$Sp(1)$ and $SO(3)$

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1 Problem statement

Let r be a point in $\mathbb{R}^3$. Let $\vec{v}$ be a unit vector in $\mathbb{R}^3$ representing the axis of rotation (passing through the origin) and let ϕ be the angle of rotation. Find the coordinates of r', the result of rotating r about v by an angle of ϕ.

2 Solution

2.1 Quaternion multiplication

One can think of a quaternion as an element of $\mathbb{R} \times \mathbb{R}^3$, where we think separately of the real part and the imaginary part. If we let

$$q_1 \coloneqq (a, \vec{v}) \text{ where } a \in \mathbb{R}, \vec{v} \in \mathbb{R}^3,$$
$$q_2 \coloneqq (b, \vec{u}) \text{ where } b \in \mathbb{R}, \vec{u} \in \mathbb{R}^3,$$

then quaternion multiplication takes quite a nice form:

$$q_1 q_2 = (ab - \vec{v} \cdot \vec{u}, a\vec{u} + b\vec{v} + \vec{v} \times \vec{u}).$$

Purely imaginary quaternions have a representation in $\mathbb{R}^3$. We let our point in question, r, be represented by a quaternion.

$$r \coloneqq (0, \vec{r}) \text{ where } \vec{r} \in \mathbb{R}^3.$$

2.2 The conjugation map is a reflection with respect to a plane in $\mathbb{R}^3$ (sort of)

Let $h = (0, \vec{h})$ be a purely imaginary *unit* quaternion. It turns out that hr is not a purely imaginary quaternion, so it doesn't have a representation in $\mathbb{R}^3$. However $hr\bar{h}$ does.

Define a map $\mathrm{ref}_h \colon \mathbb{R}^3 \to \mathbb{R}^3$ by expressing the input as a purely imaginary quaternion and then writing $\mathrm{ref}_h(r) \coloneqq -hr\bar{h}$. The result will also be a purely imaginary quaternion, which is an element of $\mathbb{R}^3$ (we are being a bit careless about the distinction between $\mathbb{R}^3$ and purely imaginary quaternions).

Using the multiplication given above it is possible to check that [1]

$$\mathrm{ref}_h(r) := -hr\bar{h} = r - 2|r|\cos\theta h,$$

where θ is the angle between h and r (in $\mathbb{R}^3$). See Figure 1 for a geometrical representation in two dimensions.

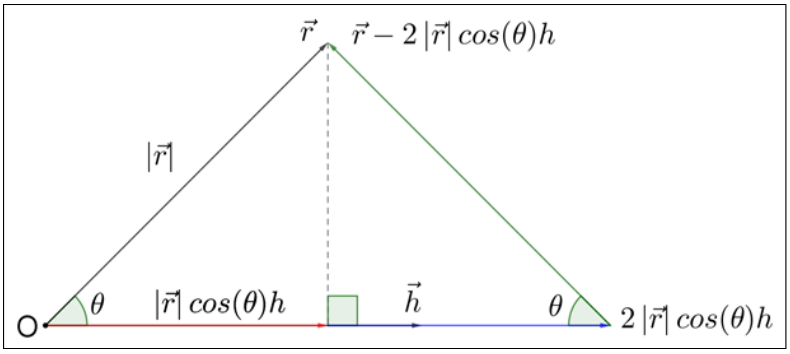

Figure 1: Representation of $\mathrm{ref}_h(r)$ in two dimensions

If we translate the resultant vector to the origin we get what we see in Figure 2.

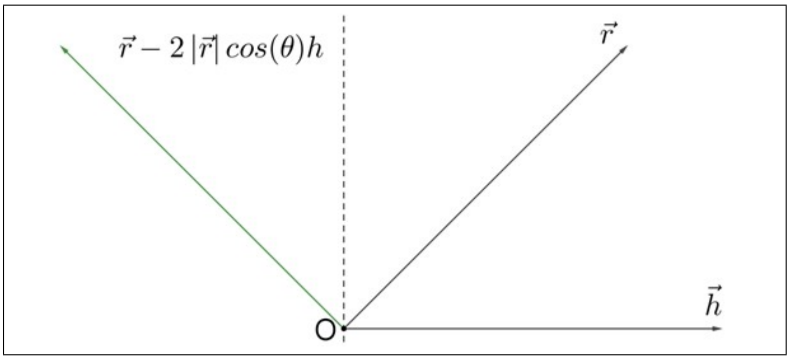

Figure 2: $\mathrm{ref}_h(r)$ is a reflection?

Of course, we only get to see the full picture in three dimensions, see Figure 3.

This is the main result of this subsection: $\mathrm{ref}_h(r)$ is a reflection of r with respect to the plane normal to h that passes through the origin.

[1] I think one needs to use the identities that $(\vec{h} \times \vec{r}) \times \vec{h} = |\vec{h}|^2\vec{r} - (\vec{h} \cdot \vec{r})\vec{h}$ and that $(\vec{h} \times \vec{r}) \cdot \vec{h} = 0$. Use the fact that h is a unit quaternion and expand the dot product near the end.

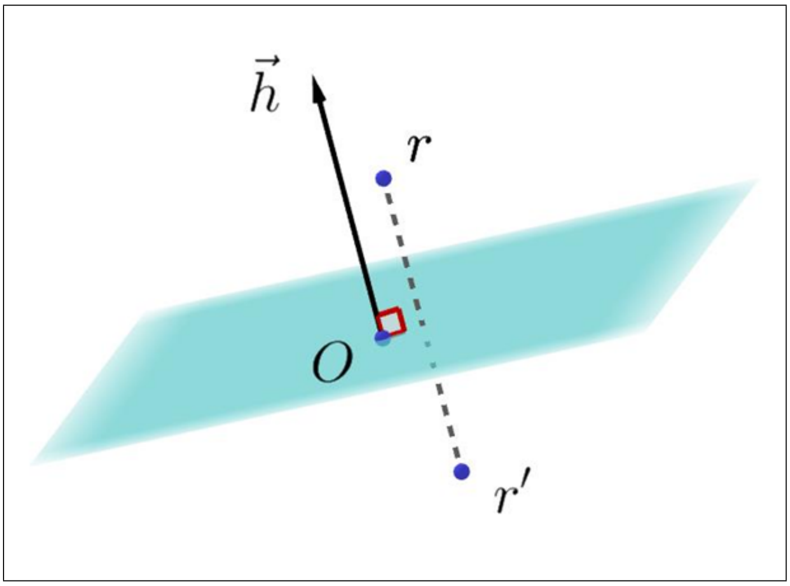

Figure 3: $\mathrm{ref}_h(r)$ *is* a reflection

2.3 Reflections can be rotations (if you do things in a smart way)

Reflections are the most natural linear transformations in $\mathbb{R}^3$ using quaternions. If one wishes to express rotations with quaternions, it is imperative to express rotations as reflections first. Consider Figure 4 (imagine the axis of rotation going through the origin instead and the angle being ϕ (sorry!)). Also see Figure 5, which is a cross-section of Figure 4.

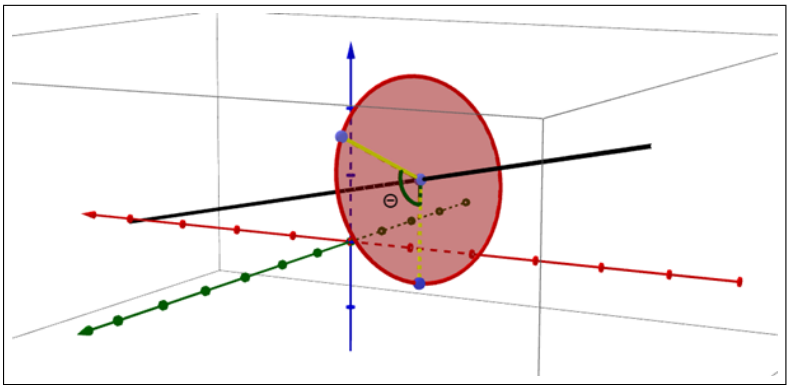

Figure 4: Rotations in three dimensions

In Figure 5 the axis of rotation $\vec{v}$ is pointing straight into the page. The idea is the following: we don't know how to rotate but we do know how to reflect, so we will reflect with respect to a plane that bisects our desired

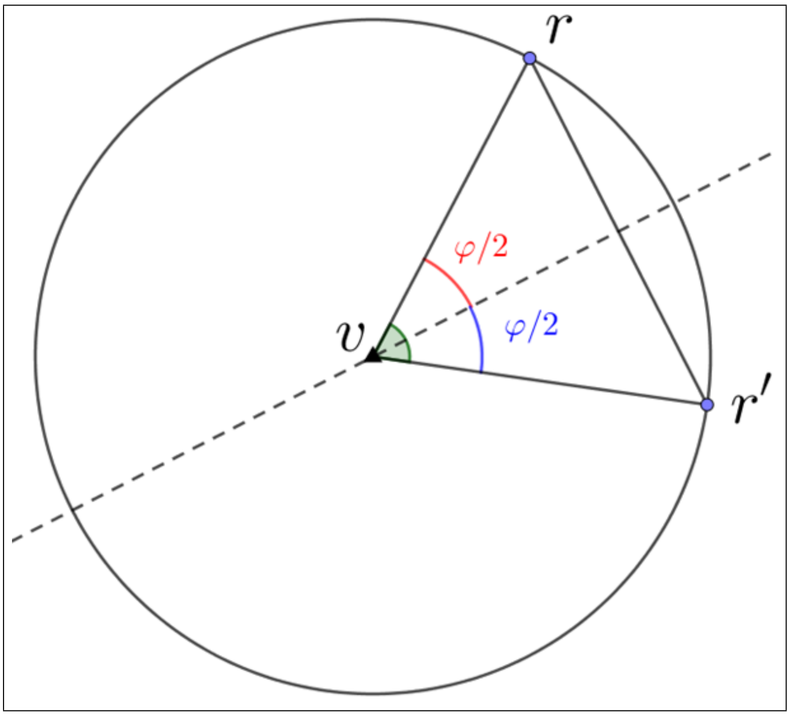

Figure 5: Rotation as a reflection

angle of rotation (see the dotted line).

In three dimensions things look like in Figure 6.

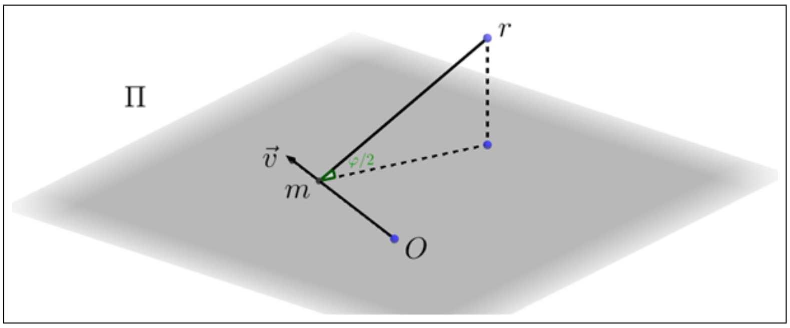

Figure 6: The rotation plane

If Π is our plane of reflection/rotation then let h the a the purely imaginary unit quaternion normal to that plane that passes through the origin. So, $r' = -hr\bar{h}$. If we can find h then we are done, but this is actually very hard to do.

Hamilton's idea (and this is why I love Hamilton) is the following. Let Π' be the plane passing through the origin and r that also contains $\vec{v}$. Let $n = (0, \vec{n})$ be the purely imaginary unit quaternion normal to Π' that passes through the origin[2]. We can conclude a couple of things about n.

Note that the angle between Π and Π' is $\phi/2$, and hence the angle between $\vec{h}$ and $\vec{n}$ is $\phi/2$. Also, since $\vec{v}$ is contained in both planes then $\vec{v}$ is perpendicular to both normals. All this allows us to deduce that

$$\vec{n} \times \vec{h} = \sin(\phi/2)\vec{v}$$
$$\vec{n} \cdot \vec{h} = \cos(\phi/2).$$

As r is contained in Π' then reflecting r with respect to that plane does nothing (!). So we can write $r = -nr\bar{n}$. Hence, to find r' we just do this:

$$r' = -hr\bar{h}$$
$$r' = -h(-nr\bar{n})\bar{h}.$$

Using associativity (and the identity for products of conjugates) we get that $r' = (hn)r(\overline{hn})$. But, using the formula for quaternion multiplication, we see that $hn = (-\vec{n} \cdot \vec{h}, -\vec{n} \times \vec{h})$.

Then, if $d = (-\cos(\phi/2), -\sin(\phi/2)\vec{v})$ we get that

$$r' = dr\bar{d}.$$

3 $Sp(1)$ and $SO(3)$

As in Tapp's, we let $C_q \colon \mathbb{H} \to \mathbb{H}$ represent the conjugation map for each $q \in Sp(1)$, defined by $C_q(V) = qV\bar{q}$.

3.1 What the conjugation map is

I think the following is a better way to think of the conjugation map. For each unit quaternion $d \in Sp(1)$ it is straightforward to check that $d = (-\cos(\phi/2), -\sin(\phi/2)\vec{v})$ for some $\phi \in [0, 2\pi)$ and $\vec{v} \in \mathbb{R}^3$ where $|\vec{v}| = 1$.

Then, the map $d \mapsto C_d$ identifies d with an element of $SO(3)$, namely the rotation about $\vec{v}$ by an angle of ϕ. As we have seen, there is such quaternion d for each rotation, so the map is surjective.

3.2 Why the conjugation map is not an isomorphism

This is because $C_d = C_{-d}$, so $d \mapsto C_d$ is not injective. What does $C_d = C_{-d}$ mean geometrically?

It comes from the following fact. Adopt a convention, say, the right-hand rule, to decide the orientation of the rotation (i.e. in which of the two possible directions the rotation around $\vec{v}$ is to happen). Then, a rotation of ϕ about $\vec{v}$ is the same as a rotation of $2\pi - \phi$ about $-\vec{v}$.

[2]Or, if you prefer, the plane passing through the origin and containing both $\vec{v}$ and $\vec{mr}$, where m is the perpendicular projection of r in $\vec{v}$ (see Figure 6).

If $d = (-\cos(\phi/2), -\sin(\phi/2)\vec{v})$, we have

$$(-\cos\left(\frac{2\pi - \phi}{2}\right), -\sin\left(\frac{2\pi - \phi}{2}\right)(-\vec{v})) =$$

$$(-\cos\left(\pi - \frac{\phi}{2}\right), -\sin\left(\pi - \frac{\phi}{2}\right)(-\vec{v})) = (\cos(\phi/2), \sin(\phi/2)\vec{v}) = -d.$$

So now it is clear that we must have $C_d = C_{-d}$.