The Kronecker Quiver

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The Kronecker quiver is the following

$$1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2 \ .$$

Fix some algebraically closed field k for the rest of this article. We will study the representations of the Kronecker quiver. First let us focus on the special case where both maps are invertible.

1 Invertible representations

We say a representation of the Kronecker is *invertible* if its of the form

$$k^n \underset{B}{\overset{A}{\rightrightarrows}} k^n \tag{1}$$

where A and B are invertible matrices. There is a classification of the isomorphism classes of these representations.

Lemma 1.1. *Let M be an invertible representation. Then M is isomorphic to one representation of the form*

$$k^n \underset{J}{\overset{1}{\rightrightarrows}} k^n$$

where J is an invertible matrix in Jordan normal form. Further, this J is unique, up to permutation of its Jordan blocks.

Proof. Let M be as in (1), where A, B are invertible. The idea is to first change bases in the domain and codomain to get A to be the identity. More formally, since A is invertible, there exist invertible matrices P, Q such that $PAQ = 1$ (think about row and column operations). After this we have transformed $A \mapsto PAQ = 1$ and $B \mapsto PBQ$. We cannot change bases so freely now as one of the maps is in the form we want. We can, however, *conjugate*. It is a fact that there exists an invertible matrix R such that $R(PBQ)R^{-1} = J$ is in Jordan normal form, and it is clear that J is invertible. So now we do $1 \mapsto R1R^{-1} = 1$

and $PBQ \mapsto R(PBQ)R^{-1} = J$. And indeed one verifies that the following gives an isomorphism.

$$\begin{tikzcd}
k^n \arrow[r, shift left, "A"] \arrow[r, shift right, "B"'] \arrow[d, "RQ^{-1}"'] & k^n \arrow[d, "RP"] \\
k^n \arrow[r, shift left, "1"] \arrow[r, shift right, "J"'] & k^n
\end{tikzcd}$$

Now suppose J' is another invertible matrix in Jordan normal form such that we have an isomorphism of representation as follows,

$$\begin{tikzcd}
k^n \arrow[r, shift left, "1"] \arrow[r, shift right, "J"'] \arrow[d, "X"'] & k^n \arrow[d, "Y"] \\
k^n \arrow[r, shift left, "1"] \arrow[r, shift right, "J'"'] & k^n
\end{tikzcd}$$

for X, Y invertible. Then commutativity in the "α" square gives that $X = Y$, and so commutativity in the "β" square gives that J is conjugate to J'. Two conjugate matrices in Jordan normal form are equal up to permutation of their Jordan blocks; thus the result follows. □

1.1 Indecomposable representations

The next natural question is what are the indecomposable invertible representations. Let

$$J_{\lambda,r} \coloneqq \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}$$

be the $r \times r$ Jordan block with eigenvalue $\lambda \in k$. Now, for $\lambda, r \neq 0$, define a representation

$$V_{\lambda,r} \coloneqq \begin{tikzcd} k^r \arrow[r, shift left, "1"] \arrow[r, shift right, "J_{\lambda,r}"'] & k^r \end{tikzcd} .$$

Remark 1.2. We don't define $V_{\lambda,r}$ when $\lambda = 0$ or $r = 0$ because then some of the results below would have trivial exceptions.

This definition is fairly natural, because then, using Lemma 1.1, the result of Exercise 1 is immediate to verify.

Exercise 1. *If M is an invertible representation then $M \cong V_{\lambda_1,r_1} \oplus \ldots \oplus V_{\lambda_n,r_n}$ in a unique way (up to permutation of the summands).*

The next lemma will help us prove what the reader already suspects about invertible indecomposable representations.

Lemma 1.3. *Invertible representations are closed under nonzero direct summands.*

Proof. Let L be an invertible representation and assume $L \cong M \oplus N$ for some representations M and N, none of which is zero.

$$M := \begin{tikzcd} k^{m_1} \arrow[r, shift left, "M_\alpha"] \arrow[r, shift right, "M_\beta"'] & k^{m_2} \end{tikzcd}$$

$$N := \begin{tikzcd} k^{n_1} \arrow[r, shift left, "N_\alpha"] \arrow[r, shift right, "N_\beta"'] & k^{n_2} \end{tikzcd} .$$

Note that, by changing bases, we can assume both M_α and N_α are of the form

$$\left(\begin{array}{c|c} I & 0 \\ \hline 0 & 0 \end{array}\right), \tag{2}$$

where I is an identity matrix of some size (possibly zero); this is a fact from linear algebra. Also notice that both matrices have a nonzero width or nonzero height since neither M nor N is the zero representation. But then consider

$$M_\alpha \oplus N_\alpha = \left(\begin{array}{c|c} M_\alpha & 0 \\ \hline 0 & N_\alpha \end{array}\right).$$

This makes it clear that both M_α and N_α

- actually have nonzero width *and* height; and they
- have no zero rows or columns.

This is because otherwise $M_\alpha \oplus N_\alpha$ would have a zero row or column, which would make it noninvertible, and this cannot be since it is equivalent to the invertible map in L corresponding to α. As we assumed that M_α and N_α were of the form in (2), this implies that they are identity matrices under the choice of some bases. And this implies they are invertible. The same argument goes through for M_β and N_β. Hence M and N are invertible representations. $\square$

Theorem 1.4. *$V_{\lambda,r}$ is indecomposable and invertible. Further, if L is an indecomposable and invertible representation, then $L \cong V_{\lambda,r}$ for some unique λ and r.*

Proof. Clearly $V_{\lambda,r}$ is invertible (remember $\lambda, r \neq 0$). To show that it is indecomposable assume we have two representations, M and N, and assume that $V_{\lambda,r} \cong M \oplus N$. For the sake of contradiction, assume both M and N are not the zero representation. Then Lemma 1.3 tells us that M and N are invertible.

By Lemma 1.1, there are some invertible matrices J_M, J_N in Jordan normal form such that

$$M \cong \begin{tikzcd} k^m \arrow[r, shift left, "1"] \arrow[r, shift right, "J_M"'] & k^m \end{tikzcd},$$

$$N \cong \begin{tikzcd} k^n \arrow[r, shift left, "1"] \arrow[r, shift right, "J_N"'] & k^n \end{tikzcd}.$$

Recall that $M \oplus N$ is also invertible since it is isomorphic to $V_{\lambda,r}$,

$$M \oplus N \cong \begin{tikzcd} k^{m+n} \arrow[r, shift left, "1"] \arrow[r, shift right, "J_M \oplus J_N"'] & k^{m+n} \end{tikzcd} \cong V_{\lambda,r},$$

and in fact it is already in the form described in Lemma 1.1. By that same lemma $J_M \oplus J_N = J_{r,\lambda}$. But this is simply not true:

$$\left(\begin{array}{c|c} J_M & 0 \\ \hline 0 & J_N \end{array}\right)$$

cannot have a diagonal of 1's above the main diagonal (look at where the blocks meet). This is a contradiction, so $V_{\lambda,r}$ is indecomposable.

If L is invertible and indecomposable then Exercise 1 implies that it is isomorphic to $V_{\lambda,r}$ for unique λ and r. $\square$

1.2 Morphisms

Next, we classify the morphisms between (invertible) indecomposable representations. Let's explore the general case: a morphism from $V_{\lambda,n}$ to $V_{\mu,m}$.

$$\begin{tikzcd} k^n \arrow[r, shift left, "1"] \arrow[r, shift right, "J_{\lambda,n}"'] \arrow[d, "A"'] & k^n \arrow[d, "B"] \\ k^m \arrow[r, shift left, "1"] \arrow[r, shift right, "J_{\mu,m}"'] & k^m \end{tikzcd}$$

Commutativity in the square of identities immediately gives that $A = B$. So now we seek $m \times n$ matrices A such that

$$J_{\mu,m} A = A J_{\lambda,n}. \tag{3}$$

Let $(A)_{ij} = a_{ij}$. Start with the left-hand side of equation (3). For $1 \le i \le m$ and $1 \le j \le n$ we have

$$(J_{\mu,m} A)_{ij} = \sum_{k=1}^{m} (J_{\mu,m})_{ik} \cdot a_{kj}.$$

Notice that we have

$$(J_{\mu,m})_{ik} = \begin{cases} \mu & \text{if } k = i \\ 1 & \text{if } k = i+1 \\ 0 & \text{otherwise} \end{cases}.$$

Thus we have two cases.

- If $1 \le i < m$ then
$$(J_{\mu,m}A)_{ij} = \mu a_{ij} + a_{(i+1)j}$$
- If $i = m$ then
$$(J_{\mu,m}A)_{mj} = \mu a_{mj}$$

Now let's consider the right-hand side of (3). In this case we get

$$(AJ_{\lambda,n})_{ij} = \sum_{k=1}^{n} a_{ik}(J_{\lambda,n})_{kj}.$$

Again, notice that we have

$$(J_{\lambda,n})_{kj} = \begin{cases} \lambda & \text{if } k = j \\ 1 & \text{if } k = j-1 \\ 0 & \text{otherwise} \end{cases}.$$

So, now the two cases are

- If $1 < j \le n$ then
$$(AJ_{\lambda,n})_{ij} = \lambda a_{ij} + a_{i(j-1)}$$
- If $j = 1$ then
$$(AJ_{\lambda,n})_{i1} = \lambda a_{i1}$$

Of course, the left and right hand side of (3) must match. These give us four necessary and sufficient conditions for A to determine a morphism.

1. For $i = m$ and $j = 1$:
$$\mu a_{m1} = \lambda a_{m1} \implies a_{m1} = 0 \text{ or } \mu = \lambda.$$
2. For $i = m$, and $1 < j \le n$:
$$a_{mj}(\mu - \lambda) = a_{m(j-1)}$$
3. For $1 \le i < m$ and $j = 1$:
$$a_{i1}(\lambda - \mu) = a_{(i+1)1}.$$

4. For $1 \leq i < m$ and $1 < j \leq n$:

$$(\lambda - \mu)a_{ij} = a_{(i+1)j} - a_{i(j-1)}$$

This is a lot of information. First, we do some relabelling to get another, more workable version of condition 4.

4'. For $1 \leq i < m$ and $1 \leq j < n$:

$$(\lambda - \mu)a_{i(j+1)} = a_{(i+1)(j+1)} - a_{ij}$$

It is natural to split our investigation into two cases.

1.2.1 $\lambda \neq \mu$

Here condition 1 implies that a_{m1}, that is, the bottom leftmost entry of A, is zero. Then, condition 2 says that each entry in the bottom row is the entry to its left, multiplied by the constant factor of $(\mu - \lambda)^{-1}$. But we said that the leftmost entry in the bottom row is zero; hence, all entries in the bottom row are zero. Similarly, condition 3 says that every entry in the first (leftmost) column of A is the entry directly below multiplied by $(\lambda - \mu)^{-1}$. But again, the bottom entry in the leftmost column is zero, hence all of the first column must be zero. In conclusion, the first (leftmost) column and the last (bottom) row of A are all zeros.

Now we apply 4'. Remember: we have shown that $a_{1x} = 0$ for all x. Plugging in $i = 1$ into condition 4' says that the second row is all zeros (except perhaps the first entry, but we already knew it to be zero). Now we have $a_{2x} = 0$ for all x. Plug in $i = 2$ in 4' and you will find that the third row is all zeros. Apply this process iteratively (justify this by induction) to conclude that A is the zero matrix! There are no nontrivial morphisms when $\lambda \neq \mu$.

1.2.2 $\lambda = \mu$

In this case conditions 2 and 3 immediately give that the first column and the last row of A must be all zeros; except perhaps a_{mn} (the rightmost bottom entry) and a_{11} (the leftmost bottom entry), of which we know nothing yet.

Here condition 4' says that $a_{ij} = a_{(i+1)(j+1)}$ for all i, j for which this expression makes sense. That is to say, given any entry in the matrix, going down one entry and then right one entry will give us the same value we started with. This immediately implies that any given diagonal of A has all of its entries equal to each other.

Recall that we had that the first column and last row were all zero, barring a_{11} and a_{mn}. If $m \geq n$ this implies that the matrix A is upper triangular (why?) and, since we have one free parameter for each nonzero diagonal, $\dim(\mathrm{Hom}(V_{\lambda,n}, V_{\lambda,m})) = n$ (why?). If $m < n$ then the situation is a bit trickier to describe. Define the *comain diagonal* to be the diagonal ending at a_{mn} (whereas the main diagonal is the one starting at a_{11}). We say that a matrix is *coupper triangular* if all entries below the comain diagonal are zero. In this case we find that A is coupper triangular (why?) and, since we get one free parameter for each nonzero diagonal, $\dim\left(\mathrm{Hom}(V_{\lambda,n}, V_{\lambda,m})\right) = m$.

Exercise 2. *Justify the claims in the preceding paragraph.*

All in all we have that $\dim(\mathrm{Hom}(V_{\lambda,n}, V_{\lambda,m})) = \min(n, m)$.