# Chapter 1

# Preliminaries: Set theory and categories

## 1 Naive set theory

### Exercises

**1.1.** Locate a discussion of Russell's paradox, and understand it.

*Solution.* □

**1.2.** ▹ Prove that if $\sim$ is a relation on a set $S$, then the corresponding family $\mathscr{P}_\sim$ defined in §1.5 is indeed a partition of $S$: that is, its elements are nonempty, disjoint, and their union is $S$. [§1.5]

*Solution.* □

**1.3.** ▹ Given a partition $\mathscr{P}$ on a set $S$, show how to define an equivalence relation $\sim$ on $S$ such that $\mathscr{P}$ is the corresponding partition. [§1.5]

*Solution.* content... □

**1.4.** How many different equivalence relations may be defined on the set $\{1, 2, 3\}$?

*Solution.* content... □

**1.5.** Give an example of a relation that is reflexive and symmetric but not transitive. What happens if you attempt to use this relation to define a partition on the set? (Hint: Thinking about the second question will help you answer the first one.)

*Solution.* content... □

**1.6.** ▷ Define a relation $\sim$ on the set $\mathbb{R}$ of real numbers by setting $a \sim b \iff b-a \in \mathbb{Z}$. Prove that this is an equivalence relation, and find a 'compelling' description for $\mathbb{R}/\sim$. Do the same for the relation $\approx$ on the plane $\mathbb{R} \times \mathbb{R}$ defined by declaring $(a_1, a_2) \approx (b_1, b_2) \iff b_1 - a_1 \in \mathbb{Z}$ and $b_2 - a_2 \in \mathbb{Z}$. [§II.8.1, II.8.10]

*Solution.* content... □

# 2 Functions between sets

## Exercises

**2.1.** ▷ How many different bijections are there between a set $S$ with $n$ elements and itself? [§II.2.1]

*Solution.* content... □

**2.2.** ▷ Prove statement (2) in Proposition 2.1. You may assume that given a family of disjoint subsets of a set, there is a way to choose one element in each member of the family. [§2.5, V.3.3]

*Solution.* content... □

**2.3.** Prove that the inverse of a bijection is a bijection and that the composition of two bijections is a bijection.

*Solution.* content... □

**2.4.** ▷ Prove that 'isomorphism' is an equivalence relation (on any set of sets). [§4.1]

*Solution.* content... □

**2.5.** ▷ Formulate a notion of *epimorphism*, in the style of the notion of *monomorphism* seen in §2.6, and prove a result analogous to Proposition 2.3, for epimorphisms and surjections. [§2.6, §4.2]

*Solution.* content... □

**2.6.** With notation as in Example 2.4, explain how any function $f\colon A \to B$ determines a section of $\pi_A$.

*Solution.* content... □

**2.7.** Let $f\colon A \to B$ be any function. Prove that the graph $\Gamma_f$ of $f$ is isomorphic to $A$.

*Solution.* content... □

**2.8.** Describe as explicitly as you can all terms in the canonical decomposition (cf. §2.8) of the function $\mathbb{R} \to \mathbb{C}$ defined by $r \mapsto e^{2\pi i r}$. (This exercise matches one assigned previously. Which one?)

*Solution.* content... □

**2.9.** ▷ Show that if $A' \cong A''$ and $B' \cong B''$, and further $A' \cap B' = \emptyset$ and $A'' \cap B'' = \emptyset$, then $A' \cup B' \cong A'' \cup B''$. Conclude that the operation $A \sqcup B$ (as described in §1.4) is well-defined up to *isomorphism* (cf. §2.9). [§2.9, 5.7]

*Solution.* content... □

**2.10.** ▷ Show that if $A$ and $B$ are finite sets, then $|B^A| = |B|^{|A|}$. [§2.1, 2.11, §II.4.1]

*Solution.* content... □

**2.11.** ▷ In view of Exercise 2.10, it is not unreasonable to use $2^A$ to denote the set of functions from an arbitrary set $A$ to a set with 2 elements (say $\{0, 1\}$. Prove that there is a bijection between $2^A$ and the *power set* of $A$ (cf. §1.2). [§1.2, III.2.3]

*Solution.* content... □

# 3 Categories

## Exercises

**3.1.** ▷ Let $\mathsf{C}$ be a category. Consider a structure $\mathsf{C}^{op}$ with

- $\mathrm{Obj}(\mathsf{C}^{op}) := \mathrm{Obj}(\mathsf{C})$;
- for A, B objects of $\mathsf{C}^{op}$ (hence objects of $\mathsf{C}$), $\mathrm{Hom}_{\mathsf{C}^{op}}(A, B) := \mathrm{Hom}_{\mathsf{C}}(B, A)$.

Show how to make this into a category (that is, define composition of morphisms in $\mathsf{C}^{op}$ and verify the properties listed in §3.1).

Intuitively, the 'opposite' category $\mathsf{C}^{op}$ is simply obtained by 'reversing all the arrows' in $\mathsf{C}$. [5.1, §VIII.1.1, §IX.1.2, IX.1.10]

*Solution.* content... □

**3.2.** If $A$ is a finite set, how large is $\mathrm{End}_{\mathsf{Set}}(A)$?

*Solution.* content... □

**3.3.** ▷ Formulate precisely what it means to say that $1_a$ is an identity with respect to composition in Example 3.3, and prove this assertion. [§3.2]

*Solution.* content... □

**3.4.** Can we define a category in the style of Example 3.3 using the relation $<$ on the set $\mathbb{Z}$?

*Solution.* content... □

**3.5.** ▷ Explain in what sense Example 3.4 is an instance of the categories considered in Example 3.3. [§3.2]

*Solution.* content... □

**3.6.** ▷ (Assuming some familiarity with linear algebra.) Define a category $\mathsf{V}$ by taking $\mathrm{Obj}(\mathsf{V}) = \mathbb{N}$ and letting $\mathrm{Hom}_{\mathsf{V}}(n, m) =$ the set of $n \times m$ matrices with real entries, for all $n, m \in \mathbb{N}$ (We will leave the reader the task of making sense of a matrix with 0 rows or columns.) Use product of matrices to define composition. Does this category 'feel' familiar? [§VI.2.1, §VIII.1.3]

*Solution.* content... □

**3.7.** ▷ Define carefully objects and morphisms in Example 3.7, and draw the diagram corresponding to composition. [§3.2]

*Solution.* content... □

**3.8.** ▷ A *subcategory* $\mathsf{C}'$ of a category $\mathsf{C}$ consists of a collection of objects of $\mathsf{C}$, with morphisms $\mathrm{Hom}_{\mathsf{C}'}(A, B) \subseteq \mathrm{Hom}_{\mathsf{C}}(A, B)$ for all objects $A, B$ in $\mathrm{Obj}(\mathsf{C}')$, such that identities and compositions in $\mathsf{C}$ make $\mathsf{C}'$ into a category. A subcategory $\mathsf{C}'$ is *full* if $\mathrm{Hom}_{\mathsf{C}'}(A, B) = \mathrm{Hom}_{\mathsf{C}}(A, B)$ for all $A, B$ in $\mathrm{Obj}(\mathsf{C}')$. Construct a category of *infinite sets* and explain how it may be viewed as a full subcategory of $\mathsf{Set}$. [4.4, §VI.1.1, §VIII.1.3]

*Solution.* content... □

**3.9.** ▷ An alternative to the notion of *multiset* introduced in §2.2 is obtained by considering sets endowed with equivalence relations; equivalent elements are taken to be multiple instances of elements 'of the same kind'. Define a notion of morphism between such enhanced sets, obtaining a category $\mathsf{MSet}$ containing (a 'copy' of) $\mathsf{Set}$ as a full subcategory. (There may be more than one reasonable way to do this! This is intentionally an open-ended exercise.) Which objects in $\mathsf{MSet}$ determine ordinary multisets as defined in §2.2 and how? Spell out what a morphism of multisets would be from this point of view. (There are several natural notions of morphisms of multisets. Try to define morphisms in $\mathsf{MSet}$ so that the notion you obtain for ordinary multisets captures your intuitive understanding of these objects.) [§2.2, §3.2, 4.5]

*Solution.* content... □

**3.10.** Since the objects of a category $\mathsf{C}$ are not (necessarily) sets, it is not clear how to make sense of a notion of 'subobject' in general. In some situations it *does* make sense to talk about subobjects, and the subobjects of any given object $A$ in $\mathsf{C}$ are in one-to-one correspondence with the morphisms $A \to \Omega$ for a fixed, special object $\Omega$ of $\mathsf{C}$, called a *subobject classifier*. Show that $\mathsf{Set}$ has a subobject classifier.

*Solution.* content... □

**3.11.** ▷ Draw the relevant diagrams and define composition and identities for the category $\mathsf{C}^{A,B}$ mentioned in Example 3.9. Do the same for the category $\mathsf{C}^{\alpha,\beta}$ mentioned in Example 3.10. [§5.5, 5.12]

*Solution.* content... □

# 4 Morphisms

## Exercises

**4.1.** ▷ Composition is defined for *two* morphisms. If more than two morphisms are given, e.g.,

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \xrightarrow{i} E$$

then one may compose them in several ways, for example:

$$(ih)(gf), \quad (i(hg))f, \quad i((hg)f), \quad \text{etc.}$$

so that at every step one is only composing two morphisms. Prove that the result of any such nested composition is independent of the placement of the parentheses. (Hint: Use induction on $n$ to show that any such choice for $f_n f_{n-1} \dots f_1$ equals

$$((\dots((f_n f_{n-1})f_{n-2})\dots)f_1).$$

Carefully working out the case $n = 5$ is helpful.) [§4.1, §II.1.3]

*Solution.* content... □

**4.2.** ▷ In Example 3.3 we have seen how to construct a category from a set endowed with a relation, provided this latter is reflexive and transitive. For what types of relations is the corresponding category a groupoid (cf. Example 4.6)? [§4.1]

*Solution.* content... □

**4.3.** Let $A, B$ be objects of a category $\mathsf{C}$, and let $f \in \mathrm{Hom}_C(A, B)$ be a morphism.

- Prove that if $f$ has a right-inverse, then $f$ is an epimorphism.
- Show that the converse does not hold, by giving an explicit example of a category and an epimorphism without a right-inverse.

*Solution.* content... □

**4.4.** Prove that the composition of two monomorphisms is a monomorphism. Deduce that one can define a subcategory $\mathsf{C}_{\mathsf{mono}}$ of a category $\mathsf{C}$ by taking the same objects as in $\mathsf{C}$ and defining $\mathrm{Hom}_{\mathsf{C}_{\mathsf{mono}}}(A, B)$ to be the subset of $\mathrm{Hom}_{\mathsf{C}}(A, B)$ consisting of monomorphisms, for all objects $A, B$. (Cf. Exercise 3.8; of course, in general $\mathsf{C}_{\mathsf{mono}}$ is not full in $\mathsf{C}$.) Do the same for epimorphisms. Can you define a subcategory $\mathsf{C}_{\mathsf{nonmono}}$ of $\mathsf{C}$ by restricting to morphisms that are *not* monomorphisms?

*Solution.* content... □

**4.5.** Give a concrete description of monomorphisms and epimorphisms in the category $\mathsf{MSet}$ you constructed in Exercise 3.9. (Your answer will depend on the notion of morphism you defined in that exercise!)

*Solution.* content... □

# 5 Universal properties

## Exercises

**5.1.** Prove that a final object in a category $\mathsf{C}$ is initial in the opposite category $\mathsf{C}^{op}$ (cf. Exercise 3.1).

*Solution.* content... □

**5.2.** ▷ Prove that $\emptyset$ is the unique initial object in $\mathsf{Set}$. [§5.1]

*Solution.* content... □

**5.3.** ▷ Prove that final objects are *unique* up to isomorphism. [§5.1]

*Solution.* content... □

**5.4.** What are initial and final objects in the category of 'pointed sets' (Example 3.8)? Are they unique?

*Solution.* content... □

**5.5.** ▷ What are the final objects in the category considered in §5.3? [§5.3]

*Solution.* content... □

**5.6.** ▷ Consider the category corresponding to endowing (as in Example 3.3) the set $\mathbb{Z}^+$ of positive integers with the *divisibility* relation. Thus there is exactly one morphism $d \to m$ in this category if and only if $d$ divides $m$ without remainder; there is no morphism between $d$ and $m$ otherwise. Show that this category has products and coproducts. What are their 'conventional' names? [§VII.5.1]

*Solution.* content... □

**5.7.** Redo Exercise 2.9, this time using Proposition 5.4.

*Solution.* content... □

**5.8.** Show that in every category $\mathsf{C}$ the products $A \times B$ and $B \times A$ are isomorphic, if they exist. (Hint: Observe that they both satisfy the universal property for the product of $A$ and $B$; then use Proposition 5.4.)

*Solution.* content... □

**5.9.** Let $\mathsf{C}$ be a category with products. Find a reasonable candidate for the universal property that the product $A \times B \times C$ of three objects of $\mathsf{C}$ ought to satisfy, and prove that both $(A \times B) \times C$ and $A \times (B \times C)$ satisfy this universal property. Deduce that $(A \times B) \times C$ and $A \times (B \times C)$ are necessarily isomorphic.

*Solution.* content... □

**5.10.** Push the envelope a little further still, and define products and coproducts for *families* (i.e., indexed sets) of objects of a category.

Do these exist in $\mathsf{Set}$?

It is common to denote the product $\underbrace{A \times \cdots \times A}_{n\text{times}}$ by $A^n$.

*Solution.* content... □

**5.11.** Let $A$, resp. $B$ be a set, endowed with an equivalence relation $\sim_A$, resp. $\sim_B$. Define a relation $\sim$ on $A \times B$ by setting

$$(a_1, b_1) \sim (a_2, b_2) \iff a_1 \sim_A a_2 \text{ and } b_1 \sim_B b_2.$$

(This is immediately seen to be an equivalence relation.)

- Use the universal property for quotients (§5.3) to establish that there are functions $(A \times B)/\sim \to A/\sim_A$, $(A \times B)/\sim \to B/\sim_B$.
- Prove that $(A\times B)/\sim$, with these two functions, satisfies the universal property for the product of $A/\sim_A$ and $B/\sim_B$.
- Conclude (without further work) that $(A \times B)/\sim \cong (A/\sim_A) \times (B/\sim_B)$.

*Solution.* content... □

**5.12.** $\neg$ Define the notions of *fibered products* and *fibered coproducts*, as terminal objects of the categories $\mathsf{C}_{\alpha,\beta}$, $\mathsf{C}^{\alpha,\beta}$ considered in Example 3.10 (cf. also Exercise 3.11), by stating carefully the corresponding universal properties.

As it happens, $\mathsf{Set}$ has both fibered products and coproducts. Define these objects 'concretely', in terms of naive set theory. [II.3.9, III.6.10, III.6.11]

*Solution.* content... □