Exercise 10.1.4. Prove Theorem 10.1.15.

Theorem 10.1.15 (Chain Rule)**.** *Let X, Y be subsets of $\mathbb{R}$, let $x_0 \in X$ be a limit point of X, and let $y_0 \in Y$ be a limit point of Y. Let $f : X \to Y$ be a function such that $f(x_0) = y_0$, and such that f is differentiable at x_0. Suppose that $g : Y \to \mathbb{R}$ is a function which is differentiable at y_0. Then the function $g \circ f : X \to \mathbb{R}$ is differentiable at x_0, and*

$$(g \circ f)'(x_0) = g'(y_0)f'(x_0).$$

Proof. By Newton's approximation (Proposition 10.1.7), it will be sufficient to show that, for every $\varepsilon > 0$, there exists some $\delta > 0$ such that for all $x \in X$ with $|x - x_0| \leq \delta$ we have

$$|g \circ f(x) - [g(y_0) + (g'(y_0)f'(x_0))(x - x_0)]| \leq \varepsilon|x - x_0|.$$

So, let $\varepsilon > 0$, and also let $\varepsilon' > 0$. As f is differentiable at x_0, we know, by Newton's approximation, that there is some $\delta_1 > 0$ such that if $x \in X$ and $|x - x_0| \leq \delta_1$ then

$$|f(x) - (f(x_0) + f'(x_0)(x - x_0))| \leq \varepsilon'|x - x_0|. \tag{1}$$

Similarly, we know that g is differentiable at y_0, hence by Newton's approximation there is some $\delta_2 > 0$ such that if $y \in Y$ and $|y - y_0| \leq \delta_2$ then

$$|g(y) - (g(y_0) + g'(y_0)(y - y_0))| \leq \frac{\varepsilon}{2(\varepsilon' + |f'(x_0)|)}|y - y_0|. \tag{2}$$

From (1) we have that, if $x \in X$ and $|x - x_0| \leq \delta_1$, then

$$f'(x_0)(x - x_0) - \varepsilon'|x - x_0| \leq f(x) - f(x_0) \leq f'(x_0)(x - x_0) + \varepsilon'|x - x_0|$$

As $a \leq |a|$, for $a \in \mathbb{R}$, we have that $f(x) - f(x_0) \leq |x - x_0|(|f'(x_0)| + \varepsilon')$. We can also see that $-|a| \leq a$ for $a \in \mathbb{R}$; thus, $-|x - x_0|(|f'(x_0)| + \varepsilon') \leq f(x) - f(x_0)$. Therefore we can conclude that

$$|f(x) - f(x_0)| \leq |x - x_0|(|f'(x_0)| + \varepsilon'). \tag{3}$$

Define $\delta := \min\{\delta_1, \delta_2, \frac{\delta_2}{|f'(x_0)| + \varepsilon'}\}$, and suppose we have some $x \in X$ such that we also have $|x - x_0| \leq \delta$. Then, notice that

$$\begin{aligned}
&|g \circ f(x) - [g(y_0) + (g'(y_0)f'(x_0))(x - x_0)]| \\
&\quad = |g \circ f(x) - (g(y_0) + g'(y_0)(f(x) - f(x_0))) + g'(y_0)(f(x) - y_0) \\
&\quad - g'(y_0)f'(x_0)(x - x_0)|.
\end{aligned}$$

So, by the triangle inequality, we have

$$\begin{aligned}
&|g \circ f(x) - [g(y_0) + (g'(y_0)f'(x_0))(x - x_0)]| \\
&\quad \leq |g \circ f(x) - (g(y_0) + g'(y_0)(f(x) - y_0))| \\
&\quad + |g'(y_0)(f(x) - y_0) - g'(y_0)f'(x_0)(x - x_0)|
\end{aligned}. \tag{4}$$

Clearly $f(x) \in Y$. Furthermore, we can combine the fact that $|x - x_0| \leq \delta \leq \frac{\delta_2}{|f'(x_0)!+\varepsilon'}$ with equation (3) to conclude that

$$|f(x) - y_0| \leq \frac{\delta_2}{|f'(x_0)|+\varepsilon'}(|f'(x_0)|+\varepsilon') = \delta_2.$$

Thus we can apply (2) to (4) and get that

$$\begin{aligned}
&|g \circ f(x) - [g(y_0) + (g'(y_0)f'(x_0))(x - x_0)]| \\
&\quad \leq \frac{\varepsilon|f(x) - y_0|}{2(\varepsilon' + |f'(x_0)|)} + |g'(y_0)(f(x) - y_0) - g'(y_0)f'(x_0)(x - x_0)| \\
&\quad = \frac{\varepsilon|f(x) - y_0|}{2(\varepsilon' + |f'(x_0)|)} + |g'(y_0)||(f(x) - y_0) - f'(x_0)(x - x_0)|
\end{aligned}$$

We can apply now (1) and (3) to get

$$|g \circ f(x) - [g(y_0) + (g'(y_0)f'(x_0))(x - x_0)]| \leq \frac{\varepsilon|x - x_0|}{2} + |g'(y_0)|\varepsilon'|x - x_0| \quad (5)$$

If $|g'(y_0)| = 0$ then clearly this is less than $\varepsilon|x - x_0|$, which is what we want to show. If $|g'(y_0)| \neq 0$ then we can assign $\varepsilon' \coloneqq \frac{\varepsilon}{2|g'(y_0)|}$. Now (5) becomes

$$|g \circ f(x) - [g(y_0) + (g'(y_0)f'(x_0))(x - x_0)]| \leq \frac{\varepsilon|x - x_0|}{2} + \frac{\varepsilon|x - x_0|}{2} = \varepsilon|x - x_0|,$$

which is what we wanted to show. $\square$