

MECÁNICA CLÁSICA

NOTAS A UN NIVEL INTERMEDIO

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Estas notas son un documento en proceso por lo que es posible que contengan errores, los cuales agradecería informaran al correo, de preferencia con un asunto: “Notas de mecánica” o algún similar. Para realizar la corrección lo más pronto posible.

Capítulos 2 y en adelante continúan en desarrollo, estos se colocaron para liberar los problemas resueltos. A pesar de esto, en algunos de ellos la solución no se ha podido completar, se encuentra incompleta o, en casos muy puntuales, parcialmente incorrecta.

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PART

I

SECTION 1

Amplitud de la Mecánica Clásica

Condiciones para la aplicabilidad de la teoría de la Mecánica Clásica:

- Las masas de los cuerpos de interés **deben ser mayores** a las masas de átomos y de partículas subatómicas.
- Las masas de los cuerpos de interés **deben ser pequeñas** en comparación a las masas de cuerpos celestes. Por ejemplo la masa del planeta Mercurio, dado que el estudio de su órbita presenta discrepancias con los datos observacionales si se realiza directamente desde la mecánica clásica ($3,285 \times 10^{23} kg$).
- Las rapideces de los cuerpos de interés **deben ser pequeñas** comparadas a la rapidez de la luz ($299,792,458 km/s$).
- La escala de tiempo en que se realiza el estudio **debe ser pequeña** en comparación a escalas de tiempo que tiendan a las astronómicas.

De lo anterior se puede concluir que la Mecánica Clásica está limitada, en el desarrollo que se le dará aquí, al estudio de fenómenos en escalas humanas. Grosso modo, **la Mecánica Clásica se emplea para estudiar cosas que los seres humanos pueden “ver”, tanto en tamaño como en escala temporal.** Las comillas al “ver” es porque es posible calcular con suficiente presición algunos fenómenos que ocurren por debajo de las escalas de visión humana con bastante presión.

PART

II

SECTION 2

Mecánica Newtoniana

Esto ya deberían saberlo y probablemente se actualice de último :)

SECTION 3

Mecánica Newtoniana para una partícula

A continuación se expresará la mecánica de partículas.

SUBSECTION 3.1

Leyes de Newton

Comenzando con algunos conceptos claves para el desarrollo de las leyes de Newton:

Definition 1

(Fuerza) Fuerza es el nombre que se le da a la interacción entre un cuerpo y su entorno, la cual es capaz de afectar el estado del cuerpo. Las fuerzas son cantidades vectoriales, por lo que poseen magnitud y dirección; su magnitud es dada en unidades de newton N .

Definition 2

(Momentum Lineal) Momentum lineal o cantidad de movimiento, ambos se refieren a una cantidad vectorial dada por la siguiente ecuación:

$$\vec{p} = m\vec{v} \quad (3.1)$$

Las leyes de Newton tal y como se expresarán a continuación **son únicamente válidas** para sistemas de referencias **inerciales**, es decir, sistemas de referencia que no poseen ningún tipo de aceleración o rotación respecto a las estrellas fijas.

Un marco de referencia inercial no es más que una construcción teórica, ya que no es posible conseguir un marco completamente inercial. No obstante, *es posible aproximarse a un marco inercial*. Por ejemplo, es posible considerar un marco de referencia en el centro del planeta Tierra, fijo a el mismo planeta (rota junto a él), como un marco inercial en un gran número de ocasiones.

Definition 3

(Primera Ley de Newton o Ley de la Inercia) Un cuerpo mantiene su estado de equilibrio a menos de que una fuerza neta diferente de cero lo perturbe. Dicho de otra forma, un cuerpo siempre mantendrá su estado de equilibrio a menos de que una fuerza neta llegue a afectarlo.

$$\sum \vec{F} = \vec{0} \quad (3.2)$$

Un estado de **equilibrio** se refiere a que el cuerpo o sistema de interés se encuentra moviéndose con velocidad lineal constante (**equilibrio dinámico**) o se encuentra en reposo (**equilibrio estático**).

Definition 4

(Segunda Ley de Newton) Un cuerpo que experimenta una fuerza neta diferente de cero, tendrá como resultado un cambio en su momentum lineal.

$$\sum \vec{F} = \frac{d\vec{p}}{dt} = \dot{\vec{p}} \quad (3.3)$$

Suponiendo que la masa es constante para el cuerpo de interés, la primera ley de Newton también se puede escribir de la forma:

$$\sum \vec{F} = m\vec{a} \quad (3.4)$$

Definition 5

(Tercera Ley de Newton o Ley de Acción-Reacción) Considere dos cuerpos denotados como A y B que presentan algún tipo de interacción entre sí, se dice que: Toda **acción** que realice el cuerpo A sobre el cuerpo B le corresponde una **reacción** que proveniente del cuerpo B. Estas **acciones y reacciones** corresponden a fuerzas

internas del sistema (cuerpos A y B) debido a su interacción, dichas fuerzas poseen la misma magnitud y su dirección es contraria.

$$\vec{F}_{AB} = -\vec{F}_{BA} \quad (3.5)$$

Para trabajar con esta ley hay que tomar cuenta cierta ambigüedad que nos lleva a los siguientes enunciados de la tercera ley:

- **Enunciado Fuerte:** Los vectores correspondientes a las fuerzas de **acción** y **reacción** se encuentran sobre una misma recta, es decir, si se conocen las direcciones de las fuerzas de **acción** y **reacción** es posible trazar una recta (conocida como **línea de acción**¹) que una los vectores de fuerzas y sea paralela a estos. Ver Figura 1.
- **Enunciado Débil:** No ocurre lo anterior. Es imposible unir los vectores de las fuerzas de **acción** y **reacción** por medio de una recta que sea paralela a ambos vectores. Ver Figura 2.

Además de lo anterior, es preciso destacar que la **Tercera Ley de Newton no es una ley general de la naturaleza** y se puede establecer que toda fuerza que dependa de velocidades no obedecerá esta ley.

SUBSECTION 3.2

Trabajo y Energía

Definition 6

(Trabajo) Corresponde a la cantidad generada al tomar el producto punto de la fuerza ejercida sobre un cuerpo a lo largo de todo su desplazamiento desplazamiento desde una punto A a un punto B.

$$W = \int_A^B \vec{F} \cdot d\vec{r} \quad (3.6)$$

Definition 7

(Fuerza conservativa) Una fuerza \vec{F} es conservativa si se puede escribir de la forma:

$$\vec{F} = -\vec{\nabla}V(\vec{r}) \quad (3.7)$$

Es decir, si el potencial que genera la fuerza depende únicamente de la posición de la partícula, será una fuerza conservativa.

A partir de las Ecuaciones (3.3) y (3.6):

$$W = \int_A^B \vec{F} \cdot d\vec{r} = \int_A^B \frac{d\vec{p}}{dt} \cdot d\vec{r}$$

Ejerciendo el producto punto y trabajando por índices:

$$W = \int_A^B \sum_{i=1}^3 \frac{dp_i}{dt} dr_i$$

Suponiendo que la masa es constante, la derivada temporal del momentum lineal es de la forma: $\frac{dp_i}{dt} = m \frac{dv_i}{dt}$:

¹ La línea de acción está en naranja en la Figura 1 y en la Figura 2.

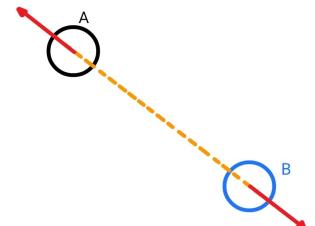


Figura 1. Situación del enunciado fuerte

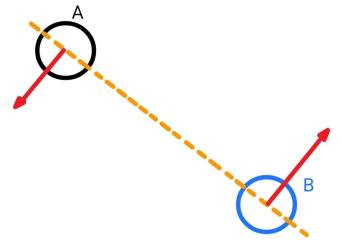


Figura 2. Situación del enunciado débil

$$\begin{aligned}
 W &= \sum_{i=1}^3 m \int_A^B \frac{dv_i}{dt} dr_i = \sum_{i=1}^3 m \int_A^B \frac{dv_i}{dt} dr_i \frac{dt}{dt} \\
 &= \sum_{i=1}^3 m \int_A^B dv_i \underbrace{\frac{dr_i}{dt}}_{=v_i} \frac{dt}{dt} \\
 &= \sum_{i=1}^3 m \int_A^B v_i dv_i = \sum_{i=1}^3 m \int_A^B \frac{1}{2} d(v_i^2) \\
 &= \sum_{i=1}^3 \frac{1}{2} mv_i^2 \Big|_A^B = \sum_{i=1}^3 \frac{1}{2} mv_{iB}^2 - \sum_{i=1}^3 \frac{1}{2} mv_{iA}^2
 \end{aligned}$$

Definition 8

(Energía Cinética Traslacional) Corresponde al trabajo necesario para comenzar a mover un cuerpo desde el reposo hasta la rapidez v .

$$T = \frac{1}{2} m \sum_{i=1}^3 v_i^2 \quad (3.8)$$

Theorem 1

(Trabajo - Energía Cinética)

$$W = \Delta T \quad (3.9)$$

Regresando a la definición Ecuación (3.6) pero ahora tomando la fuerza que es ejercida sobre el cuerpo como una fuerza conservativa, Ecuación (3.7).

$$W = \int_A^B \vec{F} \cdot d\vec{r} = \int_A^B -\vec{\nabla}V \cdot d\vec{r} = -V_B + V_A$$

Definition 9

(Energía Potencial) Corresponde a la capacidad de un cuerpo de ejercer trabajo se denomina energía potencial. Ahora se presentan algunos ejemplos de energías potenciales.

$$V = \begin{cases} mgh \\ \frac{1}{2}kx^2 \\ \frac{-GMm}{r} \\ \frac{-Kq_1q_2}{r} \\ \vdots \end{cases} \quad (3.10)$$

Theorem 2

(Trabajo - Energía Potencial)

$$W = -\Delta V \quad (3.11)$$

Definition 10

(Energía de un sistema) Ante la suposición de que el sistema a tratar tienen masa constante y es un sistema conservativo, la energía total es de la forma:

$$E = T + V \quad (3.12)$$

SUBSECTION 3.3

Análogo rotacional de las leyes de Newton

Ahora se presentarán algunos conceptos importantes y ecuaciones para una descripción sencilla de la mecánica de partículas en rotación. Nuevamente se comenzará por los conceptos básicos análogos a los usados en las leyes de Newton y posteriormente se darán las leyes análogas.

Primero deduciendo una relación entre la velocidad lineal \vec{v} y la velocidad angular $\vec{\omega}$:

- Se sabe que la velocidad angular es por definición

$$\vec{\omega} = \frac{d\theta}{dt} \quad (3.13)$$

respecto a un eje instantáneo de rotación. Siempre será posible establecer una velocidad angular para un cuerpo en movimiento arbitrario, ya que en cada instante el cuerpo se mueve con una trayectoria circular respecto a un eje de rotación; esto se puede observar en la Figura 3.

Observando la rotación infinitesimal presente en Figura 4, se puede concluir la siguiente relación ²

$$\delta\vec{r} = \delta\theta \times \vec{r}$$

Dividiendo entre δt :

$$\frac{\delta\vec{r}}{\delta t} = \frac{\delta\theta}{\delta t} \times \vec{r}$$

Lo cual, al considerar $\delta t \rightarrow 0$, se convierte en:

$$\frac{d\vec{r}}{dt} = \frac{d\theta}{dt} \times \vec{r}$$

Definition 11

(Relación velocidad Lineal - Angular) Para un cuerpo en un movimiento arbitrario se cumple lo siguiente para cada instante del movimiento.

$$\vec{v} = \vec{\omega} \times \vec{r} \quad (3.14)$$

Recuerde que el eje de rotación instantáneo puede cambiar si el movimiento no es una rotación fija.

Definition 12

(Torque) El torque es el análogo de la fuerza para las rotaciones. Por lo que el torque corresponde a una interacción del sistema con su entorno que es capaz de generar un cambio en el estado *rotacional* del sistema, dicha interacción es mediada por la presencia de una o más fuerzas y se define como:

$$\vec{N}_{\mathcal{O}} = \vec{r} \times \vec{F} \quad (3.15)$$

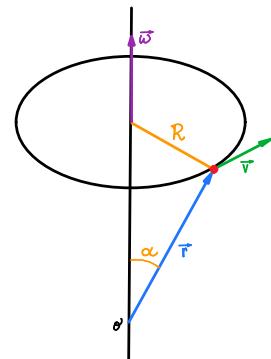


Figura 3. Relación entre \vec{v} , $\vec{\omega}$ y \vec{r}

²Recuerde que antes se era bien conocida la relación $v = \omega R$ para una partícula en un movimiento circular de radio R en un plano.

En este caso, al relacionar con \vec{r} , se tiene:

$$v = w r \operatorname{sen}(\alpha)$$

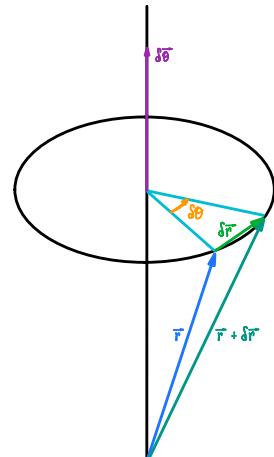


Figura 4. Relación diferencial entre \vec{v} , $\vec{\omega}$ y \vec{r}

Como tal, el torque depende del origen que se este utilizando, debido a su dependencia con el vector posición \vec{r} . Al torque es común llamarlo en algunos campos como: torca, momento de fuerza o simplemente momento.

Definition 13

(Momentum Angular) Corresponde al análogo angular del momentum lineal. Es una cantidad vectorial que para el caso de partículas se define como:

$$\vec{L}_O = \vec{r} \times \vec{p} = m \vec{r} \times \vec{v} = m \vec{r} \times (\vec{\omega} \times \vec{r})^3 = m [\vec{r}^2 \vec{\omega} - \vec{r}(\vec{r} \cdot \vec{\omega})] \quad (3.16)$$

De forma similar al torque, el momentum angular depende del origen desde el que se decida medir.

En la amplia gama de casos en que se trabaja con partículas, se podrá reconocer que el producto $\vec{r} \cdot \vec{\omega} = 0$ y los vectores \vec{r} y $\vec{\omega}$ apuntan en una única dirección, por lo que el momentum angular tomará la siguiente forma:

$$L_{Oq} = mr^2\omega_q \quad (3.17)$$

$${}^3\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{B}) - \vec{C}(\vec{A} \cdot \vec{B})$$

Definition 14

(Momentos de Inercia) La inercia es el análogo rotacional de la masa y corresponde a una medida que indica que tan difícil es girar un cuerpo respecto a cada eje (A mayor inercia más complicado es girar el objeto). Girando un cuerpo es posible concluir que se pueden generar rotaciones respecto a 3 ejes y por lo tanto **existen 3 momentos de inercia**, los cuales no necesariamente serán iguales, esto dependerá de la distribución de la masa respecto a cada eje.

En el caso de partículas puntuales, la inercia se define como:

$$I_q = \sum_i m_i r_i^2 \quad (3.18)$$

Donde r_i corresponde a la distancia que hay entre el eje de rotación y la masa puntual.

La **masa inercial** y los **momentos de inercia** están intimamente relacionados, ambos son medidas de que tan difícil es mover un cuerpo de cierta forma.

Definition 15

(Primera Ley de Newton análoga rotacional) De forma similar a la Primera Ley de Newton, este principio análogo estable que un cuerpo en un estado rotacional de equilibrio tenderá a mantener dicho estado hasta que un torque neto diferente de cero lo perturbe.

$$\sum \vec{N}_O = 0 \quad (3.19)$$

El término *inercial* es para hacer una distinción entre la **masa inercial** (La masa dada por la aceleración de un cuerpo al estar bajo el efecto de una fuerza) y la **masa gravitacional** (La masa determinada por las fuerzas gravitacionales entre el cuerpo de interés y otros cuerpos), a pesar de que ambas cantidades **son iguales** por el principio de equivalencia.

Definition 16

(Segunda Ley de Newton análoga rotacional)

$$\sum \vec{N}_O = \frac{d\vec{L}_O}{dt} = \dot{\vec{L}}_O \quad (3.20)$$

Manteniendo la inercia constante, la ecuación se escribe de la forma:

$$\sum N_q = I_q \alpha_q \quad (3.21)$$

Donde el subíndice denota el eje respectivo al cual se está realizando la suma de torques.

Definition 17

(Tercera Ley de Newton análoga rotacional) Este principio se enuncia de forma análoga a la Tercera Ley de Newton original bajo la salvedad de que en vez de trabajar con fuerzas, este trabaja con torques.

$$\vec{N}_{AB} = -\vec{N}_{BA} \quad (3.22)$$

Definition 18

(Energía Cinética Rotacional) Corresponde al trabajo necesario para hacer rotar un cuerpo desde el reposo hasta la rapidez angular ω .

$$T_{rot} = \frac{1}{2} I_q \omega^2 \quad (3.23)$$

La forma de obtener esta expresión es similar al procedimiento que se realizó con la energía cinética translacional.

SUBSECTION 3.4

Teoremas de Conservación

A continuación se van a enunciar los teoremas de conservación bajo la suposición de masa constante y que el sistema a tratar no posee fuerzas disipativas.

Theorem 3

(Conservación de la Energía) Una vez ya conocida la expresión para la energía del sistema Ecuación (3.12), basta con derivarla con respecto al tiempo para determinar que restricciones se plantean para la conservación de dicha cantidad:

$$\begin{aligned} E = T + V \Rightarrow \frac{dE}{dt} = \frac{dT}{dt} + \frac{dV}{dt}; \text{ A partir de Ecuación (3.8) y suponiendo } V = V(\vec{r}, t) \\ = \frac{d}{dt} \left(\frac{1}{2} m \dot{\vec{r}}^2 \right) + \underbrace{\sum_{i=1}^3 \frac{\partial V}{\partial x_i} \frac{dx_i}{dt}}_{= \vec{\nabla} V \cdot \dot{\vec{r}}} + \frac{\partial V}{\partial t} \stackrel{\text{Sistema}}{\rightarrow} V = V(\vec{r}) \\ = \frac{1}{2} m (\ddot{\vec{r}} \cdot \dot{\vec{r}} + \dot{\vec{r}} \cdot \ddot{\vec{r}}) + \vec{\nabla} V \cdot \dot{\vec{r}} = m \ddot{\vec{r}} \cdot \dot{\vec{r}} + \vec{\nabla} V \cdot \dot{\vec{r}} = [m \ddot{\vec{r}} + \vec{\nabla} V] \cdot \dot{\vec{r}} \\ = [\vec{F} + \vec{\nabla} V] \cdot \dot{\vec{r}} \end{aligned}$$

Para que la energía se conserve se cumple:

$$\begin{aligned} \Rightarrow \frac{dE}{dt} = [\vec{F} + \vec{\nabla} V] \cdot \dot{\vec{r}} = 0 \Rightarrow \vec{F} + \vec{\nabla} V = 0 \\ \Rightarrow \vec{F} = -\vec{\nabla} V \\ \therefore \frac{dE}{dt} = 0 \Rightarrow \vec{F} = -\vec{\nabla} V \end{aligned} \quad (3.24)$$

Theorem 4

(Conservación de Momentum Lineal) Conociendo ya la expresión de la Ecuación (3.1), se derivará con respecto al tiempo para determinar las condiciones en que se conserva dicha cantidad:

$$\vec{p} = m\dot{\vec{r}} \Rightarrow \frac{d\vec{p}}{dt} = \cancel{\frac{dm}{dt}\dot{\vec{r}}}^0 + m\ddot{\vec{r}} = m\ddot{\vec{r}} = \vec{F}$$

Para que el momentum lineal se conserve se cumple:

$$\begin{aligned} \Rightarrow \frac{d\vec{p}}{dt} &= m\ddot{\vec{r}} = \vec{F} = \vec{0} \\ \therefore \frac{d\vec{p}}{dt} &= 0 \Rightarrow \vec{F} = \vec{0} \end{aligned} \quad (3.25)$$

Theorem 5

(Conservación de Momentum de Angular) A partir de la expresión de la Ecuación (3.16) y derivandola respecto al tiempo:

$$\begin{aligned} \vec{L}_{\mathcal{O}} = \vec{r} \times \vec{p} \Rightarrow \frac{d\vec{L}_{\mathcal{O}}}{dt} &= \underbrace{\dot{\vec{r}} \times \vec{p}}_{\vec{r} \parallel \vec{p}} + \vec{r} \times \dot{\vec{p}} \\ &= \vec{r} \times \dot{\vec{p}} \ ; \text{ Por la Ecuación (3.3)} \\ &= \vec{r} \times \vec{F} \ ; \text{ Por la Ecuación (3.15)} \\ &= \vec{N}_{\mathcal{O}} \end{aligned}$$

Para que se conserve el momentum angular se debe cumplir:

$$\begin{aligned} \Rightarrow \frac{d\vec{L}_{\mathcal{O}}}{dt} &= \vec{r} \times \vec{F} = \vec{N}_{\mathcal{O}} = \vec{0} \\ \therefore \frac{d\vec{L}_{\mathcal{O}}}{dt} &= 0 \Rightarrow \vec{N}_{\mathcal{O}} = \vec{0} \end{aligned} \quad (3.26)$$

SUBSECTION 3.5

Complementos sobre la energía**Definition 19**

(Condiciones de estabilidad) Al conservarse la energía en un sistema, es posible encontrarse con que la partícula de interés se encuentra en un punto de equilibrio de algún tipo (Estables u Inestables), alrededor de estos puntos es posible obtener mucha información acerca del comportamiento del sistema. Las siguientes son las condiciones para determinar que es un punto de equilibrio y clasificarlo.

- **Punto de equilibrio:** Se define como un punto en algún potencial que cumple con:

$$\vec{\nabla}V\Big|_{\vec{r}=\vec{r}_{eq}} = \vec{0} \quad (3.27)$$

- **Punto de Equilibrio Estable:** De colocar una partícula en reposo en este punto y sin la aparición de fuerzas a su alrededor, la partícula permanecerá en equilibrio en dicho punto por siempre. Ahora, si esta es colocada en reposo en los alrededores de este punto o desde el punto de equilibrio se le ejerce una fuerza que la obligue a moverse, la partícula comenzará un movimiento en dirección a este punto y se moverá perpetuamente a su alrededor siempre que $E = \text{constante}$. La forma de determinar los puntos estables en un potencial unidimensional:

$$\frac{d^2}{dx^2}V\Big|_{x=x_{eq}} > 0 \quad (3.28)$$

- **Punto de Equilibrio Inestable:** De forma similar al anterior, si se coloca una partícula en reposo en este punto y sin la aparición de fuerzas a su alrededor, la partícula permanecerá en equilibrio en dicho punto por siempre. No obstante, de aparecer algún tipo de fuerza o colocarla en los alrededores del punto de equilibrio inestable, la partícula se encontrará en un movimiento únicamente limitado por la cantidad de energía del sistema y es posible que nunca vuelva a pasar por dicho punto. La forma de encontrar los puntos inestables en un potencial unidimensional:

$$\frac{d^2}{dx^2}V\Big|_{x=x_{eq}} < 0 \quad (3.29)$$

- **Punto No concluyente:**

$$\frac{d^2}{dx^2}V\Big|_{x=x_{eq}} = 0 \quad (3.30)$$

Es necesario comprobar derivadas de orden superior bajo los mismos criterios para determinar estabilidad u inestabilidad para un potencial unidimensional de la posición de equilibrio. De obtenerse $\frac{d^3}{dx^3}V\Big|_{x=x_{eq}} = 0$, es necesario continuar con orden 4 en la derivación, comprobar criterios y, de ser necesario, repetir los pasos en derivaciones superiores.

A continuación se muestra un ejemplo de como se aplican estos conceptos en un caso unidimensional.

Si se desea hacer el análisis de estabilidad e inestabilidad para potenciales que sean funciones de varias variables es necesario utilizar teoría de optimización en funciones de variables variables según sea el caso. A continuación se expresa el Método de la Matriz Hessiana para n variables para un punto \vec{r}_{eq} :

La matriz Hessiana para una función V se forma con las siguientes entradas:

$$H_{ij} = \left. \frac{\partial^2 V}{\partial x_i \partial x_j} \right|_{\vec{r}=\vec{r}_{eq}} ; i, j = 0, \dots, n$$

El criterio para la concavidad de la función V usando dicha matriz se basa en el uso de los determinantes (Δ) de las matrices menores de la matriz Hessiana (h_i):

- Si $\Delta h_i > 0$ con $m = 0, 1, \dots, n$ entonces es un mínimo local de V.
- Si $\Delta h_{2m+1} < 0$; $m = 0, 1, \dots, n$ y $\Delta h_{2m} > 0$ con $m = 0, 1, \dots, n$ entonces es un máximo local de V.
- Cualquier otro patrón de los determinantes de las menores corresponde a un punto de silla.
- Si $\Delta h_n = 0$, el estudio no es concluyente y hay que hacer un análisis particular. Se aclara que $h_n = H$

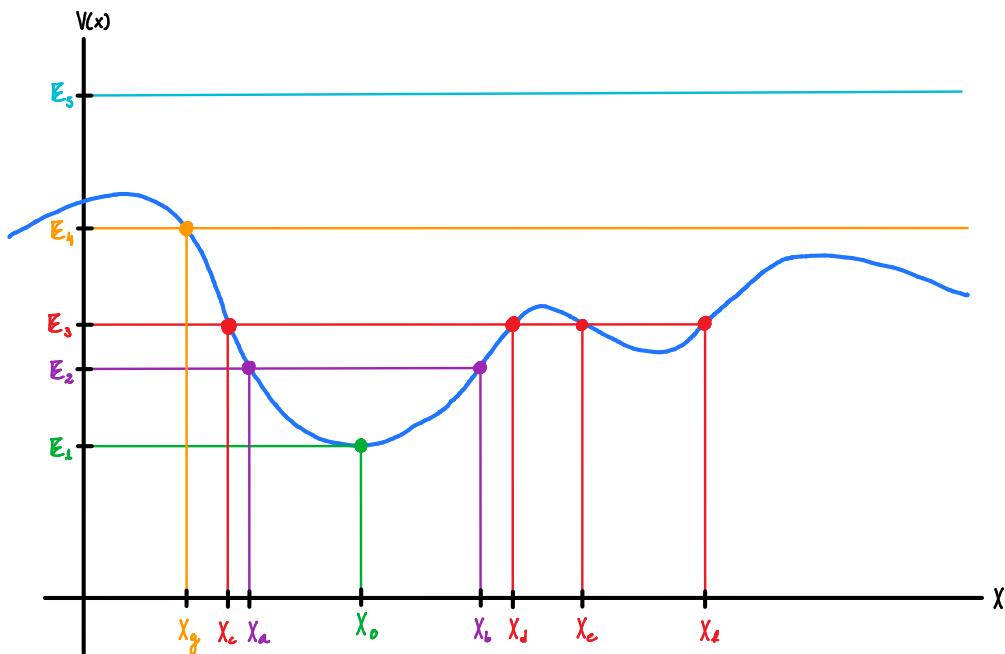


Figura 5. Ejemplo en una dimensión de las condiciones de estabilidad.

La línea azul corresponde a la energía potencial que siente la partícula en cada punto. Se colocan un total de 5 energías totales para el sistema cada una determinada por un color para mayor facilidad. Junto a esto, cada intersección que tienen las rectas de energía con la curva de potencial posee un punto y también su respectiva ubicación en el eje X acorde al color de la energía total.

Al observar la Figura 5, se determina lo siguiente:

- $x = x_0$ corresponde a un punto de equilibrio estable.
- Si se coloca una partícula en $x = x_0$ en energía E_1 , esta permanecerá en reposo.
- Para la energía E_2 , la partícula se moverá de forma periódica en $x_a < x < x_b$.
- Para la energía E_3 , la partícula se moverá de forma periódica en $x_c < x < x_d$ ó $x_e < x < x_f$. Esto es excluyente cabe destacar, la partícula se moverá en uno o en el otro foso.
- Hay un punto de equilibrio inestable entre los puntos x_d y x_e , al igual que entre $x = 0$ y x_g .
- Para la energía E_4 , la partícula se moverá de forma periódica en $x_g < x < +\infty$. La partícula va a $+\infty$, vuelve a x_g y regresa a $+\infty$, completando un ciclo.
- Para la energía E_5 , el movimiento no está limitado por la energía potencial que siente la partícula. Esta se puede mover libremente pero su velocidad va a depender de la posición en la que se encuentre.

SUBSECTION 3.6

Problemas resueltos

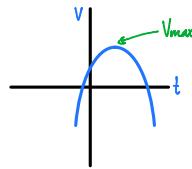
Cinematica y Leyes de Newton

Problema 1.

Un motociclista que se mueve a lo largo de un eje x dirigido hacia el este tiene una aceleración dada por $a = (6.1 - 1.2t) \text{ m/s}^2$ para $0 \leq t \leq 6.0 \text{ s}$. En $t = 0$, la velocidad y posición del motociclista son 2.7 m/s y 7.3 m .

a. ¿Cuál es la velocidad máxima alcanzada por el motociclista?

b. ¿Qué distancia total recorre el ciclista $t = 0$ y 6.0 s ?



$$a = \frac{dv}{dt} = 6.1 - 1.2t \Rightarrow v - 2.7 = \int_0^t (6.1 - 1.2t) dt = 6.1t - \frac{1.2t^2}{2} \Rightarrow v = 2.7 + 6.1t - \frac{1.2t^2}{2}$$

v es máxima cuando $a = 0 \Rightarrow 6.1 - 1.2t = 0 \Rightarrow t = 5.083 \text{ s} \Rightarrow v = 18.204 \text{ m/s}$

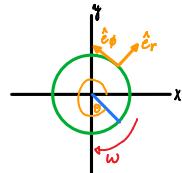
$$b) v = \frac{dx}{dt} \Rightarrow x - 7.3 = \int_0^t 2.7 + 6.1t - \frac{1.2t^2}{2} dt = 2.7t + \frac{6.1t^2}{2} - \frac{1.2t^3}{6} \Rightarrow x = 7.3 + 2.7t + \frac{6.1t^2}{2} - \frac{1.2t^3}{6}$$

x es máxima cuando $v = 0 \Rightarrow 2.7 + 6.1t - \frac{1.2t^2}{2} = 0 \Rightarrow t = 10.60 \text{ s} \Rightarrow x = 140.41 \text{ m}$ la moto no se regresa antes de $t = 10.60 \text{ s}$

$$d = x(6) - x(0) = 90.1 - 7.3 = 82.8 \text{ m} \quad = L = \int_0^6 \sqrt{\left(2.7 + 6.1t - \frac{1.2t^2}{2}\right)^2} dt$$

Problema 2.

Una partícula se encuentra en movimiento circular uniforme alrededor del origen de un sistema de coordenadas xy , moviéndose en el sentido de las manecillas del reloj con un período de 7.00 s . En un instante, su vector de posición (medido desde el origen) es $\vec{r} = (2.0 \hat{x} - 3.0 \hat{y}) \text{ m}$. En ese instante, ¿cuál es su posición, velocidad y aceleración en coordenadas cartesianas y polares?



$$\vec{r} = 2\hat{e}_x - 3\hat{e}_y = \sqrt{13} \hat{e}_r; \quad \hat{e}_r = \cos\phi \hat{e}_x + \sin\phi \hat{e}_y \quad \hat{e}_\phi = -\sin\phi \hat{e}_x + \cos\phi \hat{e}_y; \quad \phi = \tan^{-1}\left(\frac{-3}{2}\right) = 53 \text{ rad}$$

$$r = \sqrt{2^2 + 3^2} = \sqrt{13}, \quad \vec{\omega} = -\frac{2\pi}{T} \hat{e}_\phi = -\frac{2\pi}{7} \hat{e}_\phi$$

$$\vec{v} = \vec{r} \times \vec{\omega} = \sqrt{13} \hat{e}_r \times -\frac{2\pi}{7} \hat{e}_\phi = -\frac{2\pi}{7} \sqrt{13} \hat{e}_\phi = -\frac{2\pi}{7} \sqrt{13} (-\sin\phi \hat{e}_x + \cos\phi \hat{e}_y) = -\frac{2\pi}{7} \sqrt{13} (0.83 \hat{e}_x - 0.55 \hat{e}_y)$$

$$\vec{a} = -\vec{a}_{rad} \hat{e}_r = -\frac{v^2}{r} \hat{e}_r = \frac{4\pi^2 \sqrt{13}}{49} \cdot \frac{1}{\sqrt{13}} \hat{e}_r = \frac{4\pi^2 \sqrt{13}}{49} \hat{e}_r = \frac{4\pi^2 \sqrt{13}}{49} (\cos\phi \hat{e}_x + \sin\phi \hat{e}_y) = \frac{4\pi^2 \sqrt{13}}{49} (-0.55 \hat{e}_x + 0.83 \hat{e}_y)$$

Problema 3.

Determine los vectores tangente, normal y binormal para:

$$\vec{r}(t) = 2 \sin t \hat{x} + 3 \cos t \hat{y} + 4t \hat{z}, \quad \text{en } t = \pi/2.$$

$$\vec{T} = \vec{r}' = 2 \cos t \hat{e}_x - 3 \sin t \hat{e}_y + 4 \hat{e}_z$$

$$\vec{N} = \vec{T}' = -2 \sin t \hat{e}_x - 3 \cos t \hat{e}_y + 0 \hat{e}_z$$

$$\vec{B} = \vec{T} \times \vec{N} = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ 2 \cos t & -3 \sin t & 4 \\ -2 \sin t & -3 \cos t & 0 \end{vmatrix} = (-3 \sin t \cdot 0 - 3 \cos t \cdot 4) \hat{e}_x - (2 \cos t \cdot 0 - 2 \sin t \cdot 4) \hat{e}_y + (2 \cos t \cdot -3 \cos t - 2 \sin t \cdot 3 \sin t) \hat{e}_z = 3 \cos t \cdot 4 \hat{e}_x - 2 \sin t \cdot 4 \hat{e}_y - 6 \hat{e}_z$$

Problema 4.

Las componentes cartesianas de la velocidad están dadas por:

$$v_x = \left(2 + \frac{t}{2}\right) \frac{\text{m}}{\text{s}}, \quad v_y = 2 \sin\left(\frac{\pi t}{2}\right) \frac{\text{m}}{\text{s}}.$$

La posición inicial $\vec{r}(0) = \vec{0}$. En $t = 1.0$ s determine la posición y aceleración en coordenadas cartesianas.

$$\vec{V} = \begin{pmatrix} 2 + \frac{t}{2} \\ 2 \sin\left(\frac{\pi t}{2}\right) \end{pmatrix} = \frac{d\vec{r}}{dt} \implies \vec{r} = \int_0^t \begin{pmatrix} 2 + \frac{t}{2} \\ 2 \sin\left(\frac{\pi t}{2}\right) \end{pmatrix} dt = \begin{pmatrix} 2t + \frac{t^2}{2} \\ 1 - \frac{4}{\pi} \cos\left(\frac{\pi t}{2}\right) \end{pmatrix} \quad \vec{a} = \frac{d\vec{V}}{dt} = \begin{pmatrix} 0 + \frac{1}{2} \\ \frac{\pi}{2} \cos\left(\frac{\pi t}{2}\right) \end{pmatrix}$$

Problema 5. (Thornton 1.10)

1-10. A particle moves in a plane elliptical orbit described by the position vector

$$\vec{r} = 2b \sin \omega t \mathbf{i} + b \cos \omega t \mathbf{j}$$

(a) Find \vec{v} , \vec{a} , and the particle speed.

(b) What is the angle between \vec{v} and \vec{a} at time $t = \pi/2\omega$?

$$a) \frac{d\vec{r}}{dt} = \vec{V} = 2b\omega \cos(\omega t) \mathbf{i} - b\omega \sin(\omega t) \mathbf{j} \implies \frac{d^2\vec{r}}{dt^2} = \vec{a} = -2b\omega^2 \sin(\omega t) \mathbf{i} - b\omega^2 \cos(\omega t) \mathbf{j}$$

$$b) \vec{V} \cdot \vec{a} = 2b\omega \cos(\omega t) \cdot -2b\omega^2 \sin(\omega t) + -b\omega \sin(\omega t) \cdot -b\omega^2 \cos(\omega t) \\ = -4b^2 \omega^3 \cos(\omega t) \sin(\omega t) + b^2 \omega^3 \cos(\omega t) \sin(\omega t)$$

$$V = \sqrt{[2b\omega \cos(\omega t)]^2 + [b\omega \sin(\omega t)]^2} = \sqrt{[b^2 \omega^2 \cos^2(\omega t) + b^2 \omega^2 \sin^2(\omega t)]} = b\omega \sqrt{[\cos^2(\omega t) + \sin^2(\omega t)]}$$

$$a = \sqrt{[2b\omega^2 \sin(\omega t)]^2 + [b\omega^2 \cos(\omega t)]^2} = \sqrt{[b^2 \omega^4 \sin^2(\omega t) + b^2 \omega^4 \cos^2(\omega t)]} = b\omega^2 \sqrt{[\sin^2(\omega t) + \cos^2(\omega t)]}$$

$$\implies -4b^2 \omega^3 \cos(\omega t) \sin(\omega t) + b^2 \omega^3 \cos(\omega t) \sin(\omega t) = Va \cos \phi$$

$$\implies \cos \phi = \frac{b^2 \omega^3}{\frac{-4b^2 \omega^3 \cos(\omega t) \sin(\omega t) + b^2 \omega^3 \cos(\omega t) \sin(\omega t)}{b\omega \sqrt{[\cos^2(\omega t) + \sin^2(\omega t)]} \cdot b\omega^2 \sqrt{[\sin^2(\omega t) + \cos^2(\omega t)]}}} = \frac{-4(\omega \cos(\omega t) \sin(\omega t) + \omega \cos(\omega t) \sin(\omega t))}{\sqrt{[\cos^2(\omega t) + \sin^2(\omega t)]} \sqrt{[\sin^2(\omega t) + \cos^2(\omega t)]}}$$

$$\implies \phi(t = \frac{\pi}{2\omega}) = \cos^{-1} \left(\frac{-4(\omega \cos(\frac{\pi}{2}) \sin(\frac{\pi}{2}) + \omega \cos(\frac{\pi}{2}) \sin(\frac{\pi}{2}))}{\sqrt{[\cos^2(\frac{\pi}{2}) + \sin^2(\frac{\pi}{2})]} \sqrt{[\sin^2(\frac{\pi}{2}) + \cos^2(\frac{\pi}{2})]}} \right) = \cos^{-1} \left(\frac{0}{\sqrt{1} \cdot \sqrt{1}} \right) = \cos^{-1}(0) = \frac{\pi}{2}$$

Problema 6. (Taylor 1.19)

1.19 ** If \mathbf{r} , \mathbf{v} , \mathbf{a} denote the position, velocity, and acceleration of a particle, prove that

$$\dot{\mathbf{r}} = \mathbf{v} \quad \ddot{\mathbf{r}} = \dot{\mathbf{v}} = \mathbf{a}$$

$$\frac{d}{dt}[\mathbf{a} \cdot (\mathbf{v} \times \mathbf{r})] = \dot{\mathbf{a}} \cdot (\mathbf{v} \times \mathbf{r}) + \mathbf{a} \cdot \cancel{\dot{\mathbf{v}}} \cancel{(\mathbf{v} \times \mathbf{r})} = \dot{\mathbf{a}} \cdot (\mathbf{v} \times \mathbf{r}) + \mathbf{a} \cdot (\dot{\mathbf{v}} \times \mathbf{r}) + \mathbf{a} \cdot (\mathbf{v} \times \dot{\mathbf{r}}) = \dot{\mathbf{a}} \cdot (\mathbf{v} \times \mathbf{r}) + \mathbf{a} \cdot (\dot{\mathbf{a}} \times \mathbf{r}) + \cancel{\mathbf{a} \cdot (\mathbf{v} \times \mathbf{v})}$$

$$\begin{aligned} \cancel{\frac{d}{dt}[\mathbf{a} \cdot (\mathbf{v} \times \mathbf{r})]} &= \cancel{\dot{\mathbf{a}} \cdot (\mathbf{v} \times \mathbf{r})} + \cancel{\mathbf{a} \cdot (\dot{\mathbf{a}} \times \mathbf{r})} = \dot{\mathbf{a}} \cdot (\mathbf{v} \times \mathbf{r}) \\ &\stackrel{\mathbf{c} = \mathbf{a} \times \mathbf{r}; \mathbf{c} \perp \mathbf{a}}{=} \dot{\mathbf{a}} \cdot (\mathbf{v} \times \mathbf{r}) \end{aligned}$$

Problema 7. (Thornton 1.28)

1-28. Show that

$$\nabla(\ln|\mathbf{r}|) = \frac{\mathbf{r}}{r^2} \quad \frac{\partial}{\partial x_j} X_i = 0 \text{ si } i \neq j \text{ por ortogonalidad}$$

$$\mathbf{r} = \sum_i X_i \hat{e}_i \Rightarrow r = \sqrt{\sum_i X_i^2} \quad \nabla = \sum_j \hat{e}_j \frac{\partial}{\partial x_j}$$

$$\nabla(\ln|\mathbf{r}|) = \sum_j \hat{e}_j \frac{\partial}{\partial x_j} \ln(\sqrt{\sum_i X_i^2}) = \sum_j \hat{e}_j \frac{\partial}{\partial x_j} \ln(\sqrt{\sum_i X_i^2}) = \sum_i \hat{e}_i \frac{\partial}{\partial x_i} \ln(\sqrt{\sum_i X_i^2}) = \sum_i \hat{e}_i \frac{1}{\sqrt{\sum_i X_i^2}} \cdot \frac{2X_i}{\cancel{2\sqrt{\sum_i X_i^2}}} = \sum_i \hat{e}_i \frac{X_i}{\sqrt{\sum_i X_i^2}} = \frac{\mathbf{r}}{r^2}$$

Problema 8. (Thornton 1.31) Tomar en cuenta sumatorias internas y externas con diferentes índices a pesar de que no necesiten ser iguales

1-31. Show that

$$(a) \nabla r^n = nr^{(n-2)}\mathbf{r} \quad (b) \nabla f(r) = \frac{r}{r} \frac{df}{dr} \quad (c) \nabla^2(\ln r) = \frac{1}{r^2}$$

$$a) \nabla r^n = \sum_j \hat{e}_j \frac{\partial}{\partial x_j} \sqrt{\sum_i X_i^2}^n = \sum_i \hat{e}_i \frac{\partial}{\partial x_i} \sqrt{\sum_i X_i^2}^n = \sum_i \hat{e}_i n \sqrt{\sum_i X_i^2}^{n-1} \cdot \frac{2X_i}{\cancel{2\sqrt{\sum_i X_i^2}}} = \sum_i \hat{e}_i n \frac{\sqrt{\sum_i X_i^2}^{n-1}}{\sqrt{\sum_i X_i^2}} X_i \\ = \sum_i \hat{e}_i n \sqrt{\sum_i X_i^2}^{n-2} X_i = n \sqrt{r}^{n-2} \mathbf{r}$$

$$b) \nabla f(r) = \sum_i \hat{e}_i \frac{\partial}{\partial x_i} f(r) = \sum_i \hat{e}_i \frac{\partial f}{\partial r} \frac{\partial r}{\partial x_i} = \sum_i \hat{e}_i \frac{\partial f}{\partial r} \frac{\partial}{\partial x_i} \sqrt{\sum_i X_i^2} = \sum_i \frac{\partial f}{\partial r} \frac{2X_i}{\cancel{2\sqrt{\sum_i X_i^2}}} = \sum_i \frac{\partial f}{\partial r} \frac{f(r)}{\sqrt{\sum_i X_i^2}} X_i \\ = \frac{\partial f(r)}{\partial r} \frac{\mathbf{r}}{|r|^2}$$

$$c) \nabla^2(\ln|\mathbf{r}|) = \nabla \cdot \nabla(\ln|\mathbf{r}|) = \sum_i \hat{e}_i \frac{\partial}{\partial x_i} \sum_i \hat{e}_i \frac{\partial}{\partial x_i} \ln(\sqrt{\sum_i X_i^2}) = \sum_i \frac{\partial^2}{\partial x_i^2} \ln(\sqrt{\sum_i X_i^2}) = \sum_i \frac{\partial}{\partial x_i} \frac{1}{\sqrt{\sum_i X_i^2}} \cdot \frac{2X_i}{\cancel{2\sqrt{\sum_i X_i^2}}} \\ = \sum_i \frac{\partial}{\partial x_i} \frac{X_i}{\sqrt{\sum_i X_i^2}^2} = \sum_i i \cdot \frac{\sqrt{\sum_i X_i^2}^2 - X_i \cdot 2\sqrt{\sum_i X_i^2} \cdot \frac{2X_i}{\cancel{2\sqrt{\sum_i X_i^2}}}}{\sqrt{\sum_i X_i^2}^4} = \sum_i i \frac{X_i^2 - 2X_i^2}{\sqrt{\sum_i X_i^2}^4} = \sum_i \frac{X_i^2}{(\sum_i X_i^2)^2} = \frac{r^2}{r^4} = \frac{1}{r^2}$$

$$\sum_i \frac{\partial X_i}{\partial x_i} = 1 \cdot d$$

Problema 9. (Thornton 1.33)

1-33. Show that

$$\int \left(\frac{\dot{\mathbf{r}}}{r} - \frac{\mathbf{r}\dot{\phi}}{r^2} \right) dt = \frac{\mathbf{r}}{r} + \mathbf{C}$$

where \mathbf{C} is a constant vector.

$$\frac{d}{dt} \frac{\mathbf{r}}{r} = \frac{\dot{\mathbf{r}}\mathbf{r} - \mathbf{r}\dot{\mathbf{r}}}{r^2} = \frac{\dot{\mathbf{r}}}{r} - \frac{\mathbf{r}}{r^2} \dot{\phi}$$

$$\Rightarrow \int \left(\frac{\dot{\mathbf{r}}}{r} - \frac{\mathbf{r}}{r^2} \dot{\phi} \right) dt = \int \frac{d}{dt} \frac{\mathbf{r}}{r} = \frac{\mathbf{r}}{r} + \mathbf{C} //$$

Problema 10. (Taylor 1.48)

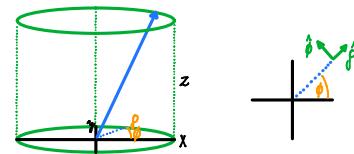
1.48 ** Find expressions for the unit vectors $\hat{\rho}$, $\hat{\phi}$, and \hat{z} of cylindrical polar coordinates (Problem 1.47) in terms of the Cartesian \hat{x} , \hat{y} , \hat{z} . Differentiate these expressions with respect to time to find $d\hat{\rho}/dt$, $d\hat{\phi}/dt$, and $d\hat{z}/dt$.

$$\hat{\rho} = \cos\phi \hat{x} + \sin\phi \hat{y} \quad \hat{\phi} = -\sin\phi \hat{x} + \cos\phi \hat{y} \quad \hat{z} = \hat{z}$$

$$\frac{d\hat{\rho}}{dt} = -\sin\phi \dot{\phi} \hat{x} + \cos\phi \dot{\phi} \hat{y} = \dot{\phi} \hat{\phi} //$$

$$\frac{d\hat{\phi}}{dt} = -\cos\phi \dot{\phi} \hat{x} - \sin\phi \dot{\phi} \hat{y} = -\dot{\phi} \hat{\rho} //$$

$$\frac{d\hat{z}}{dt} = 0 //$$



Problema 11.

La posición de una partícula está dada por $\vec{r}(t) = x(t)\hat{x} + y(t)\hat{y} + z(t)\hat{z}$ en coordenadas cartesianas.

a. Determine la velocidad y aceleración en coordenadas cartesianas.

b. Encuentre la posición, velocidad y aceleración en coordenadas cilíndricas. ¿Cuánto valen estas cantidades para una partícula que se mueve en una línea recta, paralela al eje z y sobre un casco cilíndrico de radio a y longitud muy larga.

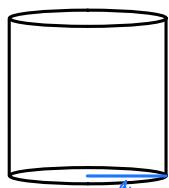
c. Encuentre la posición, velocidad y aceleración en coordenadas esféricas. ¿Cuánto valen estas cantidades para una partícula que se mueve sobre un casco esférico de radio a .

$$a) \vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k} \Rightarrow \dot{\vec{r}}(t) = \dot{x}(t)\hat{i} + \dot{y}(t)\hat{j} + \dot{z}(t)\hat{k} \quad \ddot{\vec{r}}(t) = \ddot{x}(t)\hat{i} + \ddot{y}(t)\hat{j} + \ddot{z}(t)\hat{k}$$

$$b) \vec{r}(t) = \rho\hat{e}_\rho + z\hat{k} \Rightarrow \dot{\vec{r}}(t) = \dot{\rho}\hat{e}_\rho + \dot{z}\hat{k} = \dot{\rho}\hat{e}_\rho + \dot{\rho}\hat{e}_\phi + \dot{z}\hat{k}$$

$$\Rightarrow \ddot{\vec{r}}(t) = \ddot{\rho}\hat{e}_\rho + \dot{\rho}\hat{e}_\rho + \dot{\rho}\hat{e}_\phi + \ddot{\rho}\hat{e}_\phi + \dot{z}\hat{k} = \dot{\rho}\hat{e}_\rho + \dot{\rho}\hat{e}_\phi + \dot{\rho}\hat{e}_\phi - \dot{\rho}\hat{e}_\rho + \ddot{z}\hat{k}$$

$$= (\ddot{\rho} - \dot{\rho}\dot{\phi}^2)\hat{e}_\rho + (\dot{\rho}\dot{\phi} + 2\dot{\rho}\dot{\phi})\hat{e}_\phi + \ddot{z}\hat{k}$$



$$\sim \vec{r}(t) = a\hat{e}_\rho + z\hat{k} \quad \vec{v}(t) = \cancel{\dot{\rho}\hat{e}_\rho} + \dot{\rho}\hat{e}_\phi + \dot{z}\hat{k} = a\dot{\phi}\hat{e}_\phi + \dot{z}\hat{k}$$

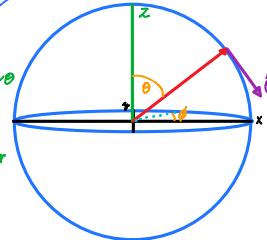
$$\vec{a}(t) = \cancel{(\ddot{\rho} - \dot{\rho}\dot{\phi}^2)\hat{e}_\rho} + \cancel{(\dot{\rho}\dot{\phi} + 2\dot{\rho}\dot{\phi})\hat{e}_\phi} + \ddot{z}\hat{k} = -a\dot{\phi}^2\hat{e}_\rho + a\ddot{\phi}\hat{e}_\phi + \ddot{z}\hat{k}$$

$$c) \vec{r}(t) = r\hat{e}_r; \quad \hat{e}_r = \sin\theta\cos\phi\hat{i} + \sin\theta\sin\phi\hat{j} + \cos\theta\hat{k}$$

$$\dot{\hat{e}}_r = \dot{\phi}\sin\theta\hat{e}_\phi + \dot{\theta}\hat{e}_\theta$$

$$\hat{e}_\theta = \cos\theta\cos\phi\hat{i} + \cos\theta\sin\phi\hat{j} - \sin\theta\hat{k} \quad \hat{e}_\phi = -\sin\phi\hat{i} + \cos\phi\hat{j} \quad \dot{\hat{e}}_\theta = \dot{\phi}\cos\theta\hat{e}_\phi - \dot{\theta}\hat{e}_\theta$$

$$\dot{\hat{e}}_\phi = -\dot{\phi}\sin\theta\hat{e}_r - \dot{\theta}\cos\theta\hat{e}_\theta$$



$$\Rightarrow \dot{\vec{r}}(t) = \vec{v}(t) = \dot{r}\hat{e}_r + r\dot{\hat{e}}_r = \dot{r}\hat{e}_r + r(\dot{\phi}\sin\theta\hat{e}_\phi + \dot{\theta}\hat{e}_\theta) = \dot{r}\hat{e}_r + r\dot{\phi}\sin\theta\hat{e}_\phi + r\dot{\theta}\hat{e}_\theta$$

$$\Rightarrow \ddot{\vec{r}}(t) = \vec{a}(t) = (\ddot{r}\hat{e}_r + \dot{r}\dot{\hat{e}}_r) + (\dot{r}\dot{\phi}\sin\theta\hat{e}_\phi + r\dot{\phi}\dot{\theta}\cos\theta\hat{e}_\phi + r\dot{\phi}\sin\theta\dot{\hat{e}}_\phi) + (\dot{r}\dot{\theta}\hat{e}_\theta + r\dot{\theta}\dot{\phi}\cos\theta\hat{e}_\phi + r\dot{\theta}\dot{\phi}\sin\theta(-\dot{\phi}\sin\theta\hat{e}_r - \dot{\theta}\cos\theta\hat{e}_\theta)) +$$

$$\dots + \dot{r}\dot{\theta}\hat{e}_\theta + r\ddot{\theta}\hat{e}_\theta + r\dot{\phi}(\dot{\phi}\cos\theta\hat{e}_\phi - \dot{\theta}\hat{e}_r)$$

$$= \underline{\ddot{r}\hat{e}_r} + \underline{\dot{r}\dot{\phi}\sin\theta\hat{e}_\phi} + \underline{\dot{r}\dot{\theta}\hat{e}_\theta} + \underline{\dot{r}\dot{\phi}\sin\theta\dot{\hat{e}}_\phi} + \underline{r\dot{\phi}\dot{\theta}\cos\theta\hat{e}_\phi} + \dots$$

$$- \underline{r\dot{\phi}\sin\theta\cdot\dot{\phi}\sin\theta\hat{e}_r} - \underline{r\dot{\phi}\sin\theta\cdot\dot{\theta}\cos\theta\hat{e}_\theta} + \underline{\dot{r}\dot{\theta}\hat{e}_\theta} + \underline{r\ddot{\theta}\hat{e}_\theta} + \underline{r\dot{\phi}\dot{\phi}\cos\theta\hat{e}_\phi} - \underline{r\dot{\phi}^2\hat{e}_r}$$

$$= (\ddot{r} - r\dot{\phi}^2\sin^2\theta - r\dot{\theta}^2)\hat{e}_r + (2\dot{r}\dot{\phi}\sin\theta + r\dot{\phi}\sin\theta + 2r\dot{\phi}\dot{\theta}\cos\theta)\hat{e}_\phi + (2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta + r\ddot{\theta})\hat{e}_\theta$$

$$\sim \vec{r}(t) = a\hat{e}_r \Rightarrow \vec{v}(t) = \cancel{\dot{r}\hat{e}_r} + r\dot{\phi}\sin\theta\hat{e}_\phi + r\dot{\theta}\hat{e}_\theta = a\dot{\phi}\sin\theta\hat{e}_\phi + a\dot{\theta}\hat{e}_\theta$$

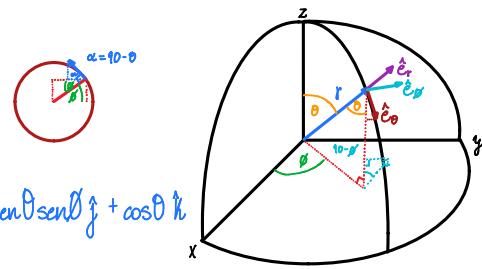
$$\Rightarrow \vec{a}(t) = (\ddot{r} - r\dot{\phi}^2\sin^2\theta - r\dot{\theta}^2)\hat{e}_r + (2\dot{r}\dot{\phi}\sin\theta + r\dot{\phi}\sin\theta + 2r\dot{\phi}\dot{\theta}\cos\theta)\hat{e}_\phi + (2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta + r\ddot{\theta})\hat{e}_\theta$$

$$= (-a\dot{\phi}^2\sin^2\theta - a\dot{\theta}^2)\hat{e}_r + (a\dot{\phi}\sin\theta + 2a\dot{\phi}\dot{\theta}\cos\theta)\hat{e}_\phi + (-a\dot{\phi}^2\sin\theta\cos\theta + a\ddot{\theta})\hat{e}_\theta$$

Posición, velocidad y aceleración en coordenadas esféricas

Hay que definir \hat{e}_r , \hat{e}_θ y \hat{e}_ϕ

$$\hat{e}_r : \hat{e}_r = \frac{\vec{r}(t)}{|\vec{r}(t)|} = \frac{r \sin \theta \cos \phi \hat{i} + r \sin \theta \sin \phi \hat{j} + r \cos \theta \hat{k}}{r} = \sin \theta \cos \phi \hat{i} + \sin \theta \sin \phi \hat{j} + \cos \theta \hat{k}$$



$$\hat{e}_\theta : \hat{e}_\theta = \cos \theta \cos \phi \hat{i} + \cos \theta \sin \phi \hat{j} - \sin \theta \hat{k}$$

$$\hat{e}_\phi : \hat{e}_\phi = -\sin(90 - \phi) \hat{i} + \sin(90 - \phi) \hat{j} = -\sin \phi \hat{i} + \cos \phi \hat{j} \quad \Rightarrow \text{Posición : } \vec{r}(t) = r \hat{e}_r //$$

$$\text{Velocidad : } \vec{v} = \dot{\vec{r}}(t) = \frac{dr}{dt} \hat{e}_r + r \frac{d\hat{e}_r}{dt} = \dot{r} \hat{e}_r + r \frac{d\hat{e}_r}{dt}$$

A partir de la definición de $\hat{e}_r \Rightarrow \frac{d\hat{e}_r}{dt} = \frac{d\hat{e}_r}{d\theta} \frac{d\theta}{dt} + \frac{d\hat{e}_r}{d\phi} \frac{d\phi}{dt} = \dot{\theta} \hat{e}_\theta + \dot{\phi} \sin \theta \hat{e}_\phi$

$$* \frac{d\hat{e}_r}{d\theta} = \cos \theta \cos \phi \hat{i} + \cos \theta \sin \phi \hat{j} - \sin \theta \hat{k} = \hat{e}_\theta \quad * \frac{d\hat{e}_r}{d\phi} = -\sin \theta \sin \phi \hat{i} + \sin \theta \cos \phi \hat{j} + \hat{k} = \sin \theta \hat{e}_\phi$$

$$\Rightarrow \vec{v} = \dot{r} \hat{e}_r + r(\dot{\theta} \hat{e}_\theta + \dot{\phi} \sin \theta \hat{e}_\phi) = \dot{r} \hat{e}_r + r \dot{\theta} \hat{e}_\theta + r \dot{\phi} \sin \theta \hat{e}_\phi //$$

Aceleración : $\vec{a} = \ddot{\vec{r}} = \ddot{\vec{r}}(t)$

$$\Rightarrow \vec{a} = \left(\frac{d\dot{r}}{dt} \hat{e}_r + \dot{r} \frac{d\hat{e}_r}{dt} \right) + \left(\frac{dr}{dt} \dot{\theta} \hat{e}_\theta + r \frac{d\dot{\theta}}{dt} \hat{e}_\theta + r \dot{\theta} \frac{d\hat{e}_\theta}{dt} \right) + \left(\frac{dr}{dt} \dot{\phi} \sin \theta \hat{e}_\phi + r \frac{d\dot{\phi}}{dt} \sin \theta \hat{e}_\phi + r \dot{\phi} \frac{d\sin \theta}{dt} \hat{e}_\phi + r \dot{\phi} \sin \theta \frac{d\hat{e}_\phi}{dt} \right)$$

$$= \underbrace{\left(\ddot{r} \hat{e}_r + \dot{r} \frac{d\hat{e}_r}{dt} \right)}_{P} + \underbrace{\left(\dot{r} \dot{\theta} \hat{e}_\theta + r \ddot{\theta} \hat{e}_\theta + r \dot{\theta} \frac{d\hat{e}_\theta}{dt} \right)}_{Q} + \underbrace{\left(\dot{r} \dot{\phi} \sin \theta \hat{e}_\phi + r \ddot{\phi} \sin \theta \hat{e}_\phi + r \dot{\phi} \frac{d\sin \theta}{dt} \hat{e}_\phi + r \dot{\phi} \sin \theta \frac{d\hat{e}_\phi}{dt} \right)}_{R}$$

A partir de la definición de $\hat{e}_\theta \Rightarrow \frac{d\hat{e}_\theta}{dt} = \frac{d\hat{e}_\theta}{d\theta} \frac{d\theta}{dt} + \frac{d\hat{e}_\theta}{d\phi} \frac{d\phi}{dt} = -\dot{\theta} \hat{e}_r + \dot{\phi} \cos \theta \hat{e}_\phi$

$$* \frac{d\hat{e}_\theta}{d\theta} = -\sin \theta \cos \phi \hat{i} - \sin \theta \sin \phi \hat{j} - \cos \theta \hat{k} = -\hat{e}_r \quad * \frac{d\hat{e}_\theta}{d\phi} = -\cos \theta \sin \phi \hat{i} + \cos \theta \cos \phi \hat{j} + \hat{k} = \cos \theta \hat{e}_\phi$$

$$\Rightarrow P = \ddot{r} \hat{e}_r + \dot{r}(\dot{\theta} \hat{e}_\theta + \dot{\phi} \cos \theta \hat{e}_\phi) = \ddot{r} \hat{e}_r + \dot{r} \dot{\theta} \hat{e}_\theta + \dot{r} \dot{\phi} \cos \theta \hat{e}_\phi$$

$$\Rightarrow Q = \dot{r} \dot{\theta} \hat{e}_\theta + r \ddot{\theta} \hat{e}_\theta + r \dot{\theta}(-\dot{r} \hat{e}_r + \dot{r} \cos \theta \hat{e}_\phi) = (\dot{r} \dot{\theta} + r \ddot{\theta}) \hat{e}_\theta - r \dot{r}^2 \hat{e}_r + r \dot{\theta} \dot{\phi} \cos \theta \hat{e}_\phi$$

$$\Rightarrow T = P + Q = (\ddot{r} - r \dot{\theta}^2) \hat{e}_r + (\dot{r} \dot{\theta} + r \ddot{\theta} + \dot{r} \dot{\phi} \cos \theta) \hat{e}_\theta + (\dot{r} \dot{\phi} \sin \theta + r \dot{\theta} \dot{\phi} \cos \theta) \hat{e}_\phi$$

Continua

A partir de la definición de $\hat{e}_\phi \Rightarrow \frac{d\hat{e}_\phi}{dt} = \frac{\partial \hat{e}_\phi}{\partial \theta} \frac{d\theta}{dt} + \frac{\partial \hat{e}_\phi}{\partial t} \frac{dt}{dt} = -\dot{\theta}(\cos\theta \hat{i} + \sin\theta \hat{j})^*$ y, Es y \hat{e}_ϕ :

$$\begin{pmatrix} \hat{e}_r \\ \hat{e}_\theta \\ \hat{e}_\phi \end{pmatrix} = \begin{pmatrix} \sin\theta \cos\phi & \sin\theta \sin\phi & \cos\theta \\ \cos\theta \cos\phi & \cos\theta \sin\phi & -\sin\phi \\ -\sin\phi & \cos\phi & 0 \end{pmatrix} \begin{pmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{pmatrix} \Rightarrow \begin{pmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{pmatrix} = \begin{pmatrix} \sin\theta \cos\phi & \cos\theta \cos\phi & -\sin\phi \\ \sin\theta \sin\phi & \cos\theta \sin\phi & \cos\phi \\ \cos\theta & -\sin\phi & 0 \end{pmatrix} \begin{pmatrix} \hat{e}_r \\ \hat{e}_\theta \\ \hat{e}_\phi \end{pmatrix}$$

$$\begin{aligned} \hat{i} &= \sin\theta \cos\phi \hat{e}_r + \cos\theta \cos\phi \hat{e}_\theta - \sin\phi \hat{e}_\phi \\ \Rightarrow \hat{j} &= \sin\theta \sin\phi \hat{e}_r + \cos\theta \sin\phi \hat{e}_\theta + \cos\phi \hat{e}_\phi \\ \Rightarrow \cos\theta \hat{i} + \sin\theta \hat{j} &= \cos\theta (\sin\theta \cos\phi \hat{e}_r + \cos\theta \cos\phi \hat{e}_\theta - \sin\phi \hat{e}_\phi) + \sin\theta (\sin\theta \sin\phi \hat{e}_r + \cos\theta \sin\phi \hat{e}_\theta + \cos\phi \hat{e}_\phi) \\ &= \sin^2\theta \cos\phi \hat{e}_r + \cos^2\theta \cos\phi \hat{e}_\theta - \cancel{\cos\theta \sin\phi \hat{e}_\phi} + \sin^2\theta \sin\phi \hat{e}_r + \cos^2\theta \sin\phi \hat{e}_\theta + \cancel{\cos\theta \sin\phi \hat{e}_\phi} \\ &= \sin^2\theta \cos\phi \hat{e}_r + \sin^2\theta \sin\phi \hat{e}_r + \cos^2\theta \cos\phi \hat{e}_\theta + \cos^2\theta \sin\phi \hat{e}_\theta \\ &= \sin\theta \hat{e}_r + \cos\theta \hat{e}_\theta \Rightarrow \frac{d\hat{e}_\phi}{dt} = -\dot{\theta}(\sin\theta \hat{e}_r + \cos\theta \hat{e}_\theta) \end{aligned}$$

Reemplazando en el parentesis de *

Reordenando

Recuperando R

$$\begin{aligned} \Rightarrow R &= \dot{r}\dot{\theta} \sin\theta \hat{e}_\phi + \ddot{r}\dot{\theta} \sin\theta \hat{e}_\phi + r\dot{\theta} \cos\theta \cdot \dot{\theta} \hat{e}_\phi + r\dot{\theta} \sin\theta \cdot -\dot{\theta}(\sin\theta \hat{e}_r + \cos\theta \hat{e}_\theta) \\ &= (\dot{r}\dot{\theta} \sin\theta + \ddot{r}\dot{\theta} \sin\theta + r\dot{\theta} \dot{\theta} \cos\theta) \hat{e}_\phi - r\dot{\theta}^2 \sin\theta (\sin\theta \hat{e}_r + \cos\theta \hat{e}_\theta) \\ &= (\dot{r}\dot{\theta} \sin\theta + \ddot{r}\dot{\theta} \sin\theta + r\dot{\theta} \dot{\theta} \cos\theta) \hat{e}_\phi - r\dot{\theta}^2 \sin^2\theta \hat{e}_r - r\dot{\theta}^2 \sin\theta \cos\theta \hat{e}_\theta \\ &= (\dot{r}\dot{\theta} \sin\theta + \ddot{r}\dot{\theta} \sin\theta + r\dot{\theta} \dot{\theta} \cos\theta) \hat{e}_\phi - r\dot{\theta}^2 \sin^2\theta \hat{e}_r - r\dot{\theta}^2 \sin\theta \cos\theta \hat{e}_\theta ; \text{ Uniendo esto a T} \\ \Rightarrow \vec{a} &= \vec{T} + R = (\ddot{r} - r\dot{\theta}^2) \hat{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta}) \hat{e}_\theta + (\dot{r}\dot{\theta} \sin\theta + r\dot{\theta} \dot{\theta} \cos\theta) \hat{e}_\phi + \dots \\ &\quad \dots + (\dot{r}\dot{\theta} \sin\theta + \ddot{r}\dot{\theta} \sin\theta + r\dot{\theta} \dot{\theta} \cos\theta) \hat{e}_\phi - r\dot{\theta}^2 \sin^2\theta \hat{e}_r - r\dot{\theta}^2 \sin\theta \cos\theta \hat{e}_\theta \\ &= (\ddot{r} - r\dot{\theta}^2 - r\dot{\theta}^2 \sin^2\theta) \hat{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta} - r\dot{\theta}^2 \sin\theta \cos\theta) \hat{e}_\theta + \dots \\ &\quad \dots + (\dot{r}\dot{\theta} \sin\theta + \underline{r\dot{\theta} \dot{\theta} \cos\theta} + \dot{r}\dot{\theta} \sin\theta + \underline{r\dot{\theta} \dot{\theta} \cos\theta}) \hat{e}_\phi \\ \Rightarrow \vec{a} &= (\ddot{r} - r\dot{\theta}^2 - r\dot{\theta}^2 \sin^2\theta) \hat{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta} - r\dot{\theta}^2 \sin\theta \cos\theta) \hat{e}_\theta + (\dot{r}\dot{\theta} \sin\theta + 2r\dot{\theta} \dot{\theta} \cos\theta + r\dot{\theta} \dot{\theta} \cos\theta) \hat{e}_\phi // \end{aligned}$$

Problema 12.

Pruebe que:

$$|\vec{v} \times \vec{a}| = \frac{v^3}{\rho},$$

donde ρ es el radio de curvatura de la trayectoria seguida por una partícula en movimiento.

Problema 13.

Un cañón situado al nivel del mar dispara una bola con velocidad inicial 82 m/s y ángulo inicial 45° . La pelota cae en el agua después de atravesar recorriendo una distancia horizontal de 686 m. ¿Cuánto mayor sería el distancia horizontal habría sido si el cañón hubiera estado 30 m más alto?



$$y_t = y_0 + V_{oy}t - \frac{1}{2}gt^2 \Rightarrow 0 = y_0 + V_{oy}t - \frac{1}{2}gt^2 \Rightarrow t = \frac{-V_{oy} \pm \sqrt{V_{oy}^2 - 4 \cdot \frac{-g}{2} \cdot y_0}}{-g} = \frac{V_{oy} \pm \sqrt{V_{oy}^2 + 2gy_0}}{g}$$

$$X_t = V_x t \Rightarrow X_t = V_x \cdot \frac{V_{oy} \pm \sqrt{V_{oy}^2 + 2gy_0}}{g}$$

$$\Rightarrow \Delta X_{y_0=30, y_0=0} = V_x \cdot \frac{V_{oy} + \sqrt{V_{oy}^2 + 2gy_0}}{g} - V_x \cdot \frac{V_{oy} - \sqrt{V_{oy}^2 + 2gy_0}}{g} = V_x \cdot \frac{V_{oy} + \sqrt{V_{oy}^2 + 2gy_0}}{g} - 2V_{oy}V_x$$

$$= \frac{V_{oy}\theta \cdot (V_{oy}\sin\theta + \sqrt{V_{oy}^2\sin^2\theta + 2gy_0} - 2V_{oy}\sin\theta)}{g} = 28,79$$

Problema 14.

Un camión agrícola se mueve hacia el este con una constante rapidez de 9.50 m/s en un tramo horizontal de carretera muy largo. Una niña que está sobre la parte trasera del camión, tira una lata de refresco hacia arriba y atrapa el proyectil en el mismo lugar en la camioneta, pero 16.0 m más adelante en la carretera.

- En el marco de referencia del camión, en qué ángulo para la vertical tira el niño la lata?
- ¿Cuál es la velocidad inicial de la lata en relación con el camión?
- ¿Cuál es la forma de la trayectoria de la lata tal como la ve el niño? ¿Y una persona en sobre la carretera?

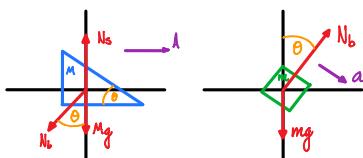
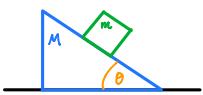
a) $\theta = \frac{\pi}{2}$ porque el proyectil se comporta como en caída libre al estar en reposo

b) Marco de referencia del suelo $y_t = v_{\text{sen}}\theta t - \frac{1}{2}gt^2 \Rightarrow v_{\text{sen}}\theta t - \frac{1}{2}gt^2 = 0 \Rightarrow v_{\text{sen}}\theta = \frac{1}{2}gt = \frac{1}{2}g \frac{X_t}{V_x} = 8.26 \text{ m/s}$

$$X_t = V_x t \Rightarrow t = \frac{X_t}{V_x}$$

c) Niña: Recta vertical

Persona en la carretera: Parábola



$$\sum F_{ay} = N_s - N_b \cos \theta - Mg = Ma_y \quad \text{la araña no se desvía}$$

$$\Rightarrow N_s = N_b \cos \theta + Mg$$

$$\sum F_{ax} = -N_b \sin \theta = MA_x$$

$$\Rightarrow -m(a_{max} + A_x) \csc \theta \sin \theta = MA_x$$

$$\Rightarrow A_x = \frac{-m a_{max}}{m+M} = \frac{-m a_{max} \cos \theta}{m+M}$$

$$\sum F_{ay} = N_b \cos \theta - mg = ma_y^*$$

$$\sum F_{ax} = N_b \sin \theta = ma_x$$

$$\Rightarrow N_b \sin \theta = m(a_{max} + A_x)$$

$$\Rightarrow N_b = m(a_{max} + A_x) \csc \theta$$

$$m(a_{max} + A_x) \csc \theta \cos \theta - mg = ma_y^*$$

$$\Rightarrow a_y = (a_{max} + A_x) \csc \theta \cos \theta - g$$

Restricción

$$\vec{a} = \vec{A} + \vec{a}_{\text{rel}} \quad \begin{matrix} \text{aceleración del} \\ \text{bloque visto} \\ \text{desde la Tierra} \end{matrix}$$

$$\Rightarrow \vec{a}_{\text{rel}} = \vec{a} - \vec{A} \quad \begin{matrix} \text{aceleración} \\ \text{del bloque visto} \\ \text{desde el punto} \end{matrix}$$

$$= (a_x - A_x)\hat{i} + a_y\hat{j}$$

$$\Rightarrow \tan \theta = \left(\frac{a_y}{a_x - A_x} \right)$$

$$\begin{cases} A_x + \frac{m}{m+M} a_{max} = 0 \\ A_x \csc \theta \cos \theta + a_{max} \csc \theta \cos \theta - a_y = g \\ a_y = a_{max} \end{cases}$$

$$\Rightarrow \begin{cases} A + \frac{m \cos \theta}{m+M} a_{max} = 0 \\ A \csc \theta \cos \theta + a_{max} \csc \theta \cos^2 \theta - a_{max} \sin \theta = g \end{cases}$$

$$\Rightarrow \begin{cases} A + \frac{m \cos \theta}{m+M} a_{max} = 0 \\ A \csc \theta \cos \theta + a_{max} (\csc \theta \cos^2 \theta - \sin \theta) = g \end{cases}$$

$$\Rightarrow \left(\begin{array}{cc|c} 1 & \frac{m \cos \theta}{m+M} & 0 \\ \csc \theta \cos \theta & \csc \theta \cos^2 \theta - \sin \theta & g \end{array} \right) \xrightarrow{\text{f}_2 \rightarrow \text{f}_2 - \csc \theta \cos \theta \text{f}_1} \left(\begin{array}{cc|c} 1 & \frac{m \cos \theta}{m+M} & 0 \\ 0 & \csc \theta \cos^2 \theta \left(\frac{M}{m+M} \right) - \sin \theta & g \end{array} \right)$$

$$\left(\begin{array}{cc|c} 1 & \frac{m \cos \theta}{m+M} & 0 \\ 0 & \csc \theta \cos^2 \theta - \sin \theta - \frac{m \cos \theta}{m+M} \cdot \csc \theta \cos \theta & g \end{array} \right) \xrightarrow{\text{f}_1 \rightarrow \text{f}_1 - \frac{m \cos \theta}{m+M} \text{f}_2} \left(\begin{array}{cc|c} 1 & \frac{m \cos \theta}{m+M} & 0 \\ 0 & \csc \theta \cos^2 \theta \left(\frac{M}{m+M} \right) - \sin \theta & g \end{array} \right) \xrightarrow{\text{f}_2 \rightarrow \frac{1}{\csc \theta \cos^2 \theta} \text{f}_2}$$

$$\left(\begin{array}{cc|c} 1 & \frac{m \cos \theta}{m+M} & 0 \\ 0 & 1 & \frac{g \left(\frac{M}{m+M} \right)}{\csc \theta \cos^2 \theta \left(\frac{M}{m+M} \right) - \sin \theta} \end{array} \right) \xrightarrow{\text{f}_1 \rightarrow \text{f}_1 - \frac{m \cos \theta}{m+M} \text{f}_2} \left(\begin{array}{cc|c} 1 & 0 & \frac{-\frac{m \cos \theta}{m+M} g}{\csc \theta \cos^2 \theta \left(\frac{M}{m+M} \right) - \sin \theta} \\ 0 & 1 & \frac{g \left(\frac{M}{m+M} \right)}{\csc \theta \cos^2 \theta \left(\frac{M}{m+M} \right) - \sin \theta} \end{array} \right)$$

$$\Rightarrow A = \frac{-\frac{m \cos \theta}{m+M} g}{\csc \theta \cos^2 \theta \left(\frac{M}{m+M} \right) - \sin \theta} \quad \text{y} \quad a_{max} = \frac{g \left(\frac{M}{m+M} \right)}{\csc \theta \cos^2 \theta \left(\frac{M}{m+M} \right) - \sin \theta} ; \quad a_x = A + a_{max} \cos \theta$$



$$\sum F_{xv} = F = m v \alpha_v$$

$$\sum F_{xb} = 18 = m_b \alpha_b \Rightarrow 18 = m_b (\alpha_v + \alpha_b \frac{v}{r})$$

$$\alpha_b = \alpha_v + \alpha_b \frac{v}{r}$$



$$T - T_c - m_2 g = m_2 a_{2r}$$

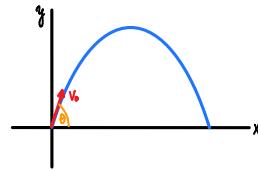


$$T_c - m_1 g = m_1 a_{1r}$$

Demostrar las ecuaciones para movimiento con resistencia del aire lineal y cuadrática.

Proyectiles: $f(v) = -bv$

$$\text{Para } X: \sum F_x = -bv_x = ma_x \Rightarrow \frac{-bv_x}{m} = \frac{dv_x}{dt} \Rightarrow \int_0^t \frac{-b}{m} dt = \int_{v_0}^v \frac{1}{v} dv_x$$



$$\Rightarrow -\frac{b}{m}t = \ln(v) - \ln(v_0 \cos \theta) = \ln\left(\frac{v}{v_0 \cos \theta}\right) \Rightarrow v_x = v_0 \cos \theta e^{-\frac{bt}{m}} = \frac{dv_x}{dt}$$

$$\Rightarrow \int_0^x dx = \int_0^t v_0 \cos \theta e^{-\frac{bt}{m}} dt *$$

$$\Rightarrow X = \frac{m v_0 \cos \theta}{b} \left(1 - e^{-\frac{bt}{m}}\right) //$$

$$\begin{aligned} & \int_0^t v_0 \cos \theta e^{-\frac{bt}{m}} dt = v_0 \cos \theta \int_0^t e^{-\frac{bt}{m}} dt ; u = -\frac{bt}{m} \Rightarrow du = -\frac{b}{m} dt \\ & \Rightarrow v_0 \cos \theta \int_0^{\frac{bt}{m}} -\frac{me^u}{b} du = -\frac{m v_0 \cos \theta}{b} e^u \Big|_0^{\frac{bt}{m}} = -\frac{m v_0 \cos \theta}{b} (e^{\frac{bt}{m}} - 1) \end{aligned}$$

$$\text{Para } Y: \sum F_y = -bv_y - mg = ma_y \Rightarrow \frac{-bv_y - mg}{m} = \frac{dv_y}{dt} \Rightarrow \int_0^t \frac{-1}{m} dt = \int_{v_0}^{v_y} \frac{1}{b v_y + mg} dv_y$$

$$\int_{v_0}^{v_y} \frac{1}{b v_y + mg} dv_y ; u = b v_y + mg \Rightarrow du = b dv_y \Rightarrow \int_{b v_0 + mg}^{b v_y + mg} \frac{1}{u} du = \int_0^t \frac{1}{b} dt = \frac{1}{b} t \Big|_0^t = \frac{1}{b} t = \ln\left(\frac{b v_y + mg}{b v_0 + mg}\right)$$

$$\Rightarrow -\frac{bt}{m} = \ln\left(\frac{b v_y + mg}{b v_0 + mg}\right) \Rightarrow b v_y + mg = (b v_0 + mg) e^{-\frac{bt}{m}} \Rightarrow v_y + \frac{mg}{b} = \left(b v_0 + mg\right) e^{-\frac{bt}{m}}$$

La máxima velocidad que puede alcanzar m en caída libre es: $v_{ter} \Rightarrow -bv_y - mg = ma_y \Rightarrow v_{ter} = -\frac{mg}{b}$

$$\Rightarrow v_y - v_{ter} = (b v_0 + mg) e^{-\frac{bt}{m}} \Rightarrow \frac{dv_y}{dt} = v_{ter} + (b v_0 + mg) e^{-\frac{bt}{m}} \Rightarrow \int_0^y dy = \int_0^t v_{ter} + (b v_0 + mg) e^{-\frac{bt}{m}} dt *$$

$$\Rightarrow y = v_{ter} t + m(b v_0 + mg) \left(e^{-\frac{bt}{m}} - 1\right) \Rightarrow y = v_{ter} t + m(b v_0 + mg) \left(1 - e^{-\frac{bt}{m}}\right) //$$

Carro Libre $f(v) = -cv^2 \hat{v}$

Movimiento rectilíneo

Problema 1. (Thornton 2.2)

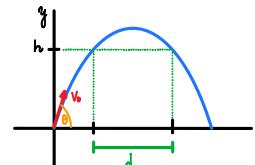
- 2-2. A particle of mass m is constrained to move on the surface of a sphere of radius R by an applied force $\mathbf{F}(\theta, \phi)$. Write the equation of motion.

$$\begin{aligned}\vec{a}(t) &= (\ddot{r} - r\dot{\phi}^2 \sin^2 \theta - r\ddot{\theta}) \hat{e}_r + (2\dot{r}\dot{\phi} \sin \theta - r\ddot{\phi} \sin \theta + 2r\dot{\phi}\dot{\theta} \cos \theta) \hat{e}_\phi + (2\dot{\theta}\dot{\phi} - r\dot{\phi}^2 \sin \theta \cos \theta + r\ddot{\theta}) \hat{e}_\theta \\ &= (-R\dot{\phi}^2 \sin^2 \theta - R\ddot{\theta}) \hat{e}_r + (R\ddot{\phi} \sin \theta + 2R\dot{\phi}\dot{\theta} \cos \theta) \hat{e}_\phi + (-R\dot{\phi}^2 \sin \theta \cos \theta + R\ddot{\theta}) \hat{e}_\theta \\ \sum \vec{F} &= m\vec{a}(t) = -mR(+\dot{\phi}^2 \sin^2 \theta + \ddot{\theta}) \hat{e}_r + mR(\ddot{\phi} \sin \theta + 2\dot{\phi}\dot{\theta} \cos \theta) \hat{e}_\phi + mR(-\dot{\phi}^2 \sin \theta \cos \theta + \ddot{\theta}) \hat{e}_\theta \quad ; \vec{F} = F_r \hat{e}_r + F_\phi \hat{e}_\phi + F_\theta \hat{e}_\theta \\ \Rightarrow F_r + F_\phi e_\phi + F_\theta e_\theta &= -mR(+\dot{\phi}^2 \sin^2 \theta + \ddot{\theta}) \hat{e}_r + mR(\ddot{\phi} \sin \theta + 2\dot{\phi}\dot{\theta} \cos \theta) \hat{e}_\phi + mR(-\dot{\phi}^2 \sin \theta \cos \theta + \ddot{\theta}) \hat{e}_\theta \quad \text{Respuesta para la pregunta} \\ \Rightarrow \vec{F}(\phi, \theta) &= mR(\ddot{\phi} \sin \theta + 2\dot{\phi}\dot{\theta} \cos \theta) \hat{e}_\phi + mR(-\dot{\phi}^2 \sin \theta \cos \theta + \ddot{\theta}) \hat{e}_\theta \quad \leftarrow \text{Respuesta según el solucionario, buscando la forma de la fuerza desconocida}\end{aligned}$$

Problema 2. (Thornton 2.8)

- 2-8. A projectile is fired with a velocity v_0 such that it passes through two points both a distance h above the horizontal. Show that if the gun is adjusted for maximum range, the separation of the points is

$$d = \frac{v_0}{g} \sqrt{v_0^2 - 4gh}$$



$$y = v_0 \sin \theta t - \frac{1}{2} g t^2 \xrightarrow{\text{Tiempo de vuelo}} 0 = v_0 \sin \theta t - \frac{1}{2} g t^2 \Rightarrow t \left(\frac{1}{2} g t - v_0 \sin \theta \right) = 0 \Rightarrow t_1 = 0 \quad t_2 = \frac{2v_0 \sin \theta}{g}$$

$$\text{Rango máximo: } R = v_0 \cos \theta t = v_0 \cos \theta \frac{2v_0 \sin \theta}{g} = \frac{v_0^2 \sin(2\theta)}{g} \Rightarrow \frac{dR}{d\theta} = \frac{2v_0^2 \cos(2\theta)}{g} = 0 \Rightarrow \theta = \frac{1}{2} \cos^{-1}(0) = \frac{\pi}{4} \quad \text{Ángulo de máxima alcance}$$

$$h = v_0 \sin \theta t - \frac{1}{2} g t^2 = \frac{v_0}{\sqrt{2}} t - \frac{1}{2} g t^2 \Rightarrow 0 = +h - \frac{v_0}{\sqrt{2}} t + \frac{1}{2} g t^2 \Rightarrow t = \frac{\frac{v_0}{\sqrt{2}} \pm \sqrt{\frac{v_0^2}{2} - \frac{1}{4} \cdot \frac{1}{2} g h}}{2 \cdot \frac{1}{2} g} = \frac{\frac{v_0}{\sqrt{2}} \pm \sqrt{\frac{v_0^2}{2} - \frac{1}{2} g h}}{g}$$

$$\Rightarrow t = \frac{1}{\sqrt{2}} \frac{v_0 \pm \sqrt{v_0^2 - \frac{1}{2} g h}}{g} \quad \text{Tiempo para alcanzar una misma altura en la parábola}$$

$$d = X(t_+) - X(t_-) = v_0 \cos \theta t_+ - v_0 \cos \theta t_- = \frac{v_0}{\sqrt{2}} \left(\frac{1}{\sqrt{2}} \frac{v_0 + \sqrt{v_0^2 - \frac{1}{2} g h}}{g} - \frac{1}{\sqrt{2}} \frac{v_0 - \sqrt{v_0^2 - \frac{1}{2} g h}}{g} \right) = \frac{v_0}{\sqrt{2}} \frac{1}{\sqrt{2}} \frac{2\sqrt{v_0^2 - \frac{1}{2} g h}}{g}$$

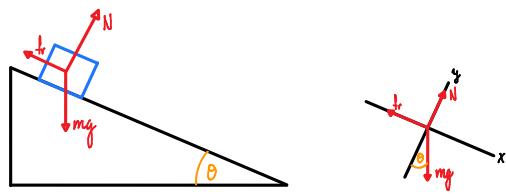
$$\Rightarrow d = \frac{v_0}{g} \sqrt{v_0^2 - \frac{1}{2} g h} //$$

Problema 3. (Thornton 2.15)

2-15. A particle of mass m slides down an inclined plane under the influence of gravity. If the motion is resisted by a force $f = kmv^2$, show that the time required to move a distance d after starting from rest is

$$t = \frac{\cosh^{-1}(e^{kd})}{\sqrt{kg \sin \theta}}$$

where θ is the angle of inclination of the plane.



$$\begin{aligned} \sum F_x = mgsen\theta - kmv^2 &= ma_x \Rightarrow gsen\theta - kv^2 = a_x = \frac{dv_x}{dt} \Rightarrow dv_x = gsen\theta - kv^2 dt \Rightarrow \frac{dv_x}{gsen\theta - kv^2} = dt \\ \Rightarrow \int_0^V \frac{dv_x}{gsen\theta - kv^2} &= \int_0^V \frac{1}{k} \frac{dv_x}{\frac{gsen\theta}{k} - v^2} = \int_0^t \frac{1}{k} dt = t \Rightarrow t = \frac{1}{k} \frac{1}{\sqrt{\frac{gsen\theta}{k}}} \tanh^{-1}\left(\frac{v}{\sqrt{\frac{gsen\theta}{k}}}\right) \\ \Rightarrow k\sqrt{\frac{gsen\theta}{k}} t &= \sqrt{gsen\theta} t = \tanh^{-1}\left(\frac{v}{\sqrt{\frac{gsen\theta}{k}}}\right) \Rightarrow \tanh\left(\sqrt{gsen\theta} t\right) = \frac{v}{\sqrt{\frac{gsen\theta}{k}}} \Rightarrow \sqrt{\frac{gsen\theta}{k}} \tanh\left(\sqrt{gsen\theta} t\right) = v \\ \Rightarrow \int_0^d \sqrt{\frac{gsen\theta}{k}} \tanh\left(\sqrt{gsen\theta} t\right) dt &= \int_0^d dx = d \Rightarrow \frac{1}{k} \ln[\cosh(\sqrt{gsen\theta} t)] = d \Rightarrow \ln[\cosh(\sqrt{gsen\theta} t)] = kd \\ \Rightarrow \cosh(\sqrt{gsen\theta} t) &= e^{kd} \Rightarrow \sqrt{gsen\theta} t = \cosh^{-1}(e^{kd}) \Rightarrow t = \frac{\cosh^{-1}(e^{kd})}{\sqrt{gsen\theta}} // \end{aligned}$$

Problema 4. (Taylor 2.14)

2.14 *** Use the method of Problem 2.7 to solve the following: A mass m is constrained to move along the x axis subject to a force $F(v) = -F_0 e^{v/V}$, where F_0 and V are constants. (a) Find $v(t)$ if the initial velocity is $v_0 > 0$ at time $t = 0$. (b) At what time does it come instantaneously to rest? (c) By integrating $v(t)$, you can find $x(t)$. Do this and find how far the mass travels before coming instantaneously to rest.

$$a) F(v) = -F_0 e^{\frac{v}{V}} = m \frac{dv}{dt} \Rightarrow \int_0^t \frac{F_0}{m} dt = \int_{v_0}^v e^{-\frac{v}{V}} dv \Rightarrow \frac{F_0 t}{m} = -V(e^{\frac{-v}{V}} - e^{\frac{-v_0}{V}}) \Rightarrow \frac{F_0 t}{V m} = e^{\frac{-v}{V}} - e^{\frac{-v_0}{V}}$$

$$\Rightarrow e^{\frac{-v}{V}} = \frac{F_0 t}{V m} + e^{\frac{-v_0}{V}} \Rightarrow -v = V \ln \left(\frac{F_0 t}{V m} + e^{\frac{-v_0}{V}} \right) \Rightarrow v = -V \ln \left(\frac{F_0 t}{V m} + e^{\frac{-v_0}{V}} \right)$$

$$b) \text{ Reposo} \Rightarrow v = 0 \Rightarrow -V \ln \left(\frac{F_0 t}{V m} + e^{\frac{-v_0}{V}} \right) = 0 \Rightarrow \frac{F_0 t}{V m} + e^{\frac{-v_0}{V}} = 1 \Rightarrow \frac{F_0 t}{V m} = 1 - e^{\frac{-v_0}{V}} \Rightarrow t_r = \frac{(1 - e^{\frac{-v_0}{V}}) V m}{F_0}$$

$$c) \frac{dx}{dt} = -V \ln \left(\frac{F_0 t}{V m} + e^{\frac{-v_0}{V}} \right) \Rightarrow \int_0^t dx = \int_0^t -V \ln \left(\frac{F_0 t}{V m} + e^{\frac{-v_0}{V}} \right) dt$$

$$* \int_0^t -V \ln \left(\frac{F_0 t}{V m} + e^{\frac{-v_0}{V}} \right) dt \quad u = \frac{F_0 t}{V m} + e^{\frac{-v_0}{V}} \quad du = \frac{F_0}{V m} dt \Rightarrow \frac{V m}{F_0} du = dt$$

$$\Rightarrow \int_0^t -V \ln \left(\frac{F_0 t}{V m} + e^{\frac{-v_0}{V}} \right) dt = \int_{e^{\frac{-v_0}{V}}}^{\frac{F_0 t}{V m} + e^{\frac{-v_0}{V}}} -\frac{V}{F_0} \ln(u) du = -\frac{V^2}{F_0} \left[\left(\frac{F_0 t}{V m} + e^{\frac{-v_0}{V}} \right) \left[\ln \left(\frac{F_0 t}{V m} + e^{\frac{-v_0}{V}} \right) - 1 \right] - e^{\frac{-v_0}{V}} [\ln(e^{\frac{-v_0}{V}}) - 1] \right]$$

$$= -\frac{V^2}{F_0} \left[\left(\frac{F_0 t}{V m} + e^{\frac{-v_0}{V}} \right) \left[\ln \left(\frac{F_0 t}{V m} + e^{\frac{-v_0}{V}} \right) - 1 \right] - e^{\frac{-v_0}{V}} \left[\frac{-v_0}{V} - 1 \right] \right]$$

$$\Rightarrow X = -\frac{V^2}{F_0} \left[\left(\frac{F_0 t}{V m} + e^{\frac{-v_0}{V}} \right) \left[\ln \left(\frac{F_0 t}{V m} + e^{\frac{-v_0}{V}} \right) - 1 \right] - e^{\frac{-v_0}{V}} \left[\frac{-v_0}{V} - 1 \right] \right]$$

$$\Rightarrow X(t_r) = -\frac{V^2}{F_0} \left[\left(\frac{F_0 (1 - e^{\frac{-v_0}{V}}) V m}{V m} + e^{\frac{-v_0}{V}} \right) \left[\ln \left(\frac{F_0 (1 - e^{\frac{-v_0}{V}}) V m}{V m} + e^{\frac{-v_0}{V}} \right) - 1 \right] - e^{\frac{-v_0}{V}} \left[\frac{-v_0}{V} - 1 \right] \right]$$

$$= -\frac{V^2}{F_0} \left\{ 1 \cdot 1 - e^{\frac{-v_0}{V}} \left[\frac{-v_0}{V} - 1 \right] \right\} = +\frac{V^2}{F_0} \left\{ 1 + e^{\frac{-v_0}{V}} \left[\frac{-v_0}{V} - 1 \right] \right\}$$

2.4** The origin of the quadratic drag force on any projectile in a fluid is the inertia of the fluid that the projectile sweeps up. (a) Assuming the projectile has a cross-sectional area A (normal to its velocity) and speed v , and that the density of the fluid is ρ , show that the rate at which the projectile encounters fluid (mass/time) is $\rho A v$. (b) Making the simplifying assumption that all of this fluid is accelerated to the speed v of the projectile, show that the net drag force on the projectile is $\rho A v^2$. It is certainly not true that all the fluid that the projectile encounters is accelerated to the full speed v , but one might guess that the actual force would have the form

$$f_{\text{quad}} = \kappa \rho A v^2 \quad (2.84)$$

where κ is a number less than 1, which would depend on the shape of the projectile, with κ small for a streamlined body, and larger for a body with a flat front end. This proves to be true, and for a sphere the factor κ is found to be $\kappa = 1/4$. (c) Show that (2.84) reproduces the form (2.3) for f_{quad} , with c given by (2.4) as $c = \gamma D^2$. Given that the density of air at STP is $\rho = 1.29 \text{ kg/m}^3$ and that $\kappa = 1/4$ for a sphere, verify the value of γ given in (2.6).

$$f_{\text{quad}} = cv^2. \quad (2.3)$$

$$c = \gamma D^2 \quad (2.4)$$

$$\gamma = 0.25 \text{ N}\cdot\text{s}^2/\text{m}^4. \quad (2.6)$$

$$\text{b)} \quad F = \frac{dp}{dt} = \frac{dm}{dt} v + m \cancel{\frac{dv}{dt}} \xrightarrow{v: \text{constante}} = \frac{dm}{dt} v = \rho v A v = \rho v^2 A \quad //$$

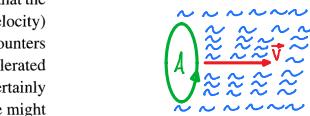
$$\text{c)} \quad f_{\text{quad}} = K \rho v^2 A \xrightarrow{\text{sphere}} \frac{1}{4} \rho v^2 \pi R^2 = \frac{1}{16} \rho v^2 \pi D^2 = c v^2 \Rightarrow c = \frac{\pi D^2}{16} = \gamma D^2 \Rightarrow \gamma = \frac{\pi}{16} \approx 0.25 \quad //$$

Problema 5. (Taylor 2.24)

2.24* Consider a sphere (diameter D , density ρ_{sph}) falling through air (density ρ_{air}) and assume that the drag force is purely quadratic. (a) Use Equation (2.84) from Problem 2.4 (with $\kappa = 1/4$ for a sphere) to show that the terminal speed is

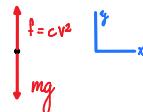
$$v_{\text{ter}} = \sqrt{\frac{8 D g \rho_{\text{sph}}}{3 \rho_{\text{air}}}}. \quad (2.88)$$

(b) Use this result to show that of two spheres of the same size, the denser one will eventually fall faster. (c) For two spheres of the same material, show that the larger will eventually fall faster.



a) Pensando que el área transversal A permanece constante y el fluido se mueve hacia v , se puede plantear

$$\begin{aligned} \cancel{F} &= \iint_S p \vec{v} \cdot d\vec{A} = \iint_S p v dA = \cancel{\rho \int_0^R} \rho v r dr db \\ &= \pi r^2 \rho v = \rho v A = \frac{dm}{dt} \end{aligned}$$



$$m = \rho_{\text{sph}} V = \rho_{\text{sph}} \frac{4}{3} \pi R^3 = \rho_{\text{sph}} \frac{4}{3} \pi D^3$$

$$\text{a)} \sum F_y = \cancel{\rho_{\text{air}} A v^2} - mg = may \Rightarrow \frac{1}{4} \rho_{\text{air}} v^2 \pi R^2 - mg = may \Rightarrow \frac{\rho_{\text{air}} v^2 \pi D^2}{16} - \rho_{\text{air}} \frac{4}{3} \pi D^3 g = \rho_{\text{sph}} \frac{4}{3} \pi D^3 a y$$

$$\text{Velocidad terminal } a_y = 0 \Rightarrow \cancel{\frac{\rho_{\text{air}} v^2 \pi D^2}{16}} - \rho_{\text{air}} \frac{4}{3} \pi D^3 g = 0 \Rightarrow \frac{\rho_{\text{air}} v^2}{16} = \rho_{\text{air}} \frac{D}{6} g \Rightarrow v_{\text{ter}} = \sqrt{\frac{8 D g \rho_{\text{sph}}}{3 \rho_{\text{air}}}} \quad //$$

$$\text{b)} \quad \rho_{\text{sph1}} > \rho_{\text{sph2}}$$

$$\Rightarrow \frac{v_{\text{ter1}}}{v_{\text{ter2}}} = \sqrt{\frac{\frac{8 D g \rho_{\text{sph1}}}{3 \rho_{\text{air}}}}{\frac{8 D g \rho_{\text{sph2}}}{3 \rho_{\text{air}}}}} = \sqrt{\frac{\rho_{\text{sph1}}}{\rho_{\text{sph2}}}} \Rightarrow v_{\text{ter1}} = v_{\text{ter2}} \sqrt{\frac{\rho_{\text{sph1}}}{\rho_{\text{sph2}}}} \text{ con } \sqrt{\frac{\rho_{\text{sph1}}}{\rho_{\text{sph2}}}} > 1 \text{ ya que } \rho_{\text{sph1}} > \rho_{\text{sph2}} \quad //$$

$$\text{c)} \quad D_1 > D_2$$

$$\Rightarrow \frac{v_{\text{ter1}}}{v_{\text{ter2}}} = \sqrt{\frac{\frac{8 D g \rho_{\text{sph1}}}{3 \rho_{\text{air}}}}{\frac{8 D g \rho_{\text{sph2}}}{3 \rho_{\text{air}}}}} = \sqrt{\frac{D_1}{D_2}} \Rightarrow v_{\text{ter1}} = v_{\text{ter2}} \sqrt{\frac{D_1}{D_2}} \text{ con } \sqrt{\frac{D_1}{D_2}} > 1 \text{ ya que } D_1 > D_2 \quad //$$

Problema 6. (Taylor 2.28)

2.28* A mass m has speed v_0 at the origin and coasts along the x axis in a medium where the drag force is $F(v) = -cv^{3/2}$. Use the “ $v \frac{dv}{dx}$ rule” (2.86) in Problem 2.12 to write the equation of motion in the separated form $m v \frac{dv}{F(v)} = dx$, and then integrate both sides to give x in terms of v (or vice versa). Show that it will eventually travel a distance $2m\sqrt{v_0/c}$.

$$\sum F_x = -cv^{\frac{3}{2}} = ma \Rightarrow -\frac{cv^{\frac{3}{2}}}{m} = \frac{dv}{dt} ; \frac{dv}{dt} = \frac{dv}{dx} \frac{dx}{dt} = v \frac{dv}{dx} \Rightarrow -\frac{cv^{\frac{3}{2}}}{m} = v \frac{dv}{dx} \Rightarrow -\frac{c}{m} dx = \frac{v}{v^{\frac{3}{2}}} dv = \frac{1}{\sqrt{v}} dv$$

$$\Rightarrow \int_0^x -\frac{c}{m} dx = \int_{v_0}^v \frac{1}{\sqrt{v}} dv \Rightarrow -\frac{c}{m} x = \int_{v_0}^v \frac{1}{\sqrt{v}} dv$$

$* \int_{v_0}^v \frac{1}{\sqrt{v}} dv = \int_{v_0}^v v^{-\frac{1}{2}} dv = \frac{v^{\frac{1}{2}+1}}{-\frac{1}{2}+1} \Big|_{v_0}^v = \frac{v^{\frac{3}{2}}}{\frac{1}{2}} \Big|_{v_0}^v = 2v^{\frac{1}{2}} \Big|_{v_0}^v = 2v^{\frac{1}{2}} - 2v_0^{\frac{1}{2}}$

$$\Rightarrow -\frac{c}{m} x = 2v^{\frac{1}{2}} - 2v_0^{\frac{1}{2}} \Rightarrow -\frac{c}{2m} x = v^{\frac{1}{2}} - v_0^{\frac{1}{2}} \Rightarrow v = \sqrt{v_0^{\frac{1}{2}} - \frac{c}{2m} x}^4$$

$$v = \sqrt{v_0^{\frac{1}{2}} - \frac{c}{2m} x}^4 = 0 \Rightarrow v_0^{\frac{1}{2}} - \frac{c}{2m} x = 0 \Rightarrow x = \frac{2m\sqrt{v_0}}{c}$$

Problema 7. (Thornton 2.32)

2.32. Two blocks of unequal mass are connected by a string over a smooth pulley (Figure 2-B). If the coefficient of kinetic friction is μ_k , what angle θ of the incline allows the masses to move at a constant speed?

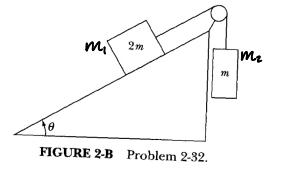
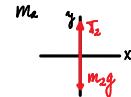
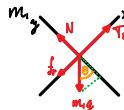


FIGURE 2-B Problem 2.32.

$$m_1 \quad \sum F_{1x} = T_1 - f_r - m_1 g \sin \theta = m_1 a_{1x}$$

$$\sum F_{1y} = N - m_1 g \cos \theta = m_1 a_{1y} \quad \text{no hay movimiento en } y$$

$$\Rightarrow T - f_r - m_1 g \sin \theta = m_1 a_{1x}$$

$$N = m_1 g \cos \theta ; f_r = \mu_k N$$

$$m_2 \quad \sum F_{2x} = 0$$

$$\sum F_{2y} = T_2 - m_2 g = m_2 a_{2y}$$

$$\Rightarrow T - m_2 g = m_2 a_{2y}$$

$$\Rightarrow T = m_2(g - a)$$

Pulea

$$\sum F_{\text{pulg}} = -R T_1 + R T_2 = T \cancel{d}$$

$$\Rightarrow T_1 = T_2$$

* Condición $a_{1x} = -a_{2y} = a$

$v: \text{constante} \Rightarrow a = 0^*$

$$\Rightarrow T = \mu_k m_1 g \cos \theta - m_1 g \sin \theta + m_1 a$$

$$= m_1(\mu_k g \cos \theta - g \sin \theta + a)$$

Uniendo ecuaciones $\Rightarrow m_1(\mu_k g \cos \theta - g \sin \theta + a) = m_2(g - a)^*$

$$\Rightarrow m_1(\mu_k g \cos \theta - g \sin \theta) = m_2 g$$

$$\Rightarrow \mu_k g \cos \theta - g \sin \theta = \frac{m_2}{m_1} ; \cos \theta = \sqrt{1 - \sin^2 \theta} \Rightarrow \mu_k \sqrt{1 - \sin^2 \theta} - \sin \theta = \frac{m_2}{m_1}$$

$$\Rightarrow \mu_k^2(1 - \sin^2 \theta) = \left(\frac{m_2}{m_1} + \sin \theta\right)^2 \Rightarrow \mu_k^2 - \mu_k^2 \sin^2 \theta = \frac{m_2^2}{m_1^2} + 2 \frac{m_2}{m_1} \sin \theta + \sin^2 \theta$$

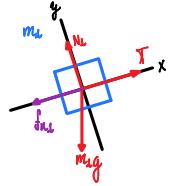
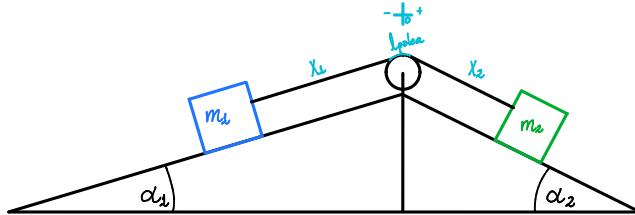
$$\Rightarrow \left(\frac{m_2^2}{m_1^2} - \mu_k^2\right) + 2 \frac{m_2}{m_1} \sin \theta + \sin^2 \theta (1 + \mu_k^2) = 0 \Rightarrow \sin \theta = \frac{-2 \frac{m_2}{m_1} \pm \sqrt{4 \frac{m_2^2}{m_1^2} - 4(1 + \mu_k^2)(\frac{m_2^2}{m_1^2} - \mu_k^2)}}{2(1 + \mu_k^2)} \quad \text{con } m_1 = 2m_2 = 2m$$

$$\Rightarrow \sin \theta = \frac{-2 \frac{m_2}{m_1} \pm \sqrt{4 \frac{m_2^2}{m_1^2} - 4(1 + \mu_k^2)(\frac{m_2^2}{m_1^2} - \mu_k^2)}}{2(1 + \mu_k^2)} = \frac{-1 \pm \sqrt{1 - 4(1 + \mu_k^2)(\frac{1}{4} - \mu_k^2)}}{2(1 + \mu_k^2)} = \frac{-1 \pm \sqrt{3\mu_k^2 + \frac{1}{4}\mu_k^2}}{2(1 + \mu_k^2)}$$

Problema 8.

Considera dos masas m_1 y m_2 colocadas en un doble plano inclinado, con ángulos α_1 y α_2 , respectivamente. Los cuerpos están unidos por medio de una cuerda y polea completamente ideales y sin masa apreciable.

1. Determine la aceleración de cada bloque sobre el plano, si no existe fricción entre ellos y el doble plano inclinando.
2. Determine la aceleración de cada bloque sobre el plano, si el coeficiente de fricción cinética entre ellos y el doble plano inclinando es el mismo (μ_k).
3. ¿Cómo cambia el resultado si el coeficiente es distinto?



$$\sum F_{1x} = T - f_{1s} - m_1 g \sin \alpha_1 = m_1 a_{1x}$$

$$\sum F_{1y} = N_1 - m_1 g \cos \alpha_1 = m_1 a_{1y}$$



$$\sum F_{2x} = -T - f_{2s} + m_2 g \sin \alpha_2 = m_2 a_{2x}$$

$$\sum F_{2y} = N_2 - m_2 g \cos \alpha_2 = m_2 a_{2y}$$

Aplicando lo anterior se obtiene:

$$N_1 - m_1 g \cos \alpha_1 = 0 \Rightarrow N_1 = m_1 g \cos \alpha_1$$

$$\Rightarrow T - \mu_k m_1 g \cos \alpha_1 - m_1 g \sin \alpha_1 = m_1 a_{1x}$$

$$N_2 - m_2 g \cos \alpha_2 = 0 \Rightarrow N_2 = m_2 g \cos \alpha_2$$

$$\Rightarrow -T - \mu_k m_2 g \cos \alpha_2 + m_2 g \sin \alpha_2 = m_2 a_{2x}$$

$$\Rightarrow m_{1x} + \mu_k m_1 g \cos \alpha_1 + m_1 g \sin \alpha_1 = -\mu_k m_2 g \cos \alpha_2 + m_2 g \sin \alpha_2 - m_{2x}$$

$$\Rightarrow a_x(m_1 + m_2) = -\mu_k m_2 g \cos \alpha_2 + m_2 g \sin \alpha_2 - \mu_k m_1 g \cos \alpha_1 - m_1 g \sin \alpha_1 = m_2 g(-\mu_k \cos \alpha_2 + \sin \alpha_2) - m_1 g(\mu_k \cos \alpha_1 + \sin \alpha_1)$$

$$\Rightarrow a_x = \frac{m_2 g(-\mu_k \cos \alpha_2 + \sin \alpha_2) - m_1 g(\mu_k \cos \alpha_1 + \sin \alpha_1)}{(m_1 + m_2)}$$

$$1) \quad \mu_{1s} = \mu_{2s} = 0 \Rightarrow a_x = \frac{m_2 g(-\mu_k \cos \alpha_2 + \sin \alpha_2) - m_1 g(\mu_k \cos \alpha_1 + \sin \alpha_1)}{(m_1 + m_2)} = \frac{m_2 g \sin \alpha_2 - m_1 g \sin \alpha_1}{(m_1 + m_2)} //$$

$$2) \quad \mu_{1s} = \mu_{2s} = \mu_k \neq 0 \Rightarrow a_x = \frac{m_2 g(-\mu_k \cos \alpha_2 + \sin \alpha_2) - m_1 g(\mu_k \cos \alpha_1 + \sin \alpha_1)}{(m_1 + m_2)} //$$

$$3) \quad \mu_{1s} \neq \mu_{2s} \neq 0 \Rightarrow a_x = \frac{m_2 g(-\mu_k \cos \alpha_2 + \sin \alpha_2) - m_1 g(\mu_k \cos \alpha_1 + \sin \alpha_1)}{(m_1 + m_2)} //$$

* Largo de la cuerda: $l = \bar{x}_1 + l_{\text{polea}} + \bar{x}_2$, derivando respecto t

$$\Rightarrow 0 = -\dot{\bar{x}}_1 + \dot{\bar{x}}_2 \xrightarrow{\text{derivando}} 0 = -\ddot{\bar{x}}_1 + \ddot{\bar{x}}_2 \Rightarrow \ddot{\bar{x}}_1 = \ddot{\bar{x}}_2$$

* Ninguno de los cuerpos se va a mover en sus ejes y, permanecen en reposo en este eje, por lo que

$$a_{1y} = a_{2y} = 0$$

* La fricción cinética se define como: $f_k = N \mu_k$

* Hay que eliminar la variable T

Problema 9.

Considera una partícula que percibe una fuerza angular solamente, de la forma $F_\theta = 2mr\dot{\theta}$. Demuestra que $r = Ae^\theta + Be^{-\theta}$, donde A y B son constantes, determinadas por las condiciones iniciales.

$$\ddot{r} = (\ddot{\rho} - \rho\dot{\theta}^2)\hat{e}_\rho + (\rho\ddot{\theta} + 2\dot{\rho}\dot{\theta})\hat{e}_\theta$$

$$\sum \vec{F} = \vec{F}_\theta \Rightarrow \vec{F}_\theta = m\ddot{r} \Rightarrow 2mr\ddot{\theta} = m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) \Rightarrow mr\ddot{\theta} = mr\frac{d\dot{\theta}}{dt} = 0 \Rightarrow \dot{\theta} = \text{constante} = c$$

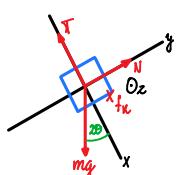
$$\Rightarrow \ddot{r} = (\ddot{\rho} - \rho c^2)\hat{e}_\rho \Rightarrow \ddot{\rho} - \rho c^2 = 0 \Rightarrow \rho = Ae^{ct} + B^{-ct} \Rightarrow \rho = Ae^{\theta} + B^{-\theta}$$

Aceleración en \hat{e}_ρ igual a cero

$$m^2 - c^2 = 0 \\ m^2 = c^2 \Rightarrow m = \pm c$$

Problema 10.

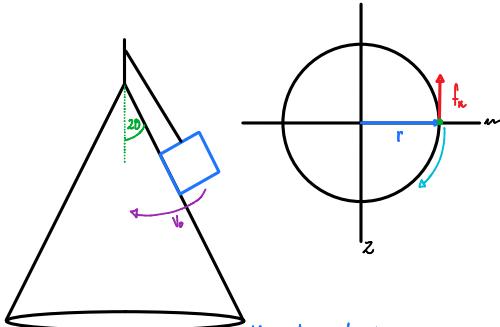
Un cono subtende un ángulo 2θ . Un bloque de masa m está conectado a la punta por una cuerda sin masa y se mueve en un círculo horizontal de radio R alrededor de la superficie externa. Si la rapidez inicial es v_0 , y si el coeficiente de fricción cinética entre el bloque y el cono es μ_k , ¿cuánto tiempo tarda el bloque en detenerse?



$$\sum F_x = -T + mg\cos(2\theta) = 0 \Rightarrow T = mg\cos(2\theta)$$

$$\sum F_y = N - mg\sin(2\theta) = 0 \Rightarrow N = mg\sin(2\theta)$$

$$\sum F_z = -f_n = ma_z \Rightarrow -\mu_k N g \sin(2\theta) = ma_z \Rightarrow a_z = -\mu_k g \sin(2\theta)$$



En el instante en que m se halla en un punto preso

Durante el giro de m , la fricción estará en la dirección tangencial al círculo horizontal por el que se mueve m .

$$\sum \vec{F} = m\ddot{r} \Rightarrow -f_n \hat{e}_\theta = m\ddot{r} \Rightarrow -\mu_k N g \sin(2\theta) \hat{e}_\theta = \ddot{r} \Rightarrow -\mu_k N g \sin(2\theta) = \frac{d\dot{\theta}}{dt} \Rightarrow \dot{\theta} - v_0 = -\mu_k g \sin(2\theta)t$$

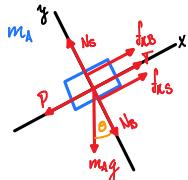
$$\Rightarrow \dot{\theta} = -\mu_k g \sin(2\theta)t + v_0; \text{ la partícula se detiene cuando } \dot{\theta} = 0$$

$$\Rightarrow 0 = -\mu_k g \sin(2\theta)t + v_0 \Rightarrow t = \frac{v_0}{\mu_k g \sin(2\theta)}$$

Problema 11.

El bloque A tiene una masa de 25 kg y el bloque B de 15 kg. El coeficiente de fricción cinética entre las superficies en contacto es de 0.15. Sabiendo que el ángulo del plano inclinado es de 25° . Una fuerza horizontal \vec{P} jala al cuadro A hacia abajo del plano inclinado, con una magnitud de 250 N y es capaz de acelerar al bloque B, determine la aceleración de cada bloque.

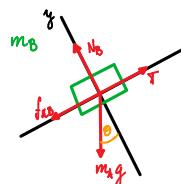
Se supone polea perfecta y cuerda perfecta



Para m_A :

$$\sum F_{Ax} = T - mg \sin \theta + f_{Ax} + f_{Ay} - P = m_A a_{Ax} \quad (1)$$

$$\sum F_{Ay} = N_A - mg \cos \theta - N_B = m_A g \cos \theta \Rightarrow N_A = m_A g \cos \theta + N_B$$



Para m_B : $\Rightarrow N_B = (m_A + m_B) g \cos \theta$

$$\sum F_{Bx} = T - mg \sin \theta - f_{Bx} = m_B a_{Bx} \quad (2)$$

$$\sum F_{By} = N_B - mg \cos \theta = m_B g \cos \theta \Rightarrow N_B = m_B g \cos \theta$$

Despejando T de 1 y 2:

$$\begin{aligned} (1) \Rightarrow T &= mg \sin \theta - f_{Ax} - f_{Ay} + P + m_A a_{Ax} \\ &= mg \sin \theta - \mu_k (m_A + m_B) g \cos \theta - \mu_k m_B g \cos \theta + P + m_A a_{Ax} \end{aligned}$$

$$(2) \Rightarrow T = mg \sin \theta + f_{Bx} + m_B a_{Bx} = mg \sin \theta + \mu_k m_B g \cos \theta + m_B a_{Bx}$$

Igualando 1 y 2:

$$mg \sin \theta - \mu_k (m_A + m_B) g \cos \theta - \mu_k m_B g \cos \theta + P + m_A a_{Ax} = mg \sin \theta + \mu_k m_B g \cos \theta + m_B a_{Bx}$$

$$(m_A - m_B) g \sin \theta - \mu_k (m_A + m_B) g \cos \theta - 2\mu_k m_B g \cos \theta + P = m_B a_{Bx} - m_A a_{Ax} = a_B (m_A + m_B)$$

$$\Rightarrow a_B = \frac{(m_A - m_B) g \sin \theta - \mu_k (m_A + m_B) g \cos \theta - 2\mu_k m_B g \cos \theta + P}{(m_A + m_B)} = 4,95 \text{ m/s}^2 //$$

$$\Rightarrow a_A = -a_B = -4,95 \text{ m/s}^2 //$$

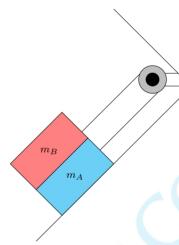


Figura 1: Configuración del problema.

Los bloques no poseen movimiento en el eje y del marco *

La fricción cinética se define como: $f_k = \mu_k N$

$$\Rightarrow f_{Ax} = \mu_k N_B = \mu_k m_B g \cos \theta$$

$$\Rightarrow f_{Ay} = \mu_k N_A = \mu_k (m_A + m_B) g \cos \theta$$

Largo de la cuerda l

$$l = x_B - x_A + c_p$$

$$\text{Derivando respecto a } t \Rightarrow 0 = -\dot{x}_B - \dot{x}_A$$

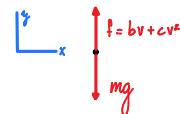
$$\text{De nuevo } \Rightarrow 0 = -\ddot{x}_B - \ddot{x}_A$$

$$\Rightarrow -\ddot{x}_A = \ddot{x}_B$$

Problema 12.

Determine la velocidad terminal cuando la fuerza de resistencia incluye los términos lineal y cuadrático.

$\sum F_y = bv + cv^2 - mg = may$ La velocidad terminal se obtiene cuando $ay = 0$.



$$\Rightarrow bv + cv^2 - mg = 0 \Rightarrow v_{ter} = \frac{-b \pm \sqrt{b^2 + 4mgc}}{2c} = \frac{-b \pm \sqrt{b^2 + 4mgc}}{2c}$$

$$\Rightarrow v_{ter} = \frac{-b + \sqrt{b^2 + 4mgc}}{2c}$$

Problema 13. (Taylor 2.40)

2.40 ** Consider an object that is coasting horizontally (positive x direction) subject to a drag force $f = -bv - cv^2$. Write down Newton's second law for this object and solve for v by separating variables. Sketch the behavior of v as a function of t . Explain the time dependence for t large. (Which force term is dominant when t is large?)

$$\sum F_x = -bv - cv^2 = max = m \frac{dv_x}{dt} \Rightarrow \frac{dv_x}{dt} = \frac{-bv - cv^2}{m} \Rightarrow \int_{v_{ter}}^{v_x} \frac{1}{bv + cv^2} dv_x = \int_0^t \frac{-1}{m} dt = \frac{-t}{m}$$

$$\int_{v_{ter}}^{v_x} \frac{1}{bv + cv^2} dv_x = \int_{v_{ter}}^{v_x} \frac{1}{v(b + cv)} dv_x = \int_{v_{ter}}^{v_x} \frac{A}{v} + \frac{B}{b + cv} dv_x \star \Rightarrow \frac{A}{v} + \frac{B}{b + cv} = \frac{1}{v(b + cv)}$$

$$\Rightarrow \frac{A(b + cv)}{v(b + cv)} + Bv = \frac{1}{v(b + cv)} \Rightarrow A(b + cv) + Bv = 1 \Rightarrow Ab + Acv + Bv = Ab + v(Ac + B) = 1$$

$$\Rightarrow \begin{cases} Ab = 1 \Rightarrow A = \frac{1}{b} \\ Ac + B = 0 \Rightarrow -\frac{1}{b} + B = 0 \Rightarrow B = \frac{1}{b} \end{cases} \star \Rightarrow \int_{v_{ter}}^{v_x} \frac{\frac{1}{b}}{v} + \frac{\frac{1}{b}}{b + cv} dv_x = \int_{v_{ter}}^{v_x} \frac{1}{bv} - \frac{1}{b(b + cv)} dv_x = \frac{1}{b} \int_{v_{ter}}^{v_x} \frac{1}{v} dv - \frac{1}{b} \int_{v_{ter}}^{v_x} \frac{1}{b + cv} dv_x$$

$$= \frac{1}{b} \left[\ln(v_x) \Big|_{v_{ter}}^{v_x} - \int_{v_{ter}}^{v_x} \frac{1}{b + cv} dv_x \right] \star; u = b + cv \Rightarrow du = c dv \Rightarrow \int \frac{1}{b + cv} dv = \int \frac{1}{b} \frac{1}{u} du = \ln(u) + C = \ln(b + cv) + C$$

$$\star \Rightarrow \frac{1}{b} \left[\ln(v_x) \Big|_{v_{ter}}^{v_x} - \int_{v_{ter}}^{v_x} \frac{1}{b + cv} dv_x \right] = \frac{1}{b} \left[\ln(v_x) - \ln(b + cv) \right] \Big|_{v_{ter}}^{v_x} = \frac{1}{b} \ln \left(\frac{v_x}{b + cv} \right) \Big|_{v_{ter}}^{v_x} = \frac{1}{b} \ln \left(\frac{v_x}{b + cv_{ter}} \right) - \frac{1}{b} \ln \left(\frac{v_{ter}}{b + cv_{ter}} \right)$$

$$\Rightarrow \frac{1}{b} \ln \left(\frac{v_x}{b + cv} \right) - \frac{1}{b} \ln \left(\frac{v_{ter}}{b + cv_{ter}} \right) = \frac{-t}{m} \Rightarrow \ln \left(\frac{v_x}{b + cv} \right) - \ln \left(\frac{v_{ter}}{b + cv_{ter}} \right) = \ln \left[\frac{(b + cv)(v_x)}{v_{ter}(b + cv_{ter})} \right] = \frac{-bt}{m}$$

$$\Rightarrow \frac{(b + cv)(v_x)}{v_{ter}(b + cv_{ter})} = e^{\frac{-bt}{m}} \Rightarrow \frac{v_x}{b + cv} = \frac{v_{ter}}{b + cv_{ter}} e^{\frac{-bt}{m}} \Rightarrow \frac{b + cv}{V_x} = \frac{b}{V_x} + c = \frac{b + cv_{ter}}{V_{ter}} e^{\frac{-bt}{m}} \Rightarrow \frac{b}{V_x} = \frac{b + cv_{ter}}{V_{ter}} e^{\frac{-bt}{m}} - c$$

$$\Rightarrow \frac{b}{V_x} = \frac{b + cv_{ter}}{V_{ter}} e^{\frac{-bt}{m}} - c \frac{V_x}{V_{ter}} \Rightarrow V_x = \frac{bV_{ter}}{b + cv_{ter} e^{\frac{-bt}{m}} - cV_{ter}}$$

$$V_x \approx \frac{b}{t}$$

Cuando t es grande V_x va a ser pequeño, restable importancia a " cV_x^{-1} " de la fuerza de arrastre

Energía y teoremas de conservación

Problema 1. (Thornton 2.25)

- 2-25. A block of mass $m = 1.62 \text{ kg}$ slides down a frictionless incline (Figure 2-A). The block is released a height $h = 3.91 \text{ m}$ above the bottom of the loop.
- What is the force of the inclined track on the block at the bottom (point A)?
 - What is the force of the track on the block at point B?
 - At what speed does the block leave the track?
 - How far away from point A does the block land on level ground?
 - Sketch the potential energy $U(x)$ of the block. Indicate the total energy on the sketch.

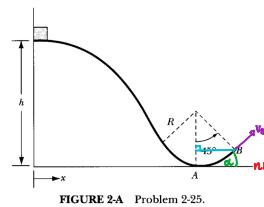


FIGURE 2-A Problem 2-25.

a)

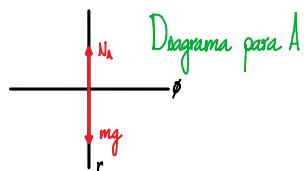


Diagrama para A

$$\sum F_r = mg - N_A = ma_{\text{rad}} \Rightarrow mg - N_A = -m \frac{v^2}{R} \Rightarrow N_A = mg + m \frac{v^2}{R}$$

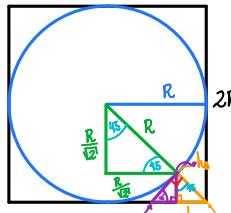
Hay que hallar v en A: $\Delta E = 0 \Rightarrow T_A + U_{g0} = T_A + U_{gA} \Rightarrow mgh = \frac{1}{2}mv_A^2 \Rightarrow v_A^2 = 2gh \star \Rightarrow N_A = mg \left(1 + \frac{2}{R} h \right)$



$$\sum F_r = m g \cos \theta - N_B = ma_{\text{rad}} \Rightarrow m g \cos \theta - N_B = -m \frac{v^2}{R} \Rightarrow N_B = m \left(\cos \theta + \frac{v^2}{R} \right) \star$$

$$\sum F_\theta = -mg \sin \theta = ma_\theta$$

Hay que hallar v en B: $\Delta E = 0 \Rightarrow T_A + U_{g0} = T_B + U_{gB} \Rightarrow mgh = \frac{1}{2}mv_B^2 + mgh_B \Rightarrow v_B^2 = 2g(h - h_B) \star$



$$R = \frac{R}{\sqrt{2}} + h_B \Rightarrow h_B = R \left(1 - \frac{1}{\sqrt{2}} \right) \star \Rightarrow N_B = mg \left(\cos \theta + \frac{2}{R} \left[h - R \left(1 - \frac{1}{\sqrt{2}} \right) \right] \right)$$

$$h_B = b_0 \text{ por triángulo } 45^\circ$$

c) $\star v_B^2 = 2g(h - h_B) \Rightarrow v_B = \sqrt{2g \left[h - R \left(1 - \frac{1}{\sqrt{2}} \right) \right]} = \sqrt{2gh - gh\sqrt{2}(\sqrt{2}-1)} = \sqrt{gh[2-\sqrt{2}(\sqrt{2}-1)]} = \sqrt{2gh}$

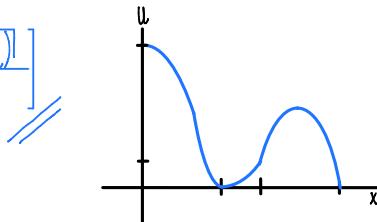
d) $v_B = v_0 \quad \alpha = 45^\circ \Rightarrow y_t = h_B + v_0 \sin \alpha \cdot t - \frac{1}{2}gt^2 \xrightarrow{t_{\text{máx}}} h_B + v_0 \sin \alpha \cdot t - \frac{1}{2}gt^2 = 0$

Es la solución "+" para $t > 0$

$$\Rightarrow t = \frac{-v_0 \sin \alpha \pm \sqrt{v_0^2 \sin^2 \alpha - 4 \cdot \frac{1}{2} g \cdot h_B}}{2 \cdot \frac{1}{2} g} = \frac{+v_0 \sin \alpha \pm \sqrt{v_0^2 \sin^2 \alpha + 2gh_B}}{g} \Rightarrow t = \frac{v_0 \sin \alpha + \sqrt{v_0^2 \sin^2 \alpha + 2gh_B}}{g}$$

$$\Rightarrow t = \frac{\sqrt{2}gh \sin \alpha + \sqrt{2}gh \sin^2 \alpha + \sqrt{2}gR(\sqrt{2}-1)}{g} = \frac{\sqrt{2}gh \sin \alpha + \sqrt{2}gh \sin^2 \alpha + \sqrt{2}gR(\sqrt{2}-1)}{g}$$

$$x_t = x_0 + v_0 t = R \sin \alpha + \sqrt{2}gh \cos \alpha \cdot \left[\frac{\sqrt{2}gh \sin \alpha + \sqrt{2}gh \sin^2 \alpha + \sqrt{2}gR(\sqrt{2}-1)}{g} \right]$$



Problema 2. (Thornton 2.27)

- 2-27. A rope having a total mass of 0.4 kg and total length 4 m has 0.6 m of the rope hanging vertically down off a work bench. How much work must be done to place all the rope on the bench?

$$\text{densidad lineal } \rho = \frac{m}{l} \Rightarrow m(l) = \rho l \Rightarrow \vec{F} = -\rho g(l-y)\hat{j}$$

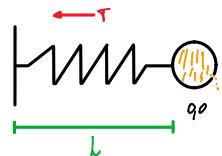
$$\text{Movimiento unidimensional} \Rightarrow d\vec{r} = dy\hat{j}$$

$$\text{Trabajo necesario para subir el extremo de la cuerda: } W = \int_c \vec{F} \cdot d\vec{r} = \int_0^{L_0} -\rho g(l-y) dy = \frac{\rho g(l-y)^2}{2} \Big|_0^{L_0} = -\frac{\rho g L^2}{2} \leftarrow \begin{array}{l} \text{Trabajo por la} \\ \text{gravedad} \end{array}$$

$$\Rightarrow W_e = -W = \frac{\rho g L^2}{2} = 0,18 \text{ J necesarios para levantar la cuerda} //$$

Problema 3.

En $t = 0$, un balde sin masa contiene una masa inicial M de arena. Está conectado a una pared por un resorte sin masa con tensión constante T (es decir, independiente de la longitud). El suelo no tiene fricción y la distancia inicial a la pared es L . El balde se suelta y, en su camino hacia la pared, pierde arena a razón de $dm/dx = M/L$, de tal forma que en un tiempo t la masa es m y x es la distancia con respecto a la pared. Por lo tanto, la tasa con la que pierde masa es constante con respecto a la distancia, no al tiempo. Además considere que el balde termina vacío justo cuando llega a la pared.



a. Determine la energía cinética del balde y la arena, como función de x . ¿Dónde es máxima?

b. ¿Cuál es el momentum lineal del balde? ¿Dónde es máximo?

a) El sistema es (balde + arena dentro) + arena fuera

$$p_T = m_b V_{ba} + m_a V_a \Rightarrow p_T = \cancel{\frac{dm_b}{dt}} V_{ba} + \cancel{m_a} \cancel{\frac{dm_a}{dt}} V_a + \cancel{\frac{dm_a}{dt}} V_a + m_a \cancel{\frac{dm_a}{dt}} \stackrel{\text{En cada instante que es expulsada tiene la velocidad del balde y es constante del acer}}{=} ; \frac{dm_a}{dt} = -\frac{dm_a}{dt} = -\frac{dm}{dt}$$

$$\Rightarrow \dot{p}_T = m_a \cancel{\frac{dV_{ba}}{dt}} = -T ; \frac{dm}{dx} = \frac{M}{L} \Rightarrow \int_0^m dm = \int_0^x \frac{M}{L} dx \Rightarrow m = \frac{M}{L} x$$

$$\Rightarrow \frac{M}{L} x \cancel{\frac{dV_{ba}}{dt}} = -T = \frac{M}{L} \cancel{\frac{dV}{dt}} \cancel{\frac{dx}{dx}} = \frac{M}{L} X v \cancel{\frac{dv}{dx}} \Rightarrow -T = \frac{M}{L} X v \cancel{\frac{dv}{dx}} \Rightarrow \int_L^X \frac{-T}{M} dx = \int_0^x v dv$$

$$\Rightarrow \frac{v^2}{2} = \frac{-T}{M} \ln\left(\frac{X}{L}\right) \Rightarrow v = \sqrt{-\frac{2T}{M} \ln\left(\frac{X}{L}\right)} \Rightarrow T_e = \frac{1}{2} m v^2 = \frac{1}{2} \cancel{M} X \cdot \cancel{\frac{-T}{M} \ln\left(\frac{X}{L}\right)} = -T X \ln\left(\frac{X}{L}\right)$$

$$\Rightarrow T_e = -T X \ln\left(\frac{X}{L}\right) //$$

$$\Rightarrow \text{Máximo } \frac{dT_e}{dx} = 0 \Rightarrow -T \ln\left(\frac{X}{L}\right) - T X \cancel{\frac{1}{X}} = -T \ln\left(\frac{X}{L}\right) - T X \cancel{\frac{1}{X}} = -T \ln\left(\frac{X}{L}\right) - T L = 0 \Rightarrow X_{e,\max} = L e^L //$$

$$b) p_{ba} = m_b v = \frac{M}{L} x \cdot \sqrt{-\frac{2T}{M} \ln\left(\frac{X}{L}\right)} = \sqrt{\frac{M^2 X^2}{L^2} \cdot \cancel{\frac{-2T}{M} \ln\left(\frac{X}{L}\right)}} = \sqrt{\frac{2TM}{L}} \cdot \sqrt{-X^2 \ln\left(\frac{X}{L}\right)} //$$

$$\frac{dp_{ba}}{dx} = \sqrt{\frac{2TM}{L}} \cdot \frac{1}{2 \sqrt{-X^2 \ln\left(\frac{X}{L}\right)}} \cdot \left(-2X \ln\left(\frac{X}{L}\right) - X \cancel{\frac{1}{X}} \right) = \sqrt{\frac{2TM}{L}} \cdot \frac{1}{2 \sqrt{-X^2 \ln\left(\frac{X}{L}\right)}} \cdot \left(-2X \ln\left(\frac{X}{L}\right) - X L \right)$$

$$\text{Máximo } \frac{dp_{ba}}{dx} = 0 \Rightarrow \sqrt{\frac{2TM}{L}} \cdot \frac{1}{2 \sqrt{-X^2 \ln\left(\frac{X}{L}\right)}} \cdot \left(-2X \ln\left(\frac{X}{L}\right) - X L \right) = 0 \Rightarrow -2X \ln\left(\frac{X}{L}\right) - X L = 0$$

$$\Rightarrow \ln\left(\frac{X}{L}\right) = -\frac{L}{2} \Rightarrow X_{p,\max} = L e^{-\frac{L}{2}} //$$

Problema 4. (Thornton 2.41)

- 2.41. A train moves along the tracks at a constant speed u . A woman on the train throws a ball of mass m straight ahead with a speed v with respect to herself. (a) What is the kinetic energy gain of the ball as measured by a person on the train? (b) by a person standing by the railroad track? (c) How much work is done by the woman throwing the ball and (d) by the train?

$$a) T_m = \frac{1}{2}mv^2 //$$

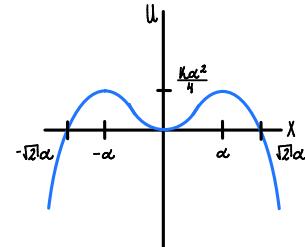
$$b) T_r = \frac{1}{2}m(u+v)^2 - \frac{1}{2}mu^2 = \frac{1}{2}mv^2 + muv //$$

$$c) W_m = \Delta T \Rightarrow W_m = \frac{1}{2}mv^2 // \quad d) W_t = W_r + W_m = \frac{1}{2}mv^2 + muv \Rightarrow W_r = \frac{1}{2}mv^2 + muv - \frac{1}{2}mv^2 = muv //$$

Problema 5. (Thornton 2.43)

- 2.43. A particle is under the influence of a force $F = -kx + kx^3/\alpha^2$, where k and α are constants and k is positive. Determine $U(x)$ and discuss the motion. What happens when $E = (1/4)k\alpha^2$?

$$F = -\nabla U \Rightarrow F = -\frac{dU}{dx} \Rightarrow -\int_0^x dU = \int_0^x -kx + \frac{kx^3}{\alpha^2} dx \Rightarrow U = \frac{1}{2}kx^2 - \frac{kx^4}{4\alpha^2}$$



Cuando $E = \frac{k\alpha^2}{4}$ es la máxima energía potencial, es un punto de equilibrio inestable. //

Problema 6.

Considera la máquina de Atwood de la Figura 4.15 (ver libro de texto), pero suponga que la polea tiene radio R y momento de inercia I .

- Escriba la energía total de las dos masas y la polea en términos de la coordenada x y \dot{x} . (Recuerde que la energía cinética de un disco es $I\omega^2/2$).
- Demuestre que puede obtener la ecuación de movimiento para la coordenada x derivando la ecuación $E = \text{cte}$. Compruebe que la ecuación de movimiento es la misma que obtendría aplicando la II Ley de Newton por separado a las dos masas y la polea, y luego eliminando las dos tensiones desconocidas de las tres ecuaciones resultantes.

$$a) \text{La polea tiene energía potencial cero} \Rightarrow T_3 = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}I\dot{\theta}_3^2 \quad y \quad U = m_1g\dot{x}_1 + m_2g\dot{x}_2$$

$$\Rightarrow E_s = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}I\dot{\theta}_3^2 + m_1g\dot{x}_1 + m_2g\dot{x}_2$$

$$b) \frac{dE_s}{dt} = m_1\dot{x}_1\ddot{x}_1 + m_2\dot{x}_2\ddot{x}_2 + I\dot{\theta}_3\ddot{\theta}_3 + m_1g\dot{x}_1 + m_2g\dot{x}_2 = 0$$

$$\Rightarrow m_1\dot{x}_1\ddot{x}_1 + m_2\dot{x}_2\ddot{x}_2 + I\dot{\theta}_3\ddot{\theta}_3 + m_1g\dot{x}_1 - m_2g\dot{x}_2 = 0 \Rightarrow \ddot{x}_1\left(m_1 + m_2 + \frac{I}{R^2}\right) + g(m_2 - m_1) = 0 //$$

$$\sum F_1 = T_1 - m_1g = m_1a_1 \Rightarrow T_1 = m_1a_1 + m_1g \quad y \quad \sum F_2 = T_2 - m_2g = m_2a_2 \Rightarrow T_2 = m_2a_2 + m_2g$$

$$\sum T = -T_1R + T_2R = I\alpha \Rightarrow -m_1a_1 - m_2a_2 + m_2g = \frac{I\alpha}{R^2} \Rightarrow -a_1\left(m_1 + m_2 + \frac{I}{R^2}\right) + g(m_2 - m_1) = 0 //$$

$$S = R\theta \\ V = R\dot{\theta}$$

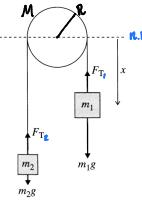


Figure 4.15 An Atwood machine consisting of two masses, m_1 and m_2 , suspended by a massless inextensible string that passes over a massless, frictionless pulley. Because the string's length is fixed, the position of the whole system is specified by the distance x of m_1 below any convenient fixed level. The forces on the two masses are their weights m_1g and m_2g , and the tension forces F_T (which are equal since the pulley and string are massless).

$$a_1 = -a_2 \quad y \quad \alpha = \frac{a}{R}$$

$$\dot{\theta}_3 = \frac{\dot{x}_2}{R} \quad y \quad \ddot{x}_1 = -\ddot{x}_2$$

(Taylor 4.36)

4.36 ** A metal ball (mass m) with a hole through it is threaded on a frictionless vertical rod. A massless string (length l) attached to the ball runs over a massless, frictionless pulley and supports a block of mass M , as shown in Figure 4.27. The positions of the two masses can be specified by the one angle θ . (a) Write down the potential energy $U(\theta)$. (The PE is given easily in terms of the heights shown as h and H . Eliminate these two variables in favor of θ and the constants b and l . Assume that the pulley and ball have negligible size.) (b) By differentiating $U(\theta)$ find whether the system has an equilibrium position, and for what values of m and M equilibrium can occur. Discuss the stability of any equilibrium positions.

$$l = H + d = H + \sqrt{b^2 + h^2} = -X_m + \sqrt{b^2 + X_m^2} = -X_m + \sqrt{b^2 + b^2 \cot^2 \theta} = -X_m + b \csc \theta$$

$$\Rightarrow X_n = b \csc \theta - l \quad y \quad X_m = -b \cot \theta \Rightarrow X_n = -X_m - l$$

$$a) U = U_m + U_n = mgX_m + MgX_n = -mgb \cot \theta + Mg(b \csc \theta - l)$$

$$b) \frac{dU}{d\theta} = mgb \csc^2 \theta - Mg b \csc \theta \cot \theta = 0 \Rightarrow \csc \theta (mg \csc \theta - Mg \cot \theta) = 0$$

$$\Rightarrow \csc \theta = 0 \vee mg \csc \theta - Mg \cot \theta = 0 \Rightarrow m \csc \theta = M \cot \theta \Rightarrow m = M \cos \theta \Rightarrow \theta = \cos^{-1}\left(\frac{m}{M}\right)$$

$$m = Mh; \text{ con } 0 < h \leq l \Rightarrow \theta = \cos^{-1}(h) \therefore \text{Hay un punto equilibrio, depende de que tanta sea la diferencia de masa}$$

c) Halle la velocidad del sistema cuando M se mueve $\frac{l}{5}$ de su posición inicial suponiendo que $M > m$

$$X_n = b \csc \theta - l \Rightarrow V_n = -b \cot \theta \csc \theta \cdot \dot{\theta} \quad y \quad X_m = -b \cot \theta \Rightarrow V_m = -b \csc^2 \theta \cdot \dot{\theta}$$

$$\Rightarrow T_t = \frac{1}{2} m b \csc^4 \theta \dot{\theta}^2 + \frac{1}{2} M b \cot^2 \theta \csc^2 \theta \dot{\theta}^2 \quad y \quad U_t = -mgb \cot \theta + Mg(b \csc \theta - l)$$

$$U_s = -mgb \cot \theta_0 + Mg(b \csc \theta_0 - l)$$

$$\Rightarrow U_s = U_t + T_t$$

$$\Rightarrow -mgb \cot \theta_0 + Mg(b \csc \theta_0 - l) = \frac{1}{2} m b \csc^4 \theta_0 \dot{\theta}_0^2 + \frac{1}{2} M b \cot^2 \theta_0 \csc^2 \theta_0 \dot{\theta}_0^2 - mgb \cot \theta_0 + Mg(b \csc \theta_0 - l)$$

$$\Rightarrow \frac{1}{2} b^2 \csc^2 \theta_0 (m \csc^2 \theta_0 + M \cot^2 \theta_0) \cdot \dot{\theta}_0^2 + mg(b \cot \theta_0 - \cot \theta_s) + Mg(b \csc \theta_0 - \csc \theta_s) = 0$$

$$\tan \theta = \frac{b}{h} \Rightarrow h = b \cot \theta$$

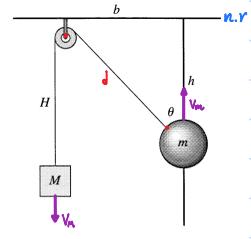


Figure 4.27 Problem 4.36 $d = l - H$

$$(d - \frac{l}{5}) \cos \theta = h$$

Problema 7. (Taylor 4.26)

4.26★ A mass m is in a uniform gravitational field, which exerts the usual force $F = mg$ vertically down, but with g varying with time, $g = g(t)$. Choosing axes with y measured vertically up and defining $U = mgy$ as usual, show that $\mathbf{F} = -\nabla U$ as usual, but, by differentiating $E = \frac{1}{2}mv^2 + U$ with respect to t , show that E is not conserved.

$$U(y, t) = - \oint_y \vec{F}(y, t) \cdot d\vec{y} = - \int_0^y -mg(t) dy = +mg(t)y \Rightarrow \vec{F}(y, t) = - \frac{\partial U}{\partial y}$$

$$E = \frac{1}{2}m\dot{y}^2 + mg(t)y \Rightarrow \frac{dE}{dt} = \frac{1}{2}m2\dot{y}\ddot{y} + m \left[\frac{dg(t)}{dt} y + g(t)\dot{y} \right]$$

$$\Rightarrow \frac{dE}{dt} = m(\dot{y} + g(t))\ddot{y} + m \frac{dg(t)}{dt} y \Rightarrow \frac{dE}{dt} = m \frac{dg(t)}{dt} y = \frac{\partial U}{\partial t}$$

Problema 8. (Thornton 2.52)

- 2-52.** A particle of mass m moving in one dimension has potential energy $U(x) = U_0[2(x/a)^2 - (x/a)^4]$, where U_0 and a are positive constants. (a) Find the force $F(x)$, which acts on the particle. (b) Sketch $U(x)$. Find the positions of stable and unstable equilibrium. (c) What is the angular frequency ω of oscillations about the point of stable equilibrium? (d) What is the minimum speed the particle must have at the origin to escape to infinity? (e) At $t = 0$ the particle is at the origin and its velocity is positive and equal in magnitude to the escape speed of part (d). Find $x(t)$ and sketch the result.

Problema 9.

Un resorte liviano de longitud natural a se coloca sobre un piso horizontal en posición vertical. Cuando un bloque de masa M descansa en equilibrio sobre el resorte, la compresión del resorte es $a/15$. El bloque ahora se levanta a una altura de $3a/2$ sobre el piso y se suelta desde el reposo. Encuentre la compresión del resorte cuando el bloque se detiene por primera vez.

$$E=0 \text{ con } X_0 = a(1 - \frac{1}{15})$$

$$\Rightarrow U_{\text{alto}} = \frac{1}{2}k\left(\frac{3a}{2} - a(1 - \frac{1}{15})\right)^2 + mg\left(\frac{3a}{2} - a(1 - \frac{1}{15})\right) = \frac{1}{2}ka^2 \frac{289}{900} + mga \frac{17}{30}$$

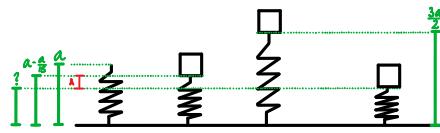
$$y \quad U_{\text{bajo}} = \frac{1}{2}k\left(X - a(1 - \frac{1}{15})\right)^2 + mg\left(X - a(1 - \frac{1}{15})\right) \Rightarrow U_{\text{alto}} = U_{\text{bajo}} \quad ; \quad A = X - a(1 - \frac{1}{15})$$

$$\Rightarrow \frac{1}{2}ka^2 \frac{289}{900} + mga \frac{17}{30} = \frac{1}{2}kA^2 + mgA \Rightarrow \frac{1}{2}kA^2 + mgA - \frac{1}{2}ka^2 \frac{289}{900} - mga \frac{17}{30} = 0$$

$$\Rightarrow A = \frac{-mg \pm \sqrt{m^2g^2 - 4 \cdot \frac{1}{2}k \cdot \frac{(289ka^2 + 1020mga)}{1800}}}{2 \cdot \frac{1}{2}k} = \frac{-mg \pm \sqrt{m^2g^2 + 2k \frac{(289ka^2 + 1020mga)}{1800}}}{k}$$

La solución buscada es la negativa porque se quiere el resorte comprimido

$$X - a(1 - \frac{1}{15}) = \frac{-mg - \sqrt{m^2g^2 + 2k \frac{(289ka^2 + 1020mga)}{1800}}}{k}$$



Problema 10.

Una partícula de masa $m = 1.0 \text{ kg}$ se mueve en un campo de fuerza dado por $\vec{F} = [(3t^2 - 4t)\hat{x} + (12t - 6)\hat{y} + (6t - 12t^2)\hat{z}] \text{ N}$. La velocidad en $t = 1.0 \text{ s}$ es $(4\hat{x} - 5\hat{y} + 10\hat{z}) \text{ m/s}$ y en $t = 0$ pasa por el origen.

- Encontrar el cambio de momentum de la partícula desde el tiempo $t = 1.0 \text{ s}$ a 2.0 s .
- ¿Cuál es la velocidad en $t = 2.0 \text{ s}$?
- Determine el torque en $t = 2.0 \text{ s}$, con respecto al origen del sistema de referencia.
- Con respecto al origen, también determine el momentum angular en $t = 2.0 \text{ s}$.

$$a) \vec{F} = \frac{d\vec{p}}{dt} \Rightarrow \int_1^2 (3t^2 - 4t)\hat{x} + (12t - 6)\hat{y} + (6t - 12t^2)\hat{z} dt = \int_{\vec{p}_1}^{\vec{p}_2} d\vec{p} \Rightarrow \Delta\vec{p} = (t^3 - 2t^2)\hat{x} + 6t^2\hat{y} + (3t^2 - 4t^3)\hat{z} \Big|_1^2 \\ = (1\hat{x} + 12\hat{y} - 19\hat{z}) \text{ kg m/s} //$$

b) Primera forma

$$\vec{p}(t=2) = \Delta\vec{p} + \vec{p}(t=1) = (1\hat{x} + 12\hat{y} - 19\hat{z}) \text{ kg m/s} + 1\text{kg} \cdot (4\hat{x} - 5\hat{y} + 10\hat{z}) = (5\hat{x} + 7\hat{y} - 9\hat{z}) \text{ kg m/s} \\ \Rightarrow \vec{v}(t=2) = \frac{\vec{p}(t=2)}{m} = (5\hat{x} + 7\hat{y} - 9\hat{z}) \text{ m/s} //$$

Segunda forma

$$\frac{d\vec{v}}{dt} = \frac{\vec{F}}{m} = (3t^2 - 4t)\hat{x} + (12t - 6)\hat{y} + (6t - 12t^2)\hat{z} \Rightarrow \int \vec{v} \, d\vec{v} = \int (3t^2 - 4t)\hat{x} + (12t - 6)\hat{y} + (6t - 12t^2)\hat{z} \, dt \\ \Rightarrow \vec{v} - (4\hat{x} - 5\hat{y} + 10\hat{z}) = (t^3 - 2t^2)\hat{x} + (6t^2 - 6t)\hat{y} + (3t^2 - 4t^3)\hat{z} \Big|_1^2 = (t^3 - 2t^2)\hat{x} + (6t^2 - 6t)\hat{y} + (3t^2 - 4t^3)\hat{z} - (-1\hat{x} + 0\hat{y} - 1\hat{z}) \\ \Rightarrow \vec{v} = (t^3 - 2t^2)\hat{x} + (6t^2 - 6t)\hat{y} + (3t^2 - 4t^3)\hat{z} + (4\hat{x} - 5\hat{y} + 10\hat{z}) - (-1\hat{x} + 0\hat{y} - 1\hat{z}) \\ \Rightarrow \vec{v} = (t^3 - 2t^2 + 5)\hat{x} + (6t^2 - 6t - 5)\hat{y} + (3t^2 - 4t^3 + 11)\hat{z} \leftarrow \text{Multiplico por } m \text{ y se convierte en vector de momento} \\ \Rightarrow \vec{v}(t=2) = 5\hat{x} + 7\hat{y} - 9\hat{z} //$$

c) Se necesita un vector que conecte el origen con la posición

$$\frac{d\vec{r}}{dt} = (t^3 - 2t^2 + 5)\hat{x} + (6t^2 - 6t - 5)\hat{y} + (3t^2 - 4t^3 + 11)\hat{z} \Rightarrow \int \vec{r} \, d\vec{r} = \int_0^t (t^3 - 2t^2 + 5)\hat{x} + (6t^2 - 6t - 5)\hat{y} + (3t^2 - 4t^3 + 11)\hat{z} \, dt \\ \Rightarrow \vec{r} = \left(\frac{t^4}{4} - \frac{2t^3}{3} + 5t \right) \hat{x} + (2t^3 - 3t^2 - 5t) \hat{y} + (t^3 - t^4 + 11t) \hat{z}$$

$$\text{Torque respecto al origen: } \vec{N}_o = \vec{r} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \left(\frac{t^4}{4} - \frac{2t^3}{3} + 5t\right) & \left(2t^3 - 3t^2 - 5t\right) & \left(t^3 - t^2 + \frac{11}{2}t\right) \\ (3t^2 - 4t) & (12t - 6) & (6t - 12t^2) \end{vmatrix}$$

$$\Rightarrow \vec{N}_o = [(2t^3 - 3t^2 - 5t) \cdot (6t - 12t^2) - (t^3 - t^2 + \frac{11}{2}t) \cdot (12t - 6)] \hat{i} + \dots$$

$$\dots - \left[\left(\frac{t^4}{4} - \frac{2t^3}{3} + 5t \right) \cdot (6t - 12t^2) - (t^3 - t^2 + \frac{11}{2}t) \cdot (3t^2 - 4t) \right] \hat{j} + \dots$$

$$\dots + \left[\left(\frac{t^4}{4} - \frac{2t^3}{3} + 5t \right) \cdot (12t - 6) - (2t^3 - 3t^2 - 5t) \cdot (3t^2 - 4t) \right] \hat{k}$$

$$\Rightarrow \vec{N}_o(t=2) = (-36 \hat{i} - 368 \hat{j} + 180 \hat{k}) \text{ Nm} //$$

$$\text{d) Momento angular: } \vec{L}_o(t=2) = \vec{r} \times \vec{p} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{26}{3} & -6 & +\frac{14}{3} \\ 5 & 7 & -9 \end{vmatrix} = \left(-44 \hat{i} + \frac{148}{3} \hat{j} + 272 \hat{k} \right) \text{ km s}^2 //$$

Problemas mixtos 1

Problema 1. (Valor: 25 pts.)

Un bote de masa m , se mueve en un lago, con una posición $x(0) = 0$ y una rapidez inicial $v(0) = v_0$. Asociado a la fuerza de fricción del bote con el agua y el viento, experimenta una fuerza de fricción, cuya magnitud está dada por:

$$|\vec{F}| = F_0 \left(\frac{v}{v_0} \right)^2,$$

donde F_0 es una constante positiva. Considere que el motor no está encendido.

- a. ¿Cuál es la aceleración del bote?
- b. Encuentre la posición del bote con respecto al tiempo.
- c. Determine el tiempo y la posición donde la velocidad se reduce a la mitad de la velocidad inicial.
- d. ¿Cuándo se detiene el bote?

$$a) \sum \vec{F} = -\vec{F}_x = m \ddot{x} \Rightarrow -F_0 \left(\frac{v}{v_0} \right)^2 = m \frac{dv}{dt} \Rightarrow -\frac{F_0 v^2}{m v_0^2} = \frac{dv}{dt} = a //$$

$$b) -\frac{F_0 v^2}{m v_0^2} = \frac{dv}{dt} \Rightarrow \int_0^t -\frac{F_0}{m v_0^2} dt = \int_{v_0}^v \frac{1}{v^2} dv \Rightarrow -v^{-1} \Big|_{v_0}^v = \frac{-F_0}{m v_0^2} t \Rightarrow \frac{1}{v} - \frac{1}{v_0} = \frac{F_0}{m v_0^2} t \Rightarrow \frac{1}{v} = \frac{1}{v_0} + \frac{F_0}{m v_0^2} t = \frac{m v_0 + F_0 t}{m v_0^2}$$

$$\Rightarrow v = \frac{m v_0^2}{m v_0 + F_0 t} = \frac{dx}{dt} \Rightarrow \int_0^x dx = \int_0^t \frac{m v_0^2}{m v_0 + F_0 t} dt$$

$$\Rightarrow x = \frac{m v_0^2}{F_0} \ln \left(\frac{m v_0 + F_0 t}{m v_0} \right) //$$

$$\int_0^t \frac{m v_0^2}{m v_0 + F_0 t} dt ; u = m v_0 + F_0 t \Rightarrow du = F_0 dt$$

$$\Rightarrow \int_{m v_0}^{m v_0 + F_0 t} \frac{m v_0^2}{F_0 u} du = \frac{m v_0^2}{F_0} \ln \left(\frac{m v_0 + F_0 t}{m v_0} \right)$$

$$c) v = \frac{v_0}{2} \Rightarrow \frac{v_0}{2} = \frac{m v_0^2}{m v_0 + F_0 t} \Rightarrow \cancel{\frac{v_0}{2}} (m v_0 + F_0 t) = m v_0^2 \Rightarrow m v_0 + F_0 t = 2 m v_0 \Rightarrow F_0 t = m v_0 \Rightarrow t = \frac{m v_0}{F_0}$$

$$\text{Tiempo en que } v = \frac{v_0}{2} : t_{\frac{v_0}{2}} = \frac{m v_0}{F_0} // \Rightarrow x_{\frac{v_0}{2}} = \frac{m v_0^2}{F_0} \ln \left(\frac{m v_0 + \cancel{\frac{m v_0}{F_0}}}{m v_0} \right) = \frac{m v_0^2}{F_0} \ln \left(\frac{2 m v_0}{m v_0} \right) = \frac{m v_0^2}{F_0} \ln(2)$$

$$\Rightarrow x_{\frac{v_0}{2}} = \frac{m v_0^2}{F_0} \ln(2) //$$

$$d) v = 0 \Rightarrow \frac{m v_0^2}{m v_0 + F_0 t} = 0 \Rightarrow t = +\infty$$

Problema 2. (Valor: 25 pts.)

Una partícula de masa m , se mueve en una dimensión. Sobre la partícula actúa una fuerza, relacionada con una energía potencial:

$$U(x) = -A + a(x - x_1)^2 - b(x - x_1)^3,$$

donde A , a y x_1 son constantes positivas. En la Figura 1 se muestra la forma cómo varía esta energía con respecto a la posición x .

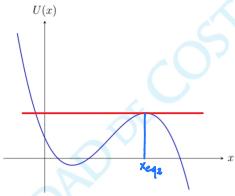


Figura 1: Energía potencial para el problema 2.

- Determine la fuerza que experimenta la partícula.
- Encuentre la posición de los puntos de equilibrio e indique qué tipo de equilibrio en cada punto, justificando su respuesta.
- La velocidad cuando pasa por $x = 0$ es $v_0 > 0$. Si la partícula se mueve en la dirección positiva, determine la velocidad crítica, que mantiene la partícula confinada cerca del origen.

a) Una fuerza conservativa en una dimensión: $F = -\frac{dU}{dx}$

$$\Rightarrow \frac{dU}{dx} = 2a(x - x_1) - 3b(x - x_1)^2 \Rightarrow F = 3b(x - x_1)^2 - 2a(x - x_1)$$

b) Puntos de equilibrio: $\frac{dU}{dx} = 0 \Rightarrow 2a(x - x_1) - 3b(x - x_1)^2 = 0 \Rightarrow [2a - 3b(x - x_1)](x - x_1) = 0$

$$\Rightarrow x - x_1 = 0 \quad \text{y} \quad 2a - 3b(x - x_1) = 0 \Rightarrow x_{eq1} = x_1 \quad \text{y} \quad x_{eq2} = x_1 + \frac{2a}{3b}$$

Clasificación: $\left. \frac{d^2U}{dx^2} \right|_{x=x_{eq}} \Rightarrow \left. \frac{d^2U}{dx^2} \right|_{x=x_{eq}} = 2a - 6b(x - x_1)$

$$\Rightarrow \left. \frac{d^2U}{dx^2} \right|_{x=x_{eq1}} = 2a - 6b(x - x_1) = 2a \Rightarrow x_{eq1} \text{ es un punto de equilibrio estable}$$

$$\Rightarrow \left. \frac{d^2U}{dx^2} \right|_{x=x_{eq2}} = 2a - 6b(x - x_1) = 2a - 6b\left(\cancel{x_1} + \frac{2a}{3b} - \cancel{x_1}\right) = 2a - 2 \cdot 2a = -2a \Rightarrow x_{eq2} \text{ es un punto de equilibrio inestable}$$

c) La mayor energía que puede poseer la partícula es $E \leq U(x_{eq})$ $U = -A + a(x - x_1)^2 + 3b(x - x_1)^3$

$$\Rightarrow U(x_{eq}) = -A + a\left(\cancel{x_1} + \frac{2a}{3b} - \cancel{x_1}\right)^2 + 3b\left(\cancel{x_1} + \frac{2a}{3b} - \cancel{x_1}\right)^3 = -A + \frac{4a^3}{9b^2} + \frac{8a^3}{9b^2} = -A + \frac{4a^3}{3b^2} \quad \text{y} \quad U(x_{eq}) = -A$$

$$E = T + U_{eq} = U_{eq2} \Rightarrow \frac{1}{2}mv_0^2 - A = -A + \frac{4a^3}{3b^2} \Rightarrow v_0 = \sqrt{\frac{8a^3}{3b^2m}}$$

Problema 3. (Valor: 25 pts.)

Un oscilador subamortiguado tiene una frecuencia angular natural $3\pi/2$ rad/s. Este oscilador es tal que la amplitud disminuye un 25% al transcurrir un período de tiempo.

- a. Determine el período de oscilación.
- b. Encuentre el parámetro de amortiguamiento.
- c. ¿Cuál es la disminución de la amplitud, con respecto a la amplitud inicial, después de haber pasado 2 períodos de tiempo?
- d. Determine el cambio de energía del sistema, con respecto a la energía en $t = 0$, al transcurrir un período de tiempo.

Nota: Se puede suponer el ángulo de desfase como cero. Además se puede considerar que el oscilador se relaciona con una masa m y un resorte de constante de elasticidad k .

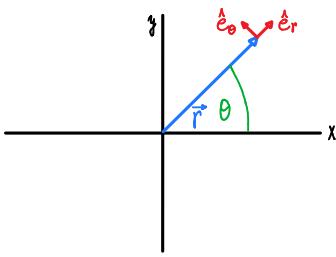
Problema 4. (Valor: 25 pts.)

Un insecto se mueve en una trayectoria en dos dimensiones, descrito por:

$$r = \frac{bt}{\tau^2}(2\tau - t), \quad \theta = \frac{t}{\tau},$$

donde $0 \leq t \leq 2\tau$, b y τ son constantes positivas.

- a. Calcule, para cualquier trayectoria descrita en coordenadas polares, los vectores \hat{e}_r y $\dot{\theta}$.
- b. Determine para cualquier trayectoria descrita en coordenadas polares, la velocidad y la aceleración de una partícula.
- c. A partir de este resultado, encuentre la velocidad y aceleración del insecto.
- d. Encuentre el tiempo cuando la aceleración tangencial del insecto es nula.
- e. Para este tiempo, determine la posición, velocidad y aceleración del insecto.



a) Primero se plantean los vectores unitarios polares: $\hat{e}_r = \cos\theta \hat{i} + \sin\theta \hat{j}$ y $\hat{e}_\theta = -\sin\theta \hat{i} + \cos\theta \hat{j}$

$$\text{Para } \hat{e}_r, \text{ hay que hallar } \frac{d\hat{e}_r}{dt} = \frac{d\hat{e}_r}{d\theta} \frac{d\theta}{dt} = \dot{\theta}(-\sin\theta \hat{i} + \cos\theta \hat{j}) = \dot{\theta} \hat{e}_\theta$$

$$\text{Para } \hat{e}_\theta, \text{ hay que hallar } \frac{d\hat{e}_\theta}{dt} = \frac{d\hat{e}_\theta}{d\theta} \frac{d\theta}{dt} = \dot{\theta}(-\cos\theta \hat{i} - \sin\theta \hat{j}) = -\dot{\theta} \hat{e}_r$$

$$\Rightarrow \dot{\hat{e}}_r = \dot{\theta} \hat{e}_\theta \quad \text{y} \quad \dot{\hat{e}}_\theta = -\dot{\theta} \hat{e}_r //$$

b) Posición en coordenadas polares: $\vec{r} = r \hat{e}_r$

$$\text{Velocidad: } \vec{v} = \frac{d\vec{r}}{dt} = \dot{r} \hat{e}_r + r \dot{\hat{e}}_r = \dot{r} \hat{e}_r + r \dot{\theta} \hat{e}_\theta \Rightarrow \vec{v} = \dot{r} \hat{e}_r + r \dot{\theta} \hat{e}_\theta //$$

$$\text{Aceleración: } \vec{a} = \frac{d\vec{v}}{dt} = \ddot{r} \hat{e}_r + \dot{r} \dot{\hat{e}}_r + \dot{r} \dot{\theta} \hat{e}_\theta + r \ddot{\theta} \hat{e}_\theta + r \dot{\theta} \dot{\hat{e}}_r = \ddot{r} \hat{e}_r + \dot{r} \dot{\theta} \hat{e}_\theta + \dot{r} \dot{\theta} \hat{e}_\theta + r \ddot{\theta} \hat{e}_\theta - r \dot{\theta} \dot{\hat{e}}_r$$

$$\Rightarrow \vec{a} = (\ddot{r} - r \dot{\theta}^2) \hat{e}_r + (r \ddot{\theta} + 2\dot{r}\dot{\theta}) \hat{e}_\theta //$$

$$c) \quad \dot{r} = \frac{b}{\tau^2}(2\tau - t) - \frac{bt}{\tau^2} = \frac{2b}{\tau} - \frac{2bt}{\tau^2} \quad ; \quad \ddot{r} = -\frac{2b}{\tau^2}; \quad \dot{\theta} = \frac{1}{\tau} \quad y \quad \ddot{\theta} = 0$$

$$\Rightarrow \vec{v}_{\text{res}} = \left(\frac{2b}{\tau} - \frac{2bt}{\tau^2} \right) \hat{e}_r + \frac{bt}{\tau^2} \left(2\tau - t \right) \frac{1}{\tau} \hat{e}_\theta = \left(\frac{2b}{\tau} - \frac{2bt}{\tau^2} \right) \hat{e}_r + \frac{bt}{\tau^3} (2\tau - t) \hat{e}_\theta$$

$$\Rightarrow \vec{a}_{\text{res}} = \left[-\frac{2b}{\tau^2} - \frac{bt}{\tau^2} (2\tau - t) \frac{1}{\tau^2} \right] \hat{e}_r + \left[\frac{bt}{\tau^2} (2\tau - t) \overset{\theta}{0} + 2 \left(\frac{2b}{\tau} - \frac{2bt}{\tau^2} \right) \frac{1}{\tau} \right] \hat{e}_\theta$$

$$= \left[-\frac{2b}{\tau^2} - \frac{bt}{\tau^4} (2\tau - t) \right] \hat{e}_r + \frac{2}{\tau} \left(\frac{2b}{\tau} - \frac{2bt}{\tau^2} \right) \hat{e}_\theta$$

Continua....

$$d) \quad a_\theta = \frac{2}{\tau} \left(\frac{2b}{\tau} - 2 \frac{b\tau}{\tau^2} \right), \text{ buscando } a_\theta = 0$$

$$\Rightarrow \frac{2}{\tau} \left(\frac{2b}{\tau} - 2 \frac{b\tau}{\tau^2} \right) = 0 \Rightarrow \frac{2b}{\tau} - 2 \frac{b\tau}{\tau^2} = 0 \Rightarrow \cancel{\frac{2b}{\tau}} = \cancel{\frac{2b\tau}{\tau^2}} \Rightarrow \tau = \tau //$$

$$e) \quad \vec{r}_{\text{ins}}(t = \tau) = \frac{b\tau}{\tau^2} (2\tau - \tau) \hat{e}_r = \frac{b}{\tau} (\tau) \hat{e}_r = b \hat{e}_r //$$

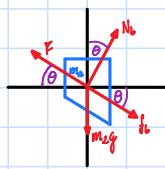
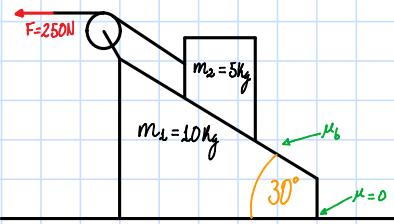
$$\vec{v}_{\text{ins}}(t = \tau) = \left(\cancel{\frac{2b}{\tau}} - 2 \cancel{\frac{b\tau}{\tau^2}} \right) \hat{e}_r + \cancel{\frac{b}{\tau^2} (2\tau - \tau)} \hat{e}_\theta = 0 \hat{e}_r + \frac{b}{\tau} \hat{e}_\theta = \frac{b}{\tau} \hat{e}_\theta //$$

$$\vec{a}_{\text{ins}}(t = \tau) = \left[-\frac{2b}{\tau^2} - \cancel{\frac{b}{\tau^2} (2\tau - \tau)} \right] \hat{e}_r + \frac{2}{\tau} \left(\cancel{\frac{2b}{\tau}} - 2 \cancel{\frac{b\tau}{\tau^2}} \right) \hat{e}_\theta = -\frac{3b}{\tau^2} \hat{e}_r + 0 \hat{e}_\theta = -\frac{3b}{\tau^2} \hat{e}_r //$$

a) Encuentre μ_b tal que los cuerpos se muevan juntos

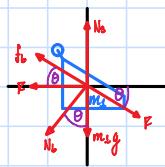
b) Tome la mitad del resultado anterior y calcule la aceleración de ambas masas y la aceleración de m_2 sobre m_1

Considera la polea sin masa y luego con masa $m_p = 2\text{kg}$ y $R = 0,5\text{m}$



$$\sum F_{xy} = N_b \cos \theta + f_b \sin \theta - m_2 g - f_b \cos \theta = m_2 a_{zy}$$

$$\sum F_{zx} = N_b \sin \theta + f_b \cos \theta - F \cos \theta = m_2 a_{zx}$$



$$\sum F_{xy} = N_b - m_1 g + f_b \sin \theta - F \sin \theta - N_b \cos \theta = m_1 a_{zy}$$

$$\sum F_{zx} = F \cos \theta - F - f_b \cos \theta - N_b \sin \theta = m_1 a_{zx}$$

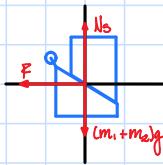
$$\vec{a}_{iz} = \vec{a}_{i1} + \vec{a}_{i2} = (a_{ix} + a_{izx})\hat{i} + a_{izy}\hat{j}$$

la rueda no se mueve en el eje y: $a_{iy} = 0$

a) Como los bloques se mueven juntos, como un solo objeto: $\Rightarrow a_{ix} = a_{zx} \wedge a_{izy} = 0 = a_{iy} ; f = \mu N$

$$\left\{ \begin{array}{l} N_b \cos \theta + f_b \sin \theta - m_2 g - f_b \cos \theta = m_2 a_{zy} \\ N_b \sin \theta + f_b \cos \theta - F \cos \theta = m_2 a_{zx} \\ N_b - m_1 g + f_b \sin \theta - F \sin \theta - N_b \cos \theta = 0 \\ F \cos \theta - F - f_b \cos \theta - N_b \sin \theta = m_1 a_{zx} \end{array} \right.$$

$$\Rightarrow \left\{ \begin{array}{l} N_b \cos \theta + f_b \sin \theta - m_2 g - \mu N_b \sin \theta = 0 \\ N_b \sin \theta + \mu N_b \cos \theta - F \cos \theta = m_2 a_{zx} \\ N_b - m_1 g + \mu N_b \sin \theta - F \sin \theta - N_b \cos \theta = 0 \\ F \cos \theta - F - \mu N_b \cos \theta - N_b \sin \theta = m_1 a_{zx} \end{array} \right.$$



$$\sum F_y = N_b - (m_1 + m_2)g = (m_1 + m_2)a_{iy} \Rightarrow N_b = (m_1 + m_2)g$$

$$\sum F_x = -F = (m_1 + m_2)a_{ix} \Rightarrow a_{ix} = -\frac{F}{m_1 + m_2}$$

Aplicando esta consideración al sistema de ecuaciones

$$\Rightarrow \left\{ \begin{array}{l} N_b \cos \theta + f_b \sin \theta - m_2 g - \mu N_b \sin \theta = 0 \\ N_b \sin \theta + \mu N_b \cos \theta - F \cos \theta = -\frac{m_2 F}{m_1 + m_2} \\ (m_1 + m_2)g - m_1 g + \mu N_b \sin \theta - F \sin \theta - N_b \cos \theta = 0 \\ F \cos \theta - F - \mu N_b \cos \theta - N_b \sin \theta = -\frac{m_1 F}{m_1 + m_2} \end{array} \right.$$

Continua

SECTION 4

Sistemas de partículas

SUBSECTION 4.1

Suposiciones iniciales

Para trabajar con sistemas de partículas N es necesario realizar ciertas suposiciones respecto a las fuerzas internas al sistema de partículas.

- Para dos partículas α y β se cumple: $\vec{f}_{\alpha\beta} = -\vec{f}_{\beta\alpha}$.
- Los vectores fuerza están sobre la línea que une a ambas partículas.

Estas dos suposiciones se pueden resumir al aceptar trabajar con el **Enunciado Fuerte de la Tercera Ley de Newton** (Figura 1). De acuerdo a esto, se procederá a construir las expresiones para sistemas de partículas para los conceptos conocidos en la sección anterior y a introducir algunos conceptos nuevos.

Definition 20

(Centro de masa) Corresponde a un punto del espacio en que se encuentra el sistema de partículas (discreto o continuo) en el cual es posible colapsar toda la masa del sistema, de modo que una fuerza arbitraria que interactue con alguna partícula del sistema se podrá transportar a dicho punto para conocer la mecánica del sistema (Figura 6). El centro de masa se define a partir de un promedio ponderado de toda la masa del sistema visto a continuación:

$$\vec{R} = \frac{1}{M} \sum_{\alpha=1}^N m_\alpha \vec{r}_\alpha \rightarrow \vec{R} = \frac{1}{M} \int_M \vec{r} dm \quad (4.1)$$

$$M = \sum_{\alpha=1}^N m_\alpha \rightarrow M = \int_M \rho dV \quad (4.2)$$

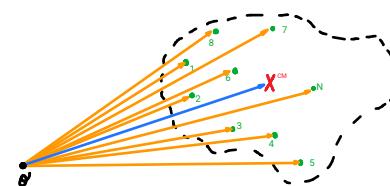


Figura 6. Un sistema arbitrario con su respectivo centro de masa

SUBSECTION 4.2

Definición de la mecánica de sistemas de partículas

Para definir la expresión del momentum lineal de un sistema de partículas hay que construirla a partir del conocimiento de la sección anterior. A partir de esto, se definen las siguientes cantidades para la partícula α del sistema:

- Fuerza neta externa al sistema sobre α : \vec{F}_α^e
- Fuerza neta interna al sistema sobre α : $\vec{f}_\alpha = \sum_{\substack{\beta=1 \\ \alpha \neq \beta}}^N \vec{f}_{\alpha\beta}$

Entonces la fuerza neta que percibe la partícula α es:

$$\vec{F}_\alpha = \vec{F}_\alpha^e + \vec{f}_\alpha$$

De acuerdo a la Segunda Ley de Newton Ecuación (3.3):

$$\begin{aligned} \dot{\vec{p}}_\alpha &= \vec{F}_\alpha = m_\alpha \ddot{\vec{r}}_\alpha \\ \Rightarrow m_\alpha \ddot{\vec{r}}_\alpha &= \vec{F}_\alpha^e + \vec{f}_\alpha \end{aligned}$$

Hasta este momento se ha estado trabajando únicamente con la partícula α , pero si ahora sumamos la expresión anterior para todas las partículas, se obtiene:

$$\begin{aligned}
 \Rightarrow \sum_{\alpha=1}^N [m_\alpha \vec{r}_\alpha] &= \sum_{\alpha=1}^N \vec{F}_\alpha^e + \sum_{\alpha=1}^N \vec{f}_\alpha \\
 &= \sum_{\alpha=1}^N \vec{F}_\alpha^e + \sum_{\alpha=1}^N \sum_{\substack{\beta=1 \\ \alpha \neq \beta}}^N \vec{f}_{\alpha\beta} \\
 &= \sum_{\alpha=1}^N \vec{F}_\alpha^e + \sum_{\substack{\alpha, \beta=1 \\ \alpha \neq \beta}}^N \vec{f}_{\alpha\beta} \\
 \Rightarrow \underbrace{\sum_{\alpha=1}^N \frac{d^2}{dt^2} [m_\alpha \vec{r}_\alpha]}_{M\vec{r}_{CM}} &= \underbrace{\sum_{\alpha=1}^N \vec{F}_\alpha^e}_{\vec{F}} + \sum_{\substack{\alpha, \beta=1 \\ \alpha \neq \beta}}^N \vec{f}_{\alpha\beta}
 \end{aligned}$$

De esta última expresión se obtiene ⁴

⁴ El término

$$\sum_{\substack{\alpha, \beta=1 \\ \alpha \neq \beta}}^N \vec{f}_{\alpha\beta}$$

Definition 21

(Momentum Lineal y Segunda Ley de Newton para un Sistema) La expresión para el momentum lineal de un sistema de partículas:

$$\vec{P} = M\dot{\vec{R}} \quad (4.3)$$

y la expresión de la Segunda Ley de Newton para el caso de un sistema de partículas en el caso translacional:

$$\vec{F} = \dot{\vec{P}} = M\ddot{\vec{R}} \quad (4.4)$$

se hace cero porque se tiene:

$$\sum_{\alpha < \beta=1}^N \vec{f}_{\alpha\beta} + \vec{f}_{\beta\alpha}$$

y es evidente que estas son fuerzas contrarias de igual magnitud debido a la Tercera Ley de Newton, por lo que el resultado de su suma es cero.

Ahora se resolverá un procedimiento similar para hallar una expresión para el momentum angular de un sistema de partículas respecto a un punto \mathcal{O} :

El momentum angular de una partícula α desde el punto \mathcal{O} (Ver Figura 7) es:

$$\vec{L}_\alpha^\mathcal{O} = \vec{r}_\alpha \times \vec{p}_\alpha$$

Si se realiza la suma de todos los momentum angulares individuales, esto da como resultado el momentum angular del sistema:

$$\begin{aligned}\vec{L}^\mathcal{O} &= \sum_{\alpha=1}^N \vec{r}_\alpha \times \vec{p}_\alpha \\ &= \sum_{\alpha=1}^N \vec{r}_\alpha \times (m_\alpha \vec{v}_\alpha) \\ &= \sum_{\alpha=1}^N (\vec{R} + \vec{r}'_\alpha) \times [m_\alpha (\dot{\vec{R}} + \dot{\vec{r}}'_\alpha)] \\ &= \sum_{\alpha=1}^N m_\alpha (\vec{r}'_\alpha \times \dot{\vec{r}}'_\alpha + \vec{r}'_\alpha \times \dot{\vec{R}} + \vec{R} \times \dot{\vec{r}}'_\alpha + \vec{R} \times \dot{\vec{R}}) \\ &= \sum_{\alpha=1}^N m_\alpha \vec{r}'_\alpha \times \dot{\vec{r}}'_\alpha + \sum_{\alpha=1}^N m_\alpha \vec{r}'_\alpha \times \dot{\vec{R}} + \sum_{\alpha=1}^N m_\alpha \vec{R} \times \dot{\vec{r}}'_\alpha + \sum_{\alpha=1}^N m_\alpha \vec{R} \times \dot{\vec{R}}\end{aligned}$$

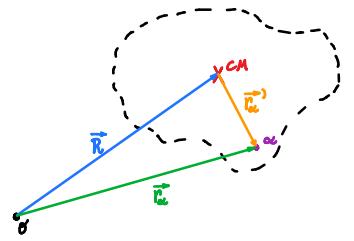


Figura 7. Posición de la partícula α y el centro de masa de un sistema respecto a \mathcal{O}

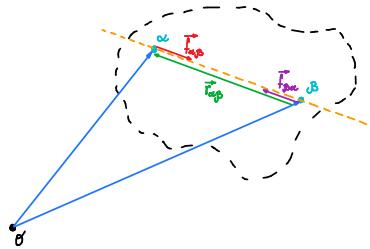


Figura 8. Interacción y posiciones de las partículas α y β de un sistema respecto a \mathcal{O}

$$\begin{aligned}&\sum_{\alpha=1}^N m_\alpha \vec{r}'_\alpha \times \dot{\vec{R}} + \sum_{\alpha=1}^N m_\alpha \vec{R} \times \dot{\vec{r}}'_\alpha \\ &= \left(\sum_{\alpha=1}^N m_\alpha \vec{r}'_\alpha \right) \times \dot{\vec{R}} + \vec{R} \times \frac{d}{dt} \left(\sum_{\alpha=1}^N m_\alpha \vec{r}'_\alpha \right) \\ &= \left(\sum_{\alpha=1}^N m_\alpha \vec{r}'_\alpha \right) \times (\dot{\vec{R}} - \vec{R}) \\ &= \left[\sum_{\alpha=1}^N m_\alpha (\vec{r}_\alpha - \vec{R}) \right] \times (\dot{\vec{R}} - \vec{R}) \\ &= \left[\sum_{\alpha=1}^N m_\alpha \vec{r}_\alpha - \left(\sum_{\alpha=1}^N m_\alpha \right) \vec{R} \right] \times (\dot{\vec{R}} - \vec{R}) \\ &= \left[M \vec{R} - M \vec{R} \right] \times (\dot{\vec{R}} - \vec{R}) \\ &= 0\end{aligned}$$

Trabajando por un momento con los terminos en azul:

Quedando claro que ambos terminos en azul son cero por como se definición del centro de masa⁵, se continuará con la expresión trasanterior:

⁵La posición del centro de masa respecto al centro de masa siempre será cero.

$$\begin{aligned}
 \vec{L}^{\mathcal{O}} &= \sum_{\alpha=1}^N m_{\alpha} \vec{r}'_{\alpha} \times \dot{\vec{r}}'_{\alpha} + \underbrace{\sum_{\alpha=1}^N m_{\alpha} \vec{r}'_{\alpha} \times \dot{\vec{R}}}_{\text{Momentum Angular alrededor del CM}} + \sum_{\alpha=1}^N m_{\alpha} \vec{R} \times \dot{\vec{r}}'_{\alpha} + \sum_{\alpha=1}^N m_{\alpha} \vec{R} \times \dot{\vec{R}} \\
 &= \sum_{\alpha=1}^N m_{\alpha} \vec{r}'_{\alpha} \times \dot{\vec{r}}'_{\alpha} + \sum_{\alpha=1}^N m_{\alpha} \vec{R} \times \dot{\vec{R}} \\
 &= \underbrace{\sum_{\alpha=1}^N \vec{r}'_{\alpha} \times \dot{\vec{p}}'_{\alpha}}_{\text{Momentum Angular alrededor del CM}} + \underbrace{M \vec{R} \times \dot{\vec{R}}}_{\text{Momentum Angular del CM alrededor a } \mathcal{O}}
 \end{aligned}$$

Ahora, buscando la expresión para la derivada respecto al tiempo del momentum angular del sistema. Para esto hay que comenzar de forma similar, analizar desde una unica partícula puntual que forme parte del sistema de partículas:

$$\dot{\vec{L}}_{\alpha}^{\mathcal{O}} = \vec{r}_{\alpha} \times \dot{\vec{p}}_{\alpha} = \vec{r}_{\alpha} \times \left(\vec{F}_{\alpha}^e + \sum_{\substack{\beta=1 \\ \alpha \neq \beta}}^N \vec{f}_{\alpha\beta} \right)$$

Sumando para todas las partículas ⁶:

$$\dot{\vec{L}}_{\mathcal{O}} = \sum_{\alpha=1}^N \vec{r}_{\alpha} \times \vec{F}_{\alpha}^e + \sum_{\substack{\alpha, \beta=1 \\ \alpha \neq \beta}}^N \vec{r}_{\alpha} \times \vec{f}_{\alpha\beta}$$

De esta última expresión se obtiene:

$$\begin{aligned}
 \dot{\vec{L}}_{\mathcal{O}} &= \sum_{\alpha=1}^N \vec{r}_{\alpha} \times \vec{F}_{\alpha}^e = \sum_{\alpha=1}^N \vec{N}_{\alpha}^e \\
 \Rightarrow \dot{\vec{L}}_{\mathcal{O}} &= \vec{N}
 \end{aligned}$$

Definition 22

(Momentum Angular y Análogo Rotacional de la Segunda Ley de Newton para Sistemas) La expresión del momentum angular para un sistema de partículas

$$\vec{L}^{\mathcal{O}} = \underbrace{\sum_{\alpha=1}^N \vec{r}'_{\alpha} \times \dot{\vec{p}}'_{\alpha}}_{\text{Momentum Angular alrededor del CM}} + \underbrace{M \vec{R} \times \dot{\vec{R}}}_{\text{Momentum Angular del CM alrededor a } \mathcal{O}} \quad (4.5)$$

La expresión del análogo rotacional para un sistema de partículas de la Segunda Ley de Newton:

$$\vec{N} = \dot{\vec{L}}_{\mathcal{O}} \quad (4.6)$$

El siguiente es el resultado ya conocido del Teorema de Ejes paralelos, el cual puede ser útil para resolver problemas concernientes a este tema.

⁶ Aquí se encuentra una situación similar a la que se tuvo con el momentum lineal, el termino que se canceló, es cero por lo siguiente:

La expresión:

$$\sum_{\alpha, \beta=1}^N \vec{r}_{\alpha} \times \vec{f}_{\alpha\beta}$$

es posible reescribirla como:

$$\sum_{\alpha<\beta=1}^N \left(\vec{r}_{\alpha} \times \vec{f}_{\alpha\beta} + \vec{r}_{\beta} \times \vec{f}_{\beta\alpha} \right)$$

y aquí tomando que se cumple el **Enunciado Fuerte** de la Tercera Ley de Newton Ecuación (3.5) y la siguiente definición (Ver Figura 8):

$$\vec{r}_{\alpha\beta} = \vec{r}_{\alpha} - \vec{r}_{\beta}$$

La expresión de interés se convierte en:

$$\begin{aligned}
 &\sum_{\alpha<\beta=1}^N \left(\vec{r}_{\alpha} \times \vec{f}_{\alpha\beta} - \vec{r}_{\beta} \times \vec{f}_{\alpha\beta} \right) \\
 &= \sum_{\alpha<\beta=1}^N \left(\vec{r}_{\alpha} - \vec{r}_{\beta} \right) \times \vec{f}_{\alpha\beta} \\
 &= \sum_{\alpha<\beta=1}^N \underbrace{\vec{r}_{\alpha\beta} \times \vec{f}_{\alpha\beta}}_{\vec{r}_{\alpha\beta} \parallel \vec{f}_{\alpha\beta}} \\
 &= 0
 \end{aligned}$$

Theorem 6

(Teorema de Ejes paralelos)

$$I_{q'} = I_q^{cm} + Md^2 \quad (4.7)$$

La demostración de este teorema tal y como se encuentra escrito aquí se le dejará al lector y se recomienda verlo como una simplificación del teorema de ejes paralelos real que se desarrollará más adelante.

Ahora hay que estudiar la forma de la energía cinética para un sistema de partículas. Para esto se va a empezar la deducción desde el trabajo de mover la partícula α de un punto a otro para la posterior definición de la energía cinética del sistema:

$$W_{\alpha}^{AB} = \int_A^B \vec{F}_{\alpha} \cdot d\vec{r}_{\alpha}$$

Sumando para todo el sistema:

$$W_{AB} = \sum_{\alpha=1}^N \int_A^B \vec{F}_{\alpha} \cdot d\vec{r}_{\alpha} = \sum_{\alpha=1}^N \int_A^B d \left(\frac{1}{2} m_{\alpha} v_{\alpha}^2 \right)$$

Entonces, en una primera instancia se puede definir la energía cinética del sistema como:

$$T = \sum_{\alpha=1}^N T_{\alpha} = \sum_{\alpha=1}^N \frac{1}{2} m_{\alpha} v_{\alpha}^2$$

Pero se está buscando una expresión que tome en cuenta el centro de masa, entonces hay que continuar trabajando con este último resultado. Recordando de la Figura 7 lo siguiente: $\vec{r}_{\alpha} = \vec{R} + \vec{r}'_{\alpha}$. De aquí se obtiene:

$$\vec{r}_{\alpha} = \vec{R} + \vec{r}'_{\alpha} \Rightarrow \dot{\vec{r}}_{\alpha} = \dot{\vec{R}} + \dot{\vec{r}}'_{\alpha}$$

$$\begin{aligned} \dot{\vec{r}}_{\alpha}^2 &= \dot{\vec{r}} \cdot \dot{\vec{r}} = (\dot{\vec{R}} + \dot{\vec{r}}'_{\alpha}) \cdot (\dot{\vec{R}} + \dot{\vec{r}}'_{\alpha}) \\ &= \dot{\vec{r}}'^2_{\alpha} + 2\dot{\vec{r}}'_{\alpha} \cdot \dot{\vec{R}} + \dot{\vec{R}}^2 \end{aligned}$$

Con esta expresión y la provisional de la energía cinética, reemplazando en la energía:

$$\begin{aligned} T &= \frac{1}{2} \sum_{\alpha=1}^N m_{\alpha} \dot{\vec{r}}^2 = \frac{1}{2} \sum_{\alpha=1}^N m_{\alpha} \left(\dot{\vec{r}}'^2_{\alpha} + 2\dot{\vec{r}}'_{\alpha} \cdot \dot{\vec{R}} + \dot{\vec{R}}^2 \right) \\ &= \frac{1}{2} \sum_{\alpha=1}^N m_{\alpha} \dot{\vec{r}}'^2_{\alpha} + \underbrace{\sum_{\alpha=1}^N m_{\alpha} \dot{\vec{r}}'_{\alpha} \cdot \dot{\vec{R}}}_0 + \frac{1}{2} \sum_{\alpha=1}^N m_{\alpha} \dot{\vec{R}}^2 \\ &= \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2} \sum_{\alpha=1}^N m_{\alpha} \dot{\vec{r}}'^2_{\alpha} \end{aligned}$$

Velocidad del CM respecto al CM

Definition 23

(Energía Cinética de un Sistema de Partículas) La expresión de la energía cinética es la siguiente:

$$T = \underbrace{\frac{1}{2} M \dot{\vec{R}}^2}_{\text{T del CM}} + \underbrace{\frac{1}{2} \sum_{\alpha=1}^N m_\alpha \dot{\vec{r}}_\alpha'^2}_{\text{T de las partículas respecto al CM}} \quad (4.8)$$

Por último, falta encontrar la expresión para la energía potencial en el caso de sistemas de partículas. Comenzando desde la expresión del trabajo de todo el sistema:

$$W_{AB} = \sum_{\alpha=1}^N \int_A^B \vec{F}_\alpha^e \cdot d\vec{r}_\alpha + \sum_{\substack{\alpha, \beta = 1 \\ \alpha \neq \beta}}^N \int_A^B \vec{f}_{\alpha\beta} \cdot d\vec{r}_\alpha$$

Será necesario considerar que \vec{F}_α y $\vec{f}_{\alpha\beta}$ son todas fuerzas conservativas por lo que se pueden escribir como ⁷:

$$\vec{F}_\alpha = -\vec{\nabla}_\alpha V_\alpha \quad y \quad \vec{f}_{\alpha\beta} = -\vec{\nabla}_\alpha \bar{V}_{\alpha\beta}$$

Para simplificar el desarrollo de la deducción se van a trabajar los términos del trabajo total del sistema por separado. Entonces, comenzando por la que contiene las fuerzas externas⁸:

$$\begin{aligned} \sum_{\alpha=1}^N \int_A^B \vec{F}_\alpha^e \cdot d\vec{r}_\alpha &= \sum_{\alpha=1}^N \int_A^B -\vec{\nabla}_\alpha V_\alpha \cdot d\vec{r}_\alpha \\ &= - \sum_{\alpha=1}^N V_\alpha|_A^B \end{aligned}$$

Como se plantearon fuerzas conservativas, se sabe que $\bar{V}_{\alpha\beta}$ solo depende de las distancias entre las partículas α y β , y por lo tanto depende de 6 cantidades (las posiciones de ambas partículas). Entonces la derivada total de $\bar{V}_{\alpha\beta}$:

$$\begin{aligned} d\bar{V}_{\alpha\beta} &= \sum_{i=1}^3 \left(\frac{\partial \bar{V}_{\alpha\beta}}{\partial x_{\alpha,i}} dx_{\alpha,i} + \frac{\partial \bar{V}_{\alpha\beta}}{\partial x_{\beta,i}} dx_{\beta,i} \right) \\ &= \left(\vec{\nabla}_\alpha \bar{V}_{\alpha\beta} \right) \cdot d\vec{r}_\alpha + \left(\vec{\nabla}_\beta \bar{V}_{\alpha\beta} \right) \cdot d\vec{r}_\beta \end{aligned}$$

Recordando que: $\bar{V}_{\alpha\beta} = \bar{V}_{\beta\alpha}$, se tiene lo siguiente:

$$\vec{\nabla}_\alpha \bar{V}_{\alpha\beta} = \vec{\nabla}_\alpha \bar{V}_{\beta\alpha} \rightarrow -\vec{f}_{\alpha\beta} = \vec{f}_{\beta\alpha}$$

Ahora el segundo término⁹:

⁷ Los subíndices α en los gradientes significa que los gradientes se realizan en las respectivas coordenadas de la partícula α

⁸ Esta expresión ya debería ser familiar de la sección anterior.

⁹ A continuación se usan variedad de definiciones que se establecieron anteriormente

$$\begin{aligned}
 & \sum_{\substack{\alpha, \beta=1 \\ \alpha \neq \beta}}^N \int_A^B \vec{f}_{\alpha\beta} \cdot d\vec{r}_\alpha = \sum_{\alpha<\beta=1}^N \int_A^B (\vec{f}_{\alpha\beta} \cdot d\vec{r}_\alpha + \vec{f}_{\beta\alpha} \cdot d\vec{r}_\beta) \\
 &= \sum_{\alpha<\beta=1}^N \int_A^B \vec{f}_{\alpha\beta} \cdot (d\vec{r}_\alpha - d\vec{r}_\beta) \\
 &= \sum_{\alpha<\beta=1}^N \int_A^B \vec{f}_{\alpha\beta} \cdot \vec{r}_{\alpha\beta} \\
 &= - \sum_{\alpha<\beta=1}^N \int_A^B d\bar{V}_{\alpha\beta} \\
 &= - \sum_{\alpha<\beta=1}^N \bar{V}_{\alpha\beta}|_A^B
 \end{aligned}$$

Definition 24

(Energía Potencial de un Sistema de Partículas) La expresión de la energía potencial de un sistema de partículas:

$$V = \underbrace{\sum_{\alpha=1}^N V_\alpha}_{\text{Energía potencial externa}} + \underbrace{\sum_{\alpha<\beta=1}^N \bar{V}_{\alpha\beta}}_{\text{Energía potencial interna}} \quad (4.9)$$

Hay casos, por ejemplo, cuerpos rígidos en los que es posible ignorar la energía potencial interna porque es constante.

Definition 25

(Energía total del Sistema de Partículas) La expresión para el sistema de partículas es la misma que para una única partícula Ecuación (3.12)

$$E = T + V$$

SUBSECTION 4.3

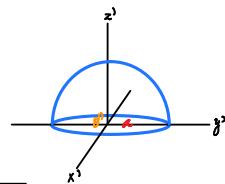
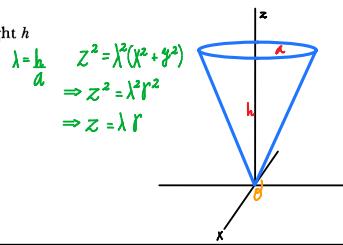
Problemas resueltos

Dinámica de partículas

- 9-3. Find the center of mass of a uniformly solid cone of base diameter $2a$ and height h and a solid hemisphere of radius a where the two bases are touching.

Cilindro

$$M_c = \int_{A_c} dm = \iiint \rho_c dV = \int_0^h \int_0^{2\pi} \int_0^{\frac{z}{h}} \rho_c r dr d\theta dz$$



$$\int_0^h \int_0^{2\pi} \int_0^{\frac{z}{h}} \rho_c r dr d\theta dz = 2\pi \rho_c \int_0^h r dr dz = 2\pi \rho_c \int_0^h \frac{a^2}{2h^2} z^2 dz = 2\pi \rho_c \frac{a^2}{2h^2} \frac{h^3}{3} = \rho_c \frac{\pi a^2 h}{3} \Rightarrow M_c = \rho_c \frac{\pi a^2 h}{3}$$

$$M_c \vec{R}_c = \int_{A_c} \vec{r} dm = \iiint \rho_c \vec{r} dV = \rho_c \int_0^h \int_0^{2\pi} \int_0^{\frac{z}{h}} (r \cos \theta \hat{i} + r \sin \theta \hat{j} + z \hat{k}) r dr d\theta dz$$

$$M_c X_{cm-c} = \rho_c \int_0^h \int_0^{2\pi} \int_0^{\frac{z}{h}} r^2 \cos \theta dr d\theta dz = \rho_c \int_0^h \int_0^{2\pi} 0 dr d\theta dz = 0$$

$$M_c Y_{cm-c} = \rho_c \int_0^h \int_0^{2\pi} \int_0^{\frac{z}{h}} r^2 \sin \theta dr d\theta dz = \rho_c \int_0^h \int_0^{2\pi} 0 dr d\theta dz = 0$$

$$M_c Z_{cm-c} = \rho_c \int_0^h \int_0^{2\pi} \int_0^{\frac{z}{h}} z r dr d\theta dz = 2\pi \rho_c \int_0^h z \int_0^{\frac{z}{h}} r dr dz = 2\pi \rho_c \int_0^h z \frac{a^2 z^2}{2h^2} dz = 2\pi \rho_c \frac{a^2}{2h^2} z^3 dz = 2\pi \rho_c \frac{a^2}{2h^2} \frac{h^4}{4} = \pi \rho_c a^2 \frac{h^4}{4}$$

$$\Rightarrow M_c \vec{R}_c = 0 \hat{i} + 0 \hat{j} + \cancel{\pi \rho_c a^2 \frac{h^2}{4} \hat{k}} = \cancel{\pi \rho_c a^2 \frac{h^2}{3} \hat{k}} \Rightarrow \vec{R}_c = \frac{3h}{4} \hat{k}$$

Hemisferio

$$M_H = \rho_H V_H = \rho_H \frac{4\pi a^3}{3} \cdot \frac{1}{2} = \rho_H \frac{2\pi a^3}{3}$$

$$M_H \vec{R}_H = \int_{A_H} \vec{r} dm = \iiint \rho_H \vec{r} dV = \rho_H \int_0^{\frac{a}{2}} \int_0^{\pi} \int_0^a (r \sin \theta \cos \phi \hat{i} + r \sin \theta \sin \phi \hat{j} + r \cos \theta \hat{k}) r^2 \sin \theta dr d\theta d\phi \quad \text{Por simetria}$$

$$M_H Z_{cm-H} = \rho_H \int_0^{\frac{a}{2}} \int_0^{\pi} \int_0^a r^3 \cos \theta \sin \theta dr d\theta d\phi = 2\pi \rho_H \int_0^{\frac{a}{2}} \int_0^a r^3 \cos \theta \sin \theta dr d\phi = 2\pi \rho_H \int_0^{\frac{a}{2}} \frac{a^4}{4} \cos \theta \sin \theta d\phi = \cancel{2\pi \rho_H \frac{a^4}{4} \frac{1}{2}} = \rho_H \frac{\pi a^4}{4}$$

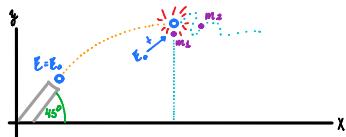
$$\Rightarrow M_H \vec{R}_H = 0 \hat{i} + 0 \hat{j} + \cancel{\rho_H \frac{\pi a^4}{4} \hat{k}} = \cancel{\rho_H \frac{2\pi a^3}{3} \vec{R}_H} \Rightarrow \vec{R}_H = \frac{3a}{8} \hat{k} \Rightarrow \vec{R}_H = \vec{R}_H + h = \left(\frac{3a}{8} + h \right) \hat{k}$$

$$\text{Sistema } M_{H+c} \vec{R}_{H+c} = M_H \vec{R}_H + M_c \vec{R}_c \Rightarrow \left(\rho_H \frac{2\pi a^3}{3} + \rho_c \frac{\pi a^2 h}{3} \right) \vec{R}_{H+c} = \rho_H \frac{2\pi a^3}{3} \left(\frac{3a}{8} + h \right) \hat{k} + \rho_c \frac{\pi a^2 h}{3} \frac{3h}{4} \hat{k}$$

$$\Rightarrow \cancel{\frac{2\pi}{3} (2\rho_H a + \rho_c h) \vec{R}_{H+c}} = \cancel{\frac{\pi a^2}{3} \left[2\rho_H a \left(\frac{3a}{8} + h \right) + \rho_c \frac{3h^2}{4} \right] \hat{k}} \Rightarrow \vec{R}_{H+c} = \frac{\left[2\rho_H a \left(\frac{3a}{8} + h \right) + \rho_c \frac{3h^2}{4} \right]}{(2\rho_H a + \rho_c h)} \hat{k}$$

Problema 2. (Thornton 9.9)

- 9.9. A projectile is fired at an angle of 45° with initial kinetic energy E_0 . At the top of its trajectory, the projectile explodes with additional energy E_0 into two fragments. One fragment of mass m_1 travels straight down. What is the velocity (magnitude and direction) of the second fragment of mass m_2 and the velocity of the first? What is the ratio of m_1/m_2 when m_1 is a maximum?



En el punto más alto de la trayectoria, justo antes de la explosión $M = m_1 + m_2$

$$V_{ay} = 0 \quad y \quad V_{ax} = V_0 \cos(45^\circ) = \frac{V_0}{\sqrt{2}} = \frac{1}{\sqrt{2}} \sqrt{2E_0} \Rightarrow P_x = MV_{ax} = \frac{M}{\sqrt{2}} \sqrt{2E_0} = \sqrt{ME_0}$$

El momentum lineal se conserva: $P_a = p_{a1} + p_{a2} \Rightarrow P_x = p_{a1} + p_{a2}$

$$\Rightarrow (\sqrt{ME_0}, 0, 0) = m_1(0, V_{ay}, 0) + m_2(V_{2x}, V_{2y}, V_{2z}) \Rightarrow \begin{cases} \sqrt{ME_0} = m_2 V_{2x} \\ 0 = m_1 V_{ay} + m_2 V_{2y} \\ 0 = m_2 V_{2z} \end{cases} \rightarrow m_2 y \text{ todo el sistema se mantiene en el plano } XY$$

Dado que entra energía al sistema tras la explosión se tiene, justo antes y después de la explosión

$$\frac{1}{2}MV_{ax}^2 + E_0 = \frac{1}{2}m_1V_{ay}^2 + \frac{1}{2}m_2(V_{2x}^2 + V_{2y}^2) \quad \text{con } * V_{ay} = -\frac{m_2V_{2y}}{m_1}$$

$$\Rightarrow \frac{1}{2}M\left(\frac{1}{\sqrt{2}}\sqrt{2E_0}\right)^2 + E_0 = \frac{1}{2}m_1\left(\frac{m_2V_{2y}^2}{m_1^2}\right) + \frac{1}{2}m_2(V_{2x}^2 + V_{2y}^2) = \frac{1}{2}\frac{m_2^2V_{2y}^2}{m_1} + \frac{1}{2}m_2(V_{2x}^2 + V_{2y}^2)$$

$$\Rightarrow E_0 + 2E_0 = 3E_0 = \frac{m_2^2V_{2y}^2}{m_1} + m_2(V_{2x}^2 + V_{2y}^2) \Rightarrow 3E_0 = m_2V_{2x}^2 + V_{2y}^2\left(1 + \frac{m_2}{m_1}\right) \Rightarrow V_{2y} = \sqrt{\frac{3E_0 - m_2V_{2x}^2}{m_2\left(1 + \frac{m_2}{m_1}\right)}}$$

$$\text{Con } * V_{2x} = \frac{\sqrt{ME_0}}{m_2} \Rightarrow V_{2y} = \sqrt{\frac{3E_0 - m_2\left(\frac{\sqrt{ME_0}}{m_2}\right)^2}{m_2\left(1 + \frac{m_2}{m_1}\right)}} = \sqrt{\frac{3E_0 - \frac{ME_0}{m_2}}{m_2\left(1 + \frac{m_2}{m_1}\right)}} = \sqrt{\frac{3E_0 m_1 m_2 - M E_0 m_1}{m_2^2 (m_1 + m_2)}}$$

$$\Rightarrow \begin{cases} V_2 = \frac{\sqrt{ME_0}}{m_2} t + \sqrt{\frac{3E_0 m_1 m_2 - M E_0 m_1}{m_2^2 (m_1 + m_2)}} \\ V_1 = -\frac{m_2 V_{2y}}{m_1} = -\frac{m_2}{m_1} \sqrt{\frac{3E_0 m_1 m_2 - M E_0 m_1}{m_2^2 (m_1 + m_2)}} \end{cases} \quad \text{Si } m_1 \text{ tiene un valor máximo } V_{ay} = 0 = \frac{m_2}{m_1} \sqrt{\frac{3E_0 m_1 m_2 - M E_0 m_1}{m_2^2 (m_1 + m_2)}} \Rightarrow \frac{m_2}{m_1} = \frac{1}{2}$$

Problema 3. (Taylor 3.4)

3.4** Two hobos, each of mass m_h , are standing at one end of a stationary railroad flatcar with frictionless wheels and mass m_{fc} . Either hobo can run to the other end of the flatcar and jump off with the same speed u (relative to the car). (a) Use conservation of momentum to find the speed of the recoiling car if the two men run and jump simultaneously. (b) What is it if the second man starts running only after the first has already jumped? Which procedure gives the greater speed to the car? [Hint: The speed u is the speed of either hobo, relative to the car just after he has jumped; it has the same value for either man and is the same in parts (a) and (b).]



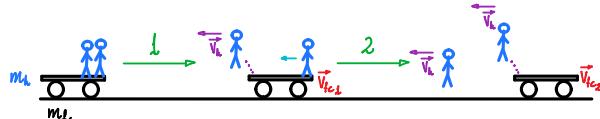
a) Justo antes de saltar: $P_0 = (m_{t_0} + 2m_h)v_{tcf}^0 = 0$

Justo después de saltar: $P_f = m_{t_0}v_{tcf} + 2m_hv_{hf}$

$$\Rightarrow P_0 = P_f \Rightarrow m_{t_0}v_{tcf} + 2m_hv_{hf} = 0 \quad y \quad v_{hf} = v_{tcf} - u \Rightarrow m_{t_0}v_{tcf} + 2m_h(v_{tcf} - u) = 0$$

$$\Rightarrow m_{t_0}v_{tcf} + 2m_hv_{tcf} - 2m_hu = 0 \Rightarrow v_{tcf}(m_{t_0} + 2m_h) = 2m_hu \Rightarrow v_{tcf} = \frac{2m_hu}{m_{t_0} + 2m_h} //$$

b) Hay 2 tráctos, cada uno definido por el salto de los hobos



① Justo antes del primer salto: $P_{01} = (m_{t_0} + 2m_h)v_{tcf}^0 = 0$

$v_{htj} = v_{tcf_j} - u *$

Justo después del primer salto: $P_{t_1} = (m_{t_0} + m_h)v_{tcf_{t_1}} + m_hv_{ht1}$

$$\Rightarrow P_0 = P_{t_1} \Rightarrow 0 = (m_{t_0} + m_h)v_{tcf_{t_1}} + m_hv_{ht1} = (m_{t_0} + m_h)v_{tcf_{t_1}} + m_h(v_{tcf} - u) \Rightarrow (m_{t_0} + m_h)v_{tcf_{t_1}} + m_hv_{tcf} - m_hu = 0$$

$$\Rightarrow (m_{t_0} + 2m_h)v_{tcf_{t_1}} = m_hu \Rightarrow v_{tcf_{t_1}} = \frac{2m_hu}{m_{t_0} + 2m_h}$$

② Justo antes del segundo salto: $P_{02} = (m_{t_0} + m_h)v_{tcf_{t_1}} + \underbrace{m_hv_{ht1}}_{\text{Primer hobo}}$

Justo después del segundo salto: $P_{t_2} = m_{t_0}v_{tcf_{t_2}} + m_hv_{ht2} + \sqrt{m_hv_{ht1}}$

$$\Rightarrow P_0 = P_{t_2} \Rightarrow (m_{t_0} + m_h)v_{tcf_{t_1}} = m_{t_0}v_{tcf_{t_2}} + m_hv_{ht2} \Rightarrow (m_{t_0} + m_h)v_{tcf_{t_1}} = m_{t_0}v_{tcf_{t_2}} + m_h(v_{tcf_{t_1}} - u) *$$

$$\Rightarrow (m_{t_0} + m_h)v_{tcf_{t_1}} = m_{t_0}v_{tcf_{t_2}} + m_hv_{tcf_{t_2}} - m_hu = (m_{t_0} + m_h)v_{tcf_{t_2}} - m_hu \Rightarrow v_{tcf_{t_2}} = \frac{(m_{t_0} + m_h)v_{tcf_{t_1}} + m_hu}{(m_{t_0} + m_h)}$$

$$\Rightarrow v_{tcf_{t_2}} = \frac{2m_hu}{m_{t_0} + 2m_h} + \frac{m_hu}{m_{t_0} + m_h} // \leftarrow \text{Mayor velocidad}$$

Problema 4. (Thornton 9.12)

9-12. Astronaut Stumblebum wanders too far away from the space shuttle orbiter while repairing a broken communications satellite. Stumblebum realizes that the orbiter is moving away from him at 3 m/s. Stumblebum and his maneuvering unit have a mass of 100 kg, including a pressurized tank of mass 10 kg. The tank includes only 2 kg of gas that is used to propel him in space. The gas escapes with a constant velocity of 100 m/s. \leftarrow Debe el marco de referencia del cohete = v_g^* \rightarrow $v_a = v_a - v_g^*$

- (a) Will Stumblebum run out of gas before he reaches the orbiter?
 (b) With what velocity will Stumblebum have to throw the empty tank away to reach the orbiter?

a) $P = \text{constante} \rightarrow \text{No hay fuerzas externas} \Rightarrow \dot{P} = 0$

$$P = m_a V_a + m_g V_g \Rightarrow \dot{P} = m_a \dot{V}_a + m_a V_a + m_g V_g + \cancel{m_g \dot{V}_g} \xrightarrow{\text{El gas tiene velocidad constante}} m_a \dot{V}_a + m_a V_a + m_g V_g = 0 ; m_a = -m_g$$

$$\Rightarrow m_a V_a + m_a V_a - m_a V_g = 0 \Rightarrow m_a (V_a - V_g) = -m_a V_a \Rightarrow m_a (V_a - V_a + V_g^*) = -m_a V_a \Rightarrow m_a V_g^* = -m_a V_a$$

$$\Rightarrow dm_a V_g^* = -m_a dV_a \Rightarrow -\int_{m_a}^{m_f} \frac{V_g^*}{m_a} dm_a = \int_{V_a}^{V_f} dV_a \Rightarrow V_f = V_a + V_g^* \ln\left(\frac{m_a}{m_f}\right)$$

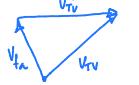
$$\Rightarrow V_f = V_a + 100 \ln\left(\frac{100}{98}\right) = 202 \text{ m/s} \quad \leftarrow \text{No puede alcanzar la nave sin antes quedarse sin gas}$$

b) Antes de lanzar el tanque: $P_0 = (m_s + m_{tv}) V_{at}$ Despues de lanzar el tanque: $P_t = m_s V_t + m_{tv} V_{tv}$

$$\Rightarrow P_0 = P_t \Rightarrow (m_s + m_{tv}) V_{at} = m_s V_t + m_{tv} V_{tv} \Rightarrow V_{tv} = \frac{(m_s + m_{tv}) V_{at} - m_s V_t}{m_{tv}}$$

$$V_t > 3 \text{ m/s}, m_s = 90 \text{ kg} \text{ y } m_{tv} = 8 \text{ kg} \Rightarrow V_{tv} \leq -9 \text{ m/s} \quad \leftarrow \text{Velocidad con la que tiene que lanzar el tanque medida por un observador externo}$$

Velocidad con la que tiene que lanzar el tanque prevista por el astronauta



$$V_{tv}^* \leq V_{tv} - V_{ta} \leq -11 \text{ m/s}$$



Problema 5. (Thornton 9.21)

- 9-21. A flexible rope of length 1.0 m slides from a frictionless table top as shown in Figure 9-E. The rope is initially released from rest with 30 cm hanging over the edge of the table. Find the time at which the left end of the rope reaches the edge of the table.

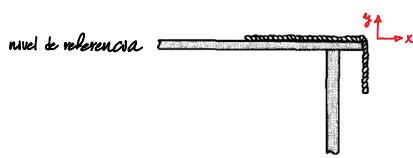


FIGURE 9-E Problem 9-21.

$$p = \frac{m\dot{r}}{L} \Rightarrow m = -p\frac{\dot{y}}{L} = -m\frac{\ddot{y}}{L} \rightarrow 0 \geq \ddot{y} \geq -\frac{L}{2} \quad y(t=0) = l_0 \quad y'(t=0) = 0$$

$$F = \dot{P} \Rightarrow -mg = p\dot{y} + m_r\dot{v} \Rightarrow -mg = m_r\ddot{y} \Rightarrow +\frac{m_r}{L}\ddot{y}g = m_r\ddot{y} \Rightarrow \ddot{y} = \frac{g}{L}y = \omega_0^2 y$$

$$\Rightarrow y = A e^{i\omega_0 t} + B e^{-i\omega_0 t} \xrightarrow[\text{condiciones iniciales}]{\text{Aplicando}} \Rightarrow y = \frac{l_0}{2} (e^{i\omega_0 t} + e^{-i\omega_0 t}) = l_0 \cosh\left(\sqrt{\frac{g}{L}}t\right) \Rightarrow y = l_0 \cosh\left(\sqrt{\frac{g}{L}}t\right)$$

$$y(t) = l_0 \cosh\left(\sqrt{\frac{g}{L}}t\right) \Rightarrow t = \sqrt{\frac{L}{g}} \cosh^{-1}\left(\frac{l_0}{l_0}\right)$$

Dispersión y choque de partículas

Problema 1. (Thornton 9.30)

9-30. A tennis player strikes an incoming tennis ball of mass 60 g as shown in Figure 9-G.

The incoming tennis ball velocity is $v_i = 8 \text{ m/s}$, and the outgoing velocity is $v_f = 16 \text{ m/s}$.

(a) What impulse was given to the tennis ball?

(b) If the collision time was 0.01 s, what was the average force exerted by the tennis racket?

$$\begin{aligned} a) \quad \vec{p}_i &= 8(\sin 45 \hat{i} - \cos 45 \hat{j}) \text{ m} \quad \vec{p}_f = 16(\sin 15 \hat{i} + \cos 15 \hat{j}) \text{ m} \\ \Rightarrow \vec{\Delta p} &= m(16 \sin 15 - 8 \sin 45) \hat{i} + m(16 \cos 15 + 8 \cos 45) \hat{j} \\ &= -0,09 \text{ N}\cdot\text{s} \hat{i} + 1,27 \text{ N}\cdot\text{s} \hat{j} // \end{aligned}$$

$$b) \quad F_{\text{impulso}} = \frac{\vec{\Delta p}}{\Delta t} = -9,09 \text{ N} \hat{i} + 126,7 \text{ N} \hat{j} //$$

Problema 2. (Thornton 9.37)

9-37. When a bullet fires in a gun, the explosion subsides quickly. Suppose the force on the bullet is $F = (360 - 10^7 t^2 \text{ s}^{-2}) \text{ N}$ until the force becomes zero (and remains zero). The mass of the bullet is 3 g.

(a) What impulse acts on the bullet?

(b) What is the muzzle velocity of the gun?

$$a) \quad \Delta P = \int_0^t 360 - 10^7 t^2 dt = 360t - \frac{10^7 t^3}{3} \Rightarrow \Delta P = 360t - \frac{10^7 t^3}{3} \quad \text{donde } t \text{ es el tiempo que dura la colisión,} \\ \text{tiempo en que la fuerza impulsiva es} \\ \text{diferente de cero.}$$

$$F=0 \Rightarrow 360 - 10^7 t^2 = 0 \Rightarrow t = 6 \cdot 10^{-3} \quad \Rightarrow \Delta P = 1,44 \text{ N}\cdot\text{s} //$$

$$b) \quad \Delta P = m v_e \Rightarrow v_e = \frac{\Delta P}{m} = 480 \text{ m/s} //$$

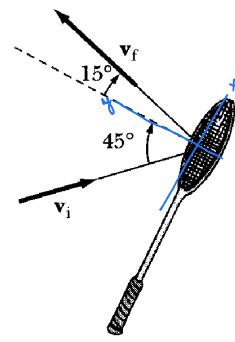


FIGURE 9-G Problem 9-30.

- 9.22. A deuteron (nucleus of deuterium atom consisting of a proton and a neutron) with speed 14.9 km/s collides elastically with a neutron at rest. Use the approximation that the deuteron is twice the mass of the neutron. (a) If the deuteron is scattered through a LAB angle $\psi = 10^\circ$, what are the final speeds of the deuteron and neutron? (b) What is the LAB scattering angle of the neutron? (c) What is the maximum possible scattering angle of the deuteron?

- 9-24.** A particle of mass m at the end of a light string wraps itself about a fixed vertical cylinder of radius a (Figure 9-F). All the motion is in the horizontal plane (disregard gravity). The angular velocity of the cord is ω_0 when the distance from the particle to the point of contact of the string and cylinder is b . Find the angular velocity and tension in the string after the cord has turned through an additional angle θ .

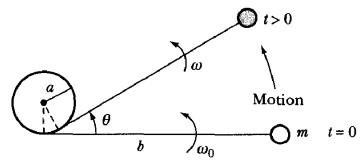


FIGURE 9-F Problem 9-24.

- 9-40. A particle of mass m_1 and velocity u_1 strikes head-on a particle of mass m_2 at rest. The coefficient of restitution is ϵ . Particle m_2 is tied to a point a distance a away as shown in Figure 9-H. Find the velocity (magnitude and direction) of m_1 and m_2 after the collision.

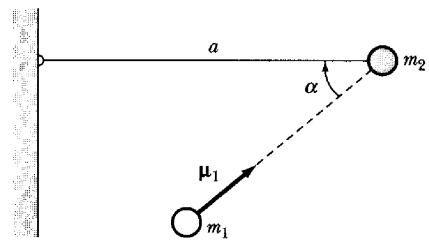


FIGURE 9-H Problem 9-40.

Problema 3. (Taylor 3.8)

3.8* A rocket (initial mass m_0) needs to use its engines to hover stationary, just above the ground. (a) If it can afford to burn no more than a mass λm_0 of its fuel, for how long can it hover? [Hint: Write down the condition that the thrust just balance the force of gravity. You can integrate the resulting equation by separating the variables t and m . Take v_{ex} to be constant.] (b) If $v_{ex} \approx 3000$ m/s and $\lambda \approx 10\%$, for how long could the rocket hover just above the earth's surface?

$$\text{Midiendo desde LAB } y \quad V_c = V_r + V_{ex}^*$$

$$\begin{aligned}
 a) \quad P &= m_r V_r + m_c V_{ex} \Rightarrow \dot{P} = -m_r g \Rightarrow \dot{m}_r V_r + m_r \dot{V}_r + \dot{m}_c V_c + m_c \dot{V}_c = -m_r g ; \quad \dot{m}_r = -\dot{m}_c \quad y \quad V_c = \text{constante} \\
 &\Rightarrow \dot{m}_r (V_r - V_c) + m_r \dot{V}_r = -m_r g \stackrel{*}{\Rightarrow} \dot{m}_r (V_r - V_r - V_{ex}) + m_r \dot{V}_r = -m_r g \Rightarrow -\dot{m}_r V_{ex} + m_r \dot{V}_r = -m_r g \\
 &\Rightarrow \frac{dm_r}{dt} V_{ex} = m_r \frac{dV_r}{dt} + m_r g = m_r \left(\frac{dV_r}{dt} + g \right) \Rightarrow \frac{dm_r}{dt} V_{ex} = m_r \left(\frac{dV_r}{dt} + g \right) \Rightarrow \frac{1}{m_r} \frac{dm_r}{dt} V_{ex} = \frac{1}{dt} \left(dV_r + g dt \right) \\
 &\Rightarrow \left(\frac{1}{m_r} \frac{dm_r}{dt} V_{ex} - g \right) dt = dV_r \Rightarrow \left(g \frac{dt}{dm_r} - \frac{1}{m_r} V_{ex} \right) \cdot \frac{dm_r}{dt} dt = dV_r ; \quad -\frac{dm_r}{dt} = \alpha \Rightarrow \left(g - \frac{V_{ex}}{m_r} \right) dm_r = dV_r \\
 &\Rightarrow \int_{m_0}^m \left(g - \frac{V_{ex}}{m_r} \right) dm_r = \int_{V_0}^{V_f} dV_r \Rightarrow V_f = V_0 - V_{ex} \ln \left(\frac{m_0}{m} \right) + \frac{g}{\alpha} (m - m_0) \quad y \quad \frac{dm_r}{dt} = -\alpha \Rightarrow m - m_0 = -\alpha t \\
 &\Rightarrow V_f - V_0 = -V_{ex} \ln \left(\frac{m_0}{m} \right) - gt = 0 \Rightarrow -V_{ex} \ln \left(\frac{m_0}{m} \right) - gt = 0 \Rightarrow gt = V_{ex} \ln \left(\frac{m_0}{m} \right) \Rightarrow t = \frac{1}{g} V_{ex} \ln \left(\frac{m_0}{m} \right) //
 \end{aligned}$$

Otra forma

$$\begin{aligned}
 &\Rightarrow \dot{m}_r V_r + m_r \dot{V}_r + \dot{m}_c V_c + m_c \dot{V}_c = -m_r g \stackrel{V_r \text{ constante}}{\Rightarrow} \dot{m}_r (V_r - V_c) = -m_r g \stackrel{*}{\Rightarrow} \dot{m}_r (V_r - V_r - V_{ex}) = -m_r g \\
 &\Rightarrow -\dot{m}_r V_{ex} = -m_r g \Rightarrow \frac{dm_r}{dt} V_{ex} = m_r g \Rightarrow -\int_{m_0}^m \frac{1}{m_r} dm_r = \int_0^t \frac{g}{V_{ex}} dt \Rightarrow gt = -V_{ex} \ln \left(\frac{m_0}{m} \right) = V_{ex} \ln \left(\frac{m_0}{m} \right) \\
 &\Rightarrow t = \frac{1}{g} V_{ex} \ln \left(\frac{m_0}{m} \right) //
 \end{aligned}$$

$$b) \quad t = \frac{1}{g} V_{ex} \ln \left(\frac{m_0}{m_0 - \lambda m_0} \right) = \frac{1}{g} V_{ex} \ln \left(\frac{1}{1-\lambda} \right) //$$

=

9-58. Consider a single-stage rocket taking off from Earth. Show that the height of the rocket at burnout is given by Equation 9.166. How much farther in height will the rocket go after burnout?

9-59. A rocket has an initial mass of m and a fuel burn rate of α (Equation 9.161). What is the minimum exhaust velocity that will allow the rocket to lift off immediately after firing?

Problema 4. (Thornton 9.50/Taylor 14.23)

9-50. A fixed force center scatters a particle of mass m according to the force law $F(r) = k/r^3$. If the initial velocity of the particle is u_0 , show that the differential scattering cross section is

$$\sigma(\theta) = \frac{k\pi^2(\pi - \theta)}{mu_0^3\theta^2(2\pi - \theta)^2 \sin \theta}$$

$$\sigma(\theta) = \frac{d\Omega}{d\Omega'} \Rightarrow \sigma(\theta) = \frac{d\Omega}{d\Omega'} = \frac{1}{I} \frac{dN}{d\Omega'} ; d\Omega = 2\pi \sin \theta d\theta$$

$$\Rightarrow \cancel{\int 2\pi b db} = - \cancel{\int 2\pi \sigma(\theta) \sin \theta d\theta}$$

$$\Rightarrow \sigma(\theta) = - \frac{b}{\sin \theta} \frac{db}{d\theta} = \frac{b}{\sin \theta} \left| \frac{db}{d\theta} \right| \quad \text{Hay que hallar la función } b(\theta)$$

$$\theta = |\pi + 2\Theta| = \left| \pi + 2 \int_{r_{\min}}^{r_0} \frac{\frac{b}{r^2}}{\sqrt{2m(E-U - \frac{K^2}{2mr^2})}} dr \right| ; F = -\frac{dU}{dr} \Rightarrow U = - \int F dr = - \int \frac{K}{r^3} dr = \frac{K}{2r^2} \quad K = mu_0 b$$

$$\text{Reescribiendo } L, \text{ primero } T_0 = \frac{1}{2} mu_0^2 \Rightarrow u_0 = \sqrt{\frac{2T_0}{m}} \Rightarrow L = mb \sqrt{\frac{2T_0}{m}} = b \sqrt{2mT_0}$$

$$\Rightarrow \theta = \int_{r_{\min}}^{r_0} \frac{b \sqrt{2mT_0}}{\sqrt{2m \left[T_0 - \frac{K}{2r^2} - \frac{2mT_0 b^2}{2mr^2} \right]}} dr = \int_{r_{\min}}^{r_0} \frac{b \sqrt{2mT_0}}{\sqrt{1 - \frac{b^2}{r^2} - \frac{K}{r^2 T_0}}} dr = \int_{r_{\min}}^{r_0} \frac{b}{\sqrt{1 - \frac{b^2}{r^2} - \frac{K}{r^2 m u_0^2}}} dr$$

$$\text{Sustituyendo } z = \frac{b}{r} \quad dz = -\frac{b}{r^2} dr \Rightarrow \theta = - \int_{\frac{b}{r_{\min}}}^0 \frac{b}{\sqrt{1 - b^2 z^2 - \frac{K}{m u_0^2 z^2}}} dz = \int_0^{\frac{b}{r_{\min}}} \frac{b}{\sqrt{1 - z^2 \left(b^2 + \frac{K}{m u_0^2} \right)}} dz$$

$$\Rightarrow \theta = \sqrt{\frac{mb^2 u_0^2}{mb^2 u_0^2 + K}} \left| b \int_0^{\frac{b}{r_{\min}}} \frac{1}{\sqrt{\frac{mb^2 u_0^2}{mb^2 u_0^2 + K} - z^2}} dz \right| = \sqrt{\frac{mb^2 u_0^2 b^2}{mb^2 u_0^2 + K}} \left| \sin^{-1} \left(\frac{z}{\sqrt{\frac{mb^2 u_0^2}{mb^2 u_0^2 + K}}} \right) \right|_0^{\frac{b}{r_{\min}}}$$

$$\Rightarrow \theta = \sqrt{\frac{mb^2 u_0^2 b^2}{mb^2 u_0^2 + K}} \left| \sin^{-1} \left(\frac{z_{\max}}{\sqrt{\frac{mb^2 u_0^2}{mb^2 u_0^2 + K}}} \right) \right| ; \frac{mb^2 u_0^2}{mb^2 u_0^2 + K} - z^2 = 0 \Rightarrow z = \sqrt{\frac{mb^2 u_0^2}{mb^2 u_0^2 + K}}$$

$$\Rightarrow \theta = \sqrt{\frac{mb^2 u_0^2}{mb^2 u_0^2 + K}} \cdot \frac{\pi}{2} \Rightarrow b = \sqrt{\frac{K}{m u_0^2}} \frac{2\theta}{\sqrt{\theta^2 - 4\theta^2}} \Rightarrow \theta = |\pi + 2\Theta| \Rightarrow \theta = \frac{1}{2} (\theta - \pi) \Rightarrow b = \sqrt{\frac{K}{m u_0^2}} \frac{\theta - \pi}{\sqrt{\theta(2\pi - \theta)}}$$

$0 \leq \theta \leq \pi$
sin el menor de negativo

$$\Rightarrow \frac{db}{d\theta} = \sqrt{\frac{K}{m u_0^2}} \frac{\sqrt{\theta(2\pi - \theta)} - (\theta - \pi)}{\sqrt{\theta(2\pi - \theta)^2}} \cdot \frac{1/(2\pi - \theta) - \theta}{2\theta(2\pi - \theta)} = \sqrt{\frac{K}{m u_0^2}} \frac{\pi^2}{\sqrt{\theta(2\pi - \theta)^3}} \Rightarrow \sigma(\theta) = \frac{b}{\sin \theta} \left| \frac{db}{d\theta} \right|$$

$$\Rightarrow \sigma(\theta) = \frac{1}{\sin \theta} \sqrt{\frac{K}{m u_0^2}} \frac{\pi - \theta}{\sqrt{\theta(2\pi - \theta)}} \cdot \sqrt{\frac{K}{m u_0^2}} \frac{\pi^2}{\sqrt{\theta(2\pi - \theta)^3}} = \frac{1}{\sin \theta} \frac{K}{m u_0^2} \frac{\pi^2 (\pi - \theta)}{\sqrt{\theta(2\pi - \theta)^4}} = \frac{1}{\sin \theta} \frac{K}{m u_0^2} \frac{\pi^2 (\pi - \theta)}{\theta^2 (2\pi - \theta)^2}$$

Problema 5. (Thornton 9.53)

9.53. The most energetic α -particles available to Ernest Rutherford and his colleagues for the famous Rutherford scattering experiment were 7.7 MeV. For the scattering of 7.7 MeV α -particles from ^{238}U (initially at rest) at a scattering angle in the lab of 90° (all calculations are in the LAB system unless otherwise noted), find the following:

- the recoil scattering angle of ^{238}U .
- the scattering angles of the α -particle and ^{238}U in the CM system.
- the kinetic energies of the scattered α -particle and ^{238}U .
- the impact parameter b .
- the distance of closest approach r_{\min} .
- the differential cross section at 90° .
- the ratio of the probabilities of scattering at 90° to that of 5° .

$$m_1: \alpha\text{-partículas} = 4 \text{ amas} \quad m_2: ^{238}\text{U} = 238 \text{ amas}$$

$$q_1 = 2e^+$$

$$q_2 = 92e^+$$

$$T_0 = 7.7 \cdot 10^6 \text{ eV} = 1,234 \cdot 10^{-12} \text{ J}$$

$\Psi = 90^\circ$: Ángulo de dispersión de m_1 en LAB



a) $\zeta = ?$ Hay que resolver la ecuación

Momentum lineal

$$\text{Antes: } \vec{P}_b = \vec{p}_{U_1} = m_1(v_1, 0) \quad \text{Después: } \vec{P}_t = \vec{p}_{U_1} + \vec{p}_{U_2} = m_1(0, v_1) + m_2(v_2 \cos \zeta, v_2 \sin \zeta)$$

$$\Rightarrow \vec{P}_b = \vec{P}_t \Rightarrow \begin{cases} m_1 v_1 = m_2 v_2 \cos \zeta \\ 0 = m_1 v_1 + m_2 v_2 \sin \zeta \end{cases} \Rightarrow \begin{aligned} m_1 v_1 &= m_2 v_2 \cos \zeta \Rightarrow m_1 (\vec{v}_1 - \vec{v}_2) = m_2 \vec{v}_2 \\ m_1^2 v_1^2 &= m_2^2 v_2^2 \Rightarrow m_1^2 (v_1^2 + v_2^2 - 2 \vec{v}_1 \cdot \vec{v}_2) = m_2^2 v_2^2 \\ m_1^2 v_1^2 + m_2^2 v_2^2 &= m_2^2 v_2^2 \Rightarrow 1 + \frac{v_2^2}{m_1^2 v_1^2} = \frac{m_2^2 v_2^2}{m_1^2 v_1^2} \Rightarrow \frac{v_2^2}{v_1^2} = \frac{m_2^2 + m_1^2}{m_1^2} \end{aligned} \star$$

Energía

$$\text{Antes: } E_{01} = T_0 = \frac{1}{2} m_1 v_1^2 \quad \text{Después: } E_{04} = T_{1+} + T_{2+} = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 \Rightarrow E_{01} = E_{04} \Rightarrow T_0 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$

$$\Rightarrow \begin{cases} m_1 v_1 = m_2 v_2 \cos \zeta : X \\ 0 = m_1 v_1 + m_2 v_2 \sin \zeta : Y \\ \frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 \end{cases} \Rightarrow \frac{v_2}{v_1} = -\tan \zeta \Rightarrow v_2 = -v_1 \tan \zeta \quad \textcircled{4}$$

$$\frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 \Rightarrow m_1 v_1^2 = m_1 v_1^2 + m_2 v_2^2 \Rightarrow 1 = \frac{v_1^2}{m_1^2 v_1^2} + \frac{m_2 v_2^2}{m_1^2 v_1^2} \Rightarrow 1 - \frac{m_2 v_2^2}{m_1^2 v_1^2} = \frac{v_1^2}{m_1^2 v_1^2} \quad \textcircled{*}$$

$$\text{De } \textcircled{*} \text{ y } \textcircled{*}: \Rightarrow \frac{v_1^2}{v_1^2} = 1 - \frac{m_2}{m_1} \left(\frac{m_2}{m_2 + m_1} + \frac{m_2 v_2^2}{m_2^2 v_1^2} \right) = 1 - \frac{m_2}{m_1} - \frac{m_2 v_2^2}{m_2^2 v_1^2} \quad y \quad \frac{v_2^2}{v_1^2} = a \Rightarrow a = 1 - \frac{m_2}{m_1} - \frac{m_2}{m_2} a = \frac{m_2 - m_1}{m_2} - \frac{m_2}{m_2} a$$

$$\Rightarrow a + \frac{m_2}{m_2} a = a \left(1 + \frac{m_2}{m_2} \right) = a \left(\frac{m_2 + m_1}{m_2} \right) = \frac{m_2 + m_1}{m_2} \Rightarrow a = \frac{\frac{m_2 - m_1}{m_2}}{\frac{m_2 + m_1}{m_2}} = \frac{m_2 - m_1}{m_2 + m_1} = \frac{v_1^2}{v_1^2} = \frac{1}{\tan^2 \zeta} \quad \textcircled{5}$$

$$\Rightarrow \tan \zeta = \sqrt{\frac{m_2 - m_1}{m_2 + m_1}} \approx 0.98 \Rightarrow \zeta \approx 44.5^\circ \Rightarrow \zeta \approx -44.5^\circ //$$

b) $\theta = ?$



* Velocidad del centro de masa: $M\vec{V} = m_1 \vec{U}_1 + m_2 \vec{U}_2 \Rightarrow \vec{V} = \frac{m_1}{m_1+m_2} \vec{U}_1$

* Transformaciones de marco de referencia, LAB \leftrightarrow CM

$$\bullet \vec{U}_1 = \vec{V} + \vec{U}'_1 \Rightarrow \vec{U}'_1 = \vec{U}_1 - \vec{V} = \vec{U}_1 - \frac{m_1}{m_1+m_2} \vec{U}_1 = \frac{m_2}{m_1+m_2} \vec{U}_1 = \frac{m_2}{m_1+m_2} u_1 \hat{i}$$

Partida 1:

$$\bullet \vec{V}_1 = \vec{V} + \vec{U}'_1 \Rightarrow \vec{V}'_1 = \vec{V}_1 - \vec{V} = -u_1 \tan \xi \hat{j} - \frac{m_1}{m_1+m_2} u_1 \hat{i}$$

$$\bullet \vec{U}_2 = \vec{V} + \vec{U}'_2 \Rightarrow \vec{U}'_2 = -\vec{V} = -\frac{m_1}{m_1+m_2} \vec{U}_1 = -\frac{m_1}{m_1+m_2} u_1 \hat{i}$$

Partida 2:

$$\bullet \vec{V}_2 = \vec{V} + \vec{U}'_2 \Rightarrow \vec{V}'_2 = \vec{V}_2 - \vec{V} = \frac{m_1 u_1}{m_2 \cos \xi} (\cos \xi \hat{i} + \sin \xi \hat{j}) - \frac{m_1}{m_1+m_2} u_1 \hat{i}$$

Buscando ϕ a partir de \vec{V}'_2

$$\tan \phi = \frac{\frac{m_1 u_1 \tan \xi}{m_2}}{\frac{m_1}{m_2} - \frac{m_1}{m_1+m_2} u_1} = \frac{\frac{m_1 \tan \xi}{m_2}}{\frac{m_1(m_1+m_2) - m_1 m_2}{(m_1+m_2)m_2}} = \frac{m_1(m_1+m_2) \tan \xi}{m_2^2} = \frac{(m_1+m_2) \tan \xi}{m_2} = \frac{(m_1+m_2)}{m_2} \sqrt{\frac{m_2^2 - m_1^2}{m_2^2 + m_1^2}} = \sqrt{\frac{m_2^2 - m_1^2}{m_1^2}}$$

$$= \sqrt{\frac{m_2^2 - m_1^2}{m_1^2}} = 59,49 \Rightarrow \phi = 89,0^\circ \Rightarrow \phi = -89,0^\circ \Rightarrow \theta = 180^\circ - \phi = 91^\circ$$

c) $T_{1_0}, T_{2_0}, T_{1_+}, T_{2_+} = ?$ $T_{1_0} = T_0 = 1,234 \cdot 10^{-12} \text{ J}$ $T_{2_0} = 0$ Utilizando la ya planteada ecuación de la conservación de la energía

$$\Rightarrow \frac{1}{2} m_1 u_1^2 = \frac{1}{2} m_1 V_1^2 + \frac{1}{2} m_2 V_2^2 \quad y \quad V_2 = -u_1 \tan \xi$$

$$\Rightarrow T_{2_+} = \frac{1}{2} m_2 V_2^2 = \frac{1}{2} m_2 u_1^2 (1 - \tan^2 \xi) = T_0 (1 - \tan^2 \xi) = 4,23 \cdot 10^{-14} \text{ J} \quad y \quad T_{1_+} = \frac{1}{2} m_1 V_1^2 = \frac{1}{2} m_1 u_1^2 \tan^2 \xi = 1,19 \cdot 10^{-12} \text{ J}$$

d) $b = ?$ Usando el resultado conocido para la dispersión de Rutherford

$$b = \frac{K}{2T_0} \cot(\frac{\theta}{2}) ; K = \frac{q_1 q_2}{4\pi \epsilon_0} = 4,25 \cdot 10^{-26} \quad y \quad T_0 = \frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2 = \frac{1}{2} m_1 \frac{m_2^2}{(m_1+m_2)^2} u_1^2 + \frac{1}{2} m_2 \frac{m_1^2}{(m_1+m_2)^2} u_1^2$$

$$\Rightarrow b = \frac{1}{2} \frac{m_1 m_2 u_1^2 (m_1+m_2)}{m_1 + m_2} = \frac{1}{2} \frac{m_1 m_2 u_1^2}{m_1 + m_2} T_0 = 1,21 \cdot 10^{-12} \text{ m}$$

e) La interacción entre partículas es de tipo columbiana $\Rightarrow U = \frac{k}{r}$, este tipo de potenciales responden a una órbita de la forma

$$\frac{\alpha}{r} = l + \epsilon \cos \theta \quad \text{en el marco de referencia CM}$$

• $r = r_{\min}$ cuando $\frac{l}{r}$ es máxima $\Rightarrow \theta = 0 \Rightarrow r_{\min} = \frac{l + \epsilon}{\alpha} \leftarrow$ Distancia mínima entre la partícula y CM

$$\alpha = \frac{l^2}{m_1 h}, \epsilon = \sqrt{\frac{2Eh^2}{m_1 h^2} + l}, h = m_1 l b^2, k = \frac{q_1 q_2}{4\pi \epsilon_0} = 4,25 \cdot 10^{-26} \text{ y } b = 1,73 \cdot 10^{-14} \text{ m}$$

$$\Rightarrow \alpha = \frac{m_1^{1/2} b^{1/2} 4\pi \epsilon_0}{q_1 q_2} = m_1^{1/2} b^{1/2} \frac{4\pi \epsilon_0}{q_1 q_2}$$

$$\epsilon = \sqrt{\frac{2T_0 m_1^{1/2} b^{1/2}}{m_1} \frac{4^2 \pi^2 \epsilon_0^2}{q_1^2 q_2^2} + l} = \sqrt{2T_0 m_1^{1/2} b^{1/2} \frac{4^2 \pi^2 \epsilon_0^2}{q_1^2 q_2^2} + l} \Rightarrow r_{\min} = \frac{m_2}{m_1 + m_2} r_{\min} = 0,93 \cdot 10^{-14} \text{ m} //$$

f) $\frac{d\sigma}{d\Omega_{\text{cm}}} = \frac{b}{\sin \theta} \left| \frac{db}{d\theta} \right| = \frac{k^2}{16 T_0^{1/2}} \frac{l}{\sin^4(\frac{\theta}{2})} \quad \frac{d\sigma(\psi)}{d\Omega_{\text{MB}}} = \frac{d\sigma(\theta)}{d\Omega_{\text{cm}}} \frac{(X \cos \psi + \sqrt{1 - X^2 \sin^2 \psi})^2}{\sqrt{1 - X^2 \sin^2 \psi}} ; \quad X = \frac{m_2}{m_1} \text{ y } \psi = 90^\circ //$

$$\theta = \arctan(X \sin \psi) + \psi$$

g) $\frac{d\sigma}{d\Omega_{\text{MB}}}(\psi) = \frac{d\sigma(\theta)}{d\Omega_{\text{cm}}} \frac{(X \cos \psi + \sqrt{1 - X^2 \sin^2 \psi})^2}{\sqrt{1 - X^2 \sin^2 \psi}} ; \quad \psi_1 = 90^\circ \text{ y } \psi_2 = 5^\circ //$

Problema 6. (Thornton 9.63)

- 9-63. A new projectile launcher is developed in the year 2023 that can launch a 10^4 kg spherical probe with an initial speed of 6000 m/s. For testing purposes, objects are launched vertically.
- Neglect air resistance and assume that the acceleration of gravity is constant. Determine how high the launched object can reach above the surface of Earth.
 - If the object has a radius of 20 cm and the air resistance is proportional to the square of the object's speed with $c_w = 0.2$, determine the maximum height reached. Assume the density of air is constant.
 - Now also include the fact that the acceleration of gravity decreases as the object soars above Earth. Find the height reached.
 - Now add the effects of the decrease in air density with altitude to the calculation. We can very roughly represent the air density by $\log_{10}(\rho) = -0.05h + 0.11$ where ρ is the air density in kg/m^3 and h is the altitude above Earth in km. Determine how high the object now goes.

$$a) \cancel{\frac{V_f^2 - V_0^2}{2}} = -2g\Delta y \Rightarrow V_0^2 = 2g\Delta y \Rightarrow \Delta y = \frac{V_0^2}{2g} = 1,83 \cdot 10^6 \text{ m}$$

$$b) \vec{F} = -\frac{1}{2} C_w \rho A V^2 \frac{\vec{V}}{V} = -\frac{1}{2} C_w \rho A V \vec{V} \quad \text{Como es trío vertical} \Rightarrow \vec{F} = -\frac{1}{2} C_w \rho A V_y^2 \hat{j}$$

$$\Rightarrow \sum \vec{F}_y = -mg + \vec{F} = m \vec{a}_y \Rightarrow m \vec{a}_y = -\frac{1}{2} C_w \rho A V_y^2 - mg = m \frac{dV_y}{dt} \Rightarrow -C_w \rho A V_y^2 - 2mg = 2m \frac{dV_y}{dt}$$

$$\Rightarrow \int_0^t \frac{1}{2m} dt = - \int_{V_0}^{V_y} \frac{1}{C_w \rho A V_y^2 + 2mg} dV_y = - \frac{1}{C_w \rho A} \int_{V_0}^{V_y} \frac{1}{V_y^2 + V_t^2} dV_y$$

$$\rho = 1,12 \text{ kg/m}^3$$

$$\Rightarrow \frac{t}{2m} = - \frac{1}{C_w \rho A} \frac{1}{V_t} \tan^{-1} \left(\frac{V_y}{V_t} \right) + \frac{1}{C_w \rho A} \frac{1}{V_t} \tan^{-1} \left(\frac{V_0}{V_t} \right)$$

$$V_t = \sqrt{\frac{2mg}{C_w \rho A}} = 2,64 \cdot 10^3$$

$$\Rightarrow \frac{C_w \rho A}{2mg} V_t g t = - \tan^{-1} \left(\frac{V_y}{V_t} \right) + \tan^{-1} \left(\frac{V_0}{V_t} \right) = \frac{gt}{V_t} \Rightarrow \tan^{-1} \left(\frac{V_y}{V_t} \right) = \tan^{-1} \left(\frac{V_0}{V_t} \right) - \frac{gt}{V_t} \Rightarrow V_y = V_t \tan \left[\tan^{-1} \left(\frac{V_0}{V_t} \right) - \frac{gt}{V_t} \right]$$

$$\Rightarrow \frac{dy}{dt} = V_t \tan \left[\tan^{-1} \left(\frac{V_0}{V_t} \right) - \frac{gt}{V_t} \right] \Rightarrow \int_0^t dy = y = \int_0^t V_t \tan \left[\tan^{-1} \left(\frac{V_0}{V_t} \right) - \frac{gt}{V_t} \right] dt \quad \text{En la altura máxima } y=0$$

$$\Rightarrow t = 3,11 \text{ s}$$

$$\Rightarrow y = \frac{V_t^2}{g} \ln \left\{ \cos \left[\tan^{-1} \left(\frac{V_0}{V_t} \right) - \frac{gt}{V_t} \right] \right\} - \frac{V_t^2}{g} \ln \left\{ \cos \left[\tan^{-1} \left(\frac{V_0}{V_t} \right) \right] \right\} \quad \text{Altura máxima} \Rightarrow y_{\max} = 6,46 \cdot 10^5 \text{ m} //$$

$$c) m \ddot{y} = -\frac{1}{2} C_w \rho A V_y^2 - mg \ddot{y} \Rightarrow m \ddot{y} = -\frac{1}{2} C_w \rho A \dot{y}^2 - m \frac{GM}{(R+y)^2} = -\frac{1}{2} C_w \rho A \dot{y}^2 - mg \frac{R^2}{(R+y)^2}$$

$$\Rightarrow \ddot{y} = -\frac{C_w \rho A \dot{y}^2}{2m} - g \frac{R^2}{(R+y)^2}$$

$$d) m \ddot{y} = -\frac{1}{2} C_w \rho \dot{y} A V_y^2 - mg \ddot{y} \Rightarrow m \ddot{y} = -\frac{1}{2} C_w 10^{-0,05y+0,11} A \dot{y}^2 - mg \frac{R^2}{(R+y)^2}$$

$$\Rightarrow \ddot{y} = \frac{C_w A 10^{-0,05y+0,11}}{2m} \dot{y}^2 - g \frac{R^2}{(R+y)^2} //$$

SECTION 5

Sistemas no inerciales

Por ahora se han trabajado solo con sistemas inerciales (los cuales poseen marcos de referencia inerciales respecto a las estrellas fijas, es decir marcos de referencia que no aceleran o rotan respecto a estas o que se considera que no lo hacen) y además se plantearon las Leyes de Newton válidas para estos sistemas, por lo que es posible pensar que los marcos de referencia no inerciales pueden ser un problema catastrófico para la toda física, mas no es así.

En esta sección se desarrollará expresiones que permitirán la generalización de la Mecánica Newtoniana, de modo que se puedan estudiar sistemas inerciales como no inerciales y se mostrarán peculiaridades que poseen los sistemas no inerciales.

SUBSECTION 5.1

Sistemas de referencia en traslación y rotación

Antes de comenzar, se va a establecer la notación presente en Figura 9, entonces:

- \mathcal{O}' : Es el origen del marco de referencia inercial.
- \mathcal{O} : Origen del marco de referencia no inercial, con aceleración translacional y rotación.
- \vec{r}' : Es el radio vector con origen en \mathcal{O}' que da la posición del punto P respecto a dicho origen.
- \vec{R} : Es el radio vector con origen en \mathcal{O}' que une los marcos de referencia **inercial** \mathcal{O}' y **no inercial** \mathcal{O} .
- \vec{r} : Radio vector con el origen en \mathcal{O} que da la posición el punto P respecto a dicho origen.
- ω_{ega} : Es la velocidad angular con que rota el marco de referencia **no inercial** \mathcal{O} .

Hecho esto, el objetivo del siguiente desarrollo es deducir la relación diferencial para la derivada temporal de un vector arbitrario (Por ejemplo, el vector posición) entre vectores con origen inercial y vectores con origen no inercial.

Ahora, por propiedades vectoriales se sabe:

$$\vec{r}' = \vec{R} + \vec{r} \quad (5.1)$$

Suponga que todos estos vectores son medidos desde el marco de referencia ¹⁰ **inercial** \mathcal{O}' y se aplica la derivada temporal:

$$\left(\frac{d\vec{r}'}{dt} \right)_{\text{fijo}} = \left(\frac{d\vec{R}}{dt} \right)_{\text{fijo}} + \left(\frac{d\vec{r}}{dt} \right)_{\text{fijo}} \quad (5.2)$$

Aquí la pregunta, ¿qué pasa si el observador se encuentra en el marco no inercial y desde este se desea realizar los análisis? Hay que obtener una relación que permita lo siguiente:

$$\left(\frac{d\vec{r}}{dt} \right)_{\text{fijo}} \leftrightarrow \left(\frac{d\vec{r}}{dt} \right)_{\text{rotacional}}$$

Entonces, desarrollando la expresión del vector $\vec{r}_{\text{rotacional}}$ y derivada temporal (Desde el marco no inercial ¹¹):

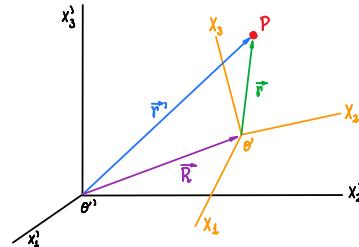


Figura 9. Un sistema de referencia **inercial** \mathcal{O}' y uno **no inercial** \mathcal{O} desde los que se estudia una partícula P .

¹⁰ Para referirse de ahora en adelante al marco de referencia desde el que es medido un vector se usará:

- **fijo** : Al medir desde el marco inercial.
- **rotacional** : al medir desde el marco de referencia no inercial sin importar la naturaleza de la no inercialidad.

¹¹ Para este tema es fundamental entender en qué lugar se encuentra el observador, donde está el origen desde el que se mide. Además, es preciso tener claro las coordenadas que se usan en todo momento.

$$\vec{r}_{rotacional} = \sum_{i=1}^3 r_i \hat{e}_i$$

$$\left(\frac{d\vec{r}}{dt} \right)_{rotacional} = \sum_{i=1}^3 \frac{dr_i}{dt} \hat{e}_i$$

Ahora la velocidad desde una perspectiva inercial ¹²:

$$\begin{aligned} \left(\frac{d\vec{r}}{dt} \right)_{fijo} &= \sum_{i=1}^3 \frac{dr_i}{dt} \hat{e}_i + \sum_{i=1}^3 r_i \left(\frac{d\hat{e}_i}{dt} \right)_{fijo} \\ &= \underbrace{\sum_{i=1}^3 \frac{dr_i}{dt} \hat{e}_i}_{\left(\frac{d\vec{r}}{dt} \right)_{rotacional}} + \sum_{i=1}^3 r_i (\vec{\omega} \times \hat{e}_i) \\ &= \left(\frac{d\vec{r}}{dt} \right)_{rotacional} + \vec{\omega} \times \sum_{i=1}^3 r_i \hat{e}_i \\ &= \left(\frac{d\vec{r}}{dt} \right)_{rotacional} + \vec{\omega} \times \vec{r} \end{aligned}$$

¹² Aquí va a ser importante recordar Ecuación (3.14), que se puede reescribir de la siguiente forma para aplicarla al caso actual de estudio:

$$\left(\frac{d\vec{r}}{dt} \right)_{fijo} = \vec{\omega} \times \vec{r}$$

Finalmente se obtuvo la relación de las derivadas temporales entre los sistemas de referencia para el caso de la posición de la partícula P :

$$\left(\frac{d\vec{r}}{dt} \right)_{fijo} = \left(\frac{d\vec{r}}{dt} \right)_{rotacional} + \vec{\omega} \times \vec{r} \quad (5.3)$$

De forma general, dicha relación se puede escribir para cualquier vector:

$$\left(\frac{d\vec{G}}{dt} \right)_{fijo} = \left(\frac{d\vec{G}}{dt} \right)_{rotacional} + \vec{\omega} \times \vec{G} \quad (5.4)$$

Continuando con la Ecuación (5.2), esta expresión se puede reescribir utilizando la Ecuación (5.3):

$$\begin{aligned} \left(\frac{d\vec{r}}{dt} \right)_{fijo} &= \left(\frac{d\vec{R}}{dt} \right)_{fijo} + \left(\frac{d\vec{r}}{dt} \right)_{fijo} \\ &= \left(\frac{d\vec{R}}{dt} \right)_{fijo} + \left(\frac{d\vec{r}}{dt} \right)_{rotacional} + \vec{\omega} \times \vec{r} \\ &= \dot{\vec{R}}_f + \dot{\vec{r}}_r + \vec{\omega} \times \vec{r} \end{aligned}$$

Definition 26

(Relación entre las velocidades inerciales y no iniciales)

$$\dot{\vec{r}}'_f = \dot{\vec{R}}_f + \dot{\vec{r}}_r + \vec{\omega} \times \vec{r} \quad (5.5)$$

Derivando la Ecuación (5.5) con respecto al tiempo, se llegará a la relación de la

aceleración entre los marcos de referencia.

$$\begin{aligned}
 \left(\frac{d\vec{r}'_f}{dt} \right)_f &= \left(\frac{d\vec{R}_f}{dt} \right)_f + \left(\frac{d\vec{r}_r}{dt} \right)_f + \left(\frac{d}{dt} \right)_f (\vec{\omega} \times \vec{r}) \\
 &= \ddot{\vec{R}}_f + \ddot{\vec{r}}_r + \vec{\omega} \times \dot{\vec{r}}_r + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times \dot{\vec{r}}_f \\
 &= \ddot{\vec{R}}_f + \ddot{\vec{r}}_r + \vec{\omega} \times \dot{\vec{r}}_r + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times (\dot{\vec{r}}_r + \vec{\omega} \times \vec{r}) \\
 &= \ddot{\vec{R}}_f + \ddot{\vec{r}}_r + \dot{\vec{\omega}} \times \vec{r} + 2\vec{\omega} \times \dot{\vec{r}}_r + \vec{\omega} \times (\vec{\omega} \times \vec{r})
 \end{aligned}$$

Definition 27

(Relación entre las aceleraciones iniciales y no iniciales)

$$\ddot{\vec{r}}'_f = \ddot{\vec{R}}_f + \ddot{\vec{r}}_r + \dot{\vec{\omega}} \times \vec{r} + 2\vec{\omega} \times \dot{\vec{r}}_r + \vec{\omega} \times (\vec{\omega} \times \vec{r}) \quad (5.6)$$

SUBSECTION 5.2

Segunda Ley de Newton para Sistemas No Inerciales

A continuación se va a establecer la forma general de la Segunda Ley de Newton y de este modo extender los casos de análisis de este formalismo. A partiendo de la Ecuación (3.4) y posteriormente haciendo uso de la Ecuación (5.6):

$$\begin{aligned}
 \sum \vec{F} &= m \ddot{\vec{r}}'_f \\
 &= m \left[\ddot{\vec{R}}_f + \ddot{\vec{r}}_r + \dot{\vec{\omega}} \times \vec{r} + 2\vec{\omega} \times \dot{\vec{r}}_r + \vec{\omega} \times (\vec{\omega} \times \vec{r}) \right] \\
 &= m \ddot{\vec{R}}_f + m \ddot{\vec{r}}_r + m \dot{\vec{\omega}} \times \vec{r} + 2m\vec{\omega} \times \dot{\vec{r}}_r + m\vec{\omega} \times (\vec{\omega} \times \vec{r}) \\
 \Rightarrow \sum \vec{F} - m \ddot{\vec{R}}_f - m \dot{\vec{\omega}} \times \vec{r} - 2m\vec{\omega} \times \dot{\vec{r}}_r - m\vec{\omega} \times (\vec{\omega} \times \vec{r}) &= m \ddot{\vec{r}}_r
 \end{aligned}$$

Definition 28

(Segunda Ley de Newton No Inercial)

$$\underbrace{\sum \vec{F}}_{\text{Fuerzas reales}} - m \ddot{\vec{R}}_f - m \dot{\vec{\omega}} \times \vec{r} - \underbrace{2m\vec{\omega} \times \dot{\vec{r}}_r}_{\text{Fuerza de Coriolis}} - \overbrace{m\vec{\omega} \times (\vec{\omega} \times \vec{r})}^{\text{Fuerza Centrífuga}} = m \ddot{\vec{r}}_r \quad (5.7)$$

Es preciso introducir el concepto de **fuerzas ficticias**, estas son aquellas “fuerzas” que resultan ser introducidas en la Segunda Ley de Newton para generalizar la expresión original (Ecuación (3.4)) de modo que sea capaz de analizar desde marcos de referencia no iniciales. Es decir, estas no son fuerzas o interacciones reales con el entorno; son más bien un artificio que permite ampliar el alcance de la Segunda Ley de Newton.

SUBSECTION 5.3

Problemas resueltos

Sistemas no inerciales

Sin rotación

Problema 1.

Una persona de 70 kg se para en una báscula de resorte de baño mientras viaja en un ascensor. Si el ascensor tiene

- aceleración hacia arriba
- hacia abajo de $g/4$, ¿cuál es el peso indicado en la balanza en cada caso?

S: Ascensor

S': Tierra móvil

$$\text{En } S': \vec{F} = m\ddot{\vec{r}} \Rightarrow F_s - mg = m\ddot{\vec{r}} ; \text{ Se sabe que } \ddot{\vec{r}} = \vec{r} + \vec{R} \Rightarrow \ddot{\vec{r}} = \dot{\vec{r}} + \ddot{\vec{R}} \Rightarrow \ddot{\vec{r}} = \ddot{\vec{r}} + \ddot{\vec{R}}$$

$$\text{Al multiplicar por } m \text{ este último resultado: } m\ddot{\vec{r}} = m\ddot{\vec{r}} + m\ddot{\vec{R}} \Rightarrow m\ddot{\vec{r}} = m\ddot{\vec{r}} - m\ddot{\vec{R}} = \vec{F} - m\ddot{\vec{R}}$$

Reemplazando \vec{F}

$$\Rightarrow F_s - mg - m\ddot{\vec{R}} = m\ddot{\vec{r}} \quad \text{Como la persona está en reposo en el marco } S$$

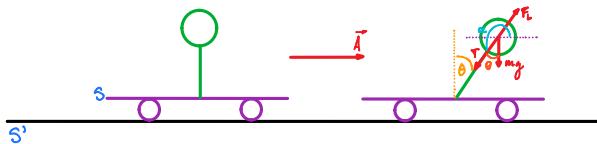
$$\Rightarrow F_s - mg - m\ddot{\vec{R}} = 0 \Rightarrow F_s = mg + m\ddot{\vec{R}}$$

$$a) \ddot{\vec{R}} = \frac{g}{4}\hat{j} \Rightarrow F_s = mg + m\frac{g}{4} = \frac{5}{4}mg //$$

$$b) \ddot{\vec{R}} = -\frac{g}{4}\hat{j} \Rightarrow F_s = mg - m\frac{g}{4} = \frac{3}{4}mg //$$

Problema 2. (Taylor 9.1)

9.1 * Be sure you understand why a pendulum in equilibrium hanging in a car that is accelerating forward tilts backward, and then consider the following: A helium balloon is anchored by a massless string to the floor of a car that is accelerating forward with acceleration A . Explain clearly why the balloon tends to tilt *forward* and find its angle of tilt in equilibrium. [Hint: Helium balloons float because of the buoyant Archimedean force, which results from a pressure gradient in the air. What is the relation between the directions of the gravitational field and the buoyant force?]



$$\text{En } S: \vec{F} = m\ddot{\vec{r}} \Rightarrow \vec{F}_B - \vec{T} - mg\hat{j} = m\ddot{\vec{r}}$$

$$\text{En } S: m\ddot{\vec{r}} = \vec{F} - m\ddot{\vec{R}} \Rightarrow m\ddot{\vec{r}} = \vec{F}_B + \vec{T} - mg\hat{j} - mAt \quad \text{como el globo termina en reposo } \ddot{\vec{r}} = 0$$

La fuerza boyante es opuesta a la dirección de la gravedad local $\Rightarrow -\vec{F}_B \parallel (-mg\hat{j} - mAt) = m\vec{g}_{local}$; $\tan\theta = \frac{g}{A}$

$$\Rightarrow m\ddot{\vec{r}} = \vec{F}_B + \vec{T} + m\vec{g}_{local} \Rightarrow \vec{T} = -(\vec{F}_B + m\vec{g}_{local})$$

Problema 3.

- a. Resuelva el problema del cuerpo en caída libre introduciendo un sistema de coordenadas de traslación con una aceleración \vec{g} . Establezca y resuelva las ecuaciones de movimiento en este sistema de coordenadas acelerado y transforme el resultado nuevamente a un sistema de coordenadas fijo en relación con la Tierra. (Desprecie la rotación de la Tierra).
- b. En el mismo sistema de coordenadas aceleradas, establezca las ecuaciones de movimiento para un cuerpo que cae sujeto a una resistencia del aire proporcional a su velocidad (relativa al aire fijo).

a) En S' : $\vec{F} = m\ddot{\vec{r}} \Rightarrow m\vec{g} = m\ddot{\vec{r}} ; m\ddot{\vec{r}} = \vec{F} - m\ddot{\vec{R}}$

En S : $m\ddot{\vec{r}} = m\vec{g} - m\vec{g} = 0 \Rightarrow m\ddot{\vec{r}} = 0 \Rightarrow \ddot{\vec{r}} = 0 \Rightarrow \dot{\vec{r}} = \vec{V} \Rightarrow \vec{r} = \vec{r}_0 + \vec{V}t \leftarrow$ Ecuación del movimiento desde S'

Transformando esta al marco S

$$\vec{F} - m\ddot{\vec{R}} = 0 \Rightarrow \vec{F} = m\ddot{\vec{R}} = m\vec{g} = m\ddot{\vec{r}} \Rightarrow \ddot{\vec{r}} = \vec{g} \Rightarrow \dot{\vec{r}} = \vec{g}t \Rightarrow \vec{r} = \vec{r}_0 + \vec{V}_0 t + \frac{1}{2}\vec{g}t^2 \leftarrow$$
 Relación buscada

Se va a utilizar la relación $\vec{r}' = \vec{r} + \vec{R}$ *, entonces hay que hallar la ecuación del movimiento del marco S respecto a S' y resolverla

$$\vec{F}_n = m\ddot{\vec{R}} = m\vec{g} \Rightarrow \ddot{\vec{R}} = \vec{g} \Rightarrow \dot{\vec{R}} = \vec{V}_0 + \vec{g}t \Rightarrow \vec{R} = \vec{R}_0 + \vec{V}_0 t + \frac{1}{2}\vec{g}t^2 \quad \text{Usando *}$$

$$\Rightarrow \vec{r}' = \vec{r} + \vec{R} = \vec{r}_0 + \vec{V}t + \vec{R}_0 + \vec{V}_0 t + \frac{1}{2}\vec{g}t^2 \Rightarrow \vec{r}' = (\vec{r}_0 + \vec{R}_0) + (\vec{V} + \vec{V}_0)t + \frac{1}{2}\vec{g}t^2 ; \text{ se obtuvo una relación de la forma de } *$$

cumpliendo con *

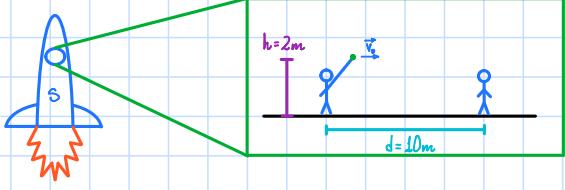
b) En S' : $\vec{F} = m\ddot{\vec{r}} \Rightarrow m\vec{g} - c\vec{V} = m\ddot{\vec{r}} ; m\ddot{\vec{r}} = \vec{F} - m\ddot{\vec{R}}$

En S : $m\ddot{\vec{r}} = m\vec{g} - c\vec{V} - m\vec{g} = -c\vec{V} \Rightarrow m\ddot{\vec{r}} = -c(\vec{V} + \vec{V}) = -c(\dot{\vec{r}} + \dot{\vec{R}}) \Rightarrow m\ddot{\vec{r}} = -c(\dot{\vec{r}} + \vec{V}_0 + \vec{g}t)$

Problema 4.

Dos astronautas están parados en una nave espacial que acelera hacia arriba con una aceleración \vec{A} . Sea la magnitud de \vec{A} igual a g . Dos astronautas están separados una distancia horizontal $d = 10 \text{ m}$. El astronauta 1 lanza una pelota directamente hacia el astronauta 2. ¿Cuál debe ser la velocidad inicial de la pelota si debe alcanzar al astronauta 2 antes de golpear el suelo, es decir la pelota debe llegar a los pies del astronauta? Supongamos que el astronauta 1 suelta la pelota a una altura $h = 2 \text{ m}$ sobre el piso de la nave. Resuelva el problema desde la perspectiva de:

- a. un observador no inercial (dentro de la nave).
- b. un observador inercial (fuera de la nave).



S'

$$\text{Condiciones iniciales } \ddot{y}(0) = \dot{y}_0 = h, \quad \ddot{x}(0) = 0, \quad x(0) = 0 \quad \text{y} \quad \ddot{x}(0) = v_0$$

$$\ddot{R} = -\vec{g}$$

$$\text{En } S': \vec{F} = m\ddot{\vec{r}}, \Rightarrow m\vec{g} = m\ddot{\vec{r}}$$

$$\text{En } S: m\ddot{\vec{r}} = \vec{F} - m\ddot{\vec{R}} \Rightarrow m\ddot{\vec{r}} = m\vec{g} - m\vec{g} = 2m\vec{g} \Rightarrow m\ddot{\vec{r}} = 2m\vec{g} \Rightarrow \ddot{y} = -2g \quad \text{y} \quad \ddot{x} = 0$$

$$\Rightarrow \ddot{y} = \dot{y}_0 - \frac{1}{2}gt^2 \quad \text{y} \quad \ddot{x} = v_0 t$$

Despejando t de * e introduciéndolo en *

$$\Rightarrow \ddot{y} = \dot{y}_0 - g \frac{\ddot{x}^2}{v_0^2} \Rightarrow v_0 = \sqrt{-g \frac{\ddot{x}^2}{\Delta y}} = 7,0 \text{ m/s} ; \Delta y = \dot{y}_0 t - \dot{y}_0 = -\dot{y}_0$$

a) $S: \vec{v}_0 = 7,0 \text{ m/s} \uparrow \diagup$

$$Y = +\frac{1}{2}gt^2$$

b) $\ddot{y} = \dot{y}_0 - \frac{1}{2}gt^2 \quad \text{y} \quad \ddot{x} = v_0 t$

$$\ddot{\vec{r}} = \vec{r} + \vec{R}$$

$$\dot{y}_t = \dot{y}_0 - \frac{1}{2}gt^2 \quad x_t = v_0 t$$

$$\ddot{\vec{r}} = \vec{r} + \vec{R}$$

$$\Delta y_t = -\frac{1}{2}g \frac{x_t^2}{v_0^2}$$

$$v_0 = \sqrt{-\frac{1}{2}g \frac{x_t^2}{\Delta y_t}}$$

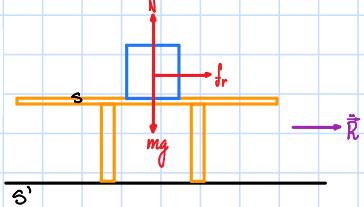
$$\ddot{\vec{r}} = \vec{r} + \vec{R}$$

$$-g - g$$

Problema 5.

Un bloque de madera descansa sobre una mesa horizontal tosca. Si la mesa se acelera en dirección horizontal, ¿en qué condiciones resbalará el bloque?

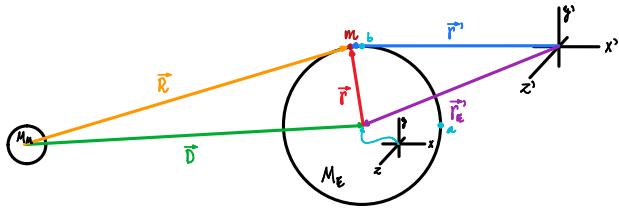
$$\text{En } S': \vec{F} = m\ddot{\vec{r}}, \Rightarrow f_r = m\ddot{x} = \mu mg$$



$$\text{En } S: m\ddot{\vec{r}} = \vec{F} - m\ddot{\vec{R}} \Rightarrow m\ddot{\vec{r}} = \mu mg - \mu \vec{X} \Rightarrow \ddot{x} = \mu g - \ddot{X} ; \text{ Para que la caja resbale } \ddot{X} = 0$$

$$\Rightarrow \ddot{X} = \mu g \quad \text{Si esto ocurre, la caja va a resbalarse inmediatamente} \quad \diagup$$

Demostración de la fuerza de marea y cálculo del cambio de altura de la marea



$$\text{En } S': \vec{F} = m\ddot{\vec{r}} \Rightarrow m\ddot{\vec{r}} = -\frac{GM_{\text{M}}m}{R^2}\hat{e}_r - \frac{GM_{\text{E}}m}{r^2}\hat{e}_r$$

$$\vec{F} = M_{\text{E}}\ddot{\vec{r}}_E \Rightarrow M_{\text{E}}\ddot{\vec{r}}_E = -\frac{GM_{\text{M}}M_{\text{E}}}{D^2}\hat{e}_r$$

Relación entre S' y S

$$\begin{aligned} \vec{r}' &= \vec{r} + \vec{r}_E \Rightarrow \ddot{\vec{r}}' = \ddot{\vec{r}} + \ddot{\vec{r}}_E \Rightarrow \ddot{\vec{r}}' - \ddot{\vec{r}}_E = \ddot{\vec{r}} \\ \Rightarrow m\ddot{\vec{r}}' - m\ddot{\vec{r}}_E &= \vec{F} - M_{\text{E}}\ddot{\vec{r}}_E = m\ddot{\vec{r}} \end{aligned}$$

$$\text{En } S: \vec{F} - m\ddot{\vec{r}}_E = m\ddot{\vec{r}} \Rightarrow -\frac{GM_{\text{M}}m}{R^2}\hat{e}_r - \frac{GM_{\text{E}}m}{r^2}\hat{e}_r + \underbrace{\frac{GM_{\text{M}}m}{D^2}\hat{e}_D}_{\substack{\text{Atracción gravitatoria} \\ \text{que ejerce la Tierra}}} = m\ddot{\vec{r}} = \underbrace{-\frac{GM_{\text{E}}m}{r^2}\hat{e}_r}_{\substack{\text{Fuerza de marea}}} - GM_{\text{M}}m\left(\frac{\hat{e}_r - \hat{e}_D}{R^2 - D^2}\right)$$

$$\Rightarrow \vec{F}_{\text{marea}} = -GM_{\text{M}}m\left(\frac{\hat{e}_r - \hat{e}_D}{R^2 - D^2}\right)$$

Ahora se va a aproximar el resultado de \vec{F}_{marea} a partir de 2 casos, uno donde m está en "a" y otro en "b"

$$* a: F_{\text{marea}-x} = -GM_{\text{M}}m\left(\frac{1}{R^2} - \frac{1}{D^2}\right) = -GM_{\text{M}}m\left(\frac{1}{(D+r)^2} - \frac{1}{D^2}\right) = -\frac{GM_{\text{M}}m}{D^2}\left(\frac{1}{(1+\frac{r}{D})^2} - 1\right)$$

$$\frac{1}{(1+\frac{r}{D})^2} \approx 1 - 2\frac{r}{D}$$

$$\Rightarrow F_{\text{marea}-x} = -\frac{GM_{\text{M}}m}{D^2}\left(1 - 2\frac{r}{D}\right) = \frac{2GM_{\text{M}}mr}{D^3} \Rightarrow F_{\text{marea}-x} = \frac{2GM_{\text{M}}mX}{D^3}$$

$$* b: \hat{e}_D \approx \hat{e}_r \text{ porque } D \approx R \Rightarrow \hat{e}_r \cdot \hat{e}_D \approx 0$$

$$\begin{array}{c} \text{Componente en} \\ \text{y de } \hat{e}_r \\ \text{y de } \hat{e}_r \end{array}$$

$$F_{\text{marea}-y} = -\frac{GM_{\text{M}}m \cdot r}{D^3} \Rightarrow F_{\text{marea}-y} = -\frac{GM_{\text{M}}mY}{D^3}$$

$$\Rightarrow \vec{F}'_{\text{marea}} = \frac{2GM_{\text{M}}mX}{D^3} \hat{i} - \frac{GM_{\text{M}}mY}{D^3} \hat{j} \quad \leftarrow \text{Aproximación de las fuerzas que sienten las mareas en un modelo no rotatorio de cuerpo celeste A y B, por ejemplo Tierra-Luna}$$

$$U_{\text{ig}(b)} - U_{\text{ig}(a)} = -W = -\int_r^R F_{\text{marea}-y} dy - \int_0^r F_{\text{marea}-x} dx = -\frac{GM_{\text{M}}m}{D^3} \left(\frac{r^2}{2} + r^2 \right) = -\frac{3}{2} \frac{GM_{\text{M}}m}{D^3} r^2 = -mgh \Rightarrow \frac{3}{2} \frac{GM_{\text{M}}m}{D^3} r^2 = mgh$$

Trabajo para mover un punto "a" hasta el centro de la Tierra y de allí al punto "b".

$$\Rightarrow h = \frac{3GM_{\text{M}}r^2}{2gD^3} \quad \leftarrow \text{Diferencia de altura de la marea causada por la luna}$$

Otra forma de probar la diferencia de altura de la marea causada por la luna

En la superficie $U = U_{\text{tg}} + U_{\text{marea}}$ y se cumple que: $U(a) = U(b) \Rightarrow U_{\text{tg}}(a) + U_{\text{marea}}(a) = U_{\text{tg}}(b) + U_{\text{marea}}(b)$

$$\Rightarrow U_{\text{tg}}(a) - U_{\text{tg}}(b) = mgh = U_{\text{marea}}(b) - U_{\text{marea}}(a) ; -\vec{\nabla} U_{\text{marea}} = \vec{F}_{\text{marea}} = -GM_{\text{um}} \left(\frac{\hat{e}_a}{R^2} - \frac{\hat{e}_b}{D^2} \right)$$

$$\Rightarrow U_{\text{marea}} = -GM_{\text{um}} \left(\frac{1}{R} + \frac{X}{D^2} \right)$$

↓ Fuerza gravitacional ↓ Fuerza constante en una dirección precisa → $X \uparrow$

* En el punto b: $X=0$ $R = \sqrt{D^2+r^2}$

$$\Rightarrow U_{\text{marea}}(b) = -GM_{\text{um}} \left(\frac{1}{R} + \frac{X}{D^2} \right) = -GM_{\text{um}} \cdot \frac{1}{\sqrt{D^2+r^2}} = -\frac{GM_{\text{um}}}{D} \cdot \frac{1}{\sqrt{1+\frac{r^2}{D^2}}} \approx -\frac{GM_{\text{um}}}{D} \left(1 - \frac{r^2}{2D^2} \right)$$

* En el punto a: $X=r$ $R=D+r$

$$\Rightarrow U_{\text{marea}}(a) = -GM_{\text{um}} \left(\frac{1}{R} + \frac{X}{D^2} \right) = -GM_{\text{um}} \left(\frac{1}{D+r} + \frac{r}{D^2} \right) = -\frac{GM_{\text{um}}}{D} \left(\frac{1}{1+\frac{r}{D}} + \frac{r}{D} \right) \approx -\frac{GM_{\text{um}}}{D} \left(1 - \frac{r}{D} + \frac{r^2}{D^2} + \frac{r}{D} \right)$$

$$\Rightarrow U_{\text{marea}}(b) - U_{\text{marea}}(a) \approx -\frac{GM_{\text{um}}}{D} \left(1 - \frac{r^2}{2D^2} \right) - -\frac{GM_{\text{um}}}{D} \left(1 + \frac{r^2}{D^2} \right) = -\frac{GM_{\text{um}}}{D} \left(1 - \frac{r^2}{2D^2} \right) + \frac{GM_{\text{um}}}{D} \left(1 + \frac{r^2}{D^2} \right)$$

$$\approx \frac{GM_{\text{um}}}{D} \frac{r^2}{2D^2} + \frac{GM_{\text{um}}}{D} \frac{r^2}{D^2} = \frac{GM_{\text{um}}}{D} \frac{3r^2}{2D^2}$$

$$\Rightarrow U_{\text{marea}}(b) - U_{\text{marea}}(a) \approx \frac{GM_{\text{um}}}{D} \frac{3r^2}{2D^2} \approx mgh$$

$$\Rightarrow \frac{GM_{\text{um}}}{D} \frac{3r^2}{2D^2} \approx mgh \Rightarrow h = \frac{GM_{\text{um}}}{gD^3} \frac{3r^2}{2} //$$

Problema 6. (Thorton 5.17)

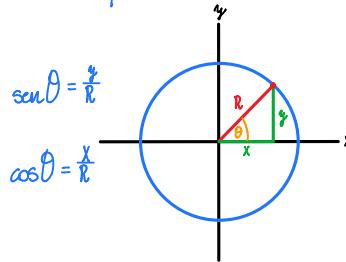
- 5-17. Newton's model of the tidal height, using the two water wells dug to the center of Earth, used the fact that the pressure at the bottom of the two wells should be the same. Assume water is incompressible and find the tidal height difference h , Equation 5.55, due to the Moon using this model. (Hint: $\int_0^{x_{\max}} \rho g_y dy = \int_0^{y_{\max}} \rho g_x dx$; $h = x_{\max} - y_{\max}$, where $x_{\max} + y_{\max} = 2R_{\text{earth}}$, and R_{earth} is Earth's median radius.)

$$h = \frac{3GM_m r^2}{2gD^3} \quad . \quad (5.55)$$

Para comenzar se requiere de g_x y g_y , siendo estos una especie de gres que incluye el campo de la fuerza de marea

$$\Rightarrow g_i = g_0 + \frac{F_{\text{marea}-i}}{m}$$

$$\text{De forma aproximada } F_{\text{marea}} = \frac{2GM_m m X \uparrow}{D^3} - \frac{GM_m m Y \uparrow}{D^3} \uparrow$$



$$\Rightarrow g_x = -\frac{GM \cos \theta}{R^2} + \frac{2GM_m X}{D^3} = -\frac{GM}{R^2} \frac{X}{R} + \frac{2GM_m X}{D^3} = -\frac{GM X}{R^3} + \frac{2GM_m X}{D^3}$$

$$\Rightarrow g_y = -\frac{GM \sin \theta}{R^2} - \frac{GM_m Y}{D^3} = -\frac{GM}{R^2} \frac{Y}{R} - \frac{GM_m Y}{D^3} = -\frac{GM Y}{R^3} - \frac{GM_m Y}{D^3}$$

$$\Rightarrow \int_0^{x_{\max}} \cancel{\int g_y dy} = \int_0^{x_{\max}} \cancel{\int g_x dx} \Rightarrow \int_0^{x_{\max}} -\frac{GM Y}{R^3} - \frac{GM_m Y dy}{D^3} = \int_0^{x_{\max}} -\frac{GM X}{R^3} + \frac{2GM_m X dx}{D^3}$$

$$\Rightarrow -\frac{GM \frac{Y^2}{R^3}}{2D^3} - \frac{GM_m \frac{Y^2}{R^3}}{2D^3} = -\frac{GM \frac{X^2}{R^3}}{2D^3} + \frac{GM_m \frac{X^2}{R^3}}{D^3}$$

$$\Rightarrow \frac{GM \frac{X^2}{R^3} - GM \frac{Y^2}{R^3}}{2R^3} = \frac{GM (X^2 - Y^2)}{2R^3} = \frac{GM (X_{\max} - Y_{\max})(X_{\max} + Y_{\max})}{2R^3} = \frac{GM_m \frac{X^2}{R^3} + GM_m \frac{Y^2}{R^3}}{2D^3}$$

$$\Rightarrow \frac{GM (X_{\max} - Y_{\max})(X_{\max} + Y_{\max})}{2R^3 h} = \frac{GM_m \frac{X^2}{R^3}}{D^3} + \frac{GM_m \frac{Y^2}{R^3}}{2D^3} \Rightarrow \frac{M}{2R^2} h \cdot \cancel{2R} = \frac{M}{R^2} h = \frac{M_m R^2}{D^3} + \frac{M_m R^2}{2D^3}$$

$$\Rightarrow h = \frac{3 M_m R^4}{2 MD^3} = \frac{3 GM_m R^2}{2 g D^3} \Rightarrow h = \frac{3 GM_m R^2}{2 g D^3} //$$

Problema 7.

Supongamos que en algún momento la Luna estuvo cubierta por un océano y rotando con respecto a la Tierra. Encuentra la razón entre las alturas de las mareas levantadas en la Luna por la Tierra y en la Tierra por la Luna. (El radio de la Luna es $R_M = 0.27R_E$.) Estima qué tan altas habrían sido las mareas en la Luna cuando la distancia entre la Tierra y la Luna era de $10R_E$.

Utilizando $h = \frac{3 M_{\text{cu}} R^4}{2 M_p D^3}$; donde M_{cu} es la masa del cuerpo que produce la fuerza de marea, M_p y R_p son masa y radio del planeta o cuerpo en que ocurre la marea

$$\Rightarrow h_M = \frac{3 M_E R_M^4}{2 M_u D^3} \quad y \quad h_E = \frac{3 M_M R_E^4}{2 M_E D^3}$$

$$\Rightarrow \frac{h_M}{h_E} = \frac{\frac{3 M_E R_M^4}{2 M_u D^3}}{\frac{3 M_M R_E^4}{2 M_E D^3}} = \frac{\frac{M_E R_M^4}{M_u R_E^4}}{\frac{M_M R_E^4}{M_E D^3}} = \frac{M_E^2 R_M^4}{M_M^2 R_E^4} = \frac{M_E^2 0.27^4 R_E^4}{M_M^2 R_E^4} \Rightarrow \frac{h_M}{h_E} = \frac{M_E^2 0.27^4}{M_M^2} = 35.06 ; M_E = 5.97 \cdot 10^{24} \text{ kg}$$

$$M_u = 7.35 \cdot 10^{22} \text{ kg}$$

$$\Rightarrow h_M = \frac{3 M_E R_M^4}{2 M_u D^3} = \frac{3 M_E 0.27^4 R_E^4}{2 M_u 10^3 R_E^4} = \frac{3 M_E R_E 0.27^4}{2 M_u 10^3} = 41.25 \text{ m}$$

Problema 8.

En clase estudiamos los efectos de la Luna y el Sol sobre las mareas terrestres. ¿Puede determinar la influencia en la marea terrestre debido a Marte y Júpiter? Nota: Para ello considere una configuración donde se de la máxima influencia de cada uno

Utilizando $h = \frac{3 M_{\text{cu}} R^4}{2 M_p D^3}$; donde M_{cu} es la masa del cuerpo que produce la fuerza de marea, M_p y R_p son masa y radio del planeta o cuerpo en que ocurre la marea

La máxima influencia de ambos planetas sobre las mareas terrestres ocurriría cuando D_{M-T} y D_{J-T} son las distancias mínimas posibles

$$D_{M-T-m} = 5.46 \cdot 10^{10} \text{ m} \quad y \quad M_u = 6.39 \cdot 10^{23} \text{ kg} \Rightarrow h_{M_m} = 1.63 \cdot 10^{-6} \text{ m}$$

$$D_{J-T-m} = 6.29 \cdot 10^{11} \text{ m} \quad y \quad M_J = 1.90 \cdot 10^{27} \text{ kg} \Rightarrow h_{J_m} = 3.16 \cdot 10^{-6} \text{ m}$$

Con Rotación

Problema 1. (Thornton 10.1)

- 10-1. Calculate the centrifugal acceleration, due to Earth's rotation, on a particle on the surface of Earth at the equator. Compare this result with the gravitational acceleration. Compute also the centrifugal acceleration due to the motion of Earth about the Sun and justify the remark made in the text that this acceleration may be neglected compared with the acceleration caused by axial rotation.

$$\omega_r = \frac{2\pi}{24 \cdot 3600} = 7,27 \cdot 10^{-5} \text{ rad/s} \Rightarrow a_{centr} = r_r \omega_r^2 = 3,38 \text{ m/s}^2$$

$$\omega_{s-r} = \frac{2\pi}{365 \cdot 24 \cdot 3600} = 2 \cdot 10^{-7} \text{ rad/s} \Rightarrow a_{centr,s} = r_{s-r} \omega_{s-r}^2 = 0,6 \text{ m/s}^2 //$$

Problema 2. (Taylor 9.12)

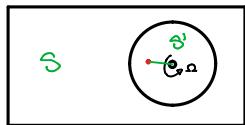
- 9.12* (a) Show that to design a static structure in a rotating frame (such as a space station) one can use the ordinary rules of statics except that one must include the extra "fictitious" centrifugal force.
 (b) I wish to place a puck on a rotating horizontal turntable (angular velocity Ω) and to have it remain at rest on the table, held by the force of static friction (coefficient μ). What is the maximum distance from the axis of rotation at which I can do this? (Argue from the point of view of an observer in the rotating frame.)

a) $S': \vec{F} = m\ddot{\vec{r}} = \vec{F}_g$

$S: \vec{F}_{eff} = 0 \leftarrow \text{Se quiere que sea estática}; \vec{F}_{eff} = \vec{F} + *$; la estación va a sentir la fuerza gravitacional de la Tierra, por lo que \vec{F} no puede ser cero y requiere que * sea algo, y debe contrarrestar a \vec{F} en S.

$$\Rightarrow \vec{F}_{eff} = \vec{F}_g - m\vec{\omega} \times (\vec{\omega} \times \vec{r}) = 0 \Rightarrow \vec{F}_g = m\vec{\omega} \times (\vec{\omega} \times \vec{r})$$

b)

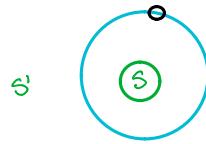


$$S': \vec{F} = m\ddot{\vec{r}} \Rightarrow m\ddot{\vec{r}} = -\mu_s mg \uparrow$$

$$S: \vec{F}_{eff} = \vec{F} - m\vec{\omega} \times (\vec{\omega} \times \vec{r}) - \cancel{m\vec{\omega} \times \vec{v}_r} = 0 \leftarrow \text{Se busca el radio máximo en que es estático}$$

$$\Rightarrow \vec{F} - m\vec{\omega} \times (\vec{\omega} \times \vec{r}) = 0 \Rightarrow -\mu_s mg = m\vec{\omega} \times (\vec{\omega} \times \vec{r}) = -m\vec{\omega}^2 \vec{r} \Rightarrow \mu_s mg = \cancel{m\vec{\omega}^2 \vec{r}}$$

$$\Rightarrow r = \frac{\mu_s g}{\omega^2} \Rightarrow r = \frac{\mu_s g}{\Omega^2} //$$



Problema 3. (Thornton 10.16)

10-16. Consider Problem 943 but include the effects of the Coriolis force on the probe. The probe is launched at a latitude of 45° straight up. Determine the horizontal deflection in the probe at its maximum height for each part of Problem 9-63.

$$\omega = 73 \cdot 10^{-5} \text{ rad/s}$$

$$90^\circ - \text{latitud} = \theta = 45^\circ$$

a) $S' : \sum \vec{F} = m\ddot{\vec{r}} \Rightarrow -mg\hat{k} = m\ddot{\vec{r}}$

$$S : \sum \vec{F}_{\text{ext}} = m\ddot{\vec{r}} \Rightarrow m\ddot{\vec{r}} - 2m\vec{\omega} \times \vec{v}_r = m\ddot{\vec{r}} \Rightarrow -mg\hat{k} - 2m\vec{\omega} \times \vec{r} = m\ddot{\vec{r}} \Rightarrow \ddot{\vec{r}} = -g\hat{k} - 2\vec{\omega} \times \vec{r}$$

$$\Rightarrow \ddot{\vec{r}} = -g\hat{k} - 2\vec{\omega} \times \vec{r} ; \quad \vec{\omega} = \omega \cos \theta \hat{k} - \omega \sin \theta \hat{i}$$

$$\vec{r} = \hat{x}\hat{i} + \hat{y}\hat{j} + \hat{z}\hat{k}$$

$$\vec{\omega} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\omega \sin \theta & 0 & \omega \cos \theta \\ \hat{x} & \hat{y} & \hat{z} \end{vmatrix} = -\hat{i}\omega \cos \theta \hat{i} - (-\hat{z}\omega \sin \theta - \hat{x}\omega \cos \theta) \hat{j} - \hat{y}\omega \sin \theta \hat{k}$$

$$\Rightarrow \ddot{\vec{r}} = -g\hat{k} - 2[-\hat{i}\omega \cos \theta \hat{i} - (-\hat{z}\omega \sin \theta - \hat{x}\omega \cos \theta) \hat{j} - \hat{y}\omega \sin \theta \hat{k}] \\ = -g\hat{k} + 2\hat{i}\omega \cos \theta \hat{i} - 2(\hat{z}\omega \sin \theta + \hat{x}\omega \cos \theta) \hat{j} + 2\hat{y}\omega \sin \theta \hat{k} \Rightarrow \ddot{\vec{r}} = \begin{cases} \hat{x} = 2\hat{i}\omega \cos \theta \\ \hat{y} = -2(\hat{z}\omega \sin \theta + \hat{x}\omega \cos \theta) \\ \hat{z} = -g + 2\hat{y}\omega \sin \theta \end{cases}$$

Ahora con la expresión para la aceleración que va a sentir el cohete, hay que aproximar la solución mediante iteración de soluciones comenzando por una solución simple conocida.

1) Suponer $\omega \approx 0 \Rightarrow \ddot{z} = -g \Rightarrow \dot{z}^* = \dot{z}_0 - gt, \ddot{y} = 0, \ddot{x} = 0 \leftarrow$ Aproximación de orden cero, caída libre

2) $\omega \neq 0$ y se tiene \star y \ddot{y} en \ddot{y} : $\Rightarrow \ddot{y} = -2(\dot{z}_0 - gt)\omega \sin \theta \Rightarrow \ddot{y}^* = -2\omega \sin \theta \dot{z}_0 t + \omega \sin \theta g t^2$

$$\Rightarrow \dot{z}^* = \dot{z}_0 - gt \quad y \quad \ddot{x} = 0 \quad \leftarrow$$

3) Con \star y \ddot{y} en \ddot{x} : $\Rightarrow \ddot{x} = 2(-2\omega \sin \theta \dot{z}_0 t + \omega \sin \theta g t^2) \omega \cos \theta \quad \leftarrow$ Aproximación de orden 2

$$\Rightarrow \ddot{x} = \left(-2\omega \sin \theta \dot{z}_0 t^2 + \frac{2}{3} \omega \sin \theta g t^3 \right) \omega \cos \theta$$

En $\ddot{z} : \Rightarrow \ddot{z} = -g + 2(-2\omega \sin \theta \dot{z}_0 t + \omega \sin \theta g t^2) \omega \sin \theta \Rightarrow \dot{z} = \dot{z}_0 - gt + \left(-2\omega \sin \theta \dot{z}_0 t^2 + \frac{2}{3} \omega \sin \theta g t^3 \right) \omega \sin \theta$

t hasta $z_{\max} \Rightarrow \dot{z} = 0 \Rightarrow t = 6.108 \text{ s}$

Usando \star $\ddot{y} = -7.71 \cdot 10^4 \text{ m/s}^2, \ddot{x} = -1.8 \cdot 10^3 \text{ m/s}^2 \neq$ las otras aproximaciones en computadora

9-63. A new projectile launcher is developed in the year 2023 that can launch a 10^4 kg spherical probe with an initial speed of 6000 m/s . For testing purposes, objects are launched vertically.

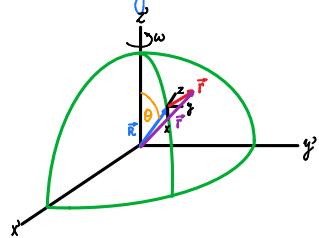
(a) Neglect air resistance and assume that the acceleration of gravity is constant.

Determine how high the launched object can reach above the surface of Earth.

(b) If the object has a radius of 20 cm and the air resistance is proportional to the square of the object's speed with $c_a = 0.2$, determine the maximum height reached. Assume the density of air is constant.

(c) Now also include the fact that the acceleration of gravity decreases as the object soars above Earth. Find the height reached.

(d) Now add the effects of the decrease in air density with altitude to the calculation. We can very roughly represent the air density by $\log_{10}(\rho) = -0.05h + 0.11$ where ρ is the air density in kg/m^3 and h is the altitude above Earth in km . Determine how high the object now goes.



Problema 4. (Thornton 10.21)

10-21. Water being diverted during a flood in Helsinki, Finland (latitude 60°N) flows along a diversion channel of width 47 m in the south direction at a speed of 3.4 m/s. On which side is the water the highest (from the standpoint of noninertial systems) and by how much?

$$S: \sum \vec{F} = m\ddot{\vec{r}} \Rightarrow -mg\hat{k} = m\ddot{\vec{r}} ; l = 47\text{m}$$

$$S: \sum \vec{F}_{\text{ext}} = m\ddot{\vec{r}} \Rightarrow m\ddot{\vec{r}} - 2m\vec{\omega} \times \vec{v}_r = m\ddot{\vec{r}} \Rightarrow \ddot{\vec{r}} = -g\hat{k} - 2\vec{\omega} \times \vec{v}$$

$$\vec{\omega} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -w \sin \theta & 0 & w \cos \theta \\ \hat{x} & 0 & 0 \end{vmatrix} = \dot{x} w \cos \theta \hat{j}$$

$$\Rightarrow \ddot{\vec{r}} = -g\hat{k} - 2\dot{x} w \cos \theta \hat{j} \Rightarrow \ddot{\vec{r}} = -g\hat{k} - 2\dot{x} w \cos \theta \hat{j} = -g\hat{k} - 4,30 \cdot 10^{-4} \text{ N}$$

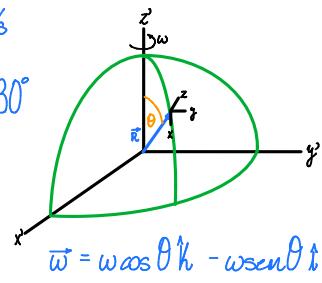
$$\Delta h = l \sin \beta \quad g \sin \beta = \frac{4,30 \cdot 10^{-4}}{\sqrt{g^2 + (4,30 \cdot 10^{-4})^2}} = 4,38 \cdot 10^{-5}$$

$$\Rightarrow \Delta h = 2,06 \cdot 10^{-3} \text{ m}$$

$$\omega = 7,3 \cdot 10^{-5} \text{ rad/s}$$

$$90^{\circ} - \text{latitud} = \theta = 30^{\circ}$$

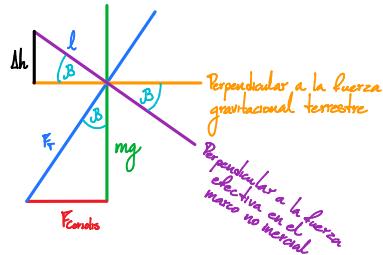
$$\dot{x} = 3,4 \text{ m/s}$$



$$\vec{r} = \dot{x} \hat{i} + \cancel{\dot{y} \hat{j}} + \cancel{\dot{z} \hat{k}} = \dot{x} \hat{i}$$

El agua va hacia el sur

El agua tiene mayor altura hacia el oeste



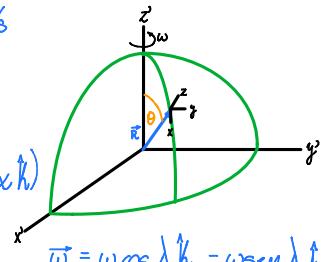
Problema 5.

Un proyectil se dispara en una colatitud λ con una rapidez v_0 dirigida hacia el oeste y formando un ángulo α con la horizontal.

- Determine el tiempo que le toma alcanzar la altura máxima, si se toma en cuenta la rotación de la Tierra.
- ¿Cuál es la altura máxima?
- ¿Qué ocurre con sus resultados si la velocidad angular se hace cero.

$$\omega = 73 \cdot 10^{-5} \text{ rad/s}$$

$$*\ddot{\vec{r}} = v_0(-\cos\alpha \hat{j} + \sin\alpha \hat{k})$$



$$S' : \sum \vec{F} = m\ddot{\vec{r}} \Rightarrow -mg\hat{k} = m\ddot{\vec{r}} ; \dot{\vec{r}} = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k} \quad \vec{\omega} = \omega \sin\lambda \hat{k} - \omega \cos\lambda \hat{i}$$

$$S : \sum \vec{F}_{\text{ext}} = m\ddot{\vec{r}} \Rightarrow m\ddot{\vec{r}} - 2m\vec{\omega} \times \vec{v}_r = m\ddot{\vec{r}} \Rightarrow -mg\hat{k} - 2m\vec{\omega} \times \dot{\vec{r}} = m\ddot{\vec{r}} \Rightarrow \ddot{\vec{r}} = -g\hat{k} - 2\vec{\omega} \times \dot{\vec{r}}$$

$$\Rightarrow \ddot{\vec{r}} = -g\hat{k} - 2\vec{\omega} \times \dot{\vec{r}}$$

$$\vec{\omega} \times \dot{\vec{r}} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\omega \cos\lambda & \hat{j} & \omega \cos\lambda \\ \hat{i} & \hat{j} & \hat{z} \end{vmatrix} = -\hat{j}\omega \cos\lambda \hat{i} - (-\hat{z}\omega \cos\lambda - \hat{x}\omega \sin\lambda) \hat{j} - \hat{y}\omega \sin\lambda \hat{k}$$

$$\Rightarrow \ddot{\vec{r}} = -g\hat{k} - 2[-\hat{j}\omega \cos\lambda \hat{i} - (-\hat{z}\omega \cos\lambda - \hat{x}\omega \sin\lambda) \hat{j} - \hat{y}\omega \sin\lambda \hat{k}]$$

$$= -g\hat{k} + 2\hat{j}\omega \cos\lambda \hat{i} - 2(\hat{z}\omega \cos\lambda + \hat{x}\omega \sin\lambda) \hat{j} + 2\hat{y}\omega \sin\lambda \hat{k} \Rightarrow \ddot{\vec{r}} = \begin{cases} \hat{x} = 2\hat{j}\omega \cos\lambda \\ \hat{y} = -2(\hat{z}\omega \cos\lambda + \hat{x}\omega \sin\lambda) \\ \hat{z} = -g + 2\hat{y}\omega \sin\lambda \end{cases}$$

Ahora con la expresión para la aceleración que va a sentir el cohete, hay que aproximar la solución mediante iteración de soluciones comenzando por una solución simple conocida.

$$1) \text{ Suponer } \omega \approx 0 \Rightarrow \ddot{z} = -g \Rightarrow \dot{z}^* = \dot{z}_0 - gt, \ddot{y} = 0, \ddot{x} = 0 \leftarrow \text{aproximación de orden cero, caída libre}$$

$$2) \omega \neq 0 \text{ y se tiene } * \text{ en } \ddot{y} : \Rightarrow \ddot{y} = -2(\dot{z}_0 - gt)\omega \sin\lambda \Rightarrow \ddot{y}^* = \ddot{y}_0 - 2\omega \sin\lambda \dot{z}_0 t + \omega \sin\lambda g t^2$$

$$\Rightarrow \dot{z}^* = \dot{z}_0 - gt \quad y \quad \ddot{x} = 0 \quad \text{aproximación de orden 1}$$

$$3) \text{ En } \ddot{z} \text{ con } * : \Rightarrow \ddot{z} = -g + 2(\ddot{y}_0 - 2\omega \sin\lambda \dot{z}_0 t + \omega \sin\lambda g t^2) \omega \sin\lambda$$

$$\Rightarrow \dot{z} = \dot{z}_0 - gt + \left(\ddot{y}_0 t - 2\omega \sin\lambda \dot{z}_0 t^2 + \frac{2}{3} \omega \sin\lambda g t^3 \right) \omega \sin\lambda$$

Recordando *:

$$\Rightarrow \dot{z}^* = v_0 \sin\alpha - gt + \left(-v_0 \cos\alpha t - 2\omega \sin\lambda v_0 \sin\alpha t^2 + \frac{2}{3} \omega \sin\lambda g t^3 \right) \omega \sin\lambda$$

Continua...

Tiempo en la altura máxima $\dot{z} = 0$, despejar el tiempo de esa ecuación puede ser muy complejo por lo que se va a optar por aproximar el tiempo de vuelo a cero y se reemplazara en las potencias de t mayores a 1

$$\Rightarrow v \sin \alpha - gt + \left(-v_0 \cos \alpha t - 2 \omega \sin \lambda v \sin \alpha t^2 + \frac{2}{3} \omega \sin \lambda g t^3 \right) \omega \sin \lambda = 0 \quad t = \frac{v \sin \alpha}{g}$$

$$\Rightarrow v \sin \alpha - gt - v_0 \cos \alpha \omega \sin \lambda t + \left(-2 \omega \sin \lambda v \sin \alpha \frac{v^3 \sin^3 \alpha}{g^2} + \frac{2}{3} \omega \sin \lambda g \frac{v^3 \sin^3 \alpha}{g^3} \right) \omega \sin \lambda = 0$$

$$\Rightarrow t(g + v_0 \cos \alpha \omega \sin \lambda) = -v \sin \alpha - \left(-\frac{6 \omega \sin \lambda g v^3 \sin^3 \alpha}{3g^3} + \frac{2}{3} \omega \sin \lambda g \frac{v^3 \sin^3 \alpha}{g^3} \right) \omega \sin \lambda$$

$$\Rightarrow -t(g + v_0 \cos \alpha \omega \sin \lambda) = -v \sin \alpha - \left(-\frac{10 \omega \sin \lambda v^3 \sin^3 \alpha}{3g^2} \right) \omega \sin \lambda$$

$$\Rightarrow -t(g + v_0 \cos \alpha \omega \sin \lambda) = -v \sin \alpha + \frac{10 \omega^2 \sin^2 \lambda v^3 \sin^3 \alpha}{3g^2} \Rightarrow -t(g + v_0 \cos \alpha \omega \sin \lambda) = \frac{-3g^2 v \sin \alpha + 10 \omega^2 \sin^2 \lambda v^3 \sin^3 \alpha}{3g^2}$$

$$\Rightarrow t = \frac{-3g^2 v \sin \alpha + 10 \omega^2 \sin^2 \lambda v^3 \sin^3 \alpha}{-3g^2(g + v_0 \cos \alpha \omega \sin \lambda)}$$

$$a) t = \frac{-3g^2 v \sin \alpha + 10 \omega^2 \sin^2 \lambda v^3 \sin^3 \alpha}{-3g^2(g + v_0 \cos \alpha \omega \sin \lambda)}$$

b) Para encontrar z_{\max} primero hay que integrar \dot{z}^* y reemplazar el tiempo en la altura máxima

$$\Rightarrow \dot{z}^* = v \sin \alpha - gt + \left(-v_0 \cos \alpha t - 2 \omega \sin \lambda v \sin \alpha t^2 + \frac{2}{3} \omega \sin \lambda g t^3 \right) \omega \sin \lambda$$

$$\Rightarrow z = v \sin \alpha t - \frac{1}{2} g t^2 + \left(-\frac{v_0 \cos \alpha t^2}{2} - \frac{2 \omega \sin \lambda v \sin \alpha t^3}{3} + \frac{2}{12} \omega \sin \lambda g t^4 \right) \omega \sin \lambda \quad \text{Reemplazar } t$$

$$c) \omega = 0 \Rightarrow t = \frac{-3g^2 v \sin \alpha + 10 \omega^2 \sin^2 \lambda v^3 \sin^3 \alpha}{-3g^2(g + v_0 \cos \alpha \omega \sin \lambda)} = \frac{v \sin \alpha}{g}$$

$$\Rightarrow z = v \sin \alpha t - \frac{1}{2} g t^2 + \left(-\frac{v_0 \cos \alpha t^2}{2} - \frac{2 \omega \sin \lambda v \sin \alpha t^3}{3} + \frac{2}{12} \omega \sin \lambda g t^4 \right) \omega \sin \lambda = v \sin \alpha t - \frac{1}{2} g t^2$$

Caida libre normal

Problema 6. (Taylor 9.26)

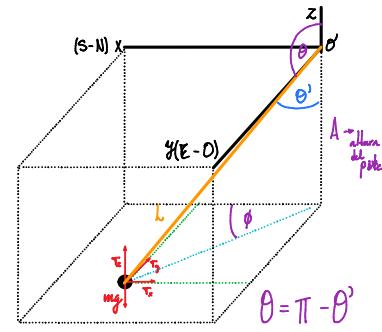
9.26** In Section 9.8, we used a method of successive approximations to find the orbit of an object that is dropped from rest, correct to first order in the earth's angular velocity Ω . Show in the same way that if an object is thrown with initial velocity v_0 from a point O on the earth's surface at colatitude θ , then to first order in Ω its orbit is

$$\left. \begin{aligned} x &= v_{x0}t + \Omega(v_{z0}\cos\theta - v_{x0}\sin\theta)t^2 + \frac{1}{3}\Omega gt^3 \sin\theta \\ y &= v_{y0}t - \Omega(v_{z0}\cos\theta)t^2 \\ z &= v_{z0}t - \frac{1}{2}gt^2 + \Omega(v_{x0}\sin\theta)t^2 \end{aligned} \right\} \quad (9.73)$$

[First solve the equations of motion (9.53) in zeroth order, that is, ignoring Ω entirely. Substitute your zeroth-order solution for \dot{x} , \dot{y} , and \dot{z} into the right side of equations (9.53) and integrate to give the next approximation. Assume that v_0 is small enough that air resistance is negligible and that g is a constant throughout the flight.]

Péndulo de Foucault con cuerda elástica y resistencia lineal y cuadrática

$$S: \sum \vec{F} = m\ddot{\vec{r}} \Rightarrow \vec{w} - \alpha \dot{\vec{r}} - \beta |\dot{\vec{r}}| \dot{\vec{r}} - k(\vec{r} - \vec{L}) = m\ddot{\vec{r}}; \vec{w} = -mg \hat{k}$$



$$S: \sum \vec{F} = m\ddot{\vec{r}} \Rightarrow \vec{w} - \alpha \dot{\vec{r}} - \beta |\dot{\vec{r}}| \dot{\vec{r}} - k(\vec{r} - \vec{L}) - 2m\omega \times \dot{\vec{r}} - m\vec{\omega} \times [\vec{\omega} \times (\vec{r} + \vec{R})] = m\ddot{\vec{r}} \star$$

$$\vec{r} = r \hat{e}_r \quad y \quad \dot{\vec{r}} = \dot{r} \hat{e}_r + r \dot{\theta} \hat{e}_{\phi} + r \dot{\theta} \operatorname{sen} \theta \hat{e}_z; \quad \vec{R} = (R_{\text{Tierra}} + A) \hat{k}$$

$$\vec{\omega} = \omega \cos \lambda \hat{k} - \omega \operatorname{sen} \lambda \hat{t}; \quad \alpha = \frac{3\pi n}{2} \eta \quad y \quad \beta = \frac{c_A p}{2} \rho$$

• Hay que pasar el vector de \vec{w} , $\vec{\omega}$ y \vec{R} a coordenadas esféricas, por definición

$$\begin{cases} \hat{e}_r = \operatorname{sen} \theta \cos \phi \hat{i} + \operatorname{sen} \theta \operatorname{sen} \phi \hat{j} + \cos \theta \hat{k} \\ \hat{e}_{\phi} = \cos \theta \cos \phi \hat{i} + \cos \theta \operatorname{sen} \phi \hat{j} - \operatorname{sen} \theta \hat{k} \\ \hat{e}_{\lambda} = -\operatorname{sen} \phi \hat{i} + \cos \phi \hat{j} \end{cases} \Rightarrow S = \begin{pmatrix} \operatorname{sen} \theta \cos \phi & \operatorname{sen} \theta \operatorname{sen} \phi & \cos \theta \\ \cos \theta \cos \phi & \cos \theta \operatorname{sen} \phi & -\operatorname{sen} \theta \\ -\operatorname{sen} \phi & \cos \phi & 0 \end{pmatrix} \leftarrow \begin{matrix} \text{Matriz de transformación} \\ \text{de cartesianas a esféricas} \end{matrix}$$

$$\Rightarrow \vec{w} = \begin{pmatrix} w_r \\ w_{\phi} \\ w_{\lambda} \end{pmatrix} = \begin{pmatrix} \operatorname{sen} \theta \cos \phi & \operatorname{sen} \theta \operatorname{sen} \phi & \cos \theta \\ \cos \theta \cos \phi & \cos \theta \operatorname{sen} \phi & -\operatorname{sen} \theta \\ -\operatorname{sen} \phi & \cos \phi & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -mg \end{pmatrix} = -mg \begin{pmatrix} \cos \theta \\ -\operatorname{sen} \theta \\ 0 \end{pmatrix} = -mg (\cos \theta \hat{e}_r - \operatorname{sen} \theta \hat{e}_{\phi})$$

$$\Rightarrow \vec{\omega} = \begin{pmatrix} \omega_r \\ \omega_{\phi} \\ \omega_{\lambda} \end{pmatrix} = \begin{pmatrix} \operatorname{sen} \theta \cos \phi & \operatorname{sen} \theta \operatorname{sen} \phi & \cos \theta \\ \cos \theta \cos \phi & \cos \theta \operatorname{sen} \phi & -\operatorname{sen} \theta \\ -\operatorname{sen} \phi & \cos \phi & 0 \end{pmatrix} \begin{pmatrix} -\omega \operatorname{sen} \lambda \\ 0 \\ \omega \cos \lambda \end{pmatrix} = \omega \begin{pmatrix} -\operatorname{sen} \lambda \operatorname{sen} \theta \cos \phi + \cos \lambda \cos \theta \\ -\operatorname{sen} \lambda \cos \theta \cos \phi - \cos \lambda \operatorname{sen} \theta \\ \operatorname{sen} \lambda \operatorname{sen} \phi \end{pmatrix}$$

$$= \omega (-\operatorname{sen} \lambda \operatorname{sen} \theta \cos \phi + \cos \lambda \cos \theta) \hat{e}_r - \omega (\operatorname{sen} \lambda \cos \theta \cos \phi + \cos \lambda \operatorname{sen} \theta) \hat{e}_{\phi} + \omega \operatorname{sen} \lambda \operatorname{sen} \phi \hat{e}_{\lambda}$$

$$\Rightarrow \vec{R} = \begin{pmatrix} R_r \\ R_{\phi} \\ R_{\lambda} \end{pmatrix} = \begin{pmatrix} \operatorname{sen} \theta \cos \phi & \operatorname{sen} \theta \operatorname{sen} \phi & \cos \theta \\ \cos \theta \cos \phi & \cos \theta \operatorname{sen} \phi & -\operatorname{sen} \theta \\ -\operatorname{sen} \phi & \cos \phi & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ R \end{pmatrix} = R \begin{pmatrix} \cos \theta \\ -\operatorname{sen} \theta \\ 0 \end{pmatrix} = R \cos \theta \hat{e}_r - R \operatorname{sen} \theta \hat{e}_{\phi}$$

Continua...

- Ahora, resolviendo los productos vectoriales

$$*\vec{\omega} \times \vec{r} = \begin{vmatrix} \hat{e}_r & \hat{e}_{\theta} & \hat{e}_{\phi} \\ w(-\operatorname{sen}\lambda \operatorname{sen}\theta \cos\phi + \operatorname{cos}\lambda \operatorname{cos}\theta) & -w(\operatorname{sen}\lambda \operatorname{cos}\theta \cos\phi + \operatorname{cos}\lambda \operatorname{sen}\theta) & w\operatorname{sen}\lambda \operatorname{sen}\phi \\ r & r\theta & r\phi \end{vmatrix}$$

$$= [-r\dot{\theta} \operatorname{sen}\theta (-\operatorname{sen}\lambda \operatorname{cos}\theta \cos\phi + \operatorname{cos}\lambda \operatorname{sen}\theta) - r\dot{\theta} w \operatorname{sen}\lambda \operatorname{sen}\phi] \hat{e}_r + \dots$$

$$\dots - [r\dot{\theta} \operatorname{sen}\theta w (-\operatorname{sen}\lambda \operatorname{cos}\theta \cos\phi + \operatorname{cos}\lambda \operatorname{cos}\theta) - r\dot{\theta} w \operatorname{sen}\lambda \operatorname{sen}\phi] \hat{e}_{\theta} + \dots$$

$$\dots + [r\dot{\theta} w (-\operatorname{sen}\lambda \operatorname{sen}\theta \cos\phi + \operatorname{cos}\lambda \operatorname{cos}\theta) + \dot{r} w (\operatorname{sen}\lambda \operatorname{cos}\theta \cos\phi + \operatorname{cos}\lambda \operatorname{sen}\theta)] \hat{e}_{\phi}$$

$$*\vec{\omega} \times \vec{r} = \begin{vmatrix} \hat{e}_r & \hat{e}_{\theta} & \hat{e}_{\phi} \\ w(-\operatorname{sen}\lambda \operatorname{sen}\theta \cos\phi + \operatorname{cos}\lambda \operatorname{cos}\theta) & -w(\operatorname{sen}\lambda \operatorname{cos}\theta \cos\phi + \operatorname{cos}\lambda \operatorname{sen}\theta) & w\operatorname{sen}\lambda \operatorname{sen}\phi \\ r & 0 & 0 \end{vmatrix}$$

$$= -r w \operatorname{sen}\lambda \operatorname{sen}\phi \hat{e}_{\theta} - r w (\operatorname{sen}\lambda \operatorname{cos}\theta \cos\phi + \operatorname{cos}\lambda \operatorname{sen}\theta) \hat{e}_{\phi}$$

$$= r \operatorname{sen}\lambda \operatorname{sen}\phi \hat{e}_{\theta} + r w (\operatorname{sen}\lambda \operatorname{cos}\theta \cos\phi + \operatorname{cos}\lambda \operatorname{sen}\theta) \hat{e}_{\phi}$$

$$*\vec{\omega} \times (\vec{\omega} \times \vec{r}) = \begin{vmatrix} \hat{e}_r & \hat{e}_{\theta} & \hat{e}_{\phi} \\ w(-\operatorname{sen}\lambda \operatorname{sen}\theta \cos\phi + \operatorname{cos}\lambda \operatorname{cos}\theta) & -w(\operatorname{sen}\lambda \operatorname{cos}\theta \cos\phi + \operatorname{cos}\lambda \operatorname{sen}\theta) & w\operatorname{sen}\lambda \operatorname{sen}\phi \\ 0 & r \operatorname{sen}\lambda \operatorname{sen}\phi & r w (\operatorname{sen}\lambda \operatorname{cos}\theta \cos\phi + \operatorname{cos}\lambda \operatorname{sen}\theta) \end{vmatrix}$$

$$= [-w (\operatorname{sen}\lambda \operatorname{cos}\theta \cos\phi + \operatorname{cos}\lambda \operatorname{sen}\theta) r w (\operatorname{sen}\lambda \operatorname{cos}\theta \cos\phi + \operatorname{cos}\lambda \operatorname{sen}\theta) - r \operatorname{sen}\lambda \operatorname{sen}\phi w \operatorname{sen}\lambda \operatorname{sen}\phi] \hat{e}_r + \dots$$

$$\dots - [w (-\operatorname{sen}\lambda \operatorname{sen}\theta \cos\phi + \operatorname{cos}\lambda \operatorname{cos}\theta) r w (\operatorname{sen}\lambda \operatorname{cos}\theta \cos\phi + \operatorname{cos}\lambda \operatorname{sen}\theta)] \hat{e}_{\theta} + \dots$$

$$\dots + [r \operatorname{sen}\lambda \operatorname{sen}\phi w (-\operatorname{sen}\lambda \operatorname{sen}\theta \cos\phi + \operatorname{cos}\lambda \operatorname{cos}\theta)] \hat{e}_{\phi}$$

$$\rightarrow -r w^2 [(\operatorname{sen}\lambda \operatorname{cos}\theta \cos\phi + \operatorname{cos}\lambda \operatorname{sen}\theta)^2 + \operatorname{sen}^2 \lambda \operatorname{sen}^2 \phi] \hat{e}_r + \dots$$

$$\dots - [r w^2 (-\operatorname{sen}\lambda \operatorname{sen}\theta \cos\phi + \operatorname{cos}\lambda \operatorname{cos}\theta)(\operatorname{sen}\lambda \operatorname{cos}\theta \cos\phi + \operatorname{cos}\lambda \operatorname{sen}\theta)] \hat{e}_{\theta} + \dots$$

$$\dots + [r w^2 \operatorname{sen}\lambda \operatorname{sen}\phi (-\operatorname{sen}\lambda \operatorname{sen}\theta \cos\phi + \operatorname{cos}\lambda \operatorname{cos}\theta)] \hat{e}_{\phi}$$

Continua...

$$*\vec{\omega} \times \vec{R} = \begin{vmatrix} \hat{e}_r & \hat{e}_{\theta} & \hat{e}_{\phi} \\ w(-\sin\lambda \cos\theta \cos\phi + \cos\lambda \cos\theta) & -w(\sin\lambda \cos\theta \cos\phi + \cos\lambda \sin\theta) & w \sin\lambda \cos\theta \\ R \cos\theta & -R \sin\theta & 0 \end{vmatrix}$$

$$= R \sin\theta w \sin\lambda \cos\theta \hat{e}_r + R \cos\theta w \sin\lambda \cos\theta \hat{e}_{\theta} + [R \sin\theta w(-\sin\lambda \cos\theta \cos\phi + \cos\lambda \cos\theta) + \dots \curvearrowleft \gamma \\ \dots + R \cos\theta w(\sin\lambda \cos\theta \cos\phi + \cos\lambda \sin\theta)] \hat{e}_{\phi}$$

$$*\gamma = -R \sin\theta w(-\sin\lambda \cos\theta \cos\phi + \cos\lambda \cos\theta) + R \cos\theta w(\sin\lambda \cos\theta \cos\phi + \cos\lambda \sin\theta)$$

$$= R w \sin\lambda \cos^2\theta \cos\phi - \cancel{R w \cos\lambda \cos\theta \sin\theta} + R w \sin\lambda \cos^2\theta \cos\phi + \cancel{R w \sin\lambda \cos\theta \sin\theta}$$

$$= R w \sin\lambda \cos\theta$$

$$*\vec{\omega} \times (\vec{\omega} \times \vec{r}) = \begin{vmatrix} \hat{e}_r & \hat{e}_{\theta} & \hat{e}_{\phi} \\ w(-\sin\lambda \cos\theta \cos\phi + \cos\lambda \cos\theta) & -w(\sin\lambda \cos\theta \cos\phi + \cos\lambda \sin\theta) & w \sin\lambda \cos\theta \\ R w \sin\lambda \cos\theta \sin\phi & R w \sin\lambda \cos\theta \sin\phi & R w \sin\lambda \cos\theta \end{vmatrix}$$

$$= [-w(\sin\lambda \cos\theta \cos\phi + \cos\lambda \sin\theta) R w \sin\lambda \cos\theta - w \sin\lambda \cos\theta R w \sin\lambda \cos\theta \sin\phi] \hat{e}_r + \dots$$

$$\dots - [w(-\sin\lambda \cos\theta \cos\phi + \cos\lambda \cos\theta) R w \sin\lambda \cos\theta - w \sin\lambda \cos\theta R w \sin\lambda \cos\theta \sin\phi] \hat{e}_{\theta} + \dots$$

$$\dots + [w(-\sin\lambda \cos\theta \cos\phi + \cos\lambda \cos\theta) R w \sin\lambda \cos\theta \sin\phi + w(\sin\lambda \cos\theta \cos\phi + \cos\lambda \sin\theta) R w \sin\lambda \cos\theta \sin\phi] \hat{e}_{\phi}$$

$$= (-R w^2 \sin^2\lambda \cos^2\theta - R w^2 \sin\lambda \cos\lambda \cos\theta \sin\theta - R w^2 \sin^2\lambda \sin^2\theta \cos\theta) \hat{e}_r + \dots$$

$$\dots - (-R w^2 \sin^2\lambda \cos^2\theta \sin\theta + R w^2 \sin\lambda \cos\lambda \cos\theta \cos\theta - R w^2 \sin^2\lambda \sin^2\theta \sin\theta) \hat{e}_{\theta} + \dots$$

$$\dots + (-\cancel{R w^3 \sin^2\lambda \cos\theta \sin\theta \cos\theta \sin\theta} + R w^3 \sin\lambda \cos\lambda \sin\theta \cos^2\theta + \cancel{R w^3 \sin^2\lambda \cos\theta \sin\theta \cos\theta} + \dots$$

$$\dots + R w^3 \sin\lambda \cos\lambda \sin\theta \sin^2\theta) \hat{e}_{\phi}$$

$$= -R w^2 (\sin^2\lambda \cos\theta + \sin\lambda \cos\lambda \cos\theta \sin\theta) \hat{e}_r - R w^2 (\sin\lambda \cos\lambda \cos\theta \cos\theta - \sin^2\lambda \sin\theta) \hat{e}_{\theta} + R w^2 \sin\lambda \cos\lambda \sin\theta \hat{e}_{\phi}$$

$$= -R w^2 [(\sin^2\lambda \cos\theta + \sin\lambda \cos\lambda \cos\theta \sin\theta) \hat{e}_r + (\sin\lambda \cos\lambda \cos\theta \cos\theta - \sin^2\lambda \sin\theta) \hat{e}_{\theta} - \sin\lambda \cos\lambda \sin\theta \hat{e}_{\phi}]$$

Continua...

Regresando a \star :

$$\left\{ \begin{array}{l} r: -mg \cos \theta - \alpha r - \beta \sqrt{\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \ddot{\theta}^2 \sin^2 \theta} \dot{r} - k(r - l) + \dots \\ \dots - 2m[-r \dot{\theta} \sin \theta (\sin \lambda \cos \theta + \cos \lambda \sin \theta) - r \dot{\theta} w \sin \lambda \sin \theta] + \dots \\ \dots + mrw^2[(\sin \lambda \cos \theta \cos \theta + \cos \lambda \sin \theta)^2 + \sin^2 \lambda \sin^2 \theta] + mRw^2(\sin^2 \lambda \cos \theta + \sin \lambda \cos \lambda \cos \theta \sin \theta) \\ \\ \theta: mg \sin \theta - \alpha r \dot{\theta} - \beta \sqrt{\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \ddot{\theta}^2 \sin^2 \theta} r \dot{\theta} + \dots \\ \dots + 2m[r \dot{\theta} \sin \theta (-\sin \lambda \sin \theta \cos \theta + \cos \lambda \cos \theta) - \dot{r} w \sin \lambda \sin \theta] \\ \dots + mrw^2(-\sin \lambda \sin \theta \cos \theta + \cos \lambda \cos \theta)(\sin \lambda \cos \theta \cos \theta + \cos \lambda \sin \theta) \\ \dots + mRw^2(\sin \lambda \cos \lambda \cos \theta \cos \theta - \sin^2 \lambda \sin \theta) \\ \\ \phi: -\alpha r \dot{\theta} \sin \theta - \beta \sqrt{\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \ddot{\theta}^2 \sin^2 \theta} r \dot{\theta} \sin \theta + \dots \\ \dots - 2m[r \dot{\theta} w (-\sin \lambda \sin \theta \cos \theta + \cos \lambda \cos \theta) + \dot{r} w (\sin \lambda \cos \theta \cos \theta + \cos \lambda \sin \theta)] \\ \dots - mrw^2 \sin \lambda \sin \theta (-\sin \lambda \sin \theta \cos \theta + \cos \lambda \cos \theta) - mRw^2 \sin \lambda \cos \lambda \sin \theta \end{array} \right.$$

$\Rightarrow m\ddot{r} =$

La aceleración en coordenadas esféricas

$$\ddot{r} = (\ddot{r} - r \dot{\theta}^2 - r \ddot{\theta}^2 \sin^2 \theta) \hat{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta} - r \dot{\theta}^2 \sin \theta \cos \theta) \hat{e}_{\theta} + (2\dot{r}\dot{\theta} \sin \theta + 2r\dot{\theta}\ddot{\theta} \cos \theta + r\ddot{\theta} \sin \theta) \hat{e}_{\phi}$$

* Por último, pasando θ a θ' y renombrando θ' como θ : $\theta = \pi - \theta'$

- $\sin \theta = +\sin \theta'$
- $\cos \theta = -\cos \theta'$
- $\dot{\theta} = -\dot{\theta}'$
- $\ddot{\theta} = -\ddot{\theta}'$

Continua

Las ecuaciones del movimiento

$$1: m\ddot{r} = mr\dot{\theta}^2 + mr\dot{\phi}^2 \sin^2\theta + mgsen\theta - air - \beta \sqrt{\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2 \sin^2\theta} \dot{r} - k(r-L) + \dots \\ \dots - 2m[-r\dot{\theta}\sin\theta w(-\sin\lambda\cos\theta\cos\phi + \cos\lambda\sin\phi) - r\dot{\phi}w\sin\lambda\sin\phi] + \dots \\ \dots + mrw^2[(-\sin\lambda\cos\theta\cos\phi + \cos\lambda\sin\phi)^2 + \sin^2\lambda\sin^2\phi] + mRw^2(-\sin^2\lambda\cos\theta + \sin\lambda\cos\lambda\cos\theta\sin\phi)$$

$$\Rightarrow 2: -mr\ddot{\theta} = -mr\dot{\phi}\sin\theta\cos\theta + 2mr\dot{\phi}\dot{\theta} + mgsen\theta + air\dot{\theta} + \beta \sqrt{\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2 \sin^2\theta} r\dot{\theta} + \dots \\ \dots + 2m[r\dot{\theta}\sin\theta w(-\sin\lambda\sin\theta\cos\phi - \cos\lambda\cos\phi) - \dot{r}w\sin\lambda\sin\phi] \\ \dots + mrw^2(-\sin\lambda\sin\theta\cos\phi - \cos\lambda\cos\phi)(-\sin\lambda\cos\theta\cos\phi + \cos\lambda\sin\phi) \\ \dots + mRw^2(-\sin\lambda\cos\lambda\cos\theta\cos\phi - \sin^2\lambda\sin\phi)$$

$$3: mr\ddot{\phi}\sin\theta = -2mr\dot{r}\dot{\phi}\sin\theta - 2mr\dot{\phi}\dot{\theta}\cos\theta - air\dot{\phi}\sin\theta - \beta \sqrt{\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2 \sin^2\theta} r\dot{\phi}\sin\theta + \dots \\ \dots - 2m[-r\dot{\phi}w(-\sin\lambda\sin\theta\cos\phi - \cos\lambda\cos\phi) + \dot{r}w(-\sin\lambda\cos\theta\cos\phi + \cos\lambda\sin\phi)] + \dots \\ \dots - mrw^2\sin\lambda\sin\phi(-\sin\lambda\sin\theta\cos\phi - \cos\lambda\cos\phi) - mRw^2\sin\lambda\cos\lambda\sin\phi$$

Problema 7.

¿Cuánto tiempo tarda en dar una revolución completa el plano de oscilación del péndulo de Foucault si está situando en una latitud de 85° ?

Reapareciendo la ecuación 3

$$\begin{aligned} m(2\dot{\theta}\dot{\phi}\cos\theta + \ddot{\phi}\sin\theta) &= -2mr[\dot{\theta}\omega(-\sin\lambda\sin\theta\cos\phi + \cos\lambda\cos\theta)] - mr[w^2\sin\lambda\sin\phi(-\sin\lambda\sin\theta\cos\phi + \cos\lambda\cos\theta)] \\ \Rightarrow 2\dot{\theta}\dot{\phi}\cos\theta + \ddot{\phi}\sin\theta &= -2\dot{\theta}\omega(-\sin\lambda\sin\theta\cos\phi + \cos\lambda\cos\theta) - w^2\sin\lambda\sin\phi(-\sin\lambda\sin\theta\cos\phi + \cos\lambda\cos\theta) \\ \Rightarrow \ddot{\phi}\sin\theta &= -2\dot{\theta}\omega(-\sin\lambda\sin\theta\cos\phi + \cos\lambda\cos\theta) - w^2\sin\lambda\sin\phi(-\sin\lambda\sin\theta\cos\phi + \cos\lambda\cos\theta) - 2\dot{\theta}\dot{\phi}\cos\theta \\ \Rightarrow \ddot{\phi} &= -2\dot{\theta}\omega(-\sin\lambda\cos\theta + \cos\lambda\cot\theta) - w^2\sin\lambda\sin\phi(-\sin\lambda\cos\theta + \cos\lambda\cot\theta) - 2\dot{\theta}\dot{\phi}\cot\theta \end{aligned}$$

Ignorando los efectos centrífuga

$$\Rightarrow \ddot{\phi} = -2\dot{\theta}\omega(-\sin\lambda\cos\theta + \cos\lambda\cot\theta) - 2\dot{\theta}\dot{\phi}\cot\theta$$

$$w_\phi = w\sin\lambda = 7,3 \cdot 10^{-5} \text{ rad/s} \sin 85^\circ = 7,27 \cdot 10^{-5} \text{ rad/s}$$

Problema 8. (Taylor 9.30)

9.30 *** The Coriolis force can produce a torque on a spinning object. To illustrate this, consider a horizontal hoop of mass m and radius r spinning with angular velocity ω about its vertical axis at colatitude θ . Show that the Coriolis force due to the earth's rotation produces a torque of magnitude $m\omega\Omega r^2 \sin\theta$ directed to the west, where Ω is the earth's angular velocity. This torque is the basis of the gyrocompass.



$$\mathbf{F} = -2m\vec{\Omega} \times \dot{\vec{r}} = -2m\vec{\Omega} \times (\vec{\omega} \times \vec{r}) = -2m\Omega r \omega \sin\theta \hat{\mathbf{z}} \quad \Rightarrow \quad \mathbf{N} = r\mathbf{F} = -2m\Omega r^2 \omega \sin\theta \hat{\mathbf{z}}$$

Mecánica Lagrangiana

Este capítulo se va a dedicar a establecer un formalismo alternativo a las Leyes de Newton, el formalismo lagrangiano aplicado a la mecánica clásica. Este nuevo formalismo produce resultados equivalentes al de Newton, como es de esperarse, no obstante permite simplificar de gran manera los análisis.

Aquí se expresará la Mecánica Lagrangiana de forma general, de modo que se aplique a sistemas conservativos (o no) y a sistemas de partículas.

SECTION 6

Conceptos Fundamentales

SUBSECTION 6.1

Grados de Libertad y Coordenadas Generalizadas

Definition 29

(Grados de Libertad) Para un sistema físico cualquiera, el número de grados de libertad corresponde al número más pequeño de cantidades escalares independientes necesarias para dar la posición (Sin contar el tiempo) de un objeto de interés dentro del sistema.

- **Partícula en una dimensión:** 1 grado de libertad, la partícula se encuentra sobre una recta está puede ir hacia adelante o atrás.
- **Partícula en 2 dimensiones:** 2 grados de libertad, la partícula tiene libertad de moverse en un plano.
- **Partícula en 3 dimensiones:** 3 grados de libertad, la partícula tiene puede moverse en un espacio tridimensional (en un ancho, largo y altura respecto a un origen).
- **n-Partículas en 3 dimensiones:** $3n$ grados de libertad, cada partícula individual tiene sus 3 grados de libertad propios y la suma es el total de grados de libertad del sistema.

Definition 30

(Fuerzas de Restricción) Las fuerzas de restricción, como su nombre lo indican, restringen el movimiento que puede tener un sistema desde la libertad total, es decir, cortan los grados de libertad que posee un sistema. Algunos ejemplos:

- La tensión debido a la cuerda en un péndulo.
- La fuerza normal.
- Fuerzas electromagnéticas entre partículas de un sólido.

Definition 31

(Restricciones) Las restricciones son las limitaciones que se le imponen al movimiento del sistema, ya sea por observación de dicha limitación o simplificación del sistema y son dadas por medios de ecuaciones que contengan los parámetros que son restringidos en el sistema (Posiciones, velocidades, ...). Las restricciones que posea

un sistema están directamente relacionadas a las fuerzas de restricción que el sistema siente, de acuerdo a los ejemplos de fuerzas de restricción:

- Un sistema atado a una cuerda.
- Un sistema sobre una o entre superficies.
- Un cuerpo, sus partículas se mueven muy poco o no lo hacen en sus posiciones relativas al CM.

Existen diferentes tipos de restricciones de acuerdo a como se expresan:

- **Holonómicas:** Son restricciones que se pueden escribir de la forma:

$$g(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_n) = 0 \quad (6.1)$$

- **No Holonómicas:** Son aquellas restricciones que no pueden ser escritas como las holonómicas.

También es posible clasificar a las restricciones a partir de su dependencia temporal:

- **Esclerónomas:** El tiempo no es una variable explícita
- **Reónomas:** El tiempo es una variable explícita

A partir de aquí se va a trabajar con restricciones holonómicas únicamente a menos de que se indique lo contrario. Esto porque este tipo de restricción permite reducir el número de coordenadas necesarias para describir el movimiento de un sistema a partir de un nuevo conjunto de coordenadas conocido como coordenadas generalizadas.

Definition 32

(Coordenadas Generalizadas) Son el conjunto de cantidades ¹³ que especifican completamente el estado de un sistema.

Ahora, suponiendo que un sistema de estudio está conformado por n partículas y al estar en un espacio tridimensional, en un principio el sistema posee $3n$ grados de libertad. Si el sistema posee m ecuaciones de restricción¹⁴, tendrá m restricciones en los grados de libertad totales del sistema, entonces los grados de libertad son:

$$s = 3n - m \leftarrow \begin{array}{l} \text{Grados de libertad de un} \\ \text{sistema en forma general} \end{array}$$

Es posible escribir las ecuaciones de transformación entre las coordenadas generalizadas y coordenadas curvilíneas cualesquiera, como se presenta a continuación:

$$\begin{array}{ccc} \text{Coordenadas Curvilíneas} & & \text{Coordenadas Generalizadas} \\ \left\{ \begin{array}{l} x_{\alpha,i} = x_{\alpha,i}(q_j, t) \\ \dot{x}_{\alpha,i} = \dot{x}_{\alpha,i}(q_j, \dot{q}_j, t) \end{array} \right. & \leftrightarrow & \left\{ \begin{array}{l} q_j = q_j(x_{\alpha,i}, t) \\ \dot{q}_j = \dot{q}_j(x_{\alpha,i}, \dot{x}_{\alpha,i}, t) \end{array} \right. \\ \left\{ \begin{array}{l} \alpha = 1, 2, \dots, n \\ i = 1, 2, 3 \\ j = 1, 2, \dots, s \end{array} \right. & & \end{array}$$

Si se tiene el conjunto más pequeño de coordenadas generalizadas, este es conocido como el **conjunto adecuado** de coordenadas¹⁵. En este caso, el conjunto de cantidades

¹³ No necesariamente tienen que ser distancias, pueden ser cantidades adimensionales o incluso con unidades de energía.

¹⁴ Asociadas o no a fuerzas de restricción. Estas pueden ser incluso cortes en la dimensionalidad del espacio en que se encuentra el sistema.

¹⁵ Va a ser común desarrollar un problema tanto con el conjunto adecuado como con un conjunto de cantidades que superen al adecuado.

(Coordenadas Generalizadas) corresponde a los grados de libertad del sistema.

SUBSECTION 6.2

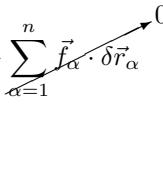
Trabajo Virtual, Fuerzas Generalizadas y Principio de D'Alambert

A partir de aquí se considerará un sistema compuesto por n partículas, cada una de ellas expuesta a su vector de fuerza neta correspondiente F_α . Siendo F_α la suma de cualquier clase de fuerzas que actúen sobre la partícula α (Fuerzas conservativas, no conservativas, de restricción) y posee sus componentes, por ejemplo, en coordenadas cartesianas F_{α_x} , F_{α_y} y F_{α_z} . Además, la fuerza se puede separar de la forma $\vec{F}_\alpha = \vec{F}_\alpha^e + \vec{f}_\alpha$, donde \vec{F}_α^e corresponde a todas las fuerzas externas aplicadas en la partícula α y \vec{f}_α corresponde a todas las fuerzas de restricción que afectan a dicha partícula.

Un trabajo infinitesimal provocado por esta fuerza:

$$\begin{aligned}\delta W &= \sum_{\alpha=1}^n \vec{F}_i \cdot \delta \vec{r}_\alpha \\ &= \sum_{\alpha=1}^n \vec{F}_\alpha^e \cdot \delta \vec{r}_\alpha + \sum_{\alpha=1}^n \vec{f}_\alpha \cdot \delta \vec{r}_\alpha\end{aligned}$$

Limitando el análisis a sistemas tales que las fuerzas de restricción no producen este tipo de trabajo infinitesimal:

$$\begin{aligned}\delta W &= \sum_{\alpha=1}^n \vec{F}_\alpha^e \cdot \delta \vec{r}_\alpha + \sum_{\alpha=1}^n \vec{f}_\alpha \cdot \delta \vec{r}_\alpha \\ &= \sum_{\alpha=1}^n \vec{F}_\alpha^e \cdot \delta \vec{r}_\alpha\end{aligned}$$


Redefiniendo el vector \vec{F}_α como la suma de todas las fuerzas aplicadas sobre la partícula α , excepto las de restricción, dado que ya se estableció que para este análisis estas no producen trabajo. se introduce el siguiente concepto:

Definition 33

(Trabajo Virtual) Corresponde al trabajo producido por un **desplazamiento virtual**¹⁶. Un desplazamiento virtual es un desplazamiento infinitesimal de un sistema, una alteración en la configuración de este, como resultado de un cambio infinitesimal arbitrario de las coordenadas $\delta \vec{r}_\alpha$, que debe ser consistente con las fuerzas y las restricciones impuestas en el sistema en un cierto instante t .

¹⁶ Al decir “virtual”, es para diferenciarlo de un desplazamiento o trabajo real del sistema.

$$\delta W = \sum_{\alpha=1}^n \vec{F}_\alpha \cdot \delta \vec{r}_\alpha \quad (6.2)$$

A partir de la definición anterior, buscando colocar el trabajo virtual en el conjunto de coordenadas generalizadas:

Es posible desarrollar $\delta \vec{r}_\alpha$ a partir de la regla de la cadena para 3 grados de libertad como:

$$\delta \vec{r}_\alpha = \sum_{j=1}^s \frac{\partial \vec{r}_\alpha}{\partial q_j} \delta q_j$$

Excluyendo la derivada parcial temporal por la definición del desplazamiento virtual, que solo considera desplazamientos en las coordenadas. Realizando el cambio de $\delta \vec{r}_\alpha$ en la Ecuación (6.2):

$$\begin{aligned}\delta W &= \sum_{\alpha=1}^n \vec{F}_\alpha \cdot \delta \vec{r}_\alpha \\ &= \sum_{\alpha=1}^n \vec{F}_\alpha \cdot \sum_{j=1}^s \frac{\partial \vec{r}_\alpha}{\partial q_j} \delta q_j \\ &= \sum_{\alpha=1}^n \sum_{j=1}^s \vec{F}_\alpha \cdot \frac{\partial \vec{r}_\alpha}{\partial q_j} \delta q_j \\ &= \sum_{j=1}^s Q_j \delta q_j\end{aligned}$$

De lo anterior se obtiene:

$$Q_j = \sum_{\alpha=1}^n \vec{F}_\alpha \cdot \frac{\partial \vec{r}_\alpha}{\partial q_j}$$

Lo cual al considerar que tanto \vec{F}_α y \vec{r}_α se encuentran inicialmente en coordenadas curvilíneas:

$$Q_j = \sum_{\alpha=1}^n \left(F_{x_{1,\alpha}} \frac{\partial x_{1,\alpha}}{\partial q_j} + F_{x_{2,\alpha}} \frac{\partial x_{2,\alpha}}{\partial q_j} + F_{x_{3,\alpha}} \frac{\partial x_{3,\alpha}}{\partial q_j} \right)$$

Definition 34

(Fuerzas Generalizadas) Corresponde a un nombre genérico para referirse, en un principio, a fuerzas y torques escritos en las coordenadas generalizadas que describen el sistema. Estas son dadas en los componentes correspondientes a cada coordenada generalizada. Comúnmente las fuerzas o torques son escritas inicialmente en coordenadas curvilíneas en un inicio para luego ser transformadas a fuerzas generalizadas.

$$Q_j = \sum_{\alpha=1}^n \left(F_{x_{1,\alpha}} \frac{\partial x_{1,\alpha}}{\partial q_j} + F_{x_{2,\alpha}} \frac{\partial x_{2,\alpha}}{\partial q_j} + F_{x_{3,\alpha}} \frac{\partial x_{3,\alpha}}{\partial q_j} \right) \quad (6.3)$$

Para conocer la naturaleza de la fuerza generalizada (Fuerza, Torque, ...), se debe revisar la dimensionalidad del trabajo virtual ejercido por tal fuerza, es decir:

$$\delta W_j = Q_j \delta q_j ; \text{ Debe poseer unidades de energía}$$

Una vez establecidas las coordenadas generalizadas, es sencillo describir a que corresponden las fuerzas generalizadas a partir de lo anterior.

Ahora, en busca de generar una forma alternativa de mecánica, se va a explotar el concepto anterior. Primero, a partir de la Ecuación (3.3):

$$\vec{F}_\alpha = \dot{\vec{p}}_\alpha \\ \Rightarrow \vec{F}_\alpha - \dot{\vec{p}}_\alpha = 0$$

Esta expresión enuncia que una partícula expuesta a una fuerza \vec{F}_α , se encontrará en equilibrio si se le ejerce una fuerza efectiva contraria a la original $-\dot{\vec{p}}_\alpha$. Entonces, con esto se puede buscar el trabajo virtual de todas estas fuerzas sobre el sistema, que por lo anterior se sabe que será cero.

$$\delta W = \sum_{\alpha=1}^n (\vec{F}_\alpha - \dot{\vec{p}}_\alpha) \cdot \delta \vec{r}_\alpha = 0$$

Definition 35

(Principio de D'Alambert) Este principio está expresando en coordenadas generalizadas.

$$\sum_{\alpha=1}^n (\vec{F}_\alpha - \dot{\vec{p}}_\alpha) \cdot \delta \vec{r}_\alpha = 0 \quad (6.4)$$

Un resultado de este principio, es el Principio de Trabajo Virtual ¹⁷:

$$\delta W = \sum_{\alpha=1}^n \vec{F}_\alpha \cdot \delta \vec{r}_\alpha = 0$$

¹⁷Es el caso estático del Principio de D'Alambert, ampliamente usado en ingeniería.

SECTION 7

Principio de Hamilton

Este principio puede ser deducido desde el Principio de D'Alambert, no obstante, la deducción no se realizará por ahora.

Definition 36

(Principio de Hamilton Extendido) El movimiento de un sistema desde un tiempo t_1 a un tiempo t_2 es tal que la integral de línea (La llamada Acción o integral de la Acción) de la energía cinética más el trabajo ejercido por las fuerzas del sistema, tenga un valor estacionario para el camino real que sigue el sistema ¹⁸.

$$I = \int_{t_1}^{t_2} T(q_j, \dot{q}_j, t) + W dt \quad (7.1)$$

$$\delta \int_{t_1}^{t_2} T(q_j, \dot{q}_j, t) + W dt = 0 \leftarrow \text{Valor estacionario}$$

¹⁸El Principio de Hamilton original se define de forma similar, donde la Acción es de la forma:

$$I = \int_{t_1}^{t_2} T - V dt$$

SECTION 8

Ecuaciones de la Mecánica de Lagrange

A continuación, en esta sección se va deducir la **ecuación de Euler - Lagrange** a partir del Principio de Hamilton Extendido de modo que sea aplicable a sistemas no conservativos y se introducirán conceptos propios de la Mecánica Lagrangiana.

SUBSECTION 8.1

Deducción del formalismo Lagrangiano

Considere un sistema compuesto por n partículas y con m restricciones, dando a lugar $s = n - m$ grados de libertad, igual a la cantidad de coordenadas generalizadas (q_1, q_2, \dots, q_s) que describen la dinámica del sistema. Además, se expone a cada una de las partículas a una fuerza neta (totalmente conservativa, no conservativa o con ambos tipos de fuerzas) Q_j con $j = 1, 2, \dots, s$.

Comenzando con la deducción de la ecuación Euler - Lagrange, se debe optimizar la Acción (Ecuación (7.1)) para conseguir un valor estacionario, hay que tomar la variación de esta e igualarla a cero:

$$\begin{aligned} \delta I &= \delta \int_{t_1}^{t_2} T(q_j, \dot{q}_j, t) + W dt = 0 \\ \delta \int_{t_1}^{t_2} T(q_j, \dot{q}_j, t) + W dt &= 0 \\ \Rightarrow \int_{t_1}^{t_2} \delta T(q_j, \dot{q}_j, t) + \delta W dt &= 0 \\ \Rightarrow \int_{t_1}^{t_2} \left[\sum_{j=1}^s \left(\frac{\partial T}{\partial q_j} \delta q_j + \frac{\partial T}{\partial \dot{q}_j} \delta \dot{q}_j \right) + \sum_{j=1}^s Q_j \delta q_j \right] dt &= 0 \\ \Rightarrow \int_{t_1}^{t_2} \sum_{j=1}^s \left(\frac{\partial T}{\partial q_j} \delta q_j + \frac{\partial T}{\partial \dot{q}_j} \delta \dot{q}_j + Q_j \delta q_j \right) dt &= 0 \end{aligned}$$

Con este resultado, ahora se tomará el término marcado en azul para transformarlo en una expresión equivalente:

$$\begin{aligned} \int_{t_1}^{t_2} \sum_{j=1}^s \frac{\partial T}{\partial \dot{q}_j} \delta \dot{q}_j dt &= \int_{t_1}^{t_2} \sum_{j=1}^s \frac{\partial T}{\partial \dot{q}_j} \delta \left(\frac{dq_j}{dt} \right) dt \\ &= \int_{t_1}^{t_2} \sum_{j=1}^s \frac{\partial T}{\partial \dot{q}_j} \frac{d}{dt} (\delta q_j) dt \\ &= \sum_{j=1}^s \int_{t_1}^{t_2} \frac{\partial T}{\partial \dot{q}_j} \frac{d}{dt} (\delta q_j) dt \end{aligned}$$

Resolviendo esta integral por integración por partes para un término arbitrario de la suma ¹⁹:

$$\int_{t_1}^{t_2} \underbrace{\frac{\partial T}{\partial \dot{q}_j} \frac{d}{dt} (\delta q_j)}_{\text{dv}} dt = \underbrace{\frac{\partial T}{\partial \dot{q}_j} \delta q_j}_{\text{u}} \Big|_{t_1}^0 - \int_{t_1}^{t_2} \delta q_j \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) dt = - \int_{t_1}^{t_2} \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) \delta q_j dt$$

De volviendo esta expresión al interior de la ecuación inicial:

¹⁹ El término:

$$\frac{\partial T}{\partial \dot{q}_j} \delta q_j \Big|_{t_1}^{t_2} = 0$$

porque los extremos están fijos, de modo que la variación en la coordenada generalizada q_j es igual a 0 cuando se está en el momento inicial t_1 y en el momento final t_2

$$\begin{aligned}
& \int_{t_1}^{t_2} \sum_{j=1}^s \left(\frac{\partial T}{\partial q_j} \delta q_j + \frac{\partial T}{\partial \dot{q}_j} \delta \dot{q}_j + Q_j \delta q_j \right) dt = 0 \\
& \Rightarrow \int_{t_1}^{t_2} \sum_{j=1}^s \left(\frac{\partial T}{\partial q_j} \delta q_j + -\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) \delta \dot{q}_j + Q_j \delta q_j \right) dt = 0 \\
& \Rightarrow \int_{t_1}^{t_2} \left\{ \sum_{j=1}^s \left[\frac{\partial T}{\partial q_j} - \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) + Q_j \right] \delta q_j \right\} dt = 0
\end{aligned}$$

Entonces, para que esta integral sea igual a cero, el contenido de la integral debe ser igual a cero para valores arbitrarios de las variaciones $\delta q_1, \delta q_2, \dots, \delta q_s$:

$$\sum_{j=1}^s \left[\frac{\partial T}{\partial q_j} - \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) + Q_j \right] \delta q_j = 0$$

Dada la arbitrariedad de dichas variaciones, es posible suponer que todas las variaciones de δq 's son cero, excepto un δq_j . Se debe cumplir para ese δq_j :

$$\frac{\partial T}{\partial q_j} - \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) + Q_j = 0$$

Reescribiendo ²⁰:

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} = Q_j$$

Para terminar esta deducción, es necesario separar las fuerzas conservativas (c) de las no conservativas (nc), entonces:

$$\begin{aligned}
Q_j &= Q_j^c + Q_j^{nc} \\
&= \sum_{\alpha}^n \vec{F}_{\alpha}^c \cdot \frac{\partial \vec{r}_{\alpha}}{\partial q_j} + \sum_{\alpha}^n \vec{F}_{\alpha}^{nc} \cdot \frac{\partial \vec{r}_{\alpha}}{\partial q_j}
\end{aligned}$$

Ahora, es necesario recordar la Ecuación (3.7) para fuerzas conservativas, lo que permite:

$$\begin{aligned}
Q_j &= \sum_{\alpha}^n -\vec{\nabla} V_{\alpha} \cdot \frac{\partial \vec{r}_{\alpha}}{\partial q_j} + \sum_{\alpha}^n \vec{F}_{\alpha}^{nc} \cdot \frac{\partial \vec{r}_{\alpha}}{\partial q_j} \\
&= -\sum_{\alpha}^n \left(\frac{\partial V_{\alpha}}{\partial x_{1,\alpha}} \frac{\partial x_{1,\alpha}}{\partial q_j} + \frac{\partial V_{\alpha}}{\partial x_{2,\alpha}} \frac{\partial x_{2,\alpha}}{\partial q_j} + \frac{\partial V_{\alpha}}{\partial x_{3,\alpha}} \frac{\partial x_{3,\alpha}}{\partial q_j} \right) + Q_j^{nc} \\
&= -\sum_{\alpha}^n \frac{\partial V_{\alpha}}{\partial q_j} + Q_j^{nc} \\
&= -\frac{\partial V}{\partial q_j} + Q_j^{nc}
\end{aligned}$$

El volver a colocar \vec{F}^{nc} como Q_j^{nc} no representa un problema por ahora, debido a que la forma de estas fuerzas puede llegar a depender del problema en que se planteen

²⁰ Esta última ecuación ya posee la forma de una ecuación de Euler

Recuerde que Q_j no posee ninguna fuerza de restricción

y no hay ninguna pérdida al dejarlas de forma general.

Con la expresión anterior, se introduce en la ecuación de Euler:

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} &= Q_j \\ \Rightarrow \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} &= -\frac{\partial V}{\partial q_j} + Q_j^{nc} \\ \Rightarrow \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \left(\frac{\partial T}{\partial q_j} - \frac{\partial V}{\partial q_j} \right) &= Q_j^{nc} \end{aligned}$$

A esta última ecuación no le hará ningún daño aprovechar encarecidamente de la definición que acompaña a la Ecuación (3.7), diciendo lo siguiente ²¹:

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{d}{dt} \left(\frac{\partial V}{\partial \dot{q}_j} \right) - \left(\frac{\partial T}{\partial q_j} - \frac{\partial V}{\partial q_j} \right) &= Q_j^{nc} \\ \Rightarrow \frac{d}{dt} \left(\frac{\partial (T - V)}{\partial \dot{q}_j} \right) - \frac{\partial (T - V)}{\partial q_j} &= Q_j^{nc} \end{aligned}$$

SUBSECTION 8.2

Ecuaciones de Euler-Lagrange

Definition 37

(Lagrangiano) Se define la función \mathcal{L} como el Lagrangiano del sistema, función que para el caso de un sistema conservativo dicta toda la dinámica del sistema al operarlo para cada coordenada generalizada en la ecuación de Euler-Lagrange.

$$\mathcal{L} = T - V \quad (8.1)$$

- T : Es la energía cinética total del sistema, que posea el sistema.
- V : Es la energía potencial total del sistema, la suma de todas las energías potenciales que afecten al sistema.

Definition 38

(Ecuaciones de Euler-Lagrange para sistemas semi-conservativos)

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j} \right) - \frac{\partial \mathcal{L}}{\partial q_j} = Q_j^{nc} \quad ; \quad j = 1, 2, \dots, s \quad (8.2)$$

Al operar esta ecuación sobre el Lagrangiano del sistema y tomando o no las fuerzas disipativas para cada coordenada generalizada, la solución que genera serán s ecuaciones diferenciales de segundo orden que dan las condiciones para que el camino que sigue el sistema en el espacio de configuración sea un valor estacionario para el Principio de Hamilton (Extendido) Ecuación (7.1).

²¹ Introducir el término

$$-\frac{d}{dt} \left(\frac{\partial V}{\partial \dot{q}_j} \right)$$

no debe ser un problema, puesto que en un inicio será cero por la limitación de fuerzas conservativas.

No obstante, fuera de la mecánica clásica será común que este término no sea cero en un intento de escribir potenciales para fuerzas que, al menos en el sentido usual, no los poseen. Por ejemplo la fuerza de Lorentz que da lugar a potenciales generalizados $V = V(q_j, \dot{q}_j)$.

SECTION 9

Ecuaciones de la Mecánica de Lagrange con Restricciones

En esta sección se va a ampliar la mecánica de Lagrange para resolver problemas con condiciones **no holónómicas de cierto tipo**²² a partir del Principio de Hamilton (Extendido). Para esto se va a comenzar de nuevo con la deducción de las ecuaciones de Euler-Lagrange, pero se van a usar resultados ya mostrados de la deducción anterior que aquí también sean válidos.

SUBSECTION 9.1

Deducción del formalismo de Lagrange con Restricciones

Esta deducción no está completa y por el momento debe considerarse como errónea.

Nuevamente considere un sistema compuesto por n partículas y con m restricciones $g_i(q_1, q_2, \dots, q_s, \dot{q}_1, \dot{q}_2, \dots, \dot{q}_s, t) = 0$, dando a lugar $s \geq 3n - m$ grados de libertad, se puede hallar un conjunto de **coordenadas generalizadas adecuado** (q_1, q_2, \dots, q_s) tal que sea **independiente de las restricciones** impuestas al sistema. Además, se expone a cada una de las partículas a una fuerza neta²³ Q_j con $j = 1, 2, \dots, s$.

Entonces, aplicando al Principio de Hamilton Extendido la técnica de Multiplicadores Indeterminados de Lagrange y suponiendo que el multiplicador es de la forma $\lambda(q_j, \dot{q}_j, t)$, se llega a:

$$I = \int_{t_1}^{t_2} \left[T(q_j, \dot{q}_j, t) + W + \sum_{i=1}^m \lambda_i(q_j, \dot{q}_j, t) g_i(q_j, \dot{q}_j, t) \right] dt$$

Buscando un punto de estacionario de la Acción, hay que tomar la variación de esta e igualarla a cero:

$$\begin{aligned} \delta I &= \delta \int_{t_1}^{t_2} \left[T(q_j, \dot{q}_j, t) + W + \sum_{i=1}^m \lambda_i(q_j, \dot{q}_j, t) g_i(q_j, \dot{q}_j, t) \right] dt = 0 \\ &\Rightarrow \int_{t_1}^{t_2} \left[\delta T(q_j, \dot{q}_j, t) + \delta W + \delta \sum_{i=1}^m \lambda_i(q_j, \dot{q}_j, t) g_i(q_j, \dot{q}_j, t) \right] dt = 0 \end{aligned}$$

Desarrollando primero los términos ya conocidos, δT y δW :

$$\begin{aligned} &\Rightarrow \int_{t_1}^{t_2} \left\{ \sum_{j=1}^s \left[\frac{\partial T}{\partial q_j} - \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) + Q_j \right] \delta q_j + \delta \sum_{i=1}^m \lambda_i(q_j, \dot{q}_j, t) g_i(q_j, \dot{q}_j, t) \right\} dt = 0 \\ &\Rightarrow \end{aligned}$$

SUBSECTION 9.2

Ecuaciones de Euler-Lagrange con Restricciones

²²El formalismo será válido para restricciones semi-holónómicas:

$$g(q_1, q_2, \dots, q_s, \dot{q}_1, \dot{q}_2, \dots, \dot{q}_s, t) = 0$$

²³Recuerde que esta fuerza no posee ningún tipo de fuerza de restricción en su interior, es una agrupación de fuerzas conservativas y no conservativas.

Definition 39**(Ecuaciones de Euler-Lagrange con Restricciones)**

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j} \right) - \frac{\partial \mathcal{L}}{\partial q_j} + \sum_{i=1}^m \lambda_i \frac{\partial g_i}{\partial q_j} = Q_j^{nc} \quad (9.1)$$

SUBSECTION 9.3

Problemas resueltos

Cálculo Variacional

Problema 1.

Determine las geodésicas de un cono recto de base circular.

$$z = \lambda \rho \Rightarrow dz = \lambda d\rho$$

$$ds^2 = \rho^2 + \rho^2 d\phi^2 + dz^2 \Rightarrow ds^2 = (1 + \lambda^2) \rho^2 + \rho^2 d\phi^2 \Rightarrow ds = \sqrt{\rho^2 + (1 + \lambda^2) \rho^2} d\phi$$

Primer forma

$$\Rightarrow J = \int_0^{10} ds = \int_0^{10} \sqrt{\rho^2 + (1 + \lambda^2) \rho^2} d\phi \Rightarrow f = \sqrt{\rho^2 + (1 + \lambda^2) \rho^2} \xrightarrow{\text{No hay dependencia explícita con } \phi} \Rightarrow \frac{df}{dx} = \frac{df}{d\rho} + \frac{df}{dy} \frac{dy}{dx} + \frac{df}{dz} \frac{dz}{dx}$$

$$\text{con } \frac{d}{dx} \frac{df}{dx} = \frac{df}{d\rho} \Rightarrow \frac{df}{dx} = \frac{df}{d\rho} + \frac{d}{dx} \frac{df}{dx} y_x + \frac{df}{dy} y_{xx} = \frac{df}{d\rho} + \frac{d}{dx} \left(\frac{df}{dx} y_x \right) \Rightarrow 0 = -\frac{df}{d\rho} + \frac{df}{dx} - \frac{d}{dx} \left(\frac{df}{dx} y_x \right)$$

$$\Rightarrow 0 = -\cancel{\frac{df}{d\rho}} + \frac{d}{dx} \left(f - \frac{df}{dx} y_x \right) \Rightarrow f - \frac{df}{dx} y_x = \text{constante} \Rightarrow f - \frac{df}{d\rho} \rho_\phi = c_1 \Rightarrow \frac{df}{d\rho} = \frac{(1 + \lambda^2) \rho_\phi}{\sqrt{\rho^2 + (1 + \lambda^2) \rho^2}}$$

$$\Rightarrow \sqrt{\rho^2 + (1 + \lambda^2) \rho^2} - \frac{(1 + \lambda^2) \rho_\phi^2}{\sqrt{\rho^2 + (1 + \lambda^2) \rho^2}} = c_1 \Rightarrow \frac{\rho^2 + (1 + \lambda^2) \rho^2 - (1 + \lambda^2) \rho_\phi^2}{\sqrt{\rho^2 + (1 + \lambda^2) \rho^2}} = c_1 \Rightarrow \frac{\rho^2}{\sqrt{\rho^2 + (1 + \lambda^2) \rho^2}} = c_1$$

$$\Rightarrow \rho^4 = c_1^2 \rho^2 + c_1^2 (1 + \lambda^2) \rho_\phi^2 \Rightarrow \rho^2 (\rho^2 - c_1^2) = c_1^2 (1 + \lambda^2) \rho_\phi^2 \Rightarrow \frac{d\rho}{d\phi} = \sqrt{\frac{\rho^2 (\rho^2 - c_1^2)}{c_1^2 (1 + \lambda^2)}}$$

$$\Rightarrow \int d\phi = \int \frac{c_1 \sqrt{(1 + \lambda^2)}}{\sqrt{\rho^2 (\rho^2 - c_1^2)}} d\rho \Rightarrow \phi - \phi_0 = c_1 \sqrt{(1 + \lambda^2)} \cos^{-1}\left(\frac{c_1}{\rho}\right)$$

Segunda forma $ds = \sqrt{\rho^2 \dot{\phi}_p^2 + (1 + \lambda^2)} d\rho \Rightarrow f = \sqrt{\rho^2 \dot{\phi}_p^2 + (1 + \lambda^2)} \Rightarrow \frac{d}{d\rho} \frac{df}{d\phi_p} - \cancel{\frac{df}{d\rho}} = 0$

$$\Rightarrow \frac{df}{d\phi_p} = \frac{\rho^2 \dot{\phi}_p}{\sqrt{\rho^2 \dot{\phi}_p^2 + (1 + \lambda^2)}} \Rightarrow \frac{d}{d\rho} \frac{df}{d\phi_p} = 0 \Rightarrow \frac{df}{d\phi_p} = \frac{\rho^2 \dot{\phi}_p}{\sqrt{\rho^2 \dot{\phi}_p^2 + (1 + \lambda^2)}} = c_1 \Rightarrow \rho^4 \dot{\phi}_p^2 = c_1^2 \rho^2 \dot{\phi}_p^2 + c_1^2 (1 + \lambda^2)$$

$$\Rightarrow \rho^2 \dot{\phi}_p^2 (\rho^2 - c_1^2) = c_1^2 (1 + \lambda^2) \Rightarrow \dot{\phi}_p = \sqrt{\frac{c_1^2 (1 + \lambda^2)}{\rho^2 (\rho^2 - c_1^2)}} = \frac{d\phi}{d\rho} \Rightarrow \int d\phi = \int \frac{c_1 \sqrt{(1 + \lambda^2)}}{\sqrt{\rho^2 (\rho^2 - c_1^2)}} d\rho$$

$$\Rightarrow \phi - \phi_0 = c_1 \sqrt{(1 + \lambda^2)} \cos^{-1}\left(\frac{c_1}{\rho}\right)$$

Tercera forma $f = \sqrt{\rho^2 + (1 + \lambda^2) \rho^2} \Rightarrow \frac{d}{d\rho} \frac{df}{d\phi_p} - \frac{df}{d\rho} = 0 \Rightarrow \frac{df}{d\rho} = \frac{\rho}{\sqrt{\rho^2 + (1 + \lambda^2) \rho^2}}$

$$\Rightarrow \frac{df}{d\phi_p} = \frac{(1 + \lambda^2) \rho_\phi}{\sqrt{\rho^2 + (1 + \lambda^2) \rho^2}} \Rightarrow \frac{d}{d\phi} \frac{df}{d\phi_p} = \text{regla de la cadena}$$

Problema 2. (Thornton 6.7/Taylor 6.4)

- 6-7. Consider light passing from one medium with index of refraction n_1 into another medium with index of refraction n_2 (Figure 6-A). Use Fermat's principle to minimize time, and derive the law of refraction: $n_1 \sin \theta_1 = n_2 \sin \theta_2$.

Principio de Fermat: La luz recorre el camino que le permite minimizar el tiempo

$$J = \int_{t_1}^{t_2} dt \rightarrow dt = \frac{ds}{v_m}; ds^2 = dx^2 + dy^2 \Rightarrow ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + X_y^2} dy$$

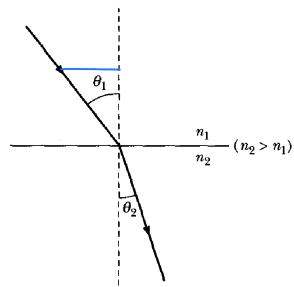


FIGURE 6-A Problem 6-7.

$$\Rightarrow J = \int_{t_1}^{t_2} dt = \int_{x_1}^{x_2} \frac{\sqrt{1 + X_y^2}}{v_m} dy \Rightarrow f = \frac{\sqrt{1 + X_y^2}}{v_m} \Rightarrow \frac{d}{dy} \frac{df}{dx} - \cancel{\frac{df}{dx}} = 0$$

$$\Rightarrow \frac{df}{dx} = \text{constante} = \frac{X_y}{\sqrt{1 + X_y^2}} = \frac{X_y}{V_m \sqrt{1 + X_y^2}} \Rightarrow \frac{X_y}{V_m \sqrt{1 + X_y^2}} = 1; X_y = \tan \theta = \frac{dx}{dy}$$

$$\Rightarrow \frac{\sin \theta}{V_m} = 1 = \frac{n \sin \theta}{c} \Rightarrow n \sin \theta = \text{constante} //$$

Problema 3. (Thornton 6.8.a)

- 6-8. Find the dimensions of the parallelepiped of maximum volume circumscribed by

- (a) a sphere of radius R ; (b) an ellipsoid with semiaxes a, b, c .

El paralelepípedo tiene volumen: $V = abc$

$$a) J = \int f(X, Y, Z) dz \Rightarrow \frac{d}{dz} \frac{\partial f}{\partial q_0} - \frac{\partial f}{\partial q_0} + \lambda \frac{\partial g}{\partial q_0} = 0 \quad \text{Restricción: } g = a^2 + b^2 + c^2 - 4R^2 = 0$$

$$\Rightarrow - \frac{\partial V}{\partial q_0} + \lambda \frac{\partial g}{\partial q_0} = 0 \quad \Rightarrow^* 3a^2 = 4R^2 \Rightarrow a = \frac{2R}{\sqrt{3}}$$

$$\left. \begin{aligned} \text{Para } a: - \frac{\partial V}{\partial a} + \lambda \frac{\partial g}{\partial a} = 0 \Rightarrow -bc + \lambda \cdot 2a = 0 \\ \text{Para } b: - \frac{\partial V}{\partial b} + \lambda \frac{\partial g}{\partial b} = 0 \Rightarrow -ac + \lambda \cdot 2b = 0 \\ \text{Para } c: - \frac{\partial V}{\partial c} + \lambda \frac{\partial g}{\partial c} = 0 \Rightarrow -ab + \lambda \cdot 2c = 0 \end{aligned} \right\} \Rightarrow a = \frac{bc}{2\lambda} * \Rightarrow \frac{bc}{2\lambda} = \frac{2\lambda b}{c} \Rightarrow c = 2\lambda *$$

$$\left. \begin{aligned} \text{Para } a: - \frac{\partial V}{\partial a} + \lambda \frac{\partial g}{\partial a} = 0 \Rightarrow -bc + \lambda \cdot 2a = 0 \\ \text{Para } b: - \frac{\partial V}{\partial b} + \lambda \frac{\partial g}{\partial b} = 0 \Rightarrow -ac + \lambda \cdot 2b = 0 \\ \text{Para } c: - \frac{\partial V}{\partial c} + \lambda \frac{\partial g}{\partial c} = 0 \Rightarrow -ab + \lambda \cdot 2c = 0 \end{aligned} \right\} \Rightarrow a = \frac{2\lambda b}{c} ** \Rightarrow \frac{2\lambda b}{c} = \frac{2\lambda c}{b} \Rightarrow b^2 = c^2 *$$

$$\left. \begin{aligned} \text{Para } a: - \frac{\partial V}{\partial a} + \lambda \frac{\partial g}{\partial a} = 0 \Rightarrow -bc + \lambda \cdot 2a = 0 \\ \text{Para } b: - \frac{\partial V}{\partial b} + \lambda \frac{\partial g}{\partial b} = 0 \Rightarrow -ac + \lambda \cdot 2b = 0 \\ \text{Para } c: - \frac{\partial V}{\partial c} + \lambda \frac{\partial g}{\partial c} = 0 \Rightarrow -ab + \lambda \cdot 2c = 0 \end{aligned} \right\} \Rightarrow a = \frac{2\lambda c}{b} ** \Rightarrow a = 2\lambda$$

$$\Rightarrow a = b = c = 2\lambda^* = \frac{2R}{\sqrt{3}} \quad y \quad \lambda = \frac{R}{\sqrt{3}} //$$

Problema 4.

Considera el funcional:

$$J[y(x), z(x)] = \int_0^\pi (2yz - 2y^2 + y_x^2 - z_x^2) dx.$$

Si $y(0) = 0$, $y(\pi) = 1$, $z(0) = 0$ y $z(\pi) = -1$. Determina los extremos de J .

$$J = \int_0^\pi 2yz - 2y^2 + y_x^2 - z_x^2 dx \Rightarrow f = f(y, z, y_x, z_x) = 2yz - 2y^2 + y_x^2 - z_x^2 \Rightarrow \frac{d}{dx} \frac{\partial f}{\partial y_x} - \frac{d}{dx} \frac{\partial f}{\partial z_x} = 0$$

$$\text{Para } y: \frac{d}{dx} \frac{\partial f}{\partial y_x} - \frac{d}{dx} \frac{\partial f}{\partial z_x} = 0 \quad * \quad \frac{\partial f}{\partial y_x} = 2z - 4y \quad * \quad \frac{\partial f}{\partial z_x} = 2y_x \Rightarrow \frac{d}{dx} \frac{\partial f}{\partial y_x} = 2y_{xx}$$

$$\Rightarrow 2y_{xx} - 2z + 4y = 0 \Rightarrow y_{xx} - z + 2y = 0 *$$

$$\text{Para } z: \frac{d}{dx} \frac{\partial f}{\partial z_x} - \frac{d}{dx} \frac{\partial f}{\partial z} = 0 \quad * \quad \frac{\partial f}{\partial z_x} = -2z_x \quad * \quad \frac{\partial f}{\partial z} = 2y \Rightarrow \frac{d}{dx} \frac{\partial f}{\partial z_x} = -2z_{xx}$$

$$\Rightarrow -2z_{xx} - 2y = 0 \Rightarrow z_{xx} + y = 0 *$$

$$\begin{cases} y_{xx} - z + 2y = 0 \\ z_{xx} + y = 0 \end{cases}$$

Problema 5.

Determine los extremos del funcional:

$$Z = X^3 - 2XY^2$$

$$J[z(x, y)] = \iint \sqrt{1 + z_x^2 + z_y^2} dx dy,$$

donde:

$$z(x, y) = x^3 - 2xy^2.$$

$$J = \iint \sqrt{1 + z_x^2 + z_y^2} dx dy \Rightarrow f(z, z_x, z_y, x, y) = f(z_x, z_y) = \sqrt{1 + z_x^2 + z_y^2}$$

Problema 6. (Thornton 6.10)

- 6-10. Find the ratio of the radius R to the height H of a right-circular cylinder of fixed volume V that minimizes the surface area A .

$$V = \pi R^2 H \rightarrow A = 2\pi RH + 2\pi R^2 = 2\pi R(H+R)$$

$$\text{Restricción: } g = V - \pi R^2 H = 0$$

$$\Rightarrow \frac{\partial A}{\partial R} - \frac{\partial A}{\partial H} + \lambda \frac{\partial g}{\partial H} = 0 \Rightarrow -2\pi R - 2\pi R^2 = 0 \Rightarrow \lambda = -\frac{2}{R}$$

$$\Rightarrow \frac{\partial A}{\partial H} - \frac{\partial A}{\partial R} + \lambda \frac{\partial g}{\partial R} = 0 \Rightarrow -2\pi(H+R) - 2\pi R - \lambda \cdot 2\pi R H = 0 \Rightarrow H+R+R+\lambda RH = 0 \\ \Rightarrow H+2R+\lambda RH = 0 \Rightarrow H+2R-2H = 0 \Rightarrow -H+2R = 0 \Rightarrow R = \frac{H}{2}$$

Problema 7. (Thornton 6.11)

- 6-11. A disk of radius R rolls without slipping inside the parabola $y = ax^2$. Find the equation of constraint. Express the condition that allows the disk to roll so that it contacts the parabola at one and only one point, independent of its position.

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + y'^2} dx = \sqrt{1 + 4a^2 x^2} dx \Rightarrow \sqrt{1 + 4a^2 x^2} dx = R d\theta$$

$$\text{Radio de curvatura: } r = \frac{(1 + y'^2)^{3/2}}{y''} \Rightarrow r = \frac{(1 + 4a^2 x^2)^{3/2}}{2a}$$

Para que el disco toque la parábola en un único punto, $r > R$, para que el disco no toque ambos lados de la parábola

$$\text{Radio mínimo de la parábola} \Rightarrow r = \frac{1}{2a} \Rightarrow R < \frac{1}{2a}$$

↑
En el vértice

Problema 8.

Para $dy/dx \equiv y_x \neq 0$, muestre la equivalencia de las dos formas de la ecuación de Euler:

$$\frac{\partial f}{\partial x} - \frac{d}{dx} \frac{\partial f}{\partial y_x} = 0$$

y

$$\frac{\partial f}{\partial x} - \frac{d}{dx} \left(f - y_x \frac{\partial f}{\partial y_x} \right) = 0.$$

$$f = f(y, y_x, x)$$

$$\eta(x_1) = \eta(x_2) = 0$$

$$J = \int_{x_1}^{x_2} f(y, y_x, x) dx ; \quad y(x_1, \alpha) = y(x_1) + \alpha \eta(x)$$

$$\Rightarrow \delta J = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} \frac{\partial y}{\partial x} + \frac{\partial f}{\partial y_x} \frac{\partial y_x}{\partial x} \right] dx = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} \eta + \frac{\partial f}{\partial y_x} \eta_x \right] dx ; \quad \int_{x_1}^{x_2} \frac{\partial f}{\partial y_x} \eta_x dx = \eta \frac{\partial f}{\partial y_x} \Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta \frac{d}{dx} \frac{\partial f}{\partial y_x} dx$$

$$\Rightarrow \delta J = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} \eta - \eta \frac{d}{dx} \frac{\partial f}{\partial y_x} \right] dx = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y_x} \right] \eta dx$$

Para $f = f(y, y_x, x)$, se define

$$\Rightarrow \frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx} + \frac{\partial f}{\partial y_x} \frac{dy_x}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y_x + \frac{\partial f}{\partial y_x} y_{xx} ; \quad \frac{\partial f}{\partial y} = \frac{d}{dx} \frac{\partial f}{\partial y_x}$$

$$\Rightarrow \frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{d}{dx} \frac{\partial f}{\partial y_x} y_x + \frac{\partial f}{\partial y_x} y_{xx} = \frac{\partial f}{\partial x} + \frac{d}{dx} \left(\frac{\partial f}{\partial y_x} y_x \right) \Rightarrow \frac{d}{dx} \left(f - \frac{\partial f}{\partial y_x} y_x \right) - \frac{\partial f}{\partial y_x} = 0$$

Problema 9. (Taylor 6.18)

6.18 ** Show that the shortest path between two given points in a plane is a straight line, using plane polar coordinates.

$$ds^2 = dr^2 + r^2 d\phi^2 \Rightarrow ds = \sqrt{1 + r^2 \phi_r^2} dr \Rightarrow J = \int_A^B ds = \int_{r_A}^{r_B} \sqrt{1 + r^2 \phi_r^2} dr \Rightarrow f = \sqrt{1 + r^2 \phi_r^2}$$

$$\Rightarrow \frac{d}{dr} \frac{df}{d\phi_r} - \cancel{\frac{d^2f}{d\phi^2}} = 0 \Rightarrow \frac{d}{dr} \frac{df}{d\phi_r} = 0 \Rightarrow \frac{df}{d\phi_r} = \text{constante} = c_1 \Rightarrow \frac{r^2 \phi_r}{\sqrt{1 + r^2 \phi_r^2}} = c_1$$

$$\Rightarrow r^4 \phi_r^2 = c_1^2 + c_1^2 r^2 \phi_r^2 \Rightarrow r^2 \phi_r^2 (r^2 - c_1^2) = c_1^2 \Rightarrow \phi_r = \frac{d\phi}{dr} = \frac{c_1}{r \sqrt{r^2 - c_1^2}} \Rightarrow \int d\phi = \int \frac{c_1}{r \sqrt{r^2 - c_1^2}} dr$$

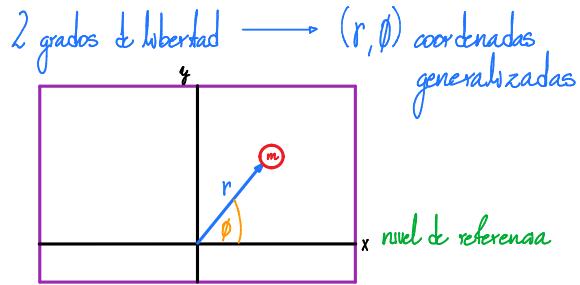
$$\Rightarrow \phi - \phi_0 = c_1 \cos^{-1}\left(\frac{r}{r_0}\right) \Rightarrow r \cos\left(\frac{\phi - \phi_0}{c_1}\right) = r_0$$

Mecánica Lagrangiana

Problema 1. (Thornton 7.5)

7-5. Consider a vertical plane in a constant gravitational field. Let the origin of a coordinate system be located at some point in this plane. A particle of mass m moves in the vertical plane under the influence of gravity and under the influence of an additional force $f = -Ar^{\alpha-1}$ directed toward the origin (r is the distance from the origin; A and α [$\neq 0$ or 1] are constants). Choose appropriate generalized coordinates, and find the Lagrangian equations of motion. Is the angular momentum about the origin conserved? Explain.

$$\Rightarrow T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) \quad y \quad U = mg r \sin\theta + U_f$$



$$\text{Considerando } f \text{ como conservativa} \Rightarrow U_f = -\int_0^r -Ar^{\alpha-1} dr = \frac{1}{\alpha} r^\alpha$$

$$\Rightarrow L = T - U = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{1}{\alpha}r^\alpha - mg r \sin\theta \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_1} - \frac{\partial L}{\partial q_1} = 0$$

$$\text{Para } r : \frac{d}{dt} \frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0$$

$$\left. \begin{aligned} * \frac{\partial L}{\partial r} &= \frac{1}{2}m2r\dot{\theta}^2 - \frac{1}{\alpha}\alpha r^{\alpha-1} - mg \sin\theta \\ * \frac{\partial L}{\partial \dot{r}} &= \frac{1}{2}m2\dot{r} = m\ddot{r} \end{aligned} \right\} \Rightarrow m\ddot{r} - mr\dot{\theta}^2 + Ar^{\alpha-1} + mg \sin\theta = 0$$

$$\text{Para } \theta : \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

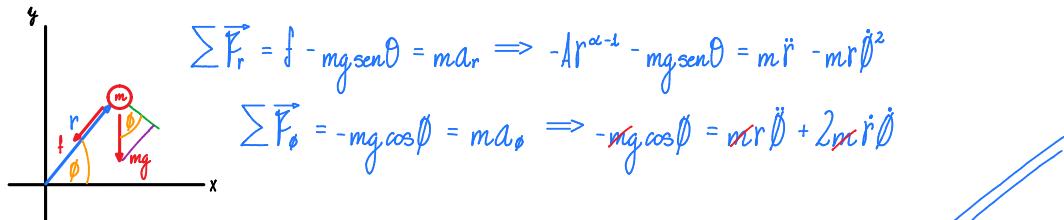
$$\left. \begin{aligned} * \frac{\partial L}{\partial \theta} &= -mg r \cos\theta \\ \frac{\partial L}{\partial \dot{\theta}} &= \frac{1}{2}m r^2 2\dot{\theta} = mr^2\dot{\theta} \end{aligned} \right\} \Rightarrow mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} + mg r \cos\theta = 0$$

El torque neto no es cero, hay torque por la gravedad

Momento angular

Es igual al torque neto \therefore El momento angular no se conserva //

Por Newton



Problema 2. (Taylor 7.17)

7.17* Use the Lagrangian method to find the acceleration of the Atwood machine of Example 7.3 (page 255) including the effect of the pulley's having moment of inertia I . (The kinetic energy of the pulley is $\frac{1}{2}I\omega^2$, where ω is its angular velocity.)

Hay un grado de libertad $\rightarrow (y)$ coordenada generalizada ;

$$\Rightarrow T = \frac{1}{2}(m_1 \dot{y}_1^2 + m_2 \dot{y}_2^2 + I_p \dot{\theta}^2); \quad \dot{\theta} = R \dot{\phi} \quad l = -y_1 - y_2 + \pi R \Rightarrow \ddot{y}_1 = -\ddot{y}_2$$

$$\Rightarrow T = \frac{1}{2} \left(m_1 \frac{\dot{y}_1^2}{R^2} + m_2 \frac{\dot{y}_2^2}{R^2} + I_p \frac{\dot{\theta}^2}{R^2} \right)$$

$$\Rightarrow U = m_1 g y_1 + m_2 g y_2 \Rightarrow U = (m_1 - m_2) g y_1 - m_2 g l + m_2 g \pi R$$

$$\Rightarrow L = \frac{1}{2} \left(m_1 \frac{\dot{y}_1^2}{R^2} + m_2 \frac{\dot{y}_2^2}{R^2} + I_p \frac{\dot{\theta}^2}{R^2} \right) - (m_1 - m_2) g y_1 + m_2 g l - m_2 g \pi R$$

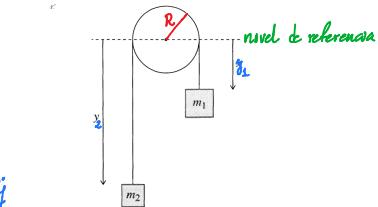


Figure 7.6 An Atwood machine consisting of two masses, m_1 and m_2 , suspended by a massless inextensible string that passes over a massless, frictionless pulley of radius R . Because the string's length is fixed, the position of the whole system can be specified by a single variable, which we can take to be the distance x .

$$*\frac{\partial L}{\partial \dot{y}_1} = -(m_1 - m_2)g$$

$$*\frac{\partial L}{\partial \dot{y}_2} = \left(m_1 \frac{\dot{y}_1^2}{R^2} + m_2 \frac{\dot{y}_2^2}{R^2} + I_p \frac{\dot{\theta}^2}{R^2} \right) \Rightarrow \frac{d}{dt} \frac{\partial L}{\partial \dot{y}_2} = \left(m_1 + m_2 + \frac{I_p}{R^2} \right) \ddot{y}_2$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{y}_1} - \frac{\partial L}{\partial y_1} = 0$$

$$\Rightarrow \left(m_1 + m_2 + \frac{I_p}{R^2} \right) \ddot{y}_1 + (m_1 - m_2)g = 0$$

Problema 3. (Taylor 7.20)

7.20 * A smooth wire is bent into the shape of a helix, with cylindrical polar coordinates $\rho = R$ and $z = \lambda\phi$, where R and λ are constants and the z axis is vertically up (and gravity vertically down). Using z as your generalized coordinate, write down the Lagrangian for a bead of mass m threaded on the wire. Find the Lagrange equation and hence the bead's vertical acceleration \ddot{z} . In the limit that $R \rightarrow 0$, what is \ddot{z} ? Does this make sense?

$$z = \lambda\phi \implies \dot{z} = \lambda\dot{\phi}$$

$$T = \frac{1}{2}m(R^2\dot{\phi}^2 + \dot{z}^2) \quad \text{y} \quad U = mgz$$

$$\Rightarrow L = \frac{1}{2}m(R^2\dot{\phi}^2 + \dot{z}^2) + mgz = \frac{1}{2}m\left(\frac{R^2}{\lambda^2}\dot{z}^2 + \dot{z}^2\right) + mgz = \frac{m\dot{z}^2(R^2 + \lambda^2)}{2\lambda^2} + mgz \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{z}} - \frac{\partial L}{\partial z} = 0$$

$$* \frac{\partial L}{\partial z} = mg$$

$$* \frac{\partial L}{\partial \dot{z}} = \cancel{2} \frac{m\dot{z}(R^2 + \lambda^2)}{2\lambda^2} = \frac{m\dot{z}(R^2 + \lambda^2)}{\lambda^2} \quad \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{z}} = \frac{m\ddot{z}(R^2 + \lambda^2)}{\lambda^2} \quad \Rightarrow \cancel{m\ddot{z}(R^2 + \lambda^2)} - \cancel{mg} = 0$$

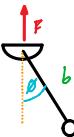
$$\Rightarrow \ddot{z} = \frac{g\lambda^2}{R^2 + \lambda^2} //$$

Problema 4. (Thornton 7.14)

- 7-14. A simple pendulum of length b and bob with mass m is attached to a massless support moving vertically upward with constant acceleration a . Determine (a) the equations of motion and (b) the period for small oscillations.

a) 1 grado de libertad $\rightarrow (\theta)$ coordenadas generalizada

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \quad y \quad U = mg\dot{y}$$



$$x = b \sin \theta$$

$$y = \frac{at^2}{2} - b \cos \theta$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

$$\Rightarrow T = \frac{1}{2}m[b^2 \cos^2 \theta \cdot \dot{\theta}^2 + (at - b \sin \theta \cdot \dot{\theta})^2] = \frac{1}{2}m(b^2 \cos^2 \theta \cdot \dot{\theta}^2 + a^2 t^2 - 2ab \sin \theta \cdot \dot{\theta} t + b^2 \sin^2 \theta \cdot \dot{\theta}^2)$$

$$= \frac{1}{2}m(b^2 \dot{\theta}^2 + a^2 t^2 - 2ab \sin \theta \cdot \dot{\theta} t)$$

$$\Rightarrow U = mg\left(\frac{at^2}{2} - b \cos \theta\right) \Rightarrow L = \frac{1}{2}m(b^2 \dot{\theta}^2 + a^2 t^2 - 2ab \sin \theta \cdot \dot{\theta} t) - mg\left(\frac{at^2}{2} - b \cos \theta\right)$$

$$* \frac{\partial L}{\partial \theta} = \cancel{\frac{1}{2}m \cdot 2ab \cos \theta \cdot \dot{\theta} t} - mg b \sin \theta$$

$$* \frac{\partial L}{\partial \dot{\theta}} = \cancel{\frac{1}{2}m(b^2 \dot{\theta} - 2ab \sin \theta t)} \Rightarrow \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = mb^2 \ddot{\theta} - ab \cos \theta \cdot \dot{\theta} t - ab \sin \theta$$

$$\Rightarrow mb^2 \ddot{\theta} - ab \cos \theta \cdot \dot{\theta} t - ab \sin \theta + mb \cos \theta \cdot \dot{\theta} t + mg \sin \theta = 0$$

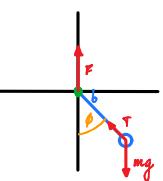
$$\Rightarrow b \ddot{\theta} + a \sin \theta + g \sin \theta = 0 \Rightarrow \ddot{\theta} + \frac{a+g \sin \theta}{b} = 0$$

$$b) \text{Oscilaciones pequeñas } \sin \theta \approx \theta \Rightarrow \ddot{\theta} + \frac{a+g}{b} \theta = 0 \Rightarrow \omega^2 = \frac{a+g}{b}$$

Por Newton

$$\sum \vec{F} = -b m(a+g) \sin \theta \Rightarrow mb^2 \ddot{\theta} = -b m(a+g) \sin \theta \Rightarrow \ddot{\theta} + \frac{a+g}{b} \sin \theta = 0 //$$

$$\sum F_y = T - mg = ma \Rightarrow T = m(a+g) \leftarrow \text{Peso aparente}$$



Problema 5. (Taylor 7.23)

7.23* A small cart (mass m) is mounted on rails inside a large cart. The two are attached by a spring (force constant k) in such a way that the small cart is in equilibrium at the midpoint of the large. The distance of the small cart from its equilibrium is denoted x and that of the large one from a fixed point on the ground is X , as shown in Figure 7.13. The large cart is now forced to oscillate such that $X = A \cos \omega t$, with both A and ω fixed. Set up the Lagrangian for the motion of the small cart and show that the Lagrange equation has the form

$$\ddot{x} + \omega_0^2 x = B \cos \omega t$$

where ω_0 is the natural frequency $\omega_0 = \sqrt{k/m}$ and B is a constant. This is the form assumed in Section 5.5, Equation (5.57), for driven oscillations (except that we are here ignoring damping). Thus the system described here would be one way to realize the motion discussed there. (We could fill the large cart with molasses to provide some damping.)

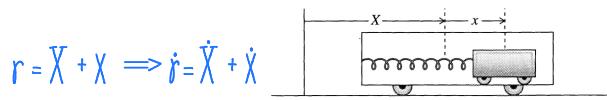


Figure 7.13 Problem 7.23

$$T = \frac{1}{2} m \dot{r}^2 = \frac{1}{2} m (\dot{x} + \dot{X})^2 \quad \text{y} \quad U = \frac{1}{2} K X^2 \Rightarrow L = \frac{1}{2} m (\dot{x} + \dot{X})^2 - \frac{1}{2} K X^2 \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0$$

$$* \frac{dL}{dx} = -\frac{1}{2} K X = -K X$$

$$* \frac{dL}{dt} = \frac{1}{2} m 2(\dot{x} + \dot{X}) = m(\dot{x} + \dot{X}) \Rightarrow \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = m(\ddot{x} + \ddot{X}) \Rightarrow m(\ddot{x} + \ddot{X}) + K X = 0 \Rightarrow \ddot{x} + \ddot{X} + \omega_0^2 X = 0 //$$

Problema 6. (Taylor 7.29)

7.29** Figure 7.14 shows a simple pendulum (mass m , length l) whose point of support P is attached to the edge of a wheel (center O , radius R) that is forced to rotate at a fixed angular velocity ω . At $t = 0$, the point P is level with O on the right. Write down the Lagrangian and find the equation of motion for the angle ϕ . [Hint: Be careful writing down the kinetic energy T . A safe way to get the velocity right is to write down the position of the bob at time t , and then differentiate.] Check that your answer makes sense in the special case that $\omega = 0$.

$$\vec{r} = (R \cos \theta + l \cos \phi) \hat{e}_x + (R \sin \theta - l \sin \phi) \hat{e}_y$$

$$\Rightarrow \dot{\vec{r}} = (-R \sin \theta \cdot \dot{\theta} + l \cos \phi \cdot \dot{\phi}) \hat{e}_x + (R \cos \theta \cdot \dot{\theta} + l \sin \phi \cdot \dot{\phi}) \hat{e}_y$$

$$\Rightarrow \dot{r}^2 = (-R \sin \theta \cdot \dot{\theta} + l \cos \phi \cdot \dot{\phi})^2 + (R \cos \theta \cdot \dot{\theta} + l \sin \phi \cdot \dot{\phi})^2 \quad \text{y} \quad U = mg \hat{y} = mg(R \sin \theta - l \cos \phi)$$

$$= R^2 \sin^2 \theta \cdot \dot{\theta}^2 - 2Rl \dot{\theta} \dot{\phi} \sin \theta \cos \phi + l^2 \cos^2 \phi \cdot \dot{\phi}^2 + R^2 \cos^2 \theta \cdot \dot{\theta}^2 + 2Rl \dot{\theta} \dot{\phi} \cos \theta \sin \phi + l^2 \sin^2 \phi \cdot \dot{\phi}^2$$

$$= R^2 \dot{\theta}^2 + l^2 \dot{\phi}^2$$

$$\Rightarrow L = \frac{1}{2} m (R^2 \dot{\theta}^2 + l^2 \dot{\phi}^2) - mg(R \sin \theta - l \cos \phi) \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

$$\text{Para } \theta: * \frac{dL}{d\theta} = -mgR \cos \theta \quad * \frac{dL}{d\dot{\theta}} = \cancel{\frac{1}{2} m 2R^2 \dot{\theta}} = mR^2 \dot{\theta} \Rightarrow \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = mR^2 \ddot{\theta} \Rightarrow mR^2 \ddot{\theta} + mgR \cos \theta = 0$$

$$\text{Para } \phi: * \frac{dL}{d\phi} = -mgl \sin \theta \quad * \frac{dL}{d\dot{\phi}} = \cancel{\frac{1}{2} m 2l^2 \dot{\phi}} = ml^2 \dot{\phi} \Rightarrow \frac{d}{dt} \frac{\partial L}{\partial \dot{\phi}} = ml^2 \ddot{\phi} \Rightarrow ml^2 \ddot{\phi} + mgl \sin \theta = 0 //$$

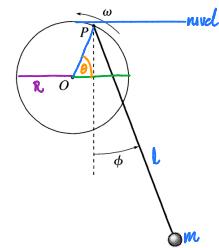


Figure 7.14 Problem 7.29

Mecánica Lagrangiana con Multiplicadores de Lagrange

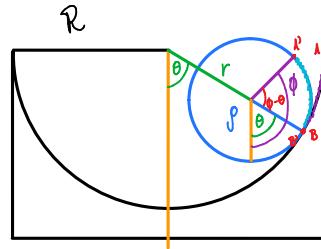
Problema 1. (Thornton 7.3)

7-3. A sphere of radius ρ is constrained to roll without slipping on the lower half of the inner surface of a hollow cylinder of inside radius R . Determine the Lagrangian function, the equation of constraint, and Lagrange's equations of motion. Find the frequency of small oscillations.

$$\vec{r} = (R - \rho) \hat{e}_r \Rightarrow \dot{\vec{r}} = (\dot{R} - \dot{\rho}) \hat{e}_r + (R - \rho) \dot{\theta} \hat{e}_{\theta}$$

$$\Rightarrow \dot{r}^2 = (\dot{R} - \dot{\rho})^2 + (R - \rho)^2 \dot{\theta}^2$$

$$I = \frac{2}{5} m \rho^2$$



$$\text{Restricciones: } g_1: R\dot{\theta} - \dot{\rho}(\phi - \theta) = 0 \quad y \quad g_2: r - (R - \rho) = 0$$

$$\widehat{AB} = \widehat{A'B'} \Rightarrow R\dot{\theta} = \dot{\rho}(\phi - \theta)$$

$$\text{Energía cinética: } T = \frac{1}{2} m [\dot{r}^2 + r^2 \dot{\theta}^2] + \frac{1}{2} I \dot{\phi}^2 \quad \text{Energía potencial: } U = mg\dot{\theta} = mg(R - r \cos \theta)$$

$$\Rightarrow L = \frac{1}{2} m [\dot{r}^2 + r^2 \dot{\theta}^2] + \frac{1}{2} m \rho^2 \dot{\phi}^2 - mg(R - r \cos \theta) \Rightarrow \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} + \lambda_1 \frac{\partial g_1}{\partial q_i} + \lambda_2 \frac{\partial g_2}{\partial q_i} = 0 \text{ con } q_i = r, \theta, \phi$$

$$\text{Para } \phi: \frac{d}{dt} \frac{\partial L}{\partial \dot{\phi}} - \cancel{\frac{\partial L}{\partial \phi}} + \lambda_1 \cancel{\frac{\partial g_1}{\partial \phi}} + \lambda_2 \cancel{\frac{\partial g_2}{\partial \phi}} = 0$$

$$* \cancel{\frac{\partial L}{\partial \phi}} = \cancel{\frac{1}{2}} \cancel{\frac{1}{5}} m \rho^2 \dot{\phi} = \frac{1}{5} m \rho^2 \dot{\phi} \Rightarrow \frac{1}{5} m \frac{d}{dt} (\rho^2 \dot{\phi}) \quad * \cancel{\frac{\partial g_1}{\partial \phi}} = -\rho \xrightarrow{\text{aceleración rotacional}}$$

$$\Rightarrow \frac{1}{5} m \frac{d}{dt} (\rho^2 \dot{\phi}) - \lambda_2 \rho = 0 \xrightarrow{\substack{\text{Aplicando} \\ \text{restricciones}}} \frac{1}{5} m \rho \cancel{\frac{d}{dt} \dot{\phi}} - \lambda_2 \rho = 0 \Rightarrow \lambda_2 = \frac{1}{5} m \rho \ddot{\phi} = \cancel{-m \frac{\ddot{\phi}(R + \rho)}{5\rho}} = -\frac{m \ddot{\phi}(R + \rho)}{5\rho}$$

$$\text{Para } r: \frac{d}{dt} \frac{\partial L}{\partial \dot{r}} - \cancel{\frac{\partial L}{\partial r}} + \lambda_1 \cancel{\frac{\partial g_1}{\partial r}} + \lambda_2 \cancel{\frac{\partial g_2}{\partial r}} = 0$$

$$* \cancel{\frac{\partial L}{\partial r}} = \cancel{\frac{1}{2}} m \cdot 2r \dot{\theta}^2 + mg \cos \theta \quad * \frac{\partial g_2}{\partial r} = 1 \quad * \cancel{\frac{\partial L}{\partial \dot{r}}} = \cancel{\frac{1}{2}} m \cancel{\frac{d}{dt} \dot{r}} = m \dot{r} \Rightarrow \frac{d}{dt} \frac{\partial L}{\partial \dot{r}} = m \ddot{r} \xrightarrow{\text{aceleración radial}}$$

$$\Rightarrow m \ddot{r} - mr \dot{\theta}^2 - mg \cos \theta + \lambda_2 = 0 \xrightarrow{\substack{\text{Aplicando} \\ \text{restricciones}}} \cancel{m \ddot{r}} - mr \dot{\theta}^2 - mg \cos \theta + \lambda_2 = 0 \Rightarrow \lambda_2 = mr \dot{\theta}^2 + mg \cos \theta$$

$$\text{Para } \theta: \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} - \cancel{\frac{\partial L}{\partial \theta}} + \lambda_1 \cancel{\frac{\partial g_1}{\partial \theta}} + \lambda_2 \cancel{\frac{\partial g_2}{\partial \theta}} = 0$$

$$* \cancel{\frac{\partial L}{\partial \theta}} = -mg r \sin \theta \quad * \cancel{\frac{\partial g_1}{\partial \theta}} = R + \rho \quad * \cancel{\frac{\partial L}{\partial \dot{\theta}}} = \cancel{\frac{1}{2}} m \cancel{2r^2 \dot{\theta}} \Rightarrow \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = 2mr \dot{r} \dot{\theta} + mr^2 \ddot{\theta}$$

$$\Rightarrow 2mr \dot{r} \dot{\theta} + mr^2 \ddot{\theta} + r \sin \theta + \lambda_1 (R + \rho) = 0 \xrightarrow{\substack{\text{Aplicando} \\ \text{restricciones}}} \cancel{2mr \dot{r} \dot{\theta}} + mr^2 \ddot{\theta} + r \sin \theta + \lambda_1 (R + \rho) = 0$$

$$\Rightarrow m(R + \rho) \ddot{\theta} + mg(R + \rho) \sin \theta + \lambda_1 (R + \rho) = 0 \Rightarrow (R + \rho) \ddot{\theta} + g \sin \theta + \frac{\lambda_1}{m} = 0 \quad \text{Algebra y reemplazar.}$$

Problema 2. (Thorton 7.34)

7-34. A particle of mass m slides down a smooth circular wedge of mass M as shown in Figure 7-C. The wedge rests on a smooth horizontal table. Find (a) the equation of motion of m and M and (b) the reaction of the wedge on m .

$$a) \quad g: r - R = 0 \Rightarrow \dot{r} = \dot{r} + r\dot{\theta} \Rightarrow \dot{r}^2 = \dot{r}^2 + r^2\dot{\theta}^2$$

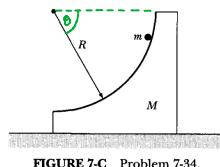


FIGURE 7-C Problem 7-34.

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + mgsen\theta \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

$$\text{Para } r: \frac{d}{dt} \frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} + \lambda \frac{\partial g}{\partial r} = 0 \quad * \frac{\partial g}{\partial r} = 1$$

$$* \cancel{\frac{\partial L}{\partial r}} = \cancel{\frac{1}{2}m2r\dot{\theta}^2} + mgsen\theta = mr\dot{\theta}^2 + mgsen\theta \quad * \cancel{\frac{\partial L}{\partial \dot{r}}} = \cancel{\frac{1}{2}m2\dot{r}} = m\ddot{r} \Rightarrow \frac{d}{dt} \cancel{\frac{\partial L}{\partial \dot{r}}} = m\ddot{r}$$

$$\text{Para } \theta: \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} + \lambda \frac{\partial g}{\partial \theta} = 0 \quad * \frac{\partial g}{\partial \theta} = 0$$



$$* \cancel{\frac{\partial L}{\partial \theta}} = +mgsr\cos\theta \quad * \cancel{\frac{\partial L}{\partial \dot{\theta}}} = \cancel{\frac{1}{2}m2r^2\dot{\theta}} = mr^2\dot{\theta} \Rightarrow \frac{d}{dt} \cancel{\frac{\partial L}{\partial \dot{\theta}}} = mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta}$$

$$\Rightarrow \begin{cases} m\ddot{r} - mr\dot{\theta}^2 - mgsen\theta + \lambda = 0 \\ mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} - mgsr\cos\theta = 0 \end{cases} \xrightarrow{\text{Assumiendo que este del reposo en } \theta=0} \begin{cases} -mR\dot{\theta}^2 - mgsen\theta + \lambda = 0 \\ mR\ddot{\theta} - mgsr\cos\theta = 0 \end{cases} \Rightarrow \begin{cases} \lambda = -mR\dot{\theta}^2 - mgsen\theta \\ R\ddot{\theta} - g\cos\theta = 0 \end{cases}$$

$$H = T + U = \frac{1}{2}mR^2\dot{\theta}^2 - mgsr\cos\theta = -mgsr\cos\theta \Rightarrow R\ddot{\theta}^2 - 2g\sin\theta = -2g\sin\theta \Rightarrow \dot{\theta}^2 = \frac{2g(\sin\theta - \sin\theta_0)}{R}$$

$$\Rightarrow \begin{cases} \lambda = -mR\frac{2g(\sin\theta - \sin\theta_0)}{R} - mgsen\theta \\ R\ddot{\theta} - g\cos\theta = 0 \end{cases} \Rightarrow \begin{cases} \lambda = -2mg(\sin\theta - \sin\theta_0) - mgsen\theta \\ R\ddot{\theta} - g\cos\theta = 0 \end{cases} //$$

- 7-37. Use the method of Lagrange undetermined multipliers to find the tensions in both strings of the double Atwood machine of Example 7.8.

Restricciones: $g_1: \dot{x}_1 + \dot{y} - \dot{b}_1 = 0$ $g_2: \dot{x}_3 + \dot{x}_2 - 2\dot{y} - \dot{b}_2 = 0$

$$\Rightarrow g: 2\dot{x}_1 + \dot{x}_2 + \dot{x}_3 - \underbrace{(2\dot{b}_1 + \dot{b}_2)}_{c} = 0 \Rightarrow 2\ddot{x}_1 + \ddot{x}_2 + \ddot{x}_3 = 0$$

$$T = \frac{1}{2}(m_1\dot{x}_1^2 + m_2\dot{x}_2^2 + m_3\dot{x}_3^2) \quad \text{y} \quad U = -g(m_1x_1 + m_2x_2 + m_3x_3)$$

$$\Rightarrow L = \frac{1}{2}(m_1\dot{x}_1^2 + m_2\dot{x}_2^2 + m_3\dot{x}_3^2) + g(m_1x_1 + m_2x_2 + m_3x_3) \Rightarrow \frac{\partial L}{\partial t} - \frac{\partial L}{\partial x_i} + \lambda \frac{\partial g}{\partial q_i} = 0 \quad \text{con} \quad q_i = x_1, x_2, x_3$$

Para \dot{x}_1 : $\frac{\partial \frac{\partial L}{\partial t}}{\partial \dot{x}_1} - \frac{\partial L}{\partial x_1} + \lambda \frac{\partial g}{\partial x_1} = 0$

$$*\frac{\partial L}{\partial x_1} = m_1g \quad * \frac{\partial g}{\partial x_1} = 2$$

$$*\frac{\partial L}{\partial x_1} = \cancel{\frac{1}{2}}m_1\dot{x}_1 = m_1\dot{x}_1 \Rightarrow \frac{\partial \frac{\partial L}{\partial t}}{\partial \dot{x}_1} = m_1\ddot{x}_1$$

Para \dot{x}_2 : $\frac{\partial \frac{\partial L}{\partial t}}{\partial \dot{x}_2} - \frac{\partial L}{\partial x_2} + \lambda \frac{\partial g}{\partial x_2} = 0$

$$*\frac{\partial L}{\partial x_2} = m_2g \quad * \frac{\partial g}{\partial x_2} = 1$$

$$*\frac{\partial L}{\partial x_2} = \cancel{\frac{1}{2}}m_2\dot{x}_2 = m_2\dot{x}_2 \Rightarrow \frac{\partial \frac{\partial L}{\partial t}}{\partial \dot{x}_2} = m_2\ddot{x}_2$$

Para \dot{x}_3 : $\frac{\partial \frac{\partial L}{\partial t}}{\partial \dot{x}_3} - \frac{\partial L}{\partial x_3} + \lambda \frac{\partial g}{\partial x_3} = 0$

$$*\frac{\partial L}{\partial x_3} = m_3g \quad * \frac{\partial g}{\partial x_3} = 1$$

$$*\frac{\partial L}{\partial x_3} = \cancel{\frac{1}{2}}m_3\dot{x}_3 = m_3\dot{x}_3 \Rightarrow \frac{\partial \frac{\partial L}{\partial t}}{\partial \dot{x}_3} = m_3\ddot{x}_3$$

$$\Rightarrow \begin{cases} m_1\ddot{x}_1 - m_1g + 2\lambda = 0 \\ m_2\ddot{x}_2 - m_2g + \lambda = 0 \quad ; \quad 2\ddot{x}_1 + \ddot{x}_2 + \ddot{x}_3 = 0 \\ m_3\ddot{x}_3 - m_3g + \lambda = 0 \end{cases} \quad 4 \text{ ecuaciones, } 4 \text{ variables}$$

$$\Rightarrow \left(\begin{array}{cccc|c} \dot{x}_1 & \ddot{x}_2 & \ddot{x}_3 & \lambda & \\ m_1 & 0 & 0 & 2 & m_1g \\ 0 & m_2 & 0 & 1 & m_2g \\ 0 & 0 & m_3 & 1 & m_3g \\ 2 & 1 & 1 & 0 & 0 \end{array} \right) \xrightarrow[m_1 = \lambda_{11}]{f_1 \rightarrow \frac{1}{m_1}f_1} \left(\begin{array}{cccc|c} 1 & 0 & 0 & \frac{2}{m_1} & g \\ 0 & 1 & 0 & \frac{1}{m_2} & g \\ 0 & 0 & 1 & \frac{1}{m_3} & g \\ 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{array} \right)$$

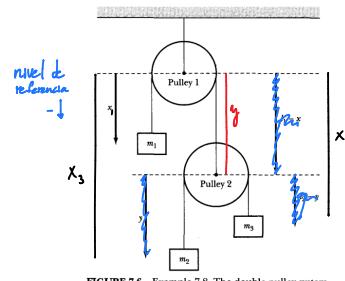


FIGURE 7-6 Example 7.8. The double pulley system.

Recuperando lo anterior

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & \frac{2}{m_1} & g \\ 0 & 1 & 0 & \frac{1}{m_2} & g \\ 0 & 0 & 1 & \frac{1}{m_3} & g \\ 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{array} \right) \xrightarrow{f_4 \rightarrow f_4 - f_1 - \frac{1}{2}f_2 - \frac{1}{2}f_3} \left(\begin{array}{cccc|c} 1 & 0 & 0 & \frac{2}{m_1} & g \\ 0 & 1 & 0 & \frac{1}{m_2} & g \\ 0 & 0 & 1 & \frac{1}{m_3} & g \\ 0 & 0 & 0 & \frac{-2}{m_1} - \frac{1}{2m_2} - \frac{1}{2m_3} & -g - \frac{f_1}{2} - \frac{f_2}{2} \end{array} \right)$$

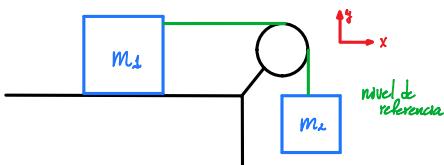
$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & \frac{2}{m_1} & g \\ 0 & 1 & 0 & \frac{1}{m_2} & g \\ 0 & 0 & 1 & \frac{1}{m_3} & g \\ 0 & 0 & 0 & \frac{-2}{m_1} - \frac{1}{2m_2} - \frac{1}{2m_3} & -2g \end{array} \right) \xrightarrow{f_4 \rightarrow \left(\frac{-2}{m_1} - \frac{1}{2m_2} - \frac{1}{2m_3} \right)^{-1} f_4} \left(\begin{array}{cccc|c} 1 & 0 & 0 & \frac{2}{m_1} & g \\ 0 & 1 & 0 & \frac{1}{m_2} & g \\ 0 & 0 & 1 & \frac{1}{m_3} & g \\ 0 & 0 & 0 & 1 & -2g \left(\frac{-2}{m_1} - \frac{1}{2m_2} - \frac{1}{2m_3} \right)^{-1} \end{array} \right)$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & \frac{2}{m_1} & g \\ 0 & 1 & 0 & \frac{1}{m_2} & g \\ 0 & 0 & 1 & \frac{1}{m_3} & g \\ 0 & 0 & 0 & 1 & -2g \left(\frac{-2}{m_1} - \frac{1}{2m_2} - \frac{1}{2m_3} \right)^{-1} \end{array} \right) \xrightarrow{\begin{array}{l} f_1 \rightarrow f_1 - \frac{2}{m_1} f_4 \\ f_2 \rightarrow f_2 - \frac{1}{m_2} f_4 \\ f_3 \rightarrow f_3 - \frac{1}{m_3} f_4 \end{array}} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & g + \frac{2}{m_1} 2g \left(\frac{-2}{m_1} - \frac{1}{2m_2} - \frac{1}{2m_3} \right)^{-1} \\ 0 & 1 & 0 & 0 & g + \frac{1}{m_2} 2g \left(\frac{-2}{m_1} - \frac{1}{2m_2} - \frac{1}{2m_3} \right)^{-1} \\ 0 & 0 & 1 & 0 & g + \frac{1}{m_3} 2g \left(\frac{-2}{m_1} - \frac{1}{2m_2} - \frac{1}{2m_3} \right)^{-1} \\ 0 & 0 & 0 & 1 & -2g \left(\frac{-2}{m_1} - \frac{1}{2m_2} - \frac{1}{2m_3} \right)^{-1} \end{array} \right)$$

$$\Rightarrow \begin{cases} \ddot{x}_1 = g + \frac{2}{m_1} 2g \left(\frac{-2}{m_1} - \frac{1}{2m_2} - \frac{1}{2m_3} \right)^{-1} \\ \ddot{x}_2 = g + \frac{1}{m_2} 2g \left(\frac{-2}{m_1} - \frac{1}{2m_2} - \frac{1}{2m_3} \right)^{-1} \\ \ddot{x}_3 = g + \frac{1}{m_3} 2g \left(\frac{-2}{m_1} - \frac{1}{2m_2} - \frac{1}{2m_3} \right)^{-1} \\ \lambda = -2g \left(\frac{-2}{m_1} - \frac{1}{2m_2} - \frac{1}{2m_3} \right)^{-1} \end{cases}$$

Problema 4. (Taylor 7.50)

7.50* A mass m_1 rests on a frictionless horizontal table. Attached to it is a string which runs horizontally to the edge of the table, where it passes over a frictionless, small pulley and down to where it supports a mass m_2 . Use as coordinates x and y the distances of m_1 and m_2 from the pulley. These satisfy the constraint equation $f(x, y) = x + y = \text{const}$. Write down the two modified Lagrange equations and solve them (together with the constraint equation) for \ddot{x} , \ddot{y} , and the Lagrange multiplier λ . Use (7.122) (and the corresponding equation in y) to find the tension forces on the two masses. Verify your answers by solving the problem by the elementary Newtonian approach.



$$L = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{y}^2 - m_2g\dot{y} \quad g: \dot{x} + \dot{y} - c = 0 \rightarrow \lambda \Rightarrow \dot{x} + \dot{y} = 0 \Rightarrow \ddot{x} + \ddot{y} = 0$$

$$\text{Hay que operar: } \frac{\partial L}{\partial t} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} + \lambda \frac{\partial g}{\partial q_i} = 0 \quad \text{con } q_1 = x, q_2 = y$$

Para X :

$$\frac{\partial L}{\partial X} = 0, \quad \frac{\partial g}{\partial X} = 1, \quad , \frac{\partial L}{\partial \dot{X}} = m_1\dot{X} \Rightarrow \frac{\partial L}{\partial t} \frac{\partial L}{\partial \dot{X}} = m_1\ddot{X} \Rightarrow m_1\ddot{X} + \lambda = 0$$

Para y :

$$\frac{\partial L}{\partial y} = -m_2g, \quad \frac{\partial g}{\partial y} = 1, \quad , \frac{\partial L}{\partial \dot{y}} = m_2\dot{y} \Rightarrow \frac{\partial L}{\partial t} \frac{\partial L}{\partial \dot{y}} = m_2\ddot{y} \Rightarrow m_2\ddot{y} + m_2g + \lambda = 0$$

$$\Rightarrow \begin{cases} \ddot{x} + \ddot{y} = 0 \\ m_1\ddot{X} + \lambda = 0 \\ m_2\ddot{y} + m_2g + \lambda = 0 \end{cases} \Rightarrow \begin{cases} \lambda = \frac{-m_1m_2g}{m_1 + m_2} \\ \ddot{X} = \frac{m_2g}{m_1 + m_2} \\ \ddot{y} = \frac{-m_2g}{m_1 + m_2} \end{cases}$$

Problemas varios

Un disco de masa M y radio R rueda sin deslizamiento hacia abajo en un plano inclinado con un ángulo α respecto a la horizontal. En su eje, el disco tiene una varilla de diámetro, largo y masa despreciables. De la varilla está suspendido un péndulo simple de largo $l < R$ y que tiene una masa m en su extremo. Consideré que el movimiento del péndulo toma lugar en el plano del disco y halle las ecuaciones de Lagrange para el sistema.

Sugerencias: La inercia rotacional del disco es $MR^2/2$; se puede usar la relación $\xi = R\theta$, con θ el ángulo con el que rota el disco sobre el plano inclinado.

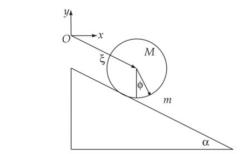
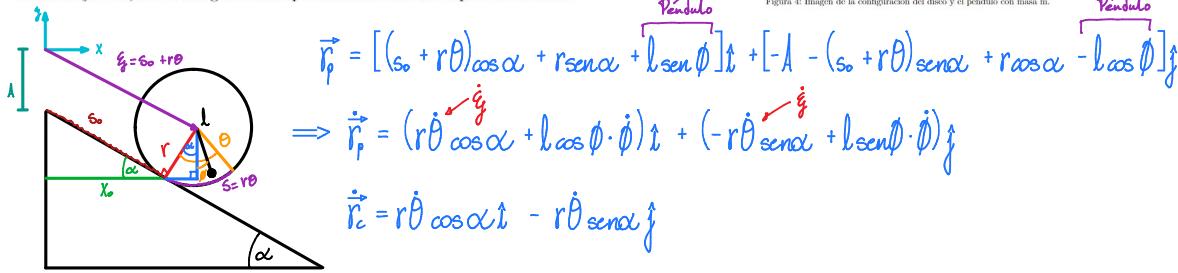


Figura 4: Imagen de la configuración del disco y el péndulo con masa m.

Péndulo:

$$\begin{aligned}\Rightarrow \dot{r}_p^2 &= r^2 \dot{\theta}^2 \cos^2 \alpha + 2rl\dot{\theta}\dot{\phi} \cos \alpha \cos \phi + l^2 \cos^2 \phi \cdot \dot{\phi}^2 + r^2 \dot{\theta}^2 \sin^2 \alpha - 2rl\dot{\theta}\dot{\phi} \sin \alpha \sin \phi + l^2 \sin^2 \phi \cdot \dot{\phi}^2 \\ &= r^2 \dot{\theta}^2 + l^2 \dot{\phi}^2 + 2rl\dot{\theta}\dot{\phi} (\cos \alpha \cos \phi - \sin \alpha \sin \phi) \\ \Rightarrow T_p &= \frac{1}{2}m[r^2 \dot{\theta}^2 + l^2 \dot{\phi}^2 + 2rl\dot{\theta}\dot{\phi} (\cos \alpha \cos \phi - \sin \alpha \sin \phi)] \\ \Rightarrow U_p &= mg \frac{q}{\dot{\phi}} = mg[-A - (s_0 + r\theta) \sin \alpha + r \cos \alpha - l \cos \phi]\end{aligned}$$

Cilindro:

$$\Rightarrow \dot{r}_c^2 = r^2 \dot{\theta}^2 \cos^2 \alpha + r^2 \dot{\theta}^2 \sin^2 \alpha = r^2 \dot{\theta}^2$$

$$\Rightarrow T_c = \frac{1}{2}Mr^2 \dot{\theta}^2 + \frac{1}{2}I \dot{\theta}^2 = \frac{1}{2}Mr^2 \dot{\theta}^2 + \frac{Mr^2}{4} \dot{\theta}^2 = \frac{3Mr^2}{4} \dot{\theta}^2$$

$$\Rightarrow U_c = Mg \frac{q}{\dot{\theta}} = Mg[-A - (s_0 + r\theta) \sin \alpha + r \cos \alpha]$$

Sistema:

$$\begin{aligned}L &= \frac{1}{2}m[r^2 \dot{\theta}^2 + l^2 \dot{\phi}^2 + 2rl\dot{\theta}\dot{\phi} (\cos \alpha \cos \phi - \sin \alpha \sin \phi)] + \frac{3Mr^2}{4} \dot{\theta}^2 - mg[-A - (s_0 + r\theta) \sin \alpha + r \cos \alpha - l \cos \phi] + \dots \\ &\quad - Mg[-A - (s_0 + r\theta) \sin \alpha + r \cos \alpha]\end{aligned}$$

$$\Rightarrow L = \frac{1}{2}m[r^2\dot{\theta}^2 + l^2\dot{\phi}^2 + 2rl\dot{\theta}\dot{\phi}\cos(\alpha + \phi)] + \frac{3Mr^2}{4}\dot{\theta}^2 - mg[-A - (s_0 + r\theta)\sin\alpha + r\cos\alpha - l\cos\phi] + \dots - Mg[-A - (s_0 + r\theta)\sin\alpha + r\cos\alpha]$$

Las coordenadas generalizadas son $\theta, \phi \Rightarrow \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$

Para ϕ :

$$*\frac{\partial L}{\partial \phi} = \frac{1}{2}m[2rl\dot{\theta}\dot{\phi}\cos(\alpha + \phi) - mglsen\phi] = -mrl\dot{\theta}\dot{\phi}\sin(\alpha + \phi) - mglsen\phi$$

$$*\frac{\partial L}{\partial \dot{\phi}} = \frac{1}{2}m[2l^2\dot{\phi} + 2rl\dot{\theta}\cos(\alpha + \phi)] = ml^2\dot{\phi} + mrl\dot{\theta}\cos(\alpha + \phi)$$

$$\Rightarrow \frac{d}{dt} \frac{\partial L}{\partial \dot{\phi}} = ml^2\ddot{\phi} + mrl\ddot{\theta}\cos(\alpha + \phi) - mrl\dot{\theta}\dot{\phi}\sin(\alpha + \phi)$$

$$\Rightarrow ml^2\ddot{\phi} + mrl\ddot{\theta}\cos(\alpha + \phi) - \cancel{mrl\dot{\theta}\dot{\phi}\sin(\alpha + \phi)} + \cancel{mrl\dot{\theta}\dot{\phi}\sin(\alpha + \phi)} + mglsen\phi = 0$$

$$\Rightarrow ml^2\ddot{\phi} + mrl\ddot{\theta}\cos(\alpha + \phi) + mglsen\phi = 0$$

$$\Rightarrow l\ddot{\theta} + r\ddot{\theta}\cos(\alpha + \phi) + gsen\phi = 0 //$$

Para θ :

$$*\frac{\partial L}{\partial \theta} = -mg[-r\sin\alpha] - Mg[-r\sin\alpha] = gr\sin\alpha(M + m)$$

$$*\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2}m[2r^2\dot{\theta} + 2rl\dot{\phi}\cos(\alpha + \phi)] + 2 \cdot \frac{3Mr^2}{4}\dot{\theta} = m[r^2\dot{\theta} + rl\dot{\phi}\cos(\alpha + \phi)] + \frac{3Mr^2\dot{\theta}}{2}$$

$$\Rightarrow \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = m[r^2\ddot{\theta} + rl\ddot{\phi}\cos(\alpha + \phi) - rl\dot{\phi}^2\sin(\alpha + \phi)] + \frac{3Mr^2\ddot{\theta}}{2}$$

$$\Rightarrow m[r^2\ddot{\theta} + rl\ddot{\phi}\cos(\alpha + \phi) - rl\dot{\phi}^2\sin(\alpha + \phi)] + \frac{3Mr^2\ddot{\theta}}{2} - gr\sin\alpha(M + m) = 0$$

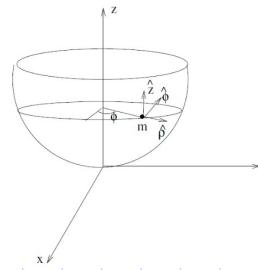
$$\Rightarrow m[r\ddot{\theta} + l\ddot{\phi}\cos(\alpha + \phi) - l\dot{\phi}^2\sin(\alpha + \phi)] + \frac{3Mr\ddot{\theta}}{2} - g\sin\alpha(M + m) = 0 //$$

Considere una partícula de masa m moviéndose sin fricción en un parabolóide dado por $x^2 + y^2 = az$, y cuya restricción por tanto se puede escribir como $2\rho\dot{\rho} - a\dot{z} = 0$.

Sea ϕ el ángulo que forma el vector posición con el eje x , ρ la distancia al eje z y z la coordenada vertical:

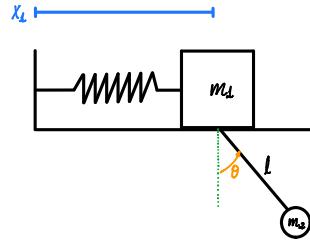
a) Encuentre el Lagrangiano del sistema y las ecuaciones de movimiento en términos del multiplicador de lagrange asociado a la restricción.

b) Considerando el caso de que la masa se mueve en un círculo horizontal de altura $z = h$, encuentre la velocidad angular $\dot{\phi}$ en términos de constantes conocidas.



El bloque m_1 está sobre una superficie sin fricción y conectado a un resorte ideal de constante de elasticidad K . Además está conectado a otro bloque de masa m_2 por medio de una cuerda ideal de largo l .

Determine ecuaciones del movimiento y simplifique a oscilaciones pequeñas



$$\vec{r}_1 = X \hat{e}_x \quad y \quad \vec{r}_2 = (X + r \sin \theta) \hat{e}_x - r \cos \theta \hat{e}_y$$

$$\Rightarrow \dot{\vec{r}}_1 = \dot{X} \hat{e}_x \quad y \quad \dot{\vec{r}}_2 = (\dot{X} + r \cos \theta \cdot \dot{\theta}) \hat{e}_x + r \sin \theta \cdot \dot{\theta} \hat{e}_y$$

$$\Rightarrow \dot{\vec{r}}_1^2 = \dot{X}^2 \quad y \quad \dot{\vec{r}}_2^2 = (\dot{X} + r \cos \theta \cdot \dot{\theta})^2 + r^2 \sin^2 \theta \cdot \dot{\theta}^2 = \dot{X}^2 + 2\dot{X}\dot{\theta}r \cos \theta + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \cdot \dot{\theta}^2 = \dot{X}^2 + 2\dot{X}\dot{\theta}r \cos \theta + r^2 \dot{\theta}^2$$

$$\Rightarrow L = \frac{1}{2} m_1 \dot{X}^2 + \frac{1}{2} m_2 (\dot{X}^2 + 2\dot{X}\dot{\theta}r \cos \theta + r^2 \dot{\theta}^2) - \frac{1}{2} K X^2 + m_2 g r \cos \theta \quad \frac{dL}{dt} \frac{\partial L}{\partial q_i} - \frac{\partial L}{\partial \dot{q}_i} = 0$$

$$\text{Para } X: \frac{d}{dt} \frac{\partial L}{\partial X} - \frac{\partial L}{\partial \dot{X}} = 0$$

$$* \frac{\partial L}{\partial X} = -\frac{1}{2} K X = -K X$$

$$* \frac{\partial L}{\partial \dot{X}} = \frac{1}{2} m_1 \cancel{2\dot{X}} + \frac{1}{2} m_2 (\cancel{2\dot{X}} + \cancel{2\dot{\theta}r \cos \theta}) = m_1 \dot{X} + m_2 \dot{X} + m_2 \dot{\theta} r \cos \theta$$

$$\Rightarrow \frac{d}{dt} \frac{\partial L}{\partial \dot{X}} = m_1 \ddot{X} + m_2 \ddot{X} + m_2 r (\ddot{\theta} \cos \theta - \ddot{\theta} \sin \theta) = (m_1 + m_2) \ddot{X} + m_2 r (\ddot{\theta} \cos \theta - \ddot{\theta} \sin \theta) \\ \Rightarrow (m_1 + m_2) \ddot{X} + m_2 r (\ddot{\theta} \cos \theta - \ddot{\theta} \sin \theta) + K X = 0$$

Para θ :

$$* \frac{\partial L}{\partial \theta} = -m_2 \dot{X} \dot{\theta} \sin \theta - m_2 g r \sin \theta$$

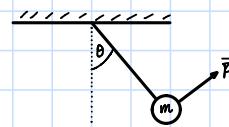
$$* \frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2} m_2 (\cancel{2\dot{X}r \cos \theta} + \cancel{2r^2 \dot{\theta}}) = m_2 (\dot{X} r \cos \theta + r^2 \dot{\theta}) \Rightarrow \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = m_2 (\ddot{X} r \cos \theta - \dot{X} \dot{\theta} \sin \theta + r^2 \ddot{\theta})$$

$$\Rightarrow m_2 (\dot{X} r \cos \theta - \cancel{\dot{X} \dot{\theta} \sin \theta} + r^2 \ddot{\theta}) + m_2 \dot{X} \dot{\theta} \sin \theta + m_2 g r \sin \theta = 0$$

$$\Rightarrow m_2 (\dot{X} r \cos \theta + r^2 \ddot{\theta}) + m_2 g r \sin \theta = 0 \Rightarrow \begin{cases} (m_1 + m_2) \ddot{X} + m_2 r (\ddot{\theta} \cos \theta - \ddot{\theta} \sin \theta) + K X = 0 \\ \dot{X} r \cos \theta + r^2 \ddot{\theta} + g r \sin \theta = 0 \end{cases}$$



Se muestra un péndulo simple, que además se le aplica una fuerza $\vec{F} = F \hat{e}_\theta$, donde \hat{e}_θ es el vector unitario en la dirección tangencial.



- a) Plantee el lagrangeano del sistema
- b) Determine las fuerzas generalizadas de la configuración
- c) ¿Cuáles son las ecuación(es) de movimiento del sistema?

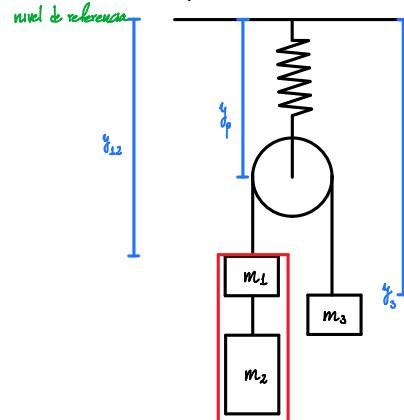
El sistema que se muestra está formado por 2 masas $m_1 + m_2 > m_3$, además de una polea ideal, de masa M , radio R y momento de inercia $J = \frac{1}{2}MR^2$. El centro de la polea y el techo están conectados por medio de un resorte ideal, con una constante de elasticidad K y una longitud sin deformación δ_0 . Considera que el resorte solo se puede mover verticalmente, que la cuerda es ideal, sin que exista fricción entre ella y la polea. En el inicio m_1 y m_3 están a la misma altura con respecto al techo, el resorte está sin deformar y el sistema se suelta desde el reposo.

$$\text{Restricciones: } q_1: \dot{q}_{12} + \dot{q}_3 - 2\dot{\phi} = 0 \quad y \quad q_2: \dot{q}_{12} - R\dot{\phi} = 0$$

$$m_1 + m_2: T_{12} = \frac{1}{2}(m_1 + m_2)\dot{q}_{12}^2 \quad y \quad U_{12} = (m_1 + m_2)g\dot{q}_{12}$$

$$M: T_p = \frac{1}{2}M\dot{\phi}^2 + \frac{1}{2}J\dot{\phi}^2 \quad y \quad U_p = Mg\dot{\phi} + \frac{1}{2}K\dot{q}_p^2$$

$$m_3: T_3 = \frac{1}{2}m_3\dot{q}_3^2 \quad y \quad U_3 = m_3g\dot{q}_3$$



$$\Rightarrow L = T - U = \frac{1}{2}(m_1 + m_2)\dot{q}_{12}^2 + \frac{1}{2}M\dot{\phi}^2 + \frac{1}{2}m_3\dot{q}_3^2 + \frac{1}{4}J\dot{\phi}^2 - (m_1 + m_2)g\dot{q}_{12} - Mg\dot{\phi} - \frac{1}{2}K\dot{q}_p^2 - m_3g\dot{q}_3$$

$$\text{Hay que operar: } \frac{d}{dt} \frac{dL}{dq_i} - \frac{dL}{dq_i} + \lambda_1 \frac{dL}{dq_1} + \lambda_2 \frac{dL}{dq_2} = 0 \quad \text{para } q = \dot{q}_{12}, \dot{\phi}, \dot{q}_3 \text{ y } \dot{\phi}$$

$$\text{Para } m_1 + m_2: * \frac{dL}{dq_{12}} = (m_1 + m_2)g * \frac{dL}{dq_1} = 1 * \frac{dL}{dq_2} = 1 * \frac{dL}{dq_{12}} = \frac{1}{2}(m_1 + m_2) * \cancel{\frac{dL}{dq_{12}}} = (m_1 + m_2) \cancel{\frac{dL}{dq_{12}}} \Rightarrow \frac{d}{dt} \frac{dL}{dq_{12}} = (m_1 + m_2) \ddot{q}_{12}$$

$$\text{Para } \dot{\phi}: * \frac{dL}{d\dot{\phi}} = -Mg - \cancel{\frac{1}{2}K\dot{q}_p} = -mg - K\dot{q}_p * \frac{dL}{d\dot{\phi}} = -2 * \frac{dL}{d\dot{\phi}} = 0 * \frac{dL}{d\dot{\phi}} = \frac{1}{2}M\dot{\phi} = M\dot{\phi} \Rightarrow \frac{d}{dt} \frac{dL}{d\dot{\phi}} = M\ddot{\phi}$$

$$\text{Para } \dot{q}_3: * \frac{dL}{dq_3} = -m_3g * \frac{dL}{dq_3} = 1 * \frac{dL}{dq_3} = 0 * \frac{dL}{dq_3} = \frac{1}{2}m_3\dot{q}_3^2 = m_3\ddot{q}_3 \Rightarrow \frac{d}{dt} \frac{dL}{dq_3} = m_3\ddot{q}_3$$

$$\Rightarrow \begin{cases} (m_1 + m_2)\ddot{q}_{12} + (m_1 + m_2)g + \lambda_1 + \lambda_2 = 0 \\ M\ddot{\phi} + Mg + K\dot{q}_p - 2\lambda_1 = 0 \\ MR^2\ddot{\phi} + 2 - R\lambda_1 = 0 \\ m_3\ddot{q}_3 + m_3g + \lambda_3 = 0 \end{cases} \Rightarrow \begin{cases} (m_1 + m_2)\ddot{q}_{12} + (m_1 + m_2)g + \lambda_1 + \lambda_2 = 0 \\ M\ddot{\phi} + Mg + K\dot{q}_p - 2\lambda_1 = 0 \\ MR^2\ddot{\phi} - 2R\lambda_1 = 0 \\ m_3\ddot{q}_3 + m_3g + \lambda_3 = 0 \end{cases}$$

$$\left\{ \begin{array}{l} (m_1+m_2)\ddot{\theta}_{12} + (m_1+m_2)g + \lambda_1 + \lambda_2 = 0 \\ M\ddot{\theta}_p + Mg + K\dot{\theta}_p - 2\lambda_1 = 0 \\ M\cancel{\dot{\theta}} - 2\cancel{R}\lambda_1 = 0 \\ m_3\ddot{\theta}_3 + m_3g + \lambda_1 = 0 \end{array} \right. ; \Rightarrow \ddot{\theta}_{12} + \ddot{\theta}_3 - 2\ddot{\theta}_p = 0 \quad \Rightarrow \left\{ \begin{array}{l} (m_1+m_2)\ddot{\theta}_{12} + (m_1+m_2)g + \lambda_1 + \lambda_2 = 0 \\ M\ddot{\theta}_p + Mg + K\dot{\theta}_p - 2\lambda_1 = 0 \\ M\ddot{\theta}_{12} - 2\lambda_1 = 0 \\ m_3\ddot{\theta}_3 + m_3g + \lambda_1 = 0 \end{array} \right.$$

$$\Rightarrow \left\{ \begin{array}{l} (m_1+m_2)\ddot{\theta}_{12} + (m_1+m_2)g + \lambda_1 + \lambda_2 = 0 \\ M\ddot{\theta}_p + Mg + K\dot{\theta}_p - 2\lambda_1 = 0 \\ M\ddot{\theta}_{12} - 2\lambda_1 = 0 \\ m_3\ddot{\theta}_3 + m_3g + \lambda_1 = 0 \end{array} \right. \quad \text{Pasando a forma matricial}$$

$$\Rightarrow \left(\begin{array}{ccccc|c} \ddot{\theta}_{12} & \ddot{\theta}_p & \ddot{\theta}_3 & \lambda_1 & \lambda_2 & \\ \hline (m_1+m_2) & 0 & 0 & 1 & 1 & -(m_1+m_2)g \\ M & 0 & 0 & -2 & 0 & 0 \\ 0 & M & 0 & -2 & 0 & -Mg - K\dot{\theta}_p \\ 0 & 0 & m_3 & 1 & 0 & -m_3g \\ 1 & -2 & 1 & 0 & 0 & 0 \end{array} \right) \xrightarrow{\begin{array}{l} f_1 \rightarrow \frac{1}{m_1+m_2}f_1 \\ f_2 \leftrightarrow f_3 \\ f_3 \leftrightarrow f_4 \end{array}} \left(\begin{array}{ccccc|c} 1 & 0 & 0 & \frac{1}{m_1+m_2} & \frac{1}{m_1+m_2} & -g \\ 0 & M & 0 & -2 & 0 & -Mg - K\dot{\theta}_p \\ 0 & 0 & m_3 & 1 & 0 & -m_3g \\ M & 0 & 0 & -2 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 \end{array} \right)$$

$$\xrightarrow{\begin{array}{l} f_4 \rightarrow f_4 - Mf_1 \\ f_5 \rightarrow f_5 - f_1 \end{array}} \left(\begin{array}{ccccc|c} 1 & 0 & 0 & \frac{1}{m_1+m_2} & \frac{1}{m_1+m_2} & -g \\ 0 & M & 0 & -2 & 0 & -Mg - K\dot{\theta}_p \\ 0 & 0 & m_3 & 1 & 0 & -m_3g \\ 0 & 0 & 0 & -2 - \frac{M}{m_1+m_2} & \frac{-M}{m_1+m_2} & Mg \\ 0 & -2 & 1 & -\frac{M}{m_1+m_2} & \frac{-M}{m_1+m_2} & g \end{array} \right)$$

$$\left(\begin{array}{cccccc} 1 & 0 & 0 & \frac{1}{m_1+m_2} & \frac{1}{m_1+m_2} & -g \\ 0 & M & 0 & -2 & 0 & -Mg - K_{dp}^y \\ 0 & 0 & m_3 & 1 & 0 & -mg \\ 0 & 0 & 0 & -2 - \frac{M}{m_1+m_2} & \frac{-M}{m_1+m_2} & Mg \\ 0 & -2 & 1 & -\frac{M}{m_1+m_2} & \frac{-M}{m_1+m_2} & g \end{array} \right) \xrightarrow{\begin{array}{l} f_2 \rightarrow \frac{1}{M} f_2 \\ f_3 \rightarrow \frac{1}{m_3} f_3 \\ f_5 \rightarrow \frac{1}{2} f_5 \end{array}} \left(\begin{array}{cccccc} 1 & 0 & 0 & \frac{1}{m_1+m_2} & \frac{1}{m_1+m_2} & -g \\ 0 & 1 & 0 & -\frac{2}{M} & 0 & -\frac{M-K_y^2}{Mm} \\ 0 & 0 & 1 & \frac{1}{m_3} & 0 & -g \\ 0 & 0 & 0 & -2 - \frac{M}{m_1+m_2} & \frac{-M}{m_1+m_2} & Mg \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \frac{M}{m_1+m_2} & \frac{1}{2} \frac{M}{m_1+m_2} & -\frac{g}{2} \end{array} \right)$$

$$\left(\begin{array}{cccccc} 1 & 0 & 0 & \frac{1}{m_1+m_2} & \frac{1}{m_1+m_2} & -g \\ 0 & 1 & 0 & -\frac{2}{M} & 0 & -\frac{M-K_y^2}{Mm} \\ 0 & 0 & 1 & \frac{1}{m_3} & 0 & -g \\ 0 & 0 & 0 & \frac{-2(m_1+m_2)-M}{m_1+m_2} & \frac{-M}{m_1+m_2} & Mg \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \frac{M}{m_1+m_2} & \frac{1}{2} \frac{M}{m_1+m_2} & -\frac{g}{2} \end{array} \right) \xrightarrow{\begin{array}{l} f_4 \rightarrow \frac{m_1+m_2}{-2(m_1+m_2)-M} f_4 \\ f_5 \rightarrow f_5 - f_2 \\ f_6 \rightarrow f_6 + \frac{1}{2} f_3 \end{array}} \left(\begin{array}{cccccc} 1 & 0 & 0 & \frac{1}{m_1+m_2} & \frac{1}{m_1+m_2} & -g \\ 0 & 1 & 0 & -\frac{2}{M} & 0 & -\frac{M-K_y^2}{Mm} \\ 0 & 0 & 1 & \frac{1}{m_3} & 0 & -g \\ 0 & 0 & 0 & 1 & \frac{-M}{-2(m_1+m_2)-M} & \frac{(m_1+m_2)Mg}{-2(m_1+m_2)-M} \\ 0 & 0 & 0 & \frac{1}{2} \frac{M}{m_1+m_2} + \frac{2}{M} - \frac{1}{2m_3} & \frac{1}{2} \frac{M}{m_1+m_2} & -g + \frac{M+K_y^2}{Mm} \end{array} \right)$$

$$\left(\begin{array}{cccccc} 1 & 0 & 0 & \frac{1}{m_1+m_2} & \frac{1}{m_1+m_2} & -g \\ 0 & 1 & 0 & -\frac{2}{M} & 0 & -\frac{M-K_y^2}{Mm} \\ 0 & 0 & 1 & \frac{1}{m_3} & 0 & -g \\ 0 & 0 & 0 & 1 & \frac{-M}{-2(m_1+m_2)-M} & \frac{(m_1+m_2)Mg}{-2(m_1+m_2)-M} \\ 0 & 0 & 0 & \frac{M^2 m_3 + 4(m_1+m_2)m_3 + (m_1+m_2)M}{2(m_1+m_2)Mm_3} & \frac{1}{2} \frac{M}{m_1+m_2} & -g + \frac{M+K_y^2}{Mm} \end{array} \right) \xrightarrow{\begin{array}{l} f_1 \rightarrow f_1 - \frac{1}{m_1+m_2} f_4 \\ f_2 \rightarrow f_2 + \frac{2}{M} f_4 \\ f_3 \rightarrow f_3 - \frac{1}{m_3} f_4 \\ f_5 \rightarrow f_5 - \frac{M^2 m_3 + 4(m_1+m_2)m_3 + (m_1+m_2)M}{2(m_1+m_2)Mm_3} f_4 \end{array}} \left(\begin{array}{cccccc} 1 & 0 & 0 & \frac{1}{m_1+m_2} & \frac{1}{m_1+m_2} & -g \\ 0 & 1 & 0 & -\frac{2}{M} & 0 & -\frac{M-K_y^2}{Mm} \\ 0 & 0 & 1 & \frac{1}{m_3} & 0 & -g \\ 0 & 0 & 0 & 1 & \frac{(m_1+m_2)Mg}{-2(m_1+m_2)-M} & -g + \frac{M+K_y^2}{Mm} \end{array} \right)$$

$$\left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & \frac{1}{m_1+m_2} - \frac{1}{m_1+m_2} \frac{-M}{-2(m_1+m_2)-M} & -g \\ 0 & 1 & 0 & 0 & \cancel{\frac{2}{M} - \frac{-M}{-2(m_1+m_2)-M}} & -g - \frac{1}{m_1+m_2} \frac{(m_1+m_2)Mg}{-2(m_1+m_2)-M} \\ 0 & 0 & 1 & 0 & -\frac{1}{m_3} \frac{-M}{-2(m_1+m_2)-M} & -g - \frac{1}{m_3} \frac{(m_1+m_2)Mg}{-2(m_1+m_2)-M} \\ 0 & 0 & 0 & 1 & \frac{-M}{-2(m_1+m_2)-M} & \frac{(m_1+m_2)Mg}{-2(m_1+m_2)-M} \\ 0 & 0 & 0 & 0 & \frac{1}{2} \frac{M}{m_1+m_2} - \frac{M^2 m_3 + 4(m_1+m_2)m_3 + (m_1+m_2)M}{2(m_1+m_2)Mm_3} & -g + \frac{M+K_y^2}{Mm} - \frac{M^2 m_3 + 4(m_1+m_2)m_3 + (m_1+m_2)M}{2(m_1+m_2)Mm_3} - \frac{(m_1+m_2)Mg}{-2(m_1+m_2)-M} \end{array} \right)$$

Recuperando la matriz y simplificando

$$\left(\begin{array}{ccccc} 1 & 0 & 0 & 0 & -\frac{1}{m_1+m_2} - \frac{1}{m_1+m_2} - \frac{-M}{2(m_1+m_2)-M} \\ 0 & 1 & 0 & 0 & \frac{2}{2(m_1+m_2)+M} \\ 0 & 0 & 1 & 0 & -\frac{1}{m_3} - \frac{-M}{2(m_1+m_2)-M} \\ 0 & 0 & 0 & 1 & \frac{-M}{2(m_1+m_2)-M} \\ 0 & 0 & 0 & 0 & \frac{1}{2m_1+m_2} - \frac{M^2m_2+4(m_1+m_2)m_3+(m_1+m_2)M}{2(m_1+m_2)m_3[2(m_1+m_2)+M]} \end{array} \right) \left(\begin{array}{c} -g - \frac{Mg}{-2(m_1+m_2)-M} \\ -\frac{Mg-Kg}{\partial M} + \frac{2(m_1+m_2)g}{-2(m_1+m_2)-M} \\ -g - \frac{1}{m_3} - \frac{(m_1+m_2)Mg}{-2(m_1+m_2)-M} \\ \frac{(m_1+m_2)Mg}{-2(m_1+m_2)-M} \\ -g + \frac{Mg+Kg}{\partial M} + g \cdot \frac{M^2m_2+4(m_1+m_2)m_3+(m_1+m_2)M}{2m_3[2(m_1+m_2)+M]} \end{array} \right)$$

$\int_5 \rightarrow \left(\frac{1}{2m_1+m_2} - \frac{M^2m_2+4(m_1+m_2)m_3+(m_1+m_2)M}{2(m_1+m_2)m_3[2(m_1+m_2)+M]} \right)^{-1} \int_5$

$$\left(\begin{array}{ccccc} 1 & 0 & 0 & 0 & -\frac{1}{m_1+m_2} - \frac{1}{m_1+m_2} - \frac{-M}{2(m_1+m_2)-M} \\ 0 & 1 & 0 & 0 & \frac{2}{2(m_1+m_2)+M} \\ 0 & 0 & 1 & 0 & -\frac{1}{m_3} - \frac{-M}{2(m_1+m_2)-M} \\ 0 & 0 & 0 & 1 & \frac{M}{2(m_1+m_2)+M} \\ 0 & 0 & 0 & 0 & 1 \end{array} \right) \left(\begin{array}{c} -g - \frac{Mg}{-2(m_1+m_2)-M} \\ -\frac{Mg-Kg}{\partial M} + \frac{2(m_1+m_2)g}{-2(m_1+m_2)-M} \\ -g - \frac{1}{m_3} - \frac{(m_1+m_2)Mg}{-2(m_1+m_2)-M} \\ \frac{(m_1+m_2)Mg}{-2(m_1+m_2)-M} \\ -g + \frac{Mg+Kg}{\partial M} + g \cdot \frac{M^2m_2+4(m_1+m_2)m_3+(m_1+m_2)M}{2m_3[2(m_1+m_2)+M]} \end{array} \right)$$

$\int_1 \rightarrow \int_1 - \left(\frac{1}{m_1+m_2} - \frac{1}{m_1+m_2} - \frac{-M}{2(m_1+m_2)-M} \right) \int_5$
 $\int_2 \rightarrow \int_2 - \frac{2}{2(m_1+m_2)+M} \int_5$
 $\int_3 \rightarrow \int_3 + \frac{1}{m_3} \frac{M}{2(m_1+m_2)+M} \int_5$
 $\int_4 \rightarrow \int_4 - \frac{M}{2(m_1+m_2)+M} \int_5$

A

$$\left(\begin{array}{ccccc} 1 & 0 & 0 & 0 & \left(-g - \frac{Mg}{-2(m_1+m_2)-M} \right) - \left(\frac{1}{m_1+m_2} - \frac{1}{m_1+m_2} - \frac{-M}{2(m_1+m_2)-M} \right) A \\ 0 & 1 & 0 & 0 & \frac{-Mg-Kg}{\partial M} + \frac{2(m_1+m_2)g}{-2(m_1+m_2)-M} - \frac{2}{2(m_1+m_2)+M} A \\ 0 & 0 & 1 & 0 & -g - \frac{1}{m_3} - \frac{(m_1+m_2)Mg}{-2(m_1+m_2)-M} + \frac{1}{m_3} \frac{M}{2(m_1+m_2)+M} A \\ 0 & 0 & 0 & 1 & \frac{(m_1+m_2)Mg}{-2(m_1+m_2)-M} - \frac{M}{2(m_1+m_2)+M} A \\ 0 & 0 & 0 & 0 & -g + \frac{Mg+Kg}{\partial M} + g \cdot \frac{M^2m_2+4(m_1+m_2)m_3+(m_1+m_2)M}{2m_3[2(m_1+m_2)+M]} = 1 \end{array} \right) \left(\begin{array}{c} \ddot{\gamma}_2 \\ \ddot{\gamma}_p \\ \ddot{\gamma}_3 \\ \lambda_1 \\ \lambda_2 \end{array} \right)$$

PART

IV

Mecánica Hamiltoneana

SUBSECTION 9.4

Problemas resueltos

Mecánica Hamiltoneana

Problema 1. (Thornton 7.17)

7-17. A particle of mass m can slide freely along a wire AB whose perpendicular distance to the origin O is h (see Figure 7-A, page 282). The line OC rotates about the origin at a constant angular velocity $\dot{\theta} = \omega$. The position of the particle can be described in terms of the angle θ and the distance q to the point C . If the particle is subject to a gravitational force, and if the initial conditions are

$$\theta(0) = 0, \quad q(0) = 0, \quad \dot{q}(0) = 0$$

show that the time dependence of the coordinate q is

$$q(t) = \frac{g}{2\omega^2} (\cosh \omega t - \cos \omega t)$$

Sketch this result. Compute the Hamiltonian for the system, and compare with the total energy. Is the total energy conserved?

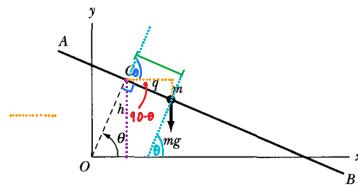


FIGURE 7-A Problem 7-17.

Trabajando en las coordenadas planteadas

$$\begin{aligned}
 \vec{r} &= (h \cos \theta + q \cos(\theta - \dot{\theta})) \hat{i} + (h \sin \theta - q \sin(\theta - \dot{\theta})) \hat{j} = (h \cos \theta + q \sin \dot{\theta}) \hat{i} + (h \sin \theta - q \cos \dot{\theta}) \hat{j} \\
 \Rightarrow \dot{\vec{r}} &= (-h \sin \theta \cdot \dot{\theta} + q \cos \theta \cdot \dot{\theta} + q \sin \dot{\theta}) \hat{i} + (h \cos \theta \cdot \dot{\theta} + q \sin \theta \cdot \dot{\theta} - q \cos \dot{\theta}) \hat{j} \\
 \Rightarrow \dot{\vec{r}}^2 &= (-h \sin \theta \cdot \dot{\theta} + q \cos \theta \cdot \dot{\theta} + q \sin \dot{\theta})^2 + (h \cos \theta \cdot \dot{\theta} + q \sin \theta \cdot \dot{\theta} - q \cos \dot{\theta})^2 \\
 &= [(-h \sin \theta \cdot \dot{\theta} + q \cos \theta \cdot \dot{\theta})^2 + 2(-h \sin \theta \cdot \dot{\theta} + q \cos \theta \cdot \dot{\theta}) q \sin \dot{\theta} + q^2 \sin^2 \dot{\theta}] + \dots \\
 &\quad \dots + [(h \cos \theta \cdot \dot{\theta} + q \sin \theta \cdot \dot{\theta})^2 - 2(h \cos \theta \cdot \dot{\theta} + q \sin \theta \cdot \dot{\theta}) q \cos \dot{\theta} + q^2 \cos^2 \dot{\theta}] \\
 \Rightarrow \dot{\vec{r}}^2 &= (-h \sin \theta \cdot \dot{\theta} + q \cos \theta \cdot \dot{\theta})^2 + 2(-h \sin \theta \cdot \dot{\theta} + q \cos \theta \cdot \dot{\theta}) q \sin \dot{\theta} + (h \cos \theta \cdot \dot{\theta} + q \sin \theta \cdot \dot{\theta})^2 - 2(h \cos \theta \cdot \dot{\theta} + q \sin \theta \cdot \dot{\theta}) q \cos \dot{\theta} + \dots \\
 &\quad \dots + q^2 \sin^2 \dot{\theta} + q^2 \cos^2 \dot{\theta} \\
 &= q^2 + 2q[(-h \sin \theta \cdot \dot{\theta} + q \cos \theta \cdot \dot{\theta}) \sin \dot{\theta} - (h \cos \theta \cdot \dot{\theta} + q \sin \theta \cdot \dot{\theta}) \cos \dot{\theta}] + (-h \sin \theta \cdot \dot{\theta} + q \cos \theta \cdot \dot{\theta})^2 + (h \cos \theta \cdot \dot{\theta} + q \sin \theta \cdot \dot{\theta})^2 \\
 &= q^2 + 2q[-h \sin^2 \theta \cdot \dot{\theta} + q \cos \theta \sin \theta \cdot \dot{\theta} - h \cos^2 \theta \cdot \dot{\theta} - q \sin \theta \cos \theta \cdot \dot{\theta}] + (-h \sin \theta \cdot \dot{\theta} + q \cos \theta \cdot \dot{\theta})^2 + (h \cos \theta \cdot \dot{\theta} + q \sin \theta \cdot \dot{\theta})^2 \\
 &= q^2 + 2q[-h \dot{\theta}] + h^2 \sin^2 \theta \cdot \dot{\theta}^2 - 2qh \sin \theta \cos \theta \cdot \dot{\theta}^2 + q^2 \cos^2 \theta \cdot \dot{\theta}^2 + h^2 \cos^2 \theta \cdot \dot{\theta}^2 + 2qh \sin \theta \cos \theta \cdot \dot{\theta}^2 + q^2 \sin^2 \theta \cdot \dot{\theta}^2 \\
 \Rightarrow \dot{\vec{r}}^2 &= q^2 - 2hq\dot{\theta} + q^2 \dot{\theta}^2 + h^2 \dot{\theta}^2
 \end{aligned}$$

• La energía cinética

$$\Rightarrow T = \frac{1}{2} m \dot{\vec{r}}^2 = \frac{1}{2} m (q^2 + q^2 \dot{\theta}^2 + h^2 \dot{\theta}^2 - 2hq\dot{\theta})$$

• La energía potencial

$$V = mg \vec{r} \cdot \hat{j} = mg(h \sin \theta - q \cos \theta)$$

Continua.....

- El lagrangiano del sistema

$$L = T - V = \frac{1}{2}m(q^2 + q^2\dot{\theta}^2 + h^2\dot{\theta}^2) - mg(h\cos\theta - q\cos\theta)$$

- Ahora, el Hamiltoniano del sistema

$$H = \sum_i \dot{q}_i p_i - L = \dot{q}p_1 + \dot{\theta}p_\theta - \frac{1}{2}m(q^2 + q^2\dot{\theta}^2 + h^2\dot{\theta}^2) - mg(h\cos\theta - q\cos\theta); \text{ con } p_i = \frac{\partial L}{\partial \dot{q}_i}$$

$$* p_1 = \frac{\partial H}{\partial \dot{q}} = \frac{1}{2}m(2\dot{q} - 2h\dot{\theta}) = m(\dot{q} - h\dot{\theta})$$

De estas ecuaciones hay que despejar para \dot{q} y $\dot{\theta}$

$$* p_\theta = \frac{\partial H}{\partial \dot{\theta}} = \frac{1}{2}m(2q^2\dot{\theta} + 2h^2\dot{\theta} - 2h\dot{q}) = m(q^2\dot{\theta} + h^2\dot{\theta} - h\dot{q})$$

$$\left. \begin{array}{l} p_1 = mq - mh\dot{\theta} \\ p_\theta = m(q^2 + h^2)\dot{\theta} - mh\dot{q} \end{array} \right\} \Rightarrow \left(\begin{array}{cc|c} \dot{q} & \dot{\theta} \\ m & -mh & p_1 \\ -mh & m(q^2 + h^2) & p_\theta \end{array} \right) \xrightarrow{\dot{f}_1 \rightarrow \frac{1}{m}\dot{f}_1} \left(\begin{array}{cc|c} 1 & -h & \frac{p_1}{m} \\ -mh & m(q^2 + h^2) & p_\theta \end{array} \right)$$

$$\xrightarrow{\dot{f}_2 \rightarrow \dot{f}_2 + mh\dot{f}_1} \left(\begin{array}{cc|c} 1 & -h & \frac{p_1}{m} \\ 0 & m(q^2 + h^2) - mh^2 & p_\theta + p_1h \end{array} \right) \rightarrow \left(\begin{array}{cc|c} 1 & -h & \frac{p_1}{m} \\ 0 & mq^2 & p_\theta + p_1h \end{array} \right) \xrightarrow{\dot{f}_2 \rightarrow \frac{1}{mq^2}\dot{f}_2} \left(\begin{array}{cc|c} 1 & -h & \frac{p_1}{m} \\ 0 & 1 & \frac{p_\theta + p_1h}{mq^2} \end{array} \right)$$

$$\xrightarrow{\dot{f}_1 \rightarrow \dot{f}_1 + h\dot{f}_2} \left(\begin{array}{cc|c} 1 & 0 & \frac{p_1}{m} + \frac{p_\theta h + p_1h^2}{mq^2} \\ 0 & 1 & \frac{p_\theta + p_1h}{mq^2} \end{array} \right) \Rightarrow \dot{q} = \frac{p_1}{m} + \frac{p_\theta h + p_1h^2}{mq^2} \quad y \quad \dot{\theta} = \frac{p_\theta + p_1h}{mq^2}$$

$$\Rightarrow H = p_1 \left(\frac{p_1}{m} + \frac{p_\theta h + p_1h^2}{mq^2} \right) + p_\theta \left(\frac{p_\theta + p_1h}{mq^2} \right) - \frac{1}{2}m \left[\left(\frac{p_1}{m} + \frac{p_\theta h + p_1h^2}{mq^2} \right)^2 + (q^2 + h^2) \left(\frac{p_\theta + p_1h}{mq^2} \right)^2 + \dots \right. \\ \left. \dots - 2h \left(\frac{p_1}{m} + \frac{p_\theta h + p_1h^2}{mq^2} \right) \left(\frac{p_\theta + p_1h}{mq^2} \right) \right] + V$$

$$= \frac{p_1}{mq^2} (q^2 p_1 + p_\theta h + p_1 h^2) + \frac{p_\theta}{mq^2} (p_\theta + p_1 h) - \frac{1}{2mq^4} [(q^2 p_1 + p_\theta h + p_1 h^2)^2 + (q^2 + h^2)(p_\theta + p_1 h)^2 + \dots]$$

$$\dots - 2h (q^2 p_1 + p_\theta h + p_1 h^2) (p_\theta + p_1 h)] + V$$

Trabajando el Hamiltoniano

$$\Rightarrow \mathcal{H} = \frac{1}{mq^2} \left\{ p_q (q^2 p_q + p_\theta h + p_\phi h^2) + p_\theta (p_\theta + p_\phi h) - \frac{1}{2q^2} [(q^2 p_q + p_\theta h + p_\phi h^2)^2 + (q^2 + h^2)(p_\theta + p_\phi h)^2 + \dots - 2h(q^2 p_q + p_\theta h + p_\phi h^2)(p_\theta + p_\phi h)] \right\} + V$$

$$\Rightarrow \mathcal{H} - V = \frac{1}{mq^2} \left(p_q^2 (q^2 + h^2) + 2p_q p_\theta h + p_\theta^2 - \frac{1}{2q^2} \{ [(q^2 + h^2)p_q + p_\theta h]^2 + (q^2 + h^2)(p_\theta + p_\phi h)^2 - 2h[(q^2 + h^2)p_q + p_\theta h](p_\theta + p_\phi h) \} \right)$$

$$\Rightarrow \mathcal{H} - V = \frac{1}{mq^2} \left(p_q^2 (q^2 + h^2) + 2p_q p_\theta h + p_\theta^2 - \frac{1}{2q^2} \{ [(q^2 + h^2)p_q + p_\theta h]^2 + (q^2 + h^2)(p_\theta + p_\phi h)^2 - 2h[(q^2 + h^2)p_q + p_\theta h](p_\theta + p_\phi h) \} \right)$$

$$\begin{aligned} & \{(q^2 + h^2)p_q^2 + 2h(q^2 + h^2)p_q p_\theta + p_\theta^2 h^2 + (q^2 + h^2)(p_\theta^2 + 2h p_\theta p_q + p_q^2 h^2) - 2h[(q^2 + h^2)p_q p_\theta + (q^2 + h^2)p_q^2 h + p_\theta^2 h + p_\theta p_q h^2]\} \\ &= \{(q^2 + h^2)p_q^2 + (q^2 + h^2)(p_\theta^2 + 2h p_\theta p_q + p_q^2 h^2) - 2h^2(q^2 + h^2)p_q^2 - p_\theta^2 h^2 - 2p_\theta p_q h^3\} \\ &= \{(q^2 + h^2)p_q^2 + (q^2 + h^2)p_\theta^2 + 2h(q^2 + h^2)p_\theta p_q + (q^2 + h^2)h p_q^2 - 2h^2(q^2 + h^2)p_q^2 - p_\theta^2 h^2 - 2p_\theta p_q h^3\} \\ &= \{p_q^2 [(q^2 + h^2)^2 + (q^2 + h^2)h - h^2(q^2 + h^2)] + q^2 p_\theta^2 + 2h(q^2 + h^2)p_\theta p_q - 2p_\theta p_q h^3\} \\ &= \{p_q^2 (q^4 + q^2 h^2 - h^4) + q^2 p_\theta^2 + 2h p_\theta p_q q^2\} \end{aligned}$$

$$\Rightarrow \mathcal{H} = \frac{1}{mq^2} \left(p_q^2 (q^2 + h^2) + 2p_q p_\theta h + p_\theta^2 - \frac{1}{2q^2} \{ p_q^2 (q^4 + q^2 h^2 - h^4) + q^2 p_\theta^2 + 2h p_\theta p_q q^2 \} \right) + V$$

$$= \frac{1}{mq^2} \left(p_q^2 (q^2 + h^2) + \cancel{\frac{1}{2}} p_q p_\theta h + \cancel{\frac{1}{2}} p_\theta^2 - \frac{p_q^2}{2} (q^2 + h^2 - \cancel{q^2}) - \cancel{\frac{p_\theta^2}{2}} - \cancel{h p_\theta p_q} \right) + V$$

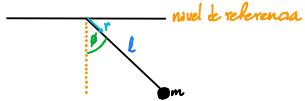
$$= \frac{1}{mq^2} \left(\frac{p_q^2}{2} (q^2 + h^2) + \frac{p_\theta^2 h^2}{2} + \frac{p_\theta^2}{2} + p_q p_\theta h \right) + V$$

$$\Rightarrow \mathcal{H} = \frac{p_q^2 (q^2 + h^2)}{2mq^2} + \frac{p_\theta^2 h^2}{2mq^2} + \frac{p_\theta^2}{2} + p_q p_\theta h + mg(h \cos \theta - q \cos \theta)$$

Problema 2. (Thornton 7.24)

- 7-24. Consider a simple plane pendulum consisting of a mass m attached to a string of length l . After the pendulum is set into motion, the length of the string is shortened at a constant rate

$$\frac{dl}{dt} = -\alpha = \text{constant} \quad \text{No holonómica y "rheonomicas"}$$



The suspension point remains fixed. Compute the Lagrangian and Hamiltonian functions. Compare the Hamiltonian and the total energy, and discuss the conservation of energy for the system.

En un inicio, la restricción es no holonómica y, además, "rheonomicas": $f: \dot{r} + \alpha = 0 \Rightarrow f: r - l_0 - \alpha t = 0 \leftarrow$ Ahora, la restricción es holonómica

- La energía cinética: $T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$
- La energía potencial: $V = -mg r \cos \theta$
- El Lagrangiano: $\Rightarrow L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + mg r \cos \theta \Rightarrow L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m(l_0 - \alpha t)^2\dot{\theta}^2 + mg(l_0 - \alpha t)\cos \theta$
- El $\mathcal{H} = \sum_i \dot{q}_i p_i - L = \dot{\theta} p_\theta - \frac{1}{2}m\dot{r}^2 - \frac{1}{2}m(l_0 - \alpha t)^2\dot{\theta}^2 - mg(l_0 - \alpha t)\cos \theta$; Hay que averiguar que es p_θ

$$* p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m(l_0 - \alpha t)\dot{\theta} \Rightarrow \dot{\theta} = \frac{p_\theta}{m(l_0 - \alpha t)}$$

$$\Rightarrow \mathcal{H} = \frac{\frac{p_\theta^2}{m(l_0 - \alpha t)^2} - \frac{1}{2}m\dot{r}^2 - \frac{1}{2}m(l_0 - \alpha t)^2\dot{\theta}^2}{m(l_0 - \alpha t)} - mg(l_0 - \alpha t)\cos \theta = \frac{1}{2}\frac{p_\theta^2}{m(l_0 - \alpha t)^2} - \frac{1}{2}m\dot{r}^2 - mg(l_0 - \alpha t)\cos \theta$$

$$\Rightarrow \mathcal{H} = \frac{1}{2}\frac{p_\theta^2}{m(l_0 - \alpha t)^2} - \frac{1}{2}m\dot{r}^2 - mg(l_0 - \alpha t)\cos \theta \quad \text{Desde aquí se nota la diferencia entre } \mathcal{H} \text{ y } E$$

- La energía total: $E = T + V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - mg r \cos \theta = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m(l_0 - \alpha t)^2\dot{\theta}^2 - mg(l_0 - \alpha t)\cos \theta$

$$\Rightarrow E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m(l_0 - \alpha t)^2\dot{\theta}^2 - mg(l_0 - \alpha t)\cos \theta$$

Ecuaciones de Hamilton

Problema 1. (Taylor 13.3)

13.3* Consider the Atwood machine of Figure 13.2, but suppose that the pulley is a uniform disc of mass M and radius R . Using x as your generalized coordinate, write down the Lagrangian, the generalized momentum p , and the Hamiltonian $\mathcal{H} = p\dot{x} - \mathcal{L}$. Find Hamilton's equations and use them to find the acceleration \ddot{x} .

- La cuerda que conecta las masas limita la posición de estos: $-X - Y - \pi R = L$

$$\Rightarrow -\dot{X} - \dot{Y} = 0 \Rightarrow \dot{X} = -\dot{Y}$$

- Restituyendo el giro de la polea: $X - R\dot{\theta} = 0 \Rightarrow \dot{X} = R\dot{\theta}$

- Entonces, la energía cinética y potencial

$$* \quad T = \frac{1}{2}m_1\dot{X}^2 + \frac{1}{2}m_2\dot{Y}^2 + \frac{1}{2}I\dot{\theta}^2 = \frac{1}{2}\left(m_1 + m_2 + \frac{MR^2}{R^2}\right)\dot{X}^2$$

$$* \quad V = m_1gX + m_2gY = m_1gX + m_2g(C - X) = (m_1 - m_2)gX + m_2gC \rightarrow V = (m_1 - m_2)gX$$

$$* \quad \mathcal{L} = \frac{1}{2}\left(m_1 + m_2 + \frac{M}{2}\right)\dot{X}^2 - (m_1 - m_2)gX$$

$$\Rightarrow p = \frac{\partial \mathcal{L}}{\partial \dot{X}} = \left(m_1 + m_2 + \frac{M}{2}\right)\dot{X} = \frac{(2m_1 + 2m_2 + M)\dot{X}}{2} \Rightarrow \dot{X} = \frac{2p}{(2m_1 + 2m_2 + M)}$$

$$* \quad \mathcal{H} = \dot{X}p - \mathcal{L} = \frac{2p^2}{(2m_1 + 2m_2 + M)} - \cancel{\frac{1}{2}(2m_1 + 2m_2 + M)} \cancel{\frac{1}{2}p^2} + (m_1 - m_2)gX = \frac{p^2}{(2m_1 + 2m_2 + M)} + (m_1 - m_2)gX$$

$$\Rightarrow \mathcal{H} = \frac{p^2}{(2m_1 + 2m_2 + M)} + (m_1 - m_2)gX //$$

- Utilizando las ecuaciones canónicas: $\dot{q} = \frac{\partial \mathcal{H}}{\partial p}$ y $\dot{p} = -\frac{\partial \mathcal{H}}{\partial q}$

$$\Rightarrow \begin{cases} \dot{X} = \frac{2p}{(2m_1 + 2m_2 + M)} \\ \dot{p} = -(m_1 - m_2)g \end{cases} \Rightarrow \dot{X} = \frac{-2(m_1 - m_2)gt}{(2m_1 + 2m_2 + M)} \Rightarrow \ddot{X} = \frac{-2(m_1 - m_2)g}{(2m_1 + 2m_2 + M)} //$$

$$\Rightarrow \begin{pmatrix} \dot{X} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0 & \frac{2}{(2m_1 + 2m_2 + M)} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} X \\ p \end{pmatrix} + \begin{pmatrix} 0 \\ -(m_1 - m_2)g \end{pmatrix}$$

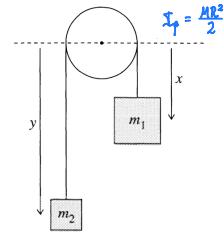


Figure 13.2 An Atwood machine consisting of two masses, m_1 and m_2 , suspended by a massless inextensible string that passes over a massless, frictionless pulley. Because the string's length is fixed, the position of the whole system is specified by the distance x of m_1 below any convenient fixed level.

Problema 2. (Taylor 13.4)

13.4★ The Hamiltonian \mathcal{H} is always given by $\mathcal{H} = pq - \mathcal{L}$ (in one dimension), and this is the form you should use if in doubt. However, if your generalized coordinate q is “natural” (relation between q and the underlying Cartesian coordinates is independent of time) then $\mathcal{H} = T + U$, and this form is almost always easier to write down. Therefore, in solving any problem you should quickly check to see if the generalized coordinate is “natural,” and if it is you can use the simpler form $\mathcal{H} = T + U$. For the Atwood machine of Example 13.2 (page 527), check that the generalized coordinate was “natural.” [Hint: There are one generalized coordinate x and two underlying Cartesian coordinates x and y . You have only to write equations for the two Cartesians in terms of the one generalized coordinate and check that they don’t involve the time, so it’s safe to use $\mathcal{H} = T + U$. This is ridiculously easy!]

- La cuerda que conecta las masas limita la posición de estas: $-X - Y - \pi R = L$

$$\Rightarrow -\dot{X} - \dot{Y} = 0 \Rightarrow \dot{X} = -\dot{Y}$$

- Restando el giro de la polea: $X - R\dot{\theta} = 0 \Rightarrow \dot{X} = R\dot{\theta}$

- Entonces, la energía cinética y potencial

$$* \quad T = \frac{1}{2}m_1\dot{X}^2 + \frac{1}{2}m_2\dot{Y}^2 + \frac{1}{2}I_p\dot{\theta}^2 = \frac{1}{2}\left(m_1 + m_2 + \frac{MR^2}{2}\right)\dot{X}^2$$

$$* \quad V = m_1gX + m_2gY = m_1gX + m_2g(C-X) = (m_1 - m_2)gX + m_2gC \rightarrow V = (m_1 - m_2)gX$$

$$* \quad L = \frac{1}{2}\left(m_1 + m_2 + \frac{M}{2}\right)\dot{X}^2 - (m_1 - m_2)gX \quad \Rightarrow \rho = \frac{\partial L}{\partial \dot{X}} = \frac{(2m_1 + 2m_2 + M)}{2}\dot{X} \Rightarrow \dot{X} = \frac{2\rho}{(2m_1 + 2m_2 + M)}$$

$$* \quad \mathcal{H}^* = T - V = \frac{1}{4}(2m_1 + 2m_2 + M)\dot{X}^2 + (m_1 - m_2)gX = \frac{1}{4}(2m_1 + 2m_2 + M) \frac{\cancel{4\rho^2}}{\cancel{(2m_1 + 2m_2 + M)^2}} + (m_1 - m_2)gX$$

$$\Rightarrow \mathcal{H} = \frac{\rho^2}{(2m_1 + 2m_2 + M)} + (m_1 - m_2)gX //$$

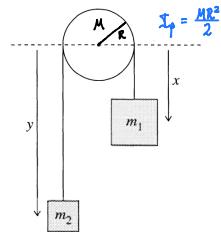


Figure 13.2 An Atwood machine consisting of two masses, m_1 and m_2 , suspended by a massless inextensible string that passes over a massless, frictionless pulley. Because the string’s length is fixed, the position of the whole system is specified by the distance x of m_1 below any convenient fixed level.

Problema 3.

Una partícula de masa m se mueve en un campo gravitacional a lo largo de una espiral $z = k\theta$, r : cte, donde k es una constante.

- Encuentre el Hamiltoniano $\mathcal{H}(z, p)$ para la partícula en movimiento.
- Determine las ecuaciones de movimiento.
- Resuelva estas ecuaciones.

$$\text{Restricción: } f: z - k\theta = 0 \leftarrow \text{Restricción holonómica}$$

- Energía cinética y potencial : $T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) = \frac{1}{2}mr^2\dot{\theta}^2$ y $V = mgz = mgk\theta$ No hay ninguna dependencia explícita con t en el sistema

$$\Rightarrow \mathcal{H}^* = T + V = \frac{1}{2}mr^2\dot{\theta}^2 + mgk\theta \rightarrow p_\theta = \frac{\partial \mathcal{H}^*}{\partial \dot{\theta}} = mr^2\dot{\theta} \rightarrow = \frac{1}{2}mr^2 \frac{p_\theta^2}{m^2r^4} + mgk\theta = \frac{p_\theta^2}{2mr^2} + mgk\theta$$

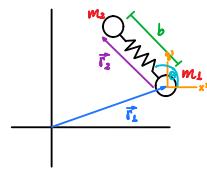
$$\Rightarrow \mathcal{H} = \frac{p_\theta^2}{2mr^2} + mgk\theta //$$

- Buscando las ecuaciones del movimiento : $\dot{q} = \frac{\partial \mathcal{H}}{\partial p}$ y $\dot{p} = -\frac{\partial \mathcal{H}}{\partial q}$

$$\Rightarrow \begin{cases} \dot{\theta} = \frac{p_\theta}{mr^2} \\ \dot{p}_\theta = -mgk \end{cases} \Rightarrow \dot{\theta} = \frac{1}{mr^2} \cdot -mgk t = -\frac{gt}{r^2} \Rightarrow \theta = -\frac{1}{2} \frac{gt^2}{r^2} \Rightarrow z = k\theta = -\frac{1}{2} gt^2 //$$

Problema 4. (Thornton 7.27)

- 7.27. A massless spring of length b and spring constant k connects two particles of masses m_1 and m_2 . The system rests on a smooth table and may oscillate and rotate.



- (a) Determine Lagrange's equations of motion.
- (b) What are the generalized momenta associated with any cyclic coordinates?
- (c) Determine Hamilton's equations of motion.

- Posición de la masa 1 : $\vec{r}_1 = X\hat{i} + Y\hat{j} \Rightarrow \dot{\vec{r}}_1 = \dot{X}\hat{i} + \dot{Y}\hat{j}$
- Posición de la masa 2 : $\vec{r}_2 = \vec{r}_1 + p\hat{e}_p = X\hat{i} + Y\hat{j} + p\cos\theta\hat{i} + p\sin\theta\hat{j}$
 $\Rightarrow \dot{\vec{r}}_2 = (\dot{X} + p\cos\theta - p\sin\theta\cdot\dot{\theta})\hat{i} + (\dot{Y} + p\sin\theta + p\cos\theta\cdot\dot{\theta})\hat{j}$

- Rapideces cuadradas :

$$*\dot{r}_1^2 = \dot{X}^2 + \dot{Y}^2$$

$$\begin{aligned} *\dot{r}_2^2 &= (\dot{X} + p\cos\theta - p\sin\theta\cdot\dot{\theta})^2 + (\dot{Y} + p\sin\theta + p\cos\theta\cdot\dot{\theta})^2 \\ &= (\dot{X}^2 + p^2\cos^2\theta + p^2\sin^2\theta\cdot\dot{\theta}^2 + 2\dot{X}\dot{p}\cos\theta - 2\dot{X}\dot{p}\sin\theta\cdot\dot{\theta} - 2\dot{p}\cos\theta\sin\theta\cdot\dot{\theta}) + (\dot{Y}^2 + p^2\sin^2\theta + p^2\cos^2\theta\cdot\dot{\theta}^2 + 2\dot{Y}\dot{p}\sin\theta + 2\dot{Y}\dot{p}\cos\theta\cdot\dot{\theta} + 2\dot{p}\cos\theta\sin\theta\cdot\dot{\theta}) \\ &= \dot{X}^2 + \dot{Y}^2 + p^2\dot{\theta}^2 + 2\dot{X}\dot{p}\cos\theta - 2\dot{X}\dot{p}\sin\theta\cdot\dot{\theta} + 2\dot{Y}\dot{p}\sin\theta + 2\dot{Y}\dot{p}\cos\theta\cdot\dot{\theta} \end{aligned}$$

- Energía cinética: $T = \frac{1}{2}m_1(\dot{X}^2 + \dot{Y}^2) + \frac{1}{2}m_2(\dot{X}^2 + \dot{Y}^2 + p^2 + p^2\dot{\theta}^2 + 2\dot{X}\dot{p}\cos\theta - 2\dot{X}\dot{p}\sin\theta\cdot\dot{\theta} + 2\dot{Y}\dot{p}\sin\theta + 2\dot{Y}\dot{p}\cos\theta\cdot\dot{\theta})$

$$\Rightarrow T = \frac{1}{2}(m_1 + m_2)(\dot{X}^2 + \dot{Y}^2) + \frac{1}{2}m_2(p^2 + p^2\dot{\theta}^2) + m_2\dot{p}(\dot{X}\cos\theta + \dot{Y}\sin\theta) + m_2\dot{\theta}(\dot{Y}\cos\theta - \dot{X}\sin\theta)$$

- Energía potencial : $V = \frac{1}{2}k(p - b)^2$

- El Lagrangiano del sistema :

$$L = \frac{1}{2}(m_1 + m_2)(\dot{X}^2 + \dot{Y}^2) + \frac{1}{2}m_2(p^2 + p^2\dot{\theta}^2) + m_2\dot{p}(\dot{X}\cos\theta + \dot{Y}\sin\theta) + m_2\dot{\theta}(\dot{Y}\cos\theta - \dot{X}\sin\theta) - \frac{1}{2}k(p - b)^2$$

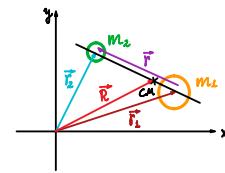
- El Hamiltoniano del sistema : $H =$

$$H = \frac{1}{2}(m_1 + m_2)(\dot{X}^2 + \dot{Y}^2) + \frac{1}{2}m_2(p^2 + p^2\dot{\theta}^2) + m_2\dot{p}(\dot{X}\cos\theta + \dot{Y}\sin\theta) + m_2\dot{\theta}(\dot{Y}\cos\theta - \dot{X}\sin\theta) + \frac{1}{2}k(p - b)^2$$

Problema 5.

Obtenga el Hamiltoniano para un planeta en órbita alrededor del Sol en coordenadas polares planas.
Determine las ecuaciones de movimiento de Hamilton.

- Buscando el centro de masa del sistema : $\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$



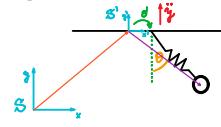
- Un vector que une ambas masas : $\vec{r} = \vec{r}_2 - \vec{r}_1$
- Colocando el origen en el centro de masa del sistema: $\vec{R} = 0 \Rightarrow m_1 \vec{r}_1 + m_2 \vec{r}_2 = 0$
- Usando las ecuaciones anteriores para hallar \vec{r}_1 y \vec{r}_2 : $\vec{r}_2 = \frac{m_1}{m_1 + m_2} \vec{r}$ y $\vec{r}_1 = \frac{-m_2}{m_1 + m_2} \vec{r}$
- El Lagrangiano del sistema: $L = \frac{1}{2} m_1 \dot{r}_1^2 + \frac{1}{2} m_2 \dot{r}_2^2 - V(r) = \frac{1}{2} m_1 \left(\frac{-m_2}{m_1 + m_2} \dot{r} \right)^2 + \frac{1}{2} m_2 \left(\frac{m_1}{m_1 + m_2} \dot{r} \right)^2 - V(r)$
 $\Rightarrow L = \frac{1}{2} m_1 \frac{m_2^2}{(m_1 + m_2)^2} \dot{r}^2 + \frac{1}{2} m_2 \frac{m_1^2}{(m_1 + m_2)^2} \dot{r}^2 - V(r) = \frac{1}{2} \frac{\dot{r}^2}{(m_1 + m_2)^2} (m_1 m_2^2 + m_2 m_1^2) - V(r)$
 $= \frac{1}{2} \frac{m_1 m_2}{(m_1 + m_2)} (\cancel{m_2^2 + m_1^2}) \dot{r}^2 - V(r) = \frac{1}{2} \frac{m_1 m_2}{(m_1 + m_2)} \dot{r}^2 - V(r) = \frac{1}{2} \mu \dot{r}^2 - V(r) ; \text{ con } \mu = \frac{m_1 m_2}{(m_1 + m_2)}$
 $\Rightarrow L = \frac{1}{2} \mu (\dot{r}^2 + r^2 \dot{\theta}^2) - V(r) //$
- Construyendo el Hamiltoniano del sistema, recordando que la energía del sistema se conserva
 $\Rightarrow H^* = T + V = \frac{1}{2} \mu (\dot{r}^2 + r^2 \dot{\theta}^2) + V(r) \quad \text{con} \quad p_r = \frac{\partial L}{\partial \dot{r}} = \mu \dot{r} \quad \text{y} \quad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \mu r^2 \dot{\theta}$
 $\Rightarrow H = \frac{1}{2} \mu \left(\frac{p_r^2}{\mu^2} + r^2 \frac{p_\theta^2}{\mu^2 r^4} \right) + V(r) \Rightarrow H = \frac{p_r^2}{2\mu} + \frac{p_\theta^2}{2\mu r^2} + V(r) //$

Problema 6. (Thornton 7.29)

7-15. A pendulum consists of a mass m suspended by a massless spring with unextended length b and spring constant k . Find Lagrange's equations of motion.

- 7-29. Consider the pendulum described in Problem 7-15. The pendulum's point of support rises vertically with constant acceleration a .

- (a) Use the Lagrangian method to find the equations of motion.
- (b) Determine the Hamiltonian and Hamilton's equations of motion.
- (c) What is the period of small oscillations?



- Vector position: $\vec{r} = r \cos \theta \hat{e}_x + \left(\frac{1}{2} a t^2 - r \cos \theta \right) \hat{e}_y \Rightarrow \dot{\vec{r}} = (r \sin \theta + r \cos \theta \cdot \dot{\theta}) \hat{e}_x + (a t - r \cos \theta + r \sin \theta \cdot \dot{\theta}) \hat{e}_y$
- $\Rightarrow \dot{r}^2 = (r \sin \theta + r \cos \theta \cdot \dot{\theta})^2 + (a t - r \cos \theta + r \sin \theta \cdot \dot{\theta})^2$
- $= (r^2 \sin^2 \theta + 2r \dot{\theta} \sin \theta \cos \theta + r^2 \cos^2 \theta \cdot \dot{\theta}^2) + (a^2 t^2 + r^2 \cos^2 \theta + r^2 \sin^2 \theta \cdot \dot{\theta}^2 - 2a t r \cos \theta + 2a t r \sin \theta \cdot \dot{\theta} - 2r \dot{\theta} \cos \theta \sin \theta)$
- $= (r^2 \sin^2 \theta + r^2 \cos^2 \theta) + (r^2 \cos^2 \theta \cdot \dot{\theta}^2 + r^2 \sin^2 \theta \cdot \dot{\theta}^2) + (a^2 t^2 - 2a t r \cos \theta + 2a t r \sin \theta \cdot \dot{\theta})$
- $= \dot{r}^2 + r^2 \dot{\theta}^2 + (a^2 t^2 - 2a t r \cos \theta + 2a t r \sin \theta \cdot \dot{\theta}) = \dot{r}^2 + r^2 \dot{\theta}^2 + a^2 t^2 + 2a t (r \sin \theta \cdot \dot{\theta} - r \cos \theta)$
- Energía cinética: $T = \frac{1}{2} m [\dot{r}^2 + r^2 \dot{\theta}^2 + a^2 t^2 + 2a t (r \sin \theta \cdot \dot{\theta} - r \cos \theta)]$
- Energía potencial: $V = mg \vec{r}_g + \frac{1}{2} k (r - b)^2 = mg \left(\frac{1}{2} a t^2 - r \cos \theta \right) + \frac{1}{2} k (r - b)^2$
- El Lagrangiano: $L = T - V = \frac{1}{2} m [\dot{r}^2 + r^2 \dot{\theta}^2 + a^2 t^2 + 2a t (r \sin \theta \cdot \dot{\theta} - r \cos \theta)] - mg \left(\frac{1}{2} a t^2 - r \cos \theta \right) - \frac{1}{2} k (r - b)^2$

a) Hay que operar: $\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_1} - \frac{\partial L}{\partial q_1} = 0$ para "r" y "θ"

$$[r]: \frac{d}{dt} \frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0$$

$$* \frac{\partial L}{\partial r} = \frac{1}{2} m [2r\dot{\theta}^2 + 2a t r \sin \theta \cdot \dot{\theta}] + mg \cos \theta - k(r - b) = m(r\dot{\theta}^2 + a t r \sin \theta \cdot \dot{\theta}) + mg \cos \theta - k(r - b)$$

$$* \frac{\partial L}{\partial \dot{r}} = \frac{1}{2} m [2\ddot{r} - 2a t \cos \theta] = m(\ddot{r} - a t \cos \theta) \Rightarrow \frac{d}{dt} \frac{\partial L}{\partial \dot{r}} = m(\ddot{r} - a t \cos \theta + a t r \sin \theta \cdot \dot{\theta})$$

$$\Rightarrow m(\ddot{r} - a t \cos \theta + a t r \sin \theta \cdot \dot{\theta}) - m(r\dot{\theta}^2 + a t r \sin \theta \cdot \dot{\theta}) - mg \cos \theta + k(r - b) = 0 \Rightarrow \ddot{r} - r\dot{\theta}^2 - (a + g) \cos \theta + \frac{k}{m}(r - b) = 0$$

Continua...

$$[\theta] : \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

$$* \frac{\partial L}{\partial \theta} = \frac{1}{2} m [2at(\cos\theta \cdot \dot{\theta} + \sin\theta)] - mg\sin\theta = m\dot{\theta}(\cos\theta \cdot \dot{\theta} + \sin\theta) - mg\sin\theta$$

$$* \frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2} m [2r^2\ddot{\theta} + 2at\sin\theta] = m(r^2\ddot{\theta} + at\sin\theta) \Rightarrow \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = m(2r\dot{r}\dot{\theta} + r^2\ddot{\theta} + ar\sin\theta + at\cos\theta + at\sin\theta)$$

$$\Rightarrow m(2r\dot{r}\dot{\theta} + r^2\ddot{\theta} + ar\sin\theta + at\cos\theta) - m\dot{\theta}(\cos\theta \cdot \dot{\theta} + \sin\theta) + mg\sin\theta = 0$$

$$\Rightarrow 2r\dot{r}\dot{\theta} + r^2\ddot{\theta} + ar\sin\theta + gr\sin\theta = 0 \Rightarrow r\ddot{\theta} + 2\dot{r}\dot{\theta} + (a + g)\sin\theta = 0 //$$

- Las ecuaciones del movimiento: $\Rightarrow \begin{cases} \ddot{r} - r\dot{\theta}^2 - (a + g)\cos\theta + \frac{k}{m}(r - b) = 0 \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} + (a + g)\sin\theta = 0 \end{cases} //$

• Generando los momentos canónicos desde el Lagrangiano:

$$L = \frac{1}{2} m [r^2 + r^2\dot{\theta}^2 + a^2t^2 + 2at(\sin\theta \cdot \dot{\theta} - \cos\theta)] - mg\left(\frac{1}{2}at^2 - r\cos\theta\right) - \frac{1}{2}k(r - b)^2$$

$$* p_r = \frac{\partial L}{\partial \dot{r}} = m(\dot{r} - at\cos\theta) \Rightarrow \dot{r} = \frac{p_r + mat\cos\theta}{m} = \frac{p_r}{m} + at\cos\theta$$

$$* p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m(r^2\dot{\theta} + at\sin\theta) \Rightarrow \dot{\theta} = \frac{p_\theta - mat\sin\theta}{mr^2}$$

• Formando el Hamiltoniano del sistema: $H = \sum_i q_i p_i - L = \dot{r}p_r + \dot{\theta}p_\theta - L$

$$\Rightarrow H = \dot{r}p_r + \dot{\theta}p_\theta - \frac{1}{2}m[r^2 + r^2\dot{\theta}^2 + a^2t^2 + 2at(\sin\theta \cdot \dot{\theta} - \cos\theta)] + mg\left(\frac{1}{2}at^2 - r\cos\theta\right) + \frac{1}{2}k(r - b)^2$$

$$* \dot{r}^2 = \left(\frac{p_r}{m} + at\cos\theta\right)^2 = \frac{p_r^2}{m^2} + 2\frac{p_r}{m}at\cos\theta + a^2t^2\cos^2\theta * \dot{\theta}^2 = \left(\frac{p_\theta}{mr^2} - mat\sin\theta\right)^2 = \frac{p_\theta^2}{mr^4} - 2\frac{p_\theta}{mr^2}mat\sin\theta + m^2a^2t^2\sin^2\theta$$

$$* (\sin\theta \cdot \dot{\theta} - \cos\theta) = \sin\theta \cdot \frac{p_\theta - mat\sin\theta}{mr^2} - \frac{p_r + mat\cos\theta}{m} \cdot \cos\theta = \frac{p_\theta \sin\theta - mat\sin^2\theta}{mr^2} - \frac{p_r \cos\theta + mat\cos^2\theta}{m}$$

$$= \frac{p_\theta \sin\theta}{mr^2} - \frac{mat\sin^2\theta}{mr^2} - \frac{p_r \cos\theta}{m} - \frac{mat\cos^2\theta}{m} = \frac{p_\theta \sin\theta}{mr^2} - \frac{p_r \cos\theta}{m} - at$$

Continua...

$$\Rightarrow \dot{H} = \dot{r} p_r + \dot{\theta} p_\theta - \frac{1}{2} m \left[\dot{r}^2 + r^2 \dot{\theta}^2 + a^2 t^2 + 2at(r \sin \theta \cdot \dot{\theta} - r \cos \theta) \right] + mg \left(\frac{1}{2} at^2 - r \cos \theta \right) + \frac{1}{2} k(r - b)^2$$

$$\bullet \quad \chi = \frac{p_r^2}{m^2} + 2 \frac{p_r a t \cos \theta}{m} + a^2 t^2 \cos^2 \theta + \cancel{r^2} \left(\frac{p_\theta^2 - 2p_\theta m a t r \sin \theta + m^2 a^2 t^2 r^2 \sin^2 \theta}{m^2 r^2} \right) + a^2 t^2 + 2at \left(\frac{p_\theta \sin \theta}{m r} - \frac{p_r \cos \theta}{m} - at \right)$$

$$= \frac{p_r^2}{m^2} + 2 \frac{p_r a t \cos \theta}{m} + a^2 t^2 \cos^2 \theta + \frac{p_\theta^2}{m^2 r^2} - \cancel{2p_\theta a t r \sin \theta} + a^2 t^2 \sin^2 \theta + a^2 t^2 + \cancel{2at p_\theta \sin \theta} - \cancel{2at p_r \cos \theta} - 2a^2 t^2$$

$$= \frac{p_r^2}{m^2} + \frac{p_\theta^2}{m^2 r^2} + a^2 t^2 \cos^2 \theta + a^2 t^2 \sin^2 \theta + a^2 t^2 - 2a^2 t^2 = \frac{p_r^2}{m^2} + \frac{p_\theta^2}{m^2 r^2} + \cancel{a^2 t^2} + \cancel{a^2 t^2} - 2a^2 t^2 = \frac{p_r^2}{m^2} + \frac{p_\theta^2}{m^2 r^2}$$

$$\Rightarrow H = \dot{r} p_r + \dot{\theta} p_\theta - \frac{1}{2} m \left[\frac{p_r^2}{m^2} + \frac{p_\theta^2}{m^2 r^2} \right] + mg \left(\frac{1}{2} at^2 - r \cos \theta \right) + \frac{1}{2} k(r - b)^2$$

$$= \frac{p_r + m a t \cos \theta}{m} \cdot p_r + \frac{p_\theta - m a t r \sin \theta}{m r^2} \cdot p_\theta - \frac{p_r^2}{2m} - \frac{p_\theta^2}{2m r^2} + mg \left(\frac{1}{2} at^2 - r \cos \theta \right) + \frac{1}{2} k(r - b)^2 \xrightarrow{\text{Ja aqui es un Hamiltoniano formal}}$$

$$= \frac{p_r^2}{m} + p_r a t \cos \theta + \frac{p_\theta^2}{m r^2} - p_\theta a t r \sin \theta - \frac{p_r^2}{2m} - \frac{p_\theta^2}{2m r^2} + mg \left(\frac{1}{2} at^2 - r \cos \theta \right) + \frac{1}{2} k(r - b)^2$$

$$= \frac{p_r^2}{2m} + \frac{p_\theta^2}{2m r^2} + p_r a t \cos \theta - \frac{p_\theta a t r \sin \theta}{r} + mg \left(\frac{1}{2} at^2 - r \cos \theta \right) + \frac{1}{2} k(r - b)^2$$

$$b) \quad H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2m r^2} + p_r a t \cos \theta - \frac{p_\theta a t r \sin \theta}{r} + mg \left(\frac{1}{2} at^2 - r \cos \theta \right) + \frac{1}{2} k(r - b)^2 //$$

• Ahora, hay que operar $\dot{q}_s = \frac{\partial H}{\partial p_s}$ y $\dot{p}_s = -\frac{\partial H}{\partial q_s}$ para "r" y "θ"

$$[r]: * \quad \dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m} + a t \cos \theta$$

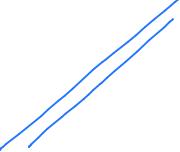
$$* \quad \dot{p}_r = -\frac{\partial H}{\partial r} = - \left[\frac{-p_\theta^2}{2m r^3} + \frac{p_\theta a t r \sin \theta}{r^2} - mg \cos \theta + \frac{1}{2} k(r - b) \right] = \frac{p_\theta^2}{m r^3} - \frac{p_\theta a t r \sin \theta}{r^2} + mg \cos \theta - \frac{1}{2} k(r - b)$$

$$[\theta]: * \quad \dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{m r^2} - \frac{a t \sin \theta}{r}$$

$$* \quad \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = - \left[-p_r a t \cos \theta - \frac{p_\theta a t r \cos \theta}{r} + mg r \sin \theta \right] = p_r a t \cos \theta + \frac{p_\theta a t r \cos \theta}{r} - mg r \sin \theta$$

Continua...

• Las ecuaciones del movimiento: \Rightarrow

$$\begin{cases} \ddot{r} = \frac{p_r}{m} + a \cos \theta \\ p_r = \frac{p_\theta^2}{mr^3} - \frac{p_r a \sin \theta}{r^2} + mg \cos \theta - k(r - b) \\ \dot{\theta} = \frac{p_\theta}{mr^2} - \frac{a \sin \theta}{r} \\ \ddot{\theta} = p_r a \sin \theta + p_\theta a \cos \theta - mg r \sin \theta \end{cases}$$


c) Recuperando las ecuaciones del movimiento de Lagrange

$$\Rightarrow \begin{cases} \ddot{r} - r\dot{\theta}^2 - (a+g)\cos\theta + \frac{k}{m}(r-b) = 0 \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} + (a+g)\sin\theta = 0 \end{cases}$$

Desplazamientos pequeños: $\begin{cases} \sin\theta \approx \theta \\ \cos \approx 1 - \frac{1}{2}\theta^2 \end{cases}$ Además: $\theta \ll 1 \Rightarrow \dot{\theta} \approx 0$

$$\Rightarrow \begin{cases} \ddot{r} - r\dot{\theta}^2 - (a+g)(1 - \frac{1}{2}\theta^2) + \frac{k}{m}(r-b) = 0 \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} + (a+g)\theta = 0 \end{cases} \Rightarrow \begin{cases} \ddot{r} - r\dot{\theta}^2 + \frac{1}{2}(a+g)\theta^2 + \frac{k}{m}r - (a+g) - \frac{k}{m}b = 0 \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} + (a+g)\theta = 0 \end{cases}$$

Eliminando los términos de segundo grado: $\Rightarrow \begin{cases} \ddot{r} + \frac{k}{m}r - (a+g + \frac{k}{m}b) = 0 \\ r\ddot{\theta} + (a+g)\theta = 0 \end{cases}$

$T = \frac{2\pi}{\omega}$

$$\Rightarrow T_o = 2\pi \sqrt{\frac{r}{a+g}} \quad y \quad T_r = 2\pi \sqrt{\frac{m}{k}}$$


Problema 7.

Obtenga el Hamiltoniano y las ecuaciones de Hamilton para un péndulo esférico.

- La energía cinética: $T = \frac{1}{2}m(r^2\dot{\theta}^2 + r^2\dot{\phi}^2 \sin^2\theta)$

la cuerda no se estira
r: constante

- La energía potencial: $V = mrg\cos\theta = mrg\cos(\pi - \theta) = -mrg\cos\theta$

- Construyendo el Lagrangiano y un Hamiltoniano provisional del sistema

- * $L = \frac{1}{2}m(r^2\dot{\theta}^2 + r^2\dot{\phi}^2 \sin^2\theta) + mrg\cos\theta$

- * $H = \frac{1}{2}m(r^2\dot{\theta}^2 + r^2\dot{\phi}^2 \sin^2\theta) - mrg\cos\theta$

- Los momentos canónicos: * $p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$ * $p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\sin^2\theta \cdot \dot{\phi}$

- El verdadero Hamiltoniano del sistema

$$\Rightarrow H = \frac{1}{2}\cancel{mr^2}\frac{p_\theta^2}{\cancel{m^2r^2}} + \frac{1}{2}\cancel{mr^2}\frac{\cancel{p_\phi^2}}{\cancel{m^2r^2}\sin^2\theta} - mrg\cos\theta$$

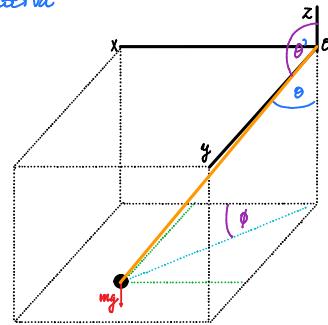
$$= \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2mr^2\sin^2\theta} - mrg\cos\theta \Rightarrow H = \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2mr^2\sin^2\theta} - mrg\cos\theta //$$

- Ecuaciones del movimiento:

- * $\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2} \quad y \quad \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -\left(\frac{-p_\theta^2}{mr^2\sin^3\theta} \cos\theta + mrg\sin\theta\right) \Rightarrow \dot{p}_\theta = \frac{p_\theta^2 \cos\theta}{mr^2\sin^3\theta} - mrg\sin\theta$

- * $\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mr^2\sin^2\theta} \quad y \quad \dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0$

la energía se conserva



Transformaciones Canónicas

SUBSECTION 9.5

Problemas resueltos

Como se puede ver, esta sección de problemas contiene una cantidad mínima de problemas y esto es debido a lo avanzado que puede ser este tema para un bachillerato. Por supuesto, se espera mejorar la sección y dar algo de luz respecto al tema.

Transformaciones Canónicas

Problema 1.

Determine si la función:

$$F(q, Q) = q^2 + Q^4 = F_1(q, Q)$$

genera una transformación canónica.

- Primera forma:

Para determinar si F es una función generatrix, F debe ser una función con diferencial exacto

$$\Rightarrow dF = \frac{\partial F}{\partial q} dq + \frac{\partial F}{\partial Q} dQ \Rightarrow \frac{\partial}{\partial Q} \left(\frac{\partial F}{\partial q} \right) = \frac{\partial}{\partial q} \left(\frac{\partial F}{\partial Q} \right) \Rightarrow 0 = 0 \quad \therefore \text{No es canónica}$$

$$*\frac{\partial F}{\partial q} = 2q$$

$$*\frac{\partial F}{\partial Q} = 4Q^3$$

- Segunda forma:

$$*P = \frac{\partial F}{\partial q} = 2q \quad *-P = \frac{\partial F}{\partial Q} = 4Q^3 \quad \Rightarrow [Q, P]_{q, p} = 1$$

Problema 2.

Considera la función generadora:

$$F_2(q, P) = (q + P)^2.$$

- a. Determine las ecuaciones de transformación.
- b. Encuentre las otras funciones generadoras.

a) Siendo la función generatrix de la forma $F_2 \Rightarrow p = \frac{\partial F_2}{\partial q} \text{ y } Q = \frac{\partial F_2}{\partial P}$

$$* p = \frac{\partial F_2}{\partial q} = 2(q+P) \quad * Q = \frac{\partial F_2}{\partial P} = 2(q+P) \quad \Rightarrow P = \frac{p}{2} - q \text{ y } Q = p$$

b) Para hallar otras funciones se realizan transformaciones de Legendre que deben cumplir con: $p\dot{q} - H = X^q - h + \frac{dF}{dt}$

$$* F_1(q, Q) = F_2(q, P) - QP$$

$$\begin{aligned} \Rightarrow F_1 &= (q+P)^2 - QP = q^2 + 2QP + P^2 - \left(\frac{p}{2} - q\right)Q = q^2 + 2q\left(\frac{p}{2} - q\right) + \left(\frac{p}{2} - q\right)^2 - \left(\frac{p}{2} - q\right)Q \\ &= q^2 + 2q\left(\frac{Q}{2} - q\right) + \left(\frac{Q}{2} - q\right)^2 - \left(\frac{Q}{2} - q\right)Q = q^2 + \left(\frac{Q}{2} - q\right)\left(q - \frac{Q}{2}\right) = qQ - \frac{Q^2}{4} \end{aligned}$$

$$\boxed{p = \frac{\partial F_1}{\partial q} = Q \text{ y } P = -\frac{\partial F_1}{\partial Q} = -q + \frac{Q}{2}}$$

$$* F_3(p, Q) = F_2(q, Q) \pm pq$$

$$\Rightarrow F_3 = qQ - \frac{Q^2}{4} \pm pq = qp - \frac{Q^2}{4} \pm pq = -\frac{Q^2}{4} = -\frac{pQ}{4}$$

Problema 3.

El Hamiltoniano de una partícula que se mueve verticalmente en un campo gravitacional uniforme \vec{g} está dado por:

$$\mathcal{H}(q, p) = \frac{p^2}{2m} + mgq$$

Considere la siguiente Transformación Canónica:

$$\begin{aligned} Q &= -p, \\ P &= q + Ap^2. \end{aligned}$$

donde A es constante.

- a. Determine $pdq - PdQ$ y demuestre que es un diferencial exacto.
- b. ¿Qué significa este resultado?
- c. Encuentre $\mathcal{H}(Q, P)$.
- d. Determine el valor de A , que haga Q una variable cíclica.
- e. Usando el resultado anterior, encuentre las ecuaciones de Hamilton para $\mathcal{H}(Q, P)$.
- f. A partir del resultado anterior, determine $q(t)$ y $p(t)$.

$$a) \begin{cases} Q = -p \\ P = q + Ap^2 \end{cases} \Rightarrow \begin{cases} p = -Q \\ q = P - AQ^2 \end{cases}$$

$$pdq - PdQ = dF_1 \Rightarrow \frac{\partial p}{\partial Q} = -\frac{\partial P}{\partial q} \Rightarrow -1 = -1 \quad //$$

$$b) \text{ Las coordenadas } Q \text{ y } P \text{ son canónicas} \quad // \quad -[Q, P]_{q,p} = 1$$

c) Para pasar de $\mathcal{H}(q, p)$ a $\mathcal{H}(Q, P)$ se cumplen las siguientes relaciones

$$\left\{ \begin{array}{l} * \mathcal{H}(Q, P) = \mathcal{H}(q, p) + \frac{\partial F}{\partial t} \\ * Q = Q(q, p) \\ * P = P(q, p) \end{array} \right.$$

Hay que construir la función generatrix

$$dF_1 = pdq - PdQ$$

$$* p = \frac{\partial F_1}{\partial q} = -Q \Rightarrow F_1 = -Qq + c_1(Q)$$

$$* -P = \frac{\partial F_1}{\partial Q} = -q - AQ^2 \Rightarrow F_1 = -Qq - \frac{1}{3}AQ^3 + c_2(Q)$$

$$\Rightarrow F_1 = -\frac{1}{3}AQ^3 - Qq$$

$$\Rightarrow \left\{ \begin{array}{l} * \mathcal{H}(Q, P) = \mathcal{H}(q, p) + \cancel{\frac{\partial F}{\partial t}} \Rightarrow \mathcal{H}(Q, P) = \mathcal{H}(q, p) \leftarrow \text{Mismo Hamiltoniano pero sustituyendo a las nuevas coordenadas} \\ * Q = Q(q, p) \rightarrow Q = -p \Rightarrow p = -Q \\ * P = P(q, p) \rightarrow P = q + Ap^2 \Rightarrow q = P - AQ^2 \end{array} \right.$$

• $\mathcal{H}(q, p) = \frac{p^2}{2m} + mgq \Rightarrow \mathcal{H}(Q, P) = \frac{Q^2}{2m} + mg(P - AQ^2) \Rightarrow \mathcal{H}(Q, P) = mgP + \frac{Q^2}{2m} - mgAQ^2 \quad //$

d) Para que Q sea una variable cíclica $\frac{\partial \mathcal{H}}{\partial Q} = 0 \rightarrow$ Que no exista dependencia explícita con Q en \mathcal{H}

$$\Rightarrow \frac{\cancel{Q}}{2m} - mgA\cancel{Q} = 0 \Rightarrow A = \frac{1}{2m^2g} \quad // \quad \Rightarrow \mathcal{H} = mgP$$

c) Con el nuevo Hamiltoniano $\mathcal{H} = mgP$

$$* \dot{Q} = \frac{\partial \mathcal{H}}{\partial P} = mg \Rightarrow Q = mgt + Q_0$$

$$* \dot{P} = -\frac{\partial \mathcal{H}}{\partial Q} = 0 \Rightarrow P: \text{constante} = \frac{E}{mg} = c$$

d) Ahora hay que regresar la transformación canónica

$$\begin{cases} p = -Q \\ q = P - AQ^2 \end{cases} \Rightarrow \begin{cases} p = -mgt - Q_0 \\ q = c - A(-mgt + Q_0)^2 = c - \frac{1}{2m^2g}(-mgt - Q_0)^2 = c - \frac{Q_0^2}{2m^2g} - \frac{Q_0 t}{m} - \frac{1}{2}gt^2 \end{cases}$$
$$= q_0 + \dot{q}_0 t - \frac{1}{2}gt^2$$

Corchetes de Poisson

Problema 1.

Muestre que las Ecuaciones de Hamilton del movimiento pueden escribirse como términos de los corchetes de Poisson:

$$\dot{q} = \{q, \mathcal{H}\}_{q,p},$$

$$\dot{p} = \{p, \mathcal{H}\}_{q,p}.$$

$$[q, \mathcal{H}]_{q,p} = \cancel{\frac{\partial q}{\partial q}} \frac{\partial \mathcal{H}}{\partial p} - \cancel{\frac{\partial q}{\partial p}} \frac{\partial \mathcal{H}}{\partial q} = \frac{\partial \mathcal{H}}{\partial p} = \dot{q}$$

$$[p, \mathcal{H}]_{q,p} = \cancel{\frac{\partial p}{\partial q}} \frac{\partial \mathcal{H}}{\partial p} - \cancel{\frac{\partial p}{\partial q}} \frac{\partial \mathcal{H}}{\partial q} = -\frac{\partial \mathcal{H}}{\partial q} = \dot{p}$$

Problema 2.

Un Hamiltoniano tiene la forma:

$$\mathcal{H}(q_1, q_2, p_1, p_2) = q_1 p_1 - q_2 p_2 + a q_1^2 - b q_2^2,$$

donde a y b son constantes:

a. Usando el método de Corchetes de Poisson, muestre que:

$$\begin{aligned} F_1 &= q_1 q_2 \\ F_2 &= \frac{1}{q_1} (p_2 + b q_2) \end{aligned}$$

son constantes del movimiento.

b. Muestre que $\{F_1, F_2\}$ es también una constante del movimiento.

c. ¿El Hamiltoniano es constante? Chequéé esto, encontrando q_1 , q_2 , p_1 y p_2 son funciones explícitas del tiempo.

Si se son constantes del movimiento

$$\frac{dF_2}{dt} = [F_2, \mathcal{H}] + \frac{\partial F_2}{\partial t} = 0 \Rightarrow [\mathcal{H}, F_2] = \frac{\partial F_2}{\partial t} \text{ como no hay dependencia explícita con } t \text{ en } F_1 \text{ y } F_2$$

$$\Rightarrow [\mathcal{H}, F_2] = 0 \Rightarrow [F_2, \mathcal{H}] = 0$$

* Para $F_1 = q_1 q_2$

$$\begin{aligned} [F_1, \mathcal{H}]_{(q_1, p_1), (q_2, p_2)} &= \frac{\partial F_1}{\partial q_1} \frac{\partial \mathcal{H}}{\partial p_1} - \cancel{\frac{\partial F_1}{\partial p_1} \frac{\partial \mathcal{H}}{\partial q_1}} + \frac{\partial F_1}{\partial q_2} \frac{\partial \mathcal{H}}{\partial p_2} - \cancel{\frac{\partial F_1}{\partial p_2} \frac{\partial \mathcal{H}}{\partial q_2}} \\ &= q_2(q_1) + q_1(-q_2) = 0 \end{aligned}$$

$$\begin{aligned} * \text{ Para } F_2 = \frac{1}{q_1} (p_2 + b q_2) &= \frac{\partial F_2}{\partial q_1} \frac{\partial \mathcal{H}}{\partial p_1} - \cancel{\frac{\partial F_2}{\partial p_1} \frac{\partial \mathcal{H}}{\partial q_1}} + \frac{\partial F_2}{\partial q_2} \frac{\partial \mathcal{H}}{\partial p_2} - \cancel{\frac{\partial F_2}{\partial p_2} \frac{\partial \mathcal{H}}{\partial q_2}} \\ &= -\frac{(p_2 + b q_2)}{q_1^2} \cdot (q_1) + \frac{b}{q_1} \cdot (-q_2) - \frac{1}{q_1} \cdot (-p_2 - 2b q_2) \\ &= -\frac{p_2 - b q_2}{q_1} - \frac{b}{q_1} q_2 + \frac{1}{q_1} \cdot (p_2 + 2b q_2) \\ &= \frac{1}{q_1} \left[-\cancel{p_2 - b q_2 - b q_2} + \cancel{p_2 + 2b q_2} \right] = 0 \end{aligned}$$

$$[F_1, F_2] : \text{constante } \frac{d}{dt} [F_1, F_2] = \left[\frac{dF_1}{dt}, F_2 \right] + \left[F_1, \frac{dF_2}{dt} \right]$$

$$\begin{aligned} &= \left[\left[F_1, \mathcal{H} \right] + \cancel{\frac{\partial F_1}{\partial t}}, F_2 \right] + \left[F_1, \left([F_2, \mathcal{H}] + \cancel{\frac{\partial F_2}{\partial t}} \right) \right] \\ &= \left[[F_1, \mathcal{H}], F_2 \right] + \left[F_1, [F_2, \mathcal{H}] \right] \\ &\stackrel{\text{O}}{=} [0, F_2] + [F_1, 0] = 0 \\ \Rightarrow [F_1, F_2] &= \text{constante} \end{aligned}$$

$$H = q_1 p_1 - q_2 p_2 + aq_1^2 - bq_2^2$$

$$\dot{q}_1 = \frac{\partial H}{\partial p_1} = q_1$$

$$\dot{q}_2 = \frac{\partial H}{\partial p_2} = -q_2$$

$$\dot{p}_1 = -\frac{\partial H}{\partial q_1} = -p_1 - 2aq_1$$

$$\dot{p}_2 = -\frac{\partial H}{\partial q_2} = +p_2 + 2bq_2$$

$$\dot{q}_1 = q_1 \Rightarrow \frac{dq_1}{dt} = q_1 \Rightarrow \int \frac{dq_1}{q_1} = \int dt \Rightarrow q_1 = ke^t$$

$$\dot{q}_2 = -q_2 \Rightarrow q_2 = Be^{-t}$$

$$\dot{p}_1 = -p_1 - 2aq_1$$

$$\dot{p}_2 = p_2 + 2bq_2$$

$$\dot{p}_1 = -p_1 - 2ake^t$$

$$\Rightarrow \dot{p}_2 = p_2 + 2bBe^{-t}$$

$$(D+1)p_1 = -2ake^t \quad | \quad q_1 = Ce^t$$

$$\Rightarrow (D-1)p_2 = 2bBe^{-t} \quad | \quad q_2 = Dc^{-t}$$

$$D(D+1)p_1 = -2ake^t$$

$$\Rightarrow D(D-1)p_2 = -2bBe^{-t}$$

$$D(D+1)p_1 - (D+1)p_1 = 0$$

$$\Rightarrow D(D-1)p_2 + (D-1)p_2 = 0$$

$$(D+1)(D-1)p_1 = 0$$

$$\Rightarrow (D-1)(D+1)p_2 = 0$$

$$S = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$q_2 = C_3 e^{-t} + C_4 e^t$$

$$p_1 = C_1 e^t$$

$$\Rightarrow p_2 = C_3 e^{-t}$$

$$\dot{p}_1 = C_1 e^t$$

$$\Rightarrow -C_3 e^{-t} = C_3 e^{-t} + 2bBe^{-t}$$

En la EDO:

$$C_1 e^t = -C_3 e^{-t} - 2ake^t$$

~~$$2C_1 e^t = -2ake^t$$~~

$$C_1 = ak$$

$$\Rightarrow p_1 = Ce^t - ake^t$$

$$\Rightarrow -2C_3 e^{-t} = 2bBe^{-t}$$

$$\Rightarrow C_3 = -bB$$

$$\Rightarrow p_2 = Dc^{-t} - bBe^{-t}$$

$$\begin{cases} q_1 = ke^t \\ q_2 = Be^{-t} \\ p_1 = Ce^t - ake^t \\ p_2 = Dc^{-t} - bBe^{-t} \end{cases}$$

$$H = q_1 p_1 - q_2 p_2 + aq_1^2 - bq_2^2$$

$$H = ke^t(Ce^t - ake^t) - Be^{-t}(Dc^{-t} - bBe^{-t}) + ake^2 - bB^2e^{-2t}$$

~~$$= Ac - ake^{2t} - BD + bB^2e^{-2t} + ake^{2t} - bB^2e^{-2t}$$~~

$= AC - BD \rightarrow H$ no depende explícitamente de $t \rightarrow H$ constante

$$\frac{dH}{dt} = \frac{\partial H}{\partial q} \dot{q} + \frac{\partial H}{\partial p} \dot{p} + \frac{\partial H}{\partial t} = \cancel{\frac{\partial H}{\partial q} \frac{\partial q}{\partial t}} - \frac{\partial H}{\partial p} \frac{\partial p}{\partial t} + \frac{\partial H}{\partial t} \Rightarrow \frac{dH}{dt} = \frac{\partial H}{\partial t}$$

Problema 3.

Sea $\varphi(\vec{r}, \vec{p})$ cualquier función que tiene simetría esférica alrededor del origen (invariante bajo rotaciones). Demuestre que si es cierto, la dependencia de función φ es con la posición y el momentum lineal es de la forma r^2 , p^2 y $\vec{r} \cdot \vec{p}$.

$$[\varphi, \vec{J}] = 0$$

Mecánica de Cuerpo Rígido

SUBSECTION 9.6

Problemas resueltos

Mecánica del cuerpo rígido ha sido uno de los capítulos más complicados a la hora de realizar los problemas resueltos, por lo que destaco la posibilidad de encontrar errores. Específicamente, a la hora de encontrar las ecuaciones de Euler, el razonamiento puede ser bastante difuso, y se espera que a la hora de plantear la parte teórica y hacer una revisión de los problemas, se pueda dar algo de luz respecto al tema.

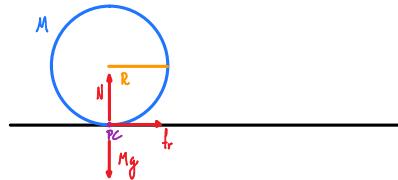
Mecánica del Sólido

Tensor de Inercia

Problema 1.

Un cilindro hueco rueda, de radio R y masa M rueda en una superficie plana con una rapidez v .

- Encuentre el coeficiente de fricción que permite rueda sin resbalar.
- Determine la energía cinética.



$$a) \quad v_{pc} = v_{cm} - R\omega = 0 \leftarrow \text{Rodar sin deslizar} \Rightarrow v_{cm} = R\omega$$

$$\sum F_x = +f_r = M_{acm_x}$$

$$\sum F_y = N - Mg = 0 \Rightarrow N = Mg ; f_r = \mu_s N \Rightarrow +\mu_s Mg = M \frac{v_{cm}^2}{R} = M \frac{v_{cm}^2}{R} \Rightarrow \cancel{\mu_s Mg} = \cancel{M \frac{v_{cm}^2}{R}}$$

$$\Rightarrow \mu_s = \frac{v_{cm}^2}{gR}$$

$$b) \quad T = T_{tra} + T_{rot} = \frac{1}{2} M v_{cm}^2 + \frac{1}{2} I \omega^2 \Rightarrow T = \frac{1}{2} M v_{cm}^2 + \frac{1}{2} M R^2 \frac{v_{cm}^2}{R^2} = M v_{cm}^2$$

$$I_{ij} = \int \lambda_m (\delta_{ij} \sum_n X_n^2 - X_i X_j) dS \Rightarrow I_{zz} = \lambda_m \iint_{x^2+y^2=R^2} (X^2 + y^2) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \Rightarrow I_{zz} = MR^2$$

Problema 2.

Un aro de radio R y masa M se usa como polea en una Máquina de Atwood, con una cuerda ideal.

- Encuentre la tensión en ambos extremos de la cuerda.
- Determine la aceleración del sistema.

Hay que calcular la inercia del aro

$$I_{ij} = \int \lambda_m (\delta_{ij} \sum_n X_n^2 - X_i X_j) dt \Rightarrow I_{zz} = \lambda_m \iint_{x^2+y^2=R^2} (X^2 + y^2 + z^2) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

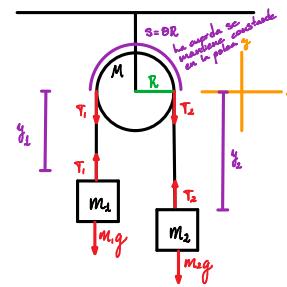
$$X = R \cos t, Y = R \sin t, Z = 0 \leq t \leq 2\pi \Rightarrow I_{zz} = \lambda_m \int_0^{2\pi} R^3 dt = \lambda_m \cdot 2\pi R^3; \lambda_m = \frac{M}{2\pi R} \Rightarrow I_{zz} = \frac{M}{2\pi R} \cdot 2\pi R^3 = MR^2$$

$$\sum F_1 = T_1 - m_1 g = m_1 a_{xy} \quad \sum F_2 = T_2 - m_2 g = m_2 a_{xy} \stackrel{*}{=} m_2 Ra \quad \sum N_{aro-cuad} = -T_2 R + T_1 R = \cancel{Ra}$$

$$y_1 + y_2 + R\theta = l \Rightarrow y_1 + y_2 = 0 \Rightarrow a_{xy} + a_{xy} = 0 \Rightarrow a_{xy} = a = -a_{xy} \quad y \quad a = -Ra$$

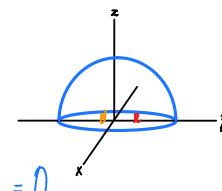
$$\Rightarrow \begin{cases} T_1 = m_1 a + m_1 g \\ T_2 = -m_2 a + m_2 g \\ T_1 - T_2 = -Ma \end{cases} \Rightarrow T_1 - T_2 = m_1 a + m_1 g + m_2 a - m_2 g = -Ma \Rightarrow a = \frac{g(m_2 - m_1)}{M + m_1 + m_2} \quad //$$

$$\Rightarrow T_1 = m_1 \frac{g(m_2 - m_1)}{M + m_1 + m_2} + m_1 g \quad y \quad T_2 = -m_2 \frac{g(m_2 - m_1)}{M + m_1 + m_2} + m_2 g \quad //$$



Problema 3. (Taylor 10.5)

10.5 ** A uniform solid hemisphere of radius R has its flat base in the xy plane, with its center at the origin. Use the result of Problem 10.4 to find the center of mass. [Comment: This and the next two problems are intended to reactivate your skills at finding centers of mass by integration. In all cases, you will need to use the integral form of the definition (10.1) of the CM. If the mass is distributed through a volume (as here), the integral will be a volume integral with $dm = \rho dV$.]



$$\vec{r}_{cm} = \frac{1}{M} \iiint_{\text{hemisphere}} (\rho_x \hat{i} + \rho_y \hat{j} + \rho_z \hat{k}) dV ; \text{ Suponiendo } \rho \text{ uniforme, por simetría } r_{x,cm} = r_{y,cm} = 0$$

$$\Rightarrow r_{z,cm} = \frac{\rho_m}{M} \iiint_{\text{hemisphere}} z dV = \frac{1}{V} \iiint_{\text{hemisphere}} z dV \Rightarrow r_{z,cm} = \frac{3}{2\pi R^3} \int_0^R \int_0^{2\pi} \int_0^\theta r^3 \cos \theta \sin \theta dr d\phi d\theta = \frac{3}{2\pi R^3} \int_0^R r^3 dr \cdot 2\pi \cdot \frac{\sin^2 \theta}{2} \Big|_0^R \\ = \frac{3}{2\pi R^3} \int_0^R r^3 dr \cdot 2\pi \cdot \frac{1}{2} = \frac{3}{2R^3} \frac{r^4}{4} \Big|_0^R = \frac{3}{2R^3} \frac{R^4}{4} = \frac{3R}{8} \Rightarrow \vec{r}_{cm} = 0\hat{i} + 0\hat{j} + \frac{3R}{8}\hat{k}$$

Problema 4.

Un cuerpo rígido formado de tres partículas de masas $2m$, m y $4m$ localizadas en los puntos $(\ell, -\ell, \ell)$, $(2\ell, 0, 2\ell)$, y $(-\ell, \ell, 0)$.

- Encontrar el momento angular del cuerpo si éste rota alrededor del origen con una velocidad angular $\vec{\omega} = 3\omega \hat{x} - 2\omega \hat{y} + 4\omega \hat{z}$.
- Determinar los momentos de inercia con respecto a los ejes x , y y z .
- Encuentre los productos de inercia del cuerpo rígido.
- ¿Cuál es la energía cinética de rotación del sistema?

$$a) \vec{l}_0 = \sum_i \vec{r}_i \times \vec{p}_i = \sum_i m_i \vec{r}_i \times \vec{v}_i = \sum_i m_i \vec{r}_i \times (\vec{\omega} \times \vec{r}_i)$$

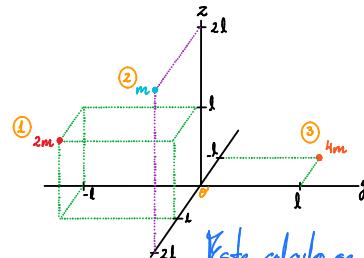
$$\Rightarrow \vec{l}_0 = m_1 \vec{r}_1 \times (\vec{\omega} \times \vec{r}_1) + m_2 \vec{r}_2 \times (\vec{\omega} \times \vec{r}_2) + m_3 \vec{r}_3 \times (\vec{\omega} \times \vec{r}_3)$$

$$\bullet \vec{r}_1 \times (\vec{\omega} \times \vec{r}_1) = l^2 \omega \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \times \left[\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right] = l^2 \omega \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix} \Rightarrow \vec{l}_1 = 2ml^2 \omega (3\hat{j} + 3\hat{k})$$

$$\bullet \vec{r}_2 \times (\vec{\omega} \times \vec{r}_2) = l^2 \omega \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} \times \left[\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} \right] = l^2 \omega \begin{pmatrix} -11 \\ -16 \\ 4 \end{pmatrix} \Rightarrow \vec{l}_2 = ml^2 \omega (-11\hat{i} - 16\hat{j} + 4\hat{k})$$

$$\bullet \vec{r}_3 \times (\vec{\omega} \times \vec{r}_3) = l^2 \omega \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \times \left[\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \times \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right] = l^2 \omega \begin{pmatrix} 1 \\ 1 \\ 8 \end{pmatrix} \Rightarrow \vec{l}_3 = 4ml^2 \omega (1\hat{i} + 1\hat{j} + 8\hat{k})$$

$$\Rightarrow \vec{l}_0 = m^2 l^2 \omega (0\hat{i} - 6\hat{j} + 42\hat{k})$$



Este cálculo se puede realizar respecto a cualquier punto, en este caso en el origen

Continua ...

$$b) c) \quad I_{ij} = \sum_{\alpha} m_{\alpha} (\delta_{ij} \sum_k X_{\alpha,k}^2 - X_{\alpha,i} X_{\alpha,j})$$

$$\begin{aligned} * \quad I_{xx} &= m_1 \left(\cancel{X_1^2} + \cancel{Y_1^2} + \cancel{Z_1^2} - \cancel{X_1^2} \right) + m_2 \left(\cancel{Y_2^2} + \cancel{Y_2^2} + \cancel{Z_2^2} - \cancel{X_2^2} \right) + m_3 \left(\cancel{X_3^2} + \cancel{Y_3^2} + \cancel{Z_3^2} - \cancel{X_3^2} \right) \\ &= m \left[2(Y_1^2 + Z_1^2) + Z_2^2 + 4Y_3^2 \right] = ml^2 \left[2(J^2 + J^2) + J^2 + 4J^2 \right] = 12ml^2 \end{aligned}$$

$$\begin{aligned} * \quad I_{yy} &= m_1 \left(\cancel{X_1^2} + \cancel{Y_1^2} + \cancel{Z_1^2} - \cancel{Y_1^2} \right) + m_2 \left(\cancel{X_2^2} + \cancel{Y_2^2} + \cancel{Z_2^2} - \cancel{Y_2^2} \right) + m_3 \left(\cancel{X_3^2} + \cancel{Y_3^2} + \cancel{Z_3^2} - \cancel{Y_3^2} \right) \\ &= m \left[2(X_1^2 + Z_1^2) + X_2^2 + Z_2^2 + 4X_3^2 \right] = ml^2 \left[2(J^2 + J^2) + J^2 + J^2 + 4J^2 \right] = 16ml^2 \end{aligned}$$

$$\begin{aligned} * \quad I_{zz} &= m_1 \left(\cancel{X_1^2} + \cancel{Y_1^2} + \cancel{Z_1^2} - \cancel{Z_1^2} \right) + m_2 \left(\cancel{X_2^2} + \cancel{Y_2^2} + \cancel{Z_2^2} - \cancel{Z_2^2} \right) + m_3 \left(\cancel{X_3^2} + \cancel{Y_3^2} + \cancel{Z_3^2} - \cancel{Z_3^2} \right) \\ &= m \left[2(X_1^2 + Y_1^2) + X_2^2 + 4(X_3^2 + Y_3^2) \right] = ml^2 \left[2(J^2 + J^2) + J^2 + 4(J^2 + J^2) \right] = 16ml^2 \end{aligned}$$

$$* \quad I_{xy} = -m_1 X_1 Y_1 - \cancel{m_2 X_2 Y_2} - m_3 X_3 Y_3 = -ml^2 (-6) = +6ml^2$$

$$* \quad I_{xz} = -m_1 X_1 Z_1 - m_2 X_2 Z_2 - \cancel{m_3 X_3 Z_3} = -ml^2 (3) = -6ml^2$$

$$* \quad I_{yz} = -m_1 Y_1 Z_1 - \cancel{m_2 Y_2 Z_2} - \cancel{m_3 Y_3 Z_3} = -ml^2 (-2) = 2ml^2$$

$$d) \quad T = T_{\text{rot}} = \frac{1}{2} \vec{\omega} \cdot \vec{I} \vec{\omega} = \frac{1}{2} \vec{\omega} \cdot \vec{I} \vec{\omega} = \frac{1}{2} \omega \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} \cdot ml^2 \omega (0 \hat{i} - 6 \hat{j} + 12 \hat{k}) = \frac{1}{2} ml^2 \omega^2 (3 \cdot 0 - 6 \cdot -2 + 4 \cdot 12)$$

$$\Rightarrow T = \frac{1}{2} ml^2 \omega^2 (3 \cdot 0 - 6 \cdot -2 + 4 \cdot 12) = 90 ml^2 \omega^2 //$$

$$\Rightarrow \vec{I} = ml^2 \begin{pmatrix} 12 & 6 & -6 \\ 6 & 16 & 2 \\ -6 & 2 & 16 \end{pmatrix} //$$

Nota: De resolver $\vec{I} = I_{ij} \vec{\omega}$ se obtiene el mismo resultado para \vec{I}

Encontrar I_1, I_2 e I_3 del tensor de inercia anterior

$$I = ml^2 \begin{pmatrix} 12 & 6 & -6 \\ 6 & 16 & 2 \\ -6 & 2 & 16 \end{pmatrix} = 2ml^2 \begin{pmatrix} 6 & 3 & -3 \\ 3 & 8 & 1 \\ -3 & 1 & 8 \end{pmatrix} ; \quad \beta = 2ml^2 \Rightarrow I = \beta \begin{pmatrix} 6 & 3 & -3 \\ 3 & 8 & 1 \\ -3 & 1 & 8 \end{pmatrix}$$

$$\det(I - \lambda \mathbb{1}) = 0 \quad I_1 = \lambda \beta$$

$$\Rightarrow \begin{vmatrix} 6-\lambda & 3 & -3 \\ 3 & 8-\lambda & 1 \\ -3 & 1 & 8-\lambda \end{vmatrix} = (6-\lambda)[(8-\lambda)^2 - 1] - 3[3(8-\lambda) + 3] - 3[3 - 3(8-\lambda)] = 0$$

$$\Rightarrow (6-\lambda)[(8-\lambda)^2 - 1] - 3[3(8-\lambda) + 3] - 3[3 - 3(8-\lambda)] = 0$$

$$\Rightarrow (6-\lambda)[(8-\lambda)^2 - 1] - 6[3(8-\lambda) + 3] = 0$$

$$\Rightarrow (6-\lambda)[(8-\lambda) - 1][(8-\lambda) + 1] - 18[(8-\lambda) + 1] = 0$$

$$\Rightarrow [(8-\lambda) + 1]\{(6-\lambda)[(8-\lambda) - 1] - 18\} = 0$$

$$\Rightarrow (9-\lambda)[(6-\lambda)(7-\lambda) - 18] = 0 \quad \Rightarrow \lambda_1 = 9 \quad \text{v} \quad (6-\lambda)(7-\lambda) - 18 = 0$$

$$\Rightarrow 42 - 13\lambda + \lambda^2 - 18 = 0$$

$$\Rightarrow 24 - 13\lambda + \lambda^2 = 0 \quad \Rightarrow \lambda_2 = \frac{13 + \sqrt{73}}{2} \quad \text{y} \quad \lambda_3 = \frac{13 - \sqrt{73}}{2}$$

$$\Rightarrow I = \beta \left\{ 9, \frac{13 + \sqrt{73}}{2}, \frac{13 - \sqrt{73}}{2} \right\}$$

Buscar la dirección de los ejes principales del problema anterior

$$I = \beta \begin{pmatrix} 6 & 3 & -3 \\ 3 & 8 & 1 \\ -3 & 1 & 8 \end{pmatrix} \text{ e } I_1 = \beta \left\{ 9, \frac{13 + \sqrt{73}}{2}, \frac{13 - \sqrt{73}}{2} \right\}$$

* Para I_1 :

$$(I - I_1) \vec{\omega}_1 = 0 \Rightarrow \beta \begin{pmatrix} 6-9 & 3 & -3 \\ 3 & 8-9 & 1 \\ -3 & 1 & 8-9 \end{pmatrix} \begin{pmatrix} \omega_{1x} \\ \omega_{1y} \\ \omega_{1z} \end{pmatrix} = \beta \begin{pmatrix} -3 & 3 & -3 \\ 3 & -1 & 1 \\ -3 & 1 & -1 \end{pmatrix} \begin{pmatrix} \omega_{1x} \\ \omega_{1y} \\ \omega_{1z} \end{pmatrix} = \vec{0}$$

$$\Rightarrow -3\omega_{1x} + 3\omega_{1y} - 3\omega_{1z} = 0 \Rightarrow \omega_{1x} + \omega_{1y} - \omega_{1z} = 0 \Rightarrow \vec{\omega}_1 = \frac{1}{2}\hat{i} + \frac{1}{2}\hat{j} - \frac{1}{2}\hat{k} //$$

* Para I_2 :

$$(I - I_2) \vec{\omega}_2 = 0 \Rightarrow \beta \begin{pmatrix} 6 - \frac{13 + \sqrt{73}}{2} & 3 & -3 \\ 3 & 8 - \frac{13 + \sqrt{73}}{2} & 1 \\ -3 & 1 & 8 - \frac{13 + \sqrt{73}}{2} \end{pmatrix} \begin{pmatrix} \omega_{2x} \\ \omega_{2y} \\ \omega_{2z} \end{pmatrix} = \beta \begin{pmatrix} -\frac{1+\sqrt{73}}{2} & 3 & -3 \\ 3 & \frac{3-\sqrt{73}}{2} & 1 \\ -3 & 1 & \frac{3-\sqrt{73}}{2} \end{pmatrix} \begin{pmatrix} \omega_{2x} \\ \omega_{2y} \\ \omega_{2z} \end{pmatrix} = \vec{0}$$

$$\Rightarrow \begin{cases} -\frac{1+\sqrt{73}}{2} \omega_{2x} + 3\omega_{2y} - 3\omega_{2z} = 0 \\ 3\omega_{2x} + \frac{3-\sqrt{73}}{2} \omega_{2y} + \omega_{2z} = 0 \\ -3\omega_{2x} + \omega_{2y} + \frac{3-\sqrt{73}}{2} \omega_{2z} = 0 \end{cases} \Rightarrow \begin{cases} -\frac{1+\sqrt{73}}{2} \omega_{2x} + 3\omega_{2y} - 3\omega_{2z} = 0 \\ \frac{5-\sqrt{73}}{2} \omega_{2y} + \frac{5-\sqrt{73}}{2} \omega_{2z} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} -\frac{1+\sqrt{73}}{2} \omega_{2x} + 3\omega_{2y} - 3\omega_{2z} = 0 \\ \omega_{2y} + \omega_{2z} = 0 \end{cases} \Rightarrow \vec{\omega}_2 = \frac{-1+\sqrt{73}}{6} \hat{i} + \frac{1}{2}\hat{j} - \frac{1}{2}\hat{k} //$$

* Para I_3

$$(I - I_3) \vec{\omega}_3 = 0 \Rightarrow \beta \begin{pmatrix} 6 - \frac{13 - \sqrt{73}}{2} & 3 & -3 \\ 3 & 8 - \frac{13 - \sqrt{73}}{2} & 1 \\ -3 & 1 & 8 - \frac{13 - \sqrt{73}}{2} \end{pmatrix} \begin{pmatrix} \omega_{3x} \\ \omega_{3y} \\ \omega_{3z} \end{pmatrix} = \beta \begin{pmatrix} \frac{-1+\sqrt{73}}{2} & 3 & -3 \\ 3 & \frac{3+\sqrt{73}}{2} & 1 \\ -3 & 1 & \frac{3+\sqrt{73}}{2} \end{pmatrix} \begin{pmatrix} \omega_{3x} \\ \omega_{3y} \\ \omega_{3z} \end{pmatrix} = \vec{0}$$

$$\Rightarrow \begin{cases} -\frac{1+\sqrt{73}}{2} \omega_{3x} + 3\omega_{3y} - 3\omega_{3z} = 0 \\ 3\omega_{3x} + \frac{3+\sqrt{73}}{2} \omega_{3y} + \omega_{3z} = 0 \\ -3\omega_{3x} + \omega_{3y} + \frac{3+\sqrt{73}}{2} \omega_{3z} = 0 \end{cases} \Rightarrow \begin{cases} -\frac{1+\sqrt{73}}{2} \omega_{3x} + 3\omega_{3y} - 3\omega_{3z} = 0 \\ \frac{5+\sqrt{73}}{2} \omega_{3y} + \frac{5+\sqrt{73}}{2} \omega_{3z} = 0 \end{cases}$$

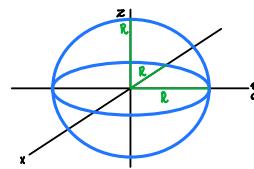
$$\Rightarrow \begin{cases} -\frac{1+\sqrt{73}}{2} \omega_{3x} + 3\omega_{3y} - 3\omega_{3z} = 0 \\ \omega_{3y} + \omega_{3z} = 0 \end{cases} \Rightarrow \vec{\omega}_3 = -\frac{1+\sqrt{73}}{6} \hat{i} + \frac{1}{2}\hat{j} - \frac{1}{2}\hat{k} //$$

Problema 5. (Thornton 10.1)

11-1. Calculate the moments of inertia I_1 , I_2 , and I_3 for a homogeneous sphere of radius R and mass M . (Choose the origin at the center of the sphere.)

Entradas del tensor de inercia $I_{ij} = \iiint pm (\delta_{ij} \sum_k x_k^2 - x_i x_j) dV$

$$pm = \frac{3M}{4\pi R^3} \Rightarrow I_{ij} = pm \iiint \delta_{ij} \sum_k x_k^2 - x_i x_j dV$$



$$* I_{xx} = pm \iiint X^2 + Y^2 + Z^2 - X^2 dxdydz = pm \iiint Y^2 + Z^2 dxdydz ; Y = r \sin \theta \cos \phi \text{ y } Z = r \cos \theta$$

$$\begin{aligned} \Rightarrow I_{xx} &= pm \int_0^R \int_0^{2\pi} \int_0^\pi (r^2 \sin^2 \theta \cos^2 \phi + r^2 \cos^2 \theta) r^2 \sin \theta d\theta d\phi dr \\ &= pm \int_0^R r^4 dr (\int_0^{2\pi} \int_0^\pi \sin^2 \theta d\theta \int_0^\pi \sin^3 \theta d\theta + \int_0^{2\pi} \int_0^\pi \cos^2 \theta d\theta \int_0^\pi \cos^3 \theta d\theta) = pm \int_0^R r^4 dr (\pi \int_0^\pi \sin^3 \theta d\theta + 2\pi \int_0^\pi \cos^3 \theta d\theta) \\ &= pm \int_0^R r^4 dr \left(\pi \frac{4}{3} + 2\pi \frac{2}{3} \right) = pm \frac{R^5}{5} \frac{8\pi}{3} \Rightarrow I_{xx} = \frac{3M}{4\pi R^3} \frac{R^5}{5} \frac{8\pi}{3} = \frac{2MR^2}{5} // \end{aligned}$$

$$* I_{yy} = pm \iiint X^2 + Y^2 + Z^2 - Y^2 dxdydz ; X = r \sin \theta \cos \phi , Y = r \sin \theta \cos \phi \text{ y } Z = r \cos \theta$$

$$\begin{aligned} \Rightarrow I_{yy} &= pm \int_0^R \int_0^{2\pi} \int_0^\pi (r^2 - r^2 \sin^2 \theta \cos^2 \phi) r^2 \sin \theta d\theta d\phi dr = pm \int_0^R r^4 dr (\int_0^{2\pi} \int_0^\pi \sin^2 \theta d\phi \int_0^\pi \sin^3 \theta d\theta - \int_0^{2\pi} \int_0^\pi \sin^2 \theta d\phi \int_0^\pi \cos^3 \theta d\theta) \\ &= pm \int_0^R r^4 dr (2\pi \int_0^\pi \sin^2 \theta d\theta - \pi \int_0^\pi \cos^3 \theta d\theta) = pm \int_0^R r^4 dr (2\pi \cdot 2 - \pi \cdot \frac{4}{3}) = pm \frac{R^5}{5} \frac{8\pi}{3} \\ \Rightarrow I_{yy} &= \frac{3M}{4\pi R^3} \frac{R^5}{5} \frac{8\pi}{3} = \frac{2MR^2}{5} \end{aligned}$$

$$* I_{xy} = -pm \iiint X Y dxdydz \Rightarrow I_{xy} = -pm \int_0^R \int_0^{2\pi} \int_0^\pi r^4 \sin \theta \cos \phi \sin \theta \cos \phi \sin \phi d\theta d\phi d\phi$$

$$\Rightarrow I_{xy} = -pm \int_0^R r^4 dr \int_0^{2\pi} \sin^2 \theta d\theta \int_0^\pi \cos \phi \sin \phi d\phi = 0 \Rightarrow I_{xy} = 0$$

* Al ser una esfera homogénea con el origen en su centro, el tensor de inercia será diagonal donde $I_x = I_y = I_z$. Los productos de inercia son cero porque en cualquiera de estos casos la integral respecto a ϕ será cero.

$$\Rightarrow I = \frac{2MR^2}{5} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} //$$

Problema 6. (Thornton 10.3)

11-3. Calculate the moments of inertia I_1 , I_2 , and I_3 for a homogeneous ellipsoid of mass M with axes' lengths $2a > 2b > 2c$.

$$\text{Entradas del tensor de inercia } I_{ij} = \iiint_{\text{vol}} \rho_m (\delta_{ij} \sum_k X_k^2 - X_i X_j) dV$$

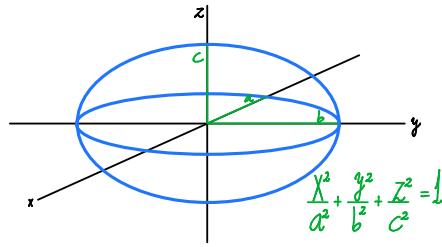
$$\rho_m = \frac{3M}{4\pi abc} \Rightarrow I_{ij} = \rho_m \iiint_{\text{vol}} (\delta_{ij} \sum_k X_k^2 - X_i X_j) dV$$

Planteando un cambio de coordenadas: $\begin{cases} X = ar \sin \theta \cos \phi \\ Y = br \sin \theta \sin \phi ; \quad 0 \leq r \leq 1, \quad 0 \leq \phi \leq 2\pi \quad 0 \leq \theta \leq \pi \\ Z = cr \cos \theta \end{cases}$

Buscando el jacobiano

$$\begin{aligned} \frac{\partial(X, Y, Z)}{\partial(r, \theta, \phi)} &= \begin{vmatrix} \frac{\partial X}{\partial r} & \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \phi} \\ \frac{\partial Y}{\partial r} & \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \phi} \\ \frac{\partial Z}{\partial r} & \frac{\partial Z}{\partial \theta} & \frac{\partial Z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} a \sin \theta \cos \phi & b r \sin \theta \sin \phi & c \cos \theta \\ -a r \sin \theta \sin \phi & b r \sin \theta \cos \phi & 0 \\ a r \cos \theta \cos \phi & b r \cos \theta \sin \phi & -c r \sin \theta \end{vmatrix} \\ &= a \sin \theta \cos \phi [b r \sin \theta \cos \phi - c r \sin \theta] - b r \sin \theta \sin \phi [a r \sin \theta \sin \phi + c r \sin \theta] + \dots \\ &\quad \dots + c \cos \theta [-a r \sin \theta \sin \phi + b r \cos \theta \sin \phi - a r \cos \theta \sin \phi] \\ &= -abc r^2 \sin^3 \theta \cos^2 \phi - abc r^2 \sin^3 \theta \sin^2 \phi + abc r^2 [-\cos^2 \theta \sin \theta \sin^2 \phi - \cos^2 \theta \cos^2 \theta \sin \theta] \\ &= abc r^2 [-\sin^3 \theta \cos^2 \phi - \cos^2 \theta \cos^2 \theta \sin \theta - \sin^3 \theta \sin^2 \phi - \cos^2 \theta \sin \theta \sin^2 \phi] \\ &= abc r^2 [-\sin \theta (\sin^2 \theta \cos^2 \phi + \cos^2 \theta \cos^2 \phi) - \sin \theta (\sin^2 \theta \sin^2 \phi + \cos^2 \theta \sin^2 \phi)] \\ &= abc r^2 [-\sin \theta (\cos^2 \phi) - \sin \theta (\sin^2 \phi)] = -abc r^2 \sin \theta \quad \Rightarrow |J(X, Y, Z)| = abc r^2 \sin \theta \end{aligned}$$

$$\begin{aligned} * I_{xx} &= \rho_m \iiint_{\text{vol}} X^2 + Y^2 + Z^2 - X^2 dX dY dZ = \rho_m \iiint_{\text{vol}} (a^2 \cancel{\sin^2 \theta \cos^2 \phi} + b^2 r^2 \sin^2 \theta \sin^2 \phi + c^2 r^2 \cos^2 \theta - \cancel{a^2 r^2 \sin^2 \theta \cos^2 \phi}) abc r^2 \sin \theta dr d\theta d\phi \\ &= abc \rho_m \int_0^{2\pi} d\theta \int_0^{\pi} \sin \theta \left(b^2 \cancel{\sin^3 \theta \sin^2 \phi} + c^2 \cos^2 \theta \sin \theta \right) dr d\phi \\ &= abc \rho_m \int_0^{\frac{\pi}{2}} r^4 dr \left(b^2 \int_0^{2\pi} \cancel{\sin^2 \theta d\theta} \int_0^{\pi} \sin^3 \theta d\theta + c^2 \int_0^{2\pi} \cancel{\cos^2 \theta d\theta} \int_0^{\pi} \cos^2 \theta \sin \theta d\theta \right) = abc \rho_m \frac{1}{5} \left(b^2 \frac{4}{3}\pi + c^2 \cdot 2\pi \frac{2}{3} \right) \\ &= abc \rho_m \frac{4\pi}{3} (b^2 + c^2) \Rightarrow I_{xx} = \cancel{abc} \frac{3M}{4\pi abc} \cancel{\frac{4\pi}{3}} (b^2 + c^2) = \frac{M}{5} (b^2 + c^2) \quad \Rightarrow I_{xx} = \frac{M}{5} (b^2 + c^2) // \end{aligned}$$



$$\begin{aligned}
 * I_{yy} &= \rho_m \iiint X^2 + Y^2 + Z^2 - Z^2 dx dy dz = \rho_m \int_0^{2\pi} \int_0^\pi \int_0^1 (a^2 r^2 \sin^2 \theta \cos^2 \phi + c^2 r^2 \cos^2 \theta) abc r^2 \sin \theta dr d\theta d\phi \\
 &= abc \rho_m \int_0^1 r^4 dr \left(a^2 \int_0^\pi \int_0^\pi \int_0^1 \sin^2 \theta \cos^2 \phi d\phi d\theta + c^2 \int_0^{\frac{2\pi}{3}} \int_0^\pi \int_0^1 \sin^2 \theta \cos^2 \theta d\phi d\theta \right) \\
 &= abc \rho_m \frac{1}{5} \left(a^2 \frac{4\pi}{3} + c^2 \frac{4\pi}{3} \right) = abc \rho_m \frac{4\pi}{3 \cdot 5} (a^2 + c^2) \Rightarrow I_{yy} = \cancel{abc} \frac{\cancel{BM}}{\cancel{Mabc}} \frac{4\pi}{3 \cdot 5} (a^2 + c^2) \Rightarrow I_{yy} = \frac{M}{5} (a^2 + c^2) //
 \end{aligned}$$

$$\begin{aligned}
 * I_{xx} &= \rho_m \iiint X^2 + Y^2 + Z^2 - Z^2 dx dy dz = \rho_m \int_0^{2\pi} \int_0^\pi \int_0^1 (a^2 r^2 \sin^2 \theta \cos^2 \phi + b^2 r^2 \sin^2 \theta \sin^2 \phi) abc r^2 \sin \theta dr d\theta d\phi \\
 &= abc \rho_m \int_0^1 r^4 dr \left(a^2 \int_0^\pi \int_0^\pi \int_0^1 \sin^2 \theta \cos^2 \phi d\phi d\theta + b^2 \int_0^{\frac{\pi}{2}} \int_0^\pi \int_0^1 \sin^2 \theta \sin^2 \phi d\phi d\theta \right) = abc \rho_m \frac{1}{5} \frac{4}{3} (a^2 \pi + b^2 \pi) = abc \rho_m \frac{4\pi}{3 \cdot 5} (a^2 + b^2) \\
 \Rightarrow I_{xx} &= \cancel{abc} \frac{\cancel{BM}}{\cancel{Mabc}} \frac{4\pi}{3 \cdot 5} (a^2 + b^2) \Rightarrow I_{xx} = \frac{M}{5} (a^2 + b^2) //
 \end{aligned}$$

$$\begin{aligned}
 * I_{xy} &= - \rho_m \iiint XY dx dy dz = - \rho_m \int_0^{2\pi} \int_0^\pi \int_0^1 a r \sin \theta \cos \phi b r \sin \theta \sin \phi abc r^2 \sin \theta dr d\theta d\phi \\
 &= a^2 b^2 c \int_0^1 r^4 dr \int_0^\pi \int_0^{\frac{\pi}{2}} \sin^2 \theta \cos \phi \sin \theta \sin \phi d\phi d\theta = 0 //
 \end{aligned}$$

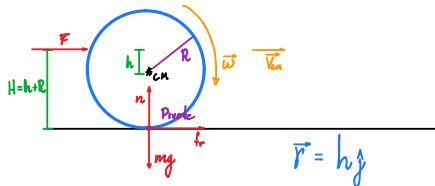
$$\begin{aligned}
 * I_{xz} &= - \rho_m \iiint XZ dx dy dz = - \rho_m \int_0^{2\pi} \int_0^\pi \int_0^1 a r \sin \theta \cos \phi c r \cos \theta abc r^2 \sin \theta dr d\theta d\phi \\
 &= - a^2 b c^2 \rho_m \int_0^1 r^4 dr \int_0^\pi \int_0^{\frac{\pi}{2}} \sin \theta \cos \theta \cos \phi d\phi d\theta = 0 //
 \end{aligned}$$

$$\begin{aligned}
 * I_{yz} &= - \rho_m \iiint YZ dx dy dz = - \rho_m \int_0^{2\pi} \int_0^\pi \int_0^1 b r \sin \theta \sin \phi c r \cos \theta abc r^2 \sin \theta dr d\theta d\phi \\
 &= - a^2 b^2 c^2 \rho_m \int_0^1 r^4 dr \int_0^\pi \int_0^{\frac{2\pi}{3}} \sin^2 \theta \cos \theta \cos \phi d\phi d\theta = 0 //
 \end{aligned}$$

$$\Rightarrow I = \frac{M}{5} \begin{pmatrix} b^2 + c^2 & 0 & 0 \\ 0 & a^2 + c^2 & 0 \\ 0 & 0 & b^2 + c^2 \end{pmatrix} //$$

Problema 7. (Thornton 10.5)

- 11-5. (a) Find the height at which a billiard ball should be struck so that it will roll with no initial slipping. (b) Calculate the optimum height of the rail of a billiard table. On what basis is the calculation predicated?



$$\text{Impulso } \bar{J} = \int_0^t \bar{F}(t') dt' \quad \text{Condiciones iniciales: } \vec{p}(t=0) = \bar{L}(t=0) = 0$$

$$\bullet \frac{d\vec{p}}{dt} = \bar{F} \Rightarrow \Delta \vec{p} = \int_0^t \bar{F}(t') dt' = \bar{J} \Rightarrow \vec{p}_f = \bar{J} = M\bar{V}_{cm}$$

$$\bullet \frac{d\bar{L}}{dt} = \bar{r} \times \bar{F} \Rightarrow \Delta \bar{L} = \int_0^t \bar{r} \times \bar{F} dt' = \bar{r} \times \bar{J} \Rightarrow \bar{L}_f = \bar{r} \times \bar{J} = h \bar{J} \hat{k} = J_z \bar{\omega}_z \Rightarrow \bar{\omega}_z = \frac{h \bar{J}}{J_z} \hat{k}$$

$$\text{La velocidad en cualquier punto de la esfera: } \bar{v}_p = \bar{V}_{cm} + \bar{r}_p \times \bar{\omega}$$

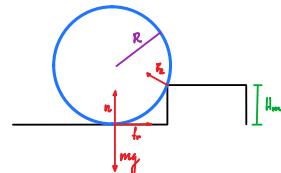
La condición de rodar sin deslizar requiere que en el punto de contacto P entre la superficie y el objeto, la velocidad sea cero.

a)

$$\Rightarrow \cancel{\bar{v}_p} = \bar{V}_{cm} + \bar{\omega} \times \bar{r}_p \Rightarrow 0 = \frac{\bar{J}}{M} \hat{i} + \frac{h \bar{J}}{J_z} \hat{k} \times \bar{R}_y \Rightarrow \cancel{\frac{\bar{J}}{M} \hat{i}} - \cancel{\frac{h \bar{J}}{J_z} \hat{k} \times \bar{R} \hat{i}} = 0 \Rightarrow \frac{1}{M} = \frac{h}{J_z} R \Rightarrow h = \frac{J_z}{MR}$$

$$b) \quad p_x = MV_{cm} \quad y \quad p_z = -MV_{cm}$$

$$L_x = h_z \quad L_z = -L_x = -h_z$$



$$\Rightarrow \Delta \vec{p} = -2MV_{cm} = -2\bar{J}$$

$$\Rightarrow \Delta \bar{L} = -2h_z = -2J_z \omega_z = -2h_m \bar{J} \Rightarrow \cancel{-2J_z \omega_z} = \cancel{-2h_m \bar{J}} \Rightarrow h_m = \frac{J_z \omega_z}{MV_{cm}}$$

Problema 8. (Thornton 10.13)

11-13. A three-particle system consists of masses m_i and coordinates (x_1, x_2, x_3) as follows:

$$\begin{aligned}m_1 &= 3m, \quad (b, 0, b) \\m_2 &= 4m, \quad (b, b, -b) \\m_3 &= 2m, \quad (-b, b, 0)\end{aligned}$$

Find the inertia tensor, principal axes, and principal moments of inertia.

Tensor de inercia:

$$* I_{xx} = 3m(b^2 + 0^2 + b^2 - b^2) + 4m(b^2 + b^2 + b^2 - b^2) + 2m(b^2 + b^2 + 0^2 - b^2)$$

$$= 3m(b^2) + 4m(b^2 + b^2) + 2m(b^2) = 3b^2m + 8b^2m + 2b^2m = 13mb^2$$

$$* I_{yy} = \begin{cases} 3m(b^2 + 0^2 + b^2 - b^2) = 6mb^2 \\ 4m(b^2 + b^2 + b^2 - b^2) = 8mb^2 \\ 2m(b^2 + b^2 + 0^2 - b^2) = 2mb^2 \end{cases}$$

$$\Rightarrow I_{yy} = 16mb^2$$

$$* I_{zz} = \begin{cases} 3m(b^2 + 0^2 + b^2 - b^2) = 3mb^2 \\ 4m(b^2 + b^2 + b^2 - b^2) = 8mb^2 \\ 2m(b^2 + b^2 + 0^2 - b^2) = 4mb^2 \end{cases}$$

$$\Rightarrow I_{zz} = 15mb^2$$

$$* I_{xy} = \begin{cases} -3m(b \cdot 0) = 0 \\ -4m(b \cdot b) = -4mb^2 \\ -2m(-b \cdot b) = 2mb^2 \end{cases}$$

$$\Rightarrow I_{xy} = -2mb^2$$

$$* I_{xz} = \begin{cases} -3m(b \cdot b) = 3mb^2 \\ -4m(b \cdot -b) = 4mb^2 \\ -2m(-b \cdot 0) = 0 \end{cases}$$

$$\Rightarrow I_{xz} = mb^2$$

$$* I_{yz} = \begin{cases} -3m(0 \cdot b) = 0 \\ -4m(b \cdot -b) = 4mb^2 \\ -2m(b \cdot 0) = 0 \end{cases} \Rightarrow I_{yz} = 4mb^2$$

$$\Rightarrow I = \begin{pmatrix} 13mb^2 & -2mb^2 & mb^2 \\ -2mb^2 & 16mb^2 & 4mb^2 \\ mb^2 & 4mb^2 & 15mb^2 \end{pmatrix} = mb^2 \begin{pmatrix} 13 & -2 & 1 \\ -2 & 16 & 4 \\ 1 & 4 & 15 \end{pmatrix} = \beta \begin{pmatrix} 13 & -2 & 1 \\ -2 & 16 & 4 \\ 1 & 4 & 15 \end{pmatrix} ; \beta = mb^2$$

Para hallar los momentos principales hay que diagonalizar el tensor de inercia, para esto se toma $I_i = \lambda \beta$

$$\begin{vmatrix} 13-\lambda & -2 & 1 \\ -2 & 16-\lambda & 4 \\ 1 & 4 & 15-\lambda \end{vmatrix} = 0 \Rightarrow (13-\lambda)[(16-\lambda)(15-\lambda)-16] - 2[2(15-\lambda)-4] - 8 - 16 + \lambda = 0$$

$$\Rightarrow (13-\lambda)[240 - 31\lambda + \lambda^2 - 16] - 68 + 4\lambda - 24 + \lambda = 0 \Rightarrow (13-\lambda)[224 - 31\lambda + \lambda^2] - 92 + 5\lambda = 0$$

$$\Rightarrow 2912 - 627\lambda + 44\lambda^2 - \lambda^3 - 92 + 5\lambda = 0 \Rightarrow 2820 - 622\lambda + 44\lambda^2 - \lambda^3 = 0 \Rightarrow \lambda = \{19,65; 14,35; 10\}$$

$$\Rightarrow I_i = \beta \{19,65; 14,35; 10\}$$

Ejes principales

$$* \quad I_1 : (I - I_{11}1) \bar{\omega}_1 = 0 \Rightarrow \left[\beta \begin{pmatrix} 13 & -2 & 1 \\ -2 & 16 & 4 \\ 1 & 4 & 15 \end{pmatrix} - \beta \begin{pmatrix} 19,65 & 0 & 0 \\ 0 & 19,65 & 0 \\ 0 & 0 & 19,65 \end{pmatrix} \right] \begin{pmatrix} \omega_{1x} \\ \omega_{1y} \\ \omega_{1z} \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} -6,65 & -2 & 1 \\ -2 & -3,65 & 4 \\ 1 & 4 & -1,65 \end{pmatrix} \begin{pmatrix} \omega_{1x} \\ \omega_{1y} \\ \omega_{1z} \end{pmatrix} = 0 \Rightarrow \begin{cases} -6,65\omega_{1x} - 2\omega_{1y} + \omega_{1z} = 0 \\ -2\omega_{1x} - 3,65\omega_{1y} + 4\omega_{1z} = 0 \\ \omega_{1x} + 4\omega_{1y} - 1,65\omega_{1z} = 0 \end{cases} \xrightarrow{f_2 \rightarrow f_2 + 2f_3} \begin{cases} -6,65\omega_{1x} - 2\omega_{1y} + \omega_{1z} = 0 \\ 4,35\omega_{1y} - 5,3\omega_{1z} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} -6,65\omega_{1x} - 2\omega_{1y} + \omega_{1z} = 0 \\ \omega_{1y} = 1,21\omega_{1z} \end{cases} \Rightarrow \bar{\omega}_1 = -0,21\mathbf{i} + 1,21\mathbf{j} + 1\mathbf{k} \Rightarrow \dot{\omega}_1 = -0,13\mathbf{i} + 0,77\mathbf{j} + 0,63\mathbf{k} //$$

$$* \quad I_2 : (I - I_{22}1) \bar{\omega}_2 = 0 \Rightarrow \left[\beta \begin{pmatrix} 13 & -2 & 1 \\ -2 & 16 & 4 \\ 1 & 4 & 15 \end{pmatrix} - \beta \begin{pmatrix} 14,35 & 0 & 0 \\ 0 & 14,35 & 0 \\ 0 & 0 & 14,35 \end{pmatrix} \right] \begin{pmatrix} \omega_{2x} \\ \omega_{2y} \\ \omega_{2z} \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} -1,35 & -2 & 1 \\ -2 & 1,65 & 4 \\ 1 & 4 & 0,65 \end{pmatrix} \begin{pmatrix} \omega_{2x} \\ \omega_{2y} \\ \omega_{2z} \end{pmatrix} = 0 \Rightarrow \begin{cases} -1,35\omega_{2x} - 2\omega_{2y} + \omega_{2z} = 0 \\ -2\omega_{2x} + 1,65\omega_{2y} + 4\omega_{2z} = 0 \\ 1\omega_{2x} + 4\omega_{2y} + 0,65\omega_{2z} = 0 \end{cases} \xrightarrow{f_2 \rightarrow f_2 + 2f_3} \begin{cases} -1,35\omega_{2x} - 2\omega_{2y} + \omega_{2z} = 0 \\ 9,65\omega_{2y} + 5,3\omega_{2z} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} -1,35\omega_{2x} - 2\omega_{2y} + \omega_{2z} = 0 \\ \omega_{2y} = -0,55\omega_{2z} \end{cases} \Rightarrow \bar{\omega}_2 = 1,56\mathbf{i} - 0,55\mathbf{j} + 1\mathbf{k} \Rightarrow \dot{\omega}_2 = 0,81\mathbf{i} - 0,28\mathbf{j} + 0,52\mathbf{k} //$$

$$* \quad I_3 : (I - I_{33}1) \bar{\omega}_3 = 0 \Rightarrow \left[\beta \begin{pmatrix} 13 & -2 & 1 \\ -2 & 16 & 4 \\ 1 & 4 & 15 \end{pmatrix} - \beta \begin{pmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{pmatrix} \right] \begin{pmatrix} \omega_{3x} \\ \omega_{3y} \\ \omega_{3z} \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} 3 & -2 & 1 \\ -2 & 6 & 4 \\ 1 & 4 & 5 \end{pmatrix} \begin{pmatrix} \omega_{3x} \\ \omega_{3y} \\ \omega_{3z} \end{pmatrix} = 0 \Rightarrow \begin{cases} 3\omega_{3x} - 2\omega_{3y} + \omega_{3z} = 0 \\ -2\omega_{3x} + 6\omega_{3y} + 4\omega_{3z} = 0 \\ \omega_{3x} + 4\omega_{3y} + 5\omega_{3z} = 0 \end{cases} \xrightarrow{f_2 \rightarrow f_2 + 2f_3} \begin{cases} 3\omega_{3x} - 2\omega_{3y} + \omega_{3z} = 0 \\ 14\omega_{3y} + 14\omega_{3z} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} 3\omega_{3x} - 2\omega_{3y} + \omega_{3z} = 0 \\ \omega_{3y} = -\omega_{3z} \end{cases} \Rightarrow \bar{\omega}_3 = -\frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k} \Rightarrow \dot{\omega}_3 = \underline{-\frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}} //$$

Problema 9.

Un hemisferio sólido uniforme de radio R y masa M , tiene la parte plana en el plano xy , con el centro en el origen. Determine el tensor de inercia.

Problema 10.

Repita el ejercicio anterior si la densidad de masa varia proporcionalmente con la coordenada z .

Propiedades del tensor de inercia y ángulos de Euler

Problema 1. (Thornton 11.16)

11-16. Consider the following inertia tensor:

$$\{I\} = \begin{pmatrix} \frac{1}{2}(A+B) & \frac{1}{2}(A-B) & 0 \\ \frac{1}{2}(A-B) & \frac{1}{2}(A+B) & 0 \\ 0 & 0 & C \end{pmatrix}$$

Perform a rotation of the coordinate system by an angle θ about the x_3 -axis. Evaluate the transformed tensor elements, and show that the choice $\theta = \pi/4$ renders the inertia tensor diagonal with elements A , B , and C .

La matriz de rotación $\lambda_\psi = \begin{pmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Rotando el tensor

$$\begin{aligned} \Rightarrow I' &= \lambda_\psi I \lambda_\psi^T = \begin{pmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1+B}{2} & \frac{1-B}{2} & 0 \\ \frac{1-B}{2} & \frac{1+B}{2} & 0 \\ 0 & 0 & C \end{pmatrix} \begin{pmatrix} \cos \psi & -\sin \psi & 0 \\ \sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1+B}{2} \cos^2 \psi + \frac{1-B}{2} \sin^2 \psi & \frac{1-B}{2} \cos \psi + \frac{1+B}{2} \sin \psi & 0 \\ -\frac{1+B}{2} \sin^2 \psi + \frac{1-B}{2} \cos^2 \psi & -\frac{1-B}{2} \sin \psi + \frac{1+B}{2} \cos \psi & 0 \\ 0 & 0 & C \end{pmatrix} \begin{pmatrix} \cos \psi & -\sin \psi & 0 \\ \sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \left(\frac{1+B}{2} \cos^2 \psi + \frac{1-B}{2} \sin^2 \psi\right) \cos \psi + \left(\frac{1-B}{2} \cos \psi + \frac{1+B}{2} \sin \psi\right) \sin \psi & -\left(\frac{1+B}{2} \cos^2 \psi + \frac{1-B}{2} \sin^2 \psi\right) \sin \psi + \left(\frac{1-B}{2} \cos \psi + \frac{1+B}{2} \sin \psi\right) \cos \psi & 0 \\ -\left(\frac{1+B}{2} \sin^2 \psi + \frac{1-B}{2} \cos^2 \psi\right) \cos \psi + \left(-\frac{1-B}{2} \sin \psi + \frac{1+B}{2} \cos \psi\right) \sin \psi & -\left(-\frac{1+B}{2} \sin^2 \psi + \frac{1-B}{2} \cos^2 \psi\right) \sin \psi + \left(-\frac{1-B}{2} \sin \psi + \frac{1+B}{2} \cos \psi\right) \cos \psi & 0 \\ 0 & 0 & C \end{pmatrix} \end{aligned}$$

Evaluando $\psi = \frac{\pi}{4} \Rightarrow \sin \psi = \cos \psi = \frac{\sqrt{2}}{2}$

$$\begin{aligned} \Rightarrow I' &= \begin{pmatrix} \frac{\sqrt{2}}{4}(A+B+A-B)\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{4}(A-B+A+B)\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{4}(A+B-A-B)\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{4}(A-B+A+B)\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{4}(-A-B+A-B)\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{4}(-A+B+A+B)\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{4}(-A-B+A-B)\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{4}(-A+B+A+B)\frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & C \end{pmatrix} \\ &= \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix} \end{aligned}$$

Problema 2. (Thornton 11.16)

11-17. Consider a thin homogeneous plate that lies in the x_1 - x_2 plane. Show that the inertia tensor takes the form

$$\{\mathbf{I}\} = \begin{Bmatrix} A & -C & 0 \\ -C & B & 0 \\ 0 & 0 & A + B \end{Bmatrix}$$

Problema 3. (Thornton 11.17)

11-18. If, in the previous problem, the coordinate axes are rotated through an angle θ about the x_3 -axis, show that the new inertia tensor is

$$\{\mathbf{I}'\} = \begin{Bmatrix} A' & -C' & 0 \\ -C' & B' & 0 \\ 0 & 0 & A' + B' \end{Bmatrix}$$

where

$$A' = A \cos^2 \theta - C \sin 2\theta + B \sin^2 \theta$$

$$B' = A \sin^2 \theta + C \sin 2\theta + B \cos^2 \theta$$

$$C' = C \cos 2\theta - \frac{1}{2}(B - A) \sin 2\theta$$

and hence show that the x_1 - and x_2 -axes become principal axes if the angle of rotation is

$$\theta = \frac{1}{2} \tan^{-1} \left(\frac{2C}{B - A} \right)$$

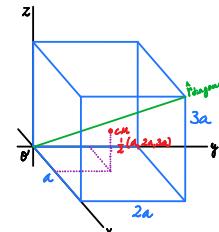
$$\begin{aligned} C' &= 0 \\ C \cos(2\theta) - \frac{1}{2}(B - A) \sin(2\theta) &= 0 \\ 2C \cos(2\theta) &= (B - A) \sin(2\theta) \\ \frac{2C}{B - A} &= \tan(2\theta) \end{aligned}$$

Problema 4.

Un bloque uniforme de masa m , tiene dimensiones a , $2a$ y $3a$.

- a. Encuentre el tensor de inercia si el origen coincide con una esquina, los ejes son paralelos a cada lado y está en el octante positivo.

$$\text{Entradas del tensor de inercia } I_{ij} = \iiint_{\text{bloque}} \rho_m (\delta_{ij} \sum_k X_k^2 - X_i X_j) dV$$



$$* I_{xx} = \iiint_{\text{bloque}} \rho_m x^2 + z^2 dx dy dz = a \rho_m \int_0^{2a} \int_0^a x^2 + z^2 dz dy = a \rho_m \left(3a \cdot \frac{2^3 a^3}{3} + 2a \cdot \frac{3^3 a^3}{3} \right) = a \rho_m (2^3 a^4 + 2 \cdot 3^2 a^4)$$

$$= 26a^5 \rho_m = 26a^2 \frac{M}{6a^3} = \frac{13}{3} Ma^2 \Rightarrow I_{xx} = \frac{13}{3} Ma^2$$

$$* I_{yy} = \iiint_{\text{bloque}} \rho_m x^2 + z^2 dx dy dz = 2a \rho_m \int_0^{2a} \int_0^a x^2 + z^2 dz dy = 2a \rho_m \left(3a \cdot \frac{a^3}{3} + a \cdot \frac{3^3 a^3}{3} \right) = 2a \rho_m (a^4 + 3^2 a^4)$$

$$= 20a^5 \rho_m = 20a^2 \frac{M}{6a^3} = \frac{10}{3} Ma^2 \Rightarrow I_{yy} = \frac{10}{3} Ma^2$$

$$* I_{zz} = \iiint_{\text{bloque}} \rho_m x^2 + y^2 dx dy dz = 3a \rho_m \int_0^{2a} \int_0^a x^2 + y^2 dx dy = 3a \rho_m \left(2a \cdot \frac{a^3}{3} + a \cdot \frac{2^3 a^3}{3} \right) = a \rho_m (2a^4 + 2^3 a^4)$$

$$= 10a^5 \rho_m = 10a^2 \frac{M}{6a^3} = \frac{5}{3} Ma^2 \Rightarrow I_{zz} = \frac{5}{3} Ma^2$$

$$* I_{xy} = - \iiint_{\text{bloque}} xy dx dy dz = -3a \rho_m \int_0^{2a} xy dx dy = -3a \rho_m \left(\frac{a^2}{2} \cdot \frac{2^2 a^2}{2} \right) = -3a \rho_m \cdot a^4 = -3a^5 \rho_m$$

$$= -3a^2 \frac{M}{2} = -\frac{1}{2} Ma^2 \Rightarrow I_{xy} = -\frac{1}{2} Ma^2$$

$$* I_{xz} = - \iiint_{\text{bloque}} XZ dx dy dz = -2a \rho_m \int_0^{2a} XZ dx dz = -2a \rho_m \left(\frac{a^2}{2} \cdot \frac{3^2 a^2}{2} \right) = -\frac{9}{2} a^5 \rho_m = -\frac{9}{2} a^2 \frac{M}{2} = -\frac{9}{4} Ma^2$$

$$\Rightarrow I_{xz} = -\frac{9}{4} Ma^2$$

$$* I_{yz} = - \iiint_{\text{bloque}} YZ dx dy dz = -a \rho_m \int_0^{2a} YZ dy dz = -a \rho_m \left(\frac{2a^2}{2} \cdot \frac{3^2 a^2}{2} \right) = -9a^5 \rho_m = -9a^2 \frac{M}{2} = -\frac{9}{2} Ma^2$$

$$\Rightarrow I_{yz} = -\frac{9}{2} Ma^2$$

$$\Rightarrow I = Ma^2 \begin{pmatrix} \frac{13}{3} & -\frac{1}{2} & -\frac{3}{4} \\ -\frac{1}{2} & \frac{10}{3} & -\frac{3}{2} \\ -\frac{3}{4} & -\frac{3}{2} & \frac{5}{3} \end{pmatrix} = \frac{Ma^2}{12} \begin{pmatrix} 52 & -6 & -9 \\ -6 & 40 & -18 \\ -9 & -18 & 20 \end{pmatrix}$$

- b. Determine las componentes de la velocidad angular, si se hace girar alrededor de la diagonal del bloque.

Si gira respecto a su diagonal se cumple: $\vec{\omega} = \omega \hat{A}_{\text{diagonal}}$

$$\hat{A}_{\text{diagonal}} = \frac{a\hat{i} + 2a\hat{j} + 3a\hat{k}}{\sqrt{a^2 + 2^2a^2 + 3^2a^2}} = \frac{a\hat{i} + 2a\hat{j} + 3a\hat{k}}{\sqrt{14a^2}} = \frac{\hat{i} + 2\hat{j} + 3\hat{k}}{\sqrt{14}} \Rightarrow \vec{\omega} = \omega \left(\frac{\hat{i} + 2\hat{j} + 3\hat{k}}{\sqrt{14}} \right)$$

- c. Encuentre el momentum angular asociado a la velocidad anterior.

$$\vec{L}_\theta = I\vec{\omega} = \frac{Ma^2}{12} \begin{pmatrix} 52 & -6 & -9 \\ -6 & 40 & -18 \\ -9 & -18 & 20 \end{pmatrix} \frac{\omega}{\sqrt{14}} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \frac{Ma^2}{12} \frac{\omega}{\sqrt{14}} \begin{pmatrix} 52 & -6 & -9 \\ -6 & 40 & -18 \\ -9 & -18 & 20 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \frac{Ma^2}{12} \frac{\omega}{\sqrt{14}} \begin{pmatrix} 13 \\ 20 \\ 15 \end{pmatrix}$$

$$\Rightarrow \vec{L}_\theta = \frac{Ma^2\omega}{12\sqrt{14}} (13\hat{i} + 20\hat{j} + 15\hat{k})$$

- d. ¿Cuál es la posición del centro de masa?

Siendo un bloque uniforme, su centro de masa se halla en su centro geométrico

$$\Rightarrow \vec{r}_{cm} = \frac{a\hat{i} + 2a\hat{j} + 3a\hat{k}}{2}$$

- e. Determine el tensor de inercia si el sistema de referencia se traslada al centro de masa.

Usando el Teorema de ejes paralelos $I_{ij-cm} = I_{ij} - M(\delta_{ij} \sum_k d_k^2 - d_i d_j)$

$$\Rightarrow I_{cm} = \frac{Ma^2}{12} \begin{pmatrix} 52 & -6 & -9 \\ -6 & 40 & -18 \\ -9 & -18 & 20 \end{pmatrix} - M \begin{pmatrix} d_y^2 + d_z^2 & -d_x \cdot d_y & -d_x \cdot d_z \\ -d_x \cdot d_y & d_x^2 + d_z^2 & -d_y \cdot d_z \\ -d_x \cdot d_z & -d_y \cdot d_z & d_x^2 + d_y^2 \end{pmatrix} \quad \text{con } d_x = \frac{a}{2}, d_y = a \text{ y } d_z = \frac{3a}{2}$$

$$= \frac{Ma^2}{12} \begin{pmatrix} 52 & -6 & -9 \\ -6 & 40 & -18 \\ -9 & -18 & 20 \end{pmatrix} - Ma^2 \begin{pmatrix} \frac{1^2 + 3^2}{2^2} & -\frac{1}{2} & -\frac{1}{2} \cdot \frac{3}{2} \\ -\frac{1}{2} & \frac{1^2 + 3^2}{2^2} & -\frac{3}{2} \\ -\frac{1}{2} \cdot \frac{3}{2} & -\frac{3}{2} & \frac{1^2 + 1^2}{2^2} \end{pmatrix} \rightarrow \begin{pmatrix} \frac{13}{4} & -\frac{1}{2} & -\frac{3}{4} \\ -\frac{1}{2} & \frac{13}{4} & -\frac{3}{2} \\ -\frac{3}{4} & -\frac{3}{2} & \frac{5}{4} \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 13 & -2 & -3 \\ -2 & 10 & -6 \\ -3 & -6 & 5 \end{pmatrix}$$

$$= \frac{Ma^2}{12} \begin{pmatrix} 52 & -6 & -9 \\ -6 & 40 & -18 \\ -9 & -18 & 20 \end{pmatrix} - \frac{Ma^2}{4} \begin{pmatrix} 13 & -2 & -3 \\ -2 & 10 & -6 \\ -3 & -6 & 5 \end{pmatrix} = \frac{Ma^2}{12} \left[\begin{pmatrix} 52 & -6 & -9 \\ -6 & 40 & -18 \\ -9 & -18 & 20 \end{pmatrix} - \begin{pmatrix} 39 & -6 & -9 \\ -6 & 30 & -18 \\ -9 & -18 & 15 \end{pmatrix} \right]$$

$$= \frac{Ma^2}{12} \begin{pmatrix} 13 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

$$\Rightarrow I_{cm} = \frac{Ma^2}{12} \begin{pmatrix} 13 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

- f. Determine las componentes de la velocidad angular, si se hace girar alrededor de la diagonal del bloque usando el nuevo sistema de coordenadas. Vuelva a encontrar el momentum angular, respecto a este sistema de referencia.

Nuevamente se hace girar al cubo sobre una de sus diagonales, trabajando esto en θ_{cm}

$$\vec{\omega} = \omega \hat{\vec{r}}_{\text{diagonal}}$$

$$\hat{\vec{r}}_{\text{diagonal}} = \frac{\vec{E}\theta}{|\vec{E}\theta|} ; \quad \vec{E} \cdot (a, 2a, 3a) - \frac{1}{2}(a, 2a, 3a) = \frac{1}{2}(a, 2a, 3a)$$

$$\theta \cdot (0, 0, 0) - \frac{1}{2}(a, 2a, 3a) = -\frac{1}{2}(a, 2a, 3a)$$

Velocidad angular

$$\Rightarrow \hat{\vec{r}}_{\text{diagonal}} = \frac{\vec{E} - \theta}{|\vec{E} - \theta|} = \frac{a\hat{i} + 2a\hat{j} + 3a\hat{k}}{\sqrt{a^2 + 2^2a^2 + 3^2a^2}} = \frac{a\hat{i} + 2a\hat{j} + 3a\hat{k}}{\sqrt{14a^2}} = \frac{1\hat{i} + 2\hat{j} + 3\hat{k}}{\sqrt{14}} \Rightarrow \vec{\omega} = \omega \left(\frac{1\hat{i} + 2\hat{j} + 3\hat{k}}{\sqrt{14}} \right)$$

Ahora, el momento angular $\vec{L}_{cm} = \vec{I}_{cm}\vec{\omega}$

$$\Rightarrow \vec{L}_{cm} = \vec{I}_{cm}\vec{\omega} = \frac{Ma^2}{12} \begin{pmatrix} 13 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 5 \end{pmatrix} \frac{\omega}{\sqrt{14}} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \frac{Ma^2}{12} \frac{\omega}{\sqrt{14}} \begin{pmatrix} 13 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \frac{Ma^2}{12} \frac{\omega}{\sqrt{14}} \begin{pmatrix} 13 \\ 20 \\ 15 \end{pmatrix}$$

$$= \frac{Ma^2\omega}{12\sqrt{14}} (13\hat{i} + 20\hat{j} + 15\hat{k})$$

g. Diagonalice ambos tensores de inercia.

De ambos tensores, solo hay que diagonalizar $\mathbf{I} = \frac{Ma^2}{12} \begin{pmatrix} 52 & -6 & -9 \\ -6 & 40 & -18 \\ -9 & -18 & 20 \end{pmatrix}$, porque $\mathbf{I}_{cm} = \frac{Ma^2}{12} \begin{pmatrix} 13 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 5 \end{pmatrix}$ ya es diagonal.

Se toma $\beta = \frac{Ma^2}{12}$; comenzando

$$\Rightarrow \mathbf{I} = \beta \begin{pmatrix} 52 & -6 & -9 \\ -6 & 40 & -18 \\ -9 & -18 & 20 \end{pmatrix} \Rightarrow \det(\mathbf{I} - \beta \lambda \mathbf{I}) = \beta^3 \begin{vmatrix} 52-\lambda & -6 & -9 \\ -6 & 40-\lambda & -18 \\ -9 & -18 & 20-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 52-\lambda & -6 & -9 \\ -6 & 40-\lambda & -18 \\ -9 & -18 & 20-\lambda \end{vmatrix} = (52-\lambda)[(40-\lambda)(20-\lambda) - 324] + 6[-6(20-\lambda) - 162] - 9[108 + 9(40-\lambda)] = 0$$

$$\Rightarrow (52-\lambda)[(40-\lambda)(20-\lambda) - 324] - 720 + 36\lambda - 972 - 972 - 3240 + 81\lambda = 0$$

$$\Rightarrow (52-\lambda)(800 - 60\lambda + \lambda^2 - 324) - 5904 + 117\lambda = 0$$

$$\Rightarrow (52-\lambda)(476 - 60\lambda + \lambda^2) - 5904 + 117\lambda = 0$$

$$\Rightarrow (24752 - 3120\lambda + 52\lambda^2 - 476\lambda + 60\lambda^2 - \lambda^3) - 5904 + 117\lambda = 0$$

$$\Rightarrow 24752 - 3596\lambda + 112\lambda^2 - \lambda^3 - 5904 + 117\lambda = 0$$

$$\Rightarrow 18848 - 3479\lambda + 112\lambda^2 - \lambda^3 = 0 \Rightarrow \lambda = \{6,83 ; 50,51 ; 54,67\} \Rightarrow \mathbf{I}_s = \beta \{6,83 ; 50,51 ; 54,67\} //$$

Buscando los ejes correspondientes

$$\bullet (\mathbf{I} - \beta \lambda \mathbf{I}) \vec{\omega}_s = 0 \Rightarrow \begin{pmatrix} 52-6,83 & -6 & -9 \\ -6 & 40-6,83 & -18 \\ -9 & -18 & 20-6,83 \end{pmatrix} \begin{pmatrix} \omega_{sx} \\ \omega_{sy} \\ \omega_{sz} \end{pmatrix} = \begin{pmatrix} 45,17 & -6 & -9 \\ -6 & 33,17 & -18 \\ -9 & -18 & 13,17 \end{pmatrix} \begin{pmatrix} \omega_{sx} \\ \omega_{sy} \\ \omega_{sz} \end{pmatrix} = \vec{0}$$

$$\Rightarrow \begin{cases} 45,17\omega_{sx} - 6\omega_{sy} - 9\omega_{sz} = 0 \\ -6\omega_{sx} + 33,17\omega_{sy} - 18\omega_{sz} = 0 \\ -9\omega_{sx} - 18\omega_{sy} + 13,17\omega_{sz} = 0 \end{cases} \quad f_1 \rightarrow f_1 - \frac{1}{2}f_2 \Rightarrow \begin{cases} 48,17\omega_{sx} - 22,59\omega_{sy} = 0 \\ -9\omega_{sx} - 18\omega_{sy} + 13,17\omega_{sz} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \omega_{sx} = 0,469\omega_{sy} \\ -9\omega_{sx} - 18\omega_{sy} + 13,17\omega_{sz} = 0 \end{cases} ; \omega_{sy} = 1 \Rightarrow \vec{\omega}_s = 0,469\hat{i} + \hat{j} + 1,687\hat{k} \Rightarrow \omega_s = 0,233\hat{i} + 0,496\hat{j} + 0,837\hat{k} //$$

$$\bullet (\mathcal{I} - \beta \lambda_1 \mathbb{1}) \vec{\omega}_1 = 0 \Rightarrow \begin{pmatrix} 52 - 50,51 & -6 & -9 \\ -6 & 40 - 50,51 & -18 \\ -9 & -18 & 20 - 50,51 \end{pmatrix} \begin{pmatrix} \omega_{1x} \\ \omega_{1y} \\ \omega_{1z} \end{pmatrix} = \begin{pmatrix} 1,49 & -6 & -9 \\ -6 & -10,51 & -18 \\ -9 & -18 & -30,51 \end{pmatrix} \begin{pmatrix} \omega_{1x} \\ \omega_{1y} \\ \omega_{1z} \end{pmatrix} = \vec{0}$$

$$\Rightarrow \begin{cases} 1,49 \omega_{1x} - 6 \omega_{1y} - 9 \omega_{1z} = 0 \\ -6 \omega_{1x} - 10,51 \omega_{1y} - 18 \omega_{1z} = 0 \\ -9 \omega_{1x} - 18 \omega_{1y} - 30,51 \omega_{1z} = 0 \end{cases} \quad f_1 \rightarrow f_1 - \frac{1}{2}f_2 \Rightarrow \begin{cases} 1,49 \omega_{1x} - 0,745 \omega_{1y} = 0 \\ -9 \omega_{1x} - 18 \omega_{1y} - 30,51 \omega_{1z} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \omega_{1x} = 0,166 \omega_{1y} ; \omega_{1y} = 1 \\ -9 \omega_{1x} - 18 \omega_{1y} - 30,51 \omega_{1z} = 0 \end{cases} \Rightarrow \vec{\omega}_1 = 0,166 \mathbf{i} + \mathbf{j} - 0,639 \mathbf{k} \Rightarrow \hat{\omega}_1 = 0,139 \mathbf{i} + 0,835 \mathbf{j} - 0,533 \mathbf{k}$$

$$\bullet (\mathcal{I} - \beta \lambda_2 \mathbb{1}) \vec{\omega}_2 = 0 \Rightarrow \begin{pmatrix} 52 - 54,67 & -6 & -9 \\ -6 & 40 - 54,67 & -18 \\ -9 & -18 & 20 - 54,67 \end{pmatrix} \begin{pmatrix} \omega_{2x} \\ \omega_{2y} \\ \omega_{2z} \end{pmatrix} = \begin{pmatrix} -2,67 & -6 & -9 \\ -6 & -14,67 & -18 \\ -9 & -18 & -34,67 \end{pmatrix} \begin{pmatrix} \omega_{2x} \\ \omega_{2y} \\ \omega_{2z} \end{pmatrix} = \vec{0}$$

$$\Rightarrow \begin{cases} -2,67 \omega_{2x} - 6 \omega_{2y} - 9 \omega_{2z} = 0 \\ -6 \omega_{2x} - 14,67 \omega_{2y} - 18 \omega_{2z} = 0 \\ -9 \omega_{2x} - 18 \omega_{2y} - 34,67 \omega_{2z} = 0 \end{cases} \quad f_1 \rightarrow f_1 - \frac{1}{2}f_2 \Rightarrow \begin{cases} 0,33 \omega_{2x} + 1,34 \omega_{2y} = 0 \\ -9 \omega_{2x} - 18 \omega_{2y} - 34,67 \omega_{2z} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \omega_{2x} = -4,06 \omega_{2y} ; \omega_{2y} = 1 \\ -9 \omega_{2x} - 18 \omega_{2y} - 34,67 \omega_{2z} = 0 \end{cases} \Rightarrow \vec{\omega}_2 = -4,06 \mathbf{i} + \mathbf{j} - 0,535 \mathbf{k} \Rightarrow \hat{\omega}_2 = -0,963 \mathbf{i} + 0,237 \mathbf{j} - 0,126 \mathbf{k}$$

$$\Rightarrow \begin{cases} \chi_1 = \hat{\omega}_1 = 0,139 \mathbf{i} + 0,835 \mathbf{j} + 0,533 \mathbf{k} \\ \chi_2 = \hat{\omega}_2 = -0,963 \mathbf{i} + 0,237 \mathbf{j} - 0,126 \mathbf{k} \\ \chi_3 = \hat{\omega}_3 = -0,963 \mathbf{i} + 0,237 \mathbf{j} - 0,126 \mathbf{k} \end{cases}$$

h. Vuelva a encontrar las componentes de la velocidad angular, pero ahora en los sistemas de referencia asociados a los tensores determinados en el ítem anterior. Luego encuentre el momentum angular en cada caso.

• Para el origen en el centro de masa, la diagonal del objeto desde la esquina o del centro de masa, posee la misma dirección, entonces, la velocidad angular es:

$$\vec{\omega} = \omega \begin{pmatrix} 1\hat{i} + 2\hat{j} + 3\hat{k} \\ \sqrt{14} \end{pmatrix} \Rightarrow \vec{L}_{cm} = I_{cm}\vec{\omega} = \frac{Ma^2}{12} \begin{pmatrix} 13 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 5 \end{pmatrix} \omega \begin{pmatrix} 1\hat{i} + 2\hat{j} + 3\hat{k} \\ \sqrt{14} \end{pmatrix} = \frac{Ma^2\omega}{12\sqrt{14}} (13\hat{i} + 20\hat{j} + 15\hat{k})$$

• Para los ejes principales en la esquina, hay que

i. Determine la energía cinética respecto a los cuatro de sistemas de referencia.

Rotación del cuerpo rígido

Problema 1. (Taylor 10.23, 10.30, 10.41)

10.23 ** Consider a rigid plane body or "lamina," such as a flat piece of sheet metal, rotating about a point O in the body. If we choose axes so that the lamina lies in the xy plane, which elements of the inertia tensor \mathbf{I} are automatically zero? Prove that $I_{zz} = I_{xx} + I_{yy}$.

$$\text{Entradas del tensor de inercia } I_{ij} = \iiint pm (\delta_{ij} \sum_k x_k^2 - x_i x_j) dV$$

$$* I_{xz} = I_{yz} = 0$$

$$* I_{xx} = \iint pm (y^2 + z^2) dx dy$$

$$* I_{zz} = \iint pm (x^2 + y^2) dx dy = I_{yy} + I_{xx}$$

$$* I_{yy} = \iint pm (x^2 + z^2) dx dy$$

$$\Rightarrow \mathbf{I} = \begin{pmatrix} I_{xx} & I_{xy} & 0 \\ I_{xy} & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}$$

10.30 * Consider a lamina, such as a flat piece of sheet metal, rotating about a point O in the body. Prove that the axis through O and perpendicular to the plane is a principal axis. [Hint: See Problem 10.23.]

Un eje principal de rotación tiene la característica de cumplir $I_i = I_i \omega_i$, entonces considerando $\vec{\omega} = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}$

$$\Rightarrow \vec{I} = \vec{I}\vec{\omega} = (I_{xx}\omega_x + I_{xy}\omega_y)\hat{i} + (I_{xy}\omega_x + I_{yy}\omega_y)\hat{j} + I_{zz}\omega_z \Rightarrow \text{Cualquier eje perpendicular al plano que pase por punto } O, \text{ es un eje principal}$$

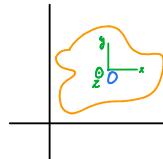
10.41 ** Consider a lamina rotating freely (no torques) about a point O of the lamina. Use Euler's equations to show that the component of ω in the plane of the lamina has constant magnitude. [Hint: Use the results of Problems 10.23 and 10.30. According to Problem 10.30, if you choose the direction \mathbf{e}_3 normal to the plane of the lamina, \mathbf{e}_3 points along a principal axis. Then what you have to prove is that the time derivative of $\omega_1^2 + \omega_2^2$ is zero.]

Suponiendo que I_x, I_y e I_z son los ejes principales del cuerpo y además $I_z = I_x + I_y$, al escribir las

Ecuaciones de Euler

$$\left\{ \begin{array}{l} (I_z - I_x)\omega_z \dot{\omega}_x - I_x \dot{\omega}_z = 0 \Rightarrow -I_z \omega_x \dot{\omega}_z - I_x \dot{\omega}_z = 0 \Rightarrow \dot{\omega}_z = -\omega_x \dot{\omega}_x \\ (I_z - I_y)\omega_z \dot{\omega}_y - I_y \dot{\omega}_z = 0 \Rightarrow I_z \omega_y \dot{\omega}_z - I_y \dot{\omega}_z = 0 \Rightarrow \dot{\omega}_z = \omega_y \dot{\omega}_y \\ (I_x - I_y)\omega_x \dot{\omega}_y - I_y \dot{\omega}_x = 0 \end{array} \right.$$

$$\Rightarrow \frac{\dot{\omega}_z}{\dot{\omega}_x} = \frac{-\omega_x \dot{\omega}_x}{\omega_y \dot{\omega}_y} = \frac{-\omega_x}{\omega_y} \Rightarrow \dot{\omega}_x \omega_y = -\dot{\omega}_y \omega_x \Rightarrow \dot{\omega}_x \omega_x + \dot{\omega}_y \omega_y = 0 \Rightarrow \frac{d}{dt} (\omega_x^2 + \omega_y^2) = 0$$



Motion of a Symmetric Top with One Point Fixed

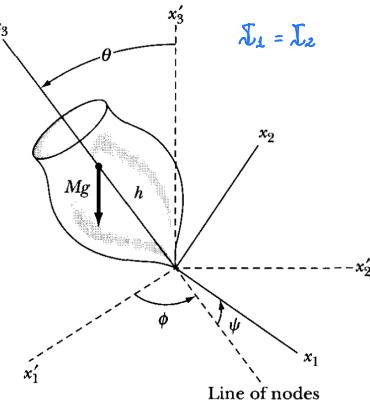
Buscando el Lagrangiano del sistema $\mathcal{L} = T - U$

$$\text{La energía cinética: } T_{\text{rot}} = \frac{1}{2} (I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2)$$

$$\Rightarrow T_{\text{rot}} = \frac{1}{2} I_1 (\dot{\theta}^2 + \dot{\psi}^2) + \frac{1}{2} I_3 \dot{\psi}^2$$

$$= \frac{1}{2} I_1 [(\dot{\phi} \sin \theta \sin \psi + \dot{\theta} \cos \psi)^2 + (\dot{\phi} \sin \theta \cos \psi - \dot{\theta} \sin \psi)] + \frac{1}{2} I_3 (\dot{\phi} \cos \theta + \dot{\psi})^2$$

$$= \frac{1}{2} I_1 [\dot{\phi}^2 \sin^2 \theta \sin^2 \psi + 2 \cancel{\dot{\phi} \dot{\theta} \sin \theta \sin \psi \cos \psi} + \dot{\theta}^2 \cos^2 \psi + \dot{\phi}^2 \sin^2 \theta \cos^2 \psi - \cancel{2 \dot{\phi} \dot{\theta} \sin \theta \sin \psi \cos \psi} + \dot{\theta}^2 \sin^2 \psi] + \dots$$



$$\boxed{\begin{aligned}\omega_1 &= \dot{\phi} \sin \theta \sin \psi + \dot{\theta} \cos \psi \\ \omega_2 &= \dot{\phi} \sin \theta \cos \psi - \dot{\theta} \sin \psi \\ \omega_3 &= \dot{\phi} \cos \theta + \dot{\psi}\end{aligned}}$$

$$\dots + \frac{1}{2} I_3 (\dot{\phi}^2 \cos^2 \theta + 2 \dot{\phi} \dot{\psi} \cos \theta + \dot{\psi}^2)$$

$$= \frac{1}{2} I_1 [\dot{\phi}^2 \sin^2 \theta \sin^2 \psi + \dot{\phi}^2 \sin^2 \theta \cos^2 \psi + \dot{\theta}^2 \cos^2 \psi + \dot{\theta}^2 \sin^2 \psi] + \frac{1}{2} I_3 (\dot{\phi}^2 \cos^2 \theta + 2 \dot{\phi} \dot{\psi} \cos \theta + \dot{\psi}^2)$$

$$= \frac{1}{2} I_1 (\dot{\phi}^2 \sin^2 \theta + \dot{\theta}^2) + \frac{1}{2} I_3 (\dot{\phi}^2 \cos^2 \theta + 2 \dot{\phi} \dot{\psi} \cos \theta + \dot{\psi}^2)$$

$$\text{La energía potencial: } U = mgh \cos \theta$$

$$\Rightarrow \mathcal{L} = \frac{1}{2} I_1 (\dot{\phi}^2 \sin^2 \theta + \dot{\theta}^2) + \frac{1}{2} I_3 (\dot{\phi}^2 \cos^2 \theta + 2 \dot{\phi} \dot{\psi} \cos \theta + \dot{\psi}^2) - mgh \cos \theta \leftarrow \text{No es función de } \phi \text{ ó } \psi, \text{ estas serán entonces variables cíclicas}$$

Buscando los momentos generalizados

$$*\ p_\psi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = I_3 \dot{\phi} \cos \theta + I_3 \dot{\psi} \Rightarrow \dot{\psi} = \frac{p_\psi - I_3 \dot{\phi} \cos \theta}{I_3}$$

$$*\ p_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = I_1 \dot{\phi} \sin^2 \theta + I_3 \dot{\phi} \cos^2 \theta + I_3 \dot{\psi} \cos \theta \Rightarrow p_\phi = I_1 \dot{\phi} \sin^2 \theta + I_3 \dot{\phi} \cos^2 \theta + I_3 \frac{p_\psi - I_3 \dot{\phi} \cos \theta}{I_3} \cdot \cos \theta$$

$$\Rightarrow \dot{\phi}^* = \frac{p_\phi - p_\psi \cos \theta}{I_1 \sin^2 \theta} \quad y \quad \Rightarrow \dot{\psi} = \frac{p_\psi - p_\phi \cos \theta}{I_3 \sin^2 \theta} \cdot \cos \theta$$

Tomando un sistema conservativo

$$E = T_{\text{rot}} + U = \frac{1}{2} I_1 (\dot{\theta}^2 \sin^2 \theta + \dot{\phi}^2) + \frac{1}{2} I_3 w_3^2 + mgh \cos \theta = \frac{1}{2} I_1 \left[\frac{(p_\theta - p_\phi \cos \theta)^2}{I_1 \sin^2 \theta} + \dot{\phi}^2 \right] + \frac{1}{2} I_3 w_3^2 + mgh \cos \theta$$
$$= \frac{1}{2} \frac{(p_\theta - p_\phi \cos \theta)^2}{I_1 \sin^2 \theta} + \frac{1}{2} I_1 \dot{\phi}^2 + \frac{1}{2} I_3 w_3^2 + mgh \cos \theta$$

Se define otra cantidad constante: $E' = E - \frac{1}{2} I_3 w_3^2$

$$\Rightarrow E' = \frac{1}{2} I_1 \dot{\phi}^2 + \frac{1}{2} \frac{(p_\theta - p_\phi \cos \theta)^2}{I_1 \sin^2 \theta} + mgh \cos \theta \Rightarrow E' = \frac{1}{2} I_1 \dot{\phi}^2 + V(\theta)$$

Con $V(\theta) = \frac{1}{2} \frac{(p_\theta - p_\phi \cos \theta)^2}{I_1 \sin^2 \theta} + mgh \cos \theta$

Problema 2. (Thornton 11.29)

11-29. Investigate the motion of the symmetric top discussed in Section 11.11 for the case in which the axis of rotation is vertical (i.e., the x_3 - and x_3 -axes coincide). Show that the motion is either stable or unstable depending on whether the quantity $4I_1Mgh/I_3^2\omega_3^2$ is less than or greater than unity. Sketch the effective potential $V(\theta)$ for the two cases, and point out the features of these curves that determine whether the motion is stable. If the top is set spinning in the stable configuration, what is the effect as friction gradually reduces the value of ω_3 ? (This is the case of the "sleeping top".)

Buscando el lagrangiano del sistema $L = T - U$

$$\text{La energía cinética: } T_{\text{rot}} = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2)$$

$$\begin{aligned} \Rightarrow T_{\text{rot}} &= \frac{1}{2}I_1(\omega_1^2 + \omega_2^2) + \frac{1}{2}I_3\omega_3^2 = \frac{1}{2}I_1[(\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi)^2 + (\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi)^2] + \frac{1}{2}I_3(\dot{\psi}\cos\theta + \dot{\psi})^2 \\ &= \frac{1}{2}I_1[\dot{\phi}^2\sin^2\theta\sin^2\psi + 2\dot{\phi}\dot{\theta}\sin\theta\sin\psi\cos\psi + \dot{\theta}^2\cos^2\psi + \dot{\phi}^2\sin^2\theta\cos^2\psi - 2\dot{\phi}\dot{\theta}\sin\theta\sin\psi\cos\psi + \dot{\theta}^2\sin^2\psi] + \dots \\ &\quad \dots + \frac{1}{2}I_3(\dot{\psi}^2\cos^2\theta + 2\dot{\phi}\dot{\psi}\cos\theta + \dot{\psi}^2) \\ &= \frac{1}{2}I_1[\dot{\phi}^2\sin^2\theta\sin^2\psi + \dot{\phi}^2\sin^2\theta\cos^2\psi + \dot{\theta}^2\cos^2\psi + \dot{\theta}^2\sin^2\psi] + \frac{1}{2}I_3(\dot{\psi}^2\cos^2\theta + 2\dot{\phi}\dot{\psi}\cos\theta + \dot{\psi}^2) \\ &= \frac{1}{2}I_1(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2) + \frac{1}{2}I_3(\dot{\psi}^2\cos^2\theta + 2\dot{\phi}\dot{\psi}\cos\theta + \dot{\psi}^2) \end{aligned}$$

$$\text{La energía potencial: } U = mgh\cos\theta$$

$$\Rightarrow L = \frac{1}{2}I_1(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2) + \frac{1}{2}I_3(\dot{\psi}^2\cos^2\theta + 2\dot{\phi}\dot{\psi}\cos\theta + \dot{\psi}^2) - mgh\cos\theta \rightarrow \text{No es función de } \phi \text{ ó } \psi, \text{ estas serán entonces variables cíclicas}$$

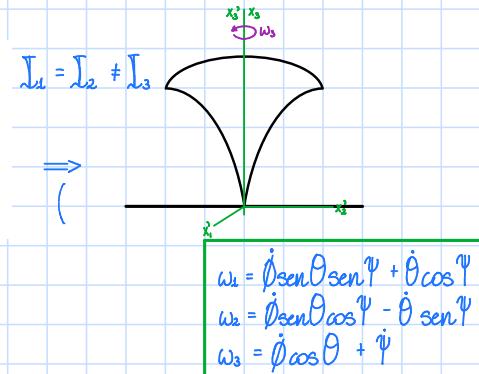
Ahora, como el trompo está fijo en $\theta = 0$, $\dot{\theta} = 0$

$$\Rightarrow L = \frac{1}{2}I_1\dot{\phi}^2 + \frac{1}{2}I_3(\dot{\psi}^2 + 2\dot{\phi}\dot{\psi} + \dot{\psi}^2) - mgh ; \quad \text{Buscando los momentos generalizados}$$

$$*\rho_\psi = \frac{\partial L}{\partial \dot{\psi}} = I_3(\dot{\phi} + \dot{\psi}) \quad * \rho_\phi = \frac{\partial L}{\partial \dot{\phi}} = I_3(\dot{\phi} + \dot{\psi}) \Rightarrow \rho_\psi = \rho_\phi = I_3(\dot{\phi} + \dot{\psi}) = I_3\omega_3$$

$$\text{La energía } E = \frac{1}{2}I_3\omega_3^2 + mgh \Rightarrow E' = E - \frac{1}{2}I_3\omega_3^2 = mgh$$

Continua...



Como se desea estudiar el comportamiento del trompo en $\theta = 0$, se puede estudiar lo que sucede en las cercanías de dicho ángulo, para esto es posible utilizar la expresión de E' para el trompo inclinado.

$$\Rightarrow E' = mgh \quad y \quad E' = \frac{1}{2} I_1 \dot{\theta}^2 + \frac{1}{2} \left(p_\theta - p_\phi \cos \theta \right)^2 + mgh \cos \theta$$

$$\Rightarrow mgh = \frac{1}{2} I_1 \dot{\theta}^2 + \frac{1}{2} \left(p_\theta - p_\phi \cos \theta \right)^2 + mgh \cos \theta = \frac{1}{2} \frac{I_1^2 \sin^2 \theta \cdot \dot{\theta}^2}{I_1 \sin^2 \theta} + \left(p_\theta - p_\phi \cos \theta \right)^2 + mgh \cos \theta$$

$$\Rightarrow mgh = \frac{1}{2} \frac{I_1^2 (\sin \theta \cdot \dot{\theta})^2}{I_1 \sin^2 \theta} + \left(p_\theta - p_\phi \cos \theta \right)^2 + mgh \cos \theta \quad \text{Tomando } z = \cos \theta \quad y \quad \dot{z} = -\sin \theta \cdot \dot{\theta}$$

$$\Rightarrow mgh = \frac{1}{2} \frac{I_1^2 \dot{z}^2 + (p_\theta - p_\phi z)^2}{I_1 (1-z^2)} + mgh z \Rightarrow 2mgh(1-z)(1+z)I_1 = I_1^2 \dot{z}^2 + (p_\theta - p_\phi z)^2$$

$$\text{Recordando que } p_\theta = p_\phi = I_3 \omega_3 \Rightarrow 2mgh(1-z)^2(1+z)I_1 = I_1^2 \dot{z}^2 + I_3^2 \omega_3^2 (1-z)^2$$

$$\Rightarrow I_1^2 \dot{z}^2 = 2mgh(1-z)^2(1+z)I_1 - I_3^2 \omega_3^2 (1-z)^2 = (1-z)^2 [2mgh(1+z)I_1 - I_3^2 \omega_3^2]$$

$$\Rightarrow \dot{z}^2 = \frac{(1-z)^2}{I_1^2} [2mgh(1+z)I_1 - I_3^2 \omega_3^2]$$

Notese que si ω_3 es muy grande, el parentesis cuadrado sería negativo, entonces la única forma en que la expresión tendría sentido es si $\theta=0$ y $z=1$, siendo entonces que el trompo gira muy rápido sin inclinación.

$$\Rightarrow \dot{z}^2 = \frac{(1-z)^2}{I_1^2} [2mgh(1+z)I_1 - I_3^2 \omega_3^2] \geq 0 \quad y \quad [2mgh(1+z)I_1 - I_3^2 \omega_3^2] < 0 \quad \text{pero } z=1$$

$$\Rightarrow \dot{z}^2 = 0 \quad y \quad \frac{4mghI_1 - I_3^2 \omega_3^2}{I_1^2} < 0 \Rightarrow 1 > \frac{4mghI_1}{I_3^2 \omega_3^2} \therefore \text{El giro sin inclinación del trompo es estable si se cumple lo anterior. //}$$

Si ω_3 no es suficientemente grande para satisfacer la condición de estabilidad, la rotación es inestable. //

$$\omega_c = \sqrt{\frac{4mghI_1}{I_3^2}} \Rightarrow \omega_3 > \omega_c$$

Problema 3. (Thornton 11.31)

11-31. Consider a thin homogeneous plate with principal moments of inertia

$$I_1 \text{ along the principal axis } x_1$$

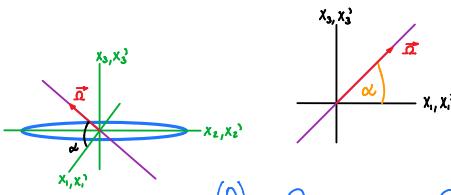
$$I_2 > I_1 \text{ along the principal axis } x_2$$

$$I_3 = I_1 + I_2 \text{ along the principal axis } x_3$$

Let the origins of the x_i and x'_i systems coincide and be located at the center of mass O of the plate. At time $t = 0$, the plate is set rotating in a force-free manner with an angular velocity Ω about an axis inclined at an angle α from the plane of the plate and perpendicular to the x_2 -axis. If $I_2/I_1 \equiv \cos 2\alpha$, show that at time t the angular velocity about the x_2 -axis is

$$\omega_2(t) = \Omega \cos \alpha \tanh(\Omega t \sin \alpha)$$

$$I_1 = I_2 \cos(2\alpha) \quad \text{y} \quad I_3 = I_1 + I_2 = I_2 [1 + \cos(2\alpha)]$$



$$\bar{\omega}(0) = \Omega \cos \alpha \hat{e}_x + \Omega \sin \alpha \hat{e}_z$$

Ecuaciones de Euler

$$\Rightarrow \begin{cases} (I_2 - I_3)\omega_2\omega_3 - I_1\dot{\omega}_2 = 0 \\ (I_3 - I_1)\omega_3\omega_1 - I_2\dot{\omega}_3 = 0 \\ (I_1 - I_2)\omega_1\omega_2 - I_3\dot{\omega}_1 = 0 \end{cases} \Rightarrow \begin{cases} -I_2 \cancel{\cos(2\alpha)}\omega_2\omega_3 - I_1\dot{\omega}_2 = 0 \\ \cancel{I_2}\omega_3\omega_1 - I_2\dot{\omega}_3 = 0 \\ I_2[\cos(2\alpha) - 1]\omega_1\omega_2 - I_2[1 + \cos(2\alpha)]\dot{\omega}_1 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} -\omega_2\omega_3 - \dot{\omega}_2 = 0 \\ \omega_3\omega_1 - \dot{\omega}_3 = 0 \\ -\cancel{\sin^2 \alpha} \cdot \omega_1\omega_2 - \cancel{\cos^2 \alpha} \cdot \dot{\omega}_1 = 0 \end{cases} \Rightarrow \begin{cases} \dot{\omega}_2 = -\omega_2\omega_3 \\ \dot{\omega}_3 = \omega_3\omega_1 \\ \dot{\omega}_1 = -\tan^2 \alpha \cdot \omega_1\omega_2 \end{cases}$$

Sistema de ecuaciones diferenciales acopladas

$$\Rightarrow \omega_1\omega_2\omega_3 = \dot{\omega}_2\omega_3 = -\dot{\omega}_2\omega_1 = -\cot^2 \alpha \cdot \dot{\omega}_3\omega_1 \Rightarrow \int_0^t \omega_2 \frac{d\omega_2}{dt} dt = \int_{\omega_2(0)}^{\omega_2} \omega_2' d\omega_2' = -\int_{\omega_3(0)}^{\omega_3} \omega_1' d\omega_1' = -\cot^2 \alpha \cdot \int_{\omega_3(0)}^{\omega_3} \omega_3' d\omega_3'$$

$$\Rightarrow \omega_2^2 - \omega_2^2(0) = \omega_3^2(0) - \omega_3^2 = \cot^2 \alpha [\omega_3^2(0) - \omega_3^2] \quad ; \text{ Evaluando las condiciones iniciales}$$

$$\Rightarrow \omega_2^2 = \Omega^2 \cos^2 \alpha - \omega_1^2 = \Omega^2 \cancel{\cos^2 \alpha} \cancel{\sin^2 \alpha} - \omega_3^2 \cot^2 \alpha$$

Como $\omega_2 = \omega_3\omega_1$, hay que solucionarlo

$$*\omega_1: \Rightarrow \cancel{\Omega^2 \cos^2 \alpha} - \omega_1^2 = \cancel{\Omega^2 \cos^2 \alpha} - \omega_3^2 \cot^2 \alpha \Rightarrow \omega_1^2 = \omega_3^2 \cot^2 \alpha$$

$$*\omega_3: \Rightarrow \omega_2^2 = \Omega^2 \cos^2 \alpha - \omega_3^2 \cot^2 \alpha \Rightarrow \omega_3^2 = \frac{\Omega^2 \cos^2 \alpha - \omega_2^2}{\cot^2 \alpha}$$

$$\Rightarrow \dot{\omega}_2^2 = \omega_1^2 \omega_3^2 = \omega_3^2 \cot^2 \alpha \omega_3^2 \Rightarrow \dot{\omega}_2 = \omega_3^2 \cot \alpha = \frac{\Omega^2 \cos^2 \alpha - \omega_2^2}{\cot^2 \alpha} \Rightarrow \int_{\omega_2(0)}^{\omega_2} \frac{1}{\Omega^2 \cos^2 \alpha - \omega_2^2} d\omega_2 = \frac{1}{\cot \alpha} \int_0^t dt$$

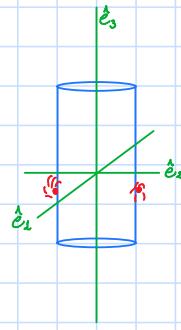
$$\Rightarrow \frac{1}{\Omega \cos \alpha} \tanh^{-1} \left(\frac{\omega_2}{\Omega \cos \alpha} \right) = \tanh^{-1} \left(\frac{\omega_2}{\Omega \cos \alpha} \right) = \Omega \sin \alpha t \Rightarrow \omega_2 = \Omega \cos \alpha \tanh(\Omega \sin \alpha t) //$$

Problema 4. (Taylor 10.44)

10.44 ★★ An axially symmetric space station (principal axis \mathbf{e}_3 , and $\lambda_1 = \lambda_2$) is floating in free space. It has rockets mounted symmetrically on either side that are firing and exert a constant torque Γ about the symmetry axis. Solve Euler's equations exactly for ω (relative to the body axis) and describe the motion. At $t = 0$ take $\omega = (\omega_{10}, 0, \omega_{30})$.

Euler's Equations

$$\Rightarrow \begin{cases} (\mathbb{J}_2 - \mathbb{J}_3)\omega_2\omega_3 - \mathbb{J}_1\dot{\omega}_1 = -N_1 \\ (\mathbb{J}_3 - \mathbb{J}_1)\omega_3\omega_1 - \mathbb{J}_2\dot{\omega}_2 = -N_2 \\ (\mathbb{J}_1 - \mathbb{J}_2)\omega_1\omega_2 - \mathbb{J}_3\dot{\omega}_3 = -N_3 \end{cases}$$



$$\Rightarrow \begin{cases} (\mathbb{J}_1 - \mathbb{J}_3)\omega_2\omega_3 - \mathbb{J}_1\dot{\omega}_1 = 0 \\ (\mathbb{J}_3 - \mathbb{J}_1)\omega_3\omega_1 - \mathbb{J}_1\dot{\omega}_2 = 0 \\ -\mathbb{J}_3\dot{\omega}_3 = -N_3 \end{cases} \Rightarrow \dot{\omega}_1 = \frac{(\mathbb{J}_3 - \mathbb{J}_1)\omega_2\omega_3}{\mathbb{J}_1} = Q\omega_2\omega_3 \text{ y } \dot{\omega}_2 = -\frac{(\mathbb{J}_1 - \mathbb{J}_3)\omega_3\omega_1}{\mathbb{J}_1} = -Q\omega_3\omega_1$$

$$-\mathbb{J}_3\dot{\omega}_3 = -N_3 \Rightarrow \mathbb{J}_3\dot{\omega}_3 = +N_3 \Rightarrow \omega_3 = \frac{N_3}{\mathbb{J}_3}t + \omega_{30} = ct + \omega_{30}$$

Problema 1. (Valor: 25 pts.)

El juguete que se muestra en la figura está formado por una barra (masa M , longitud L) y dos esferas pequeñas, que se consideran puntuales y con masa M cada una. Las dos masas están atadas a un eje perpendicular al plano de la página y que pasa por el CM del sistema, que gira con una velocidad $\vec{\omega}$. El sistema completo, está sobre una plataforma plana, que gira con la velocidad $\vec{\Omega}$, como se muestra en la figura.

a. Determine las ecuaciones de Euler del sistema. **Nota:** Recuerde especificar los sistemas de referencia utilizados con toda claridad.

b. ¿Cuáles son las limitaciones del movimiento?

Nota: El momento de inercia de una partícula, localizada a una distancia ℓ del origen es $I = M\ell^2$. El momento de inercia de una barra, con el eje pasando por el CM, es $I = 1/12M\ell^2$.

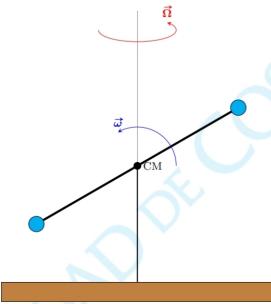


Figura 1: Configuración problema 1.

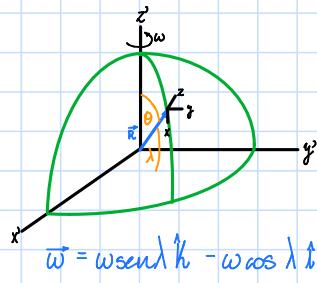
Problema 2. (Valor: 25 pts.)

Si una partícula se proyecta verticalmente hacia arriba hasta una altura h sobre un punto en la superficie de la Tierra en una latitud norte λ , demuestre que golpea el suelo en un punto al oeste, dado por:

$$d = \frac{4}{3}\omega \cos \lambda \sqrt{\frac{8h^3}{g}}$$

Nota: ω corresponde a la rapidez angular de la Tierra y g es la aceleración de la gravedad. No considere los efectos debido a la fuerza centrífuga.

$$\ddot{\vec{r}} = -g\hat{k} - 2\vec{\omega} \times \dot{\vec{r}}$$



$$\vec{\omega} \times \dot{\vec{r}} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\omega \cos \lambda & \hat{j} & \hat{g} \\ \hat{x} & \hat{g} & \hat{z} \end{vmatrix} = -\hat{j}\omega \sin \lambda \hat{i} - (-\hat{z}\omega \cos \lambda - \hat{x}\omega \sin \lambda)\hat{j} - \hat{g}\omega \cos \lambda \hat{k}$$

$$= -\hat{j}\omega \sin \lambda \hat{i} + (\hat{z}\omega \cos \lambda + \hat{x}\omega \sin \lambda)\hat{j} - \hat{g}\omega \cos \lambda \hat{k}$$

$$\ddot{x} = +2\hat{j}\omega \sin \lambda$$

$$\ddot{y} = -2\hat{z}\omega \cos \lambda - 2\hat{x}\omega \sin \lambda$$

$$\ddot{z} = -g + 2\hat{x}\omega \cos \lambda$$

$$\text{Primer orden}$$

$$\ddot{y} = -2(\dot{z}_0 - gt) \omega \cos \lambda \Rightarrow \ddot{y} = -2(\dot{z}_0 t - \frac{1}{2}gt^2) \omega \cos \lambda \Rightarrow y = -2\left(\frac{\dot{z}_0}{2}t^2 - \frac{1}{6}gt^3\right) \omega \cos \lambda$$

$$0 = 0 + \sqrt{2gh} t - \frac{1}{2}gt^2 \Rightarrow t\left(\sqrt{2gh} - \frac{1}{2}gt\right) = 0$$

$$(t = \frac{2\sqrt{2gh}}{g}) \Rightarrow \text{Tiempo vuelo completo} \approx$$

$$t = \frac{\dot{z}_0}{g}$$

$$\text{orden cero}$$

$$h = 0 + \dot{z}_0 \frac{\dot{z}_0}{g} - \frac{1}{2}g \frac{\dot{z}_0^2}{g^2} = \frac{\dot{z}_0^2}{g} - \frac{1}{2} \frac{\dot{z}_0^2}{g} = \frac{1}{2} \frac{\dot{z}_0^2}{g} \Rightarrow h = \frac{1}{2} \frac{\dot{z}_0^2}{g} \Rightarrow \dot{z}_0 = \sqrt{2gh}$$

$$\boxed{\dot{z}_0 = \sqrt{2gh}} \quad \text{Velocidad inicial}$$

$$y = -2\left(\frac{\dot{z}_0}{2}t^2 - \frac{1}{6}gt^3\right) \omega \cos \lambda$$

$$= -2 \left[\frac{\sqrt{2gh}}{2} \cdot \frac{4(2gh)}{g^2} - \frac{1}{6} g \frac{2^3 \sqrt{2gh}^3}{g^2} \right] \omega \cos \lambda$$

$$= -2 \left[\frac{2\sqrt{2gh}^3}{g^2} - \frac{8}{6} \frac{\sqrt{2gh}^3}{g^2} \right] \omega \cos \lambda$$

$$= -2 \left[\frac{2}{3} \frac{\sqrt{2gh}^3}{g^2} \right] \omega \cos \lambda$$

$$= -\frac{4}{3} \frac{\sqrt{2gh}^3}{g^2} \omega \cos \lambda = \frac{4}{3} \sqrt{\frac{8h^3}{g}} \omega \cos \lambda$$

$$\frac{\sqrt{g^2}}{g^2} = \frac{\sqrt{g}}{g} = \frac{1}{\sqrt{g}}$$

Problema 3. (Valor: 25 pts.)

Tres partículas de masa igual ($m = 2 \text{ kg}$) se localizan en $\vec{r}_1 = 2 \text{ m } \hat{x}$, $\vec{r}_2 = 2 \text{ m } \hat{y}$ y $\vec{r}_3 = 2 \text{ m } \hat{z}$, medidas con respecto a O .

- Determine el tensor de inercia con respecto a O .
- Encuentre el CM del sistema y coloque un sistema referencia con el origen allí (O') y con los ejes paralelos al inicial. Vuelva a determinar el tensor de inercia.
- Encuentre los ejes principales y sus correspondientes momentos de inercia, con respecto a O' .

a)

$$\mathcal{J}_{xx} = \begin{cases} m(x_1^2 + z_1^2) = 2(0) = 0 \\ m(x_2^2 + z_2^2) = 2(2^2 + 0) = 2^3 \\ m(x_3^2 + z_3^2) = 2(0 + 2^2) = 2^3 \end{cases} = 2^3 + 2^3 = 16$$

$$\mathcal{J}_{yy} = \begin{cases} m(x_1^2 + z_1^2) = 2(2^2 + 0) = 2^3 \\ m(x_2^2 + z_2^2) = 2(0) = 0 \\ m(x_3^2 + z_3^2) = 2(0 + 2^2) = 2^3 \end{cases} = 16$$

$$\mathcal{J}_{zz} = \begin{cases} m(x_1^2 + y_1^2) = 2(2^2 + 0) = 2^3 \\ m(x_2^2 + y_2^2) = 2(0 + 2^2) = 2^3 \\ m(x_3^2 + y_3^2) = 2(0) = 0 \end{cases} = 16$$

$$\mathcal{J} = \begin{pmatrix} 16 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 16 \end{pmatrix}$$

$$\mathcal{J}_{xy} = \begin{cases} -m x_1 y_1 = -2 \cdot 2 \cdot 0 = 0 \\ -m x_2 y_2 = -2 \cdot 0 \cdot 2 = 0 \\ -m x_3 y_3 = -2 \cdot 0 \cdot 0 = 0 \end{cases} = 0 = \mathcal{J}_{xz} = \mathcal{J}_{yz}$$

b)

$$x_{cm} = \frac{1}{3m} (m x_1 + m x_2 + m x_3) = \frac{2}{3}$$

$$y_{cm} = \frac{1}{3m} (m y_1 + m y_2 + m y_3) = \frac{2}{3}$$

$$z_{cm} = \frac{1}{3m} (m z_1 + m z_2 + m z_3) = \frac{2}{3}$$

$$\vec{R}_{cm} = \frac{2}{3} (1\hat{i} + 1\hat{j} + 1\hat{k})$$

$$I_{cm} = \begin{pmatrix} 16 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 16 \end{pmatrix} - 3m \begin{pmatrix} \frac{dx^2 + dz^2}{2} & -dydz & -dxdz \\ -dydz & \frac{dy^2 + dz^2}{2} & -dbdy \\ -dxdz & -dbdy & \frac{dx^2 + dy^2}{2} \end{pmatrix} \Bigg| \frac{1}{9} \begin{pmatrix} 8 & -4 & -4 \\ -4 & 8 & -4 \\ -4 & -4 & 8 \end{pmatrix}$$

$$\begin{pmatrix} 16 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 16 \end{pmatrix} - \frac{2}{3} \begin{pmatrix} 8 & -4 & -4 \\ -4 & 8 & -4 \\ -4 & -4 & 8 \end{pmatrix} = \begin{pmatrix} \frac{32}{3} & \frac{8}{3} & \frac{8}{3} \\ \frac{8}{3} & \frac{32}{3} & \frac{8}{3} \\ \frac{8}{3} & \frac{8}{3} & \frac{32}{3} \end{pmatrix}$$

$$I_{cm} = \begin{pmatrix} \frac{32}{3} & \frac{8}{3} & \frac{8}{3} \\ \frac{8}{3} & \frac{32}{3} & \frac{8}{3} \\ \frac{8}{3} & \frac{8}{3} & \frac{32}{3} \end{pmatrix}$$

$$\begin{vmatrix} \frac{32}{3} - \lambda & \frac{8}{3} & \frac{8}{3} \\ \frac{8}{3} & \frac{32}{3} - \lambda & \frac{8}{3} \\ \frac{8}{3} & \frac{8}{3} & \frac{32}{3} - \lambda \end{vmatrix} = 0$$

$$(\frac{32}{3} - \lambda) \left[\left(\frac{32}{3} - \lambda \right)^2 - \frac{8^2}{3^2} \right] - \frac{8}{3} \left[\frac{8}{3} \left(\frac{32}{3} - \lambda \right) - \frac{8^2}{3^2} \right] + \frac{8}{3} \left[\frac{8^2}{3^2} - \frac{8}{3} \left(\frac{32}{3} - \lambda \right) \right] = 0$$

$$(\frac{32}{3} - \lambda) \left[\left(\frac{32}{3} - \lambda \right)^2 - \frac{8^2}{3^2} \right] - \frac{8}{3} \left[\frac{8}{3} \left(\frac{32}{3} - \lambda \right) - \frac{8^2}{3^2} \right] - \frac{8}{3} \left[\frac{8}{3} \left(\frac{32}{3} - \lambda \right) - \frac{8^2}{3^2} \right] = 0$$

$$(\frac{32}{3} - \lambda) \left[\left(\frac{32}{3} - \lambda \right)^2 - \frac{8^2}{3^2} \right] + \left[\frac{8}{3} \left(\frac{32}{3} - \lambda \right) - \frac{8^2}{3^2} \right] \left(-\frac{8}{3} - \frac{8}{3} \right) = 0$$

$$(\frac{32}{3} - \lambda) \left[\left(\frac{32}{3} - \lambda \right)^2 - \frac{8^2}{3^2} \right] - \frac{16}{3} \left[\frac{8}{3} \left(\frac{32}{3} - \lambda \right) - \frac{8^2}{3^2} \right] = 0$$

$$\left(\frac{32}{3} - \lambda \right)^3 - \frac{8^2}{3^2} \left(\frac{32}{3} - \lambda \right) - \frac{16}{3} \cdot \frac{8}{3} \left(\frac{32}{3} - \lambda \right) + \frac{16 \cdot 8^2}{3^2} = 0$$

$$\left(\frac{32}{3} - \lambda \right)^3 - \frac{8^2}{3^2} \left(\frac{32}{3} - \lambda \right) - \frac{128}{9} \left(\frac{32}{3} - \lambda \right) + \frac{1024}{27} = 0$$

$$\left(\frac{32}{3} - \lambda \right)^3 - \frac{64}{3} \left(\frac{32}{3} - \lambda \right) + \frac{1024}{27} = 0$$

$$\lambda = \left(\frac{32}{3} - \lambda \right)$$

$$\lambda^3 - \frac{64}{3} \lambda + \frac{1024}{27} = 0$$

$$\lambda_1 = -\frac{16}{3} \rightarrow \lambda_1 = 16$$

$$\lambda_2 = \frac{8}{3} \rightarrow \lambda_2 = 8$$

$$\lambda_3 = \frac{8}{3} \rightarrow \lambda_3 = 8$$

$$\rightarrow \begin{pmatrix} 16 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

$$I_{cm} = \begin{pmatrix} \frac{32}{3} & \frac{8}{3} & \frac{8}{3} \\ \frac{8}{3} & \frac{32}{3} & \frac{8}{3} \\ \frac{8}{3} & \frac{8}{3} & \frac{32}{3} \end{pmatrix}$$

$$\begin{pmatrix} 16 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

$$\lambda_1 = 16$$

$$\begin{pmatrix} \frac{32}{3} - 16 & \frac{8}{3} & \frac{8}{3} \\ \frac{8}{3} & \frac{32}{3} - 16 & \frac{8}{3} \\ \frac{8}{3} & \frac{8}{3} & \frac{32}{3} - 16 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{pmatrix} -\frac{16}{3} & \frac{8}{3} & \frac{8}{3} \\ \frac{8}{3} & -\frac{16}{3} & \frac{8}{3} \\ \frac{8}{3} & \frac{8}{3} & -\frac{16}{3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{pmatrix} -16 & 8 & 8 \\ 8 & -16 & 8 \\ 8 & 8 & -16 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$-24x_1 + 24x_2 = 0$$

$$-x_1 + x_2 = 0 \quad \left| \begin{array}{l} x_2 = x_1 \\ x_1 = x_2 \end{array} \right.$$

$$x_1 + x_2 - 2x_3 = 0 \quad \left| \begin{array}{l} 2x_1 - 2x_3 = 0 \Rightarrow x_1 = x_3 \end{array} \right.$$

$$\Rightarrow x_1 = 1 \quad \Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 8$$

$$\begin{pmatrix} \frac{8}{3} & \frac{8}{3} & \frac{8}{3} \\ \frac{8}{3} & \frac{8}{3} & \frac{8}{3} \\ \frac{8}{3} & \frac{8}{3} & \frac{8}{3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$x_1 + x_2 + x_3 = 0$$

$$\vec{v}_2 = \begin{pmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\vec{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \quad \vec{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{array}{l} x_2 = 1 \\ x_3 = 0 \end{array}$$

$$\begin{array}{l} x_2 = 0 \\ x_3 = 1 \end{array}$$

Problema 4. (Valor: 25 pts.)

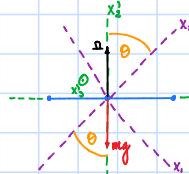
Una barra muy larga, recta y sin masa se encuentra en el plano yz , tal que pasa por el origen. La barra rota con una velocidad angular $\vec{\omega} = \omega \hat{x}$ constante. Hay una partícula de masa m , que está limitada a moverse a lo largo de la varilla, sin fuerzas de fricción y bajo la acción de la fuerza de gravedad ($\vec{g} = g \hat{z}$). Determine la posición radial y angular de la partícula con respecto al tiempo. Considere las condiciones iniciales $\theta(0) = 0$, $\rho(0) = \rho_0$ y $\dot{\rho}(0) = v_0$.



Problema 2. (Valor: 25 pts.)

Un panel publicitario de masa m , longitud $2a$ y ancho $2b$ se mantiene girando a una velocidad angular constante ω alrededor de su eje horizontal (AB) mediante un pequeño motor eléctrico conectado en A al soporte ACB (ver Figura). Este mismo marco se mantiene girando a una velocidad angular constante Ω alrededor de un eje vertical mediante un segundo motor conectado en C a la columna CD .

- Determine el tensor de inercia del panel con respecto a los ejes principales.
- Utilizando las Ecuaciones de Euler, encuentre el torque percibido en el punto D .
- ¿Cuál es la interpretación física de cada componente?



$$\vec{\omega} = \omega \hat{x}_3$$

$$\vec{\Omega} = \Omega (\cos \theta \hat{x}_2 - \sin \theta \hat{x}_1)$$

$$\theta = \omega t$$

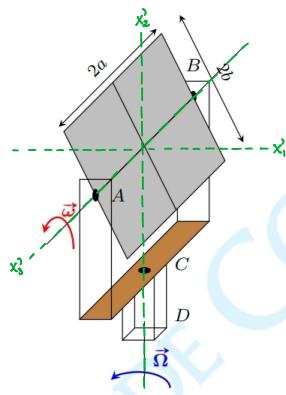


Figura 2: Configuración problema 2.

Ecuaciones de Euler

$$\Rightarrow \begin{cases} ((I_2 - I_3)\omega\omega_3 - I_3\dot{\omega}_3) = N_1 \\ (I_3 - I_1)\omega\omega_1 - I_2\dot{\omega}_2 = -N_2 \\ (I_1 - I_2)\omega\omega_2 - I_3\dot{\omega}_3 = -N_3 \end{cases} \Rightarrow \begin{cases} ((I_2 - I_3)\omega\Omega \cos(\omega t)) + I_3\omega\Omega \cos(\omega t) = -N_1 \\ (I_3 - I_1)\omega - \Omega \sin(\omega t) + I_2\omega\Omega \sin(\omega t) = -N_2 \\ (I_1 - I_2)\omega - \Omega^2 \cos(\omega t) \sin(\omega t) = -N_3 \end{cases}$$

Aquí se describe la rotación del panel

Oscilaciones

SUBSECTION 9.7

Problemas resueltos

Hay que tener cuidado con el problema 6 de oscilaciones acopladas, la solución en página cuadricula no es la mejor y falta concretar detalles, incluso podría ser incorrecta a la hora de buscar los modos normales de oscilación.

Oscilaciones simples

Problema 1. (Thornton 3.7)

3-7. A body of uniform cross-sectional area $A = 1 \text{ cm}^2$ and of mass density $\rho = 0.8 \text{ g/cm}^3$ floats in a liquid of density $\rho_0 = 1 \text{ g/cm}^3$ and at equilibrium displaces a volume $V_s = 0.8 \text{ cm}^3$. Show that the period of small oscillations about the equilibrium position is given by

$$\tau = 2\pi\sqrt{\frac{V_s}{gA}}$$

where g is the gravitational field strength. Determine the value of τ .

$$\sum F_y = F_b - mg = m\ddot{y} \Rightarrow \rho g V_s - \rho g h_s = \rho \cancel{A} h_s \Rightarrow h_s = \frac{\rho V_s}{\rho_0} = \frac{0.8}{1} = 0.8 \text{ cm}$$

Solo fluye el organo en la superficie, no impulsa

$$\sum F_y = F_b - mg = m\ddot{y} \Rightarrow \rho g V_s - \rho g y = \rho V_s \ddot{y} \Rightarrow \rho g A(h_s - y) - \rho g y = \rho V \frac{d^2 y}{dt^2} \Rightarrow \frac{d^2 y}{dt^2} - \frac{\rho g A}{\rho V} (h_s - y) + \frac{g}{V} y = 0$$

$$\Rightarrow \frac{d^2 y}{dt^2} - \frac{\rho g A}{\rho V} (h_s - y) + g = 0 \Rightarrow \frac{d^2 y}{dt^2} + \frac{\rho g A}{\rho V} y + g \left(1 - \frac{\rho g A}{\rho V} h_s\right) = 0 \Rightarrow \frac{d^2 y}{dt^2} + \frac{\rho g A}{\rho V} y + g \left(1 - \frac{\rho g A}{\rho V} h_s\right) = 0$$

$$\Rightarrow \frac{d^2 y}{dt^2} + \frac{\rho g A}{\rho V} y = 0 \Rightarrow \omega_0^2 = \frac{\rho g A}{\rho V} = \frac{\rho g}{\rho_0 h_s} = \frac{g}{h_s} = \frac{g}{V_s} \Rightarrow \omega_0 = \sqrt{\frac{g}{V_s}} \Rightarrow T = \frac{2\pi}{\omega_0} = 2\pi \sqrt{\frac{V_s}{A g}}$$

Problema 2.

Un resorte de constante k sostiene una caja de masa m_1 en la que se coloca un bloque de masa m_2 . Si el sistema se jala hacia abajo una distancia d desde la posición de equilibrio y y luego se suelta, encuentre la fuerza de reacción entre el bloque y el fondo de la caja en función del tiempo. ¿Para qué valor de d el bloque apenas comienza a dejar el fondo de la caja en la parte superior de las oscilaciones verticales? Desprecie cualquier resistencia del aire.

$$N_b - N_b - m_2 g = m_2 a_2 \Rightarrow \frac{d^2 y_1}{dt^2} + \frac{k}{m_2} y_1 + \frac{N_b}{m_2} + g = 0 \quad y_1(t=0) = -d, \quad \dot{y}_1(t=0) = 0$$

$$\Rightarrow \ddot{y}_1 = A \cos(\omega_0 t - \delta) \Rightarrow y_1 = d \cos(\omega_0 t - \pi) \Rightarrow \ddot{y}_1 = -\omega_0^2 d \cos(\omega_0 t - \pi)$$

$$N_b - m_2 g = m_2 a_2 \Rightarrow \ddot{y}_2 = A \cos(\omega_0 t - \delta) \Rightarrow y_2 = d \cos(\omega_0 t - \pi)$$

$$\sum F_{2y} = N_b - m_2 g = m_2 a_2 \Rightarrow N_b = -m_2 \omega_0^2 d \cos(\omega_0 t - \pi) + m_2 g = +m_2 \omega_0^2 d \cos(\omega_0 t) + m_2 g$$

$$\Rightarrow N_b = 0 \Rightarrow -m_2 \omega_0^2 d \cos(\omega_0 t) + m_2 g = 0 \Rightarrow \omega_0^2 d = -g \Rightarrow d = -\frac{g}{\omega_0^2} \Rightarrow \text{elongación} = \frac{g}{\omega_0^2}$$

Problema 3.

Un reloj de pie tiene un péndulo que consta de un disco delgado de latón de radio $r = 15.00$ cm y masa 1.00 kg, que está unido a una varilla larga y delgada de masa despreciable. El péndulo se balancea libremente alrededor de un eje perpendicular a la barra, en el extremo opuesto. Si el péndulo va a tener un período de 2.000 s para pequeñas oscilaciones, ¿cuál debe ser la longitud de la barra l ? Escriba su respuesta considerando hasta décimas de milímetro.

$$\sum N_z = -mgL \sin\theta \Rightarrow I_p \frac{d^2\theta}{dt^2} = -mg(r+l) \sin\theta$$

$$\text{Inercia disco } I_p = \frac{MR^2}{2}$$

$$I_c = \frac{mr^2}{2} + m(r+l)^2 = \frac{m[r^2 + 2(r+l)^2]}{2}$$

$$\Rightarrow \frac{m[r^2 + 2(r+l)^2]}{2} \frac{d^2\theta}{dt^2} = -mg(r+l) \sin\theta \Rightarrow \ddot{\theta} = -\frac{2g(r+l)}{r^2 + 2(r+l)^2} \sin\theta \Rightarrow \omega^2 = \frac{2g(r+l)}{r^2 + 2(r+l)^2}$$

$$\Rightarrow \omega_0 = \sqrt{\frac{2g(r+l)}{r^2 + 2(r+l)^2}} \Rightarrow T = \frac{2\pi}{\omega_0} = 2\pi \sqrt{\frac{r^2 + 2(r+l)^2}{2g(r+l)}} \Rightarrow \frac{T^2}{4\pi^2} = \frac{r^2 + 2(r+l)^2}{2g(r+l)}$$

$$\Rightarrow \frac{grT^2}{2\pi^2} + \frac{glT^2}{2\pi^2} = r^2 + 2(r+l)^2 = 3r^2 + 4rl + 2l^2 \Rightarrow 2l^2 + \left(4r - \frac{grT^2}{2\pi^2}\right)l + 3r^2 - \frac{grT^2}{2\pi^2} = 0 \Rightarrow l = 0.8370 \text{ m}$$

Se cumple

Problema 4. (Taylor 5.13)

5.13 ** The potential energy of a one-dimensional mass m at a distance r from the origin is

$$U(r) = U_0 \left(\frac{r}{R} + \lambda^2 \frac{R}{r} \right)$$

for $0 < r < \infty$, with U_0 , R , and λ all positive constants. Find the equilibrium position r_0 . Let x be the distance from equilibrium and show that, for small x , the PE has the form $U = \text{const} + \frac{1}{2}kx^2$. What is the angular frequency of small oscillations?

$$U = U_0 \left(\frac{r}{R} + \lambda^2 \frac{R}{r} \right) \Rightarrow \frac{dU}{dx} = U_0 \left(\frac{1}{R} - \lambda^2 \frac{R}{r^2} \right) \Rightarrow \frac{dU}{dx} = 0 \Rightarrow U_0 \left(\frac{1}{R} - \lambda^2 \frac{R}{r^2} \right) = 0 \Rightarrow \frac{1}{R} = \lambda^2 \frac{R}{r^2} \Rightarrow r^2 = \lambda^2 R^2$$

$$\Rightarrow r = \pm \lambda R \quad \frac{d^2U}{dx^2} = U_0 \left(\frac{\lambda^2 R}{r^3} \right) \Rightarrow \frac{d^2U}{dx^2} \Big|_{r=\lambda R} > 0 \quad \text{y} \quad \frac{d^2U}{dx^2} \Big|_{r=-\lambda R} < 0 \quad \therefore r_0 = +\lambda R \text{ P.E.E}$$

PEE en eje x positivo, hay que correr la función al eje x negativo para centrarla

$$r \xrightarrow{*} r + r_0 \rightarrow U \text{ con el PEE centrada } r=0: U = U_0 \left(\frac{r+r_0}{R} + \lambda^2 \frac{R}{r+r_0} \right) ; \frac{1}{1+x} \approx 1 - x + x^2$$

Oscilaciones pequeñas

$$\Rightarrow U = U_0 \left(\frac{r+r_0}{R} + \lambda^2 \frac{R}{r_0} \frac{1}{1+\frac{r}{r_0}} \right) \approx U_0 \left[\frac{r+r_0}{R} + \lambda^2 \frac{R}{r_0} \left(1 - \frac{r}{r_0} + \frac{r^2}{r_0^2} \right) \right] \stackrel{*}{=} U_0 \left[\lambda \cdot \frac{r+r_0}{r_0} + \lambda \left(1 - \frac{r}{r_0} + \frac{r^2}{r_0^2} \right) \right]$$

$$= U_0 \lambda \left[1 + \cancel{\frac{r}{r_0}} + 1 - \cancel{\frac{r}{r_0}} + \frac{r^2}{r_0^2} \right] = U_0 \lambda \left(2 + \frac{r^2}{\lambda^2 R^2} \right) = U_0 \lambda \left(2 + \frac{r^2}{\lambda^2 R^2} \right) = 2U_0 \lambda + \frac{U_0 r^2}{\lambda R^2}$$

$$\Rightarrow \text{Energía potencial oscilaciones pequeñas: } U_p = \frac{1}{2}kr^2 + \text{constante} = 2U_0 \lambda + \frac{U_0 r^2}{\lambda R^2} \Rightarrow k = \frac{U_0}{\lambda R^2} \rightarrow \omega_0 = \sqrt{\frac{k}{m}}$$

Problema 5. (Taylor 5.18)

5.18 *** The mass shown from above in Figure 5.27 is resting on a frictionless horizontal table. Each of the two identical springs has force constant k and unstretched length l_0 . At equilibrium the mass rests at the origin, and the distances a are not necessarily equal to l_0 . (That is, the springs may already be stretched or compressed.) Show that when the mass moves to a position (x, y) , with x and y small, the potential energy has the form (5.104) (Problem 5.14) for an anisotropic oscillator. Show that if $a < l_0$ the equilibrium at the origin is unstable and explain why.

$$U = \frac{1}{2}(k_x x^2 + k_y y^2). \quad (5.104)$$

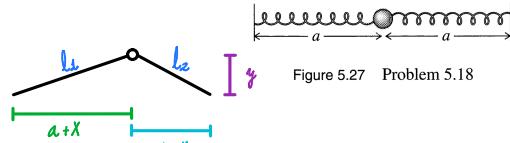
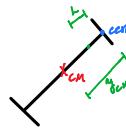


Figure 5.27 Problem 5.18

$$\begin{aligned}
 l_1 &= \sqrt{(a+x)^2 + y^2} = a \sqrt{1 + \left(\frac{2x}{a} + \frac{x^2 + y^2}{a^2}\right)} \approx a \left[1 + \frac{1}{2} \left(\frac{2x}{a} + \frac{x^2 + y^2}{a^2} \right) - \frac{1}{8} \left(\frac{2x}{a} + \frac{x^2 + y^2}{a^2} \right)^2 \right] \\
 &= a \left[1 + \frac{1}{2} \left(\frac{2x}{a} + \frac{x^2 + y^2}{a^2} \right) - \frac{1}{8} \left(\frac{4x^2}{a^2} + 2 \frac{2x}{a} \frac{x^2 + y^2}{a^2} + \frac{(x^2 + y^2)^2}{a^4} \right) \right] \\
 &= a \left[1 + \frac{x}{a} + \frac{x^2 + y^2}{2a^2} - \frac{x^2}{4a^2} \right] = a + x + \frac{y^2}{2a} \\
 x \text{ } y \text{ } \text{pequeño} & \approx 0 \\
 l_2 &= \sqrt{(a-x)^2 + y^2} = a \sqrt{1 - \left(\frac{2x}{a} - \frac{x^2 + y^2}{a^2}\right)} \approx a \left[1 - \frac{1}{2} \left(\frac{2x}{a} - \frac{x^2 + y^2}{a^2} \right) - \frac{1}{8} \left(\frac{2x}{a} - \frac{x^2 + y^2}{a^2} \right)^2 \right] \\
 &= a \left[1 - \frac{1}{2} \left(\frac{2x}{a} - \frac{x^2 + y^2}{a^2} \right) - \frac{1}{8} \left(\frac{4x^2}{a^2} - 2 \frac{2x}{a} \frac{x^2 + y^2}{a^2} + \frac{(x^2 + y^2)^2}{a^4} \right) \right] \\
 &= a \left[1 - \frac{x}{a} + \frac{x^2 + y^2}{2a^2} - \frac{x^2}{4a^2} \right] = a - x + \frac{y^2}{2a} \\
 x \text{ } y \text{ } \text{pequeño} & \approx 0 \\
 \Rightarrow U_x &= \frac{1}{2}k(l_1 - l_0)^2 = \frac{1}{2}k \left[(a-l_0) + x + \frac{y^2}{2a^2} \right]^2 = \frac{1}{2}k \left[(a-l_0)^2 + 2(a-l_0)x + \frac{(a-l_0)y^2}{a^2} + x^2 + \frac{xy^2}{a^2} + \frac{y^4}{4a^4} \right] \\
 &\approx \frac{1}{2}k \left[(a-l_0)^2 + 2(a-l_0)x + \frac{(a-l_0)y^2}{a^2} + x^2 \right] \\
 \Rightarrow U_x &= \frac{1}{2}k(l_2 - l_0)^2 = \frac{1}{2}k \left[(a-l_0) - x + \frac{y^2}{2a} \right]^2 = \frac{1}{2}k \left[(a-l_0)^2 - 2(a-l_0)x + \frac{(a-l_0)y^2}{a} + x^2 - \frac{xy^2}{a} + \frac{y^4}{4a^2} \right] \\
 &\approx \frac{1}{2}k \left[(a-l_0)^2 - 2(a-l_0)x + \frac{(a-l_0)y^2}{a} + x^2 \right] \\
 \Rightarrow U_x &= U_{l_1} + U_{l_2} = \frac{1}{2}k \left[(a-l_0)^2 + 2(a-l_0)x + \frac{(a-l_0)y^2}{a^2} + x^2 \right] + \frac{1}{2}k \left[(a-l_0)^2 - 2(a-l_0)x + \frac{(a-l_0)y^2}{a} + x^2 \right] \\
 &= k(a-l_0)^2 + \frac{k(a-l_0)y^2}{a^2} + kx^2 //
 \end{aligned}$$

Problema 6.

Muestre que una mancuerna con dos masas iguales (que se considerarán como masas puntuales) unidas a una varilla muy delgada (sin masa) de longitud ℓ .



- a. Demuestre que el período de este péndulo es mínimo cuando el punto de pivot P está en una de las masas.

- b. Encuentre el período de este péndulo físico, si la distancia entre P y la masa superior es $\ell/4$.

$$a) \quad I_{cm} = \frac{ml + m \cdot 0}{m+m} = \frac{l}{2} \quad ; \quad I_{cm} = m \frac{l^2}{4} + m \frac{l^2}{4} = m \frac{l^2}{2}$$

$$\Rightarrow \sum N_z = -mg\left(\frac{l}{2} + h\right) \sin\theta = I \ddot{\theta} \Rightarrow (I_{cm} + I_p) \ddot{\theta} = \left[m \frac{l^2}{2} + m\left(\frac{l}{2} + h\right)^2\right] \ddot{\theta} = -mg\left(\frac{l}{2} + h\right) \sin\theta$$

$$\Rightarrow \ddot{\theta} = -\frac{4mg(l+2h)}{2m[2l^2+2(l+2h)^2]} \sin\theta = -\frac{2mg(l+2h)}{l^2+4(l+2h)^2} \sin\theta = -\frac{g(l+2h)}{l^2+4(l+2h)^2} \sin\theta$$

$$\Rightarrow \omega_0^2 = \frac{g(l+2h)}{l^2+4(l+2h)^2} \Rightarrow T = \frac{2\pi}{\omega_0} \text{ si se encuentran máximos para } \omega_0, \text{ estos minimizan el período}$$

$$\Rightarrow \frac{d\omega_0^2}{dh} = 0 \Rightarrow \cancel{2[l^2 + (l+2h)^2]} - \cancel{(l+2h) \cdot 2(l+2h) \cdot 2} = 0 \Rightarrow 2l^2 + 2(l+2h)^2 - 4(l+2h)^2 = 0$$

$$\Rightarrow 2(l+2h)^2 - 2l^2 = 0 \Rightarrow (l+2h)^2 - l^2 = 0 \Rightarrow \cancel{l^2} + 4lh + 4h^2 - \cancel{l^2} = 0 \Rightarrow 4lh + 4h^2 = 0 \Rightarrow 4h(l+h) = 0$$

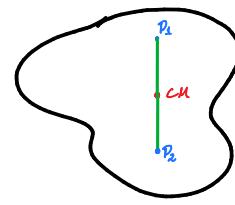
$$\Rightarrow h=0 \vee h=-l \rightarrow \text{Mínimo en el período en cualquiera de los dos extremos}$$

$$b) \quad \left[m \frac{l^2}{2} + md^2\right] \ddot{\theta} = -mgd \sin\theta \Rightarrow \ddot{\theta} = \frac{-2mgd}{m(l^2+2d^2)} \sin\theta \Rightarrow \omega_0^2 = \frac{2gd}{l^2+2d^2} \Rightarrow T = \frac{2\pi}{\omega_0}$$

$$\Rightarrow T = 2\pi \sqrt{\frac{l^2+2d^2}{2gd}} ; d \geq \frac{l}{4}$$

Problema 7.

Un objeto plano tiene un momento de inercia I con respecto a su centro de masa. Cuando se pivota en un punto P_1 , a una distancia h_1 del centro de masa, el período de oscilación es τ . Hay un segundo punto P_2 en el lado opuesto del centro de masa y en la misma línea que P_1 , localizado con respecto al centro de masa a una distancia h_2 , alrededor del cual se puede pivotar el objeto de modo que el período de oscilación también es τ . Demuestre que $h_1 + h_2 = g\tau^2/4\pi^2$.



$$(I + mh^2)\ddot{\theta} = -mgh \operatorname{sen}\theta \Rightarrow \ddot{\theta} = \frac{-mgh}{I + mh^2} \operatorname{sen}\theta \Rightarrow \omega_0^2 = \frac{mgh}{I + mh^2} \Rightarrow \tau = \frac{2\pi}{\omega_0} \Rightarrow \frac{\tau^2}{4\pi^2} = \frac{I + mh^2}{mgh}$$

$$\Rightarrow mh^2 - \frac{\tau^2}{4\pi^2}mgh + I = 0 \Rightarrow h = \frac{\frac{\tau^2}{4\pi^2}mg \pm \sqrt{\frac{\tau^2 m^2 g^2}{16\pi^4} - 4 \cdot m \cdot I}}{2m} = \frac{\frac{\tau^2}{8\pi^2}g \pm \sqrt{\frac{\tau^2 m^2 g^2}{16\pi^4} - 4 \cdot m \cdot I}}{2m}$$

$$\Rightarrow h_1 + h_2 = 2 \frac{\frac{\tau^2}{8\pi^2}g + \sqrt{\frac{\tau^2 m^2 g^2}{16\pi^4} - 4 \cdot m \cdot I}}{2m} - \frac{\sqrt{\frac{\tau^2 m^2 g^2}{16\pi^4} - 4 \cdot m \cdot I}}{2m} \Rightarrow h_1 + h_2 = \frac{\tau^2}{4\pi^2}g$$

Oscilaciones en varias dimensiones, amortiguadas y forzadas

Obtención de la ecuación sinusoidal para un oscilador suavemente amortiguado : $\ddot{X} + 2\beta\dot{X} + \omega_0^2 X = 0$ con $\beta^2 < \omega_0^2$

$$\Rightarrow \ddot{X} + 2\beta\dot{X} + \omega_0^2 X = 0 \leftarrow \text{Ecación diferencial de segundo orden homogénea}$$

$$\Rightarrow z^2 + 2\beta z + \omega_0^2 = 0 \quad \begin{matrix} 2 \text{ soluciones diferentes} \\ z_1, z_2 \end{matrix}$$

$$\Rightarrow z = \frac{-2\beta \pm \sqrt{\beta^2 - 2\omega_0^2}}{2} = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$$

$$\Rightarrow X = A_1 e^{-\beta t + \sqrt{\beta^2 - \omega_0^2} t} + A_2 e^{-\beta t - \sqrt{\beta^2 - \omega_0^2} t}$$

$$= e^{-\beta t} (A_1 e^{\sqrt{\beta^2 - \omega_0^2} t} + A_2 e^{-\sqrt{\beta^2 - \omega_0^2} t})$$

$$\Rightarrow X = e^{-\beta t} (A_1 e^{i\omega_0 t} + A_2 e^{-i\omega_0 t})$$

$$= e^{-\beta t} [A_1 \cos(\omega_0 t) + A_1 i \operatorname{sen}(\omega_0 t) + A_2 \cos(-\omega_0 t) - A_2 i \operatorname{sen}(-\omega_0 t)]$$

$$= e^{-\beta t} \left[\underbrace{(A_1 + A_2) \cos(\omega_0 t)}_B + \underbrace{(A_1 - A_2) i \operatorname{sen}(\omega_0 t)}_C \right]$$

$$= e^{-\beta t} [B \cos(\omega_0 t) + C i \operatorname{sen}(\omega_0 t)]$$

$$= e^{-\beta t} \sqrt{B^2 + C^2} \left[\frac{B}{\sqrt{B^2 + C^2}} \cos(\omega_0 t) + \frac{C}{\sqrt{B^2 + C^2}} i \operatorname{sen}(\omega_0 t) \right]$$

$$= e^{-\beta t} \sqrt{B^2 + C^2} [\cos(\delta) \cos(\omega_0 t) + \operatorname{sen}(\delta) i \operatorname{sen}(\omega_0 t)]$$

$$\Rightarrow X = \sqrt{B^2 + C^2} e^{-\beta t} \cos(\omega_0 t - \delta)$$

* Transformar a ecación característica

Solucionar la ecación cuadrática

* Armar la solución de la EDO

* Restar la condición $\beta^2 < \omega_0^2$ y realizar la sustitución

$$\omega_0^2 = \omega_0^2 - \beta^2$$

* Aplicar $e^{i\theta} = \cos\theta + i \operatorname{sen}\theta$

* Agrupar parte real e imaginaria, y sustituir

* Multiplicar arriba y abajo por $\sqrt{B^2 + C^2}$ en cada término

* De la expresión anterior, se busca utilizar propiedades trigonométricas para definir un desfase δ a partir de un triángulo rectángulo y con esto, obtener una expresión de $\cos(u+v)$ ó $\operatorname{sen}(u+v)$

* Deshaciendo sustituciones

$$\Rightarrow X = \sqrt{(A_1 + A_2)^2 + (A_1 - A_2)^2} e^{-\beta t} \cos \left\{ \sqrt{\omega_0^2 - \beta^2} t - \tan^{-1} \left[\frac{A_1 + A_2}{A_1 - A_2} \right] \right\}$$

Demonstración de la solución de un oscilador forzado simple: $\ddot{X} + 2\beta\dot{X} + \omega_0^2 X = A \cos(\omega t)$

$$\Rightarrow \ddot{X} + 2\beta\dot{X} + \omega_0^2 X = A \cos(\omega t) \leftarrow \text{Ecación diferencial de segundo orden no homogénea}$$

La ecación va a tener 2 soluciones, la solución particular y la complementaria (Ver página anterior para saber como encontrarla), siendo la suma de ambas soluciones la solución completa del sistema físico.

$$\Rightarrow D^2 X + 2\beta D X + \omega_0^2 X = A \cos(\omega t)$$

$$\Rightarrow (D^2 + 2\beta D + \omega_0^2) X = A \cos(\omega t)$$

$$\Rightarrow (D^2 + \omega_0^2)(D^2 + 2\beta D + \omega_0^2) X = (D^2 - \omega_0^2) A \cos(\omega t) = 0$$

$$\Rightarrow (D - i\omega)(D + i\omega)[D - (-\beta + \sqrt{\beta^2 - \omega_0^2})][D - (-\beta - \sqrt{\beta^2 - \omega_0^2})] X = 0$$

$$\Rightarrow D = \pm i\omega \quad \text{v} \quad D = \beta \pm \sqrt{\beta^2 - \omega_0^2} \xrightarrow{\text{Homogénea}}$$

$$\Rightarrow D = \pm i\omega$$

$$\Rightarrow X_p = B_1 e^{i\omega t} + B_2 e^{-i\omega t}$$

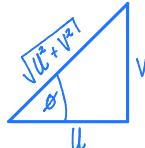
$$= B_1 \cos(\omega t) + B_1 i \operatorname{sen}(\omega t) + B_2 \cos(\omega t) + B_2 i \operatorname{sen}(\omega t)$$

$$= \underbrace{(B_1 + B_2) \cos(\omega t)}_{U} + \underbrace{(B_1 - B_2) i \operatorname{sen}(\omega t)}_{V}$$

$$= U \cos(\omega t) + V \operatorname{sen}(\omega t)$$

$$= \sqrt{U^2 + V^2} \left[\frac{U}{\sqrt{U^2 + V^2}} \cos(\omega t) + \frac{V}{\sqrt{U^2 + V^2}} \operatorname{sen}(\omega t) \right]$$

$$= \frac{\sqrt{U^2 + V^2}}{D} [\cos(\phi) \cos(\omega t) + \operatorname{sen}(\phi) \operatorname{sen}(\omega t)]$$



$$\Rightarrow X_p = D \cos(\omega t - \phi)$$

A partir de aquí, ya se obtuvo una propuesta de solución de la ecación diferencial, ya solo es necesario definir el parámetro D.

Se usa el método de aniquiladores

* Reescribir la ecación en notación de operador diferencial

* Se identifica $H = D^2 + \omega_0^2$ como el aniquilador apropiado

* Se multiplica por H a ambos lados de la ecación y se factoriza al máximo

* Se soluciona la nueva ecación diferencial homogénea, eliminando las soluciones de la ecación homogénea del sistema físico

* Aplicar $e^{i\theta} = \cos(\theta) + i \operatorname{sen}(\theta)$

* Agrupar parte real e imaginaria, y sustituir

* Multiplicar arriba y abajo por $\sqrt{B^2 + C^2}$ en cada término

* De la expresión anterior, se busca utilizar propiedades trigonométricas para definir un desfase ϕ a partir de un triángulo rectángulo y con esto, obtener una expresión de $\cos(U+V)$ ó $\operatorname{sen}(U+V)$

* Se realiza una sustitución extra y se obtiene la forma de la solución de la ecación diferencial

Continua.

Definición del parámetro D: $(D^2 + 2\beta D + \omega_0^2)X_p = A \cos(\omega t)$

$$\Rightarrow \ddot{X}_p + 2\beta \dot{X}_p + \omega_0^2 X_p = A \cos(\omega t)$$

$$\Rightarrow -D\omega^2 \cos(\omega t - \phi) - 2\beta\omega D \sin(\omega t - \phi) + \omega_0^2 D \cos(\omega t - \phi) = A \cos(\omega t)$$

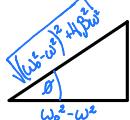
$$\Rightarrow -D\omega^2 [\cos(\omega t) \cos(\phi) + \sin(\omega t) \sin(\phi)] - 2\beta\omega D [\sin(\omega t) \cos(\phi) - \cos(\omega t) \sin(\phi)] + \omega_0^2 D [\cos(\omega t) \cos(\phi) + \sin(\omega t) \sin(\phi)] \\ = A \cos(\omega t)$$

$$\Rightarrow \cos(\omega t) [-D\omega^2 \cos(\phi) + 2\beta\omega D \sin(\phi) + \omega_0^2 D \cos(\phi) - A] + \sin(\omega t) [-D\omega^2 \sin(\phi) - 2\beta\omega D \cos(\phi) + \omega_0^2 D \sin(\phi)] = 0$$

No existe un valor de t que haga $R_1 \cos(\omega t) + R_2 \sin(\omega t) = 0$, por lo que $R_1 = R_2 = 0$, esto porque son linealmente independientes

$$\Rightarrow \begin{cases} -D\omega^2 \cos(\phi) + 2\beta\omega D \sin(\phi) + \omega_0^2 D \cos(\phi) - A = 0 \\ -D\omega^2 \sin(\phi) - 2\beta\omega D \cos(\phi) + \omega_0^2 D \sin(\phi) = 0 \end{cases} \quad * \text{Se forma un "sistema" con las partes de } \cos/\sin(\omega t)$$

$$\Rightarrow \tan(\phi) = \frac{2\beta\omega}{\omega_0^2 - \omega^2} \quad * \text{Al resolver } \tan(\phi), \text{ es posible formar un triángulo rectángulo para } \cos/\sin(\phi)$$



$$\Rightarrow D\omega^2 \cos(\phi) - 2\beta\omega D \sin(\phi) - \omega_0^2 D \cos(\phi) + A = 0$$

$$\Rightarrow D\omega^2 \frac{\omega_0^2 - \omega^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\omega^2\beta^2}} - 2\beta\omega D \frac{2\omega\beta}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\omega^2\beta^2}} - \omega_0^2 D \frac{\omega_0^2 - \omega^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\omega^2\beta^2}} + A = 0$$

$$\Rightarrow D\omega^2 \frac{(\omega_0^2 - \omega^2)}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\omega^2\beta^2}} - \frac{4\beta^2\omega^2 D}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\omega^2\beta^2}} - \omega_0^2 D \frac{(\omega_0^2 - \omega^2)}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\omega^2\beta^2}} = -A$$

$$\Rightarrow D[\omega^2(\omega_0^2 - \omega^2) - \frac{4\beta^2\omega^2 D}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\omega^2\beta^2}} - \omega_0^2(\omega_0^2 - \omega^2)] = -A \sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}$$

$$D[\omega^2(\omega_0^2 - \omega^2) + 4\beta^2\omega^2 D] = +A \sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}$$

$$\Rightarrow D = \frac{A}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}$$

$$\Rightarrow X_p = \frac{A}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}} \cos(\omega t - \phi) \quad \Rightarrow X = X_p + X_c$$

$$\Rightarrow X = B e^{-\beta t} \cos[\sqrt{\omega_0^2 - \beta^2} t - \delta] + \frac{A}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}} \cos(\omega t - \phi)$$

Frecuencia natural
Depende de ω_0 , ω y β
Frecuencia del sistema
amortiguado

* Se sustituye X_p en la ecuación original y se operan las divisiones

* Se expanden $\cos/\sin(\omega t - \phi)$ y se agrupan por $\cos/\sin(\omega t)$

* Tomando la primera ecuación del sistema y sustituyendo los resultados obtenidos

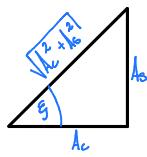
* Despejando D se logra colocar el parámetro en términos de variables del sistema físico

* Reemplazando D en la propuesta de solución, se obtiene la solución particular

* Sumandola a la solución homogénea, se obtiene la solución real del sistema físico.

Demostración de la solución de un oscilador forzado de fuerza arbitraria: $\ddot{X} + 2\beta\dot{X} + \omega_0^2 X = A(t)$

Suponiendo $A(t) = \sum_{n=0}^{+\infty} A_c \cos(n\omega t) + A_s \operatorname{sen}(n\omega t)$



$$\begin{aligned} &= \sum_{n=0}^{+\infty} \sqrt{A_c^2 + A_s^2} \left[\frac{A_c}{\sqrt{A_c^2 + A_s^2}} \cos(n\omega t) + \frac{A_s}{\sqrt{A_c^2 + A_s^2}} \operatorname{sen}(n\omega t) \right] \\ &= \sum_{n=0}^{+\infty} \sqrt{A_c^2 + A_s^2} [\cos(\phi_f) \cos(n\omega t) + \operatorname{sen}(\phi_f) \operatorname{sen}(n\omega t)] \\ &= \sum_{n=0}^{+\infty} \sqrt{A_c^2 + A_s^2} \cos(n\omega t - \phi_f) \end{aligned}$$

$$\Rightarrow \ddot{X} + 2\beta\dot{X} + \omega_0^2 X = \sum_{n=0}^{+\infty} A_n \cos(n\omega t - \phi_n)$$

$$\Rightarrow \left(\frac{d^2}{dt^2} + 2\beta \frac{d}{dt} + \omega_0^2 \right) \left(\sum_{n=0}^{+\infty} X_n \right) = \sum_{n=0}^{+\infty} A_n \cos(n\omega t - \phi_n)$$

$$\Rightarrow X = \sum_{n=0}^{+\infty} X_n = \sum_{n=0}^{+\infty} \frac{A_n}{\sqrt{(\omega_0^2 - n^2\omega^2)^2 + (\beta n \omega)^2}} \cos(n\omega t - \phi_n - \phi_f)$$

$$y \tan(\phi_n) = \left(\frac{2\beta n \omega}{\omega_0^2 - n^2 \omega^2} \right)$$

↓ Diversas frecuencias
↓ Amortiguamiento
↓ Frecuencia natural

* Se escribe el forzamiento en términos de una serie de Fourier

* Multiplicar arriba y abajo por $\sqrt{A_c^2 + A_s^2}$ en cada término

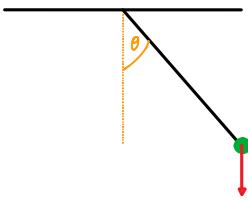
* De la expresión anterior, se busca utilizar propiedades trigonométricas para definir un desfase ϕ_f a partir de un triángulo rectángulo y con esto, obtener una expresión de $\cos(\phi_f + \phi)$ ó $\operatorname{sen}(\phi_f + \phi)$

* Se reescribe la ecuación diferencial con el forzamiento en serie

* Utilizando el método de anuladores, es posible observar que para cada término individual de la serie se obtiene una solución con la forma de un oscilador forzado armónico, dando a entender que la solución para el forzamiento arbitrario es la suma de todas las soluciones de forzamiento armónico

Problema 1. (Thornton 3.12)

- 3-12. A simple pendulum consists of a mass m suspended from a fixed point by a weightless, extensionless rod of length l . Obtain the equation of motion and, in the approximation that $\sin \theta \approx \theta$, show that the natural frequency is $\omega_0 = \sqrt{g/l}$, where g is the gravitational field strength. Discuss the motion in the event that the motion takes place in a viscous medium with retarding force $2m\sqrt{gl}\dot{\theta}$.



Péndulo simple

$$\sum N_z = -mg \text{sen} \theta \Rightarrow ml^2 \ddot{\theta} = -mg \text{sen} \theta \Rightarrow \ddot{\theta} = -\frac{g}{l} \text{sen} \theta ; \theta \text{ pequeño} \Rightarrow \text{sen} \theta \approx \theta$$

$$\Rightarrow \ddot{\theta} = -\frac{g}{l} \theta \Rightarrow \omega_b^2 = \frac{g}{l} \Rightarrow \omega_b = \sqrt{\frac{g}{l}}$$

Péndulo simple en un medio viscoso

$$\Rightarrow ml^2 \ddot{\theta} = -mg \text{sen} \theta - 2m\sqrt{gl} \dot{\theta} \Rightarrow \ddot{\theta} + \frac{g}{l} \text{sen} \theta + \frac{2\sqrt{gl}}{l} \dot{\theta} = 0 ; \theta \text{ pequeño} \Rightarrow \text{sen} \theta \approx \theta$$

$$\Rightarrow \ddot{\theta} + \frac{2\sqrt{g}}{l^{1/2}} \dot{\theta} + \frac{g}{l} \theta = 0 \Rightarrow \beta = \sqrt{\frac{g}{l}} \quad y \quad \omega^2 = \frac{g}{l} \Rightarrow \omega^2 = \beta^2 \leftarrow \text{Oscilador armónico}$$

$$\Rightarrow \lambda^2 + 2\omega_b \lambda + \omega^2 = 0 \Rightarrow \lambda = \frac{-2\omega_b \pm \sqrt{4\omega_b^2 - 4\omega^2}}{2} = -\omega_b \pm \sqrt{\omega_b^2 - \omega^2} \Rightarrow \lambda = -\omega_b \leftarrow \text{con multiplicidad 2}$$

$$\Rightarrow \theta = (A + Bt) e^{-\omega_b t}$$

Problema 2. (Thornton 3.26)

- 3-26. Figure 3-B illustrates a mass m_1 driven by a sinusoidal force whose frequency is ω . The mass m_1 is attached to a rigid support by a spring of force constant k and slides on a second mass m_2 . The frictional force between m_1 and m_2 is represented by the damping parameter b_1 , and the frictional force between m_2 and the support is represented by b_2 . Construct the electrical analog of this system and calculate the impedance.

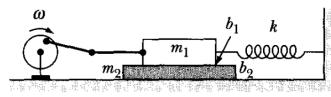


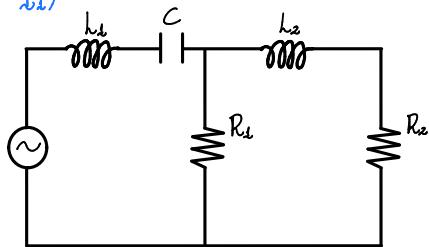
FIGURE 3-B Problem 3-26.

$$m_1 \ddot{x}_1 = -kx_1 + F \cos(\omega t) - b_1(x_1 - x_2) \quad y \quad m_2 \ddot{x}_2 = -b_2 x_2 - b_1(x_2 - x_1)$$

Entre sistemas eléctricos y mecánicos : $m \rightarrow L$, $k \rightarrow \frac{1}{C}$, $b \rightarrow R$ y $x \rightarrow q \Rightarrow \dot{x} \rightarrow I \Rightarrow \ddot{x} \rightarrow \dot{I}$

$$\Rightarrow L_1 \ddot{I}_1 = -\frac{q_1}{C} + E \cos(\omega t) - R_1(I_1 - I_2) \quad y \quad L_2 \ddot{I}_2 = -R_2 I_2 - R_1(I_2 - I_1)$$

Hay 2 mallas, una fuente, un capacitor, 2 resistencias y 2 inductores



Problema 3. (Taylor 5.23)

5.23 * A damped oscillator satisfies the equation (5.24), where $F_{\text{dmp}} = -b\dot{x}$ is the damping force. Find the rate of change of the energy $E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$ (by straightforward differentiation), and, with the help of (5.24), show that dE/dt is (minus) the rate at which energy is dissipated by F_{dmp} .

La energía se approxima así por ser oscilaciones pequeñas

$$\boxed{\text{E} = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2} \Rightarrow \frac{dE}{dt} = \cancel{\frac{1}{2}m\dot{x}\cdot\ddot{x}} + \cancel{\frac{1}{2}kx\cdot\dot{x}} = \dot{x}(m\ddot{x} + kx) = \dot{x}(-b\dot{x}) = -F_{\text{dmp}} \Rightarrow \frac{dE}{dt} = -F_{\text{dmp}}$$

$$m\ddot{x} + b\dot{x} + kx = 0. \quad (5.24)$$

$$\Rightarrow m\ddot{x} + kx = -b\dot{x}$$

Problema 4. (Taylor 5.41)

5.41 * We know that if the driving frequency ω is varied, the maximum response (A^2) of a driven damped oscillator occurs at $\omega \approx \omega_0$ (if the natural frequency is ω_0 and the damping constant $\beta \ll \omega_0$). Show that A^2 is equal to half its maximum value when $\omega \approx \omega_0 \pm \beta$, so that the full width at half maximum is just 2β . [Hint: Be careful with your approximations. For instance, it's fine to say $\omega + \omega_0 \approx 2\omega_0$, but you certainly mustn't say $\omega - \omega_0 \approx 0$.]

$$\beta \ll \omega_0$$

$$\begin{aligned} X_p &= \frac{A}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\omega_0^2\beta^2}} \cos(\omega t - \delta) \Rightarrow D^2 = \frac{A^2}{(\omega_0^2 - \omega^2)^2 + 4\omega_0^2\beta^2}; \text{ máximo } \omega \approx \omega_0 \\ \Rightarrow D^2 &\approx \frac{A^2}{(\omega_0^2 - \omega^2)^2 + 4\omega_0^2\beta^2} \approx \frac{A^2}{[(\omega_0 - \omega)(\omega_0 + \omega)]^2 + 4\omega_0^2\beta^2} \approx \frac{A^2}{(\omega_0 - \omega)^2[4\omega_0^2 + 4\omega_0^2\beta^2]} \\ &= \frac{A^2}{(\omega_0^2 - 2\omega\omega_0 + \omega^2)[4\omega_0^2 + 4\omega_0^2\beta^2]} = \frac{A^2}{4\omega_0^4 - 8\omega\omega_0^3 + 4\omega^2\omega_0^2 + 4\omega_0^2\beta^2} = \frac{A^2}{4\omega_0^4 - 8\omega_0^4 + 4\omega_0^4 + 4\omega_0^2\beta^2} \\ &= \frac{A^2}{4\omega_0^4 - 4\omega_0^4 + 4\omega_0^2\beta^2} = \frac{A^2}{4\omega_0^2\beta^2} \end{aligned}$$

Full width half maxima $\omega \approx \omega_0 \pm \beta$

$$\begin{aligned} D^2 &= \frac{A^2}{(\omega_0^2 - \omega^2)^2 + 4\omega_0^2\beta^2} = \frac{A^2}{[(\omega_0 - \omega)(\omega_0 + \omega)]^2 + 4\omega_0^2\beta^2} \approx \frac{A^2}{[(\omega_0 - \omega + \beta)(\omega_0 + \omega_0 + \beta)]^2 + 4\omega_0^2\beta^2} = \frac{A^2}{[\beta(2\omega_0 + \beta)]^2 + 4(\omega_0 + \beta)^2\beta^2} \\ &= \frac{A^2}{\beta^2(4\omega_0^2 - 4\omega_0\beta + \beta^2) + 4(\omega_0^2 - 2\omega_0\beta + \beta^2)\beta^2} = \frac{A^2}{4\omega_0^2\beta^2 - 4\omega_0\beta^3 + \beta^4 + 4\omega_0^2\beta^2 - 8\omega_0\beta^3 + 4\beta^4} \\ &= \frac{A^2}{8\omega_0^2\beta^2 - 12\omega_0\beta^3 + 5\beta^4} = \frac{A^2}{8\omega_0^2\beta^2 \left(1 - \frac{12\beta^2}{8\omega_0^2} + \frac{5\beta^2}{8\omega_0^2}\right)} = \frac{A^2}{8\omega_0^2\beta^2} \end{aligned}$$

Problema 5. (Thornton 3.18)

3-18. Show that, if a driven oscillator is only lightly damped and driven near resonance, the Q of the system is approximately

$$Q \equiv 2\pi \times \left(\frac{\text{Total energy}}{\text{Energy loss during one period}} \right)$$

$$X_p = D \cos(\omega t - \delta) \Rightarrow \dot{X}_p = -\omega D \sin(\omega t - \delta)$$

$$\Rightarrow E = \frac{1}{2} m \dot{X}^2 + \frac{1}{2} k X^2 = \frac{1}{2} m \omega^2 D^2 \sin^2(\omega t - \delta) + \frac{1}{2} k D^2 \cos^2(\omega t - \delta)$$

$$= \frac{1}{2} m D^2 [\omega^2 \sin^2(\omega t - \delta) + \omega_0^2 \cos^2(\omega t - \delta)] ; \quad \omega \approx \omega_0$$

$$\approx \frac{1}{2} m D^2 \omega_0^2 \leftarrow \text{Energia total}$$

$$W = \int \vec{F} \cdot d\vec{r} = \int 2m\beta \dot{X} \cdot d\vec{r} = \int_0^T 2m\beta \dot{X} \cdot \dot{X} dt = \int_0^T 2m\beta \omega^2 D^2 \sin^2(\omega t - \delta) dt = 2m\beta \omega^2 D^2 \cdot T = 2m\beta \omega^2 D^2 \frac{2\pi}{\omega}$$

$$= 4\pi m\beta \omega D^2 \approx 4\pi m\beta \omega_0 D^2 \leftarrow \text{Pérdida de energía}$$

$$\Rightarrow \frac{\text{Energia total}}{\text{Pérdida de energía}} = \frac{\frac{1}{2} \cdot m D^2 \omega_0^2}{4\pi m\beta \omega_0 D^2} = \frac{\frac{1}{2} \cdot \omega_0^2}{4\pi \beta} = \frac{\omega_0}{4\pi \beta} \quad y \quad Q = \frac{\omega_0}{2\beta}$$

$$\Rightarrow 2\pi \frac{\text{Energia total}}{\text{Pérdida de energía}} = \frac{\omega_0}{2\beta} = Q \quad //$$

Problema 6.

Una partícula de masa 2.0 kg se mueve en el plano xy atraída hacia el origen por una fuerza $\vec{F} = -18x \hat{x} - 50y \hat{y}$. En $t = 0$ la partícula se coloca en el punto $(3, 4)$ m y se le da una velocidad de magnitud 10 m/s en dirección perpendicular al eje x .

- a. Determine la posición y la velocidad de la partícula en cualquier tiempo.
- b. Describa la trayectoria de la partícula.

$$\vec{F} = -18x \hat{x} - 50y \hat{y} = m\vec{a} \Rightarrow m a_x = -18x \quad y \quad m a_y = -50y \Rightarrow a_x = -9x \quad y \quad a_y = -25y$$

$$* \Rightarrow \frac{dv_x}{dt} = \frac{dv_x}{dx} \frac{dx}{dt} = v_x \frac{dv_x}{dx} = -9x \Rightarrow \int_{v_0}^{v_x} v_x dv_x = \int_3^x -9x dx \Rightarrow \frac{v_x^2}{2} = \frac{-9x^2}{2} + \frac{81}{2} \Rightarrow v_x = \sqrt{81 - 9x^2}$$

$$* \Rightarrow \frac{dv_y}{dt} = \frac{dv_y}{dy} \frac{dy}{dt} = v_y \frac{dv_y}{dy} = -25y \Rightarrow \int_{v_0}^{v_y} v_y dv_y = \int_4^y -25y dy \Rightarrow \frac{v_y^2}{2} = \frac{-25y^2}{2} + \frac{100}{2}$$

$$\Rightarrow v_y = \sqrt{500 - 25y^2}$$

$$* \Rightarrow \frac{dx}{dt} = \sqrt{81 - 9x^2} \Rightarrow \int_3^x \frac{1}{\sqrt{81 - 9x^2}} dx = \int_0^t dt \Rightarrow \frac{1}{3} \operatorname{sen}^{-1}\left(\frac{x}{3}\right) - \frac{1}{3} \operatorname{sen}^{-1}(1) = t \Rightarrow \frac{1}{3} \operatorname{sen}^{-1}\left(\frac{x}{3}\right) - \frac{\pi}{6} = t$$

$$\Rightarrow \frac{1}{3} \operatorname{sen}^{-1}\left(\frac{x}{3}\right) = t + \frac{\pi}{6} \Rightarrow x = 3 \operatorname{sen}\left(3t + \frac{\pi}{2}\right) = 3 \cos(3t)$$

$$* \Rightarrow \frac{dy}{dt} = \sqrt{500 - 25y^2} \Rightarrow \int_4^y \frac{1}{\sqrt{500 - 25y^2}} dy = \int_0^t dt \Rightarrow \frac{1}{5} \operatorname{sen}^{-1}\left(\frac{y}{2\sqrt{5}}\right) - \frac{1}{5} \operatorname{sen}^{-1}(2) = t$$

$$\Rightarrow \operatorname{sen}^{-1}\left(\frac{y}{2\sqrt{5}}\right) = 5t + 63,61^\circ \Rightarrow y = 2\sqrt{5} \operatorname{sen}(5t + 63,61^\circ)$$

Problema 7.

La posición de una partícula que se mueve a lo largo del eje x se determina mediante la ecuación

$$\frac{d^2x}{dt^2} + 4 \frac{dx}{dt} + 8x = 20 \cos 2t. \quad \beta^2 = 2^2 < \omega_0^2 = 8$$

Si la partícula parte del reposo en $x = 0$, determine:

a. la posición en función del tiempo

b. la amplitud, periodo y la frecuencia de oscilación después de que ha transcurrido un tiempo largo.

$$a) \frac{d^2X}{dt^2} + 4 \frac{dX}{dt} + 8X = 20 \cos(2t) \quad \text{en } t=0 \rightarrow X_0 = 0 \quad v_0 = 0$$

$$\Rightarrow \beta = 2, \omega = 2, \omega_0^2 = 8 \quad y \quad A_3 = 20$$

$$\tan(\theta) = \left(\frac{2\omega\beta}{\omega_0^2 - \omega^2} \right)$$

$$\Rightarrow X = e^{-\beta t} (A_1 e^{\sqrt{\beta^2 - \omega^2} t} + A_2 e^{-\sqrt{\beta^2 - \omega^2} t}) + \frac{A_3}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2 \omega^2}} \cos(\omega t - \theta)$$

$$\Rightarrow \dot{X} = -\beta e^{-\beta t} (A_1 \sqrt{\beta^2 - \omega^2} e^{\sqrt{\beta^2 - \omega^2} t} - A_2 \sqrt{\beta^2 - \omega^2} e^{-\sqrt{\beta^2 - \omega^2} t}) + \frac{-A_3 \omega}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2 \omega^2}} \sin(\omega t - \theta)$$

$$\text{En } t=0 \Rightarrow \begin{cases} X = A_1 + A_2 + \frac{A_3}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2 \omega^2}} = 0 \Rightarrow 2A_1 + \frac{A_3}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2 \omega^2}} = 0 \Rightarrow A_1 = \frac{A_3}{2\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2 \omega^2}} \\ \dot{X} = -\beta(A_1 \cancel{\sqrt{\beta^2 - \omega^2}} - A_2 \cancel{\sqrt{\beta^2 - \omega^2}}) + 0 = 0 \Rightarrow A_1 = A_2 \end{cases}$$

$$\Rightarrow X = e^{-\beta t} \left(\frac{A_3}{2\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2 \omega^2}} e^{\sqrt{\beta^2 - \omega^2} t} + \frac{A_3}{2\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2 \omega^2}} e^{-\sqrt{\beta^2 - \omega^2} t} \right) + \frac{A_3}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2 \omega^2}} \cos(\omega t - \theta)$$

$$= e^{-2t} \left(\frac{20}{2\sqrt{(8-2^2)^2 + 4 \cdot 2^2 \cdot 2^2}} e^{\sqrt{2^2 - 2^2} t} + \frac{20}{2\sqrt{(8-2^2)^2 + 4 \cdot 2^2 \cdot 2^2}} e^{-\sqrt{2^2 - 2^2} t} \right) + \frac{A_3}{\sqrt{(8-2^2)^2 + 4 \cdot 2^2 \cdot 2^2}} \cos(2t - \theta)$$

$$= \frac{\sqrt{5}}{2} e^{-2t} (e^{2st} + e^{-2st}) + \sqrt{5} \cos(2t - \theta)$$

$$= \frac{\sqrt{5}}{2} e^{-2t} [2 \cos(2t)] + \sqrt{5} \cos(2t - \theta) = \sqrt{5} e^{-2t} \cos(2t) + \sqrt{5} \cos(2t - \theta)$$

Ampolitud
frecuencia
Tiempo largo

Problema 8. (Thornton 3.28)

3-28. Obtain the Fourier expansion of the function

$$F(t) = \begin{cases} -1, & -\pi/\omega < t < 0 \\ +1, & 0 < t < \pi/\omega \end{cases}$$

in the interval $-\pi/\omega < t < \pi/\omega$. Take $\omega = 1$ rad/s. In the periodical interval, calculate and plot the sums of the first two terms, the first three terms, and the first four terms to demonstrate the convergence of the series.

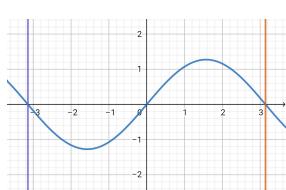
$$F(t) = \sum_{n=0}^{+\infty} a_n \cos(nwt) + b_n \sin(nwt) \Rightarrow \begin{cases} a_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} F(t) \cos(nwt) dt = \frac{\omega}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} F(t) \cos(nwt) dt \\ b_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} F(t) \sin(nwt) dt = \frac{\omega}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} F(t) \sin(nwt) dt \end{cases}$$

Importante: Es una $F(t)$ a trozos, para definir a_n y b_n hay que separar las integrales en los intervalos en que se define $F(t)$.

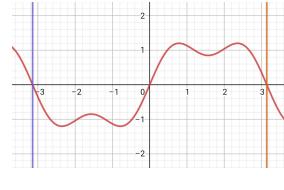
$$\Rightarrow a_n = \frac{\omega}{\pi} \int_{-\frac{\pi}{2}}^0 -1 \cdot \cos(nwt) dt + \frac{\omega}{\pi} \int_0^{\frac{\pi}{2}} 1 \cdot \cos(nwt) dt = \frac{\omega}{\pi} \left[\frac{-\sin(nwt)}{nw} \right]_{-\frac{\pi}{2}}^0 + \frac{\omega}{\pi} \left[\frac{\sin(nwt)}{nw} \right]_0^{\frac{\pi}{2}} = 0$$

$$\Rightarrow b_n = \frac{\omega}{\pi} \int_{-\frac{\pi}{2}}^0 -1 \cdot \sin(nwt) dt + \frac{\omega}{\pi} \int_0^{\frac{\pi}{2}} 1 \cdot \sin(nwt) dt = \frac{\omega}{\pi} \left[\frac{\cos(nwt)}{nw} \right]_{-\frac{\pi}{2}}^0 - \frac{\omega}{\pi} \left[\frac{\cos(nwt)}{nw} \right]_0^{\frac{\pi}{2}} \\ = \frac{1}{n\pi} [1 - \cos(n\pi) - \cos(n\pi) + 1] = \frac{2}{n\pi} [1 - \cos(n\pi)] = \frac{2}{n\pi} [1 - (-1)^n]$$

$$\Rightarrow F(t) = \sum_{n=0}^{+\infty} \frac{2}{n\pi} [1 - (-1)^n] \sin(nwt) = \sum_{n=0}^{+\infty} \frac{4}{(2n+1)\pi} \sin[(2n+1)wt]$$



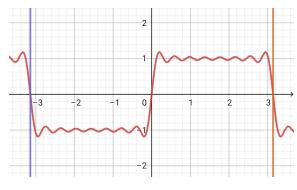
$$n=0$$



$$n=0,1$$



$$n=0,1,2$$



$$n=0,1,2,3,4,5,6,7$$

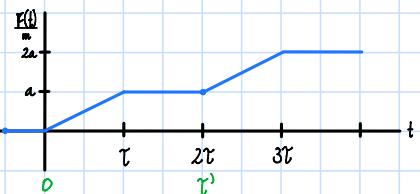
Problema 8. (Thornton 3.31)

3-31. A damped linear oscillator, originally at rest in its equilibrium position, is subjected to a forcing function given by

$$\frac{F(t)}{m} = \begin{cases} 0, & t < 0 \\ a \times (t/\tau), & 0 < t < \tau \\ a, & t > \tau \end{cases}$$

Find the response function. Allow $\tau \rightarrow 0$ and show that the solution becomes that for a step function.

$F(t)$ posee un acho de 2τ con un aumento de a por acho



$$\frac{F(t)}{m} = \sum_{n=0}^{+\infty} a_n \cos(n\omega t) + b_n \sin(n\omega t) \Rightarrow \begin{cases} a_n = \frac{2}{\tau} \int_0^{\tau} \frac{F(t)}{m} \cos(n\omega t) dt ; \omega = \frac{2\pi}{\tau} \\ b_n = \frac{2}{\tau} \int_0^{\tau} \frac{F(t)}{m} \sin(n\omega t) dt \end{cases}$$

$$\Rightarrow a_n = \frac{2}{\tau} \int_0^{\tau} \frac{at}{\tau} \cos(n\omega t) dt + \frac{2}{\tau} \int_0^{\tau} a \cos(n\omega t) dt = \frac{2}{\tau} \left[\frac{at \sin(n\omega t)}{\tau n \omega} + \frac{\cos(n\omega t)}{n^2 \omega^2} \right] \Big|_0^{\frac{\tau}{2}} + \frac{2}{\tau} \left[\frac{\sin(n\omega t)}{n \omega} \right] \Big|_0^{\frac{\tau}{2}}$$

$$= \frac{2}{\tau} \left[\frac{a \cancel{\sin(n\frac{\tau}{2})}}{2 \tau n \omega} - \frac{a}{n^2 \omega^2 \tau} \right] - \frac{2}{\tau} \left[\frac{\sin(n\frac{\tau}{2})}{n \omega} \right] = \frac{a \cancel{\sin(n\frac{\tau}{2})} \tau^3}{2 \pi \tau n} - \frac{a \tau^3}{2 \pi^2 n^2 \tau} - \frac{\sin(n\frac{\tau}{2})}{\pi \tau n}$$

$$\Rightarrow b_n = \frac{2}{\tau} \int_0^{\tau} \frac{at}{\tau} \sin(n\omega t) dt + \frac{2}{\tau} \int_0^{\tau} a \sin(n\omega t) dt = \frac{2}{\tau} \left[-\frac{at \cos(n\omega t)}{\tau n \omega} - \frac{a \sin(n\omega t)}{n^2 \omega^2} \right] \Big|_0^{\frac{\tau}{2}} - \frac{2}{\tau} \left[\cos(n\omega t) \right] \Big|_0^{\frac{\tau}{2}}$$

$$= \frac{2}{\tau} \left[-\frac{a \cancel{\cos(n\frac{\tau}{2})}}{2 \tau n \omega} + \frac{a \cancel{\cos(n\cdot 0)}}{\tau n \omega} - \frac{a \sin(n\frac{\tau}{2})}{n^2 \omega^2} \right] - \frac{2}{\tau} \left[\cos(n\omega t) \right] \Big|_0^{\frac{\tau}{2}} = -\frac{T \cancel{a \sin(n\frac{\tau}{2})}}{2 \pi^2 \tau} - \frac{1}{n \pi}$$

$$\Rightarrow \frac{F(t)}{m} = \sum_{n=0}^{+\infty} \left[\frac{a \cancel{\sin(n\frac{\tau}{2})} \tau^3}{2 \pi \tau n} - \frac{a \tau^3}{2 \pi^2 n^2 \tau} - \frac{\sin(n\frac{\tau}{2})}{\pi \tau n} \right] \cos(n\omega t) + \left[-\frac{T \cancel{a \sin(n\frac{\tau}{2})}}{2 \pi^2 \tau} - \frac{1}{n \pi} \right] \sin(n\omega t)$$

Problema 9. (Thornton 3.43)

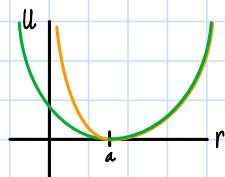
3-43. A point mass m slides without friction on a horizontal table at one end of a massless spring of natural length a and spring constant k as shown in Figure 3-C. The spring is attached to the table so it can rotate freely without friction. The net force on the mass is the central force $F(r) = -k(r - a)$. (a) Find and sketch both the potential energy $U(r)$ and the effective potential $U_{\text{eff}}(r)$. (b) What angular velocity ω_0 is required for a circular orbit with radius r_0 ? (c) Derive the frequency of small oscillations ω about the circular orbit with radius r_0 . Express your answers for (b) and (c) in terms of k , m , r_0 , and a .

$$a) \vec{F} = -k(r-a)\hat{e}_r \Rightarrow \vec{F} = -\nabla U \Rightarrow U = \frac{k(r-a)^2}{2}$$

$$\mathcal{T} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2; L_0 = rmv \Rightarrow \frac{L_0}{mr^2} = \frac{v}{r} = \dot{\theta}$$

$$\Rightarrow \mathcal{T} = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}\cancel{mr^2}\frac{\dot{L}_0^2}{\cancel{mr^2}} = \frac{1}{2}m\dot{r}^2 + \frac{\dot{L}_0^2}{2mr^2} \Rightarrow E = \mathcal{T} + U = \frac{1}{2}m\dot{r}^2 + \frac{\dot{L}_0^2}{2mr^2} + \frac{k(r-a)^2}{2}$$

$$\Rightarrow U_{\text{eff}} = \frac{k(r-a)^2}{2} + \frac{\dot{L}_0^2}{2mr^2}$$



$$b) \vec{F} = -k(r-a)\hat{e}_r = m(\ddot{r} - r\dot{\theta}^2)\hat{e}_r \Rightarrow k(r-a) = mr\dot{\theta}^2 \Rightarrow \dot{\theta} = \sqrt{\frac{k(r-a)}{mr}} \Rightarrow \dot{\theta}_0 = \sqrt{\frac{k(r_0-a)}{mr_0}}$$

c) Oscilaciones para una órbita r_0 , en $r = r_0$ la energía potencial debe poseer un mínimo

$$\begin{aligned} U_{\text{eff}} &= \frac{k(r-a)^2}{2} + \frac{\dot{L}_0^2}{2mr^2} = \frac{k(r-a)^2}{2} + \frac{1}{2}mr^2\dot{\theta}^2 \\ &\approx U_{\text{eff}}(r_0) + U_{\text{eff}}'(r_0)(r-r_0) + \frac{1}{2}U_{\text{eff}}''(r_0)(r-r_0)^2 \\ &\approx \frac{k(r_0-a)^2}{2} + \frac{1}{2}mr_0^2\dot{\theta}^2 + [k(r_0-a) + mr_0\dot{\theta}^2](r-r_0) + \frac{1}{2}[k + m\dot{\theta}^2](r-r_0)^2 \\ &= \frac{kr_0^2 - 2kr_0a + ka^2}{2} + \frac{1}{2}mr_0^2\dot{\theta}^2 + k(r_0-a)(r-r_0) + mr_0\dot{\theta}^2(r-r_0) + \frac{1}{2}k(r-r_0)^2 + \frac{1}{2}m\dot{\theta}^2(r-r_0)^2 \\ &= \frac{-kr_0^2 + kr_0a + ka^2}{2} - \frac{1}{2}mr_0^2\dot{\theta}^2 + k(r_0-a)r + mr_0\dot{\theta}^2r + \frac{1}{2}k(r-r_0)^2 + \frac{1}{2}m\dot{\theta}^2(r-r_0)^2 \end{aligned}$$

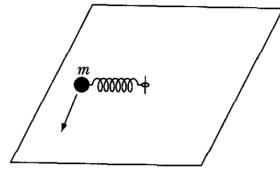


FIGURE 3-C Problem 3-43.

Oscilaciones no lineales

Problema 1. (Thornton 4.2)

4.2. Construct a phase diagram for the potential in Figure 4-1.

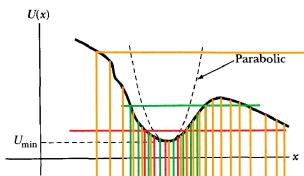


FIGURE 4-1 Arbitrary potential $U(x)$ indicating a parabolic region where simple harmonic motion is applicable.

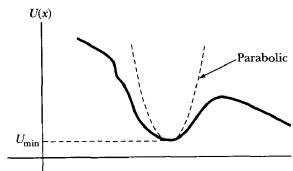
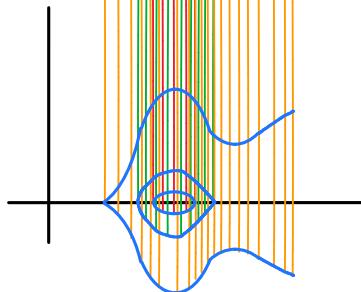


FIGURE 4-1 Arbitrary potential $U(x)$ indicating a parabolic region where simple harmonic motion is applicable.



Problema 2. (Thornton 4.4)

4.4. Lord Rayleigh used the equation

$$\ddot{x} - (a - b\dot{x}^2)\dot{x} + \omega_0^2 x = 0$$

in his discussion of nonlinear effects in acoustic phenomena.* Show that differentiating this equation with respect to time and making the substitution $y = y_0 \sqrt{3b/a} \dot{x}$ results in van der Pol's equation:

$$\ddot{y} - \frac{a}{y_0^2} (y_0^2 - y^2) \dot{y} + \omega_0^2 y = 0$$

$$\Rightarrow \ddot{X} - (-2b\dot{X}\ddot{X})\dot{X} - (a - b\dot{X}^2)\ddot{X} + \omega_0^2 \dot{X} = 0 \Rightarrow \ddot{X} + 2b\dot{X}^2\ddot{X} - a\ddot{X} + b\dot{X}^2\ddot{X} + \omega_0^2 \dot{X} = 0$$

$$\Rightarrow \ddot{X} - a\ddot{X} + 3b\dot{X}^2\ddot{X} + \omega_0^2 \dot{X} = 0$$

$$\Rightarrow \cancel{\frac{\ddot{y}}{y_0} \sqrt{\frac{a}{3b}}} - a \cancel{\frac{\ddot{y}}{y_0} \sqrt{\frac{a}{3b}}} + 3b \frac{y^2}{y_0^2} \cancel{\frac{\ddot{y}}{y_0} \sqrt{\frac{a}{3b}}} + \omega_0^2 \cancel{\frac{y}{y_0} \sqrt{\frac{a}{3b}}} = 0$$

$$\Rightarrow \ddot{y} - a\ddot{y} + \frac{y^2}{y_0^2} a\ddot{y} + \omega_0^2 y = 0 \Rightarrow \ddot{y} - \frac{a}{y_0^2} (y_0^2 - y^2) \dot{y} + \omega_0^2 y = 0$$

$$\ddot{X} - (a - b\dot{X}^2)\dot{X} + \omega_0^2 X = 0$$

$$y = y_0 \sqrt{\frac{3b}{a}} \dot{X} \Rightarrow \dot{X} = \frac{y}{y_0} \sqrt{\frac{a}{3b}}$$

$$\Rightarrow \ddot{X} = \frac{\ddot{y}}{\frac{y_0}{\sqrt{3b}}} \sqrt{\frac{a}{3b}}$$

$$\Rightarrow \ddot{X} = \frac{\ddot{y}}{\frac{y_0}{\sqrt{3b}}} \sqrt{\frac{a}{3b}}$$

Problema 3. (Taylor 12.1)

12.1* Consider the nonlinear first-order equation $\dot{x} = 2\sqrt{x - 1}$. **(a)** By separating variables, find a solution $x_1(t)$. **(b)** Your solution should contain one constant of integration k , so you might reasonably expect it to be the general solution. Show, however, that there is another solution, $x_2(t) = 1$, that is not of the form of $x_1(t)$ whatever the value of k . **(c)** Show that although $x_1(t)$ and $x_2(t)$ are solutions, neither $Ax_1(t)$, nor $Bx_2(t)$, nor $x_1(t) + x_2(t)$ are solutions. (That is, the superposition principle does not apply to this equation.)

Problema 4.

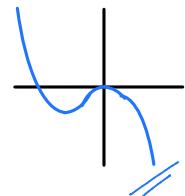
Considere la siguiente ecuación de movimiento:

$$\ddot{x} - x - \frac{1}{2}x^2 = 0$$

$$F = -\frac{dU}{dx}$$

- a. Grafique el potencial en función de la posición.
- b. Determine los puntos de equilibrio del sistema.
- c. Realice el diagrama de fase.
- d. Indique los diferentes casos en el diagrama anterior.

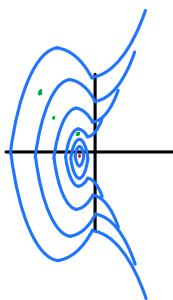
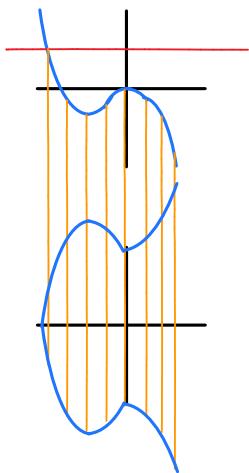
$$a) \ddot{x} - x - \frac{1}{2}x^2 = 0 \Rightarrow \ddot{x} = x + \frac{1}{2}x^2 \Rightarrow U = -\int x + \frac{1}{2}x^2 dx = -\frac{x^2}{2} - \frac{x^3}{6} = -x^2 \left(\frac{x}{6} + \frac{1}{2} \right)$$



$$b) \frac{dU}{dx} = 0 \Rightarrow x + \frac{1}{2}x^2 = 0 \Rightarrow x \left(\frac{1}{2} + x \right) = 0 \Rightarrow x = 0 \quad \text{y} \quad x = -2$$

$$c) \frac{dx}{dt} = \dot{x} + \frac{1}{2}x^2 \quad \text{y} \quad \frac{d\dot{x}}{dt} = \ddot{x} \Rightarrow \frac{\frac{d\dot{x}}{dt}}{\frac{dx}{dt}} = \frac{\dot{x} + \frac{1}{2}x^2}{\dot{x}} = \frac{d\dot{x}}{dx} \Rightarrow \frac{d\dot{x}}{dx} = \frac{\dot{x}}{x} \left(\frac{1}{2} + x \right)$$

$$\Rightarrow \int_{x_0}^x \dot{x} d\dot{x} = \int_{x_0}^x \dot{x} \left(\frac{1}{2}x^2 + \frac{1}{2}x^2 \right) dx \Rightarrow \dot{x}^2 - \dot{x}_0^2 = \frac{x^2}{2} + \frac{x^3}{6} - \frac{x_0^2}{2} + \frac{x_0^3}{6}$$



Movimiento libre entre un x_* y $+\infty$

Armónico simple

Péndulo plano y caos

Problema 1. (Thornton 4.6)

- 4.6. Derive the expression for the phase paths of the plane pendulum if the total energy is $E > 2mgl$. Note that this is just the case of a particle moving in a periodic potential $U(\theta) = mgl(1 - \cos \theta)$.

$$\text{Pendulo plano: } \ddot{\theta} + \omega^2 \sin \theta = 0 \quad \text{con } \omega^2 = \frac{g}{l} \quad \Rightarrow \frac{d\dot{\theta}}{dt} = -\omega^2 \sin \theta \quad \text{y} \quad \frac{d\theta}{dt} = \dot{\theta}$$

$$\Rightarrow \frac{\frac{d\dot{\theta}}{dt}}{\dot{\theta}} = \frac{d\dot{\theta}}{d\theta} = \frac{-\omega^2 \sin \theta}{\dot{\theta}} \quad \Rightarrow \int_{\theta_0}^{\theta} \dot{\theta} d\dot{\theta} = \int_{\theta_0}^{\theta} -\omega^2 \sin \theta' d\theta' \quad \Rightarrow \dot{\theta}^2 - \dot{\theta}_0^2 = \omega^2 \cos \theta - \omega^2 \cos \theta_0$$

$$\text{Con energía } E = T + U = \frac{1}{2} ml^2 \dot{\theta}^2 + mgl(1 - \cos \theta) \Rightarrow \frac{1}{2} ml^2 \dot{\theta}^2 = E - mgl(1 - \cos \theta)$$

$$\Rightarrow \dot{\theta} = \sqrt{\frac{2}{ml^2} [E - mgl(1 - \cos \theta)]}$$

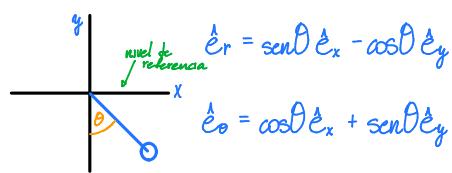
como $E > 2mgl$, $\dot{\theta}$ tiene valores reales y por tanto, el péndulo oscila y como $\dot{\theta}$ está definido para $-\pi \leq \theta \leq \pi$, el péndulo puede dar una vuelta completa

Problema 2. (Thornton 4.7)

- 4.7. Consider the free motion of a plane pendulum whose amplitude is not small. Show that the horizontal component of the motion may be represented by the approximate expression (components through the third order are included)

$$\ddot{x} + \omega_0^2 \left(1 + \frac{x_0^2}{l^2}\right)x - \varepsilon x^3 = 0$$

where $\omega_0^2 = g/l$ and $\varepsilon = 3g/2l^3$, with l equal to the length of the suspension.



$$\text{Pendulo plano: } \ddot{\theta} + \omega_0^2 \sin \theta = 0 \text{ con } \omega_0^2 = \frac{g}{l}$$

Se quiere únicamente la parte horizontal del movimiento

$$\Rightarrow \dot{x} = \sin \theta \star \Rightarrow \theta = \sin^{-1}(\frac{\dot{x}}{l}) \Rightarrow \dot{\theta} = \frac{1}{\sqrt{1-(\frac{\dot{x}}{l})^2}} \dot{x} \star$$

$$\Rightarrow \ddot{\theta} = \frac{1}{\sqrt{1-(\frac{\dot{x}}{l})^2}} \frac{\ddot{x}}{l} - \frac{\dot{x}}{l} \frac{-2(\frac{\dot{x}}{l}) \dot{x}}{2\sqrt{1-(\frac{\dot{x}}{l})^2}} \Rightarrow \ddot{\theta} = \frac{1}{\sqrt{1-(\frac{\dot{x}}{l})^2}} \frac{\ddot{x}}{l} + \frac{\dot{x}^2}{l^2} \frac{\dot{x}}{l\sqrt{1-(\frac{\dot{x}}{l})^2}} \quad \textcircled{*} \quad \text{Dada la aparición de } \dot{x}, \text{ es necesario una relación, usando la conservación}$$

$$\frac{1}{2} m l^2 \dot{\theta}^2 - mg l \cos \theta = -mg l \cos \theta \leftarrow \text{Ángulo máximo que alcanza el péndulo} \Rightarrow \dot{\theta}^2 = \frac{2g(\cos \theta - \cos 0)}{l}$$

$$\Rightarrow \dot{\theta}^2 = \frac{2g(\sqrt{1-\sin^2 \theta} - \sqrt{1-\sin^2 0})}{l} \star \Rightarrow \dot{\theta}^2 = \frac{2g(\sqrt{1-(\frac{\dot{x}}{l})^2} - \sqrt{1-(\frac{0}{l})^2})}{l} \text{ y } \star \Rightarrow \dot{\theta}^2 = \frac{1}{1-(\frac{\dot{x}}{l})^2} \dot{x}^2$$

$$\Rightarrow \dot{\theta}^2 = \frac{2g(\sqrt{1-(\frac{\dot{x}}{l})^2} - \sqrt{1-(\frac{0}{l})^2})}{l} = \frac{1}{1-(\frac{\dot{x}}{l})^2} \frac{\dot{x}^2}{l^2} \Rightarrow \frac{\dot{x}^2}{l^2} = 2\omega_0^2 [1-(\frac{\dot{x}}{l})^2] (\sqrt{1-(\frac{\dot{x}}{l})^2} - \sqrt{1-(\frac{0}{l})^2}) \quad \textcircled{a}$$

$$\textcircled{a} \Rightarrow \dot{\theta} = \frac{1}{\sqrt{1-(\frac{\dot{x}}{l})^2}} \frac{\ddot{x}}{l} + 2\omega_0^2 [1-(\frac{\dot{x}}{l})^2] (\sqrt{1-(\frac{\dot{x}}{l})^2} - \sqrt{1-(\frac{0}{l})^2}) \frac{\dot{x}}{l\sqrt{1-(\frac{\dot{x}}{l})^2}^{3/2}}$$

$$\Rightarrow \ddot{\theta} = \frac{1}{\sqrt{1-(\frac{\dot{x}}{l})^2}} \frac{\ddot{x}}{l} + 2\omega_0^2 (\sqrt{1-(\frac{\dot{x}}{l})^2} - \sqrt{1-(\frac{0}{l})^2}) \frac{\dot{x}}{l\sqrt{1-(\frac{\dot{x}}{l})^2}} = -\omega_0^2 \frac{\dot{x}}{l}$$

$$\Rightarrow \frac{1}{\sqrt{1-(\frac{\dot{x}}{l})^2}} \frac{\ddot{x}}{l} + 2\omega_0^2 \frac{\dot{x}}{l} - 2\omega_0^2 \frac{\sqrt{1-(\frac{\dot{x}}{l})^2}}{\sqrt{1-(\frac{\dot{x}}{l})^2}} \frac{\dot{x}}{l} = -\omega_0^2 \frac{\dot{x}}{l} \Rightarrow \frac{1}{\sqrt{1-(\frac{\dot{x}}{l})^2}} \frac{\ddot{x}}{l} + 3\omega_0^2 \frac{\dot{x}}{l} - 2\omega_0^2 \frac{\sqrt{1-(\frac{\dot{x}}{l})^2}}{\sqrt{1-(\frac{\dot{x}}{l})^2}} \frac{\dot{x}}{l} = 0$$

$$\Rightarrow \frac{\ddot{x}}{l} + 3\omega_0^2 \sqrt{1-(\frac{\dot{x}}{l})^2} \frac{\dot{x}}{l} - 2\omega_0^2 \sqrt{1-(\frac{\dot{x}}{l})^2} \frac{\dot{x}}{l} = 0 \Rightarrow \ddot{x} + \omega_0^2 \dot{x} [3\sqrt{1-(\frac{\dot{x}}{l})^2} - 2\sqrt{1-(\frac{\dot{x}}{l})^2}] = 0$$

$$\star \sqrt{1-(\frac{\dot{x}}{l})^2} \approx 1 - \frac{1}{2} (\frac{\dot{x}}{l})^2$$

$$\star \sqrt{1-(\frac{\dot{x}}{l})^2} \approx 1 - \frac{1}{2} (\frac{\dot{x}}{l})^2$$

$$\Rightarrow 3\sqrt{1-(\frac{\dot{x}}{l})^2} - 2\sqrt{1-(\frac{\dot{x}}{l})^2} \approx 3 - \frac{3}{2} (\frac{\dot{x}}{l})^2 - 2 + (\frac{\dot{x}}{l})^2 = 1 + (\frac{\dot{x}}{l})^2 - \frac{3}{2} (\frac{\dot{x}}{l})^2$$

$$\Rightarrow \ddot{x} + \omega_0^2 \dot{x} [1 + (\frac{\dot{x}}{l})^2 - \frac{3}{2} (\frac{\dot{x}}{l})^2] = 0 \Rightarrow \ddot{x} + \omega_0^2 \dot{x} [1 + (\frac{\dot{x}}{l})^2] - \omega_0^2 \dot{x} \cdot \frac{3}{2} (\frac{\dot{x}}{l})^2 = \ddot{x} + \omega_0^2 \dot{x} [1 + (\frac{\dot{x}}{l})^2] - \frac{3}{2} \omega_0^2 \frac{\dot{x}^3}{l^2}$$

Problema 3.

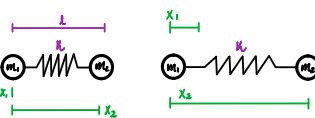
Considera un disco muy delgado de radio a , con un punto de pivote en la circunferencia. El movimiento comienza en el reposo, cuando la posición angular es θ_0 .

- a. Planteé desde el inicio, el problema de movimiento.
- b. Resuelva la ecuación del movimiento. ¿Qué ocurre en el caso de oscilaciones pequeñas?
- c. Realice el diagrama de fase. Describa el movimiento, de acuerdo a cada parte del diagrama y las consideraciones de energía. ¿Cómo se simplifica el diagrama en el caso de oscilaciones pequeñas?

Oscilaciones Acopladas

Problema 1. (Thornton 3.6)

- 3-6. Two masses $m_1 = 100 \text{ g}$ and $m_2 = 200 \text{ g}$ slide freely in a horizontal frictionless track and are connected by a spring whose force constant is $k = 0.5 \text{ N/m}$. Find the frequency of oscillatory motion for this system.



$$m_1 \ddot{x}_1 = -k(x_1 - x_2 + l) \quad \text{y} \quad m_2 \ddot{x}_2 = -k(x_2 - x_1 - l) \Rightarrow x_1 = \frac{1}{k}(m_2 \ddot{x}_2 + kx_2 + kl) \quad ; \text{ Sustituyendo en } *$$

$$\Rightarrow m_1 \ddot{x}_1 = -(m_2 \ddot{x}_2 + kx_2 + kl) + kx_2 + kl \Rightarrow m_1 \ddot{x}_1 = -m_2 \ddot{x}_2$$

$$\Rightarrow m_1 \ddot{x}_1 = m_1 \frac{d^2}{dt^2} \left[\frac{1}{k} (m_2 \ddot{x}_2 + kx_2 + kl) \right] = \frac{m_2 m_1}{k} \frac{d^2}{dt^2} (\ddot{x}_2) + m_1 \ddot{x}_2 \Rightarrow \frac{m_2 m_1}{k} \frac{d^2}{dt^2} (\ddot{x}_2) + m_1 \ddot{x}_2 = -m_2 \ddot{x}_2$$

$$\Rightarrow m_2 m_1 \frac{d^2}{dt^2} (\ddot{x}_2) + k(m_1 + m_2) \ddot{x}_2 = 0 \Rightarrow \frac{d^2}{dt^2} [m_2 m_1 \ddot{x}_2 + k(m_1 + m_2) x_2] = 0 \Rightarrow \ddot{x}_2 + \frac{k(m_1 + m_2)}{m_2 m_1} x_2 = 0$$

$$\Rightarrow \omega^2 = \frac{k(m_1 + m_2)}{m_2 m_1} \quad // \quad \text{Si se resuelve para } x_1 \text{ se obtiene la misma frecuencia.}$$

Otra forma

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 \quad \text{y} \quad U = \frac{1}{2} k(x_1 - x_2 + l)^2 \leftarrow \text{Una sola fuerza independiente en el sistema}$$

$$T = \frac{1}{2} \sum_i m_i \dot{q}_i^2 = \frac{1}{2} (m_1 \dot{x}_1^2 + m_2 \dot{x}_2^2) \Rightarrow m = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$$

$$A_{ij} = \frac{\partial^2 U}{\partial q_i \partial q_j} \rightarrow A_{11} = k, A_{22} = k, A_{12} = -k, A_{21} = -k \Rightarrow A = \begin{pmatrix} k & -k \\ -k & k \end{pmatrix}$$

$$\text{Para hallar frecuencias & oscilación: } |A - \omega^2 m| = 0 \Rightarrow \begin{vmatrix} k - \omega^2 m_1 & -k \\ -k & k - \omega^2 m_2 \end{vmatrix} = 0$$

$$\Rightarrow (k - \omega^2 m_1)(k - \omega^2 m_2) - k^2 = 0 \Rightarrow k^2 - \omega^2 m_2 k - \omega^2 m_1 k + m_2 m_1 \omega^4 - k^2 = 0 \Rightarrow m_2 m_1 \omega^4 - \omega^2 k(m_2 + m_1) = 0$$

$$\Rightarrow m_2 m_1 \omega^2 - k(m_2 + m_1) = 0 \Rightarrow \omega^2 = \frac{k(m_1 + m_2)}{m_2 m_1} //$$

Problema 2. (Thornton 3.10)

- 3-10. If the amplitude of a damped oscillator decreases to $1/e$ of its initial value after n periods, show that the frequency of the oscillator must be approximately $[1 - (8\pi^2 n^2)^{-1}]$ times the frequency of the corresponding undamped oscillator.

$$n = 1, 2, 3, \dots$$

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0 \Rightarrow x = A e^{-\beta t} \cos(\omega_0 t) \Rightarrow x(t) = A e^{-\beta t} \quad \text{y} \quad x(t+nT) = A e^{-\beta(t+nT)}; \text{ Decaimiento de la base}$$

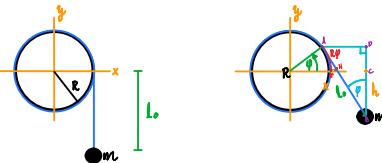
$$\Rightarrow \text{Decaimiento} = \frac{A e^{-\beta(t+nT)}}{A e^{-\beta t}} = e^{-n\beta T} \Rightarrow n\beta T = 1 = n\beta 2\pi \Rightarrow \beta = \frac{\omega_0}{2\pi n} \leftarrow \begin{array}{l} \text{Condición necesaria para} \\ \text{decaimiento exponencial} \end{array}$$

Ejemplos $\omega_0^2 = \omega_b^2 - \beta^2 \Rightarrow \omega_0^2 = \omega_b^2 - \frac{\omega_0^2}{4\pi^2 n^2} \Rightarrow \omega_b^2 = \omega_0^2 \left(1 + \frac{1}{4\pi^2 n^2}\right) \Rightarrow \frac{\omega_0^2}{\omega_b^2} = 1 + \frac{1}{4\pi^2 n^2}$

$\Rightarrow \omega_0 = \left(1 + \frac{1}{4\pi^2 n^2}\right)^{-\frac{1}{2}} \stackrel{\text{Serie de Taylor}}{\approx} 1 - \frac{1}{2} \frac{1}{4\pi^2 n^2} = 1 - \frac{1}{8\pi^2 n^2} \Rightarrow \omega_0 \approx \omega_0 \left(1 - \frac{1}{8\pi^2 n^2}\right)$

Problema 3. (Taylor 5.4)

- 5.4** An unusual pendulum is made by fixing a string to a horizontal cylinder of radius R , wrapping the string several times around the cylinder, and then tying a mass m to the loose end. In equilibrium the mass hangs a distance l_0 vertically below the edge of the cylinder. Find the potential energy if the pendulum has swung to an angle ϕ from the vertical. Show that for small angles, it can be written in the Hooke's law form $U = \frac{1}{2}k\phi^2$. Comment on the value of k .



$$\overline{BC} = h, \overline{AB} = \overline{AN} + \overline{NB} \quad \text{y} \quad \overline{AN} = R\phi, \overline{NB} = l_0 \Rightarrow \overline{AB} = R\phi + l_0 \quad ; \quad \overline{CD} = R \operatorname{sen} \phi$$

$$\overline{BD} = \overline{AB} \cos \phi = \overline{CD} + \overline{BC} \Rightarrow (R\phi + l_0) \cos \phi = R \operatorname{sen} \phi + h \Rightarrow h = (R\phi + l_0) \cos \phi - R \operatorname{sen} \phi$$

La energía potencial: $U = -mgh \Rightarrow U = -mg[(R\phi + l_0) \cos \phi - R \operatorname{sen} \phi]$

Aproximaciones pequeñas: $\operatorname{sen} \theta = \theta$ y $\cos \theta = 1 - \frac{\theta^2}{2}$

$$\Rightarrow U = -mg[(R\phi + l_0)(1 - \frac{\phi^2}{2}) - R\phi] = -mg(R\phi + l_0 - R\frac{\phi^3}{2} - l_0\frac{\phi^2}{2} - R\phi) = mg(R\frac{\phi^3}{2} + l_0\frac{\phi^2}{2} - l_0) = \frac{1}{2}mg l_0 \phi^2 - mg l_0$$

$$\Rightarrow U = \frac{1}{2}mg l_0 \phi^2$$

$$U = \frac{1}{2}K\phi^2 = \frac{1}{2}mg l_0 \phi^2 \Rightarrow K = mg l_0$$

Problema 4. (Taylor 11.1)

11.1 * In discussing the two carts of Figure 11.1, I mentioned that it is simplest to assume that when the two carts are in equilibrium the lengths L_1, L_2, L_3 of the three springs are equal to their natural, unstretched lengths l_1, l_2, l_3 . However, this assumption is not needed, and the three springs could all be in tension (or compression) at the equilibrium position. (a) Find the relations among these six lengths (and the three spring constants k_1, k_2, k_3) required for the two carts to be in equilibrium. (b) Show that the net force on either cart is exactly as given in Equation (11.2), irrespective of how L_1, L_2, L_3 compare with l_1, l_2, l_3 , just as long as x_1 and x_2 are measured from the carts' equilibrium positions.

$$\begin{cases} m_1 \ddot{x}_1 = -(k_1 + k_2)x_1^* + k_2 x_2^* \\ m_2 \ddot{x}_2 = k_2 x_1^* - (k_2 + k_3)x_2^* \end{cases}$$

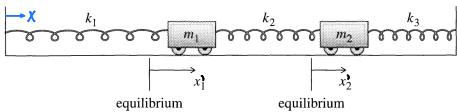


Figure 11.1 Two carts attached to fixed walls by the springs labeled k_1 and k_3 , and to each other by k_2 . The carts' positions x_1 and x_2 are measured from their respective equilibrium positions.

a) Equilibrio: Se mide en un marco distinto a la distancia de equilibrio de los resorte

$$\begin{aligned} \Rightarrow & \begin{cases} m_1 \ddot{x}_1 = -k_1(x_1 - l_1) - k_{21}(x_1 - x_2 - l_2) \\ m_2 \ddot{x}_2 = k_{21}x_1^* - (k_2 + k_3)x_2^* \end{cases} \Rightarrow \begin{cases} k_{21}(x_1 - l_1) = -k_2(x_1 - x_2 - l_2) \\ k_3(x_2 - l_3) = -k_2(x_2 - x_1 + l_2) = k_2(x_1 - x_2 - l_2) \end{cases} \\ \Rightarrow & \begin{cases} k_{21}(x_1 - l_1) = -k_2(x_1 - x_2 - l_2) \\ k_3(x_2 - l_3) = k_2(x_1 - x_2 - l_2) \end{cases} \Rightarrow k_3(x_2 - l_3) = k_2(x_1 - x_2 - l_2) = -k_1(x_1 - l_1) \end{aligned}$$

$$\text{Si } x_1 = l_1, x_2 = l_3 \text{ y } x_1 - x_2 = l_1 - l_3 = l_2 \Rightarrow k_3(l_3 - l_3) = k_2(l_2 - l_2) = -k_1(l_1 - l_1) \star$$

$$b) \text{Carro 1: } m_1 \ddot{x}_1 = -k_1(x_1 + l_1 - l_1) - k_{21}(x_1^* - x_2 + l_3 - l_2) = -k_1 x_1^* - k_{21}(l_1 - l_1) - k_{21}(x_1^* - x_2) - k_{21}(l_3 - l_2) \text{ Por } \star$$

$$\Rightarrow m_1 \ddot{x}_1 = -k_1 x_1^* - k_{21}(x_1^* - x_2)$$

$$\text{Carro 2: } m_2 \ddot{x}_2 = -k_{21}(x_2 - x_1^* - l_2 + l_2) - k_3(x_2^* + l_3 - l_3) = -k_{21}(x_2 - x_1^*) - k_{21}(l_2 - l_2) - k_3 x_2^* - k_3(l_3 - l_3) \text{ Por } \star$$

$$\Rightarrow m_2 \ddot{x}_2 = -k_{21}(x_2 - x_1^*) - k_3 x_2^*$$

Problema general

Resuelva

$$\Rightarrow \begin{cases} M \ddot{x}_1 = -K_1 x_1 - K_{12}(x_1 - x_2) \\ M \ddot{x}_2 = -K_{12}(x_2 - x_1) - K_2 x_2 \end{cases} \Rightarrow \begin{cases} M \ddot{x}_1 = -(K_1 + K_{12})x_1 + K_{12}x_2 \\ M \ddot{x}_2 = +K_{12}x_1 - (K_2 + K_{12})x_2 \end{cases}$$

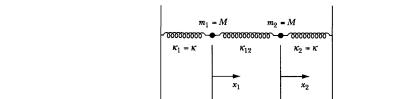


FIGURE 12-1 Two masses are connected by a spring to each other and by springs to fixed positions. This is a system of coupled motion in one dimension.

$$\Rightarrow \begin{cases} M \ddot{x}_1 + (K_1 + K_{12})x_1 - K_{12}x_2 = 0 \\ M \ddot{x}_2 - K_{12}x_1 + (K_2 + K_{12})x_2 = 0 \end{cases} \Rightarrow M \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \begin{pmatrix} -(K_1 + K_{12}) & K_{12} \\ K_{12} & -(K_2 + K_{12}) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leftarrow \text{Sistema de ecuaciones diferenciales de } 2^{\circ} \text{ orden}$$

$$\Rightarrow \begin{pmatrix} \dot{x}_1 \\ \ddot{x}_1 \\ \dot{x}_2 \\ \ddot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{(K_1 + K_{12})}{M} & 0 & \frac{K_{12}}{M} \\ 1 & 0 & 0 & 0 \\ 0 & \frac{K_{12}}{M} & 0 & -\frac{(K_2 + K_{12})}{M} \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{pmatrix} \leftarrow \begin{array}{l} \text{Se puede resolver de esta forma directamente, no obstante} \\ \text{puede ponerse muy complejo, por lo que se regresara a 1} \end{array}$$

$$\Rightarrow \begin{cases} M \ddot{x}_1 + (K_1 + K_{12})x_1 - K_{12}x_2 = 0 \\ M \ddot{x}_2 - K_{12}x_1 + (K_2 + K_{12})x_2 = 0 \end{cases} \quad \begin{array}{l} \text{Al ser un conjunto de osciladores, se puede} \\ \text{suponer una soluci\'on de la forma} \end{array} \Rightarrow \begin{cases} x_1 = B_1 e^{i\omega t} \\ x_2 = B_2 e^{i\omega t} \end{cases}$$

Sustituyendo en el sistema

$$\Rightarrow \begin{cases} M \cdot -\omega^2 B_1 e^{i\omega t} + (K_1 + K_{12})B_1 e^{i\omega t} - K_{12}B_2 e^{i\omega t} = 0 \\ M \cdot -\omega^2 B_2 e^{i\omega t} - K_{12}B_1 e^{i\omega t} + (K_2 + K_{12})B_2 e^{i\omega t} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} (K_1 + K_{12} - M\omega^2)B_1 - K_{12}B_2 = 0 \\ -K_{12}B_1 + (K_2 + K_{12} - M\omega^2)B_2 = 0 \end{cases} \quad \begin{array}{l} \text{Ahora se tiene un sistema de ecuaciones algebraicas para } B_1 \text{ y } B_2, \\ \text{pero hay que garantizar que estas no tengan soluci\'on trivial, para esto} \\ \text{el determinante del sistema debe ser igual a cero.} \end{array}$$

$$\Rightarrow \begin{vmatrix} (K_1 + K_{12} - M\omega^2) & -K_{12} \\ -K_{12} & (K_2 + K_{12} - M\omega^2) \end{vmatrix} = 0 \Rightarrow (K_1 + K_{12} - M\omega^2)^2 - K_{12}^2 = 0 \Rightarrow (K_1 + K_{12} - M\omega^2)^2 = K_{12}^2$$

$$\Rightarrow K_1 + K_{12} - M\omega^2 = \pm K_{12} \Rightarrow \omega = \pm \sqrt{\frac{K_1 + K_{12} \pm K_{12}}{M}} \Rightarrow \omega_1 = \pm \sqrt{\frac{K_1 + 2K_{12}}{M}} \quad y \quad \omega_2 = \pm \sqrt{\frac{K_1}{M}}$$

Como hay "4" frecuencias (realmente 2 diferentes), las soluciones en realidad son de la forma

$$\Rightarrow \begin{cases} x_1 = B_{11}^+ e^{i\omega_1 t} + B_{11}^- e^{-i\omega_1 t} + B_{12}^+ e^{i\omega_2 t} + B_{12}^- e^{-i\omega_2 t} \\ x_2 = B_{21}^+ e^{i\omega_1 t} + B_{21}^- e^{-i\omega_1 t} + B_{22}^+ e^{i\omega_2 t} + B_{22}^- e^{-i\omega_2 t} \end{cases}$$

Continua...

Recuperando los resultados anteriores

$$\Rightarrow \begin{cases} \chi_1 = B_{11}^+ e^{iw_1 t} + B_{11}^- e^{-iw_1 t} + B_{12}^+ e^{iw_2 t} + B_{12}^- e^{-iw_2 t} \\ \chi_2 = B_{21}^+ e^{iw_1 t} + B_{21}^- e^{-iw_1 t} + B_{22}^+ e^{iw_2 t} + B_{22}^- e^{-iw_2 t} \end{cases}$$

$$\omega_1 = \pm \sqrt{\frac{k+2k_{12}}{M}} \quad y \quad \omega_2 = \pm \sqrt{\frac{k}{M}}$$

Sustituyendo ω_1 y ω_2 en ④:

$$\Rightarrow \begin{cases} (k+k_{12}-M\sqrt{\frac{k+2k_{12}}{M}})^2 B_{11}^{\pm} - k_{12}B_{21}^{\pm} = 0 \\ -k_{12}B_{12}^{\pm} + (k+k_{12}-M\sqrt{\frac{k}{M}})^2 B_{22}^{\pm} = 0 \end{cases}$$

Esto se puede resumir por la naturaleza del sistema para B_{ij}^{\pm} siendo que al montar el sistema real, se quiere generar este sistema por su simetría y para mantener la o las igualdades a cero de todo el sistema, las igualdades individuales (la escritas) deben ser cero.

$$\Rightarrow \begin{cases} (k+k_{12}-k-2k_{12})B_{11}^{\pm} - k_{12}B_{21}^{\pm} = 0 \\ -k_{12}B_{12}^{\pm} + (k+k_{12}-k)B_{22}^{\pm} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} -k_{12}B_{11}^{\pm} - k_{12}B_{21}^{\pm} = 0 \\ -k_{12}B_{12}^{\pm} + k_{12}B_{22}^{\pm} = 0 \end{cases} \Rightarrow \begin{cases} B_{11}^{\pm} = -B_{21}^{\pm} \\ B_{12}^{\pm} = B_{22}^{\pm} \end{cases}$$

La solución entonces es de la forma

$$\Rightarrow \begin{cases} \chi_1 = B_{11}^+ e^{iw_1 t} + B_{11}^- e^{-iw_1 t} + B_{22}^+ e^{iw_2 t} + B_{22}^- e^{-iw_2 t} \\ \chi_2 = -B_{11}^+ e^{iw_1 t} - B_{11}^- e^{-iw_1 t} + B_{22}^+ e^{iw_2 t} + B_{22}^- e^{-iw_2 t} \end{cases}$$

$$\Rightarrow \begin{cases} \chi_1 = B_1^+ e^{iw_1 t} + B_1^- e^{-iw_1 t} + B_2^+ e^{iw_2 t} + B_2^- e^{-iw_2 t} \\ \chi_2 = -B_1^+ e^{iw_1 t} - B_1^- e^{-iw_1 t} + B_2^+ e^{iw_2 t} + B_2^- e^{-iw_2 t} \end{cases}$$

Problema 1. (Thornton 12.1)

- 12.1. Reconsider the problem of two coupled oscillators discussed in Section 12.2 in the event that the three springs all have different force constants. Find the two characteristic frequencies, and compare the magnitudes with the natural frequencies of the two oscillators in the absence of coupling.

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2$$

$$U = \frac{1}{2}K_{11}x_1^2 + \frac{1}{2}K_{12}(x_1 - x_2)^2 + \frac{1}{2}K_{22}x_2^2 + \frac{1}{2}K_{12}(x_2 - x_1)^2 = \frac{1}{2}K_{11}x_1^2 + \frac{1}{2}K_{22}x_2^2 + K_{12}(x_1 - x_2)^2$$

- $\mathcal{T} = \frac{1}{2} \sum_i m_{ij} \dot{q}_j \dot{q}_j = \frac{1}{2} (m_1 \dot{x}_1^2 + m_2 \dot{x}_2^2) \Rightarrow m = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$

- $A_{ij} = \frac{\partial^2 U}{\partial q_j \partial q_i} \Rightarrow A = \begin{pmatrix} K_{11} + K_{12} & -K_{12} \\ -K_{12} & K_{22} + K_{12} \end{pmatrix}$

has autofrequency: $\det(A - \omega^2 m) = 0$

$$\Rightarrow \begin{vmatrix} K_{11} + K_{12} - m_1\omega^2 & -K_{12} \\ -K_{12} & K_{22} + K_{12} - m_2\omega^2 \end{vmatrix} = 0 \Rightarrow (K_{11} + K_{12} - m_1\omega^2)(K_{22} + K_{12} - m_2\omega^2) - K_{12}^2 = 0$$

$$\Rightarrow (K_{11} + K_{12})(K_{22} + K_{12}) - (K_{11} + K_{12})m_2\omega^2 - (K_{11} + K_{12})m_1\omega^2 + m_1m_2\omega^4 - K_{12}^2 = 0$$

$$\Rightarrow m_1m_2\omega^4 - \omega^2[(K_{11} + K_{12})m_2 + (K_{22} + K_{12})m_1] + (K_{11} + K_{12})(K_{22} + K_{12}) - K_{12}^2 = 0$$

$$\Rightarrow \omega^2 = \frac{[(K_{11} + K_{12})m_2 + (K_{22} + K_{12})m_1] \pm \sqrt{[(K_{11} + K_{12})m_2 + (K_{22} + K_{12})m_1]^2 - 4m_1m_2[(K_{11} + K_{12})(K_{22} + K_{12}) - K_{12}^2]}}{2m_1m_2} //$$

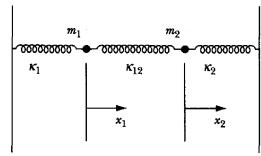


FIGURE 12.1 Two masses are connected by a spring to each other and by springs to fixed positions. This is a system of coupled motion in one dimension.

Otra forma de resolver

$$\begin{aligned} m_1 \ddot{\chi}_1 &= -k_{11}\chi_1 - k_{12}(\chi_1 - \chi_2) \Rightarrow m_1 \ddot{\chi}_1 + (k_{11} + k_{12})\chi_1 - k_{12}\chi_2 = 0 \\ m_2 \ddot{\chi}_2 &= -k_{22}\chi_2 - k_{12}(\chi_2 - \chi_1) \Rightarrow m_2 \ddot{\chi}_2 + (k_{22} + k_{12})\chi_2 - k_{12}\chi_1 = 0 \end{aligned} \Rightarrow \begin{cases} m_1 \ddot{\chi}_1 + (k_{11} + k_{12})\chi_1 - k_{12}\chi_2 = 0 \\ m_2 \ddot{\chi}_2 + (k_{22} + k_{12})\chi_2 - k_{12}\chi_1 = 0 \end{cases}$$

Se sabe que las soluciones son de la forma: $\begin{cases} \chi_1 = B_1 e^{\lambda_1 t} \\ \chi_2 = B_2 e^{\lambda_2 t} \end{cases}$ reemplazando en el sistema

$$\Rightarrow \begin{cases} m_1 B_1 \cancel{- \frac{1}{2} w^2 e^{\lambda_1 t}} + (k_{11} + k_{12}) B_1 e^{\lambda_1 t} - k_{12} B_2 e^{\lambda_1 t} = 0 \\ m_2 B_2 \cancel{- \frac{1}{2} w^2 e^{\lambda_2 t}} + (k_{22} + k_{12}) B_2 e^{\lambda_2 t} - k_{12} B_1 e^{\lambda_2 t} = 0 \end{cases} \Rightarrow \begin{cases} -m_1 B_1 w^2 + (k_{11} + k_{12}) B_1 - k_{12} B_2 = 0 \\ -m_2 B_2 w^2 + (k_{22} + k_{12}) B_2 - k_{12} B_1 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} B_1 (k_{11} + k_{12} - m_1 w^2) - k_{12} B_2 = 0 \\ B_2 (k_{22} + k_{12} - m_2 w^2) - k_{12} B_1 = 0 \end{cases} \quad \text{Para garantizar la existencia de soluciones no triviales para } B_1 \text{ y } B_2 \text{ el determinante de "m", donde } mB \text{ es igual al sistema,} \\ \text{debe ser igual a cero.}$$

$$\Rightarrow \begin{vmatrix} (k_{11} + k_{12} - m_1 w^2) & -k_{12} \\ -k_{12} & (k_{22} + k_{12} - m_2 w^2) \end{vmatrix} = 0 \Rightarrow (k_{11} + k_{12} - m_1 w^2)(k_{22} + k_{12} - m_2 w^2) - k_{12}^2 = 0$$

$$\Rightarrow (k_{11} + k_{12})(k_{22} + k_{12}) - (k_{11} + k_{12})m_2 w^2 - (k_{22} + k_{12})m_1 w^2 + m_1 m_2 w^4 - k_{12}^2 = 0$$

$$\Rightarrow m_1 m_2 w^4 - w^2 [(k_{11} + k_{12})m_2 + (k_{22} + k_{12})m_1] + (k_{11} + k_{12})(k_{22} + k_{12}) - k_{12}^2 = 0$$

$$\Rightarrow w^2 = \frac{[(k_{11} + k_{12})m_2 + (k_{22} + k_{12})m_1] \pm \sqrt{[(k_{11} + k_{12})m_2 + (k_{22} + k_{12})m_1]^2 - 4m_1 m_2 [(k_{11} + k_{12})(k_{22} + k_{12}) - k_{12}^2]}}{2m_1 m_2}$$

* Tomando $m_1 = m_2$, da la solución del sistema.

$$\text{Siguiendo el problema original, } m_1 = m_2 = M \Rightarrow \omega^2 = \frac{k_{11} + k_{22} + 2k_{12} \pm \sqrt{(k_{11} - k_{22})^2 + 4k_{12}^2}}{2M}$$

Considerando las frecuencias de un solo oscilador conectado a 2 resortes, para los casos de m_1 y m_2 , se tiene:

$$\Rightarrow \omega_1^2 = \frac{k_{11} + k_{12}}{M} \quad y \quad \omega_2^2 = \frac{k_{22} + k_{12}}{M}$$

Ahora se van a comparar estas frecuencias con las del problema original

$$\Rightarrow \omega_t^2 = \frac{k_{11} + k_{22} + 2k_{12} + \sqrt{(k_{11} - k_{22})^2 + 4k_{12}^2}}{2M} > \frac{k_{11} + k_{22} + 2k_{12} + \sqrt{(k_{11} - k_{22})^2}}{2M} = \frac{k_{11} + k_{22} + 2k_{12} + k_{11} - k_{22}}{2M}$$

Por conveniencia se elimina $\sqrt{k_{12}^2}$ de la raíz para plantear una relación entre las frecuencias acoplada y la frecuencia no acoplada

$$= \frac{2k_{11} + 2k_{12}}{2M} = \frac{k_{11} + k_{12}}{M} = \omega_i^2$$

$$\Rightarrow \omega_t^2 > \omega_i^2$$

$$\Rightarrow \omega^2 = \frac{k_{11} + k_{22} + 2k_{12} - \sqrt{(k_{11} - k_{22})^2 + 4k_{12}^2}}{2M} * > \frac{k_{11} + k_{22} + 2k_{12} - \sqrt{(k_{11} - k_{22})^2}}{2M} = \frac{k_{11} + k_{22} + 2k_{12} - k_{11} + k_{22}}{2M}$$

$$= \frac{2k_{22} + 2k_{12}}{2M} = \frac{k_{22} + k_{12}}{M} = \omega_e^2$$

$$\Rightarrow \omega_t^2 > \omega_e^2$$

$$\Rightarrow \boxed{\omega_t^2 > \omega_i^2 \quad y \quad \omega_t^2 > \omega_e^2}$$



Problema 2. (Thornton 12.2)

12.2. Continue Problem 12.1, and investigate the case of weak coupling: $\kappa_{12} \ll \kappa_1, \kappa_2$. Show that the phenomenon of beats occurs but that the energy-transfer process is incomplete.

$$\Rightarrow \begin{cases} \omega^2 = \frac{\kappa_1 + \kappa_2 + 2\kappa_{12} \pm \sqrt{(\kappa_1 - \kappa_2)^2 + 4\kappa_{12}^2}}{2M} \\ \omega_1^2 = \frac{\kappa_1 + \kappa_{12}}{M} \quad y \quad \omega_2^2 = \frac{\kappa_2 + \kappa_{12}}{M} \end{cases}$$

Frecuencias acopladas aproximadas

$$\kappa_{12} \ll \kappa_1, \kappa_2 \Rightarrow \omega^2 = \frac{\kappa_1 + \kappa_2 + 2\kappa_{12} \pm \sqrt{(\kappa_1 - \kappa_2)^2 + 4\kappa_{12}^2}}{2M} \approx \frac{\kappa_1 + \kappa_2 + 2\kappa_{12} \pm (\kappa_1 - \kappa_2)}{2M}$$

$$\Rightarrow \omega_+^2 = \frac{\kappa_1 + 2\kappa_{12}}{M} = \frac{\kappa_1}{M} \left(1 + \frac{2\kappa_{12}}{\kappa_1}\right) \Rightarrow \omega_+ = \sqrt{\frac{\kappa_1}{M}} \left(1 + \frac{2\kappa_{12}}{\kappa_1}\right)^{\frac{1}{2}} \approx \sqrt{\frac{\kappa_1}{M}} \left(1 + \frac{1}{2} \frac{2\kappa_{12}}{\kappa_1}\right) \stackrel{(1)}{=} \sqrt{\frac{\kappa_1}{M}} (1 + 2\varepsilon_1)$$

$$\Rightarrow \omega_-^2 = \frac{\kappa_2 + 2\kappa_{12}}{M} = \frac{\kappa_2}{M} \left(1 + \frac{2\kappa_{12}}{\kappa_2}\right) \Rightarrow \omega_- = \sqrt{\frac{\kappa_2}{M}} \left(1 + \frac{2\kappa_{12}}{\kappa_2}\right)^{\frac{1}{2}} \approx \sqrt{\frac{\kappa_2}{M}} \left(1 + \frac{1}{2} \frac{2\kappa_{12}}{\kappa_2}\right) \stackrel{(2)}{=} \sqrt{\frac{\kappa_2}{M}} (1 + 2\varepsilon_2)$$

Ahora, obteniendo las frecuencias no acopladas con un resorte muy débil

$$\bullet \quad \omega_1 = \sqrt{\frac{1}{M}} (\kappa_1 + \kappa_{12})^{\frac{1}{2}} = \sqrt{\frac{\kappa_1}{M}} \left(1 + \frac{\kappa_{12}}{\kappa_1}\right)^{\frac{1}{2}} \approx \sqrt{\frac{\kappa_1}{M}} \left(1 + \frac{1}{2} \frac{\kappa_{12}}{\kappa_1}\right) = \sqrt{\frac{\kappa_1}{M}} (1 + \varepsilon_1) \Rightarrow \sqrt{\frac{\kappa_1}{M}} \approx \omega_1 (1 - \varepsilon_1) \quad (3)$$

$$\bullet \quad \omega_2 = \sqrt{\frac{1}{M}} (\kappa_2 + \kappa_{12})^{\frac{1}{2}} = \sqrt{\frac{\kappa_2}{M}} \left(1 + \frac{\kappa_{12}}{\kappa_2}\right)^{\frac{1}{2}} \approx \sqrt{\frac{\kappa_2}{M}} \left(1 + \frac{1}{2} \frac{\kappa_{12}}{\kappa_2}\right) = \sqrt{\frac{\kappa_2}{M}} (1 + \varepsilon_2) \Rightarrow \sqrt{\frac{\kappa_2}{M}} \approx \omega_2 (1 - \varepsilon_2) \quad (4)$$

Planteando condiciones iniciales: $X_1(0) = D$, $X_2(0) = 0$, $\dot{X}_1(0) = 0$ y $\dot{X}_2(0) = 0$ y trayendo la solución general previamente conseguida

$$\Rightarrow \begin{cases} X_1 = B_1^+ e^{i\omega_1 t} + B_1^- e^{-i\omega_1 t} + B_2^+ e^{i\omega_2 t} + B_2^- e^{-i\omega_2 t} \\ X_2 = B_1^+ e^{i\omega_1 t} - B_1^- e^{-i\omega_1 t} + B_2^+ e^{i\omega_2 t} + B_2^- e^{-i\omega_2 t} \end{cases}$$

Solucionando para estas condiciones

$$\Rightarrow X_1(0) = B_1^+ + B_1^- + B_2^+ + B_2^- = D \quad y \quad \dot{X}_1(0) = i\omega_1 B_1^+ - i\omega_1 B_1^- + i\omega_2 B_2^+ - i\omega_2 B_2^- = 0$$

$$\Rightarrow X_2(0) = -B_1^+ - B_1^- + B_2^+ + B_2^- = 0 \quad y \quad \dot{X}_2(0) = -i\omega_1 B_1^+ + i\omega_1 B_1^- + i\omega_2 B_2^+ - i\omega_2 B_2^- = 0$$

De aquí se puede generar un sistema para hallar las constantes

$$\Rightarrow \begin{cases} B_1^+ + B_1^- + B_2^+ + B_2^- = D \\ -B_1^+ - B_1^- + B_2^+ + B_2^- = 0 \\ i\omega_1 B_1^+ - i\omega_1 B_1^- + i\omega_2 B_2^+ - i\omega_2 B_2^- = 0 \\ -i\omega_1 B_1^+ + i\omega_1 B_1^- + i\omega_2 B_2^+ - i\omega_2 B_2^- = 0 \end{cases}$$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ i\omega_1 & -i\omega_1 & i\omega_2 & -i\omega_2 \\ -i\omega_1 & i\omega_1 & i\omega_2 & -i\omega_2 \end{pmatrix} \begin{pmatrix} B_1^+ \\ B_1^- \\ B_2^+ \\ B_2^- \end{pmatrix} = \begin{pmatrix} D \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Resolviendo el sistema anterior

$$\Rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & D \\ -1 & -1 & 1 & 1 & 0 \\ \omega_1 & -\omega_1 & \omega_2 & -\omega_2 & 0 \\ -\omega_1 & \omega_1 & \omega_2 & -\omega_2 & 0 \end{array} \right) \xrightarrow{\begin{matrix} f_2 \rightarrow f_2 + f_1 \\ f_4 \rightarrow f_4 + f_3 \end{matrix}} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & D \\ 0 & 0 & 2 & 2 & 0 \\ \omega_1 & -\omega_1 & \omega_2 & -\omega_2 & 0 \\ 0 & 0 & 2\omega_2 & -2\omega_2 & 0 \end{array} \right) \xrightarrow{\begin{matrix} f_1 \rightarrow f_1 - \frac{1}{2}f_2 \\ f_3 \rightarrow f_3 - \frac{1}{2}f_4 \end{matrix}} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & \frac{D}{2} \\ 0 & 0 & 1 & 1 & 0 \\ \omega_1 & -\omega_1 & 0 & 0 & 0 \\ 0 & 0 & \omega_2 & -\omega_2 & 0 \end{array} \right)$$

$$\xrightarrow{f_3 \rightarrow \frac{1}{\omega_2}f_3} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & \frac{D}{2} \\ 0 & 0 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array} \right) \xrightarrow{f_3 \rightarrow f_2} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & \frac{D}{2} \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array} \right) \xrightarrow{\begin{matrix} f_2 \rightarrow f_2 - f_1 \\ f_4 \rightarrow f_4 - f_3 \end{matrix}} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & \frac{D}{2} \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 \end{array} \right)$$

$$\xrightarrow{f_4 \rightarrow -\frac{1}{2}f_4} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & \frac{D}{2} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right) \xrightarrow{f_1 \rightarrow f_1 - f_2} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{D}{4} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right) \Rightarrow B_1^+ = B_2^+ = \frac{D}{4}$$

$$\Rightarrow \begin{cases} X_1 = B_1^+ e^{i\omega_1 t} + B_1^- e^{-i\omega_1 t} + B_2^+ e^{i\omega_2 t} + B_2^- e^{-i\omega_2 t} = \frac{D}{4}(e^{i\omega_1 t} + e^{-i\omega_1 t} + e^{i\omega_2 t} + e^{-i\omega_2 t}) \\ X_2 = -B_1^+ e^{i\omega_1 t} - B_1^- e^{-i\omega_1 t} + B_2^+ e^{i\omega_2 t} + B_2^- e^{-i\omega_2 t} = \frac{D}{4}(-e^{i\omega_1 t} - e^{-i\omega_1 t} + e^{i\omega_2 t} + e^{-i\omega_2 t}) \end{cases}$$

$$\Rightarrow \begin{cases} X_1 = \frac{D}{2}[\cos(\omega_1 t) + \cos(\omega_2 t)] = \frac{D}{2}[2\cos(\frac{\omega_1 + \omega_2}{2}t)\cos(\frac{\omega_1 - \omega_2}{2}t)] = D \cos(\frac{\omega_1 + \omega_2}{2}t) \cos(\frac{\omega_1 - \omega_2}{2}t) \\ X_2 = \frac{D}{2}[-\cos(\omega_1 t) + \cos(\omega_2 t)] = \frac{D}{2}[2\sin(\frac{\omega_1 + \omega_2}{2}t)\sin(\frac{\omega_1 - \omega_2}{2}t)] = D \sin(\frac{\omega_1 + \omega_2}{2}t) \sin(\frac{\omega_1 - \omega_2}{2}t) \end{cases}$$

$$\Rightarrow \begin{cases} X_1 = D \cos(\frac{\omega_1 + \omega_2}{2}t) \cos(\frac{\omega_1 - \omega_2}{2}t) \\ X_2 = D \sin(\frac{\omega_1 + \omega_2}{2}t) \sin(\frac{\omega_1 - \omega_2}{2}t) \end{cases} \quad \text{← Hasta aquí se están considerando cualquier frecuencia con las condiciones iniciales dadas.}$$

Recordando 1 y 3, y 2 y 4 (ω_+ y ω_- son las frecuencias "reales" del sistema y corresponden a ω_1 y ω_2 de las ecuaciones anteriores)

$$\Rightarrow \omega_+ \approx \sqrt{\frac{k_1}{M}}(1+2E_1) \approx \omega_1(1-E_1)(1+2E_1) = \omega_1(1+E_1 - 2E_1) \xrightarrow{\approx} \omega_1(1+E_1) \Rightarrow \omega_+ \approx \omega_1(1+E_1)$$

$$\Rightarrow \omega_- \approx \sqrt{\frac{k_2}{M}}(1+2E_2) \approx \omega_2(1-E_2)(1+2E_2) = \omega_2(1+E_2 - 2E_2) \xrightarrow{\approx} \omega_2(1+E_2) \Rightarrow \omega_- \approx \omega_2(1+E_2)$$

$$\Rightarrow \begin{cases} X_1 = D \cos(\frac{\omega_1 + \omega_2}{2}t) \cos(\frac{\omega_1 - \omega_2}{2}t) \\ X_2 = D \sin(\frac{\omega_1 + \omega_2}{2}t) \sin(\frac{\omega_1 - \omega_2}{2}t) \end{cases} \Rightarrow \begin{cases} X_1 = D \cos(\frac{\omega_1 + \omega_2 + E_1\omega_1 + E_2\omega_2}{2}t) \cos(\frac{\omega_1 - \omega_2 + E_1\omega_1 - E_2\omega_2}{2}t) \\ X_2 = D \sin(\frac{\omega_1 + \omega_2 + E_1\omega_1 + E_2\omega_2}{2}t) \sin(\frac{\omega_1 - \omega_2 + E_1\omega_1 - E_2\omega_2}{2}t) \end{cases}$$

Frente a t=0, X_1 y X_2 alcanzan puros máximos y mínimos o viceversa, fuera de fase, flujo E incompleto

Problema 3. (Thornton 12.10)

- 12.10. Consider two identical, coupled oscillators (as in Figure 12-1). Let each of the oscillators be damped, and let each have the same damping parameter β . A force $F_0 \cos \omega t$ is applied to m_1 . Write down the pair of coupled differential equations that describe the motion. Obtain the solution by expressing the differential equations in terms of the normal coordinates given by Equation 12.11 and by comparing these equations with Equation 3.53. Show that the normal coordinates η_1 and η_2 exhibit resonance peaks at the characteristic frequencies ω_1 and ω_2 , respectively.

$$\left. \begin{array}{l} \text{Fuerza } m_1 = M \\ m_1 = M \\ K_1 = K_{12} \\ K_2 = K \end{array} \right| \quad \left. \begin{array}{l} m_2 = M \\ K_{12} \\ x_1 \\ x_2 \end{array} \right| \quad \left. \begin{array}{l} \eta_1 = x_1 - x_2 \\ \eta_2 = x_1 + x_2 \end{array} \right\} \quad (12.11)$$

$$\left. \begin{array}{l} x_1 = \frac{1}{2}(\eta_2 + \eta_1) \\ x_2 = \frac{1}{2}(\eta_2 - \eta_1) \end{array} \right\} \quad (12.12)$$

FIGURE 12-1 Two masses are connected by a spring to each other and by springs to fixed positions. This is a system of coupled motion in one dimension.

$$\Rightarrow \left\{ \begin{array}{l} M\ddot{x}_1 + Kx_1 + K_{12}(x_1 - x_2) + b\dot{x}_1 = F_0 \cos \omega t \\ M\ddot{x}_2 + Kx_2 + K_{12}(x_2 - x_1) + b\dot{x}_2 = 0 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} M\ddot{x}_1 + b\dot{x}_1 + (K + K_{12})x_1 - K_{12}x_2 = F_0 \cos \omega t \\ M\ddot{x}_2 + b\dot{x}_2 + (K + K_{12})x_2 - K_{12}x_1 = 0 \end{array} \right. \quad //$$

Ahora, tomando $x_1 = \frac{1}{2}(\eta_2 + \eta_1)$ y $x_2 = \frac{1}{2}(\eta_2 - \eta_1)$

$$\Rightarrow \left\{ \begin{array}{l} M\frac{1}{2}(\ddot{\eta}_2 + \ddot{\eta}_1) + b\frac{1}{2}(\dot{\eta}_2 + \dot{\eta}_1) + (K + K_{12})\frac{1}{2}(\eta_2 + \eta_1) - K_{12}\frac{1}{2}(\eta_2 - \eta_1) = 2F_0 \cos \omega t \\ M\frac{1}{2}(\ddot{\eta}_2 - \ddot{\eta}_1) + b\frac{1}{2}(\dot{\eta}_2 - \dot{\eta}_1) + (K + K_{12})\frac{1}{2}(\eta_2 - \eta_1) - K_{12}\frac{1}{2}(\eta_2 + \eta_1) = 0 \end{array} \right.$$

$$\Rightarrow \left\{ \begin{array}{l} M(\ddot{\eta}_2 + \ddot{\eta}_1) + b(\dot{\eta}_2 + \dot{\eta}_1) + (K + K_{12})(\eta_2 + \eta_1) - K_{12}(\eta_2 - \eta_1) = 2F_0 \cos \omega t \\ M(\ddot{\eta}_2 - \ddot{\eta}_1) + b(\dot{\eta}_2 - \dot{\eta}_1) + (K + K_{12})(\eta_2 - \eta_1) - K_{12}(\eta_2 + \eta_1) = 0 \end{array} \right. \quad \begin{array}{l} \text{Sumando ambas y} \\ \text{también al restarlos se} \\ \text{obtiene} \end{array}$$

$$\Rightarrow \left\{ \begin{array}{l} M\ddot{\eta}_2 + b\dot{\eta}_2 + (K + K_{12})\eta_2 - K_{12}\eta_1 = F_0 \cos \omega t \\ M\ddot{\eta}_1 + b\dot{\eta}_1 + (K + K_{12})\eta_1 + K_{12}\eta_2 = F_0 \cos \omega t \end{array} \right. \Rightarrow \left\{ \begin{array}{l} M\ddot{\eta}_2 + b\dot{\eta}_2 + K\eta_2 = F_0 \cos \omega t \\ M\ddot{\eta}_1 + b\dot{\eta}_1 + (K + 2K_{12})\eta_1 = F_0 \cos \omega t \end{array} \right. //$$

Consider three identical pendula suspended from a slightly yielding support. Because the support is not rigid, a coupling occurs between the pendula, and energy can be transferred from one pendulum to the other. Find the eigenfrequencies and eigenvectors and describe the normal mode motion. Figure 12-8 shows the geometry of the problem.

$$T = \frac{1}{2} M l^2 \dot{\theta}_1^2 + \frac{1}{2} M l^2 \dot{\theta}_2^2 + \frac{1}{2} M l^2 \dot{\theta}_3^2$$

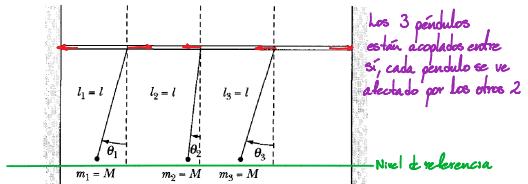


FIGURE 12-8 Example 12.6. Three identical pendula are suspended from a slightly yielding support that allows energy to be transferred between pendula. Such an experiment is easy to set up and demonstrate.

$$U = M g l [(1 - \cos \theta_1) + (1 - \cos \theta_2) + (1 - \cos \theta_3)] + \frac{1}{2} [(l \sin \theta_1 - l \sin \theta_2)^2 + (l \sin \theta_2 - l \sin \theta_3)^2 + (l \sin \theta_3 - l \sin \theta_1)^2]$$

Energía Potencial gravitacional

Potencial del acople, considerando el soporte como un resorte

- Planteando oscilaciones pequeñas: $\cos \theta_i = 1 - \frac{\theta_i^2}{2} \Rightarrow (1 - \cos \theta_i) = (1 - 1 + \frac{\theta_i^2}{2}) = \frac{\theta_i^2}{2}$; $\sin \theta_i = \theta_i$

$$\Rightarrow U = M g l \left(\frac{\theta_1^2}{2} + \frac{\theta_2^2}{2} + \frac{\theta_3^2}{2} \right) + \frac{l^2 k}{2} [(\theta_1 - \theta_2)^2 + (\theta_2 - \theta_3)^2 + (\theta_3 - \theta_1)^2]$$

$$= \frac{M g l}{2} (\theta_1^2 + \theta_2^2 + \theta_3^2) + \frac{l^2 k}{2} (2\theta_1^2 + 2\theta_2^2 + 2\theta_3^2 - 2\theta_1\theta_2 - 2\theta_1\theta_3 - 2\theta_2\theta_3)$$

$$= \frac{l}{2} [M g (\theta_1^2 + \theta_2^2 + \theta_3^2) + l^2 k (2\theta_1^2 + 2\theta_2^2 + 2\theta_3^2 - 2\theta_1\theta_2 - 2\theta_1\theta_3 - 2\theta_2\theta_3)]$$

$$= \frac{l}{2} [\theta_1^2 (M g + 2l k) + \theta_2^2 (M g + 2l k) + \theta_3^2 (M g + 2l k) - 2l k (\theta_1\theta_2 + \theta_1\theta_3 + \theta_2\theta_3)]$$

$$= \frac{l}{2} (M g + 2l k) (\theta_1^2 + \theta_2^2 + \theta_3^2) - \frac{l^2 k}{2} (2\theta_1\theta_2 + 2\theta_1\theta_3 + 2\theta_2\theta_3)$$

$$= \frac{l}{2} (M g + 2l k) \left[\theta_1^2 + \theta_2^2 + \theta_3^2 - \left(\frac{l k}{M g + 2l k} \right) (2\theta_1\theta_2 + 2\theta_1\theta_3 + 2\theta_2\theta_3) \right]; \quad \epsilon = \left(\frac{l k}{M g + 2l k} \right)$$

$$\Rightarrow U = \frac{l}{2} (M g + 2l k) (\theta_1^2 + \theta_2^2 + \theta_3^2 - 2\epsilon\theta_1\theta_2 - 2\epsilon\theta_1\theta_3 - 2\epsilon\theta_2\theta_3); \quad r = \frac{l}{2} (M g + 2l k)$$

$$= r (\theta_1^2 + \theta_2^2 + \theta_3^2 - 2\epsilon\theta_1\theta_2 - 2\epsilon\theta_1\theta_3 - 2\epsilon\theta_2\theta_3)$$

$$\Rightarrow \begin{cases} U = r (\theta_1^2 + \theta_2^2 + \theta_3^2 - 2\epsilon\theta_1\theta_2 - 2\epsilon\theta_1\theta_3 - 2\epsilon\theta_2\theta_3) \\ T = \frac{1}{2} M l^2 (\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2) \end{cases}$$

Continua ...

Recuperando lo anterior

$$\Rightarrow \begin{cases} U = r(\theta_1^2 + \theta_2^2 + \theta_3^2 - 2\epsilon\theta_1\theta_2 - 2\epsilon\theta_1\theta_3 - 2\epsilon\theta_2\theta_3) \\ T = \frac{1}{2}Ml^2(\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2) \end{cases}$$

$$\bullet T = \frac{1}{2} \sum_i m_{ij} \dot{q}_j \dot{q}_i = \frac{1}{2} Ml^2 (\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2) \Rightarrow m = Ml^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\bullet A_{ij} = \frac{\partial^2 U}{\partial q_i \partial q_j} \Rightarrow A_{11} = 2r, A_{22} = 2r, A_{33} = 2r, A_{12} = -2\epsilon, A_{13} = -2\epsilon, A_{23} = -2\epsilon$$

$$\Rightarrow A = \begin{pmatrix} 2r & -2\epsilon & -2\epsilon \\ -2\epsilon & 2r & -2\epsilon \\ -2\epsilon & -2\epsilon & 2r \end{pmatrix} = 2r \begin{pmatrix} 1 & -\epsilon & -\epsilon \\ -\epsilon & 1 & -\epsilon \\ -\epsilon & -\epsilon & 1 \end{pmatrix}$$

$$\star \det(A - \omega^2 m) = 0$$

$$\Rightarrow \left| 2r \begin{pmatrix} 1 & -\epsilon & -\epsilon \\ -\epsilon & 1 & -\epsilon \\ -\epsilon & -\epsilon & 1 \end{pmatrix} - \omega^2 Ml^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| = 0 \Rightarrow \begin{vmatrix} 2r - Ml^2 \omega^2 & -\epsilon & -\epsilon \\ -\epsilon & 2r - Ml^2 \omega^2 & -\epsilon \\ -\epsilon & -\epsilon & 2r - Ml^2 \omega^2 \end{vmatrix} = 0$$

$$\Rightarrow (2r - Ml^2 \omega^2) [(2r - Ml^2 \omega^2)^2 - \epsilon^2] + \epsilon [-\epsilon(2r - Ml^2 \omega^2) - \epsilon^2] - \epsilon [\epsilon^2 + \epsilon(2r - Ml^2 \omega^2)] = 0$$

$$\Rightarrow (2r - Ml^2 \omega^2)^3 - \epsilon^2(2r - Ml^2 \omega^2) - \epsilon^2(2r - Ml^2 \omega^2) - \epsilon^3 - \epsilon^3 - \epsilon^2(2r - Ml^2 \omega^2) = 0$$

$$\Rightarrow (2r - Ml^2 \omega^2)^3 - 3\epsilon^2(2r - Ml^2 \omega^2) - 2\epsilon^3 = 0$$

$$\downarrow \Rightarrow [(2r - Ml^2 \omega^2) - 2\epsilon][(2r - Ml^2 \omega^2) + \epsilon]^2 = 0$$

$$\Rightarrow \omega^2 = \frac{2r - 2\epsilon}{Ml^2} \quad \text{o} \quad \omega^2 = \frac{2r + \epsilon}{Ml^2}$$

\uparrow
Frecuencia cílica

\uparrow
Frecuencia con multiplicidad 2

$$\text{Tomando } u = (2r - Ml^2 \omega^2) \Rightarrow u^3 - 3\epsilon^2 u - 2\epsilon^3 = 0$$

$$\text{Por teorema del factor } u = 2\epsilon \Rightarrow \boxed{1 \ 0 \ -3\epsilon^2 \ -2\epsilon^3}$$

$$\begin{array}{cccc} & +2\epsilon & +4\epsilon^2 & +2\epsilon^2 \\ \hline 1 & 2\epsilon & 1\epsilon^2 & 0 & 2\epsilon \end{array}$$

$$\Rightarrow (u - 2\epsilon)(u^2 + 2\epsilon u + \epsilon^2) = 0$$

$$\Rightarrow (u - 2\epsilon)(u + \epsilon)^2 = 0$$

Continua...

* Primer autovector: $\omega_1^2 = \frac{2r - 2e}{Ml^2}$

$$(I - \omega_1^2 m) \vec{a}_{11} = \vec{0} \Rightarrow \begin{pmatrix} 2r - Ml^2 & \cancel{2r - 2e} & -e & -e \\ -e & 2r - Ml^2 & \cancel{2r - 2e} & -e \\ -e & -e & 2r - Ml^2 & \cancel{2r - 2e} \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{12} \\ a_{13} \end{pmatrix} = \vec{0}$$

$$\Rightarrow \begin{pmatrix} 2e & -e & -e \\ -e & 2e & -e \\ -e & -e & 2e \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{12} \\ a_{13} \end{pmatrix} = \vec{0} \Rightarrow \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{12} \\ a_{13} \end{pmatrix} = \vec{0} \xrightarrow{f_1 \rightarrow f_1 + f_2} \begin{pmatrix} 1 & 1 & -2 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

$$\xrightarrow{f_2 \rightarrow f_2 + f_1} \begin{pmatrix} 1 & 1 & -2 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{f_2 \rightarrow \frac{1}{3}f_2} \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{12} \\ a_{13} \end{pmatrix} = \vec{0} \Rightarrow \begin{cases} a_{11} + a_{12} - 2a_{13} = 0 \\ a_{12} - a_{13} = 0 \\ a_{13} \in \mathbb{R} \end{cases}$$

Con $a_{13} = 1 \Rightarrow \vec{a}_{11}^\text{T} = (1, 1, 1)$

* Segundo autovector $\omega_2^2 = \frac{2r + e}{Ml^2}$

$$(I - \omega_2^2 m) \vec{a}_{12} = \vec{0} \Rightarrow \begin{pmatrix} 2r - Ml^2 & \cancel{2r + e} & -e & -e \\ -e & 2r - Ml^2 & \cancel{2r + e} & -e \\ -e & -e & 2r - Ml^2 & \cancel{2r + e} \end{pmatrix} \begin{pmatrix} a_{21} \\ a_{22} \\ a_{23} \end{pmatrix} = \vec{0}$$

$$\Rightarrow \begin{pmatrix} -e & -e & -e \\ -e & -e & -e \\ -e & -e & -e \end{pmatrix} \begin{pmatrix} a_{21} \\ a_{22} \\ a_{23} \end{pmatrix} = \vec{0} \Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a_{21} \\ a_{22} \\ a_{23} \end{pmatrix} = \vec{0} \Rightarrow a_{21} + a_{22} + a_{23} = 0$$

$$\Rightarrow \begin{pmatrix} a_{21} \\ a_{22} \\ a_{23} \end{pmatrix} = b_1 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + b_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \xleftarrow{\text{Multiplicidad geométrica } \neq 2, \text{ igual a la multiplicidad algebraica}} \quad \begin{array}{l} \text{Para } b_1 = 1 \text{ y } b_2 = 0 \\ \Rightarrow \vec{a}_{12}^\text{T} = (-1, 1, 0) \end{array}$$

* Tercer autovector $\Rightarrow \vec{a}_{13}^\text{T} = (-1, 0, 1)$

Continua...

- \vec{a}_2^\top y \vec{a}_3^\top no son ortogonales, hay que orthogonalizar estos con el método de Gram-Schmidt

Tomando \vec{a}_2^\top como el vector escogido y \vec{a}_3^\top el vector a orthogonalizar respecto al anterior

$$\vec{a}_2^\top = (-1, 1, 0) \Rightarrow \hat{a}_2^\top = \frac{1}{\sqrt{2}}(-1, 1, 0) \quad \text{y} \quad \vec{a}_3^\top = (-1, 0, 1)$$

$$\Rightarrow \vec{a}_3^{\top\prime} = \vec{a}_3^\top - (\hat{a}_2^\top \cdot \vec{a}_3^\top) \hat{a}_2^\top = (-1, 0, 1) - \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}(-1, 1, 0) = (-1, 0, 1) + \frac{1}{2}(1, -1, 0) = \frac{1}{2}(-1, -1, 2)$$

$$\Rightarrow \vec{a}_3^{\top\prime} = \frac{1}{2}(-1, -1, 2)$$

$$\vec{a}_1^\top = \alpha_1(1, 1, 1)$$

$$\vec{a}_2^\top = \alpha_2(-1, 1, 0)$$

$$\vec{a}_3^{\top\prime} = \frac{\alpha_3}{2}(-1, -1, 2) \xrightarrow[-2]{\text{Recalando}} = \alpha_3(1, 1, -2)$$

Para normalizar autovectores

$$\vec{a}_1^\top \cdot \vec{a}_1^\top = 1$$

- Primer autovector

$$\Rightarrow \alpha_1(1, 1, 1) M\mathbb{I}^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \alpha_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1 \Rightarrow \alpha_1^2 M\mathbb{I}^2 (1, 1, 1) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1$$

$$\Rightarrow \alpha_1^2 M\mathbb{I}^2 (1, 1, 1) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1 \Rightarrow \alpha_1^2 M\mathbb{I}^2 (1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1) = \alpha_1^2 M\mathbb{I}^2 3 = 1 \Rightarrow \alpha_1 = \sqrt{\frac{1}{3M\mathbb{I}^2}}$$

- Segundo autovector

$$\Rightarrow \alpha_2(-1, 1, 0) M\mathbb{I}^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \alpha_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = 1 \Rightarrow \alpha_2^2 M\mathbb{I}^2 (-1, 1, 0) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = 1$$

$$\Rightarrow \alpha_2^2 M\mathbb{I}^2 (-1, 1, 0) \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = 1 \Rightarrow \alpha_2^2 M\mathbb{I}^2 (-1 \cdot -1 + 1 \cdot 1 + 0) = \alpha_2^2 M\mathbb{I}^2 2 = 1 \Rightarrow \alpha_2 = \sqrt{\frac{1}{2M\mathbb{I}^2}}$$

- Tercer autovector

$$\Rightarrow \alpha_3(1, 1, -2) M\mathbb{I}^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \alpha_3 \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = 1 \Rightarrow \alpha_3^2 M\mathbb{I}^2 (1, 1, -2) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = 1$$

$$\Rightarrow \alpha_3^2 M\mathbb{I}^2 (1, 1, -2) \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = 1 \Rightarrow \alpha_3^2 M\mathbb{I}^2 (1 \cdot 1 + 1 \cdot 1 - 2 \cdot -2) = 1 \Rightarrow \alpha_3^2 M\mathbb{I}^2 6 = 1 \Rightarrow \alpha_3 = \sqrt{\frac{1}{6M\mathbb{I}^2}}$$

Teniendo las α_i que normalizan al sistema, los autovectores normalizados son:

$$\Rightarrow \vec{\alpha}_1^{\text{nor}} = \sqrt{\frac{1}{3M\ell^2}} (1, 1, 1), \quad \vec{\alpha}_2^{\text{nor}} = \sqrt{\frac{1}{2M\ell^2}} (-1, 1, 0) \quad \text{y} \quad \vec{\alpha}_3^{\text{nor}} = \underbrace{\vec{\alpha}_3^{\text{nor}}}_{\text{renombrando}} = \sqrt{\frac{1}{6M\ell^2}} (1, 1, -2)$$

Construyendo la matriz de autovectores

$$E = \{\vec{\alpha}_1, \vec{\alpha}_2, \vec{\alpha}_3\} = \begin{pmatrix} \frac{1}{\sqrt{3M\ell^2}} & -\frac{1}{\sqrt{2M\ell^2}} & \frac{1}{\sqrt{6M\ell^2}} \\ \frac{1}{\sqrt{3M\ell^2}} & \frac{1}{\sqrt{2M\ell^2}} & \frac{1}{\sqrt{6M\ell^2}} \\ \frac{1}{\sqrt{3M\ell^2}} & 0 & -2\frac{1}{\sqrt{6M\ell^2}} \end{pmatrix} = \sqrt{\frac{1}{M\ell^2}} \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -2\frac{1}{\sqrt{6}} \end{pmatrix}$$

Como $\vec{\eta} = E \vec{m} \vec{q}$

Es ortogonal por formarse de vectores ortogonales y ortonormales respecto a la base m

$$\Rightarrow E^T = \sqrt{\frac{1}{M\ell^2}} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -2\frac{1}{\sqrt{6}} \end{pmatrix} \quad \text{Armando la relación de coordenadas}$$

$$\Rightarrow \vec{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = E^T \vec{m} \vec{q} = \sqrt{\frac{1}{M\ell^2}} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -2\frac{1}{\sqrt{6}} \end{pmatrix} M\ell^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} \Rightarrow \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \sqrt{M\ell^2} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -2\frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix}$$

$$\Rightarrow \begin{cases} \eta_1 = \sqrt{M\ell^2} (\frac{1}{\sqrt{3}} \theta_1 + \frac{1}{\sqrt{3}} \theta_2 + \frac{1}{\sqrt{3}} \theta_3) \\ \eta_2 = \sqrt{M\ell^2} (-\frac{1}{\sqrt{2}} \theta_1 + \frac{1}{\sqrt{2}} \theta_2) \\ \eta_3 = \sqrt{M\ell^2} (\frac{1}{\sqrt{6}} \theta_1 + \frac{1}{\sqrt{6}} \theta_2 - 2\frac{1}{\sqrt{6}} \theta_3) \end{cases} \quad \text{Además las coordenadas normales cumplen por definición} \\ \ddot{\eta}_r + \omega_r^2 \eta_r = 0 \Rightarrow \eta_r = A \cos(\omega_r t + \delta_r)$$

Finalmente se halla

$$\Rightarrow \begin{cases} \eta_1 = \sqrt{M\ell^2} (\frac{1}{\sqrt{3}} \theta_1 + \frac{1}{\sqrt{3}} \theta_2 + \frac{1}{\sqrt{3}} \theta_3) = A_1 \cos(\omega_1 t + \delta_1) \\ \eta_2 = \sqrt{M\ell^2} (-\frac{1}{\sqrt{2}} \theta_1 + \frac{1}{\sqrt{2}} \theta_2) = A_2 \cos(\omega_2 t + \delta_2) \\ \eta_3 = \sqrt{M\ell^2} (\frac{1}{\sqrt{6}} \theta_1 + \frac{1}{\sqrt{6}} \theta_2 - 2\frac{1}{\sqrt{6}} \theta_3) = A_3 \cos(\omega_3 t + \delta_3) \end{cases}$$

• $A_1, \delta_1, A_2, \delta_2$ y A_3, δ_3 se encuentran por medio de las condiciones iniciales.

• ω_1, ω_2 y ω_3 son las autofrecuencias ya encontradas y, para este caso, cumplen $\omega_1 = \omega_2 = \omega_3$



- 12-19. In the problem of the three coupled pendula, consider the three coupling constants as distinct, so that the potential energy may be written as

$$U = \frac{1}{2} (\theta_1^2 + \theta_2^2 + \theta_3^2 - 2\epsilon_{13}\theta_1\theta_2 - 2\epsilon_{13}\theta_1\theta_3 - 2\epsilon_{23}\theta_2\theta_3)$$

with $\epsilon_{12}, \epsilon_{13}, \epsilon_{23}$ all different. Show that no degeneracy occurs in such a system. Show also that degeneracy can occur only if $\epsilon_{12} = \epsilon_{13} = \epsilon_{23} = 0$

Para probar que no ocurre degeneración en el sistema los autovalores deben ser todos diferentes

$$U = Mg(l[(1-\cos\theta_1) + (1-\cos\theta_2) + (1-\cos\theta_3)] + \frac{1}{2}[\kappa_1(l\sin\theta_1 - l\sin\theta_2)^2 + \kappa_2(l\sin\theta_2 - l\sin\theta_3)^2 + \kappa_3(l\sin\theta_3 - l\sin\theta_1)^2])$$

Energía Potencial gravitacional

Potencial del resorte, considerando el soporte como un resorte

- Plantear oscilaciones pequeñas: $\cos\theta_i = 1 - \frac{\theta_i^2}{2} \Rightarrow (1-\cos\theta_i) = (1-1+\frac{\theta_i^2}{2}) = \frac{\theta_i^2}{2}; \sin\theta_i = \theta_i$

$$\begin{aligned} \Rightarrow U &= Mg\left(\frac{\theta_1^2}{2} + \frac{\theta_2^2}{} + \frac{\theta_3^2}{2}\right) + \frac{l^2}{2}[\kappa_1(\theta_1 - \theta_2)^2 + \kappa_2(\theta_2 - \theta_3)^2 + \kappa_3(\theta_3 - \theta_1)^2] \\ &= \frac{Mgl}{2}(\theta_1^2 + \theta_2^2 + \theta_3^2) + \frac{l^2}{2}[\kappa_1(\theta_1^2 - 2\theta_1\theta_2 + \theta_2^2) + \kappa_2(\theta_2^2 - 2\theta_2\theta_3 + \theta_3^2) + \kappa_3(\theta_3^2 - 2\theta_3\theta_1 + \theta_1^2)] \\ &= \frac{l}{2}[(Mg + \kappa_1l + \kappa_3l)\theta_1^2 + (Mg + \kappa_2l + \kappa_1l)\theta_2^2 + (Mg + \kappa_3l + \kappa_2l)\theta_3^2 - 2l\kappa_1\theta_1\theta_2 - 2l\kappa_2\theta_2\theta_3 - 2l\kappa_3\theta_3\theta_1] \\ &= \left[\frac{l}{2}(Mg + \kappa_1l + \kappa_3l)\theta_1^2 - \frac{2l^2\kappa_1\theta_1\theta_2}{2}\right] + \left[\frac{l}{2}(Mg + \kappa_2l + \kappa_1l)\theta_2^2 - \frac{2l^2\kappa_2\theta_2\theta_3}{2}\right] + \left[\frac{l}{2}(Mg + \kappa_3l + \kappa_2l)\theta_3^2 - \frac{2l^2\kappa_3\theta_3\theta_1}{2}\right] \\ &= \frac{l(Mg + \kappa_1l + \kappa_3l)}{2}\left[\theta_1^2 - \frac{l\kappa_1}{(Mg + \kappa_1l + \kappa_3l)}2\theta_1\theta_2\right] + \frac{l(Mg + \kappa_2l + \kappa_1l)}{2}\left[\theta_2^2 - \frac{l\kappa_2}{(Mg + \kappa_2l + \kappa_1l)}2\theta_2\theta_3\right] + \dots \\ &\quad \dots + \frac{l(Mg + \kappa_3l + \kappa_2l)}{2}\left[\theta_3^2 - \frac{l\kappa_3}{(Mg + \kappa_3l + \kappa_2l)}2\theta_3\theta_1\right] \end{aligned}$$

Tomando $\epsilon_{ij} = \frac{l\kappa_{ij}}{(Mg + \kappa_{12}l + \kappa_{31}l)}$ y $\gamma_{ij} = \frac{l(Mg + \kappa_{12}l + \kappa_{31}l)}{2}$ para $i \neq j$

$$\Rightarrow U = \gamma_{13}(\theta_1^2 - 2\epsilon_{13}\theta_1\theta_2) + \gamma_{21}(\theta_2^2 - 2\epsilon_{21}\theta_2\theta_3) + \gamma_{32}(\theta_3^2 - 2\epsilon_{32}\theta_3\theta_1)$$

$$= \gamma_{13}\theta_1^2 + \gamma_{21}\theta_2^2 + \gamma_{32}\theta_3^2 - 2\gamma_{13}\epsilon_{13}\theta_1\theta_2 - 2\gamma_{21}\epsilon_{21}\theta_2\theta_3 - 2\gamma_{32}\epsilon_{32}\theta_3\theta_1$$

$$\Rightarrow U = \gamma_{13}\theta_1^2 + \gamma_{21}\theta_2^2 + \gamma_{32}\theta_3^2 - 2\gamma_{13}\epsilon_{13}\theta_1\theta_2 - 2\gamma_{21}\epsilon_{21}\theta_2\theta_3 - 2\gamma_{32}\epsilon_{32}\theta_3\theta_1$$

$$\text{La energía anéntica es igual a la anterior : } T = \frac{1}{2}Ml^2(\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2)$$

Continua...

Recuperando las energías

$$\Rightarrow \begin{cases} U = r_{11}\dot{\theta}_1^2 + r_{22}\dot{\theta}_2^2 + r_{33}\dot{\theta}_3^2 - 2r_{12}E_{12}\dot{\theta}_1\dot{\theta}_2 - 2r_{13}E_{13}\dot{\theta}_1\dot{\theta}_3 - 2r_{23}E_{23}\dot{\theta}_2\dot{\theta}_3 \\ T = \frac{1}{2}Ml^2(\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2) \end{cases} \quad \text{Para simplificar la notación } r_{ij} = r \text{ y } E_{ij} = E.$$

$$\bullet T = \frac{1}{2} \sum_i m_{ij} q_i \ddot{q}_j = \frac{1}{2} Ml^2 (\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2) \Rightarrow m = Ml^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\bullet A_{ij} = \frac{\partial^2 U}{\partial q_i \partial q_j} \Rightarrow A_{11} = 2r_1, A_{22} = 2r_2, A_{33} = 2r_3, A_{12} = -2r_1E_2, A_{23} = -2r_2E_3 \text{ y } A_{13} = -2r_3E_1$$

$$\Rightarrow A = \begin{pmatrix} 2r_1 & -2r_1E_2 & -2r_1E_3 \\ -2r_1E_2 & 2r_2 & -2r_2E_3 \\ -2r_1E_3 & -2r_2E_3 & 2r_3 \end{pmatrix} = 2 \begin{pmatrix} r_1 & -r_1E_2 & -r_1E_3 \\ -r_1E_2 & r_2 & -r_2E_3 \\ -r_1E_3 & -r_2E_3 & r_3 \end{pmatrix}$$

$$\star \det(A - \omega^2 m) = 0$$

$$\Rightarrow \left| 2 \begin{pmatrix} r_1 & -r_1E_2 & -r_1E_3 \\ -r_1E_2 & r_2 & -r_2E_3 \\ -r_1E_3 & -r_2E_3 & r_3 \end{pmatrix} - \omega^2 Ml^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| = 0 \Rightarrow \begin{vmatrix} 2r_1 - \omega^2 Ml^2 & -2r_1E_2 & -2r_1E_3 \\ -2r_1E_2 & 2r_2 - \omega^2 Ml^2 & -2r_2E_3 \\ -2r_1E_3 & -2r_2E_3 & 2r_3 - \omega^2 Ml^2 \end{vmatrix} = 0$$

$$\Rightarrow (2r_1 - \omega^2 Ml^2)[(2r_2 - \omega^2 Ml^2)(2r_3 - \omega^2 Ml^2) - 4r_2^2 E_2^2] - 2r_1 E_1 [2r_2(2r_3 - \omega^2 Ml^2) - 2r_2 E_2 - 2r_3 E_3] + \dots - 2r_1 E_1 [2r_3(2r_2 - \omega^2 Ml^2) - 2r_2 E_2 - 2r_3 E_3] = 0$$

$$\Rightarrow (2r_1 - \omega^2 Ml^2)[(2r_2 - \omega^2 Ml^2)(2r_3 - \omega^2 Ml^2) - 4r_2^2 E_2^2] - 4r_1^2 E_1^2 (2r_3 - \omega^2 Ml^2) - 8r_1 r_2 r_3 E_1 E_2 E_3 + \dots$$

$$\dots - 8r_1 r_2 r_3 E_1 E_2 E_3 - 4r_3^2 E_3^2 (2r_2 - \omega^2 Ml^2) = 0$$

$$\Rightarrow (2r_1 - \omega^2 Ml^2)(2r_2 - \omega^2 Ml^2)(2r_3 - \omega^2 Ml^2) - 4r_1^2 E_1^2 (2r_3 - \omega^2 Ml^2) - 4r_2^2 E_2^2 (2r_1 - \omega^2 Ml^2) - 4r_3^2 E_3^2 (2r_2 - \omega^2 Ml^2) + \dots$$

$$\dots - 16r_1 r_2 r_3 E_1 E_2 E_3 = 0$$

$$\Rightarrow [4r_1 r_2 - (r_1 + r_2)Ml^2 \omega^2 + M^2 l^4 \omega^4](2r_3 - \omega^2 Ml^2) - 4r_1^2 E_1^2 (2r_3 - \omega^2 Ml^2) - 4r_2^2 E_2^2 (2r_1 - \omega^2 Ml^2) + \dots$$

$$\dots - 4r_3^2 E_3^2 (2r_2 - \omega^2 Ml^2) - 16r_1 r_2 r_3 E_1 E_2 E_3 = 0$$

Continua...

Reapareando lo anterior

$$\Rightarrow [4\gamma_1\gamma_2 - (\gamma_1 + \gamma_2)Ml^2\omega^2 + M^3l^4\omega^4](2\gamma_3 - \omega^2Ml^2) - 4\gamma_1^2\epsilon_1^2(2\gamma_3 - \omega^2Ml^2) - 4\gamma_2^2\epsilon_2^2(2\gamma_1 - \omega^2Ml^2) + \dots - 4\gamma_3^2\epsilon_3^2(2\gamma_2 - \omega^2Ml^2) - 16\gamma_1\gamma_2\gamma_3\epsilon_1\epsilon_2\epsilon_3 = 0$$

$$\Rightarrow 8\gamma_1\gamma_2\gamma_3 + 2(\gamma_1 + \gamma_2)\gamma_3 Ml^2\omega^2 + 2\gamma_3 M^3l^4\omega^4 - 4\gamma_1\gamma_2 Ml^2\omega^2 - M^3l^6\omega^6 + \dots = 0$$

$$\Rightarrow -M^3l^6\omega^6 + 2\gamma_3 M^3l^4\omega^4 + 2Ml^2(\gamma_1\gamma_3 + \gamma_2\gamma_3 - 2\gamma_1\gamma_2)\omega^2 + 8\gamma_1\gamma_2\gamma_3 - 4\gamma_1^2\epsilon_1^2(2\gamma_3 - \omega^2Ml^2) - 4\gamma_2^2\epsilon_2^2(2\gamma_1 - \omega^2Ml^2) + \dots - 4\gamma_3^2\epsilon_3^2(2\gamma_2 - \omega^2Ml^2) - 16\gamma_1\gamma_2\gamma_3\epsilon_1\epsilon_2\epsilon_3 = 0$$

$$\Rightarrow -M^3l^6\omega^6 + 2\gamma_3 M^3l^4\omega^4 + 2Ml^2(\gamma_1\gamma_3 + \gamma_2\gamma_3 - 2\gamma_1\gamma_2 + 2\gamma_1^2\epsilon_1^2 + 2\gamma_2^2\epsilon_2^2 + 2\gamma_3^2\epsilon_3^2)\omega^2 - 8\gamma_1^2\epsilon_1^2\gamma_3 - 8\gamma_2^2\epsilon_2^2\gamma_1 - 8\gamma_3^2\epsilon_3^2\gamma_2 + \dots + \dots 8\gamma_1\gamma_2\gamma_3(1 - 2\epsilon_1\epsilon_2\epsilon_3) = 0$$

$$\Rightarrow -M^3l^6\omega^6 + 2\gamma_3 M^3l^4\omega^4 + 2Ml^2(\gamma_1\gamma_3 + \gamma_2\gamma_3 - 2\gamma_1\gamma_2 + 2\gamma_1^2\epsilon_1^2 + 2\gamma_2^2\epsilon_2^2 + 2\gamma_3^2\epsilon_3^2)\omega^2 + \dots + 8[\gamma_1\gamma_2\gamma_3(1 - 2\epsilon_1\epsilon_2\epsilon_3) - \gamma_1^2\epsilon_1^2\gamma_3 - \gamma_2^2\epsilon_2^2\gamma_1 - \gamma_3^2\epsilon_3^2\gamma_2] = 0$$

$$\text{Tomando } c = (\gamma_1\gamma_3 + \gamma_2\gamma_3 - 2\gamma_1\gamma_2 + 2\gamma_1^2\epsilon_1^2 + 2\gamma_2^2\epsilon_2^2 + 2\gamma_3^2\epsilon_3^2) \quad y \quad d = [\gamma_1\gamma_2\gamma_3(1 - 2\epsilon_1\epsilon_2\epsilon_3) - \gamma_1^2\epsilon_1^2\gamma_3 - \gamma_2^2\epsilon_2^2\gamma_1 - \gamma_3^2\epsilon_3^2\gamma_2]$$

$$\Rightarrow -M^3l^6\omega^6 + 2\gamma_3 M^3l^4\omega^4 + 2Ml^2c\omega^2 + 8d = 0 \Rightarrow M^3l^6\omega^6 - 2\gamma_3 M^3l^4\omega^4 - 2Ml^2c\omega^2 - 8d = 0$$

Problema 4. (Thornton 12.20)

12-20. Construct the possible eigenvectors for the degenerate modes in the case of the three coupled pendula by requiring $a_{11} = 2a_{23}$. Interpret this situation physically.

Recuperando resultados del problema original

$$\begin{cases} a_{11} + a_{12} - 2a_{13} = 0 \\ a_{12} - a_{13} = 0 \\ a_{13} \in \mathbb{R} \end{cases} \quad y \quad \{ a_{21} + a_{22} + a_{23} = 0$$

$$\Rightarrow \begin{cases} a_{11} + a_{12} - 2a_{13} = 0 \\ a_{12} - a_{13} = 0 \\ a_{13} \in \mathbb{R} \end{cases} \quad y \quad \{ 2a_{11} + a_{22} + a_{23} = 0$$

$$\text{Con } a_{13} = 1 \Rightarrow \vec{a}_1 = (1, 1, 1), \text{ con } a_{23} = 1 \Rightarrow \vec{a}_2 = (2, -3, 1) \text{ y con } a_{23} = 0 \Rightarrow \vec{a}_3 = (2, -2, 0)$$

Hay que ortogonalizar \vec{a}_2 y \vec{a}_3 , escogiendo como base a \vec{a}_2

$$\vec{a}_3' = \vec{a}_3 - (\vec{a}_3 \cdot \hat{a}_2) \hat{a}_2 = (2, -2, 0) - \frac{10}{\sqrt{14}} \hat{a}_2 = \left(\frac{6}{14}, \frac{2}{14}, \frac{-10}{14} \right) \xrightarrow{\text{Recalculando}} (8, 2, -10)$$

$$\Rightarrow \vec{a}_1^T = (1, 1, 1), \vec{a}_2 = (2, -3, 1), \vec{a}_3' = (8, 2, -10)$$

Problema 5. (Taylor 11.24)

11.24 ** Two equal masses m move on a frictionless horizontal table. They are held by three identical strings (each of length L , tension T), as shown in Figure 11.19, so that their equilibrium position is a straight line between the anchors at A and B . The two masses move in the transverse (y) direction, but not in the longitudinal (x) direction. Write down the Lagrangian for small displacements, and find and describe the motion in the corresponding normal modes. [Hint: "Small" displacements have y_1 and y_2 much less than L , which means that you can treat the tensions as constant. Therefore the PE of each string is just Td , where d is the amount by which its length has increased from equilibrium.]

$$T = \frac{1}{2}m\dot{y}_1^2 + \frac{1}{2}m\dot{y}_2^2 = \frac{1}{2}m(\dot{y}_1^2 + \dot{y}_2^2)$$

$$U = \frac{k}{2}\dot{y}_1^2 + \frac{k}{2}(y_1 - y_2)^2 + \frac{k}{2}\dot{y}_2^2 = \frac{k}{2}\dot{y}_1^2 + \frac{k}{2}(\dot{y}_1^2 - 2\dot{y}_1\dot{y}_2 + \dot{y}_2^2) + \frac{k}{2}\dot{y}_2^2 = \frac{k}{2}(2\dot{y}_1^2 - 2\dot{y}_1\dot{y}_2 + 2\dot{y}_2^2)$$

$$\Rightarrow L = \frac{1}{2}m(\dot{y}_1^2 + \dot{y}_2^2) - \frac{k}{2}(2\dot{y}_1^2 - 2\dot{y}_1\dot{y}_2 + 2\dot{y}_2^2) //$$

- $\mathcal{T} = \frac{1}{2} \sum_i M_{ij} \dot{q}_i \dot{q}_j = \frac{1}{2}m(\dot{y}_1^2 + \dot{y}_2^2) \Rightarrow M = m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

- $A_{ij} = \frac{\partial^2 U}{\partial q_i \partial q_j} \Rightarrow A_{11} = 2k, A_{22} = 2k, A_{12} = A_{21} = -k \Rightarrow A = \begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix}$

Buscando las frecuencias de oscilación

$$\det(A - \omega^2 M) = 0 \Rightarrow \left| \begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix} - \omega^2 m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = \begin{vmatrix} 2k - \omega^2 m & -k \\ -k & 2k - \omega^2 m \end{vmatrix} = 0$$

$$\Rightarrow (2k - \omega^2 m)^2 - k^2 = 0 \Rightarrow 2k - \omega^2 m = \pm k \Rightarrow \omega^2 = \frac{2k \pm k}{m} \Rightarrow \omega^2 = \frac{3k}{m} \text{ y } \omega^2 = \frac{k}{m}$$

Los modos de vibración

$$1^o (A - \omega_1^2 M) \vec{a}_1 = 0 \Rightarrow \left(\begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix} - \frac{3k}{m} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} 2k - 3k/m & -k \\ -k & 2k - 3k/m \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} -k & -k \\ -k & -k \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix} = 0 \Rightarrow -ka_{11} - ka_{12} = 0 \Rightarrow a_{11} = -a_{12} \Rightarrow \vec{a}_1 = (-1, 1) //$$

$$2^o (A - \omega_2^2 M) \vec{a}_2 = 0 \Rightarrow \left(\begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix} - \frac{k}{m} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \begin{pmatrix} a_{21} \\ a_{22} \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} 2k - k/m & -k \\ -k & 2k - k/m \end{pmatrix} \begin{pmatrix} a_{21} \\ a_{22} \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} k & -k \\ -k & k \end{pmatrix} \begin{pmatrix} a_{21} \\ a_{22} \end{pmatrix} = 0 \Rightarrow ka_{21} - ka_{22} = 0 \Rightarrow a_{21} = a_{22} \Rightarrow \vec{a}_2 = (1, 1) //$$

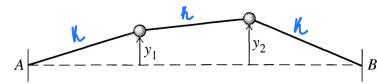


Figure 11.19 Problem 11.24

$$k = \frac{T}{L}$$

Problema 6. (Thornton 12.16)

12-16. A thin hoop of radius R and mass M oscillates in its own plane hanging from a single fixed point. Attached to the hoop is a small mass m constrained to move (in a frictionless manner) along the hoop. Consider only small oscillations, and show that the eigenfrequencies are

$$\omega_1 = \sqrt{2} \sqrt{\frac{g}{R}}, \quad \omega_2 = \frac{\sqrt{2}}{2} \sqrt{\frac{g}{R}}$$

Find the two sets of initial conditions that allow the system to oscillate in its normal modes. Describe the physical situation for each mode.

Las posiciones del centro de masa del aro y la posición Q de la masa puntual son:

$$X_{\text{hoop}} = R \cos \theta \quad \text{y} \quad Y_{\text{hoop}} = -R \sin \theta$$

$$Q_x = X_{\text{hoop}} + R \cos \phi = R \cos \theta + R \cos \phi \quad \text{y} \quad Q_y = Y_{\text{hoop}} - R \sin \phi = -R \sin \theta - R \sin \phi$$

Las energías cinéticas para los objetos individuales

$$\bullet T_a = \frac{1}{2} M (\dot{Q}_x^2 + \dot{Q}_y^2) = \frac{1}{2} M [(\dot{R} \cos \theta \dot{\theta} + R \cos \phi \dot{\phi})^2 + (\dot{R} \sin \theta \dot{\theta} + R \sin \phi \dot{\phi})^2]$$

$$= \frac{1}{2} M R^2 [(\cos \theta \dot{\theta} + \cos \phi \dot{\phi})^2 + (\sin \theta \dot{\theta} + \sin \phi \dot{\phi})^2]$$

$$= \frac{1}{2} M R^2 (\cos^2 \theta \dot{\theta}^2 + 2 \dot{\theta} \dot{\phi} \cos \theta \cos \phi + \cos^2 \phi \dot{\phi}^2 + \sin^2 \theta \dot{\theta}^2 + 2 \dot{\theta} \dot{\phi} \sin \theta \sin \phi + \sin^2 \phi \dot{\phi}^2)$$

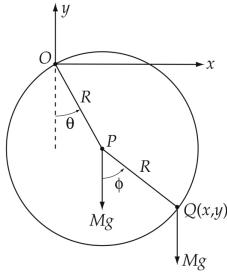
$$= \frac{1}{2} M R^2 (\cos^2 \theta \dot{\theta}^2 + \sin^2 \theta \dot{\theta}^2 + \cos^2 \phi \dot{\phi}^2 + \sin^2 \phi \dot{\phi}^2 + 2 \dot{\theta} \dot{\phi} \cos \theta \cos \phi + 2 \dot{\theta} \dot{\phi} \sin \theta \sin \phi)$$

$$= \frac{1}{2} M R^2 (\dot{\theta}^2 + \dot{\phi}^2 + 2 \dot{\theta} \dot{\phi} \cos \theta \cos \phi + 2 \dot{\theta} \dot{\phi} \sin \theta \sin \phi) ; \quad \begin{matrix} \text{Considerando oscilaciones pequeñas, eliminando términos} \\ \downarrow \text{la forma } q_i q_j \end{matrix}$$

$$\approx \frac{1}{2} M R^2 (\dot{\theta}^2 + \dot{\phi}^2 + 2 \dot{\theta} \dot{\phi}) \Rightarrow T_a = \frac{1}{2} M R^2 (\dot{\theta}^2 + \dot{\phi}^2 + 2 \dot{\theta} \dot{\phi})$$

$$\bullet T_{\text{hoop}} = \frac{1}{2} I_o \dot{\theta}^2 ; \quad I_o = I_{\text{cm}} + MR^2 = 2MR^2 \Rightarrow T_{\text{hoop}} = \frac{1}{2} 2MR^2 \dot{\theta}^2$$

$$\Rightarrow T = T_{\text{hoop}} + T_a = \frac{1}{2} 2MR^2 \dot{\theta}^2 + \frac{1}{2} M R^2 (\dot{\theta}^2 + \dot{\phi}^2 + 2 \dot{\theta} \dot{\phi}) = \frac{1}{2} M R^2 (3 \dot{\theta}^2 + \dot{\phi}^2 + 2 \dot{\theta} \dot{\phi})$$



Ahora la energía potencial

- $U_{\text{hoop}} = Mg \cdot R \cos \theta = -MgR \cos \theta$; Aproximación a ángulos pequeños
 $\approx -MgR(1 - \frac{\theta^2}{2})$

- $U_a = Mg(-R \cos \theta - R \cos \phi) = -MgR(\cos \theta + \cos \phi)$; Aproximación a ángulos pequeños
 $\approx -MgR(1 - \frac{\theta^2}{2} + 1 - \frac{\phi^2}{2}) = -MgR(2 - \frac{\theta^2}{2} - \frac{\phi^2}{2})$

$$\Rightarrow U = U_{\text{hoop}} + U_a = -MgR(1 - \frac{\theta^2}{2}) - MgR(2 - \frac{\theta^2}{2} - \frac{\phi^2}{2}) = -MgR(3 - \theta^2 - \frac{\phi^2}{2})$$

$$\Rightarrow \begin{cases} T = \frac{1}{2}MR^2(3\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}) \\ U = -MgR(3 - \theta^2 - \frac{\phi^2}{2}) \end{cases}$$

- $T = \frac{1}{2} \sum_i M_{ij} \dot{q}_i \dot{q}_j = \frac{1}{2}MR^2(3\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}) \Rightarrow m = \begin{pmatrix} 3MR^2 & MR^2 \\ MR^2 & MR^2 \end{pmatrix}$

- $A_{ij} = \frac{\partial^2 U}{\partial q_i \partial q_j} \Rightarrow A_{11} = 2MgR, A_{22} = MgR, A_{12} = A_{21} = 0 \Rightarrow A = \begin{pmatrix} 2MgR & 0 \\ 0 & MgR \end{pmatrix}$

Para hallar las autofrecuencias se resuelve: $\det(A - \omega^2 m) = 0$

$$\Rightarrow \left| \begin{pmatrix} 2MgR & 0 \\ 0 & MgR \end{pmatrix} - \omega^2 \begin{pmatrix} 3MR^2 & MR^2 \\ MR^2 & MR^2 \end{pmatrix} \right| = \begin{vmatrix} 2MgR - 3MR^2\omega^2 & -\omega^2 MR^2 \\ \omega^2 MR^2 & MgR - MR^2\omega^2 \end{vmatrix} = 0$$

$$\Rightarrow (2MgR - 3MR^2\omega^2)(MgR - MR^2\omega^2) - \omega^4 M^2 R^4 = 0$$

$$\Rightarrow 2M^2 g^2 R^2 - 2M^2 g R^3 \omega^2 - 3M^2 g R^3 \omega^2 + 3M^2 R^4 \omega^4 - \omega^4 M^2 R^4 = 0$$

$$\Rightarrow 2M^2 g^2 R^2 - 5M^2 g R^3 \omega^2 + 2M^2 R^4 \omega^4 = 0 \Rightarrow 2g^2 - 5gR\omega^2 + 2R^2\omega^4 = 0$$

$$\Rightarrow \omega^2 = \frac{5gR \pm \sqrt{25g^2R^2 - 4 \cdot 2R^2 \cdot 2g^2}}{2 \cdot 2R^2} \Rightarrow \omega^2 = \frac{5gR \pm \sqrt{25g^2R^2 - 16g^2R^2}}{4R^2}$$

$$\Rightarrow \omega^2 = \frac{5gR \pm \sqrt{9g^2R^2}}{4R^2} = \frac{5gR \pm 3gR}{4R^2} \Rightarrow \omega^2 = \frac{5g \pm 3g}{4R} \Rightarrow \omega_1^2 = \frac{2g}{R} \quad y \quad \omega_2^2 = \frac{8g}{2R} //$$

Buscando los modos normales, hay que resolver $(A - \omega^2 m) \vec{a}_n = 0$

$$\bullet \omega_1^2 = \frac{2g}{R} \Rightarrow \left(\begin{pmatrix} 2MgR & 0 \\ 0 & MgR \end{pmatrix} - \frac{2g}{R} \begin{pmatrix} 3MR^2 & MR^2 \\ MR^2 & MR^2 \end{pmatrix} \right) \begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix} = 0$$

$$\Rightarrow \left(\begin{pmatrix} 2MgR & 0 \\ 0 & MgR \end{pmatrix} - \begin{pmatrix} 6MgR & 2MgR \\ 2MgR & 2MgR \end{pmatrix} \right) \begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} -4MgR & -2MgR \\ -2MgR & -MgR \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix} = 0$$

$$\Rightarrow -MgR \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix} = 0 \Rightarrow 2a_{11} + a_{12} = 0 \Rightarrow \vec{a}_1^\top = \alpha_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$\bullet \omega_2^2 = \frac{g}{2R} \Rightarrow \left(\begin{pmatrix} 2MgR & 0 \\ 0 & MgR \end{pmatrix} - \frac{g}{2R} \begin{pmatrix} 3MR^2 & MR^2 \\ MR^2 & MR^2 \end{pmatrix} \right) \begin{pmatrix} a_{21} \\ a_{22} \end{pmatrix} = 0$$

$$\Rightarrow \left(\begin{pmatrix} 4MgR & 0 \\ 0 & 2MgR \end{pmatrix} - \begin{pmatrix} 3MgR & MgR \\ MgR & MgR \end{pmatrix} \right) \begin{pmatrix} a_{21} \\ a_{22} \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} MgR & -MgR \\ -MgR & MgR \end{pmatrix} \begin{pmatrix} a_{21} \\ a_{22} \end{pmatrix} = 0$$

$$\Rightarrow MgR \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} a_{21} \\ a_{22} \end{pmatrix} = 0 \Rightarrow a_{21} - a_{22} = 0 \Rightarrow \vec{a}_2^\top = \alpha_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

\vec{a}_1 no es ortogonal a \vec{a}_2 ,
hay que orthonormalizar

$$\vec{a}_1^\top = \vec{a}_1^\top - (\vec{a}_1^\top \cdot \vec{a}_2^\top) \vec{a}_2^\top = (-1, 2) - \frac{1}{2} \cdot 1(1, 1) = \left(\frac{3}{2}, \frac{3}{2} \right) \xrightarrow{\text{Rescalando}} \vec{a}_1^\top = \alpha_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Ahora hay que orthonormalizar los autovectores: $\vec{a}_1^\top \vec{m} \vec{a}_1 = 1$

$$\vec{a}_1: \vec{a}_1^\top \vec{m} \vec{a}_1 = 1 \Rightarrow \alpha_1^2 (1, -1) \begin{pmatrix} 3MR^2 & MR^2 \\ MR^2 & MR^2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1 \Rightarrow \alpha_1^2 MR^2 (1, -1) \begin{pmatrix} 3 \\ 1 \\ 1 \\ -1 \end{pmatrix} = 1$$

$$\Rightarrow \alpha_1^2 MR^2 (2, 0) \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1 \Rightarrow \alpha_1^2 MR^2 (2 \cdot 1 + 1 \cdot 0) = 1 \Rightarrow \alpha_1 = \sqrt{\frac{1}{2MR^2}}$$

$$\vec{a}_2: \vec{a}_2^\top \vec{m} \vec{a}_2 = 1 \Rightarrow \alpha_2^2 (1, 1) \begin{pmatrix} 3MR^2 & MR^2 \\ MR^2 & MR^2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 \Rightarrow \alpha_2^2 MR^2 (1, 1) \begin{pmatrix} 3 \\ 1 \\ 1 \\ 1 \end{pmatrix} = 1$$

$$\Rightarrow \alpha_2^2 MR^2 (4, 2) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 \Rightarrow \alpha_2^2 MR^2 (4 + 2) = 1 \Rightarrow \alpha_2 = \sqrt{\frac{1}{6MR^2}}$$

Construyendo la matriz de modos normales

$$E = \{\vec{a}_1, \vec{a}_2\} = \begin{pmatrix} \frac{1}{\sqrt{2MR^2}} & \frac{1}{\sqrt{6MR^2}} \\ -\frac{1}{\sqrt{2MR^2}} & \frac{1}{\sqrt{6MR^2}} \end{pmatrix} = \sqrt{\frac{1}{2MR^2}} \begin{pmatrix} 1 & \frac{1}{\sqrt{3}} \\ -1 & \frac{1}{\sqrt{3}} \end{pmatrix} \Rightarrow E^\top = \sqrt{\frac{1}{2MR^2}} \begin{pmatrix} 1 & -1 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

Reapareando lo anterior y la relación de coordenadas $\vec{r} = E^T m \vec{q}$

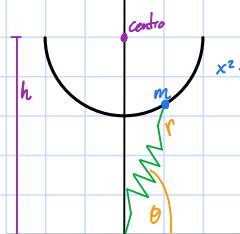
$$\Rightarrow \vec{r} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = E^T m \vec{q} = \sqrt{\frac{1}{2MR^2}} \begin{pmatrix} 1 & -1 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} MR^2 \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} (\theta) = \sqrt{\frac{MR^2}{2}} \begin{pmatrix} 1 & -1 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} (\theta)$$

$$= \sqrt{\frac{MR^2}{2}} \begin{pmatrix} 2 & 0 \\ \frac{4}{\sqrt{3}} & 2\sqrt{3} \end{pmatrix} (\theta) = \sqrt{2MR^2} \begin{pmatrix} 1 & 0 \\ 2\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} (\theta)$$

$$\Rightarrow \begin{cases} \eta_1 = \sqrt{2MR^2} \theta \\ \eta_2 = 2\sqrt{2MR^2} \theta + \sqrt{2MR^2} \phi \end{cases}$$

Problema 7 (Thomson 12.18)

12. A mass M moves horizontally along a smooth rail. A pendulum of length R hangs from the rail with a weightless string and mass m at its lower end. Find the natural frequency and describe the normal modes.



$$V = \dot{r} \hat{e}_r + r \dot{\theta} \hat{e}_{\theta}$$

$$x^2 + (y-h)^2 = R^2$$

Giroscopio

$$x^2 + y^2 = R^2$$

$$x = R \cos \theta$$

$$y = R \sin \theta$$

Mayo hago lagrange normal

Moviéndose a el origen

$$x = R \cos \theta$$

$$y = R \sin \theta + h$$

$$r^2 = R^2 \cos^2 \theta + (R \sin \theta + h)^2$$

$$= R^2 \cos^2 \theta + R^2 \sin^2 \theta + 2Rh \sin \theta + h^2$$

$$r^2 = R^2 + h^2 + 2Rh \sin \theta$$

$$T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) = \frac{1}{2} m \left[\frac{R^2 h^2 \cos^2 \theta \cdot \dot{\theta}^2}{R^2 + h^2 + 2Rh \sin \theta} + (R^2 + h^2 + 2Rh \sin \theta) \dot{\theta}^2 \right]$$

$$U = \frac{1}{2} K r^2 + mgh \sin \theta$$

$$r = \sqrt{R^2 + h^2 + 2Rh \sin \theta}$$

$$\dot{r} = \frac{R h \cos \theta \cdot \dot{\theta}}{\sqrt{R^2 + h^2 + 2Rh \sin \theta}}$$

$$\dot{r}^2 = \frac{R^2 h^2 \cos^2 \theta \cdot \dot{\theta}^2}{R^2 + h^2 + 2Rh \sin \theta}$$

Con multiplicadores

$$T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) \quad y \quad U = \frac{1}{2} K r^2 + mgh \sin \theta$$

$$f: r^2 - (R^2 + h^2 + 2Rh \sin \theta) = 0$$

$$\frac{\partial f}{\partial r} = 2r$$

$$L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) - \frac{1}{2} K r^2 - mgh \sin \theta$$

$$\frac{\partial f}{\partial \theta} = -2Rh \cos \theta$$

$$\frac{\partial}{\partial t} \frac{\partial L}{\partial q_i} - \frac{\partial L}{\partial q_i} + \sum_j \lambda_j \frac{\partial f}{\partial q_j} = 0 \xrightarrow{\text{restricciones}} \frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} + \lambda \frac{\partial f}{\partial q_i} = 0$$

Para r

$$\frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} + \lambda (2r) = 0$$

$$\frac{\partial L}{\partial \dot{r}} = m \ddot{r} \Rightarrow \frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{r}} = m \ddot{r}$$

$$\frac{\partial L}{\partial r} = m \dot{r}^2 - Kr - mgs \sin \theta$$

$$\Rightarrow m \ddot{r} - m \dot{r} \dot{\theta}^2 + Kr + mgs \sin \theta + 2r\lambda = 0$$

Para θ

$$\frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} + \lambda \frac{\partial f}{\partial \theta} = 0$$

$$\frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta} \Rightarrow \frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{\theta}} = m (2r \cdot \dot{r} + r^2 \ddot{\theta}) \Rightarrow m r^2 \ddot{\theta} + 2mr \dot{r} + m g r \cos \theta - 2Rh \cos \theta \lambda = 0$$

$$\frac{\partial L}{\partial \theta} = -m g r \cos \theta$$

$$\left\{ \begin{array}{l} m \ddot{r} - m \dot{r} \dot{\theta}^2 + Kr + mgs \sin \theta + 2r\lambda = 0 \\ m r^2 \ddot{\theta} + 2mr \dot{r} + m g r \cos \theta - 2Rh \cos \theta \lambda = 0 \end{array} \right.$$

$$\left. \begin{array}{l} m \ddot{r} - m \dot{r} \dot{\theta}^2 + Kr + mgs \sin \theta + 2r\lambda = 0 \\ m r^2 \ddot{\theta} + 2mr \dot{r} + m g r \cos \theta - 2Rh \cos \theta \lambda = 0 \end{array} \right.$$

11.18 ** Two equal masses m are constrained to move without friction, one on the positive x axis and one on the positive y axis. They are attached to two identical springs (force constant k) whose other ends are attached to the origin. In addition, the two masses are connected to each other by a third spring of force constant k' . The springs are chosen so that the system is in equilibrium with all three springs relaxed (length equal to unstretched length). What are the normal frequencies? Find and describe the normal modes.

- 12-6. Two identical harmonic oscillators are placed such that the two masses slide against one another, as in Figure 12-A. The frictional force provides a coupling of the motions proportional to the instantaneous relative velocity. Discuss the coupled oscillations of the system.

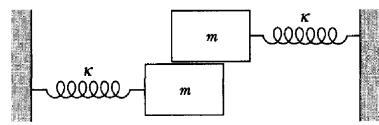


FIGURE 12-A Problem 12-6.

Gravitación

SUBSECTION 9.8

Problemas resueltos

Hay que revisar el problema del tunel que no atraviesa por el centro de la Tierra.

Gravitación

Halle el campo gravitacional en todas las regiones del sistema calculando directamente el campo y por medio del potencial

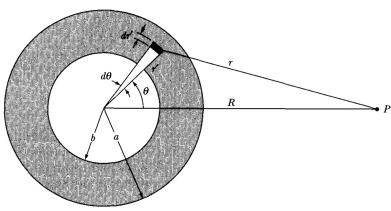


FIGURE 5-3 The geometry for finding the gravitational potential at point P due to a spherical shell of mass.

Problema 1. (Thornton 5.4)

- 5-4. A particle at rest is attracted toward a center of force according to the relation $F = -mk^2/x^3$. Show that the time required for the particle to reach the force center from a distance d is d^2/k .

$$F = -\frac{mk^2}{x^3} = -\frac{dU}{dx} \Rightarrow U = \int \frac{mk^2}{x^3} dx = -\frac{mk^2}{2x^2} \Rightarrow \text{la fuerza es conservativa, entonces la energía es constante}$$

$$\Rightarrow E = \frac{1}{2}mv^2 - \frac{mk^2}{2x^2} = \text{constante} = -\frac{mk^2}{2d^2} \Rightarrow \frac{1}{2}mv^2 - \frac{mk^2}{2x^2} = -\frac{mk^2}{2d^2} \Rightarrow v^2 = -\frac{k^2}{d^2} + \frac{k^2}{x^2} = k^2 \left(\frac{1}{d^2} + \frac{1}{x^2} \right)$$

$$\Rightarrow \frac{dx}{dt} = \pm \sqrt{k^2 \left(\frac{1}{d^2} + \frac{1}{x^2} \right)} = \pm k \sqrt{\frac{1}{d^2} + \frac{1}{x^2}} \Rightarrow \frac{\pm 1}{k \sqrt{\frac{1}{d^2} + \frac{1}{x^2}}} dx = dt \Rightarrow dt = \pm \frac{dx}{k \sqrt{d^2 - x^2}} = \pm \frac{dx}{k \sqrt{d^2 - x^2}}$$

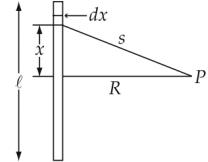
$$\Rightarrow \int_0^t dt = \int_0^d \frac{dx}{k \sqrt{d^2 - x^2}} \Rightarrow t = \pm \int_0^d \frac{dx}{k \sqrt{d^2 - x^2}} = \pm \frac{1}{k} \cdot -\sqrt{d^2 - x^2} \Big|_0^d \Rightarrow t = -\frac{1}{k} \sqrt{d^2 - x^2} \Big|_0^d = -0 + \frac{d^2}{k} = \frac{d^2}{k}$$

Problema 2. (Thornton 5.7)

- 5-7. Calculate the gravitational potential due to a thin rod of length l and mass M at a distance R from the center of the rod and in a direction perpendicular to the rod.

$$V_g = G \int_M \frac{dm}{r} ; dm = \lambda dx \quad \lambda = \frac{M}{l} \Rightarrow V_g = G \frac{M}{l} \int_{-\frac{l}{2}}^{\frac{l}{2}} \frac{1}{\sqrt{R^2 + x^2}} dx = G \frac{M}{l} \ln(x + \sqrt{x^2 + R^2}) \Big|_{-\frac{l}{2}}^{\frac{l}{2}}$$

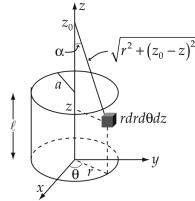
$$= G \frac{M}{l} \left[\ln\left(\frac{\frac{l}{2}}{-\frac{l}{2}} + \sqrt{\frac{l^2}{4} + R^2}\right) - \ln\left(-\frac{l}{2} + \sqrt{\frac{l^2}{4} + R^2}\right) \right] = G \frac{M}{l} \ln\left(\frac{\frac{l}{2} + \sqrt{\frac{l^2}{4} + R^2}}{-\frac{l}{2} + \sqrt{\frac{l^2}{4} + R^2}}\right) = G \frac{M}{l} \ln\left(\frac{l + \sqrt{l^2 + 4R^2}}{-l + \sqrt{l^2 + 4R^2}}\right)$$



Problema 3. (Thornton 5.8)

- 5-8. Calculate the gravitational field vector due to a homogeneous cylinder at exterior points on the axis of the cylinder. Perform the calculation (a) by computing the force directly and (b) by computing the potential first.

$$dm = \rho dV = \frac{M}{V} r dr d\theta dz$$



a) $\vec{r}_1 = r \cos\theta \hat{i} + r \sin\theta \hat{j}$ $\vec{r}_2 = z_0 \hat{k}$ $\Rightarrow \vec{r} = r \cos\theta \hat{i} + r \sin\theta \hat{j} + (z_0 - z) \hat{k}$

$$\Rightarrow |\vec{r}| = \sqrt{r^2 \cos^2\theta + r^2 \sin^2\theta + (z_0 - z)^2} = \sqrt{r^2 + (z_0 - z)^2}$$

$$\vec{g} = G \int_{-l}^{l} \frac{1}{r^3} dm \vec{r} = -G \frac{M}{V} \int_{0}^{2\pi} \int_{0}^{a} \int_{-l}^{l} \frac{r \cos\theta \hat{i} + r \sin\theta \hat{j} + (z_0 - z) \hat{k}}{\sqrt{r^2 + (z_0 - z)^2}^3} dr d\theta dz$$

$$g_x = -G \frac{M}{V} \int_{0}^{l} \int_{0}^{2\pi} \int_{0}^{a} \frac{r \cos\theta}{\sqrt{r^2 + (z_0 - z)^2}^3} dr d\theta dz = -G \frac{M}{V} \int_{0}^{l} \int_{0}^{2\pi} \frac{r \cos\theta \cancel{d\theta}}{\sqrt{r^2 + (z_0 - z)^2}^3} dr dz \Big|_0^{2\pi} = 0$$

$$g_y = -G \frac{M}{V} \int_{0}^{l} \int_{0}^{2\pi} \int_{0}^{a} \frac{r \sin\theta}{\sqrt{r^2 + (z_0 - z)^2}^3} dr d\theta dz = -G \frac{M}{V} \int_{0}^{l} \int_{0}^{2\pi} \frac{-r \sin\theta \cancel{d\theta}}{\sqrt{r^2 + (z_0 - z)^2}^3} dr dz \Big|_0^{2\pi} = 0$$

$$g_z = -G \frac{M}{V} \int_{0}^{l} \int_{0}^{2\pi} \int_{0}^{a} \frac{r(z_0 - z)}{\sqrt{r^2 + (z_0 - z)^2}^3} dr d\theta dz = -2\pi G \frac{M}{V} \int_{0}^{l} \int_{0}^{a} \frac{r(z_0 - z)}{\sqrt{r^2 + (z_0 - z)^2}^3} dr dz = -2\pi G \frac{M}{V} \int_{0}^{l} \frac{-(z_0 - z)}{\sqrt{r^2 + (z_0 - z)^2}} dz \Big|_0^l$$

$$= -2\pi \frac{MG}{V} \int_0^l \frac{-(z_0 - z)}{\sqrt{R^2 + U^2}} + l dz; \quad U = z_0 - z \Rightarrow du = -dz \Rightarrow u_1 = z_0 \quad u_2 = z_0 - l$$

$$\Rightarrow g_z = -2\pi \frac{MG}{V} \left[\int_{z_0}^{l-z_0} \frac{u}{\sqrt{R^2 + U^2}} du + \int_0^l dz \right] = -2\pi \frac{MG}{V} [\sqrt{U^2 + R^2}] \Big|_{z_0}^{l-z_0} + [z] \Big|_0^l$$

$$= -2\pi \frac{MG}{V} [l + \sqrt{(l-z_0)^2 + R^2} - \sqrt{z_0^2 + R^2}]$$

$$\Rightarrow \vec{g} = g_x \hat{i} + g_y \hat{j} + g_z \hat{k} = -2\pi \frac{MG}{V} [l + \sqrt{(l-z_0)^2 + R^2} - \sqrt{z_0^2 + R^2}] \hat{k}$$

$$b) \text{ Recuperando la norma del vector anterior} \Rightarrow |\vec{r}| = \sqrt{r^2 + (z_o - z)^2}$$

$$V_g = -G \frac{\int_0^L \int_0^{2\pi} \int_0^R M}{r} \frac{1}{\sqrt{r^2 + (z_o - z)^2}} dr d\phi dz = -G \frac{M}{V} \int_0^L \int_0^{2\pi} \int_0^R \frac{r}{\sqrt{r^2 + (z_o - z)^2}} dr d\phi dz$$

$$\int_0^L \int_0^{2\pi} \int_0^R \frac{r}{\sqrt{r^2 + (z_o - z)^2}} dr d\phi dz = 2\pi \int_0^L \int_0^R \frac{r}{\sqrt{r^2 + (z_o - z)^2}} dr dz = 2\pi \int_0^L \sqrt{r^2 + (z_o - z)^2} \Big|_0^R dz$$

$$= 2\pi \int_0^L \sqrt{R^2 + (z_o - z)^2} - (z_o - z) dz ; u = (z_o - z) \Rightarrow du = -dz \Rightarrow -2\pi \int_{z_o}^{z_o+L} \sqrt{R^2 + u^2} - u du$$

$$= -2\pi \left[\frac{u \sqrt{u^2 + R^2}}{2} + \frac{R^2}{2} \ln(u + \sqrt{u^2 + R^2}) - \frac{|u|^2}{2} \right] \Big|_{z_o}^{z_o+L}$$

$$= -2\pi \left\{ \frac{(z_o - L) \sqrt{(z_o - L)^2 + R^2}}{2} + \frac{R^2}{2} \ln[(z_o - L) + \sqrt{(z_o - L)^2 + R^2}] - \frac{(z_o - L)^2}{2} - \frac{z_o \sqrt{z_o^2 + R^2}}{2} - \frac{R^2}{2} \ln(z_o + \sqrt{z_o^2 + R^2}) + \frac{z_o^2}{2} \right\}$$

$$\Rightarrow V_g = \frac{-GM}{V} + 2\pi \left\{ \frac{(z_o - L) \sqrt{(z_o - L)^2 + R^2}}{2} + \frac{R^2}{2} \ln[(z_o - L) + \sqrt{(z_o - L)^2 + R^2}] - \frac{(z_o - L)^2}{2} - \frac{z_o \sqrt{z_o^2 + R^2}}{2} - \frac{R^2}{2} \ln(z_o + \sqrt{z_o^2 + R^2}) + \frac{z_o^2}{2} \right\}$$

$$\Rightarrow \vec{E} = \frac{\partial V_g}{\partial z} = -\frac{2\pi MG}{V} \left\{ \frac{\sqrt{(z_o - L)^2 + R^2}}{2} + \frac{(z_o - L)^2}{2 \sqrt{(z_o - L)^2 + R^2}} + \frac{R^2}{2} \frac{1}{(z_o - L) + \sqrt{(z_o - L)^2 + R^2}} \right\} \hat{z} + \dots$$

$$- \frac{\sqrt{z_o^2 + R^2}}{2} - \frac{z_o^2}{2 \sqrt{z_o^2 + R^2}} - \frac{R^2}{2} \frac{1 + \frac{z_o}{\sqrt{z_o^2 + R^2}}}{z_o + \sqrt{z_o^2 + R^2}} + z_o \hat{z}$$

$$= -\frac{2\pi MG}{V} \left\{ \frac{\sqrt{(z_o - L)^2 + R^2}}{2} + \frac{(z_o - L)^2}{2 \sqrt{(z_o - L)^2 + R^2}} + \frac{R^2}{2} \frac{1 + \frac{(z_o - L)}{\sqrt{(z_o - L)^2 + R^2}}}{(z_o - L) + \sqrt{(z_o - L)^2 + R^2}} - \frac{\sqrt{z_o^2 + R^2}}{2} - \frac{z_o^2}{2 \sqrt{z_o^2 + R^2}} - \frac{R^2}{2} \frac{1 + \frac{z_o}{\sqrt{z_o^2 + R^2}}}{z_o + \sqrt{z_o^2 + R^2}} + L \right\} \hat{z}$$

$$= -\frac{2\pi MG}{V} \left\{ \frac{\sqrt{(z_o - L)^2 + R^2}}{2} + \frac{(z_o - L)^2}{2 \sqrt{(z_o - L)^2 + R^2}} + \frac{R^2}{2} \frac{1}{\sqrt{(z_o - L)^2 + R^2}} - \frac{\sqrt{z_o^2 + R^2}}{2} - \frac{z_o^2}{2 \sqrt{z_o^2 + R^2}} - \frac{R^2}{2} \frac{1}{\sqrt{z_o^2 + R^2}} + L \right\} \hat{z}$$

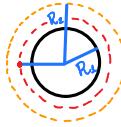
$$= -\frac{2\pi MG}{V} \left\{ \frac{2(z_o - L)^2 + R^2}{2 \sqrt{(z_o - L)^2 + R^2}} + \frac{R^2}{2 \sqrt{(z_o - L)^2 + R^2}} - \frac{2z_o^2 + R^2}{2 \sqrt{z_o^2 + R^2}} - \frac{R^2}{2 \sqrt{z_o^2 + R^2}} + L \right\} \hat{z}$$

$$= -\frac{2\pi MG}{V} \left\{ \frac{2(z_o - L)^2 + 2R^2}{2 \sqrt{(z_o - L)^2 + R^2}} - \frac{2z_o^2 + 2R^2}{2 \sqrt{z_o^2 + R^2}} + L \right\} \hat{z} = -\frac{2\pi MG}{V} \left\{ \frac{(z_o - L)^2 + R^2}{\sqrt{(z_o - L)^2 + R^2}} - \frac{z_o^2 + R^2}{\sqrt{z_o^2 + R^2}} + L \right\} \hat{z} \quad \text{Racionalizando}$$

$$\Rightarrow \vec{E} = -2\pi \frac{MG}{V} [L + \sqrt{(L - z_o)^2 + R^2} - \sqrt{z_o^2 + R^2}] \hat{z}$$

Problema 4. (Thornton 5.13)

- 5-13. A planet of density ρ_1 (spherical core, radius R_1) with a thick spherical cloud of dust (density ρ_2 , radius R_2) is discovered. What is the force on a particle of mass m placed within the dust cloud?



La partícula de masa m se encuentra $R_1 < R_o < R_2$

- * La fuerza que ejerce el planeta es de la forma de una partícula con masa
- * La parte de la nube que se haya por dentro de R_o también posee forma de una partícula con masa
- * La fuerza que provoca el cascarón estérico de nube es cero para cualquier objeto en su interior

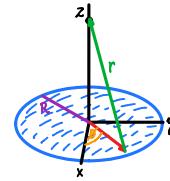
$$\Rightarrow \vec{F}_T = \vec{F}_{\text{planeta}} + \vec{F}_{\text{nube interior}} + \cancel{\vec{F}_{\text{nube exterior}}} = -\frac{GM_{\text{planeta}}m}{R_o^2} \hat{e}_r - \frac{GM_{\text{nube interior}}m}{R_o^2} \hat{e}_r = -\frac{Gm}{R_o^2} (M_p + M_{\text{nub}}) \hat{e}_r$$

$$M_p = \rho_1 V_p = \frac{4}{3} \pi R_1^3 \rho_1 \quad y \quad M_{\text{nub}} = \rho_2 V_{\text{nub}} = \frac{4}{3} \pi (R_o^3 - R_1^3) \rho_2 \quad \Rightarrow \vec{F}_T = -\frac{Gm}{R_o^2} \frac{4}{3} \pi (R_1^3 \rho_1 + R_o^3 \rho_2 - R_1^3 \rho_2) \hat{e}_r //$$

Problema 5. (Thornton 5.20)

- 5-20. A thin disk of mass M and radius R lies in the (x, y) plane with the z -axis passing through the center of the disk. Calculate the gravitational potential $\Phi(z)$ and the gravitational field $\mathbf{g}(z) = -\nabla \Phi(z) = -\hat{k} d\Phi(z)/dz$ on the z -axis.

$$V_g = -G \int \frac{dm}{r} ; dm = \sigma dA = \frac{M}{\pi R^2} r dr d\phi \quad y \quad r = \sqrt{r^2 + z^2}$$



$$\Rightarrow V_g = -G \frac{M}{\pi R^2} \int_0^{2\pi} \int_0^R \frac{r}{\sqrt{r^2 + z^2}} dr d\phi \quad \boxed{\int_0^{2\pi} \int_0^R \frac{r}{\sqrt{r^2 + z^2}} dr d\phi = 2\pi \int_0^R \frac{r}{\sqrt{r^2 + z^2}} dr = 2\pi \sqrt{r^2 + z^2} \Big|_0^R = 2\pi (\sqrt{R^2 + z^2} - z)}$$

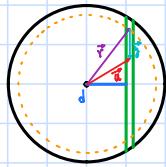
$$\Rightarrow V_g = -G \frac{M}{\pi R^2} 2\pi (\sqrt{R^2 + z^2} - z) = -\frac{2GM}{R^2} (\sqrt{R^2 + z^2} - z)$$

$$\Rightarrow \vec{g} = -\frac{\partial V_g}{\partial z} \hat{k} = +\frac{2GM}{R^2} \left(\frac{z}{\sqrt{R^2 + z^2}} - 1 \right) \hat{k} //$$

Problema 6.

Considere un túnel recto que atraviesa la Tierra y que en su punto más cercano al centro de la misma, se localiza a una distancia d . Determine la energía potencial de una masa puntual m que pasa por él, con respecto a la distancia recorrida de uno de los extremos del túnel.

$$M = \frac{4\pi M_{\text{Tierra}}}{3} \frac{4\pi r^3}{R_{\text{Tierra}}^3} = M_{\text{Tierra}} r^3$$



$$\vec{r} = r \hat{e}_r \quad y \quad \vec{u} = \sqrt{r^2 + d^2} \hat{e}_r$$

$$\vec{r} = \vec{u} - \vec{d} \star$$

$$\oint \vec{g} \cdot d\vec{A} = -4\pi G M_{\text{menc}} \Rightarrow g \iint dA = -4\pi G \frac{M_{\text{menc}}}{R_{\text{Tierra}}^3} \Rightarrow g 4\pi r^2 = -4\pi G \frac{M_{\text{Tierra}} r^3}{R_{\text{Tierra}}^3} \Rightarrow g = -\frac{M_{\text{Tierra}} r}{R_{\text{Tierra}}^3}$$

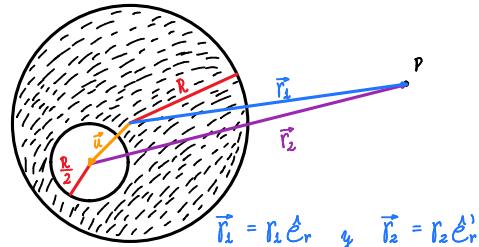
$$\Rightarrow \vec{g} = -\frac{M_{\text{Tierra}} r}{R_{\text{Tierra}}^3} \hat{e}_r \leftarrow \text{Campo respecto al centro de la esfera} \Rightarrow \vec{g} = -\frac{M_{\text{Tierra}} r}{R_{\text{Tierra}}^3} \frac{\vec{r}}{r} = -\frac{M_{\text{Tierra}}}{R_{\text{Tierra}}^3} (\sqrt{r^2 + d^2} \hat{e}_r - \vec{d})$$

Problema 7.

Una esfera sólida de masa uniforme M y radio R , tiene una cavidad esférica de radio $R/2$, de tal forma que su centro está colocado a la mitad del radio de la esfera. Determine el campo gravitacional en un punto P , localizado a una distancia radial $r > R$.

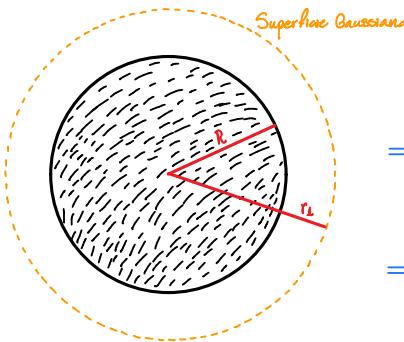
$$\text{Densidad de la esfera sólida } \rho_m = \frac{3M}{4\pi R^3}$$

$$\text{Masa de la cavidad esférica: } m_{\text{cav}} = -\rho_m V_c = \frac{3M}{4\pi R^3} \cdot \frac{4\pi}{3} \cdot \frac{R^3}{8} = \frac{M}{8}$$



$$\vec{r}_1 = r_1 \hat{e}_r \quad y \quad \vec{r}_2 = r_2 \hat{e}_r \\ \vec{u} = \frac{R}{2} \hat{e}_r \Rightarrow \vec{r}_2 = \vec{r}_1 + \vec{u}^*$$

* Considerando primero la esfera sólida

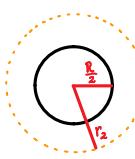


$$\oint \vec{g}_1 \cdot d\vec{A} = -4\pi G m_{\text{enc}} ; \quad m_{\text{enc}} = M \quad y \quad \vec{g}_1 = g_1 \hat{e}_r$$

$$\Rightarrow g_1 \iint dA = -4\pi GM \Rightarrow g_1 4\pi r_1^2 = -4\pi GM \Rightarrow g_1 = -\frac{GM}{r_1^2}$$

$$\Rightarrow \vec{g}_1 = -\frac{GM}{r_1^2} \hat{e}_r$$

* Considerando la cavidad



$$\oint \vec{g}_2 \cdot d\vec{A} = -4\pi G m_{\text{enc}} ; \quad m_{\text{enc}} = \frac{M}{8} \quad y \quad \vec{g}_2 = g_2 \hat{e}_r$$

$$\Rightarrow g_2 \iint dA = +4\pi G \frac{M}{8} \Rightarrow g_2 4\pi r_2^2 = +4\pi G \frac{M}{8} \Rightarrow g_2 = \frac{GM}{8r_2^2}$$

$$\Rightarrow \vec{g}_2 = \frac{GM}{8r_2^2} \hat{e}_r = \frac{GM}{8r_2^2} \frac{\vec{r}_2}{r_2} = \frac{GM}{8} \frac{\vec{r}_1 + \vec{u}}{\sqrt{r_1^2 + u^2}} = \frac{GM}{8} \frac{r_1 + \frac{R}{2}}{\sqrt{r_1^2 + \frac{R^2}{4}}} \hat{e}_r = \frac{GM}{8} \frac{1}{\sqrt{r_1^2 + \frac{R^2}{4}}} \hat{e}_r$$

* El campo gravitatorio total es la suma vectorial de ambos campos por el principio de superposición.

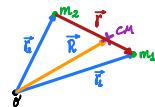
$$\Rightarrow \vec{g} = \vec{g}_1 + \vec{g}_2 = -\frac{GM}{r_1^2} \hat{e}_r + \frac{GM}{8} \frac{1}{\sqrt{r_1^2 + \frac{R^2}{4}}} \hat{e}_r = -GM \left(\frac{1}{r_1^2} - \frac{1}{8\sqrt{r_1^2 + \frac{R^2}{4}}} \right) \hat{e}_r$$

Realizar el problema anterior con la integral de campo

Movimiento de 2 cuerpos

Problema 1. (Thornton 8.1)

- 8-1. In section 8.2, we showed that the motion of two bodies interacting only with each other by central forces could be reduced to an equivalent one-body problem. Show by explicit calculation that such a reduction is also possible for bodies moving in an external uniform gravitational field.



$$\vec{r} = \vec{r}_1 - \vec{r}_2^* \quad y \quad \vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$$

Tanto m_1 como m_2 sienten un potencial externo uniforme

$$U_g = m_2 V^* \rightarrow -\vec{F}_g^* r_2 = -m_2 \alpha r_2 \hat{e}_r$$

Escribiendo \vec{r}_1 y \vec{r}_2 en términos de \vec{r} y \vec{R} :

$$\begin{cases} m_1 \vec{r}_1 + m_2 \vec{r}_2 = (m_1 + m_2) \vec{R} \\ \vec{r}_1 - \vec{r}_2 = \vec{r} \end{cases} \Rightarrow \begin{cases} \vec{r}_1 = \frac{m_2}{m_1 + m_2} \vec{r} + \vec{R} \\ \vec{r}_2 = \frac{-m_1}{m_1 + m_2} \vec{r} + \vec{R} \end{cases}$$

El lagrangiano del sistema $L = \frac{1}{2} m_1 \dot{\vec{r}}_1^2 + \frac{1}{2} m_2 \dot{\vec{r}}_2^2 - U(r) - U_g^*(r) - U_{gg}(r)$

$$\Rightarrow L = \frac{1}{2} m_1 \left(\frac{m_2}{m_1 + m_2} \vec{r} + \vec{R} \right)^2 + \frac{1}{2} m_2 \left(\frac{-m_1}{m_1 + m_2} \vec{r} + \vec{R} \right)^2 - U(r) + m_1 \alpha \left(\frac{m_2}{m_1 + m_2} \vec{r} + \vec{R} \right) + m_2 \alpha \left(\frac{-m_1}{m_1 + m_2} \vec{r} + \vec{R} \right)$$

$$= \frac{1}{2} m_1 \left[\left(\frac{m_2}{m_1 + m_2} \right)^2 \vec{r}^2 + 2 \left(\frac{m_2}{m_1 + m_2} \right) \vec{r} \cdot \vec{R} + \vec{R}^2 \right] + \frac{1}{2} m_2 \left[\vec{R}^2 - 2 \left(\frac{m_1}{m_1 + m_2} \right) \vec{R} \cdot \vec{r} + \vec{r}^2 \right] + \dots$$

$$\dots - U(r) + \cancel{\alpha \frac{m_1 m_2}{m_1 + m_2} \vec{r}^2} + \cancel{m_1 \alpha \vec{R}^2} - \cancel{\alpha \frac{m_1 m_2}{m_1 + m_2} \vec{r}^2} + \cancel{m_2 \alpha \vec{R}^2}$$

$$= \frac{1}{2} \left[m_1 \left(\frac{m_2}{m_1 + m_2} \right)^2 \vec{r}^2 + 2 \left(\frac{m_2}{m_1 + m_2} \right) \vec{r} \cdot \vec{R} + m_1 \vec{R}^2 + m_2 \vec{R}^2 - 2 \left(\frac{m_1 m_2}{m_1 + m_2} \right) \vec{R} \cdot \vec{r} + m_2 \left(\frac{m_1}{m_1 + m_2} \right)^2 \vec{r}^2 \right] - U(r) + \alpha (m_1 + m_2) \vec{R}$$

$$= \frac{1}{2} \left[m_1 \left(\frac{m_2}{m_1 + m_2} \right)^2 \vec{r}^2 + m_1 \vec{R}^2 + m_2 \vec{R}^2 + m_2 \left(\frac{m_1}{m_1 + m_2} \right)^2 \vec{r}^2 \right] - U(r) + \alpha (m_1 + m_2) \vec{R}$$

$$= \frac{1}{2} \left[m_1 \left(\frac{m_2}{m_1 + m_2} \right)^2 \vec{r}^2 + (m_1 + m_2) \vec{R}^2 + m_2 \left(\frac{m_1}{m_1 + m_2} \right)^2 \vec{r}^2 \right] - U(r) + \alpha (m_1 + m_2) \vec{R}$$

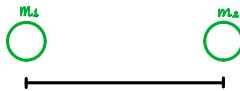
$$= \frac{1}{2} \frac{\vec{r}^2}{(m_1 + m_2)^2} [m_1 m_2^2 + m_2 m_1^2] + \frac{1}{2} (m_1 + m_2) \vec{R}^2 - U(r) - U(r) + \alpha (m_1 + m_2) \vec{R}$$

$$= \frac{1}{2} \frac{\vec{r}^2}{(m_1 + m_2)^2} [m_2 + m_1] + \frac{1}{2} (m_1 + m_2) \vec{R}^2 - U(r) + \alpha (m_1 + m_2) \vec{R} = \frac{1}{2} \mu \vec{r}^2 + \frac{1}{2} (m_1 + m_2) \vec{R}^2 - U(r) + \alpha (m_1 + m_2) \vec{R}$$

Problema 2. (Thornton 8.6)

- 8-6. Two gravitating masses m_1 and m_2 ($m_1 + m_2 = M$) are separated by a distance r_0 and released from rest. Show that when the separation is $r (< r_0)$, the speeds are

$$v_1 = m_2 \sqrt{\frac{2G}{M} \left(\frac{1}{r} - \frac{1}{r_0} \right)}, \quad v_2 = m_1 \sqrt{\frac{2G}{M} \left(\frac{1}{r} - \frac{1}{r_0} \right)}$$



Al inicio solo hay energía potencial gravitatoria:

$$E_0 = -\frac{Gm_1m_2}{r_0^2}$$

Momentum lineal: $\vec{P}_0 = 0$

Al acercarse la energía es:

$$E = \frac{1}{2} m_1 V_1^2 + \frac{1}{2} m_2 V_2^2 - \frac{Gm_1m_2}{r^2}$$

Momentum lineal: $\vec{P} = m_1 V_1 + m_2 V_2$

* Plantearon conservación de la energía y el momentum

$$\Rightarrow -\frac{Gm_1m_2}{r_0^2} = \frac{1}{2} m_1 V_1^2 + \frac{1}{2} m_2 V_2^2 - \frac{Gm_1m_2}{r^2} \quad y \quad 0 = m_1 V_1 + m_2 V_2 \Rightarrow m_1 V_1 = -m_2 V_2 \star$$

Reemplazando V_2 en la conservación de la energía

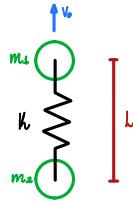
$$\Rightarrow -\frac{Gm_1m_2}{r_0^2} = \frac{1}{2} m_1 V_1^2 + \frac{1}{2} m_2 \frac{m_1^2 V_1^2}{m_2} - \frac{Gm_1m_2}{r^2} \Rightarrow \frac{1}{2} m_1 V_1^2 \left(1 + \frac{m_1}{m_2} \right) = Gm_1m_2 \left(\frac{1}{r^2} - \frac{1}{r_0^2} \right) \Rightarrow V_1^2 = \frac{2Gm_2}{1 + \frac{m_1}{m_2}} \left(\frac{1}{r^2} - \frac{1}{r_0^2} \right)$$

$$\Rightarrow V_1^2 = \frac{2Gm_2^2}{m_1 + m_2} \left(\frac{1}{r^2} - \frac{1}{r_0^2} \right) \Rightarrow V_1 = m_2 \sqrt{\frac{2G}{m_1 + m_2} \left(\frac{1}{r^2} - \frac{1}{r_0^2} \right)} \quad y \text{ con } \star \Rightarrow V_2 = -m_1 \sqrt{\frac{2G}{m_1 + m_2} \left(\frac{1}{r^2} - \frac{1}{r_0^2} \right)} //$$

Problema 3. (Taylor 8.3)

8.3** Two particles of masses m_1 and m_2 are joined by a massless spring of natural length L and force constant k . Initially, m_2 is resting on a table and I am holding m_1 vertically above m_2 at a height L . At time $t = 0$, I project m_1 vertically upward with initial velocity v_0 . Find the positions of the two masses at any subsequent time t (before either mass returns to the table) and describe the motion. [Hints: See Problem 8.2. Assume that v_0 is small enough that the two masses never collide.]

$$y_{cm} = \frac{m_1 y + m_2 y}{m_1 + m_2} \quad y = y_1 - y_2 \Rightarrow \ddot{y}_1 = \ddot{y}_{cm} + \frac{m_2}{m_1 + m_2} \ddot{y} \quad \ddot{y}_2 = \ddot{y}_{cm} - \frac{m_1}{m_1 + m_2} \ddot{y}$$



$$\begin{aligned} \text{Energía cinética: } T &= \frac{1}{2} m_1 \dot{y}_1^2 + \frac{1}{2} m_2 \dot{y}_2^2 = \frac{1}{2} m_1 \left(\ddot{y}_{cm} + \frac{m_2}{m_1 + m_2} \ddot{y} \right)^2 + \frac{1}{2} m_2 \left(\ddot{y}_{cm} - \frac{m_1}{m_1 + m_2} \ddot{y} \right)^2 \\ &= \frac{1}{2} m_1 \left[\ddot{y}_{cm}^2 + 2 \left(\frac{m_2}{m_1 + m_2} \right) \ddot{y}_{cm} \ddot{y} + \left(\frac{m_2}{m_1 + m_2} \right)^2 \ddot{y}^2 \right] + \frac{1}{2} m_2 \left[\ddot{y}_{cm}^2 - 2 \left(\frac{m_1}{m_1 + m_2} \right) \ddot{y}_{cm} \ddot{y} + \left(\frac{m_1}{m_1 + m_2} \right)^2 \ddot{y}^2 \right] \\ &= \frac{1}{2} \left[m_1 \ddot{y}_{cm}^2 + 2 m_1 \left(\frac{m_2}{m_1 + m_2} \right) \ddot{y}_{cm} \ddot{y} + m_1 \left(\frac{m_2}{m_1 + m_2} \right)^2 \ddot{y}^2 + m_2 \ddot{y}_{cm}^2 - 2 m_2 \left(\frac{m_1}{m_1 + m_2} \right) \ddot{y}_{cm} \ddot{y} + m_2 \left(\frac{m_1}{m_1 + m_2} \right)^2 \ddot{y}^2 \right] \\ &= \frac{1}{2} (m_1 + m_2) \ddot{y}_{cm}^2 + \frac{1}{2} \left[m_1 \left(\frac{m_2}{m_1 + m_2} \right)^2 \ddot{y}^2 + m_2 \left(\frac{m_1}{m_1 + m_2} \right)^2 \ddot{y}^2 \right] = \frac{1}{2} (m_1 + m_2) \ddot{y}_{cm}^2 + \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} \ddot{y}^2 \end{aligned}$$

$$\text{Energía potencial: } U = m_1 g y_1 + m_2 g y_2 + \frac{1}{2} k (y - L)^2 = (m_1 + m_2) g y_{cm} + \frac{1}{2} k (y - L)^2$$

$$\text{Lagrangiano del sistema: } \mathcal{L} = T - U = \frac{1}{2} (m_1 + m_2) \ddot{y}_{cm}^2 + \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} \ddot{y}^2 - (m_1 + m_2) g y_{cm} - \frac{1}{2} k (y - L)^2$$

Hay que operar $\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_1} - \frac{\partial \mathcal{L}}{\partial q_1} = 0$ para y_{cm} y y .

$$\text{Para } y_{cm}: * \frac{\partial \mathcal{L}}{\partial y_{cm}} = -(m_1 + m_2) g \quad * \frac{\partial \mathcal{L}}{\partial \dot{y}_{cm}} = (m_1 + m_2) \ddot{y}_{cm} \Rightarrow \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{y}_{cm}} = (m_1 + m_2) \ddot{y}_{cm} \Rightarrow \ddot{y}_{cm} + g = 0 *$$

$$\text{Para } y: * \frac{\partial \mathcal{L}}{\partial y} = -k(y - L) \quad * \frac{\partial \mathcal{L}}{\partial \dot{y}} = \frac{m_1 m_2}{m_1 + m_2} \ddot{y} \Rightarrow \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{y}} = \frac{m_1 m_2}{m_1 + m_2} \ddot{y} \Rightarrow \frac{m_1 m_2}{m_1 + m_2} \ddot{y} + k(y - L) = 0 *$$

$$* y_{cm} = y_{cm0} + \ddot{y}_{cm} t - \frac{1}{2} g t^2 \Rightarrow y_{cm} = \frac{m_1 L}{m_1 + m_2} + \frac{m_2 v_0}{m_1 + m_2} t - \frac{1}{2} g t^2 \quad \left\{ \begin{array}{l} \ddot{y}_1 = \ddot{y}_{cm} + \frac{m_2}{m_1 + m_2} \ddot{y} \\ \ddot{y}_2 = \ddot{y}_{cm} - \frac{m_1}{m_1 + m_2} \ddot{y} \end{array} \right.$$

$$* y = A \cos(\sqrt{\frac{k(m_1 + m_2)}{m_1 m_2}} t - \delta) + L \xrightarrow{A = \frac{v_0}{\omega}, \delta = \frac{\pi}{2}} y = v_0 \sqrt{\frac{m_1 + m_2}{k m_1 m_2}} \sin\left(\sqrt{\frac{k(m_1 + m_2)}{m_1 m_2}} t\right) \quad \left\{ \begin{array}{l} \ddot{y}_1 = \ddot{y}_{cm} + \frac{m_2}{m_1 + m_2} \ddot{y} \\ \ddot{y}_2 = \ddot{y}_{cm} - \frac{m_1}{m_1 + m_2} \ddot{y} \end{array} \right.$$

$$\Rightarrow \ddot{y}_1 = \frac{m_1 L}{m_1 + m_2} + \frac{m_2 v_0}{m_1 + m_2} t - \frac{1}{2} g t^2 + \frac{m_2}{m_1 + m_2} v_0 \sqrt{\frac{m_1 + m_2}{k m_1 m_2}} \sin\left(\sqrt{\frac{k(m_1 + m_2)}{m_1 m_2}} t\right) \quad y \text{ similar para } \ddot{y}_2 //$$

Problema 4. (Taylor 8.10)

8.10 ** Two particles of equal masses $m_1 = m_2$ move on a frictionless horizontal surface in the vicinity of a fixed force center, with potential energies $U_1 = \frac{1}{2}kr_1^2$ and $U_2 = \frac{1}{2}kr_2^2$. In addition, they interact with each other via a potential energy $U_{12} = \frac{1}{2}\alpha kr^2$, where r is the distance between them and α and k are positive constants. (a) Find the Lagrangian in terms of the CM position \mathbf{R} and the relative position $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$. (b) Write down and solve the Lagrange equations for the CM and relative coordinates X, Y and x, y . Describe the motion.

Problema 5.

Una partícula de masa m , se mueve en un potencial central:

$$U(r) = -U_0 \frac{r_0}{3r^3},$$

donde U_0 y r_0 son constantes positivas.

- a. Encuentre la fuerza que experimenta la partícula.
- b. Dada la magnitud del momentum angular L , encuentre el valor máximo del potencial efectivo.
- c. Describa el movimiento de la partícula. Realice los diagramas necesarios.

PART

IX

Campos de Fuerzas Centrales

SUBSECTION 9.9

Problemas resueltos

Movimiento en campos de fuerzas centrales

Demostración del problema de Kepler

$$\Rightarrow U_{ext} = -\frac{\mu}{r} + \frac{L^2}{2\mu r^2}$$

$$\theta = \int \frac{\frac{L}{r^2}}{\sqrt{2\mu(E - U_g - \frac{L^2}{2\mu r^2})}} dr + C$$

$$\Rightarrow \theta = \int \frac{\frac{L}{r^2}}{\sqrt{2\mu(E + \frac{\mu}{r} - \frac{L^2}{2\mu r^2})}} dr + C \quad u = \frac{L}{r} \Rightarrow du = -\frac{L}{r^2} dr$$

$$\Rightarrow \theta + \frac{\pi}{2} = - \int \frac{1}{\sqrt{2\mu(E + \frac{\mu}{r} - \frac{1}{2\mu L^2}))}} du = \frac{1}{\sqrt{2\mu}} \cdot \frac{-1}{\sqrt{+\frac{1}{2\mu}}} \cdot \sin^{-1} \left[\frac{2 \cdot \frac{-1}{\sqrt{2\mu}} u + \frac{\mu}{L}}{\sqrt{\frac{\mu^2}{L^2} - \frac{1}{2\mu} \cdot E}} \right] = - \sin^{-1} \left[\frac{-\frac{(L\mu + \mu)L}{\mu L}}{\sqrt{\frac{\mu^2 \mu + 2E L^2}{\mu L^2}}} \right]$$

$$\Rightarrow \sin(-\theta - \frac{\pi}{2}) = \sin[-(\theta + \frac{\pi}{2})] = -\sin(\theta + \frac{\pi}{2}) = -\cos(\theta) = \frac{-\frac{(L\mu + \mu)L}{\mu L}}{\sqrt{\frac{\mu^2 \mu + 2E L^2}{\mu L^2}}} = \frac{-\cancel{\mu} \cancel{L} \left(\frac{L\mu + \mu}{\mu} - 1 \right)}{\cancel{\mu} \cancel{L}} \cdot \sqrt{1 + \frac{2E L^2}{\mu^2 \mu}} = \frac{-\left(\frac{L\mu + \mu}{\mu} - 1 \right)}{\sqrt{1 + \frac{2E L^2}{\mu^2 \mu}}}$$

$$\Rightarrow +\cos(\theta) = \frac{\left(\frac{L\mu + \mu}{\mu} - 1 \right)}{\sqrt{1 + \frac{2E L^2}{\mu^2 \mu}}} \Rightarrow \sqrt{1 + \frac{2E L^2}{\mu^2 \mu}} \cos(\theta) + 1 = \frac{L}{\mu} u \Rightarrow \sqrt{1 + \frac{2E L^2}{\mu^2 \mu}} \cos(\theta) + 1 = \frac{L^2}{\mu^2} \frac{1}{r}$$

$$\Rightarrow E \cos(\theta) + 1 = \frac{\alpha}{r} \quad \text{con} \quad E = \sqrt{1 + \frac{2E L^2}{\mu^2 \mu}} \quad y \quad \alpha = \frac{L^2}{\mu^2} \quad //$$

Problema 1. (Thornton 8.17)

8-17. A particle moves in an elliptical orbit in an inverse-square-law central-force field. If the ratio of the maximum angular velocity to the minimum angular velocity of the particle in its orbit is n , then show that the eccentricity of the orbit is

$$\epsilon = \frac{\sqrt{n} - 1}{\sqrt{n} + 1}$$

$$\alpha = \frac{l^2}{\mu h} \quad \epsilon = \sqrt{\frac{2El^2}{\mu h^2} + 1}$$

Ecuación de la órbita para fuerzas centrales de cuadrado inverso: $\frac{dr}{r} = 1 + \epsilon \cos \theta \Rightarrow r = \frac{\alpha}{1 + \epsilon \cos \theta}$

Como la órbita es elíptica: $\dot{r} \neq 0 \Rightarrow r_{\max} = \frac{\alpha}{1 - \epsilon} \quad y \quad r_{\min} = \frac{\alpha}{1 + \epsilon}$

El momento angular es constante y dado por: $l = \mu r^2 \dot{\theta}$ y se según el enunciado se cumple $\frac{\dot{\theta}_{\max}}{\dot{\theta}_{\min}} = n$

$$\Rightarrow l = \cancel{\mu r_{\min}^2 \dot{\theta}_{\max}} = \cancel{\mu r_{\max}^2 \dot{\theta}_{\min}} \Rightarrow r_{\min}^2 \dot{\theta}_{\max} = r_{\max}^2 \dot{\theta}_{\min} \Rightarrow \frac{\dot{\theta}_{\max}}{\dot{\theta}_{\min}} = \frac{r_{\max}^2}{r_{\min}^2} = n \Rightarrow n = \frac{\left(\frac{\alpha}{1 - \epsilon}\right)^2}{\left(\frac{\alpha}{1 + \epsilon}\right)^2} = \frac{(1 + \epsilon)^2}{(1 - \epsilon)^2}$$

$$\Rightarrow \sqrt{n} = \frac{1 + \epsilon}{1 - \epsilon} \Rightarrow \sqrt{n} - \epsilon \sqrt{n} = 1 + \epsilon \Rightarrow \sqrt{n} - 1 = \epsilon (\sqrt{n} + 1) \Rightarrow \epsilon = \frac{\sqrt{n} - 1}{\sqrt{n} + 1} //$$

Problema 2. (Thornton 8.22)

8-22. Discuss the motion of a particle moving in an attractive central-force field described by $F(r) = -k/r^3$. Sketch some of the orbits for different values of the total energy. Can a circular orbit be stable in such a force field?

$$F = -\frac{dU}{dr} = \frac{-h}{r^3} \Rightarrow U_g = \int_{r=\infty}^r \frac{h}{r^3} dr = \frac{-h}{2r^2} \Rightarrow U_g = \frac{-h}{2r^2} \Rightarrow U_{\text{eff}} = \frac{-h}{2r^2} + \frac{l^2}{2\mu r^2}$$

La ecuación de la órbita es el resultado de: $\theta = \int \frac{\frac{l}{r^2}}{\sqrt{2\mu(E - U_g - \frac{l^2}{2\mu r^2})}} dr + \delta$

$$\Rightarrow \theta = \int \frac{\frac{l}{r^2}}{\sqrt{2\mu \left[E + \frac{1}{2r^2} \left(h - \frac{l^2}{\mu} \right) \right]}} dr + \delta ; \text{ Realizando el cambio de variable: } u = \frac{l}{r} \Rightarrow du = -\frac{l}{r^2} dr$$

$$\Rightarrow \theta = \int \frac{-\frac{1}{u}}{\sqrt{2\mu \left[E + \frac{u^2}{2} \left(\frac{h}{u^2} - \frac{1}{\mu} \right) \right]}} du + \delta = \frac{-1}{\sqrt{2\mu}} \int \frac{1}{\sqrt{E + \frac{u^2}{2} \left(\frac{h}{u^2} - \frac{1}{\mu} \right)}} du + \delta \quad \begin{array}{l} \text{Hay que resolver esto según el} \\ \text{comportamiento de } U_{\text{eff}} \text{ para} \\ \text{obtener la ecuación de la trayectoria} \end{array}$$

Caso 1: $h/\mu - l^2 = 0 \Rightarrow U_{\text{eff}} = 0$

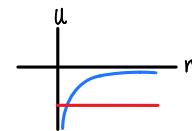
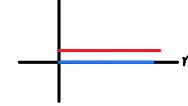
$$\Rightarrow \theta + \delta = -\frac{1}{\sqrt{2\mu}} \int \frac{1}{\sqrt{E}} du = \frac{-1}{\sqrt{2\mu E}} u \Rightarrow -\sqrt{2\mu E}(\theta + \delta) = \frac{l}{r} \Rightarrow \frac{l}{r} = -\sqrt{2\mu E} \theta + \phi \quad * \text{Para valores de } E > 0, \text{ la trayectoria} \\ \text{describe "hélices" planas y para } E = 0, \\ \text{no hay movimiento}$$

Caso 2: $\frac{h/\mu - l^2}{\mu l^2} = -\frac{B^2}{\mu} < 0 \Rightarrow U_{\text{eff}} = -\frac{B^2}{2\mu} U^2$

$$\Rightarrow \theta + \delta - \frac{\pi}{B^2} = \frac{-1}{\sqrt{2\mu}} \int \frac{1}{\sqrt{E - \frac{B^2}{2\mu} U^2}} du = \frac{-1}{\sqrt{2\mu}} \int \frac{1}{\sqrt{\frac{B^2}{2\mu} - \frac{B^2}{2\mu} U^2}} du = \frac{-1}{B} \sin^{-1} \left(\frac{U}{\sqrt{\frac{B^2}{2\mu}}} \right) = \frac{-1}{B} \sin^{-1} \left(\frac{u}{\sqrt{2\mu E}} \right)$$

$$\Rightarrow \sin \left[-\beta(\theta + \delta) + \frac{\pi}{2} \right] = \cos \left[-\beta(\theta + \delta) \right] = \cos(\beta\theta + \phi) = \frac{uB}{\sqrt{2\mu E}} \Rightarrow u = \frac{l}{r} = \frac{\sqrt{2\mu E}}{B} \cos(\beta\theta + \phi)$$

$$\Rightarrow \frac{l}{r} = \frac{\sqrt{2\mu E}}{Bl} \cos(\beta\theta + \phi) \quad * \text{Se tiene un movimiento no limitado para } E < 0$$



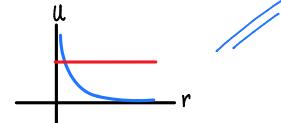
$$\text{Caso 3: } \frac{h\mu - l^2}{\mu r^2} = \frac{G^2}{\mu} < 0 \Rightarrow U_{\text{eff}} = \frac{G^2 U^2}{2\mu} ; G = g$$

$$\Rightarrow \theta + \delta - \frac{l^2}{G^2} = \frac{-1}{\sqrt{2^{\mu E}}} \int \frac{1}{\sqrt{\frac{E + G^2 U^2}{2^{\mu E}}}} du = \frac{-1}{\sqrt{2^{\mu E}}} \int \frac{1}{\sqrt{\frac{G^2}{2^{\mu E}}}} \frac{1}{\sqrt{\frac{2^{\mu E} + U^2}{G^2}}} du = \frac{-1}{\sqrt{G}} \operatorname{sech}^{-1} \left(\frac{u}{\sqrt{\frac{2^{\mu E}}{G^2}}} \right)$$

$$\Rightarrow \operatorname{sech}^{-1} \left[\pm \sqrt{G(\theta + \delta) + \frac{l^2}{2^{\mu E}}} \right] = \pm \cosh^{-1} \left(\frac{u}{\sqrt{G}} \right) - \theta = \pm \cosh^{-1} \left(\frac{u}{\sqrt{2^{\mu E}}} \right) = \frac{uG}{\sqrt{2^{\mu E}}} \Rightarrow \cosh \left(\frac{u}{\sqrt{2^{\mu E}}} \right) = \frac{u}{\sqrt{2^{\mu E}}}$$

$$\Rightarrow u = \frac{l}{r} = \sqrt{2^{\mu E}} \cosh \left(\frac{u}{\sqrt{2^{\mu E}}} \right) \cosh \left(\frac{u}{\sqrt{2^{\mu E}}} \right) + \theta * \text{Se tiene un movimiento cerrado alrededor del centro de fuerza}$$

Estabilidad de órbita circular



Para que haya una órbita circular $\dot{r} = 0 = \ddot{r}$, además puede ser estable o inestable según la función U_{eff}

$$U_{\text{eff}} = \frac{-h}{2r^2} + \frac{l^2}{2^{\mu} r^2} \quad \text{La órbita circular es estable si cumple: } \frac{dU_{\text{eff}}}{dr} \Big|_{r=r_{\text{eq}}} = 0 \quad \text{y} \quad \frac{d^2U_{\text{eff}}}{dr^2} \Big|_{r=r_{\text{eq}}} > 0$$

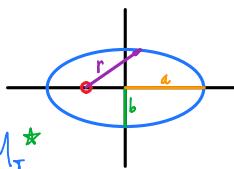
$$\frac{dU_{\text{eff}}}{dr} \Big|_{r=r_{\text{eq}}} = + \frac{h}{r^3} - \frac{l^2}{\mu r^3} \Big|_{r=r_{\text{eq}}} = \frac{h}{r_{\text{eq}}^3} - \frac{l^2}{\mu r_{\text{eq}}^3} \Rightarrow \frac{dU_{\text{eff}}}{dr} \Big|_{r=r_{\text{eq}}} = 0 \Rightarrow \frac{h}{r_{\text{eq}}^3} - \frac{l^2}{\mu r_{\text{eq}}^3} = \frac{l}{r_{\text{eq}}^3} \left(h - \frac{l^2}{\mu} \right) = 0 \Rightarrow h = \frac{l^2}{\mu}$$

$$\frac{d^2U_{\text{eff}}}{dr^2} \Big|_{r=r_{\text{eq}}} = -\frac{3h}{r_{\text{eq}}^4} + \frac{3l^2}{\mu r_{\text{eq}}^4} = -\frac{3h}{r_{\text{eq}}^4} + \frac{3l^2}{\mu r_{\text{eq}}^4} \Rightarrow \frac{d^2U_{\text{eff}}}{dr^2} \Big|_{r=r_{\text{eq}}} > 0 \Rightarrow -\frac{3h}{r_{\text{eq}}^4} + \frac{3l^2}{\mu r_{\text{eq}}^4} = -\frac{3}{r_{\text{eq}}^4} \left(h - \frac{l^2}{\mu} \right) > 0 \Rightarrow h < \frac{l^2}{\mu}$$

Como ambas propiedades no se pueden cumplir simultáneamente, la órbita circular es inestable

Otra forma de solucionar el problema anterior

8.18 ** An earth satellite is observed at perigee to be 250 km above the earth's surface and traveling about 8500 m/s. Find the eccentricity of its orbit and its height above the earth at apogee. [Hint: The earth's radius is $R_e \approx 6.4 \times 10^6$ m. You will also need to know GM_e , but you can find this if you remember that $GM_e/R_e^2 = g$.]



Se desconoce la masa m del satélite, pero es posible asumir $m \ll M_e$ *

$$\text{Masa reducida } \mu = \frac{M_e m}{M_e + m} \approx m$$

Para la fuerza gravitacional se tiene: $\mathbf{F} = GM_e m$

$$\text{Según las consideraciones del problema de 2 cuerpos: } e = \sqrt{\frac{2El^2}{\mu h^2} + 1} \quad y \quad \alpha = \frac{l^2}{\mu h}$$

$$\text{Los radios de órbita elíptica están dados por: } r_{\min} = a(1 - e) = \frac{\alpha}{1 + e} \quad y \quad r_{\max} = a(1 + e) = \frac{\alpha}{1 - e}; \quad b = \sqrt{a \alpha}$$

$$r_{\min} = h_{\min} + R_e \quad y \quad \dot{r}_{\min} = V_{r_{\min}}$$

$$\text{Tomando en cuenta lo anterior: } L = \mu r^2 \dot{\theta} = mr^2 \dot{\theta} \quad y \quad \alpha = \frac{l^2}{\mu h} = \frac{mr^2 \dot{\theta}^2}{\mu GM_e} = \frac{r^2 \dot{\theta}^2}{GM_e} = \frac{r^2 V^2}{GM_e} \Rightarrow \alpha \approx \frac{r_{\min}^2 V_{r_{\min}}^2}{GM_e}$$

$$r_{\min} = \frac{\alpha}{1 + e} \approx \frac{\frac{r_{\min}^2 V_{r_{\min}}^2}{GM_e}}{1 + e} \Rightarrow r_{\min} \approx \frac{r_{\min}^2 V_{r_{\min}}^2}{(1 + e)GM_e} \Rightarrow 1 \approx \frac{r_{\min} V_{r_{\min}}^2}{(1 + e)GM_e} \Rightarrow 1 + e \approx \frac{r_{\min} V_{r_{\min}}^2}{GM_e}$$

$$\Rightarrow e \approx \frac{r_{\min} V_{r_{\min}}^2}{GM_e} - 1 = 0,20 //$$

$$r_{\max} = h_{\max} + R_e \quad y \quad r_{\max} = \frac{\alpha}{1 - e} \Rightarrow h_{\max} + R_e \approx \frac{\frac{r_{\min} V_{r_{\min}}^2}{GM_e}}{1 - \frac{r_{\min} V_{r_{\min}}^2}{GM_e} + 1} = \frac{\frac{r_{\min} V_{r_{\min}}^2}{GM_e}}{\frac{2GM_e - r_{\min} V_{r_{\min}}^2}{GM_e}} = \frac{r_{\min} V_{r_{\min}}^2}{2GM_e - r_{\min} V_{r_{\min}}^2}$$

$$\Rightarrow h_{\max} \approx \frac{r_{\min} V_{r_{\min}}^2}{2GM_e - r_{\min} V_{r_{\min}}^2} - R_e = 9,89 \cdot 10^6 - R_e = 3,49 \cdot 10^6 \text{ m} //$$

Problema 3. (Taylor 8.19)

8.19 ** The height of a satellite at perigee is 300 km above the earth's surface and it is 3000 km at apogee. Find the orbit's eccentricity. If we take the orbit to define the xy plane and the major axis in the x direction with the earth at the origin, what is the satellite's height when it crosses the y axis? [See the hint for Problem 8.18.]

$$r_{\min} = \frac{\alpha}{1 + e} \quad y \quad r_{\max} = \frac{\alpha}{1 - e}$$

$$\Rightarrow \alpha = (1 + e)r_{\min} = (1 - e)r_{\max} \Rightarrow e = \frac{\frac{r_{\max}}{r_{\min}} - 1}{\frac{r_{\max}}{r_{\min}} + 1} = \frac{r_{\max} - r_{\min}}{r_{\max} + r_{\min}} = \frac{h_{\max} + R_e - h_{\min} - R_e}{h_{\max} + R_e + h_{\min} + R_e} = \frac{h_{\max} - h_{\min}}{h_{\max} + h_{\min} + 2R_e} = 0,168 //$$

$$\alpha = (1 + e)r_{\min} = 7,83 \cdot 10^6 \text{ m} \Rightarrow h_{y_0} = \alpha - R_e = 1,43 \cdot 10^6 \text{ m} //$$

Problema 4. (Taylor 8.32)

8.32 • Prove that for circular orbits around a given gravitational force center (such as the sun) the speed of the orbiting body is inversely proportional to the square root of the orbital radius.

$$\vec{F}_g = \frac{GM_s m}{r^2} = \mu a_c \Rightarrow a_c = \frac{v^2}{r} = \frac{GM_s + m}{r^2} \Rightarrow v^2 = \frac{G(M_s + m)}{r} \Rightarrow v = \sqrt{\frac{G(M_s + m)}{r}}$$

$$\begin{aligned} L &= \frac{1}{2} \mu \dot{r}^2 - \frac{GM_s m}{r} + \frac{l^2}{2\mu r^2} \longrightarrow \frac{dL}{dr} = \frac{GM_s m}{r^2} - \frac{l^2}{\mu r^3}, \quad \frac{dL}{dt} = \mu \dot{r} \quad \frac{d}{dt} \frac{dL}{dr} = \mu \ddot{r} \\ &\stackrel{\text{circular}}{\Rightarrow} \mu \ddot{r} - \frac{GM_s m}{r^2} + \frac{l^2}{\mu r^3} = 0 \Rightarrow GM_s m = \frac{l^2}{\mu r^2} \Rightarrow GM_s m = \mu r^3 \dot{\theta}^2 = \frac{M_s m r v^2}{M_s + m} \\ &\Rightarrow GM_s m = \frac{M_s m r v^2}{M_s + m} \Rightarrow v^2 = \frac{G(M_s + m)}{r} \Rightarrow v = \sqrt{\frac{G(M_s + m)}{r}} \end{aligned}$$

Problema 5.

Discutir el movimiento de una partícula que se mueve en un campo de fuerza dado por:

$$\vec{F} = \left(\frac{\alpha}{r^2} + \frac{\beta}{r^3} \right) \hat{r},$$

donde α y β son constantes.

$$\begin{aligned} \frac{d^2}{d\theta^2} \left(\frac{1}{r} \right) + \frac{1}{r} &= -\frac{\mu r^2}{l^2} F(r); \quad U = \frac{1}{r} \Rightarrow \frac{d^2 U}{d\theta^2} + U = -\frac{\mu}{l^2 U^2} F(\frac{1}{U}) \Rightarrow \frac{d^2 U}{d\theta^2} + U = -\frac{\mu}{l^2 U^2} (\alpha U^2 + \beta U^3) \\ &\Rightarrow \frac{d^2 U}{d\theta^2} + U = -\frac{\mu \beta}{l^2} U - \frac{\mu \alpha}{l^2} \Rightarrow \frac{d^2 U}{d\theta^2} + U + \frac{\mu \beta}{l^2} U + \frac{\mu \alpha}{l^2} = 0 \Rightarrow \frac{d^2 U}{d\theta^2} + U \left(1 + \frac{\mu \beta}{l^2} \right) + \frac{\mu \alpha}{l^2} = 0 \end{aligned}$$

$$\text{Caso 1: } 1 + \frac{\mu \beta}{l^2} = 0$$

$$\Rightarrow \frac{d^2 U}{d\theta^2} + U \left(1 + \frac{\mu \beta}{l^2} \right) + \frac{\mu \alpha}{l^2} = 0 \Rightarrow \frac{d^2 U}{d\theta^2} + \frac{\mu \alpha}{l^2} = 0 \Rightarrow U = C_2 + C_1 \theta - \frac{\mu \alpha \theta^2}{2l^2} = \frac{1}{r}$$

$$\text{Caso 2: } 1 + \frac{\mu \beta}{l^2} = k^2 > 0 \quad \text{Órbita recta abierta}$$

$$\Rightarrow \frac{d^2 U}{d\theta^2} + U \left(1 + \frac{\mu \beta}{l^2} \right) + \frac{\mu \alpha}{l^2} = 0 \Rightarrow \frac{d^2 U}{d\theta^2} + U k^2 + \frac{\mu \alpha}{l^2} = 0 \Rightarrow U = C_1 \cos(k\theta + C_2) - \frac{\mu \alpha}{l^2 k^2} = \frac{1}{r}$$

$$\text{Caso 3: } 1 + \frac{\mu \beta}{l^2} = -G^2 < 0 \quad \text{Órbita abierta} \rightarrow \text{hiperbólica}$$

$$\Rightarrow \frac{d^2 U}{d\theta^2} + U \left(1 + \frac{\mu \beta}{l^2} \right) + \frac{\mu \alpha}{l^2} = 0 \Rightarrow \frac{d^2 U}{d\theta^2} - U G^2 + \frac{\mu \alpha}{l^2} = 0 \Rightarrow U = A_1 \cosh(G\theta + C_2) + \frac{\mu \alpha}{l^2 G^2} = \frac{1}{r}$$

Colisiones y Dispersión

SUBSECTION 9.10

Problemas resueltos

Está es una de las secciones más escuetas en cuanto a problemas resueltos, lo cual se espera mejorar con el paso del tiempo y concretar como mínimo los problemas planteados en el documento.

Recomiendo consultar libros externos a [1] y [2] dado que en mi opinión no introducen el tema de dispersión de buena manera.

Dispersión y choque de partículas

Problema 1. (Thornton 9.30)

9-30. A tennis player strikes an incoming tennis ball of mass 60 g as shown in Figure 9-G.

The incoming tennis ball velocity is $v_i = 8 \text{ m/s}$, and the outgoing velocity is $v_f = 16 \text{ m/s}$.

(a) What impulse was given to the tennis ball?

(b) If the collision time was 0.01 s, what was the average force exerted by the tennis racket?

$$a) \vec{p}_i = 8(\sin 45 \hat{i} - \cos 45 \hat{j}) \text{ m} \quad \vec{p}_f = 16(\sin 15 \hat{i} + \cos 15 \hat{j}) \text{ m}$$

$$\Rightarrow \vec{\Delta p} = m(16 \sin 15 - 8 \sin 45) \hat{i} + m(16 \cos 15 + 8 \cos 45) \hat{j}$$

$$= -0,09 \text{ N}\cdot\text{s} \hat{i} + 1,27 \text{ N}\cdot\text{s} \hat{j} //$$

$$b) \vec{F}_{\text{impulso}} = \frac{\vec{\Delta p}}{\Delta t} = -9,09 \text{ N} \hat{i} + 126,7 \text{ N} \hat{j} //$$

Problema 2. (Thornton 9.37)

9-37. When a bullet fires in a gun, the explosion subsides quickly. Suppose the force on the bullet is $F = (360 - 10^7 t^2 \text{ s}^{-2}) \text{ N}$ until the force becomes zero (and remains zero). The mass of the bullet is 3 g.

(a) What impulse acts on the bullet?

(b) What is the muzzle velocity of the gun?

$$a) \Delta P = \int_0^t 360 - 10^7 t^2 dt = 360t - \frac{10^7 t^3}{3} \Rightarrow \Delta P = 360t - \frac{10^7 t^3}{3} \quad \text{donde } t \text{ es el tiempo que dura la colisión,}$$

$$\text{tiempo en que la fuerza impulsiva es diferente de cero.}$$

$$F=0 \Rightarrow 360 - 10^7 t^2 = 0 \Rightarrow t = 6 \cdot 10^{-3} \quad \Rightarrow \Delta P = 1,44 \text{ N}\cdot\text{s} //$$

$$b) \Delta P = m v_e \Rightarrow v_e = \frac{\Delta P}{m} = 480 \text{ m/s} //$$

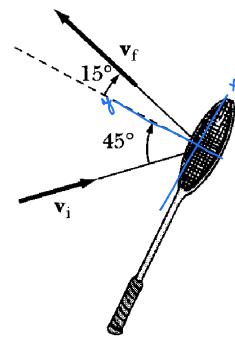


FIGURE 9-G Problem 9-30.

- 9.22. A deuteron (nucleus of deuterium atom consisting of a proton and a neutron) with speed 14.9 km/s collides elastically with a neutron at rest. Use the approximation that the deuteron is twice the mass of the neutron. (a) If the deuteron is scattered through a LAB angle $\psi = 10^\circ$, what are the final speeds of the deuteron and neutron? (b) What is the LAB scattering angle of the neutron? (c) What is the maximum possible scattering angle of the deuteron?

- 9-24.** A particle of mass m at the end of a light string wraps itself about a fixed vertical cylinder of radius a (Figure 9-F). All the motion is in the horizontal plane (disregard gravity). The angular velocity of the cord is ω_0 when the distance from the particle to the point of contact of the string and cylinder is b . Find the angular velocity and tension in the string after the cord has turned through an additional angle θ .

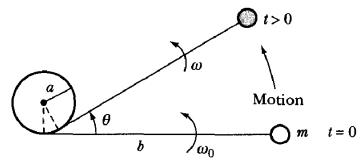


FIGURE 9-F Problem 9-24.

- 9-40. A particle of mass m_1 and velocity u_1 strikes head-on a particle of mass m_2 at rest. The coefficient of restitution is ϵ . Particle m_2 is tied to a point a distance a away as shown in Figure 9-H. Find the velocity (magnitude and direction) of m_1 and m_2 after the collision.

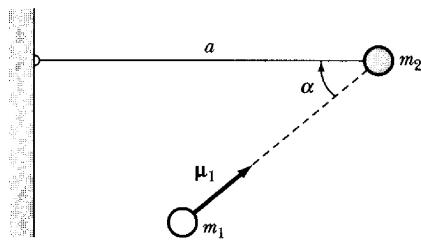


FIGURE 9-H Problem 9-40.

Problema 3. (Taylor 3.8)

3.8* A rocket (initial mass m_0) needs to use its engines to hover stationary, just above the ground. (a) If it can afford to burn no more than a mass λm_0 of its fuel, for how long can it hover? [Hint: Write down the condition that the thrust just balance the force of gravity. You can integrate the resulting equation by separating the variables t and m . Take v_{ex} to be constant.] (b) If $v_{ex} \approx 3000$ m/s and $\lambda \approx 10\%$, for how long could the rocket hover just above the earth's surface?

$$\text{Midiendo desde LAB } y \quad V_c = V_r + V_{ex}^*$$

$$\begin{aligned}
 a) \quad P &= m_r V_r + m_c V_{ex} \Rightarrow \dot{P} = -m_r g \Rightarrow \dot{m}_r V_r + m_r \dot{V}_r + \dot{m}_c V_c + m_c \dot{V}_c = -m_r g ; \quad \dot{m}_r = -\dot{m}_c \quad y \quad V_c = \text{constante} \\
 &\Rightarrow \dot{m}_r (V_r - V_c) + m_r \dot{V}_r = -m_r g \stackrel{*}{\Rightarrow} \dot{m}_r (V_r - V_r - V_{ex}) + m_r \dot{V}_r = -m_r g \Rightarrow -\dot{m}_r V_{ex} + m_r \dot{V}_r = -m_r g \\
 &\Rightarrow \frac{dm_r}{dt} V_{ex} = m_r \frac{dV_r}{dt} + m_r g = m_r \left(\frac{dV_r}{dt} + g \right) \Rightarrow \frac{dm_r}{dt} V_{ex} = m_r \left(\frac{dV_r}{dt} + g \right) \Rightarrow \frac{1}{m_r} \frac{dm_r}{dt} V_{ex} = \frac{1}{dt} \left(dV_r + g dt \right) \\
 &\Rightarrow \left(\frac{1}{m_r} \frac{dm_r}{dt} V_{ex} - g \right) dt = dV_r \Rightarrow \left(g \frac{dt}{dm_r} - \frac{1}{m_r} V_{ex} \right) \cdot \frac{dm_r}{dt} dt = dV_r ; \quad -\frac{dm_r}{dt} = \alpha \Rightarrow \left(g - \frac{V_{ex}}{m_r} \right) dm_r = dV_r \\
 &\Rightarrow \int_{m_0}^m \left(g - \frac{V_{ex}}{m_r} \right) dm_r = \int_{V_0}^{V_f} dV_r \Rightarrow V_f = V_0 - V_{ex} \ln \left(\frac{m_0}{m} \right) + \frac{g}{\alpha} (m - m_0) \quad y \quad \frac{dm_r}{dt} = -\alpha \Rightarrow m - m_0 = -\alpha t \\
 &\Rightarrow V_f - V_0 = -V_{ex} \ln \left(\frac{m_0}{m} \right) - gt = 0 \Rightarrow -V_{ex} \ln \left(\frac{m_0}{m} \right) - gt = 0 \Rightarrow gt = V_{ex} \ln \left(\frac{m_0}{m} \right) \Rightarrow t = \frac{1}{g} V_{ex} \ln \left(\frac{m_0}{m} \right) //
 \end{aligned}$$

Otra forma

$$\begin{aligned}
 &\Rightarrow \dot{m}_r V_r + m_r \dot{V}_r + \dot{m}_c V_c + m_c \dot{V}_c = -m_r g \stackrel{V_r \text{ constante}}{\Rightarrow} \dot{m}_r (V_r - V_c) = -m_r g \stackrel{*}{\Rightarrow} \dot{m}_r (V_r - V_r - V_{ex}) = -m_r g \\
 &\Rightarrow -\dot{m}_r V_{ex} = -m_r g \Rightarrow \frac{dm_r}{dt} V_{ex} = m_r g \Rightarrow -\int_{m_0}^m \frac{1}{m_r} dm_r = \int_0^t \frac{g}{V_{ex}} dt \Rightarrow gt = -V_{ex} \ln \left(\frac{m_0}{m} \right) = V_{ex} \ln \left(\frac{m_0}{m} \right) \\
 &\Rightarrow t = \frac{1}{g} V_{ex} \ln \left(\frac{m_0}{m} \right) //
 \end{aligned}$$

$$b) \quad t = \frac{1}{g} V_{ex} \ln \left(\frac{m_0}{m_0 - \lambda m_0} \right) = \frac{1}{g} V_{ex} \ln \left(\frac{1}{1-\lambda} \right) //$$

=

9-58. Consider a single-stage rocket taking off from Earth. Show that the height of the rocket at burnout is given by Equation 9.166. How much farther in height will the rocket go after burnout?

9-59. A rocket has an initial mass of m and a fuel burn rate of α (Equation 9.161). What is the minimum exhaust velocity that will allow the rocket to lift off immediately after firing?

Problema 4. (Thornton 9.50/Taylor 14.23)

9-50. A fixed force center scatters a particle of mass m according to the force law $F(r) = k/r^3$. If the initial velocity of the particle is u_0 , show that the differential scattering cross section is

$$\sigma(\theta) = \frac{k\pi^2(\pi - \theta)}{mu_0^3\theta^2(2\pi - \theta)^2 \sin \theta}$$

$$\sigma(\theta) = \frac{d\Omega}{d\Omega'} \Rightarrow \sigma(\theta) = \frac{d\Omega}{d\Omega'} = \frac{1}{I} \frac{dN}{d\Omega'} ; d\Omega = 2\pi \sin \theta d\theta$$

$$\Rightarrow \cancel{\int 2\pi b db} = - \cancel{\int 2\pi \sigma(\theta) \sin \theta d\theta}$$

$$\Rightarrow \sigma(\theta) = - \frac{b}{\sin \theta} \frac{db}{d\theta} = \frac{b}{\sin \theta} \left| \frac{db}{d\theta} \right| \quad \text{Hay que hallar la función } b(\theta)$$

$$\theta = |\pi + 2\theta| = \left| \pi + 2 \int_{r_{\min}}^{r_0} \frac{\frac{b}{r^2}}{\sqrt{2m(E-U - \frac{K^2}{2mr^2})}} dr \right| ; F = -\frac{dU}{dr} \Rightarrow U = - \int F dr = - \int \frac{K}{r^3} dr = \frac{K}{2r^2} \quad K = mu_0 b$$

$$\text{Reescribiendo } L, \text{ primero } T_0 = \frac{1}{2} mu_0^2 \Rightarrow u_0 = \sqrt{\frac{2T_0}{m}} \Rightarrow L = mb \sqrt{\frac{2T_0}{m}} = b \sqrt{2mT_0}$$

$$\Rightarrow \theta = \int_{r_{\min}}^{r_0} \frac{b \sqrt{2mT_0}}{\sqrt{2m \left[T_0 - \frac{K}{2r^2} - \frac{2mT_0 b^2}{2mr^2} \right]}} dr = \int_{r_{\min}}^{r_0} \frac{b \sqrt{2mT_0}}{\sqrt{1 - \frac{b^2}{r^2} - \frac{K}{r^2 T_0}}} dr = \int_{r_{\min}}^{r_0} \frac{b}{\sqrt{1 - \frac{b^2}{r^2} - \frac{K}{r^2 m u_0^2}}} dr$$

$$\text{Sustituyendo } z = \frac{b}{r} \quad dz = -\frac{b}{r^2} dr \Rightarrow \theta = - \int_{\frac{b}{r_{\min}}}^0 \frac{b}{\sqrt{1 - b^2 z^2 - \frac{K}{m u_0^2 z^2}}} dz = \int_0^{\frac{b}{r_{\min}}} \frac{b}{\sqrt{1 - z^2 \left(b^2 + \frac{K}{m u_0^2} \right)}} dz$$

$$\Rightarrow \theta = \sqrt{\frac{mb^2 u_0^2}{mb^2 u_0^2 + K}} \left| b \int_0^{\frac{b}{r_{\min}}} \frac{1}{\sqrt{\frac{mb^2 u_0^2}{mb^2 u_0^2 + K} - z^2}} dz \right| = \sqrt{\frac{mb^2 u_0^2 b^2}{mb^2 u_0^2 + K}} \left| \sin^{-1} \left(\frac{z}{\sqrt{\frac{mb^2 u_0^2}{mb^2 u_0^2 + K}}} \right) \right|_0^{\frac{b}{r_{\min}}}$$

$$\Rightarrow \theta = \sqrt{\frac{mb^2 u_0^2 b^2}{mb^2 u_0^2 + K}} \left| \sin^{-1} \left(\frac{z_{\max}}{\sqrt{\frac{mb^2 u_0^2}{mb^2 u_0^2 + K}}} \right) \right| ; \frac{mb^2 u_0^2 b^2}{mb^2 u_0^2 + K} - z^2 = 0 \Rightarrow z = \theta \sqrt{\frac{mb^2 u_0^2}{mb^2 u_0^2 + K}}$$

$$\Rightarrow \theta = \sqrt{\frac{mb^2 u_0^2 b^2}{mb^2 u_0^2 + K}} \cdot \frac{\pi}{2} \Rightarrow b = \sqrt{\frac{K}{m u_0^2}} \frac{2\theta}{\sqrt{\theta^2 - 4\theta^2}} \Rightarrow \theta = |\pi + 2\theta| \Rightarrow \theta = \frac{1}{2} (\theta - \pi) \Rightarrow b = \sqrt{\frac{K}{m u_0^2}} \frac{\theta - \pi}{\sqrt{\theta(2\pi - \theta)}}$$

$0 \leq \theta \leq \pi$
sin el menor de negativo

$$\Rightarrow \frac{db}{d\theta} = \sqrt{\frac{K}{m u_0^2}} \frac{\sqrt{\theta(2\pi - \theta)} - (\theta - \pi)}{\sqrt{\theta(2\pi - \theta)^2}} \cdot \frac{1/(2\pi - \theta) - \theta}{2\theta(2\pi - \theta)} = \sqrt{\frac{K}{m u_0^2}} \frac{\pi^2}{\sqrt{\theta(2\pi - \theta)^3}} \Rightarrow \sigma(\theta) = \frac{b}{\sin \theta} \left| \frac{db}{d\theta} \right|$$

$$\Rightarrow \sigma(\theta) = \frac{1}{\sin \theta} \sqrt{\frac{K}{m u_0^2}} \frac{\pi - \theta}{\sqrt{\theta(2\pi - \theta)}} \cdot \sqrt{\frac{K}{m u_0^2}} \frac{\pi^2}{\sqrt{\theta(2\pi - \theta)^3}} = \frac{1}{\sin \theta} \frac{K}{m u_0^2} \frac{\pi^2 (\pi - \theta)}{\sqrt{\theta(2\pi - \theta)^4}} = \frac{1}{\sin \theta} \frac{K}{m u_0^2} \frac{\pi^2 (\pi - \theta)}{\theta^2 (2\pi - \theta)^2}$$

Problema 5. (Thornton 9.53)

9.53. The most energetic α -particles available to Ernest Rutherford and his colleagues for the famous Rutherford scattering experiment were 7.7 MeV. For the scattering of 7.7 MeV α -particles from ^{238}U (initially at rest) at a scattering angle in the lab of 90° (all calculations are in the LAB system unless otherwise noted), find the following:

- the recoil scattering angle of ^{238}U .
- the scattering angles of the α -particle and ^{238}U in the CM system.
- the kinetic energies of the scattered α -particle and ^{238}U .
- the impact parameter b .
- the distance of closest approach r_{\min} .
- the differential cross section at 90° .
- the ratio of the probabilities of scattering at 90° to that of 5° .

$$m_1: \alpha\text{-partículas} = 4 \text{ amas} \quad m_2: ^{238}\text{U} = 238 \text{ amas}$$

$$q_1 = 2e^+$$

$$q_2 = 92e^+$$

$$T_0 = 7.7 \cdot 10^6 \text{ eV} = 1,234 \cdot 10^{-12} \text{ J}$$

$\Psi = 90^\circ$: Ángulo de dispersión de m_1 en LAB



a) $\zeta = ?$ Hay que resolver la ecuación

Momentum lineal

$$\text{Antes: } \vec{P}_b = \vec{p}_{U_1} = m_1(v_1, 0) \quad \text{Después: } \vec{P}_t = \vec{p}_{U_1} + \vec{p}_{U_2} = m_1(0, v_1) + m_2(v_2 \cos \zeta, v_2 \sin \zeta)$$

$$\Rightarrow \vec{P}_b = \vec{P}_t \Rightarrow \begin{cases} m_1 v_1 = m_2 v_2 \cos \zeta \\ 0 = m_1 v_1 + m_2 v_2 \sin \zeta \end{cases} \Rightarrow \begin{aligned} m_1 v_1 &= m_2 v_2 \cos \zeta & \Rightarrow m_1 \vec{v}_1 &= m_2 \vec{v}_2 \Rightarrow m_1(\vec{v}_1 - \vec{v}_2) = m_2 \vec{v}_2 \\ 0 &= m_1 v_1 + m_2 v_2 \sin \zeta & \Rightarrow m_1^2 (\vec{v}_1)^2 &= m_2^2 \vec{v}_2^2 \Rightarrow m_1^2 (v_1^2 + v_2^2 - 2 \vec{v}_1 \cdot \vec{v}_2) = m_2^2 v_2^2 \\ && \Rightarrow m_1^2 v_1^2 + m_1^2 v_2^2 &= m_2^2 v_2^2 \Rightarrow \frac{1}{v_1^2} + \frac{v_2^2}{m_1^2 v_1^2} = \frac{m_2^2 v_2^2}{m_1^2 v_1^2} \Rightarrow \frac{v_2^2}{v_1^2} = \frac{m_2^2 + m_1^2 v_2^2}{m_1^2 v_1^2} \end{aligned} \quad \star$$

Energía

$$\text{Antes: } E_{01} = T_0 = T_{10} = \frac{1}{2} m_1 v_1^2 \quad \text{Después: } E_{04} = T_{1+} + T_{2+} = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 \Rightarrow E_{01} = E_{04} \Rightarrow T_0 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$

$$\Rightarrow \begin{cases} m_1 v_1 = m_2 v_2 \cos \zeta : X \\ 0 = m_1 v_1 + m_2 v_2 \sin \zeta : Y \end{cases} \Rightarrow \frac{v_1}{v_2} = -\tan \zeta \Rightarrow v_2 = -v_1 \tan \zeta \quad \textcircled{4}$$

$$\frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 \Rightarrow m_1 v_1^2 = m_1 v_1^2 + m_2 v_2^2 \Rightarrow \frac{1}{v_1^2} = \frac{v_2^2}{m_1^2 v_1^2} + \frac{m_2 v_2^2}{m_1^2 v_1^2} \Rightarrow \frac{1}{v_1^2} - \frac{m_2 v_2^2}{m_1^2 v_1^2} = \frac{v_2^2}{m_1^2 v_1^2} \quad \textcircled{*}$$

$$\text{De } \textcircled{*} \text{ y } \textcircled{*}: \Rightarrow \frac{v_2^2}{v_1^2} = 1 - \frac{m_2}{m_1} \left(\frac{m_2}{m_2 + m_1} + \frac{m_2 v_2^2}{m_1^2 v_1^2} \right) = 1 - \frac{m_2}{m_1} - \frac{m_2 v_2^2}{m_2 v_1^2} \quad y \quad \frac{v_2^2}{v_1^2} = a \Rightarrow a = 1 - \frac{m_2}{m_1} - \frac{m_2}{m_2} a = \frac{m_2 - m_1}{m_2} - \frac{m_2}{m_2} a$$

$$\Rightarrow a + \frac{m_2}{m_2} a = a \left(1 + \frac{m_2}{m_2} \right) = a \left(\frac{m_2 + m_1}{m_2} \right) = \frac{m_2 + m_1}{m_2} \Rightarrow a = \frac{\frac{m_2 - m_1}{m_2}}{\frac{m_2 + m_1}{m_2}} = \frac{m_2 - m_1}{m_2 + m_1} = \frac{v_2^2}{v_1^2} \stackrel{\textcircled{4}}{=} \tan^2 \zeta$$

$$\Rightarrow \tan \zeta = \sqrt{\frac{m_2 - m_1}{m_2 + m_1}} \approx 0.98 \Rightarrow \zeta \approx 44.5^\circ \Rightarrow \zeta \approx -44.5^\circ //$$

b) $\theta = ?$



* Velocidad del centro de masa: $M\vec{V} = m_1 \vec{u}_1 + m_2 \vec{u}_2 \Rightarrow \vec{V} = \frac{m_1}{m_1+m_2} \vec{u}_1$



* Transformaciones de marco de referencia, LAB \leftrightarrow CM

$$\bullet \vec{u}_1 = \vec{V} + \vec{v}_1 \Rightarrow \vec{u}'_1 = \vec{u}_1 - \vec{V} = \vec{u}_1 - \frac{m_1}{m_1+m_2} \vec{u}_1 = \frac{m_2}{m_1+m_2} \vec{u}_1 = \frac{m_2}{m_1+m_2} u_1 \hat{i}$$

Partícula 1:

$$\bullet \vec{v}_1 = \vec{V} + \vec{v}'_1 \Rightarrow \vec{v}'_1 = \vec{v}_1 - \vec{V} = -u_1 \tan \xi \hat{j} - \frac{m_1}{m_1+m_2} u_1 \hat{i}$$

$$\bullet \vec{u}_2 = \vec{V} + \vec{v}'_2 \Rightarrow \vec{u}'_2 = -\vec{V} = -\frac{m_1}{m_1+m_2} \vec{u}_1 = -\frac{m_1}{m_1+m_2} u_1 \hat{i}$$

Partícula 2:

$$\bullet \vec{v}_2 = \vec{V} + \vec{v}'_2 \Rightarrow \vec{v}'_2 = \vec{v}_2 - \vec{V} = \frac{m_1 u_1}{m_2 \cos \xi} (\cos \xi \hat{i} + \sin \xi \hat{j}) - \frac{m_1}{m_1+m_2} u_1 \hat{i}$$

Buscando ϕ a partir de \vec{v}_2'

$$\tan \phi = \frac{\frac{m_1 u_1 \tan \xi}{m_2}}{\frac{m_1}{m_2} - \frac{m_1}{m_1+m_2} u_1} = \frac{\frac{m_1 \tan \xi}{m_2}}{\frac{m_1(m_1+m_2)-m_1 m_2}{(m_1+m_2)m_2}} = \frac{m_1(m_1+m_2) \tan \xi}{m_2^2} = \frac{(m_1+m_2) \tan \xi}{m_2} = \frac{(m_1+m_2)}{m_2} \sqrt{\frac{m_2^2 - m_1^2}{m_2^2 + m_1^2}} = \sqrt{\frac{m_2^2 - m_1^2}{m_1^2}}$$

$$= \sqrt{\frac{m_2^2 - m_1^2}{m_1^2}} = 59,49 \Rightarrow \phi = 89,0^\circ \Rightarrow \phi = -89,0^\circ \Rightarrow \theta = 180^\circ - \phi = 91^\circ$$

c) $T_{1_0}, T_{2_0}, T_{1_+}, T_{2_+} = ?$ $T_{1_0} = T_0 = 1,234 \cdot 10^{-12} \text{ J}$ $T_{2_0} = 0$ Utilizando la ya planteada ecuación de la conservación de la energía

$$\Rightarrow \frac{1}{2} m_1 u_1^2 = \frac{1}{2} m_1 V_1^2 + \frac{1}{2} m_2 V_2^2 \quad y \quad V_2 = -u_1 \tan \xi$$

$$\Rightarrow T_{2_+} = \frac{1}{2} m_2 V_2^2 = \frac{1}{2} m_2 u_1^2 (1 - \tan^2 \xi) = T_0 (1 - \tan^2 \xi) = 4,23 \cdot 10^{-14} \text{ J} \quad y \quad T_{1_+} = \frac{1}{2} m_1 V_1^2 = \frac{1}{2} m_1 u_1^2 \tan^2 \xi = 1,19 \cdot 10^{-12} \text{ J}$$

d) $b = ?$ Usando el resultado conocido para la dispersión de Rutherford

$$b = \frac{K}{2T_0} \cot(\frac{\theta}{2}) ; K = \frac{q_1 q_2}{4\pi \epsilon_0} = 4,25 \cdot 10^{-26} \quad y \quad T_0 = \frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2 = \frac{1}{2} m_1 \frac{m_2^2}{(m_1+m_2)^2} u_1^2 + \frac{1}{2} m_2 \frac{m_1^2}{(m_1+m_2)^2} u_1^2$$

$$\Rightarrow b = \frac{1}{2} \frac{m_1 m_2 u_1^2 (m_1+m_2)}{m_1 + m_2} = \frac{1}{2} \frac{m_1 m_2 u_1^2}{m_1 + m_2} T_0 = 1,21 \cdot 10^{-12} \text{ m}$$

e) La interacción entre partículas es de tipo columbiana $\Rightarrow U = \frac{k}{r}$, este tipo de potenciales responden a una órbita de la forma

$$\frac{\alpha}{r} = 1 + \epsilon \cos \theta \quad \text{en el marco de referencia del CM}$$

• $r = r_{\min}$ cuando $\frac{1}{r}$ es máxima $\Rightarrow \theta = 0 \Rightarrow r_{\min} = \frac{1 + \epsilon}{\alpha} \leftarrow$ Distancia mínima entre la partícula y CM

$$\alpha = \frac{b^2}{m_1 k}, \epsilon = \sqrt{\frac{2E_0 b^2}{m_1 k^2} + 1}, k = \frac{q_1 q_2}{4\pi \epsilon_0} = 4,25 \cdot 10^{-26} \text{ y } b = 1,73 \cdot 10^{-14} \text{ m}$$

$$\Rightarrow \alpha = \frac{m_1 b^{12}}{m_1} \frac{4\pi \epsilon_0}{q_1 q_2} = m_1 b^{12} \frac{4\pi \epsilon_0}{q_1 q_2}$$

$$\epsilon = \sqrt{\frac{2T_0 m_1 b^{12}}{m_1} \frac{4^2 \pi^2 \epsilon_0^2}{q_1^2 q_2^2} + 1} = \sqrt{2T_0 m_1 b^{12} \frac{4^2 \pi^2 \epsilon_0^2}{q_1^2 q_2^2} + 1} \Rightarrow r_{\min} = \frac{m_2}{m_1 + m_2} r_{\min} = 0,93 \cdot 10^{-14} \text{ m} //$$

$$f) \frac{d\sigma}{d\Omega_{\text{cm}}} = \frac{b}{\sin \theta} \left| \frac{db}{d\theta} \right| = \frac{k^2}{16 T_0^{12}} \frac{1}{\sin^4(\frac{\theta}{2})} \quad \frac{d\sigma(\psi)}{d\Omega_{\text{MB}}} = \frac{d\sigma(\theta)}{d\Omega_{\text{cm}}} \frac{(X \cos \psi + \sqrt{1 - X^2 \sin^2 \psi})^2}{\sqrt{1 - X^2 \sin^2 \psi}} ; \quad X = \frac{m_2}{m_1} \text{ y } \psi = 90^\circ //$$

$$\theta = \arctan(X \sin \psi) + \psi$$

$$g) \frac{d\sigma(\psi)}{d\Omega_{\text{MB}}} = \frac{d\sigma(\theta)}{d\Omega_{\text{cm}}} \frac{(X \cos \psi + \sqrt{1 - X^2 \sin^2 \psi})^2}{\sqrt{1 - X^2 \sin^2 \psi}} ; \quad \psi_1 = 90^\circ \text{ y } \psi_2 = 5^\circ //$$

Problema 6. (Thornton 9.63)

- 9-63. A new projectile launcher is developed in the year 2023 that can launch a 10^4 kg spherical probe with an initial speed of 6000 m/s. For testing purposes, objects are launched vertically.
- Neglect air resistance and assume that the acceleration of gravity is constant. Determine how high the launched object can reach above the surface of Earth.
 - If the object has a radius of 20 cm and the air resistance is proportional to the square of the object's speed with $c_w = 0.2$, determine the maximum height reached. Assume the density of air is constant.
 - Now also include the fact that the acceleration of gravity decreases as the object soars above Earth. Find the height reached.
 - Now add the effects of the decrease in air density with altitude to the calculation. We can very roughly represent the air density by $\log_{10}(\rho) = -0.05h + 0.11$ where ρ is the air density in kg/m^3 and h is the altitude above Earth in km. Determine how high the object now goes.

$$a) \cancel{\frac{V_f^2 - V_0^2}{2}} = -2g\Delta y \Rightarrow V_0^2 = 2g\Delta y \Rightarrow \Delta y = \frac{V_0^2}{2g} = 1,83 \cdot 10^6 \text{ m}$$

$$b) \vec{F} = -\frac{1}{2} C_w \rho A V^2 \frac{\vec{V}}{V} = -\frac{1}{2} C_w \rho A V \vec{V} \quad \text{Como es trío vertical} \Rightarrow \vec{F} = -\frac{1}{2} C_w \rho A V_y^2 \hat{j}$$

$$\Rightarrow \sum \vec{F}_y = -mg + \vec{F} = m \vec{a}_y \Rightarrow m \vec{a}_y = -\frac{1}{2} C_w \rho A V_y^2 - mg = m \frac{dV_y}{dt} \Rightarrow -C_w \rho A V_y^2 - 2mg = 2m \frac{dV_y}{dt}$$

$$\Rightarrow \int_0^t \frac{1}{2m} dt = - \int_{V_0}^{V_y} \frac{1}{C_w \rho A V_y^2 + 2mg} dV_y = - \frac{1}{C_w \rho h} \int_{V_0}^{V_y} \frac{1}{V_y^2 + V_t^2} dV_y$$

$$\rho = 1,12 \text{ kg/m}^3$$

$$\Rightarrow \frac{t}{2m} = - \frac{1}{C_w \rho A} \frac{1}{V_t} \tan^{-1} \left(\frac{V_y}{V_t} \right) + \frac{1}{C_w \rho A} \frac{1}{V_t} \tan^{-1} \left(\frac{V_0}{V_t} \right)$$

$$V_t = \sqrt{\frac{2mg}{C_w \rho h}} = 2,64 \cdot 10^3$$

$$\Rightarrow \frac{C_w \rho h}{2mg} V_t g t = - \tan^{-1} \left(\frac{V_y}{V_t} \right) + \tan^{-1} \left(\frac{V_0}{V_t} \right) = \frac{gt}{V_t} \Rightarrow \tan^{-1} \left(\frac{V_y}{V_t} \right) = \tan^{-1} \left(\frac{V_0}{V_t} \right) - \frac{gt}{V_t} \Rightarrow V_y = V_t \tan \left[\tan^{-1} \left(\frac{V_0}{V_t} \right) - \frac{gt}{V_t} \right]$$

$$\Rightarrow \frac{dy}{dt} = V_t \tan \left[\tan^{-1} \left(\frac{V_0}{V_t} \right) - \frac{gt}{V_t} \right] \Rightarrow \int_0^t dy = y = \int_0^t V_t \tan \left[\tan^{-1} \left(\frac{V_0}{V_t} \right) - \frac{gt}{V_t} \right] dt \quad \text{En la altura máxima } y=0$$

$$\Rightarrow t = 3,11 \text{ s}$$

$$\Rightarrow y = \frac{V_t^2}{g} \ln \left\{ \cos \left[\tan^{-1} \left(\frac{V_0}{V_t} \right) - \frac{gt}{V_t} \right] \right\} - \frac{V_t^2}{g} \ln \left\{ \cos \left[\tan^{-1} \left(\frac{V_0}{V_t} \right) \right] \right\} \quad \text{Altura máxima} \Rightarrow y_{\max} = 6,46 \cdot 10^5 \text{ m} //$$

$$c) m \vec{a}_y = -\frac{1}{2} C_w \rho A V_y^2 - mg \vec{g} \Rightarrow m \vec{g} = -\frac{1}{2} C_w \rho A \vec{g}^2 - m \frac{GM}{(R+y)^2} = -\frac{1}{2} C_w \rho A \vec{g}^2 - mg \frac{R^2}{(R+y)^2}$$

$$\Rightarrow \vec{g} = -\frac{C_w \rho A \vec{g}^2}{2m} - g \frac{R^2}{(R+y)^2}$$

$$d) m \vec{a}_y = -\frac{1}{2} C_w \rho \vec{g}(y) A V_y^2 - mg \vec{g} \Rightarrow m \vec{g} = -\frac{1}{2} C_w 10^{-0,05y+0,11} A \vec{g}^2 - mg \frac{R^2}{(R+y)^2}$$

$$\Rightarrow \vec{g} = \frac{C_w A 10^{-0,05y+0,11} \vec{g}^2}{2m} - g \frac{R^2}{(R+y)^2} //$$

PART

XI

Ondas y Medios Continuos

SUBSECTION 9.11

Problemas resueltos

Cuerdas con pesos

Problema 1. (Thornton 12.24)

12-24. Show that the equations of motion for longitudinal vibrations of a loaded string are of exactly the same form as the equations for transverse motion (Equation 12.131), except that the factor τ/d must be replaced by κ , the force constant of the string.

$$\ddot{q}_j = \frac{\tau}{md}(q_{j-1} - 2q_j + q_{j+1}) \quad (12.131)$$

La cuerda se puede tratar como resortes que unen las masas puntuales, por lo que se le puede dotar de una constante de rigidez o elasticidad de la cuerda, se le llamará κ .

Trabajando con mecánica de Lagrange

$$T = \dots + \frac{1}{2}m\dot{q}_{j-1}^2 + \frac{1}{2}m\dot{q}_j^2 + \frac{1}{2}m\dot{q}_{j+1}^2 + \dots = \frac{1}{2}m \sum_{j=1}^N \dot{q}_j^2$$

Cambiando índices
 $\dot{q}_0^2 = 0$ entonces se puede mover en la suma sin producir cambios

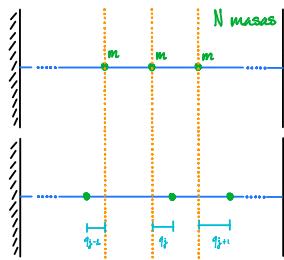
$$\frac{1}{2}m \sum_{j=0}^N \dot{q}_j^2$$

$$U = \frac{1}{2}\kappa q_1^2 + \dots + \frac{1}{2}\kappa(q_{j-1} - q_j)^2 + \frac{1}{2}\kappa(q_j - q_{j+1})^2 + \dots + \frac{1}{2}\kappa q_N^2 = \frac{1}{2}\kappa \sum_{j=0}^N (q_j - q_{j+1})^2$$

$$\Rightarrow L = \frac{1}{2}m \sum_{j=0}^N \dot{q}_j^2 - \frac{1}{2}\kappa \sum_{j=0}^N (q_j - q_{j+1})^2 = \sum_{j=0}^N \frac{1}{2} [m\dot{q}_j^2 - \kappa(q_j - q_{j+1})^2]$$

Resolviendo para masa j :

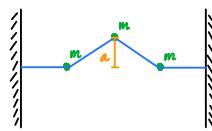
$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0 \Rightarrow m\ddot{q}_j - \kappa(q_{j-1} - 2q_j + q_{j+1}) = 0 \Rightarrow \ddot{q}_j = \frac{\kappa}{m}(q_{j-1} - 2q_j + q_{j+1})$$



Una cuerda sin masa, cargando masas puntuales idénticas, igualmente separadas en el equilibrio.

Los extremos fijos hacen $q_0 = q_{N+1} = 0 = \dot{q}_0 = \dot{q}_{N+1}$

Consider a loaded string consisting of three particles regularly spaced on the string. At $t = 0$ the center particle (only) is displaced a distance a and released from rest. Describe the subsequent motion.



$$\text{Condiciones iniciales} \quad \begin{cases} q_1(0) = q_3(0) = 0 \quad y \quad q_2(0) = a \\ \dot{q}_1(0) = \dot{q}_3(0) = \dot{q}_2(0) = 0 \end{cases}$$

La frecuencia de un sistema de cuerda con masas puntuales es : $\omega_r = 2 \sqrt{\frac{T}{md}} \sin\left(\frac{r\pi}{n+1}\right)$ donde d es la distancia entre masas, n el número de masas

La solución es de la forma : $q_j(t) = \sum_{r=1}^n \sin(j \cdot \frac{r\pi}{n+1}) (\mu_r \cos \omega_r t - v_r \sin \omega_r t)$

$$\text{donde : } \mu_r = \frac{2}{n+1} \sum_{j=1}^n q_j(0) \sin(j \cdot \frac{r\pi}{n+1}) \quad y \quad v_r = \frac{-2}{\omega_r(n+1)} \sum_{j=1}^n \dot{q}_j(0) \sin(j \cdot \frac{r\pi}{n+1})$$

En este caso: $n=3$

$$\bullet \Rightarrow \omega_r = 2 \sqrt{\frac{T}{md}} \sin\left(\frac{r\pi}{2(n+1)}\right) = 2 \sqrt{\frac{T}{md}} \sin\left(\frac{r\pi}{2(3+1)}\right) = 2 \sqrt{\frac{T}{md}} \sin\left(\frac{r\pi}{8}\right)$$

$$\bullet \Rightarrow \mu_r = \frac{2}{n+1} \sum_{j=1}^n q_j(0) \sin(j \cdot \frac{r\pi}{n+1}) = \frac{2}{3+1} \sum_{j=1}^3 q_j(0) \sin(j \cdot \frac{r\pi}{3+1}) = \frac{2}{4} \left[\cancel{q_1(0) \sin(1 \cdot \frac{r\pi}{4})} + q_2(0) \sin(2 \cdot \frac{r\pi}{4}) + \cancel{q_3(0) \sin(3 \cdot \frac{r\pi}{4})} \right]$$

$$= \frac{1}{2} a \sin\left(\frac{r\pi}{2}\right)$$

$$\bullet \Rightarrow v_r = \frac{-2}{\omega_r(n+1)} \sum_{j=1}^n \dot{q}_j(0) \sin(j \cdot \frac{r\pi}{n+1}) = \frac{-2}{\omega_r(3+1)} \sum_{j=1}^3 \cancel{j \cdot \dot{q}_j(0)} \sin(j \cdot \frac{r\pi}{3+1}) = 0$$

$$\Rightarrow q_j(t) = \sum_{r=1}^n \sin(j \cdot \frac{r\pi}{n+1}) (\mu_r \cos \omega_r t - v_r \sin \omega_r t) = \sum_{r=1}^3 \sin(j \cdot \frac{r\pi}{3+1}) (\mu_r \cos \omega_r t - v_r \sin \omega_r t)$$

$$= \sum_{r=1}^3 \sin(j \cdot \frac{r\pi}{4}) \left(\frac{1}{2} a \sin\left(\frac{r\pi}{2}\right) \cos \omega_r t - \cancel{0 \sin \omega_r t} \right) = \sum_{r=1}^3 \sin(j \cdot \frac{r\pi}{4}) \frac{1}{2} a \sin\left(\frac{r\pi}{2}\right) \cos \omega_r t$$

$$= \sum_{r=1}^3 \frac{1}{2} a \sin\left(\frac{r\pi}{2}\right) \sin(j \cdot \frac{r\pi}{4}) \cos \omega_r t$$

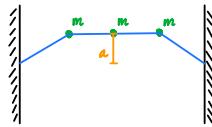
$$= \frac{1}{2} a \left[\sin\left(\frac{\pi}{2}\right) \sin(j \cdot \frac{\pi}{4}) \cos \omega_r t + \cancel{\sin\left(\frac{2\pi}{2}\right) \sin(j \cdot \frac{2\pi}{4}) \cos \omega_r t} + \sin\left(\frac{3\pi}{2}\right) \sin(j \cdot \frac{3\pi}{4}) \cos \omega_r t \right]$$

$$= \frac{1}{2} a \left[\sin\left(\frac{\pi}{2}\right) \sin(j \cdot \frac{\pi}{4}) \cos \omega_r t + \sin\left(\frac{3\pi}{2}\right) \sin(j \cdot \frac{3\pi}{4}) \cos \omega_r t \right] //$$

Problema 2. (Thornton 12.25)

12-25. Rework the problem in Example 12.7, assuming that all three particles are displaced a distance a and released from rest.

Condiciones iniciales $\begin{cases} q_1(0) = q_2(0) = q_3(0) = a \\ \dot{q}_1(0) = \dot{q}_2(0) = \dot{q}_3(0) = 0 \end{cases}$



* $n = 3$

$$\begin{aligned}
 & \bullet \Rightarrow \omega_r = 2\sqrt{\frac{T}{md}} \sin\left(\frac{r\pi}{2(n+1)}\right) = 2\sqrt{\frac{T}{md}} \sin\left(\frac{r\pi}{2(3+1)}\right) = 2\sqrt{\frac{T}{md}} \sin\left(\frac{\pi}{8}\right) \\
 & \bullet \Rightarrow \mu_r = \frac{2}{n+1} \sum_{j=1}^n q_j(0) \sin(j \cdot \frac{r\pi}{n+1}) = \frac{2}{3+1} \sum_{j=1}^3 q_j(0) \sin(j \cdot \frac{r\pi}{3+1}) = \frac{2}{4} [q_1(0) \sin(1 \cdot \frac{r\pi}{4}) + q_2(0) \sin(2 \cdot \frac{r\pi}{4}) + q_3(0) \sin(3 \cdot \frac{r\pi}{4})] \\
 & = \frac{1}{2} [a \sin\left(\frac{r\pi}{4}\right) + a \sin\left(\frac{r\pi}{2}\right) + a \sin\left(\frac{3r\pi}{4}\right)] = \frac{a}{2} [\sin\left(\frac{r\pi}{4}\right) + \sin\left(\frac{r\pi}{2}\right) + \sin\left(\frac{3r\pi}{4}\right)] \\
 & \bullet \Rightarrow v_r = \frac{-2}{\omega_r(n+1)} \sum_{j=1}^n \dot{q}_j(0) \sin(j \cdot \frac{r\pi}{n+1}) = \frac{-2}{\omega_r(3+1)} \sum_{j=1}^3 \dot{q}_j(0) \sin(j \cdot \frac{r\pi}{3+1}) = 0 \\
 \Rightarrow q_j(t) &= \sum_{r=1}^n \sin(j \cdot \frac{r\pi}{n+1})(\mu_r \cos \omega_r t - v_r \sin \omega_r t) = \sum_{r=1}^3 \sin(j \cdot \frac{r\pi}{3+1})(\mu_r \cos \omega_r t - \cancel{v_r \sin \omega_r t}) \\
 &= \sum_{r=1}^3 \sin(j \cdot \frac{r\pi}{4}) \frac{a}{2} [\sin\left(\frac{r\pi}{4}\right) + \sin\left(\frac{r\pi}{2}\right) + \sin\left(\frac{3r\pi}{4}\right)] \cos \omega_r t = \sum_{r=1}^3 \frac{a}{2} [\sin\left(\frac{r\pi}{4}\right) + \sin\left(\frac{r\pi}{2}\right) + \sin\left(\frac{3r\pi}{4}\right)] \sin(j \cdot \frac{r\pi}{4}) \cos \omega_r t \\
 &= \frac{a}{2} \{ [\cancel{\sin\left(\frac{1\pi}{4}\right)} + \cancel{\sin\left(\frac{1\pi}{2}\right)} + \cancel{\sin\left(\frac{3\cdot 1\pi}{4}\right)}] \sin(j \cdot \frac{1\pi}{4}) \cos \omega_r t + [\cancel{\sin\left(\frac{2\pi}{4}\right)} + \cancel{\sin\left(\frac{2\pi}{2}\right)} + \cancel{\sin\left(\frac{3\cdot 2\pi}{4}\right)}] \sin(j \cdot \frac{2\pi}{4}) \cos \omega_r t + \dots \\
 &\quad \dots + [\cancel{\sin\left(\frac{3\pi}{4}\right)} + \cancel{\sin\left(\frac{3\pi}{2}\right)} + \cancel{\sin\left(\frac{3\cdot 3\pi}{4}\right)}] \sin(j \cdot \frac{3\pi}{4}) \cos \omega_r t \} \\
 &= \frac{a}{2} \{ [\sin\left(\frac{\pi}{4}\right) + 1 + \sin\left(\frac{3\pi}{4}\right)] \sin(j \cdot \frac{\pi}{4}) \cos \omega_r t + 0 \sin(j \cdot \frac{2\pi}{4}) \cos \omega_r t + [\sin\left(\frac{3\pi}{4}\right) + \sin\left(\frac{3\pi}{2}\right) + \sin\left(\frac{9\pi}{4}\right)] \sin(j \cdot \frac{3\pi}{4}) \cos \omega_r t \} \\
 &= \frac{a}{2} \{ [\sin\left(\frac{\pi}{4}\right) + 1 + \sin\left(\frac{3\pi}{4}\right)] \sin(j \cdot \frac{\pi}{4}) \cos \omega_r t + [\sin\left(\frac{3\pi}{4}\right) + \sin\left(\frac{3\pi}{2}\right) + \sin\left(\frac{9\pi}{4}\right)] \sin(j \cdot \frac{3\pi}{4}) \cos \omega_r t \} \\
 \Rightarrow q_j(t) &= \frac{a}{2} \{ [\sin\left(\frac{\pi}{4}\right) + 1 + \sin\left(\frac{3\pi}{4}\right)] \sin(j \cdot \frac{\pi}{4}) \cos \omega_r t + [\sin\left(\frac{3\pi}{4}\right) + \sin\left(\frac{3\pi}{2}\right) + \sin\left(\frac{9\pi}{4}\right)] \sin(j \cdot \frac{3\pi}{4}) \cos \omega_r t \} \\
 &= \frac{a}{2} [(1 + \sqrt{2}) \sin(j \cdot \frac{\pi}{4}) \cos \omega_r t + (-1 + \sqrt{2}) \sin(j \cdot \frac{3\pi}{4}) \cos \omega_r t]
 \end{aligned}$$

Medios continuos

Find the displacement $q(x, t)$ for a "plucked string," where one point of the string is displaced (such that the string assumes a triangular shape) and then released from rest. Consider the case shown in Figure 13-1, in which the center of the string is displaced a distance h .

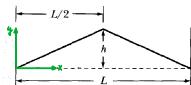


FIGURE 13-1 Example 13.1. A string is "plucked" by pulling the center of the string a distance h from equilibrium so that the string has a triangular shape. The string is released from rest in this position.

Condiciones iniciales

$$\begin{cases} q(X, 0) = \begin{cases} \frac{2h}{L} X & \rightarrow 0 < X < \frac{L}{2} \\ -\frac{2h}{L} X + 2h & \rightarrow \frac{L}{2} < X < L \end{cases} \\ \dot{q}(X, 0) = 0 \end{cases}$$

La solución es de la forma $q(X, t) = \sum_{r=1}^{+\infty} (\mu_r + \nu_r) e^{i\omega_r t} \sin\left(\frac{r\pi}{L} X\right)$ Dejando la parte real

Por definición:

- $\omega_r = \frac{r\pi}{L} \sqrt{\frac{T}{\lambda_m}}$
- $\nu_r = -2 \int_0^L \dot{q}(X, 0) \sin\left(\frac{r\pi}{L} X\right) dX = 0$
- $\mu_r = \int_0^L q(X, 0) \sin\left(\frac{r\pi}{L} X\right) dX = \int_0^L \left[\int_0^{L/2} \frac{2h}{L} X \sin\left(\frac{r\pi}{L} X\right) dX + \int_{L/2}^L (-\frac{2h}{L} X + 2h) \sin\left(\frac{r\pi}{L} X\right) dX \right] dX$
 $= \frac{2}{L^2} \left[\int_0^{L/2} 2hX \sin\left(\frac{r\pi}{L} X\right) dX + \int_{L/2}^L -2hX \sin\left(\frac{r\pi}{L} X\right) dX + 2h \int_{L/2}^L \sin\left(\frac{r\pi}{L} X\right) dX \right]$
 $= \frac{2}{L^2} \left\{ 2h \left[\frac{L^2}{r^2\pi^2} \left(\frac{\sin(r\pi/2)}{2} - \frac{L \cos(r\pi/2)}{r\pi} \right) \right] \Big|_0^{L/2} - 2h \left[\frac{L^2}{r^2\pi^2} \left(\frac{\sin(r\pi/2)}{2} - \frac{L \cos(r\pi/2)}{r\pi} \right) \right] \Big|_{L/2}^L + 2h \left[\frac{-L \cos(r\pi/2)}{r\pi} \right] \Big|_{L/2}^L \right\}$
 $= \frac{2}{L^2} \left\{ 2h \left[\frac{L^2}{r^2\pi^2} \left(\frac{\sin(r\pi/2)}{2} - \frac{L \cos(r\pi/2)}{r\pi} \right) - 0 \right] - 2h \left[\frac{L^2}{r^2\pi^2} \left(\frac{\sin(r\pi/2)}{2} - \frac{L \cos(r\pi/2)}{r\pi} \right) - \frac{L^2}{r^2\pi^2} \left(\frac{\sin(r\pi/2)}{2} - \frac{L \cos(r\pi/2)}{r\pi} \right) + \frac{L}{2} L \cos\left(\frac{r\pi}{2}\right) \right] \dots \right.$
 $\dots + 2h \left[\frac{-L \cos(r\pi/2)}{r\pi} + \frac{L \cos(r\pi/2)}{r\pi} \right] \right\}$
 $= \frac{2}{L^2} \left\{ 2h \left[\frac{L^2}{r^2\pi^2} \left(\frac{\sin(r\pi/2)}{2} - \frac{L \cos(r\pi/2)}{2r\pi} \right) - 2h \left[\frac{-L \cos(r\pi/2)}{r\pi} - \frac{L^2 \sin(r\pi/2)}{r^2\pi^2} + \frac{L^2 \cos(r\pi/2)}{2r\pi} \right] + 2h \left[\frac{-L \cos(r\pi/2)}{r\pi} + \frac{L \cos(r\pi/2)}{r\pi} \right] \right\}$
 $= \frac{2}{L^2} \left\{ 2h \left[2 \frac{L^2}{r^2\pi^2} \left(\frac{\sin(r\pi/2)}{2} - \frac{L \cos(r\pi/2)}{r\pi} \right) + \frac{L^2 \cos(r\pi/2)}{r\pi} \right] + 2h \left[\frac{-L \cos(r\pi/2)}{r\pi} + \frac{L \cos(r\pi/2)}{r\pi} \right] \right\}$
 $= \frac{2}{L^2} \cdot 2hL^2 \left[2 \frac{\sin(r\pi/2)}{r^2\pi^2} - \frac{\cos(r\pi/2)}{r\pi} + \cancel{\frac{\cos(r\pi/2)}{r\pi}} - \cancel{\frac{\cos(r\pi/2)}{r\pi}} + \cancel{\frac{\cos(r\pi/2)}{r\pi}} \right] = \frac{2}{L^2} \cdot 2hL^2 \cdot 2 \frac{\sin(r\pi/2)}{r^2\pi^2} = \frac{8h \sin(r\pi/2)}{r^2\pi^2}$
 $\Rightarrow q(X, t) = \sum_{r=1}^{+\infty} (\mu_r \cos \omega_r t - \cancel{\nu_r \sin \omega_r t}) \sin\left(\frac{r\pi}{L} X\right) = \sum_{r=1}^{+\infty} \frac{8h \sin(r\pi/2)}{r^2\pi^2} \cos \omega_r t \sin\left(\frac{r\pi}{L} X\right)$

Reconsider Example 13.1. A sinusoidal driving force of angular frequency ω drives the string at $x = b/2$. Find the displacement.

$$F(X, t) = \begin{cases} F_0 \cos \omega t & \rightarrow X = \frac{L}{2} \\ 0 & \rightarrow X \neq \frac{L}{2} \end{cases} \quad \text{Para convertirla a las coordenadas normales} \quad f(t) = \int_0^L F(X, t) \sin\left(\frac{n\pi X}{L}\right) dx$$

$$\Rightarrow f(t) = \int_0^L F_0 \cos \omega t \sin\left(\frac{n\pi X}{L}\right) dx = F_0 \cos \omega t \sin\left(\frac{n\pi L}{2}\right) = F_0 \cos \omega t \sin\left(\frac{n\pi}{2}\right)$$

La ecuación de onda en coordenadas normales generalizada pasa a una coordenada normal: $\ddot{\eta}_s + \frac{D}{\rho m} \dot{\eta}_s + \frac{s^2 \pi^2}{\rho L^2} \eta_s = \frac{2}{\rho L} f(t)$

- Como no hay amortiguamiento $\Rightarrow D = 0$ $\Rightarrow \ddot{\eta}_s + \frac{s^2 \pi^2}{\rho L^2} \eta_s = \frac{2}{\rho L} F_0 \cos \omega t \sin\left(\frac{n\pi}{2}\right)$; donde $\omega_s^2 = \frac{s^2 \pi^2}{\rho L^2}$ y $A = \frac{2F_0 \sin\left(\frac{n\pi}{2}\right)}{\rho L}$

- La solución de un oscilador armónico forzado con una fuerza sinusoidal: $X = B \cos(\omega_s t - \delta) + \frac{A}{\omega_s^2 - \omega^2} \cos(\omega t)$

Para este caso de estudio: $\eta_s = B \cos(\omega_s t - \delta) + \frac{A}{\omega_s^2 - \omega^2} \cos(\omega t) = B \cos\left(\sqrt{\frac{s^2 \pi^2}{\rho L^2}} t - \delta\right) + \frac{2F_0 \sin\left(\frac{n\pi}{2}\right)}{\rho L \left(\frac{s^2 \pi^2}{\rho L^2} - \omega^2\right)} \cos(\omega t)$

Trabajando solo con la solución particular $\Rightarrow \eta_s = \frac{2F_0 \sin\left(\frac{n\pi}{2}\right)}{\rho L \left(\frac{s^2 \pi^2}{\rho L^2} - \omega^2\right)} \cos(\omega t)$

- La relación entre las coordenadas normales y la generalizada: $g(X, t) = \sum_{r=1}^{+\infty} \eta_r(t) \sin\left(\frac{r\pi X}{L}\right)$

$$\Rightarrow g(X, t) = \sum_{r=1}^{+\infty} \frac{2F_0 \sin\left(\frac{r\pi}{2}\right)}{\rho L \left(\frac{r^2 \pi^2}{\rho L^2} - \omega^2\right)} \cos(\omega t) \sin\left(\frac{r\pi X}{L}\right)$$

Problema 1. (Thornton 13.6)

13-6. A string is set into motion by being struck at a point $L/4$ from one end by a triangular hammer. The initial velocity is greatest at $x = L/4$ and decreases linearly to zero at $x = 0$ and $x = L/2$. The region $L/2 \leq x \leq L$ is initially undisturbed. Determine the subsequent motion of the string. Why are the fourth, eighth, and related harmonics absent? How many decibels down from the fundamental are the second and third harmonics?

Condiciones iniciales

$$\begin{cases} q(X, 0) = 0 \\ \dot{q}(X, 0) = \begin{cases} \frac{4L}{t} X & \rightarrow 0 < X < \frac{L}{4} \\ \frac{4L}{t} \left(-X + \frac{L}{2} \right) & \rightarrow \frac{L}{4} < X < \frac{L}{2} \\ 0 & \rightarrow X > \frac{L}{2} \end{cases} \end{cases}$$

Por definición:

- $\omega_r = \frac{r\pi}{L} \sqrt{\frac{T}{\lambda_m}}$

- $\mu_r = \frac{2}{L} \int_0^L q(X, 0) \sin\left(\frac{r\pi}{L} X\right) dX = 0$

- $v_r = \frac{-2}{\omega_r L} \int_0^L \dot{q}(X, 0) \sin\left(\frac{r\pi}{L} X\right) dX = \frac{-2}{\omega_r L} \left[\int_0^{L/4} \frac{4L}{t} X \sin\left(\frac{r\pi}{L} X\right) dX + \int_{L/4}^{L/2} \frac{4L}{t} (-X + \frac{L}{2}) \sin\left(\frac{r\pi}{L} X\right) dX + \int_{L/2}^L 0 \sin\left(\frac{r\pi}{L} X\right) dX \right]$

$$= \frac{-2}{\omega_r L} \left[\int_0^{L/4} \frac{4L}{t} X \overset{(1)}{\sin}\left(\frac{r\pi}{L} X\right) dX + \int_{L/4}^{L/2} \frac{4L}{t} X \overset{(2)}{\sin}\left(\frac{r\pi}{L} X\right) dX + \int_{L/4}^{L/2} \frac{4L}{t} \overset{(3)}{\frac{1}{2}} \sin\left(\frac{r\pi}{L} X\right) dX \right]$$

$$(1) \int_0^{L/4} \frac{4L}{t} X \sin\left(\frac{r\pi}{L} X\right) dX = \frac{4L}{t} \int_0^{L/4} X \sin\left(\frac{r\pi}{L} X\right) dX = \frac{4L}{t} \left[\frac{L^2}{r^2 \pi^2} \sin\left(\frac{r\pi}{L} X\right) - \frac{L}{r\pi} \cos\left(\frac{r\pi}{L} X\right) \right] \Big|_0^{L/4}$$

$$= \frac{4L}{t} \left\{ \left[\frac{L^2}{r^2 \pi^2} \sin\left(\frac{r\pi}{4}\right) - \frac{L}{r\pi} \cos\left(\frac{r\pi}{4}\right) \right] - \left[\frac{L^2}{r^2 \pi^2} \sin\left(\frac{r\pi}{2}\right) - \frac{L}{r\pi} \cos\left(\frac{r\pi}{2}\right) \right] \right\} = \frac{4L}{t} \left[\frac{L^2}{r^2 \pi^2} \sin\left(\frac{r\pi}{4}\right) - \frac{L}{4r\pi} \cos\left(\frac{r\pi}{4}\right) \right] = 4V_0 \left[\frac{L \sin\left(\frac{r\pi}{4}\right)}{r^2 \pi^2} - \frac{L \cos\left(\frac{r\pi}{4}\right)}{4r\pi} \right]$$

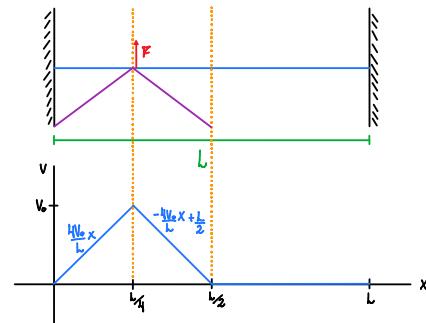
$$(2) \int_{L/4}^{L/2} \frac{4L}{t} X \sin\left(\frac{r\pi}{L} X\right) dX = -\frac{4L}{t} \int_0^{L/4} X \sin\left(\frac{r\pi}{L} X\right) dX = -\frac{4L}{t} \left[\frac{L^2}{r^2 \pi^2} \sin\left(\frac{r\pi}{L} X\right) - \frac{L}{r\pi} \cos\left(\frac{r\pi}{L} X\right) \right] \Big|_0^{L/4}$$

$$= -\frac{4L}{t} \left\{ \left[\frac{L^2}{r^2 \pi^2} \sin\left(\frac{r\pi}{2}\right) - \frac{L}{r\pi} \cos\left(\frac{r\pi}{2}\right) \right] - \left[\frac{L^2}{r^2 \pi^2} \sin\left(\frac{r\pi}{4}\right) - \frac{L}{r\pi} \cos\left(\frac{r\pi}{4}\right) \right] \right\} = -\frac{4L}{t} \left[\left[\frac{L^2}{r^2 \pi^2} \sin\left(\frac{r\pi}{2}\right) - \frac{L}{2r\pi} \cos\left(\frac{r\pi}{2}\right) \right] - \left[\frac{L^2}{r^2 \pi^2} \sin\left(\frac{r\pi}{4}\right) - \frac{L}{4r\pi} \cos\left(\frac{r\pi}{4}\right) \right] \right]$$

$$= 4V_0 \left\{ \left[\frac{L \sin\left(\frac{r\pi}{2}\right)}{r^2 \pi^2} - \frac{L \cos\left(\frac{r\pi}{2}\right)}{2r\pi} \right] - \left[\frac{L \sin\left(\frac{r\pi}{4}\right)}{r^2 \pi^2} - \frac{L \cos\left(\frac{r\pi}{4}\right)}{4r\pi} \right] \right\}$$

$$(3) \int_{L/2}^L \frac{4L}{t} \frac{1}{2} \sin\left(\frac{r\pi}{L} X\right) dX = 2V_0 \int_{L/2}^L \sin\left(\frac{r\pi}{L} X\right) dX = 2V_0 \left[-\frac{L}{r\pi} \cos\left(\frac{r\pi}{L} X\right) \right] \Big|_{L/2}^L = 2V_0 \left[-\frac{L}{r\pi} \cos\left(\frac{r\pi}{2}\right) + \frac{L}{r\pi} \cos\left(\frac{r\pi}{4}\right) \right]$$

$$= 2V_0 \left[-\frac{L \cos\left(\frac{r\pi}{2}\right)}{r\pi} + \frac{L \cos\left(\frac{r\pi}{4}\right)}{r\pi} \right]$$



La solución es de la forma:

$$q(X, t) = \sum_{r=1}^{+\infty} (\mu_r \cos \omega_r t - v_r \sin \omega_r t) \sin\left(\frac{r\pi}{L} X\right)$$

Continua.....

Reapareando lo anterior

$$\Rightarrow V_r = \frac{-2}{\omega r} (① + ② + ③)$$

$$\begin{aligned}
 ① + ② + ③ &= 4V_0 \left[\frac{L \sin(\frac{r\pi}{4}) - L \cos(\frac{r\pi}{4})}{r^2 \pi^2} \right] + 4V_0 \left\{ \left[\frac{L \sin(\frac{r\pi}{4}) - L \cos(\frac{r\pi}{4})}{r^2 \pi^2} \right] - \left[\frac{L \sin(\frac{r\pi}{2}) - L \cos(\frac{r\pi}{2})}{r^2 \pi^2} \right] \right\} + 2V_0 \left[\frac{-L \cos(\frac{r\pi}{2}) + L \cos(\frac{r\pi}{4})}{r \pi} \right] \\
 &= 8V_0 \left[\frac{L \sin(\frac{r\pi}{4}) - L \cos(\frac{r\pi}{4})}{r^2 \pi^2} \right] - 4V_0 \left[\frac{L \sin(\frac{r\pi}{2}) - L \cos(\frac{r\pi}{2})}{r^2 \pi^2} \right] + 2V_0 \left[\frac{-L \cos(\frac{r\pi}{2}) + L \cos(\frac{r\pi}{4})}{r \pi} \right] \\
 &= 8V_0 \left[\frac{L \sin(\frac{r\pi}{4}) - L \cos(\frac{r\pi}{4})}{r^2 \pi^2} \right] + 2V_0 \left[\frac{-2L \sin(\frac{r\pi}{2}) + L \cos(\frac{r\pi}{2}) - L \cos(\frac{r\pi}{2}) + L \cos(\frac{r\pi}{4})}{r \pi} \right] \\
 &= 8V_0 \left[\frac{L \sin(\frac{r\pi}{4}) - L \cos(\frac{r\pi}{4})}{r^2 \pi^2} \right] + 2V_0 \left[\frac{-2L \sin(\frac{r\pi}{2}) + L \cos(\frac{r\pi}{4})}{r \pi} \right] \\
 &= 2V_0 \left[\frac{4L \sin(\frac{r\pi}{4}) - L \cos(\frac{r\pi}{4}) - 2L \sin(\frac{r\pi}{2}) + L \cos(\frac{r\pi}{4})}{r^2 \pi^2} \right] = 2V_0 \left[\frac{4L \sin(\frac{r\pi}{4}) - 2L \sin(\frac{r\pi}{2})}{r^2 \pi^2} \right] \\
 &= -4V_0 \left[\frac{L \sin(\frac{r\pi}{2}) - 2L \sin(\frac{r\pi}{4})}{r^2 \pi^2} \right] = \frac{-4V_0}{r^2 \pi^2} [\sin(\frac{r\pi}{2}) - 2 \sin(\frac{r\pi}{4})]
 \end{aligned}$$

Finalmente

$$\Rightarrow q(X, t) = \sum_{n=1}^{+\infty} (\cancel{\mu_n \cos \omega_n t} - \nu_n \sin \omega_n t) \sin \left(\frac{n\pi}{L} X \right) = \sum_{n=1}^{+\infty} -\nu_n \sin \omega_n t \sin \left(\frac{n\pi}{L} X \right)$$

$$\Rightarrow q(X, t) = \sum_{n=1}^{+\infty} -\frac{8V_0}{r^2 \pi^2 \omega_r} [\sin(\frac{r\pi}{2}) - 2 \sin(\frac{r\pi}{4})] \sin \omega_r t \sin \left(\frac{n\pi}{L} X \right) \cancel{=}$$

Problema 2. (Thornton 13.9)

- 13-9. In Example 13.2, the complementary solution (transient part) was omitted. If transient effects are included, what are the appropriate conditions for over-damped, critically damped, and underdamped motion? Find the displacement $q(x, t)$ that results when underdamped motion is included in Example 13.2 (assume that the motion is underdamped for all normal modes).

Problema 3. (Thornton 13.10)

- 13-10. Consider the string of Example 13.1. Show that if the string is driven at an arbitrary point, none of the normal modes with nodes at the driving point will be excited.

$$F(X, t) = \begin{cases} f(X, t) & \rightarrow X = \frac{L}{a} \\ 0 & \rightarrow X \neq \frac{L}{a} \end{cases}$$

Para convertirla a las coordenadas normales $f(t) = \int_0^L F(X, t) \sin\left(\frac{s\pi X}{L}\right) dx$

$$\Rightarrow f(t) = \int_0^L F(X, t) \sin\left(\frac{s\pi X}{L}\right) \delta(X - \frac{L}{a}) dx = F\left(\frac{L}{a}, t\right) \sin\left(\frac{s\pi}{a}\right) = F\left(\frac{L}{a}, t\right) \sin(s)$$

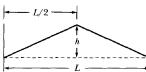


FIGURE 13-1 Example 13.1. A string is "plucked" by pulling the center of the string a distance h from equilibrium so that the string has a triangular shape. The string is released from rest in this position.

La ecuación de onda en coordenadas normales generalizada pasa a una coordenada normal: $\ddot{\eta}_s + \frac{D}{\rho I} \dot{\eta}_s + \frac{s^2 \pi^2 I}{\rho L^2} \eta_s = \frac{2}{\rho L} f(t)$

- Como no hay amortiguamiento $D=0 \Rightarrow \ddot{\eta}_s + \frac{s^2 \pi^2 I}{\rho L^2} \eta_s = \frac{2}{\rho L} f(t)$

- La relación entre las coordenadas normales y la generalizada: $g(X, t) = \sum_{s=1}^{+\infty} \eta_s(t) \sin\left(\frac{s\pi X}{L}\right)$

- Dejando de lado los efectos transitorios, la solución en coordenadas normales y generalizadas

$$\Rightarrow \eta_s = \frac{2F\left(\frac{L}{a}, t\right) \sin\left(\frac{s\pi}{a}\right)}{\rho L \left(\frac{s\pi}{a} - \omega^2\right)} \cos(\omega t) \Rightarrow g(X, t) = \sum_{s=1}^{+\infty} \frac{2F\left(\frac{L}{a}, t\right) \sin\left(\frac{s\pi}{a}\right) \cos(\omega t)}{\rho L \left(\frac{s\pi}{a} - \omega^2\right)} \sin\left(\frac{s\pi X}{L}\right)$$

- Los nodos de un modo normal se dan: $\dot{\eta}_s(X, t) = 0 \Rightarrow \frac{2F\left(\frac{L}{a}, t\right) \sin\left(\frac{s\pi}{a}\right) \cos(\omega t)}{\rho L \left(\frac{s\pi}{a} - \omega^2\right)} \sin\left(\frac{s\pi X}{L}\right) = 0$

$\Rightarrow F\left(\frac{L}{a}, t\right) \sin\left(\frac{s\pi}{a}\right) \sin\left(\frac{s\pi X}{L}\right) = 0$ dando la restricción de que el nodo debe estar en el lugar que se ejerce la fuerza $X = \frac{L}{a}$

$$\Rightarrow \underbrace{F\left(\frac{L}{a}, t\right) \sin\left(\frac{s\pi}{a}\right)}_{f(t)} \sin\left(\frac{s\pi}{a}\right) = 0 \Rightarrow f(t) \sin\left(\frac{s\pi}{a}\right) = 0 \Rightarrow n = \frac{s}{a} \text{ un entero y } \sin\left(\frac{s\pi}{a}\right) = 0$$

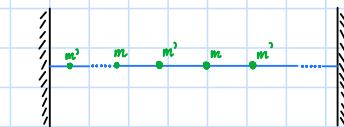
$$\Rightarrow \dot{\eta}_s\left(\frac{L}{a}, t\right) = 0 \text{ para } n = \frac{s}{a} \text{ un entero} //$$

Ondas Mecánicas

Problema 1. (Thornton 13.17)

13-17. Treat the problem of wave propagation along a string loaded with particles of two different masses, m' and m'' , which alternate in placement; that is,

$$m_j = \begin{cases} m', & \text{for } j \text{ even} \\ m'', & \text{for } j \text{ odd} \end{cases}$$



Show that the $\omega - k$ curve has two branches in this case, and show that there is attenuation for frequencies between the branches as well as for frequencies above the upper branch.

Para las partículas pares

$$m' \ddot{q}_{2n} = -\frac{\gamma}{d} (q_{2n} - q_{2n-1}) - \frac{\gamma}{d} (q_{2n} - q_{2n+1}) \Rightarrow m' \ddot{q}_{2n} = -\frac{\gamma}{d} (-q_{2n-1} + 2q_{2n} - q_{2n+1}) \Rightarrow \ddot{q}_{2n} = \frac{\gamma}{m'd} (q_{2n-1} - 2q_{2n} + q_{2n+1})$$

$$\text{Para las impares: } \ddot{q}_{2n+1} = \frac{\gamma}{m'd} (q_{2n+1} - 2q_{2n+2} + q_{2n+3})$$

- La solución de estas ecuaciones: $q_{2n} = a_n e^{i\omega t}$ y $q_{2n+1} = b_{n+1} e^{i\omega t}$, donde $a_n = A e^{i[(2n+1)\delta_1 - \delta_2]}$ y $b_{n+1} = B e^{i[(2n+2)\delta_1 - \delta_2]}$

$$\Rightarrow q_{2n} = A e^{i(2n+1)\delta_1} e^{i\omega t} \quad \text{y} \quad q_{2n+1} = B e^{i[(2n+2)\delta_1 - \delta_2]} e^{i\omega t} \quad \text{sustituyendo esto en las ecuaciones de movimiento}$$

Para $2n$:

Para $2n+1$:

Problema 2. (Thornton 13.19)

- 13-19. Consider an infinitely long continuous string with linear mass density ρ_1 for $x < 0$ and for $x > L$, but density $\rho_2 > \rho_1$ for $0 < x < L$. If a wave train oscillating with an angular frequency ω is incident from the left on the high-density section of the string, find the reflected and transmitted intensities for the various portions of the string. Find a value of L that allows a maximum transmission through the high-density section. Discuss briefly the relationship of this problem to the application of nonreflective coatings to optical lenses.



La frecuencia de oscilación ω no cambia al cruzar de un medio a otro, por otro lado el número de onda sí, debido al cambio en la densidad del medio y a la relación entre la velocidad de propagación, la densidad del medio y K .

$$K = \frac{\omega}{V} \quad V^2 = \frac{I}{\rho}$$

Al conocer la solución de la ecuación de onda para ondas viajeras, se plantea:

$$\Psi_I = A e^{i(K_1 x + \omega t)}, \quad \Psi_{T1} = B e^{i(K_1 x + \omega t)}, \quad \Psi_{R1} = C e^{i(K_1 x + \omega t)}, \quad \Psi_{T2} = D e^{i(K_2 x + \omega t)} \quad y \quad \Psi_{R2} = E e^{i(K_2 x + \omega t)}$$

Plantear la función de onda de modo que siempre quede un factor $e^{i\omega t}$ por facilidad

las ondas en cada sección de la cuerda

$$\begin{cases} \Psi_1 = \Psi_I + \Psi_{R1} = A e^{i(K_1 x + \omega t)} + B e^{i(K_1 x + \omega t)} \rightarrow x < 0 \\ \Rightarrow \Psi_2 = \Psi_{T1} + \Psi_{R2} = C e^{i(K_2 x + \omega t)} + D e^{i(K_2 x + \omega t)} \rightarrow 0 < x < L \\ \Psi_3 = \Psi_{T2} = E e^{i(K_2 x + \omega t)} \rightarrow L < x \end{cases}$$

Ahora es necesario utilizar condiciones de frontera para garantizar que la onda se propaga de forma continua en las discontinuidades de los medios, para la función Ψ y su derivada, de modo que el comportamiento sea físicamente correcto.

● Para $X = 0$:

$$\Psi_1|_{x=0} = \Psi_2|_{x=0} \Rightarrow A e^{i(K_1 \cdot 0 + \omega t)} + B e^{i(K_1 \cdot 0 + \omega t)} = C e^{i(K_2 \cdot 0 + \omega t)} + D e^{i(K_2 \cdot 0 + \omega t)} \Rightarrow A + B = C + D$$

$$\begin{aligned} \frac{\partial \Psi_1}{\partial x}|_{x=0} &= \frac{\partial \Psi_2}{\partial x}|_{x=0} \Rightarrow -i K_1 A e^{i(K_1 \cdot 0 + \omega t)}|_{x=0} + i K_1 B e^{i(K_1 \cdot 0 + \omega t)}|_{x=0} = -i K_2 C e^{i(K_2 \cdot 0 + \omega t)}|_{x=0} + i K_2 D e^{i(K_2 \cdot 0 + \omega t)}|_{x=0} \\ &\Rightarrow -i K_1 A e^{i(0)} + i K_1 B e^{i(0)} = -i K_2 C e^{i(0)} + i K_2 D e^{i(0)} \Rightarrow -K_1 A + K_1 B = -K_2 C + K_2 D \end{aligned}$$

● Para $X = L$:

$$\Psi_2|_{x=L} = \Psi_3|_{x=L} \Rightarrow C e^{i(K_2 L + \omega t)} + D e^{i(K_2 L + \omega t)} = E e^{i(K_2 L + \omega t)} \Rightarrow C e^{-i K_2 L} + D e^{i K_2 L} = E e^{-i K_2 L}$$

$$\begin{aligned} \frac{\partial \Psi_2}{\partial x}|_{x=L} &= \frac{\partial \Psi_3}{\partial x}|_{x=L} \Rightarrow -i K_2 C e^{i(K_2 L + \omega t)}|_{x=L} + i K_2 D e^{i(K_2 L + \omega t)}|_{x=L} = -i K_2 E e^{i(K_2 L + \omega t)}|_{x=L} \\ &\Rightarrow -K_2 C e^{-i K_2 L} + K_2 D e^{i K_2 L} = -K_2 E e^{-i K_2 L} \end{aligned}$$

Recuperando las condiciones de frontera

$$\begin{aligned}
 & \left\{ \begin{array}{l} A+B = C+D \\ -k_1 A + k_1 B = -k_2 C + k_2 D \\ C e^{-j\omega t} + D e^{j\omega t} = E e^{-j\omega t} \\ -k_2 C e^{-j\omega t} + k_2 D e^{j\omega t} = -k_1 E e^{-j\omega t} \end{array} \right. \\
 \Rightarrow & \left\{ \begin{array}{l} C e^{-j\omega t} + D e^{j\omega t} = E e^{-j\omega t} \\ -k_2 C e^{-j\omega t} + k_2 D e^{j\omega t} = -k_1 E e^{-j\omega t} \end{array} \right. \Rightarrow -k_2 C e^{-j\omega t} + k_2 D e^{j\omega t} = -k_1 (C e^{-j\omega t} + D e^{j\omega t}) \\
 & = -k_1 C e^{-j\omega t} - k_1 D e^{j\omega t} \\
 \Rightarrow & C e^{-j\omega t} (k_1 - k_2) = -D e^{j\omega t} (k_1 + k_2) \Rightarrow D = \underbrace{e^{\frac{-j2\omega t}{(k_2+k_1)}}}_{C} \quad \text{Despejando para } D \\
 * & C e^{-j\omega t} + D e^{j\omega t} = E e^{-j\omega t} \Rightarrow C e^{-j\omega t} + e^{\frac{j2\omega t}{(k_2+k_1)}} C = E e^{-j\omega t} = C e^{-j\omega t} \left[1 + e^{\frac{j2\omega t}{(k_2+k_1)}} \right] \\
 \Rightarrow & E = e^{\frac{j(k_1-k_2)t}{(k_2+k_1)}} \left[1 + e^{\frac{j2\omega t}{(k_2+k_1)}} \right] C \\
 \bullet & \left\{ \begin{array}{l} A+B = C+D \\ -k_1 A + k_1 B = -k_2 C + k_2 D \end{array} \right. \Rightarrow -k_1 (C+D-B) + k_1 B = -k_2 C + k_2 D \Rightarrow -k_1 (C+D-B) + k_1 B = -k_2 C + k_2 D \\
 & \Rightarrow C(k_2 - k_1) - D(k_2 + k_1) = -k_1 B \\
 \Rightarrow & C(k_2 - k_1) - \cancel{e^{\frac{-j2\omega t}{(k_2+k_1)}}} \cancel{C(k_2 + k_1)} = -k_1 B \Rightarrow C(k_2 - k_1)(1 - \cancel{e^{\frac{-j2\omega t}{(k_2+k_1)}}}) = -k_1 B \quad \text{Usandolo para despejar } A \\
 * & k_1 A - k_1 B = k_2 C - k_2 D \Rightarrow k_1 A = k_2 C - k_2 e^{\frac{-j2\omega t}{(k_2+k_1)}} C - C(k_2 - k_1)(1 - \cancel{e^{\frac{-j2\omega t}{(k_2+k_1)}}}) \\
 \Rightarrow & k_1 A = C \left[k_2 - k_2 e^{\frac{-j2\omega t}{(k_2+k_1)}} - (k_2 - k_1)(1 - \cancel{e^{\frac{-j2\omega t}{(k_2+k_1)}}}) \right] = C \left\{ k_2 - (k_2 - k_1) \left[\frac{k_2 e^{\frac{-j2\omega t}{(k_2+k_1)}}}{k_2 + k_1} + 1 - \cancel{e^{\frac{-j2\omega t}{(k_2+k_1)}}} \right] \right\} \\
 & = C \left\{ k_2 - (k_2 - k_1) \left[\frac{k_2 e^{\frac{-j2\omega t}{(k_2+k_1)}} + k_2 + k_1 - k_2 e^{\frac{-j2\omega t}{(k_2+k_1)}} - k_1 e^{\frac{-j2\omega t}{(k_2+k_1)}}}{k_2 + k_1} \right] \right\} = C \left[k_2 - (k_2 - k_1) \left(1 - \frac{k_1 e^{\frac{-j2\omega t}{(k_2+k_1)}}}{k_2 + k_1} \right) \right] \\
 & = C \left[\cancel{k_2} - \cancel{(k_2 - k_1)} - k_1 \left(\frac{k_2 - k_1}{k_2 + k_1} \right) e^{\frac{-j2\omega t}{(k_2+k_1)}} \right] = C k_1 \left[1 - \left(\frac{k_2 - k_1}{k_2 + k_1} \right) e^{\frac{-j2\omega t}{(k_2+k_1)}} \right] \Rightarrow A = C \left[1 - \left(\frac{k_2 - k_1}{k_2 + k_1} \right) e^{\frac{-j2\omega t}{(k_2+k_1)}} \right]
 \end{aligned}$$

Los coeficientes de las funciones donda son:

$$A = C \left[1 - \left(\frac{k_2 - k_1}{k_2 + k_1} \right) e^{-j2k_1 L} \right] ; \quad B = \frac{(k_1 - k_2)(1 - e^{-j2k_1 L})}{k_1} C ; \quad C ; \quad D = e^{-j2k_1 L} \left(\frac{k_2 - k_1}{k_2 + k_1} \right) C$$

$$\text{y } E = e^{j(k_1 - k_2)L} \left[1 + e^{-j2k_1 L} \left(\frac{k_2 - k_1}{k_2 + k_1} \right) \right] C$$

Coeficiente de reflexión: $R = \left| \frac{\text{(reflejado)}}{\text{(incidente)}} \right|^2$

Coeficiente de transmisión: $T = \left| \frac{\text{(transmitido)}}{\text{(incidente)}} \right|^2$

Problema 3. (Thornton 13.20)

13-20. Consider an infinitely long continuous string with tension τ . A mass M is attached to the string at $x = 0$. If a wave train with velocity ω/k is incident from the left, show that reflection and transmission occur at $x = 0$ and that the coefficients R and T are given by

$$R = \sin^2 \theta, \quad T = \cos^2 \theta$$

where

$$\tan \theta = \frac{M\omega^2}{2k\tau}$$

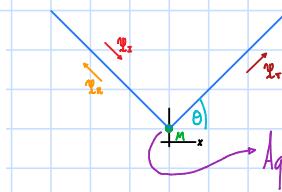
Consider carefully the boundary condition on the derivatives of the wave functions at $x = 0$. What are the phase changes for the reflected and transmitted waves?

$$\Psi_I = Ae^{i(kx+\omega t)}, \quad \Psi_R = Be^{i(kx+\omega t)}, \quad \Psi_T = Ce^{i(-kx+\omega t)}$$

$$\Rightarrow \begin{cases} \Psi_I = \Psi_R + \Psi_T = Ae^{i(kx+\omega t)} + Be^{i(kx+\omega t)} \rightarrow x < 0 \\ \Psi_R = \Psi_T = Ce^{i(-kx+\omega t)} \rightarrow x > 0 \end{cases}$$

Condiciones de Frontera:

$$\Psi_I|_{x=0} = \Psi_R|_{x=0} \Rightarrow Ae^{i\omega t} + Be^{i\omega t} = Ce^{i\omega t} \Rightarrow A + B = C$$



Aquí la derivada no es continua

Problema 4. (Thornton 13.22)

13-22. Consider a wave packet with a Gaussian amplitude distribution

$$A(k) = B \exp[-\sigma(k - k_0)^2]$$

where $2\sqrt{\sigma}$ is equal to the $1/e$ width* of the packet. Using this function for $A(k)$, show that

$$\Psi(x, t) = B \int_{-\infty}^{+\infty} A(k) e^{i\omega t - ikx} dk = B \int_{-\infty}^{+\infty} B e^{-\sigma(k-k_0)^2} e^{i\omega t - ikx} dk$$

Sketch the shape of this wave packet. Next, expand $\omega(k)$ in a Taylor series, retain the first two terms, and integrate the wave packet equation to obtain the general result

$$\Psi(x, t) = B \sqrt{\frac{\pi}{\sigma}} \exp[-(\omega'_0 t - x)^2/4\sigma] \exp[i(\omega'_0 t - k_0 x)]$$

Finally, take one additional term in the Taylor series expression of $\omega(k)$ and show that σ is now replaced by a complex quantity. Find the expression for the $1/e$ width of the packet as a function of time for this case and show that the packet moves with the same group velocity as before but spreads in width as it moves. Illustrate this result with a sketch.

$$\rightarrow \text{Shaunee 18.75*} \Rightarrow \Psi(x, 0) = B e^{-ik_0 x} \sqrt{\frac{\pi}{\sigma}} e^{\frac{i^2 \omega'^2}{4\sigma}} = B \sqrt{\frac{\pi}{\sigma}} e^{\frac{i^2 \omega'^2}{4\sigma}} e^{-ik_0 x}$$

*At the points $k = k_0 \pm 1/\sqrt{\sigma}$, the amplitude distribution is $1/e$ of its maximum value $A(k_0)$. Thus $2\sqrt{\sigma}$ is the width of the curve at the $1/e$ height.

$$\begin{aligned} \Psi(x, t) &= \int_{-\infty}^{+\infty} A(k) e^{i\omega t - ikx} dk \Rightarrow \Psi(x, 0) = \int_{-\infty}^{+\infty} A(k) e^{i\omega t - ikx} dk \\ &= \int_{-\infty}^{+\infty} B e^{-\sigma(k-k_0)^2} e^{-ikx} dk \\ \Rightarrow \Psi(x, 0) &= B \int_{-\infty}^{+\infty} e^{-\sigma(k-k_0)^2 - ikx + ik_0 x - ik_0 x} dk \\ &= B e^{-ik_0 x} \int_{-\infty}^{+\infty} e^{-\sigma(k-k_0)^2 - iku} dk = B e^{-ik_0 x} \int_{-\infty}^{+\infty} e^{-(\sigma u^2 + iku)} du \\ &\text{Illustration: } \Psi(x, 0) = B \sqrt{\frac{\pi}{\sigma}} e^{-\frac{(x-x_0)^2}{4\sigma}} \quad x_0 = k_0 \end{aligned}$$

$$\begin{aligned} \bullet \quad \Psi(x, t) &= \int_{-\infty}^{+\infty} A(k) e^{i\omega t - ikx} dk = \int_{-\infty}^{+\infty} A(k) e^{i[(\omega t - k_0 x) + \omega'_0(k - k_0) - (k - k_0)x]} dk = \int_{-\infty}^{+\infty} A(k) e^{i[(\omega t - k_0 x) + (k - k_0)(\omega t - x)]} dk \\ &= e^{i(\omega t - k_0 x)} \int_{-\infty}^{+\infty} A(k) e^{i(k - k_0)(\omega t - x)} dk = e^{i(\omega t - k_0 x)} \int_{-\infty}^{+\infty} B e^{-\sigma(k-k_0)^2} e^{i(k - k_0)(\omega t - x)} dk \quad u = (k - k_0) \quad du = dk \\ &= e^{i(\omega t - k_0 x)} \int_{-\infty}^{+\infty} B e^{-\sigma u^2} e^{i u (\omega t - x)} du = e^{i(\omega t - k_0 x)} \int_{-\infty}^{+\infty} B e^{-[\sigma u^2 - i u (\omega t - x)]} du \quad * = e^{i(\omega t - k_0 x)} B \sqrt{\frac{\pi}{\sigma}} e^{\frac{i^2 (\omega t - x)^2}{4\sigma}} \\ &= B \sqrt{\frac{\pi}{\sigma}} e^{-\frac{(\omega t - x)^2}{4\sigma}} e^{i(\omega t - k_0 x)} \end{aligned}$$

Problema 5.

Una cuerda tiene extremos en $x = 0$ y $x = L$. Resuelva la ecuación de onda considerando que cada extremo es fijo.

$$\begin{array}{ll} \text{Condiciones iniciales} & \left\{ \begin{array}{l} \Psi(x, 0) = y(x) \\ \dot{\Psi}(x, 0) = v(x) \end{array} \right. \\ & \text{Condiciones de frontera} \quad \left\{ \begin{array}{l} \Psi(0, t) = 0 \\ \Psi(L, t) = 0 \end{array} \right. \end{array}$$



Hay que resolver la ecuación de onda $\frac{\partial^2 \Psi(x, t)}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 \Psi(x, t)}{\partial t^2}$ bajo las condiciones anteriores

$$\text{Para esto se plantea } \Psi(x, t) = \psi(x) \cdot \varphi(t) \Rightarrow \psi \frac{\partial^2 \varphi}{\partial x^2} = \frac{1}{v^2} \varphi \frac{\partial^2 \psi}{\partial t^2} \Rightarrow \frac{v^2}{\psi} \frac{\partial^2 \psi}{\partial x^2} = \frac{1}{\varphi} \frac{\partial^2 \varphi}{\partial t^2} = -\omega^2$$

- $\frac{\partial^2 \psi}{\partial x^2} = -\frac{\omega^2}{v^2} \psi = -k^2 \psi \Rightarrow \psi = A \cos(kx) + B \sin(kx)$

- $\frac{\partial^2 \varphi}{\partial t^2} = -\omega^2 \varphi \Rightarrow \varphi = C \cos(\omega t) + D \sin(\omega t) \Rightarrow \Psi = [A \cos(kx) + B \sin(kx)][C \cos(\omega t) + D \sin(\omega t)]$

- $\Psi(0, t) = A[C \cos(\omega t) + D \sin(\omega t)] = 0 \Rightarrow A = 0$ Comenzando a aplicar condiciones de frontera

- $\Psi(L, t) = [A \cos(kL) + B \sin(kL)][C \cos(\omega t) + D \sin(\omega t)] = 0 \Rightarrow B \sin(kL)[C \cos(\omega t) + D \sin(\omega t)] = 0$

$$\Rightarrow B \sin(kL)[C \cos(\omega t) + D \sin(\omega t)] = 0 \Rightarrow kL = n\pi \Rightarrow k = \frac{n\pi}{L} \quad \text{Colapsando las constantes restantes y aplicando el principio de superposición}$$

$$\Rightarrow \Psi = \sum_{n=1}^{+\infty} \mu_n \cos(\omega t) \sin\left(\frac{n\pi}{L} x\right) + \nu_n \sin(\omega t) \cos\left(\frac{n\pi}{L} x\right) \quad \text{Con las condiciones iniciales}$$

- $\Psi(x, 0) = y(x) = \sum_{n=1}^{+\infty} \mu_n \sin\left(\frac{n\pi}{L} x\right) \Rightarrow \int_0^L \sin\left(\frac{n\pi}{L} x\right) y(x) dx = \sum_{n=1}^{+\infty} \mu_n \int_0^L \sin\left(\frac{n\pi}{L} x\right) \sin\left(\frac{n\pi}{L} x\right) dx$

$$\Rightarrow \int_0^L \sin\left(\frac{n\pi}{L} x\right) y(x) dx = \int_0^L \sum_{n=1}^{+\infty} \mu_n \sin\left(\frac{n\pi}{L} x\right) \sin\left(\frac{n\pi}{L} x\right) dx = \sum_{n=1}^{+\infty} \mu_n \int_0^L \sin\left(\frac{n\pi}{L} x\right) \sin\left(\frac{n\pi}{L} x\right) dx = \mu_n \frac{1}{2} \int_0^L ds$$

$$\Rightarrow \mu_n = 2 \int_0^L \sin\left(\frac{n\pi}{L} x\right) y(x) dx$$

- $\dot{\Psi}(x, 0) = v(x) = \sum_{n=1}^{+\infty} \nu_n \sin\left(\frac{n\pi}{L} x\right) \Rightarrow \int_0^L \sin\left(\frac{n\pi}{L} x\right) v(x) dx = \int_0^L \sum_{n=1}^{+\infty} \nu_n \sin\left(\frac{n\pi}{L} x\right) \sin\left(\frac{n\pi}{L} x\right) dx = \sum_{n=1}^{+\infty} \nu_n \int_0^L \sin\left(\frac{n\pi}{L} x\right) \sin\left(\frac{n\pi}{L} x\right) dx$

$$\Rightarrow \int_0^L \sin\left(\frac{n\pi}{L} x\right) v(x) dx = \nu_n \frac{1}{2} \int_0^L ds \Rightarrow \nu_n = 2 \int_0^L \sin\left(\frac{n\pi}{L} x\right) v(x) dx$$

$$\therefore \Psi(x, t) = \sum_{n=1}^{+\infty} \mu_n \cos\left(\omega t\right) \sin\left(\frac{n\pi}{L} x\right) + \nu_n \sin\left(\omega t\right) \cos\left(\frac{n\pi}{L} x\right) \quad \text{con } \mu_n = 2 \int_0^L \sin\left(\frac{n\pi}{L} x\right) y(x) dx \quad \text{y} \quad \nu_n = 2 \int_0^L \sin\left(\frac{n\pi}{L} x\right) v(x) dx$$

Problema 6.

Una onda que viaja en una cuerda dentro de un medio viscoso verifica la ecuación de onda:

$$A \frac{\partial^2 \Psi}{\partial x^2} - B \frac{\partial \Psi}{\partial t} + C \frac{\partial^2 \Psi}{\partial t^2} = 0.$$

El sistema oscila con una frecuencia angular ω y la cuerda tiene una tensión τ .

- ¿Qué representa físicamente las constantes A , B y C ?
- Determine la solución general de la ecuación
- Verifique los resultados vistos en clase. ¿Cuánto es $k(\omega)$?
- ¿Qué sucede con sus resultados cuando la viscosidad se hace muy pequeña? Y muy grande?

a) C está relacionada a la densidad del medio, es su densidad

B se refiere a la viscosidad del medio en que se encuentra

A se relaciona con la resistencia a la deformación de la cuerda, es su tensión

b) Se propone $\Psi(x, t) = \psi(x) \cdot \varphi(t)$ y se reemplaza en: $\frac{\partial^2 \Psi}{\partial x^2} + 2B \frac{\partial \Psi}{\partial t} + \frac{1}{\tau^2} \frac{\partial^2 \Psi}{\partial t^2} = 0$; $\frac{B}{A} = 2B$

$$\Rightarrow \frac{\psi \frac{\partial^2 \varphi}{\partial x^2}}{\frac{\partial \varphi}{\partial t}} - 2B \frac{\psi \frac{\partial \varphi}{\partial t}}{\varphi} - \frac{1}{\tau^2} \frac{\psi \frac{\partial^2 \varphi}{\partial t^2}}{\varphi} = 0 \quad \text{Dividiendo por } \Psi \cdot \varphi \Rightarrow \frac{\frac{\partial^2 \varphi}{\partial x^2}}{\varphi} - 2B \frac{\frac{\partial \varphi}{\partial t}}{\varphi} - \frac{1}{\tau^2} \frac{\frac{\partial^2 \varphi}{\partial t^2}}{\varphi} = 0$$

$$\Rightarrow \frac{\psi \frac{\partial^2 \varphi}{\partial x^2}}{\varphi} = \frac{2B \psi \frac{\partial \varphi}{\partial t}}{\varphi} + \frac{1}{\tau^2} \frac{\partial^2 \varphi}{\partial t^2} = -\omega^2$$

$$\bullet \frac{\frac{\partial^2 \varphi}{\partial x^2}}{\varphi} + \frac{\omega^2 \varphi}{\tau^2} = 0 \Rightarrow \varphi = c_1 e^{ikx} + c_2 e^{-ikx}$$

$$\bullet \frac{\frac{\partial^2 \varphi}{\partial t^2}}{\varphi} + 2B \frac{\frac{\partial \varphi}{\partial t}}{\varphi} + \omega^2 \varphi = 0 \Rightarrow \varphi = e^{-Bt^2} (c_3 e^{i\sqrt{\omega^2 - B^2} t} + c_4 e^{-i\sqrt{\omega^2 - B^2} t})$$

$$m = -Bt^2 \pm \sqrt{B^2 t^4 - \omega^2}$$

$$\Rightarrow \Psi = (c_1 e^{ikx} + c_2 e^{-ikx}) e^{-Bt^2} (c_3 e^{i\sqrt{\omega^2 - B^2} t} + c_4 e^{-i\sqrt{\omega^2 - B^2} t})$$

$$\text{Real} \left(\Psi \right) = e^{-Bt^2} [a e^{i(Kx + \sqrt{\omega^2 - B^2} t)} + b e^{i(-Kx + \sqrt{\omega^2 - B^2} t)} + c e^{-i(Kx + \sqrt{\omega^2 - B^2} t)} + d e^{-i(-Kx + \sqrt{\omega^2 - B^2} t)}]$$

$$= e^{-Bt^2} [\cos(Kx + \sqrt{\omega^2 - B^2} t) + i \sin(Kx + \sqrt{\omega^2 - B^2} t) + \cos(-Kx + \sqrt{\omega^2 - B^2} t) + i \sin(-Kx + \sqrt{\omega^2 - B^2} t)]$$

$$= e^{-Bt^2} [\sum \cos(Kx + \sqrt{\omega^2 - B^2} t) + \sum \cos(-Kx + \sqrt{\omega^2 - B^2} t)]$$

$$\Rightarrow \Psi = e^{-Bt^2} [\sum \cos(Kx + \sqrt{\omega^2 - B^2} t) + \sum \cos(-Kx + \sqrt{\omega^2 - B^2} t)]$$

c)

Problema 1. (Valor: 25 pts.)

Una cuerda muy larga está formada por tres segmentos de densidades distintas:

$$\lambda_m(x) = \begin{cases} 3\lambda_0 & x < 0, \\ 2\lambda_0 & 0 < x < L, \\ \lambda_0 & L < x, \end{cases}$$

donde λ_0 es una constante. Los segmentos de cuerda están unidos por nudos completamente ideales. Un tren de ondas, sin desfase, viaja desde $-\infty$ con una frecuencia angular ω y una rapidez v_1 .

- Determine el número de la onda y la rapidez de onda en cada segmento de la cuerda.
- ¿Cómo es la fase de la onda reflejada y transmitida en cada nudo? Justifique su respuesta.
- Encuentre la amplitud de la onda transmitida en el segmento más denso, en términos de la amplitud de la onda incidente.

a) $v_i^2 = \frac{\gamma}{\lambda_i}$

$$\Rightarrow v_i^2 = \frac{\gamma}{3\lambda_0}$$

$$v_i = \sqrt{\frac{\gamma}{\lambda_0}}$$

$$v_2^2 = \frac{\gamma}{2\lambda_0} = \frac{3\lambda_0 v_1^2}{2\lambda_0} = \frac{3}{2} v_1^2$$

$$v_1^2 = \frac{\gamma}{3\lambda_0} \Rightarrow \gamma = 3\lambda_0 v_1^2 \quad v_3^2 = \frac{\gamma}{\lambda_0} = \frac{3\lambda_0 v_1^2}{\lambda_0} = 3v_1^2$$

$$k_i = \frac{\omega}{v_i} = \frac{\omega}{\sqrt{\frac{\gamma}{\lambda_i}}} = \frac{\omega}{\sqrt{3\lambda_0 v_i^2}} = \frac{\omega \sqrt{x_i}}{\sqrt{3\lambda_0 v_i^2}} = \frac{\omega \sqrt{x_i}}{\sqrt{3\lambda_0 v_i^2}}$$

$$k_i = \frac{\omega \sqrt{x_i}}{\sqrt{3\lambda_0 v_i^2}}$$

$$k_1 = \frac{\omega \sqrt{3x_1}}{\sqrt{3\lambda_0 v_1^2}} = \frac{\omega}{v_1}$$

$$k_2 = \frac{\omega \sqrt{2x_2}}{\sqrt{3\lambda_0 v_1^2}} = \sqrt{\frac{2}{3}} \frac{\omega}{v_1}$$

$$k_3 = \frac{\omega \sqrt{x_3}}{\sqrt{3\lambda_0 v_1^2}} = \frac{1}{\sqrt{3}} \frac{\omega}{v_1}$$

b) $\phi_x = -k_i x + \omega t \rightarrow$ Se mueve de izquierda a derecha

$$\phi_{R_1} = k_1 x + \omega t$$

derecha a izquierda

$$\phi_{T_1} = -k_1 x + \omega t$$

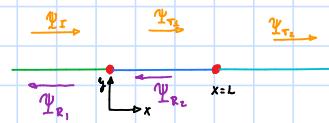
izquierda a derecha

$$\phi_{R_2} = k_2 x + \omega t$$

derecha a izquierda

$$\phi_{T_2} = -k_2 x + \omega t$$

izquierda a derecha



$$V_1 = V_1 \quad K_1 = \frac{\omega}{V_1}$$

$$V_2 = \frac{3}{2} V_1^2 \quad K_2 = \sqrt{\frac{2}{3}} \frac{\omega}{V_1}$$

$$V_3 = 3 V_1^2 \quad K_3 = \frac{1}{\sqrt{3}} \frac{\omega}{V_1}$$

Todas son ondas viajeras, entonces se obtienen de la forma $\Psi(x, t) = f(\pm x + vt)$

$$\Psi_I = A e^{i(-k_1 x + \omega t)} + B e^{i(k_1 x + \omega t)}$$

$$\Psi_{II} = C e^{i(-k_2 x + \omega t)} + D e^{i(k_2 x + \omega t)}$$

$$\Psi_{III} = E e^{i(-k_3 x + \omega t)}$$

$$c) \quad \Psi_I = A e^{i(-k_1 x + \omega t)} + B e^{i(k_1 x + \omega t)} \quad \Psi_{II} = C e^{i(-k_2 x + \omega t)} + D e^{i(k_2 x + \omega t)} \quad \Psi_{III} = E e^{i(-k_3 x + \omega t)}$$

Para $x=0$

$$\bullet \quad \Psi_I|_{x=0} = \Psi_{II}|_{x=0} \Rightarrow A e^{i \omega t} + B e^{i \omega t} = C e^{i \omega t} + D e^{i \omega t}$$

$$\Rightarrow A + B = C + D$$

$$\bullet \quad \frac{\partial \Psi_I}{\partial x}|_{x=0} = \frac{\partial \Psi_{II}}{\partial x}|_{x=0} \Rightarrow -i k_1 A e^{i \omega t} + i k_1 B e^{i \omega t} = -i k_2 C e^{i \omega t} + i k_2 D e^{i \omega t}$$

$$\Rightarrow -k_1 A + k_1 B = -k_2 C + k_2 D$$

Para $x=L$

$$\bullet \quad \Psi_{II}|_{x=L} = \Psi_{III}|_{x=L} \Rightarrow C e^{i(-k_2 L + \omega t)} + D e^{i(k_2 L + \omega t)} = E e^{i(-k_3 L + \omega t)}$$

$$\Rightarrow C e^{-i k_2 L} + D e^{i k_2 L} = E e^{-i k_3 L}$$

$$\frac{\partial \Psi_I}{\partial x} = -i k_1 A e^{i(-k_1 x + \omega t)} + i k_1 B e^{i(k_1 x + \omega t)}$$

$$\frac{\partial \Psi_{II}}{\partial x} = -i k_2 C e^{i(-k_2 x + \omega t)} + i k_2 D e^{i(k_2 x + \omega t)}$$

$$\bullet \quad \frac{\partial \Psi_{II}}{\partial x}|_{x=L} = \frac{\partial \Psi_{III}}{\partial x}|_{x=L} \Rightarrow -i k_2 C e^{i(-k_2 L + \omega t)} + i k_2 D e^{i(k_2 L + \omega t)} = -i k_3 E e^{i(-k_3 L + \omega t)}$$

$$\Rightarrow -k_2 C e^{-i k_2 L} + k_2 D e^{i k_2 L} = -k_3 E e^{-i k_3 L}$$

$$\begin{cases} A+B = C+D \\ -K_1 A + K_1 B = -K_2 C + K_2 D \\ C e^{iK_2 h} + D e^{iK_2 h} = E e^{-iK_3 h} \\ -K_2 C e^{-iK_3 h} + K_2 D e^{iK_3 h} = -K_3 F e^{-iK_3 h} \end{cases} \Rightarrow \begin{cases} A+B = C+D \\ A-B = \frac{K_2}{K_1}(C-D) \\ e^{iK_2 h}(C+D e^{2iK_2 h}) = E e^{-iK_3 h} \\ -K_2 e^{-iK_3 h}(C-D e^{2iK_3 h}) = -K_3 F e^{-iK_3 h} \end{cases}$$

$$\Rightarrow \begin{cases} A+B = C+D \\ A-B = \frac{K_2}{K_1}(C-D) \\ (C+D e^{2iK_2 h}) = E e^{-iK_3 h} \\ (C-D e^{2iK_3 h}) = \frac{K_3}{K_2} F e^{-iK_3 h} \end{cases} \Rightarrow \begin{cases} A+B = C+D & \textcircled{1} \\ A-B = \frac{K_2}{K_1}(C-D) & \textcircled{2} \\ (C+D e^{2iK_2 h}) = E e^{i(K_2-K_3)h} & \textcircled{3} \\ (C-D e^{2iK_3 h}) = \frac{K_3}{K_2} F e^{i(K_2-K_3)h} & \textcircled{4} \end{cases}$$

$$\Rightarrow \textcircled{1} + \textcircled{2} = 2A = C\left(1 + \frac{K_2}{K_1}\right) + D\left(1 - \frac{K_2}{K_1}\right) = C\left(\frac{K_1+K_2}{K_1}\right) + D\left(\frac{K_1-K_2}{K_1}\right)$$

$$\text{Reemplazo partes iguales} \Rightarrow (C-D e^{2iK_3 h}) = \frac{K_3}{K_2} (C+D e^{2iK_2 h})$$

(3) y (4)

$$\Rightarrow C\left(1 - \frac{K_2}{K_2}\right) = D e^{2iK_2 h}\left(1 + \frac{K_3}{K_2}\right)$$

$$\Rightarrow C\left(\frac{K_2-K_3}{K_2}\right) = D e^{2iK_2 h}\left(\frac{K_2+K_3}{K_2}\right)$$

$$\Rightarrow D = \frac{C(K_2-K_3)}{(K_2+K_3)} e^{-2iK_2 h}$$

$$K_2 \neq 2A = C\left(\frac{K_1+K_2}{K_1}\right) + C\left(\frac{K_1-K_2}{K_1}\right) \frac{(K_2-K_3)}{(K_2+K_3)} e^{-2iK_2 h}$$

$$= C \left[\frac{K_1+K_2}{K_1} + \frac{(K_1-K_2)(K_2-K_3)}{K_1(K_2+K_3)} e^{-2iK_2 h} \right]$$

$$C \left[\frac{(K_1+K_2)(K_2+K_3) + (K_1-K_2)(K_2-K_3)e^{-2iK_2 h}}{K_1(K_2+K_3)} \right]$$

$$C = 2A \left[\frac{K_1(K_2+K_3)}{(K_1+K_2)(K_2+K_3) + (K_1-K_2)(K_2-K_3)e^{-2iK_2 h}} \right]$$

Redundante

Problema 2. (Valor: 25 pts.)

Considere el sistema que se muestra en la figura, formado por tres masas puntuales y tres resortes ideales.

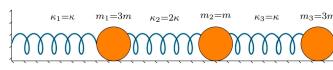


Figura 1: Configuración problema 2.

- Determine las frecuencias características para este sistema.
- Encuentre las coordenadas normales correspondientes. Trabaje su respuesta en términos de las frecuencias angulares ω_i , esto es: no es necesario escribir los valores numéricos.
- Describa el movimiento general a partir de los modos normales. Explique el razonamiento detrás de su respuesta.

$$U = \frac{k_1}{2}x_1^2 + \frac{k_2}{2}(x_1 - x_2)^2 + \frac{k_3}{2}(x_2 - x_3)^2$$

$$\frac{k_1}{2}x_1^2 + k(x_1 - x_2)^2 + \frac{k_3}{2}(x_2 - x_3)^2$$

$$U = \frac{k_1}{2}x_1^2 + k(x_1 - x_2)^2 + \frac{k_3}{2}(x_2 - x_3)^2 = \frac{k}{2}x_1^2 + kx_1^2 - 2kx_1x_2 + kx_2^2 + \frac{k}{2}x_2^2 - kx_2x_3 + \frac{k}{2}x_3^2$$

$$= \frac{3kx_1^2}{2} + \frac{3kx_2^2}{2} + \frac{kx_3^2}{2} - 2kx_1x_2 - kx_2x_3 = \frac{1}{2}(3kx_1^2 + 3kx_2^2 + kx_3^2 - 4kx_1x_2 - 2kx_2x_3)$$

$$A = \begin{pmatrix} 3k & -2k & 0 \\ -2k & 3k & -k \\ 0 & -k & k \end{pmatrix}$$

$$\det(A - \omega^2 M) = 0$$

$$\begin{vmatrix} 3k - \omega^2 \cdot 3m & -2k & 0 \\ -2k & 3k - \omega^2 \cdot m & -k \\ 0 & -k & k - \omega^2 \cdot 3m \end{vmatrix} = 0$$

$$\Rightarrow (3k - 3m\omega^2) [(3k - \omega^2 m)(k - 3m\omega^2) - k^2] + 2k [-2k(k - 3m\omega^2) - 0] + 0 = 0$$

$$\Rightarrow (3k - 3m\omega^2) [(3k - \omega^2 m)(k - 3m\omega^2) - k^2] - 4k^2(k - 3m\omega^2) = 0$$

$$\Rightarrow (3k - m\omega^2)(k - 3m\omega^2)(3k - 3m\omega^2) - k^2(3k - 3m\omega^2) - 4k^2(k - 3m\omega^2) = 0$$

$$\Rightarrow (3k^2 - 9km\omega^2 - km\omega^2 + 3m^2\omega^4)(3k - 3m\omega^2) - 3k^3 + 3k^2m\omega^2 - 4k^3 + 12k^2m\omega^2 = 0$$

$$(3k^2 - 10km\omega^2 + 3m^2\omega^4)(3k - 3m\omega^2) - 7k^3 + 15k^2m\omega^2 = 0$$

$$\Rightarrow \underline{9k^3} - \underline{30k^2m\omega^2} + \underline{9km^2\omega^4} - \underline{9k^2m\omega^2} + \underline{30km^2\omega^4} - \underline{9m^3\omega^6} - \underline{7k^3} + \underline{15k^2m\omega^2} = 0$$

$$2k^3 - 24k^2m\omega^2 + 39km^2\omega^4 - 9m^3\omega^6 = 0$$

$$9m^3\omega^6 - 39km^2\omega^4 + 24k^2m\omega^2 - 2k^3 = 0$$

$$\begin{aligned} T &= \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}m_3\dot{x}_3^2 \\ &= \frac{1}{2}3m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 + \frac{1}{2}3m\dot{x}_3^2 \\ &= \frac{3}{2}m\dot{x}_1^2 + \frac{m}{2}\dot{x}_2^2 + \frac{3}{2}m\dot{x}_3^2 \end{aligned}$$

$$T = \frac{1}{2}(3m\dot{x}_1^2 + m\dot{x}_2^2 + 3m\dot{x}_3^2)$$

$$M = \begin{pmatrix} 3m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 3m \end{pmatrix} = m \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$A = \begin{pmatrix} 3k & -2k & 0 \\ -2k & 3k & -k \\ 0 & -k & k \end{pmatrix}$$

$$q m^3 w^6 - 3 q k m^2 w^4 + 24 k^2 m w^2 - 2 k^3 = 0$$

$$q \frac{w^6}{m^3} - \frac{3 q k m^2 w^4}{m^3} + 24 \frac{k^2 m}{m^3} w^2 - \frac{2 k^3}{m^3} = 0$$

$$w^6 - \frac{3 q k m^2 w^4}{q m^3} + \frac{24 k^2 m}{q m^3} w^2 - \frac{2 k^3}{q m^3} = 0$$

$$w^6 - \frac{3 q}{q} \frac{k}{m} w^4 + \frac{24}{q} \frac{k^2}{m^2} w^2 - \frac{2}{q} \frac{k^3}{m^3} = 0$$

$$\frac{k}{m} = a \Rightarrow q w^3 - 3 a w^2 + 24 a^2 w - 2 a^3 = 0$$

$$w^2 = u$$

$$u = + 2 a^3$$

$$q \cdot 8 a^9 - 39 a \cdot 4 a^6 + 24 a^2 \cdot 2 a - 2 a^3 = 0$$

$$a^9 - 28 a^6 - 2 a^3$$

Problema 3. (Valor: 25 pts.)

Una cuerda de densidad de masa λ_m constante y largo $2L$ está fija en sus dos extremos, de tal forma que tensión en ella es τ . En $t = 0$, la cuerda está en equilibrio pero tiene una rapidez vertical dada por:

$$v(x, 0) = \begin{cases} v_0 & 0 \leq x \leq L, \\ -v_0 & L \leq x \leq 2L, \end{cases}$$

donde v_0 es una constante.

- a. Determine el desplazamiento $y(x, t)$ de la cuerda.
- b. Encuentre la velocidad correspondiente $v(x, t)$.

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