Theorem 0.1. *Prove the claim in the proof of Lemma 8.5.14, namely that every element of $Y' \setminus Y$ is an upper bound for Y and vice versa.*

Proof. We first show that, for any $n \in Y \cap Y'$ we have

$$\{ y \in Y : y \leq n \} = \{ y \in Y' : y \leq n \} = \{ y \in Y \cap Y' : y \leq n \},$$

i.e. that any element less than or equal to n that belongs to either Y or Y' belongs to both.

For the sake of contradiction, assume there is some element of $Y \cup Y'$, less than or equal to n, that does not belong to $Y \cap Y'$. In other words, assume that the set $E \coloneqq \{ y \in Y \cup Y' : y \leq n \text{ and } y \notin Y \cap Y' \}$ is non-empty.

E is not necessarily well-ordered. Yet, we claim that E has at least one minimal element. Notice that $E \cap Y$ and $E \cap Y'$ are well-ordered and at least one of them must be non-empty since their union is E. So, at least one of $\min E \cap Y$ and $\min E \cap Y'$ must exist. One of them must be a minimal element of E because of the following.

Without loss of generality assume $E \cap Y$ is non-empty, and define $e_Y \coloneqq \min E \cap Y$. Suppose that e_Y is not a minimal element of E, i.e. there exists some $e \in E$ such that $e < e_Y$. Clearly $e \notin Y$ since otherwise we would have $e \in E \cap Y$ and $e < e_Y = \min E \cap Y$. So instead we must have $e \in Y'$ and hence $e \in E \cap Y'$ and this set is non-empty.

Define $e_{Y'} \coloneqq \min E \cap Y'$. Let $m \in E$; hence either $m \in Y$ or $m \in Y'$. If $m \in Y'$ then $m \in E \cap Y'$ and clearly $e_{Y'} \leq m$. If $m \in Y$ then we have the relations: $e_{Y'} \leq e$, $e < e_Y$, and $e_Y < m$, which together imply $e_{Y'} < m$ by transitivity. In all cases $e_{Y'} \leq m$. Thus assuming that e_Y existed and that it was not a minimal element of E led us to conclude that $e_{Y'}$ was a minimal element of E. In the same fashion we can prove that $e_{Y'}$ existing and not being a minimal element of E implies e_Y is. As at least one of e_Y or $e_{Y'}$ must exist, this proves E has a minimal element, call it z.

Since $z \in E$, z belongs to one of Y or Y' but not both. Without loss of generality, let $z \in Y$ and $z \notin Y'$, which implies $z \neq x_0$. So, $z \in Y \setminus \{x_0\}$ and, as Y is good, this means $z = s(\{ y \in Y : y < z \})$. Notice that for any $y < z$ we have $y \in Y$ if and only if $y \in Y'$: this is because if y belonged to one set but not to the other, together with the fact that $y < z \leq n$, means that $y \in S$ and then we cannot have $y < z$ in the first place because z is a minimal element of S. So we have $\{ y \in Y : y < z \} = \{ y \in Y' : y < z \}$ and hence $z = s(\{ y \in Y' : y < z \})$. This equality suggests that $z \in Y'$, which would yield our contradiction.

Consider $\{ y \in Y' : y \text{ is an strict upper bound of } \{ y \in Y' : y < z \} \}$. As n belongs to this set, it is a non-empty subset of Y', which is well-ordered, so it has a minimum. Let w be this minimum. Note $w \neq x_0$ since x_0 is not a strict upper bound of $\{ y \in Y' : y < z \}$, rather it's contained in it. Thus $w = s(\{ y \in Y' : y < w \})$ since $w \in Y' \setminus \{x_0\}$ and Y' is good. Next we show that $w = z$.

Let $y \in Y'$. If $y < z$ then $y < w$ by definition of w. Conversely, suppose $y < w$; then y cannot be an strict upper bound of $\{ y \in Y' : y < z \}$ because w

is the least of all such upper bounds. So there is some $y' \in Y'$ such that $y' < z$ and $y \leq y'$, but transitivity says that then $y < z$. Hence $y < z$ iff $y < w$, so $\{ y \in Y' \colon y < z \} = \{ y \in Y' \colon y < w \}$ and, finally, this is

$$z = s(\{ y \in Y' \colon y < z \}) = s(\{ y \in Y' \colon y < w \}) = w,$$

which is a contradiction because $z \notin Y'$ but $w \in Y'$.

We have proven that, for all $n \in Y \cap Y'$ we have

$$\{ y \in Y \colon y \leq n \} = \{ y \in Y' \colon y \leq n \} = \{ y \in Y \cap Y' \colon y \leq n \}. \tag{1}$$

Clearly $Y \cap Y'$ is well-ordered, since it is the subset of a well-ordered set. Furthermore, its minimum element is x_0. Let $x \in Y \cap Y' \setminus \{x_0\}$. Since $x \in Y \setminus \{x_0\}$ we have $x = s(\{ y \in Y \colon y < x \})$, which is equal to $s(\{ y \in Y \cap Y' \colon y < x \})$ by our previous result (1), hence proving that $Y \cap Y'$ is good.

As $Y \cap Y'$ is good then $s(Y \cap Y')$ exists. Suppose $Y' \setminus Y$ is non-empty (so it has a minimum); then we claim $s(Y \cap Y') = \min Y' \setminus Y$.

Let $M \coloneqq \min Y' \setminus Y$. Then we see that $M \in Y' \setminus \{x_0\}$, and so $M = s(\{ y \in Y' \colon y < M \})$ since Y' is good. Note that if $y \in Y'$ and $y < M$ then $y \in Y$ because otherwise $y \in Y' \setminus Y$ which means we can't have $y < M = \min Y' \setminus Y$; hence $y \in Y \cap Y'$. Conversely, if we start by assuming $y \in Y \cap Y'$ we have that $y < M$, or $y \geq M$ because $y, M \in Y'$ and Y' is totally ordered. But we can't have $y \geq M$ because all elements less than or equal to $y \in Y \cap Y'$ that belong to either Y or Y' belong to $Y \cap Y'$ (see (1)), and $M \in Y' \setminus Y$. So, $y < M$ instead, and also $y \in Y'$. This all shows that $\{ y \in Y' \colon y < M \} = Y \cap Y'$ and hence $M = \min Y' \setminus Y = s(Y \cap Y')$.

Similarly we can show that if $Y \setminus Y'$ is non-empty we have $s(Y \cap Y') = \min Y \setminus Y'$. But if both $Y \setminus Y'$ and $Y' \setminus Y$ are non-empty we would have $s(Y \cap Y') = \min Y \setminus Y' = \min Y' \setminus Y$, which is impossible since $Y \setminus Y'$ and $Y' \setminus Y$ are disjoint. Hence, one of $Y \setminus Y'$ and $Y' \setminus Y$ must be empty, and it easily follows that either $Y \subseteq Y'$ or $Y' \subseteq Y$.

Finally, let $x \in Y' \setminus Y$, assuming this set is non-empty. Then, $Y \setminus Y'$ must be empty, and hence $Y \subseteq Y'$ which means $Y \cap Y' = Y$. We have $x \geq \min Y' \setminus Y = s(Y \cap Y') = s(Y)$. So $x \geq s(Y)$; by definition of s this of course means that x is an strict upper bound of Y. Every element of $Y' \setminus Y$ is an strict upper bound for Y (and vice versa too, proven similarly) which is what we wanted to show. $\square$