Torsion classes and torsion theories

Hernán Ibarra

Throughout this article, let C be the category of finite-dimensional (left) modules of some finite-dimensional algebra. Note that, especially at the beginning, one could pretend we are working in a general abelian category and most of the results will hold.[1]

By a subcategory of C we mean a strictly full additive subcategory of C, i.e. subcategories are full, closed under isomorphisms, sums and summands. When we specify a subcategory by its objects, we always mean the subcategory generated by those objects under the above conditions.

1 Definitions

The main definition is the following. We will see that all other definitions are a by-product of this one.

Definition 1.1 (Torsion theory)**.** A torsion theory is a pair $(\mathcal{T}, \mathcal{F})$ of subcategories of C, such that the following two conditions hold.

1. (Orthogonality) For all $X \in \mathcal{T}$ and $Y \in \mathcal{F}$ we have $\mathrm{Hom}_{\mathsf{C}}(X, Y) = 0$
2. (Extensionality) For all $A \in \mathsf{C}$ there exists some $X \in \mathcal{T}$ and $Y \in \mathcal{F}$ such that
$$0 \longrightarrow X \longrightarrow A \longrightarrow Y \longrightarrow 0$$
is a short exact sequence.

Exercise 1. Let Ab be the category of abelian groups. Prove that if $\mathcal{T}$ is the subcategory of torsion abelian groups (i.e. every element has finite order) and $\mathcal{F}$ is the subcategory of torsion-free abelian groups (i.e. every non-identity element has infinite order) then $(\mathcal{T}, \mathcal{F})$ is a torsion theory in Ab. This is a special case of Exercise 2.

Exercise 2. Let R-Mod be the category of (left) over R, an integral domain.[2] Prove that if $\mathcal{T}$ is the subcategory of torsion modules and $\mathcal{F}$ is the subcategory of torsion-free modules then $(\mathcal{T}, \mathcal{F})$ is a torsion theory in R-Mod.

[1] We will use terms like 'injective' and 'surjective' with their usual definitions (which are not available in general abelian categories), rather than their categorical counterparts. Occasionally we will assume that our objects have elements. This is done to shorten some of the proofs, though that is not to say that the results don't hold more generally.

[2] The hypothesis that R is an integral domain is needed so that the set of all torsion modules is a submodule

The idea is to generalize the above examples and distinguish two classes in the category, such that every object is an extension of a torsion free object by a torsion object. In general there are many ways to do this, which gives rise to different torsion theories.

Let $\mathcal{S}$ be a subcategory of C. We say $\mathcal{S}$ is *closed under quotients* if for all $Y \in \mathcal{S}$ and $Z \in \mathsf{C}$, if there is some surjective morphism $Y \twoheadrightarrow Z$ then $Z \in \mathcal{S}$. Similarly, we say $\mathcal{S}$ is *closed under subobjects* if for all $Y \in \mathcal{S}$ and $Z \in \mathsf{C}$, if there is some injective morphism $Z \twoheadrightarrow Y$ then $Z \in \mathcal{S}$.

Finally, we say $\mathcal{S}$ is *closed under extensions* if for all $X, Z \in \mathcal{S}$ and for all $Y \in \mathsf{C}$, if there is an exact sequence of the form

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

then $Y \in \mathcal{S}$.

Definition 1.2 (Torsion and torsion-free classes)**.** Let $\mathcal{S}$ be a subcategory of C. Then $\mathcal{S}$ is a *torsion class* if it is closed under quotients and closed under extensions. Dually, we say $\mathcal{S}$ is a *torsion-free class* if it is closed under subobjects and under extensions.

2 Equivalence of definitions

2.1 Torsion Correspondence. Part I

Surprisingly, Definition 1.1 is all we need to develop the theory of torsion and torsion-free classes, since these can be defined in terms of torsion theories.

Theorem 2.1 (Torsion correspondence. Part I)**.** *Let $(\mathcal{T}, \mathcal{F})$ be a torsion theory. Then $\mathcal{T}$ is a torsion class and $\mathcal{F}$ is a torsion-free class.*

Proof. We prove that $\mathcal{T}$ is a torsion class; the proof that $\mathcal{F}$ is a torsion-free class is completely analogous and is left to the reader. Suppose there is some $X \in \mathcal{T}$ and some $Z \in \mathsf{C}$ with some surjective morphism $X \twoheadrightarrow Z$. Notice that for any $Y \in \mathcal{F}$, if $f : Z \to Y$ is a non-zero morphism then $X \twoheadrightarrow Z \xrightarrow{f} Y$ would give a non-zero morphism from X to Y, which contradicts orthogonality. Hence $\mathrm{Hom}_C(Z, Y) = 0$ for all $Y \in F$.

Using extensionality, and our result above, we have the following short exact sequence

$$0 \longrightarrow X' \longrightarrow Z \xrightarrow{0} Y \longrightarrow 0$$

where $X' \in T$ and $Y \in F$. Using exactness at Z, we see the map from X' to Z is surjective. Using exactness at X', we know it is also injective. Hence $X' \cong Z$ and so $Z \in \mathcal{T}$. We have shown $\mathcal{T}$ is closed under quotients.

Now let $X, Z \in \mathcal{T}$ and let $W \in \mathsf{C}$ such that we have a short exact sequence

$$0 \longrightarrow X \xrightarrow{f} W \xrightarrow{g} Z \longrightarrow 0 \ .$$

By extensionality, there is some $Y \in \mathcal{F}$ and some $X' \in \mathcal{T}$ such that both row and column are exact in the following diagram.

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & X' \arrow[d, "s"] & & \\
0 \arrow[r] & X \arrow[r, "f"] & W \arrow[r, "g"] \arrow[d, "t"] & Z \arrow[r] \arrow[dl, dashed, "\exists!\sigma=0"] & 0 \\
 & & Y \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

By exactness of the horizontal sequence, Z is isomorphic to the cokernel of f. Note that $t \circ f = 0$ by orthogonality. By the universal property of the cokernel there is a unique map σ from Z to Y such that the diagram commutes, i.e. $\sigma \circ g = t$. But by orthogonality we must have $\sigma = 0$, and hence $t = 0$. Thus, by exactness of the column, s is an isomorphism and $W \in \mathcal{T}$ as required. We have shown that $\mathcal{T}$ is closed under quotients and extensions, hence it is a torsion class. □

Exercise 3. Complete the proof of Theorem 2.1.

Part II of this theorem says, roughly, that every torsion class (and every torsion free class) is of this form. However, we need some more preliminaries before stating and proving it.

2.2 Generating torsion and torsion free classes

Throughout this subsection let $\mathcal{X}$ be an arbitrary subcategory of C and consider the following question. How can we get a torsion class (resp. torsion free class) from $\mathcal{X}$? Looking at the definitions, we need to close $\mathcal{X}$ under quotients (resp. subobjects) and under extensions. The former is easy.

Define $\mathrm{Quot}(\mathcal{X})$ to be the subcategory of all quotients of objects of $\mathcal{X}$, i.e. all objects Y such that there exists some $X \in \mathcal{X}$ with a surjection $X \twoheadrightarrow Y$. Analogously, define $\mathrm{Sub}(\mathcal{X})$ to be the subcategory of all subobjects of objects of $\mathcal{X}$. Then $\mathrm{Quot}(\mathcal{X})$ is closed under quotients because the quotient of a quotient of some $X \in \mathcal{X}$ is a quotient of X, since the compositions of surjections is still a surjection. Similarly, $\mathrm{Sub}(\mathcal{X})$ is closed under subobjects. Both of these subcategories clearly contain all objects of X.

The situation for extensions is not as straightforward. If we define a subcategory $\mathrm{Filt}_2(\mathcal{X})$ of all extensions of objects of $\mathcal{X}$ (the notation will be clear later on) then there is no good reason why this should be closed under extensions. In a moment we will see how to fix this problem by adding extensions, extensions of extensions, extensions of extensions of extensions, and so on.

The reader should note that all of this discussion can be obviated, since we have the following results.

Proposition 2.2. *The intersection of an arbitrary collection of torsion (resp. torsion free) classes is still a torsion class (resp. torsion free class).*

Proof. This is completely trivial; we give a small part of the argument for torsion classes. Suppose X is an object of the intersection and let Y be a quotient of X. Then, for any torsion class in the collection, X belongs to that class and hence Y does as well. So Y is in the intersection. Hence we have closure under quotients. $\square$

Corollary 2.3. *We have that $\bigcap_{\substack{\mathcal{T}\ t.c.;\\ \mathcal{X}\subseteq\mathcal{T}}}\mathcal{T}$ is the smallest torsion class containing $\mathcal{X}$. Dually, $\bigcap_{\substack{\mathcal{F}\ t.f.c.;\\ \mathcal{X}\subseteq\mathcal{F}}}\mathcal{F}$ is the smallest torsion free class containing $\mathcal{X}$.*

This is perfectly valid. Yet, this approach is unenlightening and it is worth exploring the other, admittedly equivalent, construction since we will use similar constructions later on.

2.2.1 Filtering

Recall that $\mathcal{X}$ is an arbitrary subcategory of C.

Definition 2.4. For $n \in \mathbb{N}$ define $\mathrm{Filt}_n(\mathcal{X})$ to be the subcategory of all objects M such that, for some $k \leq n$, there are some objects $M_0, M_1, \ldots M_k$ such that $0 = M_0 \subsetneq M_1 \subsetneq \ldots \subsetneq M_k = M$ and M_{i+1}/M_i is in $\mathcal{X}$ for all i.

Remark 2.5. This definition is not standard in the literature. However, we shall shortly use it to define the subcategory $\mathrm{Filt}(\mathcal{X})$, which is standard.

Let's explore this notion. $\mathrm{Filt}_0(\mathcal{X})$ just consists of the zero object. $\mathrm{Filt}_1(\mathcal{X})$ is all of $\mathrm{Filt}_0(\mathcal{X})$ (so 0) along with objects M such that $0 \subsetneq M$ (i.e. $M \neq 0$) and $M/0 \cong M$ is in $\mathcal{X}$. So $\mathrm{Filt}_1(\mathcal{X}) = \mathcal{X}$.

Filt_2 is then the subcategory containing all of X along with objects M such that $0 \subsetneq M' \subsetneq M$ for some M' such that successive quotients are in $\mathcal{X}$. By the argument in above paragraph, $M' \in \mathcal{X}$. Hence M is an extension of objects of $\mathcal{X}$ as follows.

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M/M' \longrightarrow 0 \ .$$

Hence $\mathrm{Filt}_2(\mathcal{X})$ adds all of the extensions of objects of $\mathcal{X}$.

Exercise 4. Convince yourself that $\mathrm{Filt}_{i+1}(\mathcal{X})$ is the subcategory containing $\mathrm{Filt}_i(\mathcal{X})$ and all of the extensions of an object of $\mathcal{X}$ by an object of $\mathrm{Filt}_i(\mathcal{X})$. (Hint: Stare at Figure 1.)

$$
\begin{tikzcd}[column sep=tiny]
0 \arrow[ddrr] & & & & 0 \arrow[ddrr] & & & & 0 \arrow[ddrr] & & & & 0 \arrow[ddrr] & & & & 0 \arrow[ddrr] & & & & 0 \\
& & & & & & & M_1/M_0 \arrow[ur] & & & & M_2/M_1 \arrow[ur] & & & & M_3/M_2 \arrow[ur] & & & & M_4/M_3 \arrow[ur] & \\
& & 0 = M_0 \arrow[rrrr, hook, "d_0"] \arrow[ddrr] & & & & M_1 \arrow[ur] \arrow[rrrr, hook, "d_1"] \arrow[ddrr] & & & & M_2 \arrow[ur] \arrow[rrrr, hook, "d_2"] \arrow[ddrr] & & & & M_3 \arrow[ur] \arrow[rrrr, hook, "d_3"] \arrow[ddrr] & & & & M_4 \hookrightarrow \dots \arrow[ur] & & \\
& & & & & & & & & & & & & & & & & & & & \\
& & & & \operatorname{im} d_0 \cong M_0 \arrow[uurr] \arrow[ddrr] & & & & \operatorname{im} d_1 \cong M_1 \arrow[uurr] \arrow[ddrr] & & & & \operatorname{im} d_2 \cong M_2 \arrow[uurr] \arrow[ddrr] & & & & \operatorname{im} d_3 \cong M_3 \arrow[uurr] & & & & \\
& & & & & & & & & & & & & & & & & & & & \\
& & 0 \arrow[uurr] & & & & 0 \arrow[uurr] & & & & 0 \arrow[uurr] & & & & 0 \arrow[uurr] & & & & & &
\end{tikzcd}
$$

Figure 1: A filtering $0 \subsetneq M_1 \subsetneq \ldots \subsetneq M$, where the d's are inclusions and all diagonals are exact sequences.

Definition 2.6. Define $\mathrm{Filt}(\mathcal{X})$ to be the subcategory containing $\mathrm{Filt}_n(\mathcal{X})$ for all n. In other words, $\mathrm{Filt}(\mathcal{X})$ contains the objects M such that there exists a filtration $0 = M_0 \subsetneq M_1 \subsetneq \ldots \subsetneq M_n = M$ for some n and M_{i+1}/M_i is in $\mathcal{X}$ for all i.

Proposition 2.7. $\mathrm{Filt}(\mathcal{X})$ *is the smallest subcategory containing* $\mathcal{X}$ *that is closed under extensions.*

Proof. Clearly $\mathrm{Filt}(\mathcal{X})$ contains $\mathrm{Filt}_1(\mathcal{X}) \supseteq \mathcal{X}$. We will show that $\mathrm{Filt}(\mathcal{X})$ is closed under extensions. Let $M, L \in \mathrm{Filt}(\mathcal{X})$ with filtrations $0 = M_0 \subsetneq M_1 \subsetneq \ldots \subsetneq M_j = M$ and $0 = L_0 \subsetneq L_1 \subsetneq \ldots \subsetneq L_k = L$. Let N be an extension of L by M as depicted below.

$$0 \longrightarrow M \longrightarrow N \longrightarrow L \longrightarrow 0$$

We want to show that $N \in \mathrm{Filt}(\mathcal{X})$. If $M \cong N$ this is trivial. For $k = 0$ there is nothing to prove, since it implies $M \cong N$, so assume $k \geq 1$. Then we have the following diagram, where the row and column are exact.

$$
\begin{tikzcd}
& & 0 & & \\
0 \arrow[r] & L_{k-1} \arrow[r] & L \cong N/M \arrow[r] \arrow[u] & L_k/L_{k-1} \arrow[r] & 0 \\
& & N \arrow[u] & & \\
& & M \arrow[u] & & \\
& & 0 \arrow[u] & &
\end{tikzcd}
$$

As L_{k-1} is a submodule of N/M, then, by the third isomorphism theorem, we have that $L_{k-1} \cong S/M$ for some submodule S of N such that we have $M \subseteq S \subseteq N$. Furthermore, this implies that

$$L_k/L_{k-1} \cong \frac{N/M}{S/M} \cong N/S \in \mathcal{X}, \tag{1}$$

where the last isomorphism holds by the third isomorphism theorem.

If $S = M$ then (1) says that $N/M \in \mathcal{X}$ so $0 \subsetneq M_1 \subsetneq \ldots M \subsetneq N$ is the required filtration of N, so we are done. Note that $S \neq N$ because this would imply that $L_{k-1} \cong L_k$ which cannot happen since $L_{k-1} \subsetneq L_k$.

So instead assume $M \subsetneq S \subsetneq N$. Then $0 \subsetneq M_1 \subsetneq \ldots M \subsetneq S \subsetneq N$ gives the filtration of N. This closes all cases, so $N \in \mathrm{Filt}(\mathcal{X})$ and $\mathrm{Filt}(\mathcal{X})$ is closed under extensions. The reader will prove that it is the smallest subcategory containing $\mathcal{X}$ and closed under extensions. □

Exercise 5. Complete the proof of Proposition 2.7. (Hint: Suppose $\mathcal{S}$ is a subcategory closed under extensions and containing $\mathcal{X}$. Prove that $\mathcal{S}$ contains $\mathrm{Filt}_n(\mathcal{X})$ for all n by induction.)

2.2.2 Torsion classes in terms of filters

Define the following, for an arbitrary subcategory $\mathcal{X}$.

$$\mathbf{T}(\mathcal{X}) := \mathrm{Filt}(\mathrm{Quot}(\mathcal{X}))$$
$$\mathbf{F}(\mathcal{X}) := \mathrm{Filt}(\mathrm{Sub}(\mathcal{X})).$$

Proposition 2.8. *For an arbitrary subcategory $\mathcal{X}$, $\mathbf{T}(\mathcal{X})$ is the smallest torsion class containing $\mathcal{X}$. Dually, $\mathbf{F}(\mathcal{X})$ is the smallest torsion free class containing $\mathcal{X}$.*

Proof. As usual, we will give the proof for one half of the statement, leaving the other half to the reader. Let's do torsion free classes this time.

$\mathbf{F}(\mathcal{X})$ clearly contains $\mathcal{X}$ and is closed under extensions. To prove it is closed under subobjects let us □