1 Problem statement

Let A and B be two non-empty sets such that A does not have lesser or equal cardinality to B. Using the principle of transfinite induction, prove that B has lesser or equal cardinality to A. (Hint: for every subset $X \subseteq B$, let $P(X)$ denote the property that there exists an injective map from X to A.) This exercise (combined with Exercise 8.3.3) shows that the cardinality of any two sets is comparable, as long as one assumes the axiom of choice.

2 Solution

Consider the set

$$S := \bigcup_{X \in 2^B} \{\, f \in A^X : f \text{ is an injection} \,\},$$

i.e. the set of all injective functions that take a subset of B to A. We are given that A and B are non-empty, so there exists some $a \in A$ and $b \in B$. If we define a function $h \colon \{b\} \to A$ by $h(b) = a$ it is clear that h is injective and hence $h \in S$. This all proves that S is non-empty.

We partially order S as follows. For some $X, Y \subseteq B$ let $f \colon X \to A$ and $f' \colon Y \to A$ be injective functions (i.e. $f, f' \in S$). We say that $f \leq f'$ iff $X \subseteq Y$ and $f(x) = f'(x)$ for all $x \in X$. Verify that this is indeed a partial ordering. We will use Zorn's lemma to prove that S then has a maximal element, which will turn out to be an injection from B to A, completing the proof.

Suppose T is a totally ordered subset of S; we will construct a function u that will be an upper bound of T. The domain of u will be the union of the domains of all the functions in T; this is

$$D := \bigcup \{\, X \in 2^B : \text{there exists } f \colon X \to A \text{ where } f \in T \,\},$$

which happens to equal

$$\bigcup \{\, X \in 2^B : A^X \cap T \neq \emptyset \,\}.$$

Notice that this set is the union of subsets of B, hence a subset of B. Also notice that this construction guarantees that the domain of any function of T is a subset of D. The range of u will be A.

We define $u \colon D \to A$ as follows. For each $x \in D$ note that, by construction of D, there is a function $f \colon X \to A$ in T where $x \in X$. Using the axiom of choice we choose one such function f_x for each $x \in D$ and define $u(x) := f_x(x)$.

To prove that u is injective, suppose that $u(x) = u(y)$ for some $x, y \in D$. Then, by definition of u, we have $f_x(x) = f_y(y)$ for some $f_x, f_y \in T$. Since

T it totally ordered we can assume, without loss of generality, that $f_x \leq f_y$. Then we have $f_x(x) = f_y(x)$. Therefore $f_y(x) = f_y(y)$, but because $f_y \in T$ and T is a set of injections we have $x = y$, proving that u is injective.

So far we have that $u\colon D \to A$, where $D \subseteq B$, is an injection. This is enough to show that $u \in S$. Now we show that u is an upper bound of T.

Let $f\colon X \to A$, for some $X \subseteq B$, be a function in T. We know $X \subseteq D$: we remarked this when we constructed D. Let $x \in X$ and consider $f_x \in T$. Regardless of whether we have $f \leq f_x$ or $f_x \leq f$ (and one of them is true since T is totally ordered) we must have $f(x) = f_x(x)$ by the definition of our ordering. Then, $u(x) = f(x)$ for all $x \in X$. This means that $f \leq u$ for all $f \in T$, and u is an upper bound of T.

Since any totally ordered subset of S has an upper bound, by Zorn's Lemma, S, the set of all injections from subsets of B to A, must have a maximal element. Let $F\colon B' \to A$, where $B' \subseteq B$, be an injection which is a maximal element of S. We aim to show that $B' = B$.

Notice that F cannot be a surjection, because if it were a surjection from B' to A then there would be an injection from A to B' and hence an injection from A to B which is not supposed to exist since A does not have less than or equal cardinality to B. This means that there is some $a' \in A$ such that $F(x) \neq a'$ for all $x \in B'$.

Suppose, for the sake of contradiction, that B' is a proper subset of B, i.e. there exists some $b' \in B \setminus B'$. Define a function $F'\colon B' \cup \{b'\} \to A$ by the following rule.

$$F'(x) = \begin{cases} F(x) & \text{if } x \in B' \\ a' & \text{if } x = b' \end{cases}$$

It is easy to show, given that F is an injection and that $F(x) \neq a'$ for all $x \in B'$, that F' is an injection, and hence $F' \in S$. Looking at the definition of F', it is clear that $F < F'$; a contradiction since F is the maximal element of S. Thus $B' \subseteq B$ but B' cannot be a proper subset of B, which means $B = B'$, and F is an injection from B to A, as desired.