Exercise 10.4.3. Let α be a real number, and let $f\colon (0,\infty) \to \mathbf{R}$ be the function $f(x) = x^\alpha$.

(a) Show that $\lim_{x\to 1; x\in(0,\infty)\setminus\{1\}} \frac{f(x)-f(1)}{x-1} = \alpha$.

(b) Show that f is differentiable on $(0,\infty)$ and that $f'(x) = \alpha x^{\alpha-1}$.

Solution.

(a) It is enough to consider the right and left limits, prove that they exist, and that they equal α. We will focus on the right limit (the proof for the left limit is similar), i.e. we will show that $\lim_{x\to 1; x\in(1,\infty)} \frac{x^\alpha - 1}{x-1} = \alpha$.

It is enough to show that, if $(a_m)_{m=0}^\infty$ is a sequence converging to 1, with $1 < a_m$ for all $m \in \mathbf{N}$, then $\lim_{m\to\infty} \frac{(a_m)^\alpha - 1}{a_m - 1} = \alpha$.

Notice that for every positive integer n there is some rational number in the interval $\left(\alpha - \frac{1}{n}, \alpha\right)$. Using the axiom of (countable) choice, we can pick a rational from each of those intervals and have a sequence of rational numbers $(q_n^-)_{n=1}^\infty$. Similarly we can have a rational sequence $(q_n^+)_{n=1}^\infty$ by picking from the intervals $\left(\alpha, \alpha + \frac{1}{n}\right)$.

It is worth noting that

$$\alpha - \frac{1}{n} < q_n^- < \alpha < q_n^+ < \alpha + \frac{1}{n},$$

for every positive integer n. By two applications of the squeeze theorem we see that $(q_n^-)_{n=1}^\infty$ and $(q_n^+)_{n=1}^\infty$ both converge to α. Furthermore, for all n, m, as $a_m > 1$, we have the following chain of reasoning

$$\begin{gathered} a_m^{q_n^-} \leq a_m^\alpha \leq a_m^{q_n^-}, \\ a_m^{q_n^-} - 1 \leq a_m^\alpha - 1 \leq a_m^{q_n^+} - 1, \\ \frac{a_m^{q_n^-} - 1}{a_m - 1} \leq \frac{a_m^\alpha - 1}{a_m - 1} \leq \frac{a_m^{q_n^+} - 1}{a_m - 1}. \end{gathered}$$

The second part of Exercise 10.4.2 implies that $\lim_{m\to\infty} \frac{a_m^{q_n^-} - 1}{a_m - 1} = q_n^-$ and that $\lim_{m\to\infty} \frac{a_m^{q_n^+} - 1}{a_m - 1} = q_n^+$. Thus, by the comparison principle (Lemma 6.4.13), we get that

$$q_n^- \leq \liminf_{m\to\infty} \frac{a_m^\alpha - 1}{a_m - 1} \leq \limsup_{m\to\infty} \frac{a_m^\alpha - 1}{a_m - 1} \leq q_n^+.$$

Then, by two applications of the squeeze test, we see that the limit superior and the limit inferior coincide and equal α. Thus the sequence is converging to α, i.e. $\lim_{m\to\infty} \frac{a_m^\alpha - 1}{a_m - 1} = \alpha$, which is what we wanted.

We have shown that the right limit $\lim_{x\to 1; x\in(1,\infty)} \frac{x^\alpha - 1}{x-1}$ is α, and a similar argument will show that the left limit exists and also equals α. Then we have $\lim_{x\to 1; x\in(0,\infty)\setminus\{1\}} \frac{f(x)-f(1)}{x-1} = \alpha$.

(b) Let $x \in (0, \infty)$. We will show that $\lim_{n\to\infty} \frac{(x_n)^\alpha - x^\alpha}{x_n - x} = \alpha x^{\alpha-1}$, where $(x_n)_{n=0}^\infty$ is a sequence in $(0, \infty) \setminus \{x\}$ that converges to x; this is enough to show what we want.

Notice that $(\frac{x_n}{x})_{n=0}^\infty$ is a sequence in $(0, \infty) \setminus \{x\}$ that converges to 1, by limit laws. Hence, by part (a),

$$\lim_{n\to\infty} \frac{\left(\frac{x_n}{x}\right)^\alpha - 1}{\frac{x_n}{x} - 1} = \alpha. \tag{1}$$

Multiplying both sides of this equation by $x^{\alpha-1}$ and applying limit laws yields our result (to make this easier to see, multiply the numerator and denominator of the fraction by x).

$\square$