Introduction to Coxeter Groups

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Abstract

In these notes we will develop the basic theory of Coxeter groups. A lot of the material here is taken from [1] word-by-word. However, I have added my own commentary and intuition to aid the reader. The presentation has also been changed somewhat, by splitting up proofs of theorems into lemmas, providing different proofs, putting emphasis on other topics, etc. In any case, the reader should *not* assume any original contribution (mathematical or expository) is mine and should refer to [1] instead. Mistakes, of course, should all be attributed to me.

1 Basic definitions and properties

Let S be a set and let $m\colon S \times S \to \{1, 2, \ldots, \infty\}$ be a matrix. Then m is a *Coxeter matrix* if it is symmetric and has 1's on, and only on, the main diagonal.

Given a set S with a Coxeter matrix m we can define a *Coxeter group* W with the presentation $(S|\mathscr{R})$ where we define

$$\mathscr{R} := \left\{ (ss')^{m(s,s')} \mid s, s' \in S \text{ and } m(s, s') < \infty \right\}.$$

Intuitively, S is a set of reflections, hence why we immediately can deduce from the definition that $s^2 = e$ for all $s \in S$.[1] Furthermore, ss' is *not* a reflection, as can be seen with the dihedral groups, hence why we allow the possibility for it to have an order other than 2. In fact a relation $(ss')^k = e$ corresponds to hyperplanes meeting at an angle of $\frac{\pi}{k}$. We define reflections in Coxeter groups later. By multiplying on the left and on the right by s', we can see that from $(ss')^{m(s,s')} = e$ we deduce $(s's)^{m(s,s')} = e$.

If a group W, with a corresponding set S, has a presentation as described above we say that (W, S) is a *Coxeter system*. Interestingly, S is not determined by W, that is, a group can be a Coxeter group in more than one way (see Exercise 1.1).

The next proposition is useful for working with groups given by presentations.

Proposition 1.1 (Universality Property)**.** *Let $W = (S|\mathscr{R})$ be a group with a presentation and let G be a group. To specify a homomorphism from W to G it is enough to give a set-function $f\colon S \to G$ that respects the relations in $\mathscr{R}$.*

[1] We are abusing notation by identifying S with its copy in W.

Proof. As f is a function from S to G then, by the universal property of free groups, it extends uniquely to a group homomorphism $f' \colon F(S) \to G$ where $F(S)$ is the free group on S, and it is clear that f' satisfies the relations in $\mathscr{R}$. By the universal property of quotients, f' extends uniquely to a group homomorphism from $W = F(S)/N$ to G, where N is the normal subgroup of $F(S)$ generated by $\mathscr{R}$. □

Earlier it was mentioned that S cannot be reconstructed from W. Say we fix some set S and consider possible Coxeter matrices on S. Do different Coxeter matrices give us different Coxeter groups? The answer is yes.

Proposition 1.2. *Let (W,S) be a Coxeter system determined by a Coxeter matrix m. Let s, s' be distinct elements of S. Then, the following hold:*

1. *s and s' are distinct in W.*
2. *The order of ss' in W is $m(s, s')$*

So, given a Coxeter system it is possible to reconstruct the corresponding Coxeter matrix. This leads to an important conclusion.

Theorem 1.3. *Up to isomorphism, there is a one-to-one correspondence between Coxeter matrices and Coxeter systems.*

Proof. Missing! □

2 Realizing Coxeter systems as permutation groups

In this section we will learn how to construct an injective homomorphism from a Coxeter group to a symmetry group. We fix a Coxeter system (W, S).

Define $T \coloneqq \{ wsw^{-1} \mid s \in S, w \in W \}$. Intuitively T is set of all *reflections* in W, while elements of S are called *simple reflections.* Clearly $S \subseteq T$ and

$$t^2 = e \text{ for all } t \in T.$$

What is the most natural way of realizing W as a permutation group? Questions of these kind are usually answered by "have the group act on itself" as in the proof of Cayley's theorem. A similar approach will be used, but involving only the reflections of the group.

Imagine W acting on a space, say $\mathbb{R}^n$ with the standard inner product, with elements of T acting as genuine reflections about some hyperplanes. We will see that under any reflection in T these hyperplanes are permuted. This can be deduced from basic considerations of general group actions. Then if we associate every reflection with its hyperplane, that is its set of fixed points, we can extend to W permuting its own reflections.

Let G be a group acting on a set X via an action $*$. For any $g \in G$ define

$$\operatorname{fix}(g) \coloneqq \{x \in X \colon g * x = x\}.$$

Proposition 2.1. *Let $g, h \in G$. Then*

$$g * \mathrm{fix}(h) = \mathrm{fix}(ghg^{-1}).$$

Proof. Let $x \in X$ be such that $h*x = x$. Then $(ghg^{-1})*(g*x) = (ghg^{-1}g)*x = g*(h*x) = g*x$. Hence $g*\mathrm{fix}(h) \subseteq \mathrm{fix}(ghg^{-1})$.

Now let $x \in X$ be such that $(ghg^{-1})*x = x$. Then, multiplying by g^{-1} on both sides $h*((g^{-1})*x) = (g^{-1})*x$ and so $g^{-1}*x \in \mathrm{fix}(h)$ which means $x \in g*\mathrm{fix}(h)$. Thus, $\mathrm{fix}(ghg^{-1}) \subseteq g*\mathrm{fix}(h)$. □

If we read the above proposition for reflection groups, we see that reflecting by g sends the hyperplane associated to h to the hyperplane associated with ghg^{-1}. Thus one could define a function $S \to \mathrm{Aut}(T)$, which is just an action $S \times T \to T$, by sending $s*t \mapsto sts^{-1} = sts$. Then we could extend, using the universality (Proposition 1.1), to a homomorphism $W \to \mathrm{Aut}(T)$.

However, this action is not, in general, faithful. Take, for example, the symmetries of a square, also known as D_4, which is generated by reflections. Rotating by 180° leaves all the lines of reflection stationary, yet this is obviously not a trivial symmetry. The problem here is that, while the lines of symmetry are not permuted, their orientation does change. To keep track of this information we do the following. First, imagine associating to each hyperplane a certain normal vector. Then a symmetry permutes the hyperplanes as before but it also might send a normal vector to its negative; we can already see that with the example given before.

Remark 2.2. In fact, for a reflection r, we can see that the vector normal to the hyperplane associated with r flips its orientation, while all other normal vectors keep their relative orientation, even if their respective hyperplanes are shuffled around.

So, we will consider permutations of the set

$$R = T \times \{+1, -1\}.$$

Each element of R can be associated with one side of a hyperplane; if we pick one normal vector to each hyperplane then we can define a positive and negative side to each one.

For $s \in S$ and $t \in T$ let

$$\eta(s;t) \coloneqq \begin{cases} -1, & \text{if } s = t \\ +1, & \text{if } s \neq t \end{cases}.$$

Then for $s \in S$, define a mapping π_S of R to itself by

$$\pi_s(t, \varepsilon) \coloneqq (sts, \varepsilon\eta(s;t)).$$

Remark 2.3. As a sanity check, we can see that

$$\pi_s^2(t,\varepsilon) = \pi_s(sts, \varepsilon\eta(s;t)) = (sstss, \varepsilon\eta(s;t)\eta(s;sts)) = (t, \varepsilon(\eta(s;t))^2) = (t, \varepsilon).$$

Hence, π_s is its own inverse and so it is an automorphism of R. Remember that π_s is just permuting hyperplanes by a reflection s, while also keeping track of the orientation of each hyperplane. It makes sense that reflecting twice gives us the identity.

We aim for the following theorem. It will confirm that we have constructed an injective homomorphism from an arbitrary Coxeter group to a symmetry group, and also it will make precise the discussion in Remark 2.2.

Theorem 2.4. *The mapping $s \mapsto \pi_s$ extends uniquely to an injective homomorphism $w \mapsto \pi_w$ from W to $\mathrm{Aut}(R)$.*

Before proving this theorem, we will need a lemma. Firstly, we need to prove that the map $s \mapsto \pi_s$ indeed extends by Proposition 1.1, and to do that we need to check that it satisfies the relations of the Coxeter system. So, $s, s' \in S$ and $m(s, s') = p \neq \infty$, we need to prove that

$$(\pi_s \pi_{s'})^p = \mathrm{id}_R. \tag{1}$$

Note that this is intuitively obvious: applying the reflections $(ss')^p$ will not act on $\mathbb{R}^n$ and in particular it will leave the hyperplanes and their orientations unchanged.

So, if we have some $(t, \varepsilon) \in R$ we consider its image under $(\pi_s \pi_{s'})^p$, call it (t', ε'). Then $t' = ss' \dots s'ts's \dots s = (ss')^p t (s's)^p = t$, as expected.

But now we find that calculating ε' is not nearly as straightforward.

$$\varepsilon' = \varepsilon \prod_{i=1}^{2p} \eta(s_i; s_{i-1} \dots s_1 t s_1 \dots s_{i-1}), \tag{2}$$

where $s_i = s'$ if i is odd and $s_i = s$ if i is even. We want to know when $s_i = s_{i-1} \dots s_1 t s_1 \dots s_{i-1}$ holds, which is equivalent to asking when $t = s_1 \dots s_{i-1} s_i s_{i-1} \dots s_1$ holds, and in particular if it holds an even or odd number of times, for $1 \leq i \leq 2p$. This is a common situation and we will treat it a bit more generally to solve similar problems later on.

Intermezzo: Working with products of η's

Let S^* be the free monoid generated by S, i.e. the set of words with an alphabet S. There is an obvious morphism from S^* to W that just evaluates words with the group multiplication in W, so we often consider elements of S^* as elements of W without further comment.

For each word $s_1 s_2 \dots s_k \in S^*$ we define

$$t_i \coloneqq s_1 s_2 \dots s_{i-1} s_i s_{i-1} \dots s_2 s_1, \text{ for } 1 \leq i \leq k.$$

Clearly $t_i = (s_1 \dots s_{i-1}) s_i (s_1 \dots s_{i-1})^{-1}$ so t_i is always a reflection. Intuitively, if we are acting on $\mathbb{R}^n$ then t_1 can be thought of as the hyperplane associated to s_1, then t_2 is where the hyperplane associated to s_2 is sent to after the action of s_1, and in general t_i is where the hyperplane associated to s_i lands after performing s_{i-1}, then s_{i-2}, etc. until s_1. And hence if $t = t_i$ is such a hyperplane then performing all the reflections in reverse shows that the hyperplane associated to t lands in the hyperplane associated to s_i after the action of s_1, then s_2, etc. until s_{i-1}. And if we were to perform s_i next, then t would not move, but its orientation would change: exactly the phenomenon we are tracking.

If $t \in T$ we let

$$\begin{aligned} n(s_1 s_2 \dots s_k; t) &\coloneqq |\{\, i \in \{1, \dots, k\} \colon t = t_i \,\}| \\ &= \text{number of times } t = t_i \text{ holds.} \end{aligned}$$

By our previous discussion it is clear that we have the identity

$$\prod_{i=1}^{k} \eta(s_i; s_{i-1} \dots s_1 t s_1 \dots s_{i-1}) = (-1)^{n(s_1 s_2 \dots s_k; t)} \tag{3}$$

We also have a useful lemma.

Lemma 2.5. *If $w = s_1 s_2 \dots s_k$, with k minimal, then $t_i \neq t_j$ for all $1 \leq i < j \leq k$.*

Proof. If $t_i = t_j$ for some $i < j$ then $t_i t_j = e$ and so $w = t_i t_j s_1 \dots s_k$ which means $w = s_1 \dots \hat{s_i} \dots \hat{s_j} \dots s_k$ (i.e. s_i and s_j deleted). This contradicts the minimality of k. $\square$

Remark 2.6. The t_i's have a geometric interpretation involving chambers: they are the shortest sequence of adjacent hyperplanes needed to go from wB to B where B is the base chamber. Since we aren't focusing too much on the geometry of Coxeter groups, we do not pursue these observations.

(End of Intermezzo)

Coming back to equation (2), we can apply equation (3) to get

$$\varepsilon' = (-1)^{n(s_1 s_2 \dots s_{2p}; t)} \varepsilon, \tag{4}$$

where we have $s_i = s'$ if i is odd and $s_i = s$ if i is even. In particular,

$$t_i = s_1 s_2 \dots s_i \dots s_2 s_1 = (s's)^{i-1} s', \text{ for } 1 \leq i \leq 2p.$$

And from this we deduce $t_{p+i} = t_i$ for $1 \leq i \leq p$ since $(s's)^p = e$. So the number of times $t = t_i$ holds, i.e. $n(s_1 s_2 \dots s_{2p}; t)$, must be necessarily even. Thus, by equation (4), we have $\varepsilon' = \varepsilon$ as desired. We have shown equation (1) for arbitrary $s, s' \in S$ and $p = m(s, s') \neq \infty$.

Lemma 2.7. *The mapping $s \mapsto \pi_s$ respects the relations of the Coxeter system.* $\square$

Now we are ready for the theorem.

Proof of Theorem 2.4. By Lemma 2.7 and universality the mapping $s \mapsto \pi_s$ extends to a homomorphism $w \mapsto \pi_w$ from W to $\mathrm{Aut}(R)$. If we have $w = s_k \dots s_1 \in W$ and $(t, \varepsilon) \in R$ then

$$\begin{aligned}
\pi_w(t, \varepsilon) &= \pi_{s_k} \pi s_{k-1} \dots \pi_{s_1}(t, \varepsilon) \\
&= \left(s_k \dots s_1 t s_1 \dots s_k, \varepsilon \prod_{i=1}^{k} \eta(s_i; s_{i-1} \dots s_1 t s_1 \dots s_{i-1}) \right) \\
&= \left(wtw^{-1}, \varepsilon(-1)^{n(s_1 s_2 \dots s_k; t)} \right). \qquad (5)
\end{aligned}$$

Now we check that this homomorphism is injective.

Let $w = s_k \dots s_1 \in W$ be a non-identity element, where k is minimal. By Lemma 2.5 all the t_i's are distinct and hence $n(s_1 s_2 \dots s_k; t)$ is at most 1 for all $t \in T$. In particular $n(s_1 s_2 \dots s_k; t_i) = 1$ for all $1 \leq i \leq k$. Hence by equation (5) we see that $\pi_w(t_i, \varepsilon)$ maps ε to $-\varepsilon$ and so π_w is not the identity. We have shown that the kernel of $w \mapsto \pi_w$ is just e, and hence the mapping is injective. □

Corollary 2.8. *For all reflections $t \in T$, $\pi_t(t, \varepsilon) = (t, -\varepsilon)$.*

Proof. We can write t as $s_1 s_2, \dots s_p \dots s_2 s_1$, for $s_i \in S$. We use induction on p.

The case $p = 1$ is clear by definition. Then, by the inductive hypothesis,

$$\begin{aligned}
\pi_{s_1 \dots s_p \dots s_1}(s_1 \dots s_p \dots s_1, \varepsilon) &= \pi_{s_1} \pi_{s_2 \dots s_p \dots s_2}(s_2 \dots s_p \dots s_2, \varepsilon \eta(s_1; s_1 \dots s_p \dots s_1)) \\
&= \pi_{s_1}(s_2 \dots s_p \dots s_2, -\varepsilon \eta(s_1; s_1 \dots s_p \dots s_1)) \\
&= (s_1 \dots s_p \dots s_1, -\varepsilon \eta^2(s_1; s_1 \dots s_p \dots s_1)) \\
&= (t, -\varepsilon).
\end{aligned}$$

□

For $w \in W$ and $t \in T$, let

$$\eta(w; t) := (-1)^{n(s_1 s_2 \dots s_k; t)}, \qquad (6)$$

where $w = s_1 s_2 \dots s_k$ is an arbitrary expression, $s_i \in S$. This definition agrees with our old one when $w \in S$. And this expression is well defined in general since, by equation (5), the parity of $n(s_1 s_2 \dots s_k; t)$ depends only on w and t. Then we can rewrite equation (5) as

$$\pi_w(t, \varepsilon) = \left(wtw^{-1}, \varepsilon \eta(w^{-1}; t) \right). \qquad (7)$$

3 Reduced words and the exchange property

Let (W, S) be a Coxeter system. Each element $w \in W$ can be written as a product of generators:

$$w = s_1 s_2 \dots s_k, \text{ for } s_i \in S.$$

If k is minimal among all such expressions for w, then k is called the *length* of w, written as $\ell(w) = k$, and the word $s_1 s_2 \dots s_k$ is called a *reduced word* for w. The length of the identity is zero.

Lemma 3.1. *The map $\varepsilon \colon s \mapsto -1$, for all $s \in S$, extends to a group homomorphism $\varepsilon : W \to \{+1, -1\}$.*

Proof. This is immediate from the Universality Property. □

As a consequence of Lemma 3.1, the elements of even length form a subgroup of W of index 2. This is called the *alternating subgroup* (following the terminology of the symmetric group) or the *rotation subgroup* (following the terminology of finite reflection groups) of W.

Lemma 3.2 (Basic properties of lengths)**.** *For all $u, w \in W$:*

(i) $\varepsilon(w) = (-1)^{\ell(w)}$,

(ii) $\ell(uw) \equiv \ell(u) + \ell(w) \pmod 2$,

(iii) $\ell(sw) = \ell(w) \pm 1$, *for all* $s \in S$,

(iv) $\ell(w^{-1}) = \ell(w)$

(v) $|\ell(u) - \ell(w)| \leq \ell(uw) \leq \ell(u) + \ell(w)$

(vi) $\ell(uw^{-1})$ *is a metric on* W.

Proof. (i) This is immediate from Lemma 3.1.

(ii) This is true if and only if $(-1)^{\ell(uw)} = (-1)^{\ell(u)+\ell(w)}$ which is true iff $\varepsilon(uw) = \varepsilon(u)\varepsilon(w)$ which is true since ε is a homomorphism.

(iii) Let $w = s_1 \dots s_k$ and let $sw = s'_1 \dots s'_p$ be such that k and p are minimal, i.e. $\ell(w) = k$ and $\ell(sw) = p$. Then from $s(s_1 \dots s_k) = s'_1 \dots s'_p$ we deduce $k + 1 \geq p$. Multiplying the equality by s yields $s_1 \dots s_k = s(s'_1 \dots s'_p)$ which means $p + 1 \geq k$. All in all $k - 1 \leq p \leq k + 1$ and hence $-1 \leq \ell(sw) - \ell(w) \leq 1$. By (ii) we see that $\ell(sw) - \ell(w) \equiv \ell(s) \equiv 1 \pmod 2$. In other words, $\ell(sw) - \ell(w) \neq 0$ and so $\ell(sw) - \ell(w) = \pm 1$.

(iv) Let $w = s_1 \dots s_k$ and let $w^{-1} = s'_1 \dots s'_p$ be such that k and p are minimal. Then $w^{-1} = s_k \dots s_1$ so $k \geq p$. But also $w = s'_p \dots s'_1$ so $p \geq k$. Then $k = p$ and the claim follows.

(v) First notice that, by an argument analogous to the one given in (iii), $\ell(vs) = \ell(w) \pm 1$ for all $v \in W$ and $s \in S$ (a quicker way to see this is to use (iii) together with (iv)).

Now by writing u as a reduced word, a repeated application of (ii) yields

$$\ell(uw) = \ell(w) \underbrace{\pm 1 \pm 1 \pm \ldots \pm 1}_{\ell(u) \text{ times}} \geq \ell(w) - \ell(u). \tag{8}$$

A similar argument, using the analogous of (ii) mentioned earlier, yields $\ell(uw) \geq \ell(u) - \ell(w)$. In conclusion, $|\ell(u) - \ell(w)| \leq \ell(uw)$. Equation (8) can be also used to deduce that $\ell(uw) \leq \ell(w) + \ell(u)$.

(vi) We have $\ell(uw^{-1}) = 0$ iff $uw^{-1} = e$, iff $u = w$. To verify symmetry notice that $\ell(wu^{-1}) = \ell((wu^{-1})^{-1})$ by (iv), and this is equal to $\ell(uw^{-1})$. As for the triangle inequality, if $v \in W$ then by (v) we have

$$\ell(uv^{-1}) = \ell((uw^{-1})(wv^{-1})) \leq \ell(uw^{-1}) + \ell(wv^{-1}). \qquad \square$$

Now we come to a property of Coxeter systems which is quite important, in a sense that will be made precise later on. First we prove a lemma.

Lemma 3.3. *Let $w \in W$ and let $t \in T$. Then $\ell(tw) < (w)$ if and only if $\eta(w;t) = -1$.*

Proof. First, assume $\eta(w;t) = -1$ and let $w = s_1' \ldots s_d'$ be a reduced expression for w. We find that $n(s_1' \ldots s_d'; t)$ is odd and so $t = s_1' s_2' \ldots s_i' \ldots s_2' s_1'$ for some $1 \leq i \leq d$. Hence,

$$\ell(tw) = \ell(s_1' \ldots \hat{s_i'} \ldots s_d') < d = \ell(w).$$

Conversely, suppose $\eta(w;t) = 1$. Then

$$\begin{aligned} \pi_{(tw)^{-1}}(t, \varepsilon) &= \pi_{w^{-1}} \pi_t(t, \varepsilon) = \pi_{w^{-1}}(t, -\varepsilon) \\ &= (w^{-1}tw, -\varepsilon\eta(w;t)) = (w^{-1}tw, -\varepsilon). \end{aligned}$$

Hence $\eta(tw;t) = -1$. By what we have shown before, $\ell(ttw) < \ell(tw)$, that is $\ell(tw) > \ell(w)$. $\square$

Now we are ready to prove the exchange property for Coxeter groups.

Theorem 3.4 (Strong Exchange Property)**.** *Suppose $w = s_1 s_2 \ldots s_k$ with $s_i \in S$ and let $t \in T$. If $\ell(tw) < \ell(w)$, then $tw = s_1 \ldots \hat{s}_i \ldots s_k$ (i.e. s_i removed) for some $1 \leq i \leq k$.*

Proof. By Lemma 3.3, we have that $\eta(w;t) = -1$, which means $n(s_1 s_2 \ldots s_k; t)$ is odd and so there is at least one i so that $t = t_i = s_1 s_2 \ldots s_i \ldots s_2 s_1$. The claim follows from computing tw. $\square$

This characterizes Coxeter groups among groups generated by involutions.

Next we define, for all $w \in W$,

$$\begin{aligned} T_L(w) &:= \{\, t \in T : \ell(tw) < \ell(w) \,\} \\ T_R(w) &:= \{\, t \in T : \ell(wt) < \ell(w) \,\} \end{aligned} \tag{9}$$

These are the *left associated reflections* to w and the *right associated reflections* to w respectively. We now give some useful characterizations of these sets.

Corollary 3.5. *If $w = s_1 s_2 \dots s_k$ is reduced and $t \in T$, then the following are equivalent.*

(a) $\ell(tw) < \ell(w)$

(b) $tw = s_1 \dots \hat{s}_i \dots s_k$ *for some* $1 \leq i \leq k$.

(c) $t = t_i = s_1 s_2 \dots s_i \dots s_2 s_1$ *for some* $1 \leq i \leq k$.

Further the index i appearing in (b) and (c) is uniquely determined

Proof. The equivalence $(b) \Leftrightarrow (c)$ is an easy calculation, which would be unenlightening to write in full here. As the word is reduced, all the t_i's are different, by Lemma 2.5, so the index is unique. That $(a) \Rightarrow (b)$ is precisely Theorem 3.4, while $(b) \Rightarrow (a)$ is clear. □

It should be clear that there are similar statements for the case $\ell(wt) < \ell(w)$ since

$$T_R(w) = T_L(w^{-1}).$$

Corollary 3.6. $|T_L(w)| = \ell(w)$.

Proof. Let $k = \ell(w)$ and $w = s_1 \dots s_k$. By Corollary 3.5, $T_L(w)$ is the set of of t_i's for $1 \leq i \leq k$, and by Lemma 2.5 these are all distinct. □

Next we define the associated *simple* reflections.

$$\begin{aligned} D_L(w) &:= T_L(w) \cap S \\ D_R(w) &:= T_R(w) \cap S \end{aligned} \tag{10}$$

These are the sets of left and right *descents*, respectively. Again, by symmetry, $D_R(w) = D_L(w^{-1})$.

Proposition 3.7. *For all $s \in S$ and $w \in W$, we have that $s \in D_L(w)$ if and only if there is some reduced expression for w which begins with the letter s.*

Proof. The "if" direction is clear. Now assume $s \in D_L(w)$. and let $k := \ell(w)$. Clearly $\ell(sw) < \ell(w)$. By Lemma 3.2, $\ell(sw) = k - 1$. Then if $s_1 \dots s_{k-1}$ is a reduced word for sw then $ss_1 \dots s_{k-1}$ is a reduced word for w beginning with s. □

4 Realizing permutation groups as Coxeter systems

We seek to prove that S_n is a Coxeter group. Indeed, we will see it is generated by adjacent transpositions, i.e. the set of transpositions of the form $(i, i+1)$ for all $1 \leq i \leq n-1$. It is intuitively obvious that, having n objects arranged in a line, it is possible to achieve all their permutations by only swapping adjacent objects. Furthermore, these transpositions satisfy the following relations.

1. For all $1 \leq i \leq n-1$,
$$((i, i+1)(i, i+1))^1 = (i, i+1)^2 = e.$$

2. For all $1 \leq i, j \leq n-1$ such that $j \neq i-1, i+1$,
$$((i, i+1)(j, j+1))^2 = (i, i+1)^2(j, j+1)^2 = e.$$

3. For all $1 \leq i \leq n-1$,
$$((i, i+1)(i+1, i+2))^3 = (i, i+1, i+2)^3 = e,$$
and
$$((i, i+1)(i-1, i))^3 = (i-1, i, i+1)^3 = e.$$

From these identities, we construct our Coxeter system. Start with any set $S = \{s_1, \ldots, s_{n-1}\}$ of cardinality $n-1$. Define the Coxeter matrix $m \colon S \times S \to \{1, 2, \ldots, \infty\}$ by

$$m(s_i, s_j) \coloneqq \begin{cases} 1, \text{ if } i = j \\ 2, \text{ if } j \notin \{i-1, i+1\} \\ 3, \text{ if } j \in \{i-1, i+1\} \end{cases}$$

We will identify each s_i with adjacent transpositions, hence the relations given above. Call the group generated by this Coxeter system A_{n-1}. These are called the *Coxeter groups of type A*.

Lemma 4.1. *A_{n-1} is a subgroup of A_n in a canonical way (so $A_1 \subseteq A_2 \subseteq \ldots$ is a chain of subgroups).*

Proof. Send each s_i, as a generator of A_{n-1}, to s_i, as an element of A_n. This assignment clearly respects the relations of A_{n-1} and hence extends to a group homomorphism that sends each word in A_{n-1} to essentially the same word in A_n. □

Theorem 4.2. *The groups A_{n-1} and S_n are isomorphic. In particular, S_n is a Coxeter group.*

Proof. Let $f \colon S \to S_n$ be a set-function defined by $f(s_i) = (i, i+1)$. By our previous discussion, we know that this function respects the relations given by the Coxeter matrix. By the Universality Property, the function extends to a homomorphism $\bar{f} \colon A_{n-1} \to S_n$ in a unique way. □

References

[1] Anders Bjorner and Francesco Brenti. *Combinatorics of Coxeter groups.* Vol. 231. Springer Science & Business Media, 2006.