Model Theory and Non-Classical Logic

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This is a set of lecture notes taken by me from the Part III course "Model Theory and Non-Classical Logic", lectured by Dr J. Siqueira in Michaelmas, 2023. I take full responsibility for any mistakes in these notes. Chapter 0 is my summary/expansion of [**NOLAST**]

Chapter 0

Logic Background (INCOMPLETE)

Definition 0.0.1 (Signature)**.** A *signature* Σ is a triplet (Ω, Π, α), where Ω and Π are disjoint sets and $\alpha\colon \Omega \cup \Pi \to \mathbb{N}$. We call the elements of Ω *function symbols*, those of Π we call *predicate symbols*, and if $s \in \Omega \cup \Pi$ we call $\alpha(s)$ the *arity* of s.

For the rest of this chapter, assume $\Sigma = (\Omega, \Pi, \alpha)$ is an arbitrary signature and that we are given a countable set $X = \{x_1, x_2 \ldots\}$, which we call the set of *variables*. This set does not contain any symbols in our signature (nor in the set of strings on our signature, see below).

0.1 Terms, formulae, and structures

Definition 0.1.1 (Terms)**.** The set of Σ-*terms* is a subset of the set of strings on $\Omega \cup X$, defined inductively as follows.

(a) If $x \in X$ then x is a term

(b) If $t_1, \ldots t_n$ are terms, and $\omega \in \Omega$ with $\alpha(\omega) = n$ then $\omega(t_1, \ldots, t_n)$ is a term.

(c) That is all.

Remark 0.1.2. Now assume that $X \cup \Omega \cup \Pi$ do not contain the symbols '$=$', '(','')', '$\perp$', '$\forall$' nor '$\Rightarrow$' (nor commas).

Definition 0.1.3 (Atomic formulae)**.** Let T be the set of Σ-terms. We define the *atomic formulae* of Σ as certain strings on $T \cup \Pi \cup \{(,), =, ,\}$ (note that the last comma is not a typo) according to the following rules.

(a) If s and t are terms then $(s = t)$ is an atomic formula.

(b) If $\phi \in \Pi$, $\alpha(\phi) = n$ and $t_1, \ldots, t_n$ are terms then $\phi(t_1, \ldots, t_n)$ is an atomic formula.

(c) That is all.

Definition 0.1.4 (Pre-formulae)**.** Let T be the set of terms of Σ. We inductively define the set of Σ-*pre-formulae* as a subset of the set of strings on $T \cup \Pi \cup \{=, \perp, \forall, \Rightarrow, (,)\}$ satisfying the following.

(a) Atomic formulae are pre-formulae

(b) $\perp$ is a pre-formula.

(c) If p and q are pre-formulae then so is $(p \Rightarrow q)$.

(d) If p is a pre-formula and $x \in X$ is a variable then $(\forall x)p$ is a pre-formula.

(e) That's all.

Now we can define a function PFV (for pre-free variables) on the set of terms union with the set of pre-formulae by the following rules

$$\begin{aligned}
&\mathrm{PFV}(x) = \{x\} \\
&\mathrm{PFV}(\omega t_1 \cdots t_n) = \bigcup_{i=1}^{n} \mathrm{PFV}(t_i) \\
&\mathrm{PFV}(s = t) = \mathrm{PFV}(s) \cup \mathrm{PFV}(t) \\
&\mathrm{PFV}(\phi(t_1, \ldots, t_n)) = \bigcup_{i=1}^{n} \mathrm{PFV}(t_i) \\
&\mathrm{PFV}(\perp) = \emptyset \\
&\mathrm{PFV}(p \Rightarrow q) = \mathrm{PFV}(p) \cup \mathrm{PFV}(q) \\
&\mathrm{PFV}((\forall x)p) = \mathrm{PFV}(p) \setminus \{x\}
\end{aligned}$$

Finally, we can define Σ-*formulae* to be all pre-formulae of Σ except those of the form $(\forall x)p$ where $x \notin \mathrm{PFV}(p)$. Define FV to be the restriction of PFV so that it only applies to terms and formulae.

By the *language* $\mathcal{L}$ of a signature Σ we mean the set of all terms and formulae of Σ. Instead of saying Σ-terms and Σ-formulae we say $\mathcal{L}$-terms and $\mathcal{L}$-formulae to mean the same thing.

Definition 0.1.5 (Language structures)**.** An $\mathcal{L}$-*structure* is a set A together with functions $\omega_A \colon A^{\alpha(w)} \to A$ for each $\omega \in \Omega$ and relations $\phi_A \subseteq A^{\alpha(\phi)}$ for each $\phi \in \Pi$. We use the convention that S^0 is a singleton set (say $\{0\}$) for all sets S.

0.2 Derived symbols

Definition 0.2.1 (Derived operations)**.** Let A be an $\mathcal{L}$-structure and t a term. In addition, suppose n is an integer with $\mathrm{FV}(t) \subseteq \{x_1, \ldots, x_n\}$. We define, $t_A(n)$ to be a function $A^n \to A$ as follows.

(a) If $t \in X$ then $t = x_i$ for some $i \le n$. Let $t_A \colon A^n \to A$ be the i-th projection function.

(b) Suppose $t = \omega t_1 \ldots t_m$ where $\omega \in \Omega$ with $\alpha(\omega) = m$, and the t_i's are terms for which we have defined $(t_i)_A(n)$. Then t_A is the composite

$$A^n \xrightarrow{((t_1)_A(n),(t_2)_A(n),\ldots,(t_m)_A(n))} A^m \xrightarrow{\omega_A} A$$

Note that in the empty structure all derived operations are the empty function (there can't be any constant symbols).

Lemma 0.2.2 (Variable redundancy in terms)**.** *Let A be a structure and t a term with $\mathrm{FV}(t) \subseteq \{x_1, \ldots, x_n\}$ for some n. Suppose we have two sequences of elements of A^n, say a and b, such that a and b agree on free variables, i.e. $a_k = b_k$ whenever $x_k \in \mathrm{FV}(t)$. Then $t_A(n)(a) = t_A(n)(b)$.*

Proof. Induction on t. Suppose $t = x_i \in X$. Then $t_A(n)(a)$ and $t_A(n)(b)$ are a_i and b_i respectively. We have assumed these are the same, as $x_i \in \mathrm{FV}(t)$. Thus $t_A(n)(a) = t_A(n)(b)$.

Now suppose $t = \omega t_1 \cdots t_m$ where $\omega \in \Omega$ with $\alpha(\omega) = m$, and the t_i's are terms. By inductive hypothesis, we may assume that for all $1 \le i \le n$ we have

$$(t_i)_A(n)(a) = (t_i)_A(n)(b).$$

It follows that

$$((t_1)_A(n), \ldots, (t_m)_A(m))(a) = ((t_1)_A(n), \ldots, (t_m)_A(n))(b).$$

Applying ω_A to both sides gives the result. □

Definition 0.2.3. (Derived formulae) Let A be an $\mathcal{L}$-structure and p a formula. In addition, suppose n is an integer with $\mathrm{FV}(p) \subseteq \{x_1, \ldots, x_n\}$. We define $p_A(n)$ to be subset of A^n, or equivalently a function $A^n \to 2$, as follows.

(a) If p is the formula $(s = t)$ for terms s and t then

$$p_A(n) \coloneqq \{\, a \in A^n \mid s_A(n)(a) = t_A(n)(a) \,\}$$

(b) Suppose $p = \phi(t_1, \ldots, t_m)$ for some $\phi \in \Pi$, with $\alpha(\phi) = m$, and terms $t_1, \ldots, t_m$. Then (the characteristic function of) $p_A(n)$ is defined by

$$A^n \xrightarrow{((t_1)_A(n),(t_2)_A(n),\ldots,(t_m)_A(n))} A^m \xrightarrow{\phi_A} 2$$

(c) If $p = \perp$ then $p_A(n)$ is the empty set (i.e. its characteristic function is constant with value zero).

(d) Suppose p is $(q \Rightarrow r)$ for formulas q and r, where $q_A(n)$ and $r_A(n)$ have already been defined. Then we define p_A by the composition

$$A^n \xrightarrow{(q_A(n), r_A(n))} 2 \times 2 \xrightarrow{\Rightarrow_2} 2$$

(e) Suppose $p = (\forall x_m)q$ for some formula q with $x_m \in \mathrm{FV}(q)$. Define $N := \max(n, m)$. We can assume that $q_A(N)$ is defined. Let $a \in A^n$. We say that $a \in p_A(n)$ if and only if for all $a' \in A^N$ so that a' agrees with a in the first n terms—except possibly on the m-th term—we have that $a' \in q_A(N)$.

Lemma 0.2.4 (Variable redundancy in formulae)**.** *Let A be a structure and p a formula with $\mathrm{FV}(p) \subseteq \{x_1, \ldots, x_n\}$ for some n. Suppose we have two elements of A^n, say a and b such that a and b agree on free variables, i.e. $a_k = b_k$ whenever $x_k \in \mathrm{FV}(t)$. Then $a \in p_A(n)$ if and only if $b \in p_A(n)$.*

Proof. By induction on p. If p is the formula $(s = t)$ for terms s and t we have

$$\begin{aligned} a \in p_A(n) &\iff s_A(n)(a) = t_A(n)(a) \\ &\iff s_A(n)(b) = t_A(n)(b) \\ &\iff b \in p_A(n), \end{aligned}$$

where we have used Lemma 0.2.2.

Now suppose $p = \phi(t_1, \ldots, t_m)$ for some $\phi \in \Pi$ with $\alpha(\phi) = m$, and terms $t_1, \ldots, t_m$. Again by Lemma 0.2.2 we have

$$((t_1)_A(n), \ldots, (t_m)_A(m))(a) = ((t_1)_A(n), \ldots, (t_m)_A(n))(b).$$

Applying ϕ_A to both sides gives the result. This finishes the induction in the case that p is an atomic formula.

If $p = \perp$ then $a, b \notin p_A(n)$ so the claim holds. Now let q, r be formulae with p being $(q \Rightarrow r)$. By inductive hypothesis

$$(q_A(n), r_A(n))(a) = (q_A(n), r_A(n))(b),$$

and applying $\Rightarrow_2$ to both sides gives the result.

Finally, we consider the case where $p = (\forall x_m)q$ for some formula q and some variable $x_m \in \mathrm{FV}(q)$. Write $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$. By symmetry, we only need to show that $a \in p_A(n)$ implies $b \in p_A(n)$. So, suppose $a \in p_A$.

Let $N = \max(n, m)$ and let $b' \in A^N$ be a sequence agreeing with b in the first value except possibly on the m-th value, and call this value c_m. If we show that $b' \in q_A(N)$ then we are done.

Define $a' \in A^N$ to be the sequence b' but with its first n values replaced by a *except* the m-th value, which remains c_m. As $a \in p_A(n)$ it is clear by definition that $a' \in q_A(N)$. But a' and b' agree on free variables of q, which are $\mathrm{FV}(p) \cup \{x_m\}$: the first n variables are just a and b, which agree on $\mathrm{FV}(p)$, only that we have specified that they agree on the m-th value c_m, and otherwise a' and b' are identical. Hence, by the inductive hypothesis, $b' \in q_A(N)$. $\square$

Variable redundancy implies that free variables are the only thing that affects the values of $t_A(n)$ and $p_A(n)$. Hence we will write t_A and p_A, without specifying n, we take n to be the minimum so that $\mathrm{FV}(t)$ (resp. $\mathrm{FV}(p)$) is a subset of $\{x_1, \dots, x_n\}$. And in any case, this only function only requires inputs a_k where x_k is a free variable of t (resp. p).

0.3 First-order theories

Definition 0.3.1 (Satisfying a formula)**.** Let A be an $\mathcal{L}$-structure, and let p be a formula. We say that p is *satisfied* in A if the indicator function of p_A is constant with value 1. In this case we write $A \models p$.

Definition 0.3.2 (Sentences and universal closure)**.** Let p be a formula. We say that p is a *sentence* if it has no free variables. (In this case p_A is a constant function since $n = 0$). In any case, we can obtain a sentence $\bar{p}$, called the *universal closure* of p, by prefixing p with universal quantifiers for each of the free variables of p (say, in decreasing order of subscripts).

We remark that if A is empty and p is not a sentence then the indicator $p_\emptyset \colon \emptyset^n \to 2$ is constant with value 1 since it sends all elements of $\emptyset^{\mathbb{N}}$ (which there are none) to 1 (this is a vacuous truth).

Proposition 0.3.3. *For all $\mathcal{L}$-structures A and formulas p,*

$$A \models p \quad \text{if and only if} \quad A \models \bar{p}.$$

Proof. We prove a weaker statement first. Let q be a formula with a free variable x_m. Then we claim that $A \models q$ if and only if $A \models (\forall x_m)q$. Indeed, $A \models \forall x_n q$ if and only if for all $a \in A^n$ (where n is the minimum so that $\mathrm{FV}((\forall x_m)q) \subseteq \{x_1, \dots, x_n\}$) we have $a \in ((\forall x_m)q)_A(n)$. And this happens iff for all $a \in A^n$ and for all $a' \in A^N$ (where $N \coloneqq \max(n, m)$) that agrees with a on the first n values except possibly in the m-th value we have $a' \in q_A(N)$.

This is clearly equivalent to saying that for all $a' \in A^N$ we have $a \in q_A(N)$, i.e. $A \models q_A$.

Now if p is a formula with k free variables we can use induction on k, together with the above result, to deduce the claim about $\bar{p}$. $\square$

Definition 0.3.4 (First-order theory)**.** Let T be a set of $\mathcal{L}$-formulae and A a structure. We write $A \models T$ if $A \models p$ for all $p \in T$.

In the special case where T is a set of sentences we call it a *first-order theory*, and its formulae are called *axioms*. If $A \models T$ in this case we would say that A *models* the theory T.

0.4 Semantics and syntax

Recall that we had an arbitrary signature Σ that generated a language $\mathcal{L}$. We would like to add things to the signature (which will generate a difference language) from time to time. Here I will give some notation for a typical situation. Let S be a set. we denote by Σ_S the signature Σ but with $|S|$ new constant symbols (i.e. function symbols of arity zero) added. Similarly, we denote by $\mathcal{L}_S$ the language generated by Σ_S. Note that in the special case that S is an $\mathcal{L}$-structure we clearly have that S is an $\mathcal{L}_S$-structure: just interpret the new constant symbols as the elements of S.

Definition 0.4.1 (Semantic entailment)**.** Let T be a theory and let p be a sentence. We say that T *semantically entails* p, written as $T \models p$, to mean that every model of T satisfies p.

In the case where T and p are not sentences simply consider the language

$$\mathcal{L}' \coloneqq \mathcal{L}_{\mathrm{FV}(T)\cup\mathrm{FV}(p)},$$

where $\mathrm{FV}(T)$ is just the set of all free variables appearing in a formulae of T. Let T' and p' be the same formulae but with free variables replaced by the corresponding constants in $\mathcal{L}'$. Then $T' \cup \{p'\}$ is just a set of sentences in $\mathcal{L}'$, so declare that $T \models p$ in $\mathcal{L}$ if and only if $T' \models p'$ in $\mathcal{L}'$.

This seems roundabout: why not define semantic entailment $T \models p$ as "for all $A \models T$ we have $A \models p$"? This certainly makes sense when $T \cup \{p\}$ is not a set of sentences. The problem is that if we adopted this alternate definition we would have undesired consequences with the empty structure. For example, if $T = \{\neg(x_1 = x_1)\}$ and $p = \{\bot\}$ then the only model for T is the empty structure but $\emptyset \not\models p$. However, since T is clearly a contradictory statement we would like to have $T \models p$ in this case, which is guaranteed by the real definition since the addition of constants invalidate the empty structure.

Now we turn to our system of deduction. If w is a formula, t is a term and x a variable, we define $w[t/x]$ to be the formula obtained from w on replacing each free occurrence of x by t, *provided* no free variable of t occurs bound in w.

More formally, we define

$$y[t/x] = \begin{cases} y & \text{if } x \neq y \\ t & \text{if } x = y \end{cases}$$

$$(\omega t_1 \dots t_n)[t/x] = \omega(t_1[t/x]) \cdots (t_n[t/x])$$

$$(s = s')[t/x] = (s[t/x] = s'[t/x])$$

$$\phi(t_1, \dots, t_n)[t/x] = \phi((t_1[t/x]), \dots, (t_n[t/x]))$$

$$\perp[t/x] = \perp$$

$$(p \Rightarrow q)[t/x] = (p[t/x] \Rightarrow q[t/x])$$

$$((\forall y)p)[t/x] = \begin{cases} (\forall y)(p[t/x]) & \text{if } x \neq y \\ (\forall y)p & \text{if } x = y. \end{cases}$$

Lemma 0.4.2. *Let w be a term or a formula, let x_m be a variable and let t be a term such that all free variables in t do not appear bound in w. Suppose A is an $\mathcal{L}$-structure and $a \in A^n$ where n is the minimum nonnegative integer so that $\mathrm{FV}(w) \subseteq \{x_1, \dots, x_n\}$. Denote by a' the sequence a but with the m-th value replaced by $t_A(a)$. Then we have*

$$(w[t/x_n])_A(a) = w_A(a').$$

If $A = \emptyset$ then $(w[t/x_n])_A = w_A$

Proof. Suppose $A = \emptyset$. If w does not have x_n as a free variable then $w[t/x_n] = w$. It follows that $(w[t/x_n])_A = w_A$.

Now assume that x_n is a free variable of w. Clearly w is not a sentence. We claim that neither is $w[t/x_n]$. This is easily seen from the fact that t is not a constant (since otherwise the empty set could not be an $\mathcal{L}$-structure) and thus has free variables and *in addition* we assumed that no free variables of t are being bound in w. Thus, as neither w nor $w[t/x_n]$ are sentences, they are indicators $\emptyset^{\mathbb{N}} \to 2$ and thus equal. This proves the claim for the empty structure, so from now on assume $A \neq \emptyset$.

First suppose that w is a term. We use induction, so assume $w = x_m$ for some m. If $m \neq n$ then $w[t/x_n] = w$ and so we only need to show that $w_A(a) = w_A(a')$. This is immediate by variable redundancy: a and a' agree on the free variable x_m. Now suppose $m = n$. Then $w[t/x_n] = t$ and we need to show that $t_A(a) = (x_n)_A(a')$. Again, this is obvious: the right-hand side of the equation is the n-th value of a', which we assumed is $t_A(a)$. This closes the base case.

Now, suppose $w = \omega t_1 t_2 \cdots t_m$ for some $\omega \in \Omega$ with $\alpha(\omega) = m$ and where the t_i's are terms. Clearly

$$w[t/x_n] = \omega(t_1[t/x_n])(t_2[t/x_n]) \cdots (t_m[t/x_n]).$$

It follows that

$$\begin{aligned}(w[t/x_n])_A(a) &= \omega_A((t_1[t/x_n])_A(a), (t_2[t/x_n])_A(a), \dots, (t_m[t/x_n])_A(a)) \\ &= \omega_A((t_1)_A(a'), (t_2)_A(a'), \dots, (t_m)_A(a')) \\ &= w_A(a'),\end{aligned}$$

where we have used the inductive hypothesis. This closes the induction and proves the statement when w is a term.

Suppose now that w is a formula. We again use induction. If w is the formula $(s = s')$ for terms s and s' we have that

$$w[t/x_n] = (s[t/x_n] = s'[t/x_n]).$$

Then,

$$\begin{aligned}a \in (s[t/x_n] = s'[t/x_n])_A &\iff (s[t/x_n])_A(a) = (s'[t/x_n])_A(a) \\ &\iff s_A(a') = t_A(a') \\ &\iff a' \in w_A,\end{aligned}$$

where we used the result for terms. Now suppose w is $\phi(t_1, \dots, t_m)$ for some $\phi \in \Pi$ with $\alpha(\phi) = m$, and terms $t_1, \dots, t_m$. Again, using the claim for terms, we have

$$((t_1[t/x_n])_A(a), \dots, (t_m[t/x_n])_A(a)) = ((t_1)_A(a'), \dots, (t_m)_A(a'))$$

and applying ϕ_A to both sides gives the result. This closes the base case, i.e. the case where w is an atomic formula.

Clearly $(\perp[t/x_n])_A(a) = \perp_A(a) = 0 = \perp_A(a')$. Now, if w is $(p \Rightarrow q)$ then, by the inductive hypothesis

$$((p[t/x_n])_A(a), (q[t/x_n])_A(a)) = (p_A(a'), q_A(a')).$$

Applying $\Rightarrow_2$ to both sides gives the result.

Finally, suppose $w = (\forall x_m)p$. Then we have two cases. If $m = n$ then $w[t/x_n] = w$ and so we need to show that $w_A(a) = w_A(a')$. But in this case clearly x_n is not a free variable of w, so a and a' agree on free variables, and the claim follows by variable redundancy.

Now assume $m \neq n$. Then $w[t/x_n] = (\forall x_m)(p[t/x_n])$. First, note that that

$$a = (a_1, a_2, \dots) \in ((\forall x_m)(p[t/x_n]))_A$$

if and only if

$$(a_1, \dots, a_{m-1}, c, a_{m+1}, \dots) \in (p[t/x_n])_A \text{ for all } c \in A.$$

For $c \in A$ let $\alpha(c)$ be the sequence above, i.e. a but the m-th value replaced by c. Similarly, let $\alpha'(c)$ be the sequence a' but replacing the m-th value with

c. Finally, let $\alpha^*(c)$ be the sequence $\alpha(c)$ but with the n-th value replaced by $t_A(\alpha(c))$ Then we can reformulate our statement as so:

$$\alpha(c) \in (p[t/x_n])_A \text{ for all } c \in A.$$

By the inductive hypothesis, this new statement is equivalent to

$$\alpha^*(c) \in p_A \text{ for all } c \in A.$$

Note that, for all c, we have that $\alpha^*(c)$ and $\alpha'(c)$ agree on all values (including the m-th) except possibly on the n-th value, where we have $t_A(\alpha(c))$ and $t_A(a)$ fro $\alpha^*(c)$ and $\alpha'(c)$ respectively. We claim that in fact *they do* agree on the n-th value, i.e. $t_A(\alpha(c)) = t_A(a)$.

Indeed, by definition, a and $\alpha(c)$ agree on all values except possibly on the m-th value. However, we assumed (and this is the first and only time we use the assumption when $A \neq \emptyset$) that the free variables of t do not appear bound in w. Clearly x_m is bound in w (recall that we insist that variables that are being bound appear in the formula). Thus x_m cannot be a free variable of t, which implies that a and $\alpha(c)$ agree on free variables; hence $t_A(\alpha(c)) = t_A(a)$ by variable redundancy. Thus $\alpha^*(c)$ is the same sequence as $\alpha'(c)$. Therefore we can, once again, reformulate our statement:

$$\alpha'(c) \in p_A \text{ for all } c \in A.$$

This is manifestly equivalent to $a' \in ((\forall x_m)p)_A$, as desired. □

We now postulate our axioms to be substitution instances of these propositions.

(a) $(p \Rightarrow (q \Rightarrow p))$

(b) $((p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r)))$

(c) $(\neg\neg p \Rightarrow p)$
(here p, q, r may be any formulae of $\mathcal{L}$)

(d) $((\forall x)p \Rightarrow p[t/x])$
(here p is any formula with $x \in \mathrm{FV}(p)$, t any term whose free variables don't occur bound in p)

(e) $((\forall x)(p \Rightarrow q) \Rightarrow (p \Rightarrow (\forall x)q))$
(p, q formulae, $x \notin \mathrm{FV}(p)$)

(f) $(\forall x)(x = x)$

(g) $(\forall x, y)((x = y) \Rightarrow (p \Rightarrow p[y/x]))$
(p any formula with $x \in \mathrm{FV}(p)$, y not bound in p and distinct from x)

Proposition 0.4.3. *All the axioms above are tautologies.*

Proof. Let p, q, r be formulae in $\mathcal{L}$ and let A be an $\mathcal{L}$-structure.

(a) First suppose that $A = \emptyset$. Then, if there are free variables in p or q then it is clear that $\emptyset \models (p \Rightarrow (q \Rightarrow p))$. Otherwise, p and q are sentences and so they have a truth value. Case-by-case analysis reveals that $\emptyset \models (p \Rightarrow (q \Rightarrow p))$. Now assume that A is nonempty.

Note that for all $a \in A^{\mathbb{N}}$ we have

$$(p \Rightarrow (q \Rightarrow p))_A(a) = (\Rightarrow_2)(p_A(a), (\Rightarrow_2)(p_A(a), q_A(a)))$$

as elements of $2 = \{0, 1\}$. Plugging in the possible values for $p_A(a)$ and $q_A(a)$ we conclude that in all cases $(p \Rightarrow (q \Rightarrow p))_A(a) = 1$.

(b) Similar to (a).

(c) Similar to (a).

(d) Suppose $x \in \mathrm{FV}(p)$ and t is any term whose free variables don't occur bound in p. It is easy to see that the axiom is never a sentence, so $\emptyset$ models it. Assume now that $A \neq \emptyset$.

Let $a = (a_1, a_2, \ldots) \in A^{\mathbb{N}}$ and consider

$$(\Rightarrow_2)(((\forall x)p)_A(a), p[t/x]_A(a))$$

If $((\forall x)p)_A(a) = 0$ then the above equals 1, clearly. Now suppose $((\forall x)p)_A(a) = 1$ and let $x = x_n$ for some n. This means that, for all $a'_n \in A$ we have

$$(a_1, \ldots, a_{n-1}, a'_n, a_{n+1}, \ldots) \in p_A.$$

Set $a'_n \coloneqq t_A(a)$. By Lemma 0.4.2, the above implies that $p[t/x]_A(a) = 1$, as desired.

(e) Let $x \notin \mathrm{FV}(p)$. If the axiom is not a sentence then it has $\emptyset$ as a model. Suppose now that the axiom is a sentence; this is easily seen to imply that q has x as its only free variable. Clearly $(\forall x)(p \Rightarrow q)$ is satisfied in $\emptyset$. Note that $(\forall x)q$ is also a satisfied sentence in $\emptyset$. Therefore the whole axiom is seen to be satisfied in $\emptyset$. Now assume $A \neq \emptyset$.

Let $a = (a_1, a_2, \ldots) \in A^{\mathbb{N}}$. If $((\forall x)(p \Rightarrow q))_A(a) = 0$ then the formula is true for a. So, assume that $a \in ((\forall x)(p \Rightarrow q))_A$. Let $x = x_n$ for some n. We have that, for all $a'_n \in A$:

$$(a_1, \ldots, a_{n-1}, a'_n, a_{n+1}, \ldots) \in (p \Rightarrow q)_A.$$

In other words, for all $a'_n \in A$:

$$(\Rightarrow_2)(p_A, q_A)(a_1, \ldots, a_{n-1}, a'_n, a_{n+1}, \ldots) = 1$$

But, as $x_n \notin \mathrm{FV}(p)$, the value $p_A(a_1, \ldots, a_{n-1}, a'_n, a_{n+1}, \ldots)$ does not depend on a'_n by variable redundancy. Thus we conclude that for all $a'_n \in A$.

$$(\Rightarrow_2)(p_A(a), q_A((a_1, \ldots, a_{n-1}, a'_n, a_{n+1}, \ldots))) = 1.$$

From this, it is easy to deduce that $a \in (p \Rightarrow (\forall x)q)_A$, as desired.

(f) The empty set is easily seen to model this axiom. Let $a = (a_1, a_2, \ldots) \in A^{\mathbb{N}}$ and let $x = x_n$l Then $a \in ((\forall x)(x = x))_A$ iff for all $a'_n \in A$ we have

$$(a_1, \ldots, a_{n-1}, a'_n, a_{n+1}, \ldots) \in (x = x)_A.$$

This happens iff for all a'_n we have

$$x_A(a_1, \ldots, a_{n-1}, a'_n, a_{n+1}, \ldots) = x_A(a_1, \ldots, a_{n-1}, a'_n, a_{n+1}, \ldots),$$

which is manifestly true.

(g) The empty set is easily seen to model this axiom. Let $x = x_n \in \mathrm{FV}(p)$ and $y = x_m$ be not bound in p with $n \neq m$. Have some $a = (a_1, a_2, \ldots) \in A^{\mathbb{N}}$. We need to show that

$$a \in ((\forall x_n)(\forall x_m)((x_n = x_m) \Rightarrow (p \Rightarrow p[x_m/x_n])))_A$$

For $c_n, c_m \in A$ define $\alpha(c_n, c_m)$ to be the sequence a but with the i-th value replaced by c_i for $i \in \{n, m\}$ (recall that $n \neq m$). Then the above proposition is equivalent to

$$\alpha(c_n, c_m) \in ((x_n = x_m) \Rightarrow (p \Rightarrow p[x_m/x_n]))_A \text{ for all } c_n, c_m \in A$$

We need to prove the above. To that end, let $c_n, c_m \in A$ be arbitrary. If $\alpha(c_n, c_m) \notin (x_n = x_m)_A$ then we do have the inclusion above. So, assume $\alpha(c_n, c_m) \in (x_n = x_m)_A$; this clearly implies that $c \coloneqq c_n = c_m$. Now, we need to show that

$$\alpha(c, c) \in (p \Rightarrow p[x_m/x_n])_A.$$

If $\alpha(c, c) \notin p_A$ then the above is true. Therefore we can suppose $\alpha(c, c) \in p_A$. We want to prove that $\alpha(c, c) \in (p[x_m/x_n])_A$. As x_m is not bound in p we can apply Lemma 0.4.2 which tells us that it suffices to show that $\alpha(c, c)' \in p_A$, where $\alpha(c, c)'$ denotes the sequence $\alpha(c, c)$ but replacing the n-th value by $(x_m)_A(\alpha(c, c)) = c$. Clearly $\alpha(c, c)' = \alpha(c, c)$ and we supposed at the start that $\alpha(c, c) \in A$. Thus we are done.

□

To our deductive system we add the following rules of inference.

(MP) From p and $(p \Rightarrow q)$, we may infer q, *provided* either q has a free variable or p is a sentence.

(Gen) From p we may infer $(\forall x)p$, *provided* x does not occur free in any premiss which has been used in the proof of p (but is a free variable of p).

Formally, we define our concept of deduction as follows.

Definition 0.4.4 (Deduction sequence)**.** Let S be a set of formulae. A *deduction sequence* on S is a finite sequence on the set of formulae of $\mathcal{L}$, defined inductively below.

(i) The empty sequence is a deduction sequence.

(ii) If $(p_1, \ldots, p_n)$ is a deduction sequence and p is an axiom or an element of S, then $(p_1, \ldots, p_n, p)$ is a deduction sequence.

(iii) Let $(p_1, \ldots, p_n)$ be a deduction sequence. Suppose there are $1 \leq i, j \leq n$ so that p_j is the formula $(p_i \Rightarrow p)$ for some p, and, in addition either p_i is a sentence or p has a free variable. Then $(p_1, \ldots, p_n, p)$ is a deduction sequence.

(iv) Suppose $(p_1, \ldots, p_n)$ is a deduction sequence so that p_n has a free variable x but x is not a free variable of p_i for $i < n$, and $p_n \notin S$. Then, if $p = (\forall x) p_n$, we have that $(p_1, \ldots, p_n, p)$ is a deduction sequence.

(v) That is all.

Definition 0.4.5 (Syntactic entailment)**.** Let S be a set of formulae and p a formula. We say that S *syntactically entails* p, written as $S \vdash p$, if there is a deduction sequence terminating at p.

0.5 Properties of first-order languages

Again, we fix a language $\mathcal{L}$ with a set of variables $X = \{x_1, x_2, \ldots\}$

0.6 Completeness

The aim of this section is to prove the Completeness theorem. Before that, we need to prove the Soundness theorem. First, a couple of lemmata.

Lemma 0.6.1. *Let S be a set of formulae and let p and q be formulae so that either q has a free variable or p is a sentence. If $S \models p$ and $S \models (p \Rightarrow q)$, then $S \models q$.*

Proof. Let A be an $\mathcal{L}$-structure. We want to show that one of the two following statements holds:

(I) A is empty and there is a non-sentence in $S \cup \{q\}$.

(II) $\bigcap_{r \in S} r_A \subseteq q_A$.

We know that one of these two statements holds:

(a) A is empty and there is a non-sentence in $S \cup \{p\}$.

(b) $\bigcap_{r \in S} r_A \subseteq p_A$.

Suppose (a) holds. If there is a non-sentence in S then there is a non-sentence in $S \cup \{q\}$ and A is empty, i.e. (I) holds. Otherwise, if p is a non-sentence, then, by the premiss of the statement q is a non-sentence and again (I) holds. So, from now on, assume (b) holds

Similarly, we also know that one of the two statements below holds:

(a') A is empty and there is a non-sentence in $S \cup \{p, q\}$.

(b') $\bigcap_{r \in S} r_A \subseteq (p \Rightarrow q)_A$.

Suppose (a') holds. If q is a non-sentence then (I) holds, and if instead the non-sentence is in $S \cup \{p\}$ we have reduced to case (a). So, we can assume (b') holds. But (b) and (b') are easily seen to imply (II), even when A is empty. $\square$

Lemma 0.6.2. *Let S be a set of formulae, p a formula, and x a variable so that x does not occur free in any formulae of S. If $S \models p$ then $S \models (\forall x)p$.*

Proof. Let A be an $\mathcal{L}$-structure. As before, we want to show that one of the two following statements holds:

(I) A is empty and there is a non-sentence in $S \cup \{(\forall x)p\}$.

(II) $\bigcap_{r \in S} r_A \subseteq ((\forall x)p)_A$.

The hypothesis is that one of the two following statements holds. (We know that p is a non-sentence already)

(a) A is empty

(b) $\bigcap_{r \in S} r_A \subseteq p_A$.

Suppose (a) is true. If there is a non-sentence in $S \cup \{(\forall x)p\}$ then (I) holds, so assume that S is a set of sentences, and that $(\forall x)p$ is a sentence. But then, as A is empty, we have that the indicator of $((\forall x)p)_A$ is constant with value 1, implying that (II) holds.

Now suppose (a) is not true. Then A is nonempty and (b) holds. Let $a = (a_1, a_2, \ldots) \in r_A \subseteq A^{\mathbb{N}}$ for all $r \in S$. We would like to show that $a \in ((\forall x)p)_A$, which, if $x = x_n$, is equivalent to the statement

$$(a_1, \ldots, a_{n-1}, a'_n, a_{n+1}, \ldots) \in p_A \text{ for all } a'_n \in A.$$

So, fix some $a'_n \in A$ and set $a' := (a_1, \ldots, a_{n-1}, a'_n, a_{n+1}, \ldots)$. By assumption x_n is not a free variable of r for all $r \in S$. By variable redundancy, we conclude that $a' \in r_A$ for all $r \in S$. Finally, (b) implies that $a' \in p_A$, as desired. $\square$

Proposition 0.6.3 (the Soundness Theorem)**.** *Let S be a set of formulae and p a formula. If $S \vdash p$ then $S \models p$.*

Proof. It is enough to show that, for all deduction sequences σ, all formulae of σ are semantically entailed by S. We use induction on the set of deduction sequences.

The claim is vacuously true when σ is the empty sequence. Suppose $\sigma = (p_1, \ldots, p_n, p)$, where $S \models p_i$ for all i, and p is an axiom or an element of S. It easily follows that $S \models p$ (recall axioms are tautologies).

Now suppose that $\sigma = (p_1, \ldots, p_n, p)$, where $S \models p_i$ for all i, and there are $1 \leq i, j, \leq n$ so that p_j is the formula $(p_i \Rightarrow p)$, and, in addition, either p_i is a sentence or p has a free variable. Then Lemma 0.6.1 says that $S \models p$.

Finally, suppose that $\sigma = (p_1, \ldots, p_n, p)$, where $S \models p_i$ for all i, and that $p_n \notin S$ has a free variable x but x is not a free variable of p_i for $i < n$. In addition, we suppose $p = (\forall x)p_n$. Let $S' = \{p_1, \ldots, p_{n-1}\}$. We claim that $S' \models p_n$ $\square$

Chapter 1

Model Theory

1.1 Substructures and diagrams

Definition 1.1.1 ($\mathcal{L}$-homomorphism)**.** Let M and N be $\mathcal{L}$-structures. An $\mathcal{L}$-*homomorphism* is a map $\eta\colon M \to N$ such that given $\bar{a} = (a_1, \dots, a_n) \in M^n$:

- for all function symbols f of arity n we have that

$$\eta(f^M(\bar{a})) = f^N(\eta^n(\bar{a})),$$

  in other words the diagram

$$\begin{tikzcd} M^n \arrow[r, "\eta^n"] \arrow[d, "f^M"'] & N^n \arrow[d, "f^N"] \\ M \arrow[r, "\eta"'] & N \end{tikzcd}$$

  commutes;

- for all relation symbols R of arity n we have that

$$\bar{a} \in R^M \text{ iff } \eta^n(\bar{a}) \in R^N.$$

An injective $\mathcal{L}$-homomorphism is an $\mathcal{L}$-*embedding* and an invertible one is an $\mathcal{L}$-*isomorphism*. If M and N are isomorphic we write $M \cong N$. If $M \subseteq N$ and the inclusion map is an $\mathcal{L}$-homomorphism we say that M is a *substructure* of N, and N is an *extension* of M.

We are going to stop writing $\bar{m} \in M^n$ where n is the length of $\bar{m}$ and just write $\bar{m} \in M$ when n can be inferred or its unimportant.

Examples 1.1.2.

(a) Let $\mathcal{L}$ be the language of groups. Then $(\mathbb{N}, +, 0)$ is a subset of the the integers $(\mathbb{Z}, +, 0)$, but it is not a substructure.

(b) If M is an $\mathcal{L}$-structure and $X \subseteq M$ then X is the domain of a substructure of M iff it is closed under the interpretation of all function symbols.

Indeed, the inclusion $\iota\colon X \to M$ clearly preserves relations. But if it is not closed under some function f^M then there is no way to interpret f^X.

(c) It follows from the previous point that the intersection of a family of substructures is a substructure: indeed, applying a function f^M to anything in the intersection will land on all substructures (since these are closed under function symbols) and thus in the intersection.

The substructure generated by $X \subseteq M$ is defined to be the intersection of all substructures of M containing X; it is denoted by $\langle X \rangle_M$. Again, by the previous point, $\langle X \rangle_M$ is also the intersection of all subsets of M that are closed under function symbols.

Hence
$$\langle X \rangle_M = X \cup \{t^M(\bar{m}) \mid t \text{ a term and } \bar{m} \in X\}.$$

Indeed, the RHS is obviously closed under function symbols and no strict subset of it could possibly be. Therefore $|\langle X \rangle_M| \leq |X| + |\mathcal{L}|$.

We say a structure M is *finitely generated* if $M = \langle X \rangle_M$ for some finite $X \subseteq M$.

What kind of sentence is preserved under substructures?

Proposition 1.1.3. *Let $\varphi(\bar{x})$ be a quantifier-free $\mathcal{L}$-formula with n variables, M be an $\mathcal{L}$-structure and $\bar{a} \in M$. For every extension N of M we have $M \models \varphi(\bar{a})$ iff $N \models \varphi(\bar{a})$.*

Proof. First we show that if $t(\bar{x})$ is a term with k free variables then $t^M(\bar{b}) = t^N(\bar{b})$ for all $\bar{b} \in M$.

This is clearly the case if $t = x_i$ is a variable since then $t^M(\bar{b}) = b_i = t^N(\bar{b})$. Now suppose $t = f(q_1, \ldots, q_l)$ for a function symbol f of arity l and the q_i's are terms. By the inductive hypothesis we can assume $q_i^M(\bar{b}) = q_i^N(\bar{b})$ for all i. Then,

$$t^M(\bar{b}) = f^M(q_1^M(\bar{b}), \ldots, q_l^M(\bar{b})) = f^N(q_1^N(\bar{b}), \ldots, q_l^N(\bar{b}))) = t^N(\bar{b})$$

where we have used the fact that M is a substructure of N.

Now onto the main result. Let t_1 and t_2 be terms with at most n free variables. Then
$$M \models (t_1(\bar{a}) = t_2(\bar{a}))$$
if and only if $t^M(\bar{a}) = t^M(\bar{a})$. But this happens iff $t^N(\bar{a}) = t^N(\bar{a})$, and this is equivalent to $N \models (t_1(\bar{a}) = t_2(\bar{a}))$.

Next, let $R(t_1, \ldots, t_l)$ be an l-ary relation, where all the t_i's have at most n free variables. We have that $M \models R(t_1(\bar{a}), \ldots, t_l(\bar{a}))$ iff $(t_1^M(\bar{a}), \ldots, t_l^M(\bar{a})) \in R^M$. As N is an extension (and by the result for terms) the latter happens iff $(t_1^N(\bar{a}), \ldots, t_l^N(\bar{a})) \in R^N$. And this is of course the same as $N \models R(t_1(\bar{a}), \ldots, t_l(\bar{a}))$.

This finishes the induction for atomic formulae. Now if φ is a formula satisfying the claim then clearly

$$M \models \neg\varphi(\bar{a}) \quad \text{iff} \quad M \not\models \varphi(\bar{a}) \quad \text{iff} \quad N \not\models \varphi(\bar{a}) \quad \text{iff} \quad N \models \varphi(\bar{a}).$$

If the proposition is true for φ and ψ then $M \models \varphi \vee \psi$ iff $M \models \varphi$ or $M \models \psi$, which happens iff $N \models \varphi$ or $N \models \psi$, which is clearly equivalent to $N \models \varphi \vee \psi$. □

A *universal formula* is one of the form $\forall \bar{x}\varphi(\bar{x}, \bar{y})$ where φ is quantifier free. A *universal theory* is one whose axioms are universal sentences.

Definition 1.1.4. Structures M and N are *elementary equivalent* if for every $\mathcal{L}$-sentence φ we have $M \models \varphi$ iff $N \models \varphi$.

A homomorphism $f\colon M \to N$ is an *elementary embedding* if it is injective and for all $\mathcal{L}$-formulae $\varphi(x_1, \dots, x_n)$ and elements $m_1, \dots, m_n \in M$ we have

$$M \models \varphi(m_1, \dots, m_n) \quad \text{iff} \quad N \models \varphi(f(m_1), \dots, f(m_n)).$$

We denote 'M and N are elementary equivalent' by $M \equiv N$.

Remark. If M and N are $\mathcal{L}$-structures and we have some tuples of the same size $\bar{m} \in M$ and $\bar{n} \in N$ then by $(M, \bar{m}) \equiv (N, \bar{n})$ we mean that the expanded structures are elementary equivalent as $\mathcal{L}_{\bar{c}}$-structures, where $\bar{c}$ has the same size as $\bar{m}$ and $\bar{n}$.

Proposition 1.1.5. *If $M \cong N$ then $M \equiv N$.*

Proof. Let $F\colon M \to N$ be an isomorphism. By symmetry, we only need to show that all formulaes modelled by M are also modelled by N.

First we would like to show that if t is a term then for all $\bar{m} \in M$ we have $t^M(\bar{m}) = t^N(F(\bar{m}))$.

We use induction over the structure of formulae.

If $t_1(\bar{x})$ and $t_2(\bar{x})$ are terms with the same free variables, then $M \models (t_1(\bar{x}) = t_2(\bar{x}))$ iff for all $\bar{m} \in M$ we have $t_1^M(\bar{m}) = t_2^M(\bar{m})$. But then for $\bar{n} \in N$ we have that

$$\begin{aligned} t_1^N(\bar{n}) &= t_1^N(FF^{-1}\bar{n}) \\ &= F(t_1^M(F^{-1}(\bar{n}))) \\ &= F(t_2^M(F^{-1}(\bar{n}))) \\ &= t_2^N(FF^{-1}\bar{n}) \\ &= t_2^N(\bar{n}). \end{aligned}$$

Now suppose that $R(x_1, \dots, x_l)$ is a relation so that $M \models R(x_1, \dots, x_l)$. Then $\bar{m} \in R^M$ for all $\bar{m} \in M$. But then if $\bar{n} \in N$ we have that $\bar{n} \in R^N$ iff $F^{-1}(n) \in R^M$ which is clearly true.

Let φ be a formulae for which this holds. Then $M \models \neg\varphi$ iff $M \not\models \varphi$ iff $N \not\models \varphi$ iff $N \models \neg\varphi$. Similarly for disjunction.

Now suppose $M \models (\forall x)\varphi(x, \bar{y})$. That means that for all $\bar{m}, m' \in M$ we have that $\varphi^M(m', \bar{m})$ is true which means that $M \models \varphi(x, \bar{y})$. Apply inductive hypothesis and we are done. □

Recall that a theory $\mathcal{T}$ is *complete* if $T \models \varphi$ or $T \models \neg\varphi$ for every sentence φ. Any two models of the same complete theory are elementary equivalent: indeed the formulae satisfied by them are completely determined by the theory. However, the models can have different cardinalities (and hence be non-isomorphic), see Examples 1.1.10.

Definition 1.1.6. A substructure $M \subseteq N$ is an *elementary substructure* if the inclusion map $M \hookrightarrow N$ is an elementary embedding.

Definition 1.1.7. A theory $\mathcal{T}$ is *model-complete* if every embedding between models of $\mathcal{T}$ is elementary.

Definition 1.1.8. Let κ be an infinite cardinal. We say that a theory $\mathcal{T}$ is κ-categorical if all models of $\mathcal{T}$ of cardinality κ are isomorphic.

Proposition 1.1.9 (Vaught's Test)**.** *Let $\mathcal{T}$ be a consistent $\mathcal{L}$-theory with no finite models. If $\mathcal{T}$ is κ-categorical for some infinite $\kappa \geq |\mathcal{L}|$ then $\mathcal{T}$ is a complete theory.*

Proof. For the sake of contradiction, suppose $\mathcal{T}$ is not complete, so that there is a sentence φ with $T \not\models \varphi$ and $T \not\models \neg\varphi$. It follows (by the Deduction Theorem) that $T \cup \{\varphi\}$ and $T \cup \{\neg\varphi\}$ are consistent, i.e. they have a model. These two models cannot be finite since then they would be finite models of $\mathcal{T}$. Thus they are infinite.

By the Upwards Löwenheim-Skolem Theorem (together with the Downwards version if necessary) we get models of $T \cup \{\varphi\}$ and $T \cup \{\neg\varphi\}$ of cardinality κ. In particular, these are models of $\mathcal{T}$ so, as $\mathcal{T}$ is κ-categorical, they must be isomorphic. But they disagree on the valuation of φ, contradicting Proposition 1.1.5. $\square$

Examples 1.1.10.

(a) Any two countable dense linear orders without endpoints are isomorphic to $\mathbb{Q}$. So the theory is $\aleph_0$-categorical and hence complete.

(b) For every field F the theory of infinite F-vector spaces is κ-categorical for some $\kappa > |F|$ (Exercise), hence complete.

Proposition 1.1.11 (Tarski-Vaught test)**.** *Let N be an $\mathcal{L}$-structure and $M \subseteq N$. Then M is the domain of an elementary substructure of N if and only if the following condition is satisfied.*

- *Let $\varphi(x, \bar{t})$ be a formula and $\bar{m} \in M$. If there is some $n \in N$ such that $N \models \varphi(n, \bar{m})$, then there is $\hat{m} \in M$ such that $N \models \varphi(\hat{m}, \bar{m})$.*

Proof. Suppose M is an elementary substructure of N. Let $\varphi(x, \bar{t})$ be a formula and $\bar{m} \in M$. Furthermore, assume that there is some $n \in N$ so that $N \models \varphi(n, \bar{m})$. It follows that $N \models (\exists x)\varphi(x, \bar{m})$. Thus, as M is an elementary substructure, we have $M \models (\exists x)\varphi(x, \bar{m})$. Hence, there is some $\hat{m} \in M$ such that $M \models \varphi(\hat{m}, \bar{m})$. But then clearly $N \models \varphi(\hat{m}, \bar{m})$.

Conversely, assume that the property holds. Consider the formulae $\varphi_f(x, \bar{t}) :=$ $(x = f(\bar{t}))$ for each function symbol f in $\mathcal{L}$.

For any $\bar{m} \in M$ there is $n \in N$ such that $N \models (n = f(\bar{m}))$ (say $n := f^N(\bar{m})$). By hypothesis there is $\hat{m}$ such that $N \models (\hat{m} = f(\bar{m}))$. It follows that M is closed under function symbols. Interpreting relation symbols as $R^M := R^N \cap M$ we turn M into an $\mathcal{L}$-structure that is clearly a substructure of N.

We need to show that N and M satisfy the same formulae when the parameters are in M. Let $\varphi(\bar{x})$ be a formula and $\bar{m} \in M$. Note that if $N \models \varphi(\bar{m})$ iff $M \models \varphi(\bar{m})$ whenever φ is quantifier free by Proposition 1.1.3. So, by induction, we only need to check that the case when φ is an existential formula, say $\varphi(\bar{x}) = (\exists t)\psi(t, \bar{x})$.

If $M \models (\exists t)\psi(t, \bar{m})$ then clearly $N \models (\exists t)\psi(t, \bar{x})$ since a witness in M is also a witness in M. Conversely, if $N \models (\exists t)\psi(t, \bar{m})$ then there is some $n \in N$ so that $N \models \psi(n, \bar{m})$. By the condition, there is some $\hat{m}$ such that $N \models \psi(\hat{m}, \bar{m})$. Using the inductive hypothesis, we get that $M \models \psi(\hat{m}, \bar{m})$ and thus $M \models \varphi(\bar{m})$ as desired. Hence M is an elementary substructure of N. $\square$

Definition 1.1.12. Let N be an $\mathcal{L}$-structure. We define the *diagram* of N as

$$\mathrm{Diag}(N) := \{\varphi(n_1, \dots, n_k) \mid \varphi \text{ is a q.f. } \mathcal{L}_N\text{-formula and } N \models \varphi(n_1, \dots, n_k)\}.$$

The *elementary diagram* of N is defined as

$$\mathrm{Diag}_{\mathrm{el}}(N) := \{\varphi(n_1, \dots, n_k) \mid \varphi \text{ is an } \mathcal{L}_N\text{-formula and } N \models \varphi(n_1, \dots, n_k)\}$$

Basically, diagrams are all (q.f.) formulas the structure believes in. For the difference between diagrams of structures and the theory of a structure (the latter we'll discuss later) see this link.

The most important thing about diagrams is their models. Indeed, let M be a model of $\mathrm{Diag}(N)$. In particular, M is an $\mathcal{L}$-structure with a constant $m_n \in M$ for each element $n \in N$. Hence we have a function $\eta \colon N \to M$ given by $\eta(n) := m_n$. But $N \models \neg(n = n')$ for any distinct $n, n' \in N$. Thus, $M \models \neg(m_n = m_{n'})$ for distinct $n, n' \in N$, i.e. η is injective.

Let f be a function symbol. Then (bare with me on this) for all $\bar{n} \in N$ we have $N \models (f(\bar{n}) = f^N(\bar{n}))$ where the LHS is the function symbol f applied to the constants $\bar{n}$ and the RHS is just the constant $f^N(\bar{n})$. Hence, $M \models$ $(f(\eta(\bar{n})) = \eta(f^N(\bar{n})))$, i.e. we have

$$f^M(\eta(\bar{n})) = \eta(f^N(\bar{n})).$$

Also, if R is a relation symbol, then for all $\bar{n} \in N$, if we have $N \models R(\bar{n})$ then $M \models R(\eta(\bar{n}))$ then we have $N \models R(\bar{n})$. Conversely, if $N \not\models R(\bar{n})$ then $N \models \neg R(\bar{n})$ and we apply the same argument. This all shows that f is an injective homomorphism, so we can see M as an extension of N. In summary, models of $\mathrm{Diag}(N)$ are just extensions of N.

Similarly, models $\mathrm{Diag}_{\mathrm{el}}(N)$ are elementary extensions of N. Indeed, if M is such a model then M is an extension of N by the previous argument. Let $\varphi(\bar{x})$

be a formula and let $\bar{m} \in M$. If $N \models \varphi(\bar{m})$ then $M \models \varphi(\bar{m})$ by definition of $\mathrm{Diag}_{\mathrm{el}}(N)$. Conversely, if $M \models \varphi(\bar{m})$ we have that $N \models \varphi(\bar{m})$ by Proposition 1.1.3 if φ is quantifier free, and otherwise it is obvious for universal formulae.

Lemma 1.1.13. *Let $\mathcal{T}$ be a consistent theory and let $\mathcal{T}_\forall$ be the theory of all universal sentences that follow from $\mathcal{T}$. If N is a model of $\mathcal{T}_\forall$, then $\mathcal{T} \cup \mathrm{Diag}(N)$ is consistent.*

Proof. For the sake of contradiction, suppose it is inconsistent. By Compactness, there is some finite subset of $\mathcal{T} \cup \mathrm{Diag}(N)$ that is inconsistent. As $\mathcal{T}$ is consistent, there must be a finite subset of $\mathrm{Diag}(N)$ that is inconsistent with $\mathcal{T}$.

Take the conjunction of all of these sentences and call it $\varphi(\bar{n})$. Then $\mathcal{T} \cup \{\varphi(\bar{n})\}$ is inconsistent, i.e. $\mathcal{T} \vdash \neg\varphi(\bar{n})$. But $\mathcal{T}$ is an $\mathcal{L}$-theory, so it contains none of the constants $\bar{n}$. Hence, by generalization, $\mathcal{T} \vdash (\forall x)\neg\varphi(\bar{x})$. However N is a model of $\mathcal{T}_\forall$ so $N \models (\forall x)\neg\varphi(\bar{x})$, and thus $N \models \neg\varphi(\bar{n})$, a contradiction. $\square$

We say a theory $\mathcal{T}$ has an *universal axiomatization* if there is an universal theory that has exactly the same models as $\mathcal{T}$

Theorem 1.1.14 (Tarski, Łoś)**.** *An $\mathcal{L}$-theory $\mathcal{T}$ has a universal axiomatization iff whenever N is a substructure of M and $M \models T$, then $N \models T$.*

Proof. One direction is obvious (see Proposition 1.1.3). For the converse, suppose $\mathcal{T}$ is preserved under taking substructures. We would like to say that $\mathcal{T}_\forall$ is an universal axiomatization of $\mathcal{T}$. To prove this we need to show that if $N \models T_\forall$ then $N \models \mathcal{T}$ (since the converse is obvious).

By Lemma 1.1.13, we have that $\mathcal{T} \cup \mathrm{Diag}(N)$ is consistent, so let M be a model for it. Then M is a model of $\mathrm{Diag}(N)$ and hence an extension of N. But M is a model of $\mathcal{T}$ so taking substructures we conclude that $N \models \mathcal{T}$. $\square$

The method of diagrams is powerful. We can show much more with the same method.

1. Finding a common elementary extension to given structures

Theorem 1.1.15 (Elementary amalgamation)**.** *Let M and N be $\mathcal{L}$-structures, $\bar{m} \in M$ and $\bar{n} \in N$ be of the same length such that $(M, \bar{m}) \equiv (N, \bar{n})$. Then there is an $\mathcal{L}$-structure K and elementary embeddings $g\colon N \hookrightarrow K$ and $h\colon M \hookrightarrow K$ such that $g(\bar{n}) = h(\bar{m})$.*

Proof. Form the disjoint union of M and N and quotient it out by the smallest equivalence relation containing (m_i, n_i) for all i. Then the resulting set is basically copies of the two sets M and N that only intersect at $\bar{m} = \bar{n}$. Hence we may assume, without loss of generality that $\bar{m} = \bar{n}$ and otherwise M and N are disjoint.

We would like to show that $\mathcal{T} \coloneqq \mathrm{Diag}(N)_{\mathrm{el}} \cup \mathrm{Diag}_{\mathrm{el}}(M)$ is consistent; of course, we do so by Compactness. Let Φ be a finite subset of $\mathcal{T}$, which of course contains only finitely many sentences of $\mathrm{Diag}_{\mathrm{el}}(N)$.

Let $\varphi(\bar{n}, \bar{k})$ be the conjunction of all these sentences, where $\bar{k}$ does not contain any elements of $\bar{n}$ and its elements are pairwise disjoint. Define $\varphi(\bar{x}, \bar{y})$ to be the corresponding $\mathcal{L}_{\bar{N}}$ formula. If Φ is inconsistent then $\mathrm{Diag}_{\mathrm{el}} \vdash \neg\varphi(\bar{m}, \bar{k})$. Since the elements in $\bar{k}$ are distinct and not in M we in fact have $\mathrm{Diag}_{\mathrm{el}}(M) \vdash (\forall \bar{y})\neg\varphi(\bar{m}, \bar{y})$ by generalization.

In particular, $(M, \bar{m}) \models (\forall \bar{y})\neg\varphi(\bar{m}, \bar{y})$ and so by hypothesis $(N, \bar{n}) \models (\forall \bar{y})\neg\varphi(\bar{m}, \bar{y})$, contradicting the fact that $\varphi(\bar{m}, \bar{k}) \in \mathrm{Diag}_{\mathrm{el}}(N)$.

By compactness T must be consistent and a model for it would be an elementary extension of both N and M. □

2. Controlling the size of a model

Theorem 1.1.16 (Löwenheim–Skolem)**.** *Let M be an infinite $\mathcal{L}$-structure and $\kappa \geq |\mathcal{L}|$ be an infinite cardinal.*

(↓) If $\kappa < |M|$ then M admits an elementary substructure of size κ.

(↑) If $\kappa > |M|$ then M admits an elementary extension of size κ.

Proof of (↑). Expand the language $\mathcal{L}$ by adding one constant symbol for each $m \in M$ and $c \in \kappa$. Let $\mathcal{T} := \mathrm{Diag}_{\mathrm{el}}(M) \cup \{\neg(c = c')\}_{c,c' \in \kappa, c \neq c'}$ be a theory in such a language. By compactness $\mathcal{T}$ has a model, i.e. an elementary extension of size at least κ. By (↓), there must be one of size exactly κ. □

1.2 Existentially closed structures and quantifier-elimination

Definition 1.2.1. Let $\mathcal{T}$ be an $\mathcal{L}$-theory and $\varphi(\bar{x}, y)$ be an $\mathcal{L}$-formula with $\bar{x}$ nonempty. A *Skolem function* for φ is an $\mathcal{L}$-term $t(\bar{x})$ such that

$$\mathcal{T} \models \forall \bar{x} : ((\exists y : \varphi(\bar{x}, y)) \Rightarrow \varphi(\bar{x}, t(\bar{x}))).$$

A *skolemization* of an $\mathcal{L}$-theory $\mathcal{T}$ is a language $\mathcal{L}^+ \supseteq \mathcal{L}$ together with an $\mathcal{L}^+$-theory $\mathcal{T}^+ \supseteq \mathcal{T}$ such that :

(1) Every $\mathcal{L}$-structure that models $\mathcal{T}$ can be expanded to a model of $\mathcal{T}^+$.

(2) The $\mathcal{L}^+$-theory $\mathcal{T}^+$ admits Skolem functions for every $\mathcal{L}^+$-formula $\varphi(\bar{x}, y)$ with $\bar{x} \neq \emptyset$.

Finally, a theory $\mathcal{T}$ is a *Skolem theory* if it is a skolemization of itself

We say two $\mathcal{L}$-formulae φ and ψ are *equivalent modulo $\mathcal{T}$*, where $\mathcal{T}$ is an $\mathcal{L}$-theory, iff $T \vdash (\varphi \iff \psi)$.

Proposition 1.2.2. *Let $\mathcal{T}$ be an $\mathcal{L}$-theory and F be a collection of $\mathcal{L}$-formulae that includes all atomic formulae and is closed under Boolean combinations. If for every formula $\psi(\bar{x}, y)$ in F we have $\varphi(\bar{x})$ in F such that*

$$\mathcal{T} \vdash \forall \bar{x} : ((\exists y : \psi(\bar{x}, y)) \iff \varphi(\bar{x})),$$

then every $\mathcal{L}$-formula is equivalent modulo $\mathcal{T}$ to one in F with the same free variables.

Proof. By induction on the structure of formulae. Atomic formulae are in F by hypothesis and F is closed under Boolean combinations, so we only need to check the case for existential statements. But that's exactly the hypothesis. $\square$

If M is an elementary substructure of N we denote this by $M \preccurlyeq N$.

Proposition 1.2.3. *Let $\mathcal{T}$ be an Skolem $\mathcal{L}$-theory. Then*

(1) Every $\mathcal{L}$-formula $\varphi(\bar{x})$ with $\bar{x} \neq \emptyset$ is equivalent to some quantifier-free $\psi(\bar{x})$ modulo $\mathcal{T}$.

(2) If $N \models \mathcal{T}$ and X is a subset of N, then either $\langle X \rangle_N = \emptyset$ or $\langle X \rangle_N \preccurlyeq N$.

Proof.

(1) This just follows from Proposition 1.2.2 by taking F to be the set of all quantifier-free formulae, and by the definition of Skolem theory.

(2) Assume X is nonempty (otherwise this is trivial). Let $M \coloneqq \langle X \rangle_N$; we will use the Tarski-Vaught test. Let $\varphi(x, \bar{y})$ be an $\mathcal{L}$-formula and $\bar{m} \in M$. Suppose that there is some $n \in N$ with $N \models \varphi(n, \bar{m})$. Then $N \models (\exists y)(\varphi(y, \bar{m}))$. Hence there is a Skolem function $t(\bar{x})$ such that $N \models \varphi(t(\bar{m}), \bar{m})$. As M is a substructure of N it is closed under interpretations of terms, hence $t^N(\bar{m}) \in M$. We are done. $\square$

Theorem 1.2.4 (Skolemization Theorem)**.** *Every (first-order) language $\mathcal{L}$ can be expanded to some $\mathcal{L}^+ \supseteq \mathcal{L}$ that includes an $\mathcal{L}^+$-theory Σ such that*

(1) Σ is a Skolem $\mathcal{L}^+$-theory.

(2) Every $\mathcal{L}$-structure can be expanded to an $\mathcal{L}^+$-structure that models Σ.

(3) $|\mathcal{L}| = |\mathcal{L}^+|$.

Proof. For an arbitrary language $\mathcal{L}$ and for every $\mathcal{L}$-formula of the form $\chi(\bar{x}, y)$ with $\bar{x} \neq \emptyset$ we create an function symbol F_χ of arity $|\bar{x}|$. By adding all such function symbols to $\mathcal{L}$ we get a new language $\mathcal{L}^*$. We define $\Sigma(\mathcal{L})$ to be the $\mathcal{L}^*$-theory:

$$\Sigma(\mathcal{L}) \coloneqq \{\forall \bar{x}((\exists y : \chi(\bar{x}, y)) \Rightarrow \chi(\bar{x}, F_\chi(\bar{x}))) \mid \chi(\bar{x}, y) \text{ is an } \mathcal{L}\text{-formula and } \bar{x} \neq \emptyset\}.$$

Intuitively, Σ says that $\mathcal{L}^*$ has Skolem functions *for all $\mathcal{L}$-formulae*. We would be basically done if this included $\mathcal{L}^*$ formulae too. The way to fix this is to iterate the construction.

Let $\mathcal{L}$ be a language. Define a sequence of languages and theories as follows. Start with $\mathcal{L}_0 \coloneqq \mathcal{L}$ and $\Sigma_0 \coloneqq \emptyset$. For $n \geq 1$ define

$$\mathcal{L}_n \coloneqq \mathcal{L}^*_{n-1} \quad \text{and} \quad \Sigma_n \coloneqq \Sigma(\mathcal{L}_{n-1}) \cup \Sigma_{n-1}.$$

Set $\mathcal{L}^+ := \bigcup_{n\in\mathbb{N}} \mathcal{L}_n$ and $\Sigma := \bigcup_{n\in\mathbb{N}} \Sigma_n$.

It is easy to see that Σ is an $\mathcal{L}^+$-theory. Furthermore, it is Skolem since every $\mathcal{L}^+$-formula of the required form is in some $\mathcal{L}_n$ and thus there is an Skolem function for it by Σ_{n+1}. Also $|\mathcal{L}| = |\mathcal{L}^+|$ by cardinal multiplication (countable cardinalities are absorbed).

We now check the structure expansion property. We first check it step-by-step down the chain. Let M be a nonempty $\mathcal{L}$-structure. Say we have $\chi(\bar{x}, y)$ with $\bar{x} \neq \emptyset$ and a tuple $\bar{m} \in M$. If there is a b such that $M \models \chi(\bar{m}, b)$, choose one and define $F_\chi^M(\bar{m}) := b$. If there is no such b interpret it as m_0 (or whatever). $\square$