  – *e.g.*, Universal set
– ⊓ (meet) merges flow values
  – *e.g.*, Set union
– If x ⊑ y, then x is a conservative approximation of y
  – *e.g.*, Superset



Data-Flow Analysis Frameworks

Data-flow analysis framework

– A set of **flow values** (V)
– A binary **meet operator** (⊓)
– A set of **flow functions** (F) (also known as **transfer functions**)

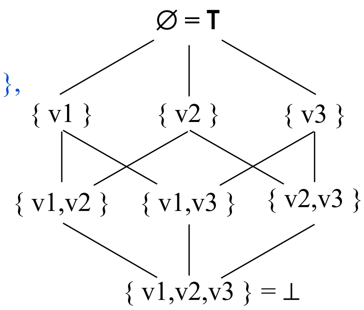
Flow Functions

– F = {f: V→V}
  f describes how each node in CFG affects the flow values
– Flow functions map program behavior onto lattices

Visualizing DFA Frameworks as Lattices

Example: Liveness analysis with 3 variables
S = {v1, v2, v3}

- V: $\mathbf{2^S =}$ {{v1,v2,v3}, {v1,v2},{v1,v3},{v2,v3}, {v1},{v2},{v3}, ∅}
- Meet (⊓): ∪
  - ⊑: ⊇
  - Top(⊤): ∅
  - Bottom (⊥): **V**
- F: $\{f_n(X) = Gen_n \cup (X - Kill_n), \forall n\}$



Inferior solutions are lower on the lattice
More conservative solutions are lower on the lattice

More Examples

Reaching definitions
- V: $\mathbf{2^S}$ **(S = set of all defs)**
- ⊓: ∪
  - ⊑: ⊇
  - Top(⊤): ∅
  - Bottom (⊥): **V**
- F: . . .

Reaching Constants
- V: $\mathbf{2^{v \times c}}$**, variables v and constants c**
- ⊓: ∩
  - ⊑: ⊆
  - Top(⊤): **V**
  - Bottom (⊥): ∅
- F: . . .

Tuples of Lattices

Problem

- Simple analyses may require very complex lattices (*e.g.,* Reaching constants)

Solution

- Use a tuple of lattices, one per variable

$L = (V, \sqcap) \;\equiv\; (L_T = (V_T, \sqcap_T))^N$

- $V = (V_T)^N$
- Meet ($\sqcap$): point-wise application of $\sqcap_T$
- $(\ldots, v_i, \ldots) \sqsubseteq (\ldots, u_i, \ldots) \;\equiv\; v_i \sqsubseteq u_i, \forall\, i$
- Top ($\top$): tuple of tops ($\top_T$)
- Bottom ($\perp$): tuple of bottoms ($\perp_T$)
- Height (L) = N * height(L_T)

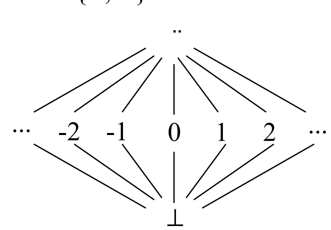
Tuples of Lattices Example

Reaching constants (previously)

- $P = v \times c$, for variables v & constants c
- V: 2^P

Alternatively

- $V = c \cup \{\top, \perp\}$



The whole problem is a tuple of lattices, one for each variable

Examples of Lattice Domains

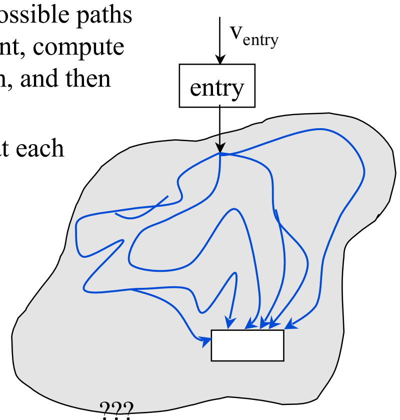
Two-point lattice ($\top$ and $\perp$)
- Examples?
- Implementation?

Set of incomparable values (and $\top$ and $\perp$)
- Examples?

Powerset lattice (2^S)
- $\top = \emptyset$ and $\perp = S$, or vice versa
- Isomorphic to tuple of two-point lattices

Solving Data-Flow Analyses

Goal
- For a forward problem, consider all possible paths from the entry to a given program point, compute the flow values at the end of each path, and then meet these values together
- Meet-over-all-paths (MOP) solution at each program point
- $\sqcap_{\text{all paths } n1, n2, ..., ni} (f_{ni}(...f_{n2}(f_{n1}(v_{entry}))))$



Solving Data-Flow Analyses (cont)

Problems

- Loops result in an infinite number of paths
- Statements following merge must be analyzed for all preceding paths
  - Exponential blow-up

Solution

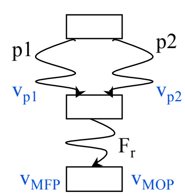
- Compute meets early (at merge points) rather than at the end
- Maximum fixed-point (MFP)

Questions

- Is this legal?
- Is this efficient?
- Is this accurate?

Legality

"Is v_{MFP} legal?" ≡ "Is $v_{MFP} \sqsubseteq v_{MOP}$?"



Look at Merges

$v_{MOP} = F_r(v_{p1}) \sqcap F_r(v_{p2})$
$v_{MFP} = F_r(v_{p1} \sqcap v_{p2})$
$v_{MFP} \sqsubseteq v_{MOP} \equiv F_r(v_{p1} \sqcap v_{p2}) \sqsubseteq F_r(v_{p1}) \sqcap F_r(v_{p2})$

Observation

$\forall x,y \in V$

$$\mathbf{f(x \sqcap y) \sqsubseteq f(x) \sqcap f(y) \quad \Leftrightarrow \quad x \sqsubseteq y \Rightarrow f(x) \sqsubseteq f(y)}$$

$\therefore$ v_{MFP} legal when F_r (really, the flow functions) are monotonic

Monotonicity

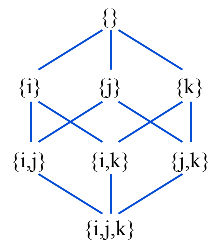
Monotonicity: $(\forall x,y \in V)[x \sqsubseteq y \Rightarrow f(x) \sqsubseteq f(y)]$

- If the flow function f is applied to two members of V, the result of applying f to the "lesser" of the two members will be under the result of applying f to the "greater" of the two
- Giving a flow function more conservative inputs leads to more conservative outputs (never more optimistic outputs)

Why else is monotonicity important?

For monotonic F over domain V

- The maximum number of times F can be applied to self w/o reaching a fixed point is height(V) – 1
- IDFA is guaranteed to terminate if the flow functions are monotonic and the lattice has finite height



Efficiency

Parameters

- n: Number of nodes in the CFG
- k: Height of lattice
- t: Time to execute one flow function

Complexity

- O(nkt)

Example

- Reaching definitions?

Accuracy

Distributivity

- $f(u \sqcap v) = f(u) \sqcap f(v)$
- $v_{MFP} \sqsubseteq v_{MOP} \equiv F_r(v_{p1} \sqcap v_{p2}) \sqsubseteq F_r(v_{p1}) \sqcap F_r(v_{p2})$
- If the flow functions are distributive, MFP = MOP

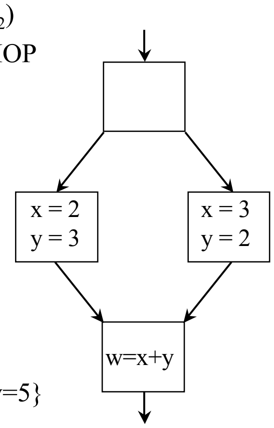
Examples

- Reaching definitions?
- Reaching constants?

$f(u \sqcap v) = f(\{x=2,y=3\} \sqcap \{x=3,y=2\})$
$= f(\emptyset) = \emptyset$

$f(u) \sqcap f(v) = f(\{x=2,y=3\}) \sqcap f(\{x=3,y=2\})$
$= [\{x=2,y=3,w=5\} \sqcap \{x=2,y=2,w=5\}] = \{w=5\}$

$\Rightarrow$ MFP $\neq$ MOP



Bitwidth Analysis Paper

Why did we read this paper?

Can all dataflow analyses be defined in terms of Gen and Kill?

Do all dataflow analysis problems operate on sets?

Concepts

Lattices

- Conservative approximation
- Optimistic (initial guess)
- Data-flow analysis frameworks
- Tuples of lattices

Data-flow analysis

- Fixed point
- Meet-over-all-paths (MOP)
- Maximum fixed point (MFP)
- Legal/safe (monotonic)
- Efficient
- Accurate (distributive)

Next Time

Reading

- Ch 8.11 in Muchnick
- all Muchnick readings are for main ideas and examples
- start reading the SSA paper, it is LONG!!

Lecture

- Program representations (static single assignment)