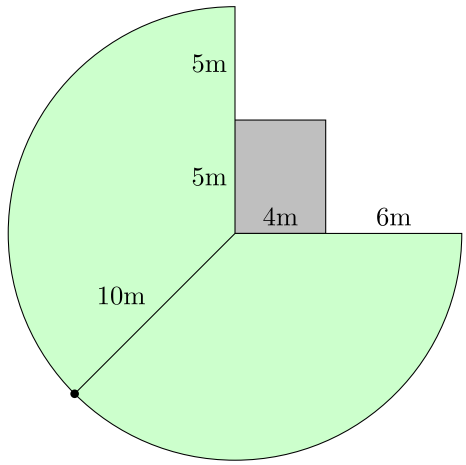
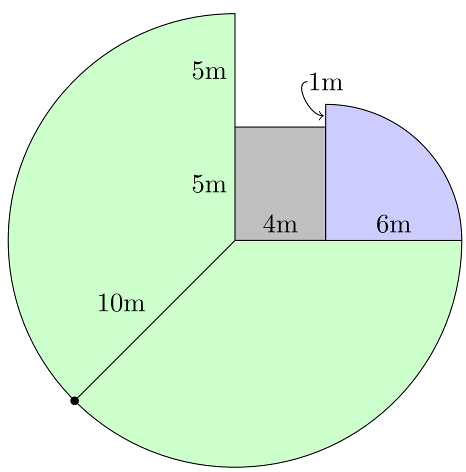
4. (b) A goat is tethered by a 10m rope to an outside corner of a 4m by 5m shed. What is the area of the ground it can reach?

Solution[1]. Even if the shed were a lot bigger, we can all agree that the goat can at least cover three quarters of a circle of radius 10m, shown below.

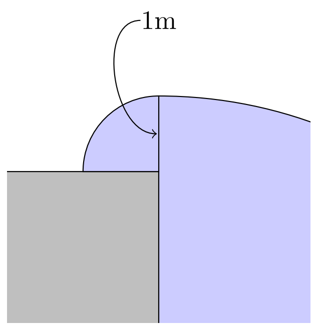


Suppose the goat wants to go further in the counter-clockwise direction. Then we could describe a quarter of a circle of radius 6m on the east wall of the shed, as shown.

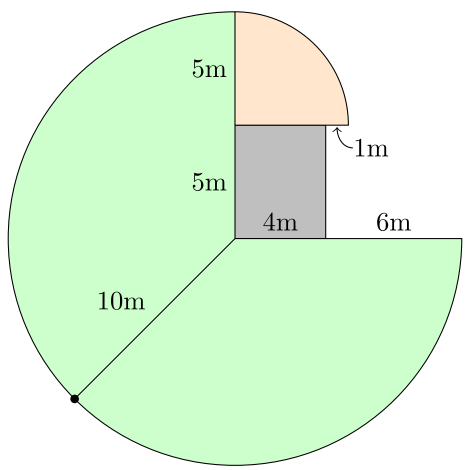


Of course, the goat can keep going counter-clockwise since it still has a bit of rope left. The goat would then describe a quarter of a circle of radius 1m on the north wall (we better zoom in for this one).

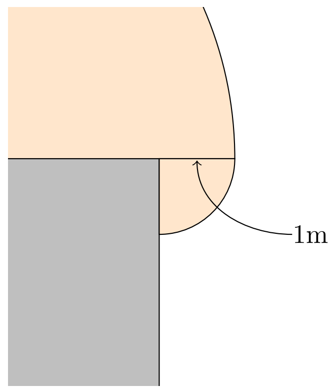
[1]Written by Hernan Ibarra Mejia. Thanks to Theodora Violari for hearing me out while I rambled about this problem.



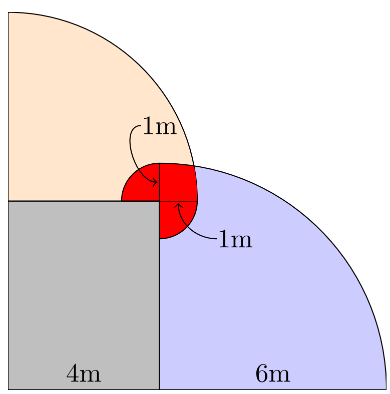
That's how long the rope will go in the counter-clockwise direction. Now let's see what happens when the goat goes clockwise. In that case, the goat would first describe quarter a circle of radius 5 on the north wall.



Then, as before, we can describe a quarter of a circle of radius 1m, this time on the east wall.

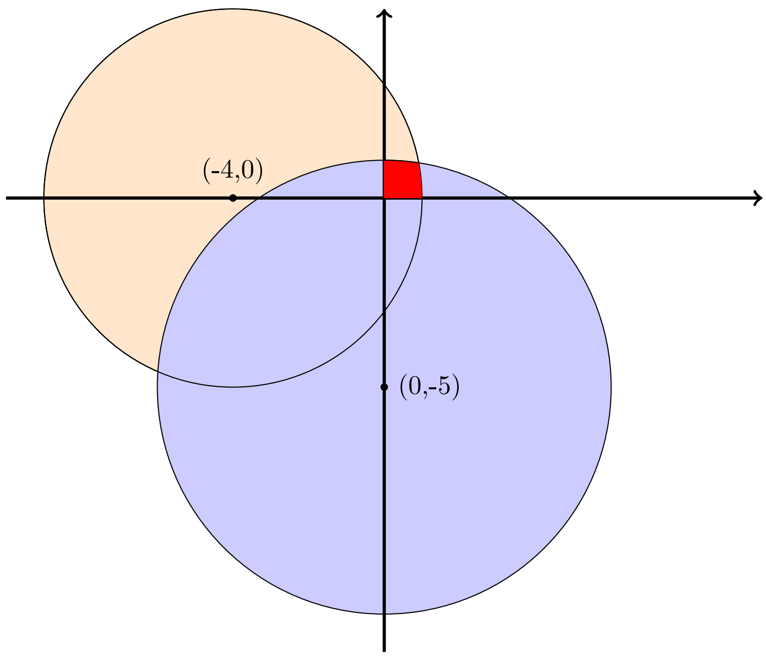


The issue here, and what makes this problem really difficult, is that the areas we found when going counter-clockwise (orange) and the ones corresponding to clockwise movement (blue) overlap. This means we can't just add all the areas we have found so far, because we will be double-counting. Here's a helpful picture of the situation, where we are disregarding the green area for now. The area of concern is coloured red.



In this context we would have that the answer to our question, the size of the area of the ground the goat can reach, is the sum of the green, orange, and blue areas, minus the red area. All of these areas are easily calculated, but the red one, so we will focus our attention on that one.

From the picture we can see that the red area is comprised of two quarters of circles of radius 1m, and a weird difficult-to-describe shape. If we introduce some coordinate axes and we make the north-east corner of the shed the origin, it is possible to describe this shape more precisely. Indeed, in these coordinates, the orange circle (if we draw it whole) is just the circle of radius 5m centred at (-4,0), while the blue circle corresponds to the circle of radius 6m centred at (0,-5). Hence, the area of the weird blob is just the area on the first quadrant enclosed by the two circles and the coordinate axes.



Before we zoom in, we will calculate the x-coordinate of the intersection in the first quadrant of the two circles, since this will be useful later on. We have the algebraic descriptions:

$$\text{Orange circle: } (x+4)^2 + y^2 = 5^2$$
$$\text{Blue circle: } x^2 + (y+5)^2 = 6^2.$$

Solving this exactly is a necessary chore. You can skip this calculation if you want to.

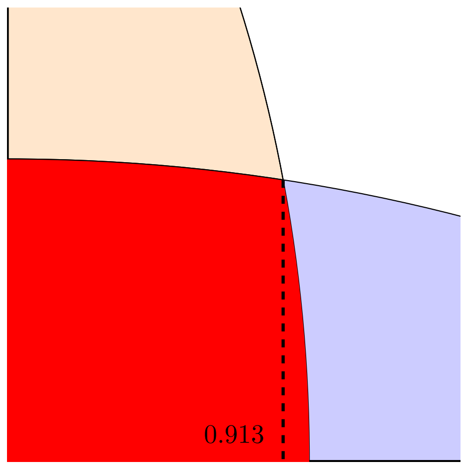
From the orange circle's description we have that $y^2 = 5^2 - (x+4)^2$. Replacing this into the blue circle's description gives the following.

$$x^2 + y^2 + 10y + 5^2 = 6^2$$
$$x^2 + 5^2 - (x+4)^2 + 10\sqrt{5^2 - (x+4)^2} + 5^2 = 6^2$$
$$10\sqrt{5^2 - (x+4)^2} = 2 + 8x$$
$$5\sqrt{5^2 - (x+4)^2} = 4x + 1$$

Notice that we have taken the positive square root of y, since we know our intersection lies on the first quadrant. Now, squaring both sides gives

$$25(5^2 - (x-4)^2) = (4x+1)^2$$
$$5^4 - 25x^2 - 200x - 400x = 16x^2 + 8x + 1$$
$$41x^2 + 208x - 224 = 0.$$

The reader can easily check that the (positive) solution to this equation is $\frac{100\sqrt{2}-104}{41} \approx 0.913$. As we know, by the diagram, that there is exactly one positive solution, this must be it. Zooming in now. $\square$



The dashed line splits up the red blob into two regions, whose areas are given by integrating the expressions describing the circles (!) Namely, we have

$$\text{Area of red blob} = \underbrace{\int_0^{\frac{100\sqrt{2}-104}{41}} \sqrt{6^2 - x^2} - 5\,\mathrm{d}x}_{\substack{\text{Left region, by integrating the} \\ \text{(top half of the) blue circle}}} + \underbrace{\int_{\frac{100\sqrt{2}-104}{41}}^{1} \sqrt{5^2 - (x+4)^2}\,\mathrm{d}x}_{\substack{\text{Right region, by integrating the} \\ \text{(top half of the) orange circle}}}.$$

Again, this is routine. Let's jump straight to the final answer.

$$\text{Total Area} = \text{Green Area} + \text{Blue Area} + \text{Orange Area} - \text{Red Area}$$

$$\text{TA} = \left(\frac{3}{4}\pi 10^2\right) + \left(\frac{1}{4}\pi 6^2 + \frac{1}{4}\pi 1^2\right) + \left(\frac{1}{4}\pi 5^2 + \frac{1}{4}\pi 1^2\right) - \left(\frac{1}{4}\pi 1^2 + \frac{1}{4}\pi 1^2 + \text{Area of red blob}\right)$$

$$\text{TA} = \frac{361\pi}{4} - \text{Area of red blob}.$$

In the next section we compute a numerical answer.

Finale

Firstly, we assume that the reader knows how to derive the identity

$$\int \sqrt{1-x^2}\,\mathrm{d}x = \frac{1}{2}\left(x\sqrt{1-x^2} + \arcsin(x)\right) + C,$$

for some $C \in \mathbb{R}$, via a trigonometric substitution or some other method (or by differentiating the above). Then, we can reduce both of our integrals to this form by some easy u-substitutions. Hence we get that

$$\int_0^{\frac{100\sqrt{2}-104}{41}} \sqrt{6^2-x^2} - 5\,\mathrm{d}x = 18\arcsin\left(\frac{2}{123}(25\sqrt{2}-26)\right) - \frac{80(227\sqrt{2}-282)}{1681},$$

and that

$$\int_{\frac{100\sqrt{2}-104}{41}}^{1} \sqrt{5^2-(x+4)^2}\,\mathrm{d}x = \frac{25(-460+108\sqrt{2}+1681\arccos(\frac{4}{41}(3+5\sqrt{2})))}{3362}.$$

We used Wolfram Alpha for this. We can't recommend doing it by hand. It's possible to write our final answer as

$$\frac{361\pi}{4} - 18\arcsin\left(\frac{2}{123}(25\sqrt{2}-26)\right) + \frac{80(227\sqrt{2}-282)}{1681} - \frac{25(-460+108\sqrt{2}+1681\arccos(\frac{4}{41}(3+5\sqrt{2})))}{3362}\mathrm{m}^2.$$

This is a bit unhelpful. Know that this number is approximately $282.583\mathrm{m}^2$.