My Part III Essay

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1 Introduction

The search for necessary and sufficient conditions for a monoid to be embeddable into a group. The more general problem of embedding categories into a groupoid is no harder but provides some perspective that is incredibly useful.

Point out that adding formal inverses does not work in general, but we will see a situation where it does work.

Definition 1.1 (Cancellable monoids)**.** Say that a monoid M is *left-cancellable* (respectively *right cancellable*) if $ab = ac$ (respectively $ba = ca$) implies $b = c$ for all $a, b, c \in M$. We say that a monoid is *cancellable* if it is both left and right-cancellable

There are monoids that are not cancellable. Indeed, note that for any set S there is the monoid $\mathrm{End}(S)$ of functions $S \to S$ under composition. If we take $S = \{0, 1\}$ then $\mathrm{End}(S)$ has two distinct constant functions c_0, c_1 and we see that $c_0c_0 = c_0c_1$ even though $c_0 \neq c_1$.

While being cancellable is obviously a necessary condition, it is not sufficient as Malcev (reference missing) pointed out.

Example 1.2. Give a simpler example than Malcev's.

However, if we assume the monoid is, in addition, commutative, then we can add formal inverses without issue.

Proposition 1.3. *A commutative cancellable monoid can be embedded into a group.*

Proof. We mimic the proof that an integral domain can be embedded into its field of fractions. Let M be a commutative cancellable monoid. Define a relation on $M \times M$ by the rule $(a, b) \sim (a', b')$ if and only if $ab' = a'b$. This relation is clearly reflexive and symmetric. Transitivity holds precisely because of commutativity and cancellability: if $(a, b) \sim (a', b')$ and $(a', b') \sim (a'', b'')$ then

$$(ab'')a' = a(a'b'') = a(a''b') = a''(ab') = a''(a'b) = (a''b)a',$$

which implies $ab'' = a''b$, i.e., $(a, b) \sim (a'', b'')$. Let G be the quotient $M \times M/ \sim$.

Define a binary operation on G extending the operation of M pointwise, that is the operation $(a,b)(c,d) = (ac,bd)$. It is easy to check that this operation respects the equivalence relation and thus is well-defined. Clearly G is a group with identity $(1,1)$ and inverses defined by $(a,b)^{-1} = (b,a)$. Furthermore, M embeds into G via the function $m \mapsto (m,1)$. $\square$

But being commutative and cancellable, while a sufficient condition, is not necessary: just take any non-commutative subgroup of a group.

Here is another set of sufficient conditions. Call a monoid M *epimorphic* if for all $a, c \in M$ there is some $b \in M$ such that $ab = c$. Clearly all groups are epimorphic (just take $b = a^{-1}c$).

Proposition 1.4. *A left-cancellable epimorphic monoid can be embedded into a group.*

The proof is a nice application of Cayley's theorem for monoids, which says that a monoid M acts faithfully on its underlying set by left-multiplication. This version of Cayley's theorem can be proven in the same way as the version for groups (in particular, it follows from Yoneda's Lemma).

Proof. Let M be a monoid with underlying set S. Then the set of functions $S \to S$, denoted by $\mathrm{End}(S)$, is a monoid under composition of functions. Cayley's theorem provides and embedding of monoids $M \hookrightarrow \mathrm{End}(S)$ defined by $m \mapsto \lambda x.mx$. But the hypotheses ensure that left-multiplication by m is injective and surjective for all $m \in M$. Thus we have an embedding $M \hookrightarrow \mathrm{Aut}(S)$, where $\mathrm{Aut}(S)$ is the set of bijections $S \to S$. $\square$

Unfortunately, these are not necessary conditions. For instance, take the free group on two generators $F(\{a,b\})$, and consider the submonoid M generated by the words a and ab. Then M is not epimorphic since $ax = ab$ has no solution for $x \in M$.

The reader is encouraged to spend some time (but not too much time!) exploring different conditions that make a monoid embeddable. There are some good and bad news in this direction. The bad news come first.

Theorem 1.5 (Malcev)**.** *There is no finite list of first-order axioms (in the language of monoids) that axiomatize monoids that embed into groups.*

There are two good news. First, Malcev proved the existence of a recursively enumerable list of necessary and sufficient axioms, that is, there are infinitely many axioms and an algorithm for enumerating all of them.

Secondly, these axioms have a geometric interpretation that clarify the situation immensely.

2 Substructures and Horn axioms

2.1 Model-theoretic background

Definition 2.1. Terms, structures, substructures, homomorphisms, universal sentences, etc.

Obviously for any language $\mathcal{L}$ the collection of $\mathcal{L}$-structures together with structure homomorphisms form a category, which we denote as $\mathcal{C}_{\mathcal{L}}$. If $\mathcal{T}$ is an $\mathcal{L}$-theory then $\mathcal{C}_{\mathcal{T}}$ denotes the full subcategory of $\mathcal{C}_{\mathcal{L}}$ whose objects are models of $\mathcal{T}$.

Examples 2.2. Algebraic theories (theory of groups, theory of monoids). The elementary theory of an abstract category (ETAC, see Maclane CTFTWM chapter 2).

2.2 Theories of substructures

Characterizing monoids that embed into groups is equivalent to axiomatizing the class of all submonoids of groups. More generally, if $\mathcal{T}$ is a theory, we would like a theory $\mathcal{T}'$ whose models are precisely the substructures of $\mathcal{T}$-models. Fortunately, it is straightforward to find such a theory. Define

$$\mathcal{T}_\forall := \{\varphi \mid \varphi \text{ is a universal sentence and } \mathcal{T} \vdash \varphi\}.$$

Proposition 2.3. *Let $\mathcal{T}$ be a theory. Then $\mathcal{T}_\forall$ axiomatizes substructures of models of $\mathcal{T}$.*

Proof. Using the method of diagrams (see Model Theory notes Lemma 1.1.13.) $\square$

This proposition is not satisfactory, since $\mathcal{T}_\forall$ has no simple description a priori. We improve our results in the following way. Recall that a literal is formula which is either atomic or the negation of an atomic formula. For formulae of the form $p := \forall \bar{x}.\, (\psi_1 \vee \cdots \vee \psi_n)$ where ψ_i if a literal for all i, define

$$p^{(i)} := \forall \bar{x}.\, (\psi_1 \vee \cdots \vee \psi_{i-1} \vee \psi_{i+1} \vee \cdots \vee \psi_n).$$

Lemma 2.4. *content...*

Proof. Sentences in $\mathcal{T}$ are of the form $\forall \bar{x}.\varphi(\bar{x})$. By using the conjunctive normal form, can assume $\varphi(\bar{x})$ is the formula $\bigwedge_i \bigvee_j \psi_{i,j}(\bar{x})$, where the $\psi_{i,j}$ are literals (either atomic formulae or negations of them). As the universal quantifier distributes over conjunctions, we see that $\forall \bar{x}.\varphi(\bar{x})$ is equivalent to a conjunction of formulae of the form $\forall \bar{x}. \bigvee_j \psi_j(\bar{x})$ and thus we can assume $\mathcal{T}$ only contains formulae of this form.

If $p = \forall \bar{x}.\, (\psi_1 \vee \cdots \vee \psi_n)$ is in $\mathcal{T}$ define

$$p^{(i)} := \forall \bar{x}.\, (\psi_1 \vee \cdots \vee \psi_{i-1} \vee \psi_{i+1} \vee \cdots \vee \psi_n)$$

$\square$

However, the situation is improved in the specific case of groups and groupoids, since we can restrict the universal sentences to only *Horn axioms*.

Definition 2.5 (Universal Horn axioms)**.** content...

Theorem 2.6. *Let $\mathcal{T}$ be a universal theory such that $\mathcal{C}_{\mathcal{T}}$ has finite products.*[1] *Then $\mathcal{T}$ is equivalent to a theory which only contains universal Horn axioms.*

[1]The hypothesis in Cohn's book I think are slightly different: they only require that $C_{\mathcal{T}}$ is closed under direct product of (finitely many) structures. I'm not sure if my claim is still true.