Chapter 1

Preliminaries: Set theory and categories

1 Naive set theory

Exercises

1.1. Locate a discussion of Russell's paradox, and understand it.

Solution. There are many available options. I first read about Russell's paradox in Section 3.2 of [1]. $\square$

1.2. $\triangleright$ Prove that if $\sim$ is a relation on a set S, then the corresponding family $\mathscr{P}_\sim$ defined in §1.5 is indeed a partition of S: that is, its elements are nonempty, disjoint, and their union is S. [§1.5]

Solution. Let $[a]_\sim \in \mathscr{P}_\sim$ for some $a \in S$. Then $[a]_\sim$ is nonempty since it contains a, by reflexivity.

Let $[b]_\sim$ be another equivalence class, where $b \in S$. Suppose $c \in [a]_\sim \cap [b]_\sim$, that is $c \sim a$ and $c \sim b$. By symmetry we have $a \sim c$ and $c \sim b$. By transitivity this implies $a \sim b$. From this, it is not hard to show that $[a]_\sim = [b]_\sim$. In conclusion, equivalence classes are either equal or disjoint.

Clearly $\bigcup_{s \in S} [s]_\sim \subseteq S$ since we are taking the union of subsets of S. Conversely, $S \subseteq \bigcup_{s \in S} [s]_\sim$ since $s \in [s]_\sim$ for all $s \in S$ by reflexivity. $\square$

1.3. $\triangleright$ Given a partition $\mathscr{P}$ on a set S, show how to define an equivalence relation $\sim$ on S such that $\mathscr{P}$ is the corresponding partition. [§1.5]

Solution. For $a, b \in S$ we say $a \sim b$ iff there exists some $X \in \mathscr{P}$ such that $a, b \in X$. $\square$

1.4. How many different equivalence relations may be defined on the set $\{1, 2, 3\}$?

Solution. This is equivalent to finding all partitions of $\{1, 2, 3\}$.

$$\{\{1\}, \{2\}, \{3\}\} \quad \{\{1\}, \{2, 3\}\} \quad \{\{1, 3\}, \{2\}\}$$

$$\{\{1, 2\}, \{3\}\} \quad \{\{1, 2, 3\}\}.$$

□

1.5. Give an example of a relation that is reflexive and symmetric but not transitive. What happens if you attempt to use this relation to define a partition on the set? (Hint: Thinking about the second question will help you answer the first one.)

Solution. Consider the relation in the set $\mathbb{R}$, defined by the following rule. Let $a, b \in \mathbb{R}$. We say $a \sim b$ iff $|a - b| \leq 1$. This relation is clearly reflexive and symmetric, but it is not transitive (why?).

A reflexive, symmetric, non-transitive relation will result in a "partition" in which the classes are no longer disjoint.

□

1.6. ▷ Define a relation $\sim$ on the set $\mathbb{R}$ of real numbers by setting $a \sim b \iff b - a \in \mathbb{Z}$. Prove that this is an equivalence relation, and find a 'compelling' description for $\mathbb{R}/\sim$. Do the same for the relation $\approx$ on the plane $\mathbb{R} \times \mathbb{R}$ defined by declaring $(a_1, a_2) \approx (b_1, b_2) \iff b_1 - a_1 \in \mathbb{Z}$ and $b_2 - a_2 \in \mathbb{Z}$. [§II.8.1, II.8.10]

Solution. Let $a, b, c \in \mathbb{R}$. We have that $a - a = 0 \in \mathbb{Z}$, so $a \sim a$ and the relation is reflexive. If $a \sim b$ then $b - a \in \mathbb{Z}$ and so $-(b - a) = a - b \in \mathbb{Z}$, which means $b \sim a$; hence the relation is symmetric. Finally, if $a \sim b$ and $b \sim c$ then $b - a$ and $c - b$ are integers and so $(b - a) + (c - b) = c - a \in \mathbb{Z}$, which means $a \sim c$ and the relation is transitive. We have shown that this is an equivalence relation. Notice that we are identifying real numbers that differ by an integer. In particular, we are identifying each (positive) real number with its fractional part (a similar thing is true for negative real numbers). So we can think of the quotient $\mathbb{R}/\sim$ as the interval $[0, 1)$ since every number in this interval is a representative of a unique equivalence class, and these are all the equivalence classes.

For $\approx$ a similar discussion applies: it is analogously showed it is an equivalence relation, and one can think of the quotient as the unit square $[0, 1) \times [0, 1)$. This result also follows from Exercise 5.11.

□

2 Functions between sets

Exercises

2.1. ▷ How many different bijections are there between a set S with n elements and itself? [§II.2.1]

Solution. $n!$. □

2.2. ▷ Prove statement (2) in Proposition 2.1. You may assume that given a family of disjoint subsets of a set, there is a way to choose one element in each member of the family. [§2.5, V.3.3]

Solution. First we prove that if $f: A \to B$ has a right inverse then it is a surjection. Let $g: B \to A$ the the right inverse of f. Let $b \in B$. Then clearly $f(g(b)) = b$, so f is surjective.

Conversely, assume f is surjective. Then, if $b \in B$, we have that $f^{-1}(\{b\})$ is nonempty. For $b \in B$ we pick some $a_b \in f^{-1}(\{b\})$ and define $g: B \to A$ by $g(b) \coloneqq a_b$. By construction, $f \circ g = \mathrm{id}_B$. □

2.3. Prove that the inverse of a bijection is a bijection and that the composition of two bijections is a bijection.

Solution. Let $f: A \to B$ and $g: B \to C$ be two bijections. Then f^{-1} is also a bijection since it has a two-sided inverse, namely f. Also, $g \circ f$ is a bijection since $f^{-1} \circ g^{-1}$ is a two-sided inverse. □

2.4. ▷ Prove that 'isomorphism' is an equivalence relation (on any set of sets). [§4.1]

Solution. See Exercise 2.3 for some necessary results. Clearly for any set A we have that A is isomorphic to A since $\mathrm{id}_A: A \to A$ is a bijection. If A is isomorphic to B then there is some bijection $f: A \to B$. But then $f^{-1}: B \to A$ is also a bijection and hence B is isomorphic to A. Now suppose A is isomorphic to B and B is isomorphic to C, where $f: A \to B$ and $g: B \to C$ are bijections. Then $g \circ f: A \to C$ is also a bijection and so A is isomorphic to C. □

2.5. ▷ Formulate a notion of *epimorphism*, in the style of the notion of *monomorphism* seen in §2.6, and prove a result analogous to Proposition 2.3, for epimorphisms and surjections. [§2.6, §4.2]

Solution. We say a function $f: A \to B$ is an *epimorphism* (or *epic*) if the following holds:

$$\textit{for all sets } Z \textit{ and all functions } \beta', \beta'': B \to Z$$

$$\beta' \circ f = \beta'' \circ f \implies \beta' = \beta''.$$

Now we claim that a function is an epimorphism iff it is a surjection. Indeed, if $f: A \to B$ is a surjection then it has a right inverse $g: B \to A$. Suppose Z is a set and let $\beta', \beta'': B \to Z$ be functions with $\beta' \circ f = \beta'' \circ f$. Then $(\beta' \circ f) \circ g = (\beta'' \circ f) \circ g$ which implies $\beta' \circ (f \circ g) = \beta'' \circ (f \circ g)$, and this in turn implies $\beta' \circ \mathrm{id}_B = \beta'' \circ \mathrm{id}_B$, and so $\beta = \beta''$ as desired.

Now suppose $f: A \to B$ is an epimorphism. For the sake of contradiction, suppose there is some $b_0 \in B$ such that $f(a) \neq b_0$ for all $a \in A$. Let $Z = \{0, 1\}$ and define $\beta', \beta'': B \to Z$ by

$$\beta'(b) \coloneqq 0 \text{ and } \beta''(b) \coloneqq \begin{cases} 0 & \text{if } b \neq b_0 \\ 1 & \text{if } b = b_0 \end{cases},$$

for all $b \in B$. Clearly $\beta' \circ f = \beta'' \circ f$ but $\beta' \neq \beta''$, a contradiction. So f must be a surjection. $\square$

2.6. With notation as in Example 2.4, explain how any function $f\colon A \to B$ determines a section of π_A.

Solution. Define $\pi_A^*\colon A \to A \times B$ by the rule $\pi_A^*(a) := (a, f(a))$. This is manifestly a section of π_A. $\square$

2.7. Let $f\colon A \to B$ be any function. Prove that the graph Γ_f of f is isomorphic to A.

Solution. Define $f^*\colon A \to \Gamma_f$ by the rule $f^*(a) := (a, f(a))$. We claim that f^* is a bijection. Indeed, we will see that the natural projection restricted to Γ_f, written as $\pi_A\big|_{\Gamma_f}$, is a two-sided inverse. Let $a \in A$ and consider

$$\begin{aligned}(\pi_A\big|_{\Gamma_f} \circ f^*)(a) &= \pi_A\big|_{\Gamma_f}(a, f(a)) \\ &= a = \mathrm{id}_A(a).\end{aligned}$$

Similarly, let $(a, f(a))$ be an arbitrary element of Γ_f. Then

$$\begin{aligned}(f^* \circ \pi_A\big|_{\Gamma_f})(a, f(a)) &= f^*(a) \\ &= (a, f(a)) = \mathrm{id}_{\Gamma_f}(a, f(a)).\end{aligned}$$

Thus f^* is a bijection. $\square$

2.8. Describe as explicitly as you can all terms in the canonical decomposition (cf. §2.8) of the function $\mathbb{R} \to \mathbb{C}$ defined by $r \mapsto e^{2\pi i r}$. (This exercise matches one assigned previously. Which one?)

Solution. Let $f\colon \mathbb{R} \to \mathbb{C}$ be defined by $f(r) := e^{2\pi i r}$. We define an equivalence relation on $\mathbb{R}$ by $a \sim a' \iff f(a') = f(a'')$. This is easily seen to be equivalent the same as the equivalence relation defined in Exercise 1.6. In that exercise we saw that the quotient can be identified with the interval $[0, 1)$ and the projection $\pi\colon \mathbb{R} \to \mathbb{R}/\sim$ is assigning to each real number its (positive) fractional part. Then the canonical decomposition gives a bijection from the quotient, i.e. $[0, 1)$, to the image of f, i.e. the unit circle in the complex plane, by assigning to each $x \in [0, 1)$ the point $e^{2\pi i x}$. Finally, this unit circle is included in the whole complex plane, in the obvious sense. $\square$

2.9. $\triangleright$ Show that if $A' \cong A''$ and $B' \cong B''$, and further $A' \cap B' = \emptyset$ and $A'' \cap B'' = \emptyset$, then $A' \cup B' \cong A'' \cup B''$. Conclude that the operation $A \sqcup B$ (as described in §1.4) is well-defined up to *isomorphism* (cf. §2.9). [§2.9, 5.7]

Solution. Let $f\colon A' \to A''$ and $g\colon B \to B'$ be bijections. Let us define $f \oplus g\colon A' \cup B' \to A'' \cup B''$ by the rule

$$f \oplus g(x) := \begin{cases} f(x) & \text{if } x \in A' \\ g(x) & \text{if } x \in B' \end{cases}.$$

This is well defined since if $x \in A' \cup B'$ then $x \in A'$ or $x \in B'$ but not both, since $A' \cap B' = \emptyset$.

Now we prove $f \oplus g$ is a bijection. Define $f^{-1} \oplus g^{-1} \colon A'' \cup B'' \to A' \cup B'$ by

$$f^{-1} \oplus g^{-1}(y) \coloneqq \begin{cases} f^{-1}(y) & \text{if } y \in A'' \\ g^{-1}(y) & \text{if } y \in B'' \end{cases}.$$

This is well defined since if $y \in A'' \cup B''$ then $y \in A''$ or $y \in B''$ but not both, since $A'' \cap B'' = \emptyset$. It is immediately verified that $f \oplus g$ and $f^{-1} \oplus g^{-1}$ are inverses of each other and hence they are bijections.

In conclusion, no matter how we make disjoint copies of A and B, the resulting disjoint unions will be isomorphic. Hence, it makes some sense to talk about *the* disjoint union of A and B. $\square$

2.10. $\triangleright$ Show that if A and B are finite sets, then $|B^A| = |B|^{|A|}$. [§2.1, 2.11, §II.4.1]

Solution. Let $|A| = n$. We use induction on n. If $n = 0$ then $A = \emptyset$ and there is only one function $\emptyset \to B$, and further $1 = |B|^0$. This closes the base case.

Suppose the claim is true for some natural number n. Defining a function from a set of $n+1$ elements to B is the same as first defining it for n elements, for which there are $|B|^n$ choices by inductive hypothesis, and then figuring out where the last element goes, for which there are $|B|$ choices. Overall, there must be $|B|^n|B| = |B|^{n+1}$ ways of defining the function when $|A| = n+1$. This closes the induction. $\square$

2.11. $\triangleright$ In view of Exercise 2.10, it is not unreasonable to use 2^A to denote the set of functions from an arbitrary set A to a set with 2 elements (say $\{0,1\}$. Prove that there is a bijection between 2^A and the *power set* of A (cf. §1.2). [§1.2, III.2.3]

Solution. Define $F \colon 2^A \to \mathscr{P}(A)$ by the rule

$$F(f) \coloneqq g^{-1}(\{1\}), \text{ for all } f \colon A \to \{0,1\}.$$

Now define $G \colon \mathscr{P}(A) \to 2^A$ by saying that, for all $S \subseteq A$ we have

$$G(S) \coloneqq \mathbf{1}_S,$$

where $\mathbf{1}_S$ is the indicator function of S, defined on A. It is readily seen that F and G are inverses of each other and hence bijections. $\square$

3 Categories

Exercises

3.1. $\triangleright$ Let C be a category. Consider a structure C^{op} with

- $\mathrm{Obj}(\mathsf{C}^{op}) := \mathrm{Obj}(\mathsf{C})$;

- for A, B objects of C^{op} (hence objects of C), $\mathrm{Hom}_{\mathsf{C}^{op}}(A,B) := \mathrm{Hom}_{\mathsf{C}}(B,A)$.

Show how to make this into a category (that is, define composition of morphisms in C^{op} and verify the properties listed in §3.1).

Intuitively, the 'opposite' category C^{op} is simply obtained by 'reversing all the arrows' in C. [5.1, §VIII.1.1, §IX.1.2, IX.1.10]

Solution. Let A, B, C be objects of C^{op} and let $f \in \mathrm{Hom}_{\mathsf{C}^{op}}(A,B)$ and let $g \in \mathrm{Hom}_{\mathsf{C}^{op}}(B,C)$. Then $f \in \mathrm{Hom}_{\mathsf{C}}(B,A)$ and $g \in \mathrm{Hom}_{\mathsf{C}}(C,B)$. As C is a category, let us write the composition of g and f as $f \circ_{\mathsf{C}} g \in \mathrm{Hom}_{\mathsf{C}}(C,A)$. Then we can define

$$g \circ_{\mathsf{C}^{op}} f := f \circ_{\mathsf{C}} g \in \mathrm{Hom}_{\mathsf{C}}(C,A) = \mathrm{Hom}_{\mathsf{C}^{op}}(A,C).$$

For simplicity, we omit the symbol $\circ_{\mathsf{C}}$ from now on. This law of composition is associative since, if we let D be an object of C^{op} and $h \in \mathrm{Hom}_{\mathsf{C}^{op}}(C,D)$, then

$$h \circ_{\mathsf{C}^{op}} (g \circ_{\mathsf{C}^{op}} f) = h \circ_{\mathsf{C}^{op}} (fg) = (fg)h \stackrel{!}{=} f(gh) = f(h \circ_{\mathsf{C}^{op}} g) = (h \circ_{\mathsf{C}^{op}} g) \circ_{\mathsf{C}^{op}} f.$$

Notice we used the associativity in C at $\stackrel{!}{=}$.

If A is an object of C^{op} then it is an object of C and hence we must have an identity in C, denoted $1_A \in \mathrm{Hom}_{\mathsf{C}}(A,A)$. But then $1 \in \mathrm{Hom}_{\mathsf{C}^{op}}(A,A)$ and thus it works as an identity in C^{op} as well. Indeed, if B is an object of C^{op} then $1_B \in \mathrm{Hom}_{\mathsf{C}^{op}}(B,B)$ and we have, for all $f \in \mathrm{Hom}_{\mathsf{C}^{op}}(A,B)$,

$$\begin{aligned} f \circ_{\mathsf{C}^{op}} 1_A &= 1_A f = f, \\ 1_B \circ_{\mathsf{C}^{op}} f &= f 1_B = f. \end{aligned}$$

□

3.2. If A is a finite set, how large is $\mathrm{End}_{\mathsf{Set}}(A)$?

Solution. $|A|^{|A|}$. □

3.3. ▷ Formulate precisely what it means to say that 1_a is an identity with respect to composition in Example 3.3, and prove this assertion. [§3.2]

Solution. Let $a, b \in S$, and let $f \in \mathrm{Hom}(a,b)$. Then $f = (a,b)$ by necessity. Furthermore, $1_a = (a,a)$ and $1_b = (b,b)$. Thus

$$\begin{aligned} 1_a f &= (a,a)(a,b) = (a,b) = f, \\ f 1_b &= (a,b)(b,b) = (a,b) = f. \end{aligned}$$

□

3.4. Can we define a category in the style of Example 3.3 using the relation $<$ on the set $\mathbb{Z}$?

Solution. No because $<$ is not reflexive; hence the resulting "category" would not have identities. □

3.5. ▷ Explain in what sense Example 3.4 is an instance of the categories considered in Example 3.3. [§3.2]

Solution. In the power set of a set (or more generally in any set of sets), the relation $\subseteq$ is reflexive and transitive. □

3.6. ▷ (Assuming some familiarity with linear algebra.) Define a category V by taking $\mathrm{Obj}(\mathsf{V}) = \mathbb{N}$ and letting $\mathrm{Hom}_{\mathsf{V}}(n, m) =$ the set of $n \times m$ matrices with real entries, for all $n, m \in \mathbb{N}$ (We will leave the reader the task of making sense of a matrix with 0 rows or columns.) Use product of matrices to define composition. Does this category 'feel' familiar? [§VI.2.1, §VIII.1.3]

Solution. In this category the product of matrices is associative, hence they form a valid composition law. Furthermore, identity matrices clearly behave like identities with respect to this composition. □

3.7. ▷ Define carefully objects and morphisms in Example 3.7, and draw the diagram corresponding to composition. [§3.2]

Solution. Objects of C^A are morphisms from A to Z for some object Z of C, i.e. $A \to Z$. Then if we have some $A \xrightarrow{f} Z_1$ and some $A \xrightarrow{g} Z_2$ then a morphism from the former to the latter is a commutative diagram

$$\begin{tikzcd} & Z_1 \arrow[d, "\sigma"] \\ A \arrow[ur, "f"] \arrow[dr, "g"'] & \\ & Z_2 \end{tikzcd}$$

where $\sigma \in \mathrm{Hom}_{\mathsf{C}}(Z_1, Z_2)$. □

3.8. ▷ A *subcategory* C' of a category C consists of a collection of objects of C, with morphisms $\mathrm{Hom}_{\mathsf{C}'}(A, B) \subseteq \mathrm{Hom}_{\mathsf{C}}(A, B)$ for all objects A, B in $\mathrm{Obj}(\mathsf{C}')$, such that identities and compositions in C make C' into a category. A subcategory C' is *full* if $\mathrm{Hom}_{\mathsf{C}'}(A, B) = \mathrm{Hom}_{\mathsf{C}}(A, B)$ for all A, B in $\mathrm{Obj}(\mathsf{C}')$. Construct a category of *infinite sets* and explain how it may be viewed as a full subcategory of Set. [4.4, §VI.1.1, §VIII.1.3]

Solution. Define $\mathrm{Obj}(\mathsf{C}')$ to be all infinite sets, and define $\mathrm{Hom}_{\mathsf{C}'}(A, B) := B^A = \mathrm{Hom}_{\mathsf{Set}}(A, B)$. Under these definitions, it is clear that C' is a full subcategory of Set. □

3.9. ▷ An alternative to the notion of *multiset* introduced in §2.2 is obtained by considering sets endowed with equivalence relations; equivalent elements are taken to be multiple instances of elements 'of the same kind'. Define a notion of morphism between such enhanced sets, obtaining a category MSet containing (a 'copy' of) Set as a full subcategory. (There may be more than one reasonable way to do this! This is intentionally an open-ended exercise.) Which objects in MSet determine ordinary multisets as defined in §2.2 and how? Spell out what a morphism of multisets would be from this point of view. (There are several natural notions of morphisms of multisets. Try to define morphisms in MSet so that the notion you obtain for ordinary multisets captures your intuitive understanding of these objects.) [§2.2, §3.2, 4.5]

Solution. If $(A, \sim_A)$ and $(B, \sim_B)$ are multisets as described in the question, then morphisms from $(A, \sim_A)$ to $(B, \sim_B)$ are functions $f\colon A \to B$ such that

$$a' \sim_A a'' \implies f(a') \sim_B f(a'') \text{ for all } a', a'' \in A.$$

It is trivial to check the axioms of a category. An isomorphism between to multisets would be a bijection between the underlying sets that preserves the equivalence classes.

Set is a full subcategory of MSet in the sense that it is the same as the full subcategory consisting of multisets of the form $(S, =)$.

The objects of MSet that are multisets in the sense of §2.2 are the ones in which the equivalence classes contain finitely many elements. □

3.10. Since the objects of a category C are not (necessarily) sets, it is not clear how to make sense of a notion of 'subobject' in general. In some situations it *does* make sense to talk about subobjects, and the subobjects of any given object A in C are in one-to-one correspondence with the morphisms $A \to \Omega$ for a fixed, special object Ω of C, called a *subobject classifier*. Show that Set has a subobject classifier.

Solution. Indeed, $\{0, 1\}$ is such a subobject classifier; this is the content of Exercise 2.11. □

3.11. ▷ Draw the relevant diagrams and define composition and identities for the category $\mathsf{C}^{A,B}$ mentioned in Example 3.9. Do the same for the category $\mathsf{C}^{\alpha,\beta}$ mentioned in Example 3.10. [§5.5, 5.12]

Solution. Objects in $\mathsf{C}^{A,B}$ are diagrams of the form

$$\begin{tikzcd} A \arrow[dr] & \\ & Z \\ B \arrow[ur] & \end{tikzcd},$$

where Z is an object of C and arrows correspond to morphisms in C in the obvious way.

Given two objects of $\mathsf{C}^{A,B}$

$$\begin{tikzcd} A \arrow[dr, "f_1"] & \\ & Z_1 \\ B \arrow[ur, "g_1"'] & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[dr, "f_2"] & \\ & Z_2 \\ B \arrow[ur, "g_2"] & \end{tikzcd}.$$

A morphism from the leftmost one to the other one is given by a commutative diagram of the form

$$\begin{tikzcd}
A \arrow[dr, "f_1"] \arrow[drr, bend left, "f_2"] & & \\
& Z_1 \arrow[r, "\sigma"] & Z_2 \\
B \arrow[ur, "g_1"'] \arrow[urr, bend right, "g_2"'] & &
\end{tikzcd},$$

for some $\sigma \in \mathrm{Hom}_{\mathsf{C}}(Z_1, Z_2)$. Given two morphisms

$$\begin{tikzcd}
A \arrow[dr, "f_1"] \arrow[drr, bend left, "f_2"] & & \\
& Z_1 \arrow[r, "\sigma"] & Z_2 \\
B \arrow[ur, "g_1"'] \arrow[urr, bend right, "g_2"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[dr, "f_2"] \arrow[drr, bend left, "f_3"] & & \\
& Z_2 \arrow[r, "\tau"] & Z_3 \\
B \arrow[ur, "g_2"'] \arrow[urr, bend right, "g_3"'] & &
\end{tikzcd},$$

we can combine them to form one big commutative diagram.

$$\begin{tikzcd}
A \arrow[dr, "f_1"] \arrow[drr, bend left, "f_2"] \arrow[drrr, bend left, "f_3"] & & & \\
& Z_1 \arrow[r, "\sigma"] & Z_2 \arrow[r, "\tau"] & Z_3 \\
B \arrow[ur, "g_1"'] \arrow[urr, bend right, "g_2"'] \arrow[urrr, bend right, "g_3"'] & & &
\end{tikzcd}.$$

It is immediate that the diagram obtained by removing the middle object is commutative; that is to say that the diagram

$$\begin{tikzcd}
A \arrow[dr, "f_1"] \arrow[drr, bend left, "f_3"] & & \\
& Z_1 \arrow[r, "\tau\sigma"] & Z_2 \\
B \arrow[ur, "g_1"'] \arrow[urr, bend right, "g_3"'] & &
\end{tikzcd}$$

commutes. Identities in $\mathsf{C}^{A,B}$ are just diagrams of the form

$$\begin{tikzcd}
A \arrow[dr, "f"] \arrow[drr, bend left, "f"] & & \\
& Z \arrow[r, "1_Z"] & Z \\
B \arrow[ur, "g"'] \arrow[urr, bend right, "g"'] & &
\end{tikzcd}.$$

Again, it is immediate that this diagram commutes and it behaves like an identity with respect to composition.

For the category $\mathsf{C}^{\alpha,\beta}$ we are given some fixed objects of C, call them A, B, C, and fixed morphisms $\alpha\colon C \to A$ and $\beta\colon C \to B$. An object in $\mathsf{C}^{\alpha,\beta}$ is a commutative diagram of the form

$$\begin{tikzcd}[row sep=small] & A \arrow[dr] & \\ C \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & Z \\ & B \arrow[ur] & \end{tikzcd},$$

where Z is an object of C and arrows correspond to morphisms in C in the obvious way. □

4 Morphisms

Exercises

4.1. ▷ Composition is defined for *two* morphisms. If more than two morphisms are given, e.g.,

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \xrightarrow{i} E$$

then one may compose them in several ways, for example:

$$(ih)(gf), \quad (i(hg))f, \quad i((hg)f), \quad \text{etc.}$$

so that at every step one is only composing two morphisms. Prove that the result of any such nested composition is independent of the placement of the parentheses. (Hint: Use induction on n to show that any such choice for $f_n f_{n-1} \dots f_1$ equals

$$((\dots((f_n f_{n-1}) f_{n-2}) \dots) f_1).$$

Carefully working out the case $n = 5$ is helpful.) [§4.1, §II.1.3]

Solution. content... □

4.2. ▷ In Example 3.3 we have seen how to construct a category from a set endowed with a relation, provided this latter is reflexive and transitive. For what types of relations is the corresponding category a groupoid (cf. Example 4.6)? [§4.1]

Solution. content... □

4.3. Let A, B be objects of a category C, and let $f \in \mathrm{Hom}_C(A, B)$ be a morphism.

- Prove that if f has a right-inverse, then f is an epimorphism.
- Show that the converse does not hold, by giving an explicit example of a category and an epimorphism without a right-inverse.

Solution. content... □

4.4. Prove that the composition of two monomorphisms is a monomorphism. Deduce that one can define a subcategory $\mathsf{C}_{\mathsf{mono}}$ of a category C by taking the same objects as in C and defining $\mathrm{Hom}_{\mathsf{C}_{\mathsf{mono}}}(A, B)$ to be the subset of $\mathrm{Hom}_{\mathsf{C}}(A, B)$ consisting of monomorphisms, for all objects A, B. (Cf. Exercise 3.8; of course, in general $\mathsf{C}_{\mathsf{mono}}$ is not full in C.) Do the same for epimorphisms. Can you define a subcategory $\mathsf{C}_{\mathsf{nonmono}}$ of C by restricting to morphisms that are *not* monomorphisms?

Solution. content... □

4.5. Give a concrete description of monomorphisms and epimorphisms in the category MSet you constructed in Exercise 3.9. (Your answer will depend on the notion of morphism you defined in that exercise!)

Solution. content... □

5 Universal properties

Exercises

5.1. Prove that a final object in a category C is initial in the opposite category C^{op} (cf. Exercise 3.1).

Solution. content... □

5.2. $\triangleright$ Prove that $\emptyset$ is the unique initial object in Set. [§5.1]

Solution. content... □

5.3. $\triangleright$ Prove that final objects are *unique* up to isomorphism. [§5.1]

Solution. content... □

5.4. What are initial and final objects in the category of 'pointed sets' (Example 3.8)? Are they unique?

Solution. content... □

5.5. $\triangleright$ What are the final objects in the category considered in §5.3? [§5.3]

Solution. content... □

5.6. $\triangleright$ Consider the category corresponding to endowing (as in Example 3.3) the set $\mathbb{Z}^+$ of positive integers with the *divisibility* relation. Thus there is exactly one morphism $d \to m$ in this category if and only if d divides m without remainder; there is no morphism between d and m otherwise. Show that this category has products and coproducts. What are their 'conventional' names? [§VII.5.1]

Solution. content... □

5.7. Redo Exercise 2.9, this time using Proposition 5.4.

Solution. content... $\square$

5.8. Show that in every category C the products $A \times B$ and $B \times A$ are isomorphic, if they exist. (Hint: Observe that they both satisfy the universal property for the product of A and B; then use Proposition 5.4.)

Solution. content... $\square$

5.9. Let C be a category with products. Find a reasonable candidate for the universal property that the product $A \times B \times C$ of three objects of C ought to satisfy, and prove that both $(A \times B) \times C$ and $A \times (B \times C)$ satisfy this universal property. Deduce that $(A \times B) \times C$ and $A \times (B \times C)$ are necessarily isomorphic.

Solution. content... $\square$

5.10. Push the envelope a little further still, and define products and coproducts for *families* (i.e., indexed sets) of objects of a category.

Do these exist in Set?

It is common to denote the product $\underbrace{A \times \cdots \times A}_{n \text{times}}$ by A^n.

Solution. content... $\square$

5.11. Let A, resp. B be a set, endowed with an equivalence relation $\sim_A$, resp. $\sim_B$. Define a relation $\sim$ on $A \times B$ by setting

$$(a_1, b_1) \sim (a_2, b_2) \iff a_1 \sim_A a_2 \text{ and } b_1 \sim_B b_2.$$

(This is immediately seen to be an equivalence relation.)

- Use the universal property for quotients (§5.3) to establish that there are functions $(A \times B)/\sim \to A/\sim_A$, $(A \times B)/\sim \to B/\sim_B$.
- Prove that $(A \times B)/\sim$, with these two functions, satisfies the universal property for the product of $A/\sim_A$ and $B/\sim_B$.
- Conclude (without further work) that $(A \times B)/\sim \cong (A/\sim_A) \times (B/\sim_B)$.

Solution. content... $\square$

5.12. $\neg$ Define the notions of *fibered products* and *fibered coproducts*, as terminal objects of the categories $\mathsf{C}_{\alpha,\beta}$, $\mathsf{C}^{\alpha,\beta}$ considered in Example 3.10 (cf. also Exercise 3.11), by stating carefully the corresponding universal properties.

As it happens, Set has both fibered products and coproducts. Define these objects 'concretely', in terms of naive set theory. [II.3.9, III.6.10, III.6.11]

Solution. content... $\square$

Bibliography

[1] Terence Tao. *Analysis*. Third edition. Vol. I. Hindustan Book Agency, 2014. 368 pp. ISBN: 81-85931-62-3.