Model Theory and Non-Classical Logic

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This is a set of lecture notes taken by me from the Part III course "Model Theory and Non-Classical Logic", lectured by Dr J. Siqueira in Michaelmas, 2023. I take full responsibility for any mistakes in these notes. Chapter 0 is my summary/expansion of [**NOLAST**]

Chapter 0

Logic Background

Definition 0.0.1 (Signature)**.** A *signature* Σ is a triplet (Ω, Π, α), where Ω and Π are disjoint sets and $\alpha \colon \Omega \cup \Pi \to \mathbb{N}$. We call the elements of Ω *function symbols*, those of Π we call *predicate symbols*, and if $s \in \Omega \cup \Pi$ we call $\alpha(s)$ the *arity* of s.

For the rest of this chapter, assume $\Sigma = (\Omega, \Pi, \alpha)$ is an arbitrary signature and that we are given a countable set $X = \{x_1, x_2 \ldots\}$, which we call the set of *variables*. This set does not contain any symbols in our signature (nor in the set of strings on our signature, see below).

0.1 Terms, formulae, and structures

Definition 0.1.1 (Terms)**.** The set of Σ-*terms* is a subset of the set of strings on $\Omega \cup X$, defined inductively as follows.

(a) If $x \in X$ then x is a term

(b) If $t_1, \ldots t_n$ are terms, and $\omega \in \Omega$ with $\alpha(\omega) = n$ then $\omega(t_1, \ldots, t_n)$ is a term.

(c) That is all.

Remark 0.1.2. Now assume that $X \cup \Omega \cup \Pi$ do not contain the symbols '$=$', '(','$)$', '$\perp$', '$\forall$' nor '$\Rightarrow$' (nor commas).

Definition 0.1.3 (Atomic formulae)**.** Let T be the set of Σ-terms. We define the *atomic formulae* of Σ as certain strings on $T \cup \Pi \cup \{(,), =, ,\}$ (note that the last comma is not a typo) according to the following rules.

(a) If s and t are terms then $(s = t)$ is an atomic formula.

(b) If $\phi \in \Pi$, $\alpha(\phi) = n$ and $t_1, \ldots, t_n$ are terms then $\phi(t_1, \ldots, t_n)$ is an atomic formula.

(c) That is all.

Definition 0.1.4 (Pre-formulae)**.** Let T be the set of terms of Σ. We inductively define the set of Σ-*pre-formulae* as a subset of the set of strings on $T \cup \Pi \cup \{=, \perp, \forall, \Rightarrow, (,)\}$ satisfying the following.

(a) Atomic formulae are pre-formulae

(b) $\perp$ is a pre-formula.

(c) If p and q are pre-formulae then so is $(p \Rightarrow q)$.

(d) If p is a pre-formula and $x \in X$ is a variable then $(\forall x)p$ is a pre-formula.

(e) That's all.

Now we can define a function PFV (for pre-free variables) on the set of terms union with the set of pre-formulae by the following rules

$$\begin{aligned}
&\mathrm{PFV}(x) = \{x\} \\
&\mathrm{PFV}(\omega t_1 \cdots t_n) = \bigcup_{i=1}^{n} \mathrm{PFV}(t_i) \\
&\mathrm{PFV}(s = t) = \mathrm{PFV}(s) \cup \mathrm{PFV}(t) \\
&\mathrm{PFV}(\phi(t_1, \ldots, t_n)) = \bigcup_{i=1}^{n} \mathrm{PFV}(t_i) \\
&\mathrm{PFV}(\perp) = \emptyset \\
&\mathrm{PFV}(p \Rightarrow q) = \mathrm{PFV}(p) \cup \mathrm{PFV}(q) \\
&\mathrm{PFV}((\forall x)p) = \mathrm{PFV}(p) \setminus \{x\}
\end{aligned}$$

Finally, we can define Σ-*formulae* to be all pre-formulae of Σ except those of the form $(\forall x)p$ where $x \notin \mathrm{PFV}(p)$. Define FV to be the restriction of PFV so that it only applies to terms and formulae.

By the *language* $\mathcal{L}$ of a signature Σ we mean the set of all terms and formulae of Σ. Instead of saying Σ-terms and Σ-formulae we say $\mathcal{L}$-terms and $\mathcal{L}$-formulae to mean the same thing.

Definition 0.1.5 (Language structures)**.** An $\mathcal{L}$-*structure* is a set A together with functions $\omega_A \colon A^{\alpha(w)} \to A$ for each $\omega \in \Omega$ and relations $\phi_A \subseteq A^{\alpha(\phi)}$ for each $\phi \in \Pi$. We use the convention that S^0 is a singleton set (say $\{0\}$) for all sets S.

0.2 Derived symbols

Definition 0.2.1 (Derived operations)**.** Let A be an $\mathcal{L}$-structure and t a term. In addition, suppose n is an integer with $\mathrm{FV}(t) \subseteq \{x_1, \ldots, x_n\}$. We define, $t_A(n)$ to be a function $A^n \to A$ as follows.

(a) If $t \in X$ then $t = x_i$ for some $i \leq n$. Let $t_A \colon A^n \to A$ be the i-th projection function.

(b) Suppose $t = \omega t_1 \dots t_m$ where $\omega \in \Omega$ with $\alpha(\omega) = m$, and the t_i's are terms for which we have defined $(t_i)_A(n)$. Then t_A is the composite

$$A^n \xrightarrow{((t_1)_A(n),(t_2)_A(n),\dots,(t_m)_A(n))} A^m \xrightarrow{\omega_A} A$$

Note that in the empty structure all derived operations are the empty function (there can't be any constant symbols).

Lemma 0.2.2 (Variable redundancy in terms)**.** *Let A be a structure and t a term with $\mathrm{FV}(t) \subseteq \{x_1, \dots, x_n\}$ for some n. Suppose we have two sequences of elements of A^n, say a and b, such that a and b agree on free variables, i.e. $a_k = b_k$ whenever $x_k \in \mathrm{FV}(t)$. Then $t_A(n)(a) = t_A(n)(b)$.*

Proof. Induction on t. Suppose $t = x_i \in X$. Then $t_A(n)(a)$ and $t_A(n)(b)$ are a_i and b_i respectively. We have assumed these are the same, as $x_i \in \mathrm{FV}(t)$. Thus $t_A(n)(a) = t_A(n)(b)$.

Now suppose $t = \omega t_1 \cdots t_m$ where $\omega \in \Omega$ with $\alpha(\omega) = m$, and the t_i's are terms. By inductive hypothesis, we may assume that for all $1 \leq i \leq n$ we have

$$(t_i)_A(n)(a) = (t_i)_A(n)(b).$$

It follows that

$$((t_1)_A(n), \dots, (t_m)_A(m))(a) = ((t_1)_A(n), \dots, (t_m)_A(n))(b).$$

Applying ω_A to both sides gives the result. □

Definition 0.2.3. (Derived formulae) Let A be an $\mathcal{L}$-structure and p a formula. In addition, suppose n is an integer with $\mathrm{FV}(p) \subseteq \{x_1, \dots, x_n\}$. We define $p_A(n)$ to be subset of A^n, or equivalently a function $A^n \to 2$, as follows.

(a) If p is the formula $(s = t)$ for terms s and t then

$$p_A(n) \coloneqq \{\, a \in A^n \mid s_A(n)(a) = t_A(n)(a) \,\}$$

(b) Suppose $p = \phi(t_1, \dots, t_m)$ for some $\phi \in \Pi$, with $\alpha(\phi) = m$, and terms $t_1, \dots, t_m$. Then (the characteristic function of) $p_A(n)$ is defined by

$$A^n \xrightarrow{((t_1)_A(n),(t_2)_A(n),\dots,(t_m)_A(n))} A^m \xrightarrow{\phi_A} 2$$

(c) If $p = \perp$ then $p_A(n)$ is the empty set (i.e. its characteristic function is constant with value zero).

(d) Suppose p is $(q \Rightarrow r)$ for formulas q and r, where $q_A(n)$ and $r_A(n)$ have already been defined. Then we define p_A by the composition

$$A^n \xrightarrow{(q_A(n),r_A(n))} 2 \times 2 \xrightarrow{\Rightarrow_2} 2$$

(e) Suppose $p = (\forall x_m)q$ for some formula q with $x_m \in \mathrm{FV}(q)$. Define $N := \max(n, m)$. We can assume that $q_A(N)$ is defined. Let $a \in A^n$. We say that $a \in p_A(n)$ if and only if for all $a' \in A^N$ so that a' agrees with a in the first n terms—except possibly on the m-th term—we have that $a' \in q_A(N)$.

Lemma 0.2.4 (Variable redundancy in formulae)**.** *Let A be a structure and p a formula with $\mathrm{FV}(p) \subseteq \{x_1, \ldots, x_n\}$ for some n. Suppose we have two elements of A^n, say a and b such that a and b agree on free variables, i.e. $a_k = b_k$ whenever $x_k \in \mathrm{FV}(t)$. Then $a \in p_A(n)$ if and only if $b \in p_A(n)$.*

Proof. By induction on p. If p is the formula $(s = t)$ for terms s and t we have

$$\begin{aligned} a \in p_A(n) &\iff s_A(n)(a) = t_A(n)(a) \\ &\iff s_A(n)(b) = t_A(n)(b) \\ &\iff b \in p_A(n), \end{aligned}$$

where we have used Lemma 0.2.2.

Now suppose $p = \phi(t_1, \ldots, t_m)$ for some $\phi \in \Pi$ with $\alpha(\phi) = m$, and terms $t_1, \ldots, t_m$. Again by Lemma 0.2.2 we have

$$((t_1)_A(n), \ldots, (t_m)_A(m))(a) = ((t_1)_A(n), \ldots, (t_m)_A(n))(b).$$

Applying ϕ_A to both sides gives the result. This finishes the induction in the case that p is an atomic formula.

If $p = \perp$ then $a, b \notin p_A(n)$ so the claim holds. Now let q, r be formulae with p being $(q \Rightarrow r)$. By inductive hypothesis

$$(q_A(n), r_A(n))(a) = (q_A(n), r_A(n))(b),$$

and applying $\Rightarrow_2$ to both sides gives the result.

Finally, we consider the case where $p = (\forall x_m)q$ for some formula q and some variable $x_m \in \mathrm{FV}(q)$. Write $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$. By symmetry, we only need to show that $a \in p_A(n)$ implies $b \in p_A(n)$. So, suppose $a \in p_A$.

Let $N = \max(n, m)$ and let $b' \in A^N$ be a sequence agreeing with b in the first value except possibly on the m-th value, and call this value c_m. If we show that $b' \in q_A(N)$ then we are done.

Define $a' \in A^N$ to be the sequence b' but with its first n values replaced by a *except* the m-th value, which remains c_m. As $a \in p_A(n)$ it is clear by definition that $a' \in q_A(N)$. But a' and b' agree on free variables of q, which are $\mathrm{FV}(p) \cup \{x_m\}$: the first n variables are just a and b, which agree on $\mathrm{FV}(p)$, only that we have specified that they agree on the m-th value c_m, and otherwise a' and b' are identical. Hence, by the inductive hypothesis, $b' \in q_A(N)$. $\square$

Variable redundancy implies that free variables are the only thing that affects the values of $t_A(n)$ and $p_A(n)$. Hence we will write t_A and p_A, without specifying n, we take n to be the minimum so that $\mathrm{FV}(t)$ (resp. $\mathrm{FV}(p)$) is a subset of $\{x_1, \ldots, x_n\}$. And in any case, this only function only requires inputs a_k where x_k is a free variable of t (resp. p).

0.3 First-order theories

Definition 0.3.1 (Satisfying a formula)**.** Let A be an $\mathcal{L}$-structure, and let p be a formula. We say that p is *satisfied* in A if the indicator function of p_A is constant with value 1. In this case we write $A \models p$.

Definition 0.3.2 (Sentences and universal closure)**.** Let p be a formula. We say that p is a *sentence* if it has no free variables. (In this case p_A is a constant function since $n = 0$). In any case, we can obtain a sentence $\bar{p}$, called the *universal closure* of p, by prefixing p with universal quantifiers for each of the free variables of p (say, in decreasing order of subscripts).

We remark that if A is empty and p is not a sentence then the indicator $p_\emptyset \colon \emptyset^n \to 2$ is constant with value 1 since it sends all elements of $\emptyset^{\mathbb{N}}$ (which there are none) to 1 (this is a vacuous truth).

Proposition 0.3.3. *For all $\mathcal{L}$-structures A and formulas p,*

$$A \models p \quad \text{if and only if} \quad A \models \bar{p}.$$

Proof. We prove a weaker statement first. Let q be a formula with a free variable x_m. Then we claim that $A \models q$ if and only if $A \models (\forall x_m)q$. Indeed, $A \models \forall x_n q$ if and only if for all $a \in A^n$ (where n is the minimum so that $\mathrm{FV}((\forall x_m)q) \subseteq \{x_1, \dots, x_n\}$) we have $a \in ((\forall x_m)q)_A(n)$. And this happens iff for all $a \in A^n$ and for all $a' \in A^N$ (where $N \coloneqq \max(n, m)$) that agrees with a on the first n values except possibly in the m-th value we have $a' \in q_A(N)$.

This is clearly equivalent to saying that for all $a' \in A^N$ we have $a \in q_A(N)$, i.e. $A \models q_A$.

Now if p is a formula with k free variables we can use induction on k, together with the above result, to deduce the claim about $\bar{p}$. □

Definition 0.3.4 (First-order theory)**.** Let T be a set of $\mathcal{L}$-formulae and A a structure. We write $A \models T$ if $A \models p$ for all $p \in T$.

In the special case where T is a set of sentences we call it a *first-order theory*, and its formulae are called *axioms*. If $A \models T$ in this case we would say that A *models* the theory T.

0.4 Semantics and syntax

Recall that we had an arbitrary signature Σ that generated a language $\mathcal{L}$. We would like to add things to the signature (which will generate a difference language) from time to time. Here I will give some notation for a typical situation. Let S be a set. we denote by Σ_S the signature Σ but with $|S|$ new constant symbols (i.e. function symbols of arity zero) added. Similarly, we denote by $\mathcal{L}_S$ the language generated by Σ_S. Note that in the special case that S is an $\mathcal{L}$-structure we clearly have that S is an $\mathcal{L}_S$-structure: just interpret the new constant symbols as the elements of S.

Definition 0.4.1 (Semantic entailment)**.** Let T be a theory and let p be a sentence. We say that T *semantically entails* p, written as $T \models p$, to mean that every model of T satisfies p.

In the case where T and p are not sentences simply consider the language

$$\mathcal{L}' \coloneqq \mathcal{L}_{\mathrm{FV}(T) \cup \mathrm{FV}(p)},$$

where $\mathrm{FV}(T)$ is just the set of all free variables appearing in a formulae of T. Let T' and p' be the same formulae but with free variables replaced by the corresponding constants in $\mathcal{L}'$. Then $T' \cup \{p'\}$ is just a set of sentences in $\mathcal{L}'$, so declare that $T \models p$ in $\mathcal{L}$ if and only if $T' \models p'$ in $\mathcal{L}'$.

This seems roundabout: why not define semantic entailment $T \models p$ as "for all $A \models T$ we have $A \models p$"? This certainly makes sense when $T \cup \{p\}$ is not a set of sentences. The problem is that if we adopted this alternate definition we would have undesired consequences with the empty structure. For example, if $T = \{\neg(x_1 = x_1)\}$ and $p = \{\perp\}$ then the only model for T is the empty structure but $\emptyset \not\models p$. However, since T is clearly a contradictory statement we would like to have $T \models p$ in this case, which is guaranteed by the real definition since the addition of constants invalidate the empty structure.

Now we turn to our system of deduction. If w is a formula, t is a term and x a variable, we define $w[t/x]$ to be the formula obtained from w on replacing each free occurrence of x by t, *provided* no free variable of t occurs bound in w. More formally, we define

$$y[t/x] = \begin{cases} y & \text{if } x \neq y \\ t & \text{if } x = y \end{cases}$$

$$(\omega t_1 \dots t_n)[t/x] = \omega(t_1[t/x]) \cdots (t_n[t/x])$$

$$(s = s')[t/x] = (s[t/x] = s'[t/x])$$

$$\phi(t_1, \dots, t_n)[t/x] = \phi((t_1[t/x]), \dots, (t_n[t/x]))$$

$$\perp[t/x] = \perp$$

$$(p \Rightarrow q)[t/x] = (p[t/x] \Rightarrow q[t/x])$$

$$((\forall y)p)[t/x] = \begin{cases} (\forall y)(p[t/x]) & \text{if } x \neq y \\ (\forall y)p & \text{if } x = y. \end{cases}$$

Lemma 0.4.2. *Let w be a term or a formula, let x_m be a variable and let t be a term such that all free variables in t do not appear bound in w. Suppose A is an $\mathcal{L}$-structure and $a \in A^n$ where n is the minimum nonnegative integer so that $\mathrm{FV}(w) \subseteq \{x_1, \dots, x_n\}$. Denote by a' the sequence a but with the m-th value replaced by $t_A(a)$. Then we have*

$$(w[t/x_n])_A(a) = w_A(a').$$

If $A = \emptyset$ then $(w[t/x_n])_A = w_A$

Proof. Suppose $A = \emptyset$. If w does not have x_n as a free variable then $w[t/x_n] = w$. It follows that $(w[t/x_n])_A = w_A$.

Now assume that x_n is a free variable of w. Clearly w is not a sentence. We claim that neither is $w[t/x_n]$. This is easily seen from the fact that t is not a constant (since otherwise the empty set could not be an $\mathcal{L}$-structure) and thus has free variables and *in addition* we assumed that no free variables of t are being bound in w. Thus, as neither w nor $w[t/x_n]$ are sentences, they are indicators $\emptyset^{\mathbb{N}} \to 2$ and thus equal. This proves the claim for the empty structure, so from now on assume $A \neq \emptyset$.

First suppose that w is a term. We use induction, so assume $w = x_m$ for some m. If $m \neq n$ then $w[t/x_n] = w$ and so we only need to show that $w_A(a) = w_A(a')$. This is immediate by variable redundancy: a and a' agree on the free variable x_m. Now suppose $m = n$. Then $w[t/x_n] = t$ and we need to show that $t_A(a) = (x_n)_A(a')$. Again, this is obvious: the right-hand side of the equation is the n-th value of a', which we assumed is $t_A(a)$. This closes the base case.

Now, suppose $w = \omega t_1 t_2 \cdots t_m$ for some $\omega \in \Omega$ with $\alpha(\omega) = m$ and where the t_i's are terms. Clearly

$$w[t/x_n] = \omega(t_1[t/x_n])(t_2[t/x_n]) \cdots (t_m[t/x_n]).$$

It follows that

$$\begin{aligned}(w[t/x_n])_A(a) &= \omega_A((t_1[t/x_n])_A(a), (t_2[t/x_n])_A(a), \dots, (t_m[t/x_n])_A(a)) \\ &= \omega_A((t_1)_A(a'), (t_2)_A(a'), \dots, (t_m)_A(a')) \\ &= w_A(a'),\end{aligned}$$

where we have used the inductive hypothesis. This closes the induction and proves the statement when w is a term.

Suppose now that w is a formula. We again use induction. If w is the formula $(s = s')$ for terms s and s' we have that

$$w[t/x_n] = (s[t/x_n] = s'[t/x_n]).$$

Then,

$$\begin{aligned}a \in (s[t/x_n] = s'[t/x_n])_A &\iff (s[t/x_n])_A(a) = (s'[t/x_n])_A(a) \\ &\iff s_A(a') = t_A(a') \\ &\iff a' \in w_A,\end{aligned}$$

where we used the result for terms. Now suppose w is $\phi(t_1, \dots, t_m)$ for some $\phi \in \Pi$ with $\alpha(\phi) = m$, and terms $t_1, \dots, t_m$. Again, using the claim for terms, we have

$$((t_1[t/x_n])_A(a), \dots, (t_m[t/x_n])_A(a)) = ((t_1)_A(a'), \dots, (t_m)_A(a'))$$

and applying ϕ_A to both sides gives the result. This closes the base case, i.e. the case where w is an atomic formula.

Clearly $(\perp[t/x_n])_A(a) = \perp_A(a) = 0 = \perp_A(a')$. Now, if w is $(p \Rightarrow q)$ then, by the inductive hypothesis

$$((p[t/x_n])_A(a), (q[t/x_n])_A(a)) = (p_A(a'), q_A(a')).$$

Applying $\Rightarrow_2$ to both sides gives the result.

Finally, suppose $w = (\forall x_m)p$. Then we have two cases. If $m = n$ then $w[t/x_n] = w$ and so we need to show that $w_A(a) = w_A(a')$. But in this case clearly x_n is not a free variable of w, so a and a' agree on free variables, and the claim follows by variable redundancy.

Now assume $m \neq n$. Then $w[t/x_n] = (\forall x_m)(p[t/x_n])$. First, note that that

$$a = (a_1, a_2, \ldots) \in ((\forall x_m)(p[t/x_n]))_A$$

if and only if

$$(a_1, \ldots, a_{m-1}, c, a_{m+1}, \ldots) \in (p[t/x_n])_A \text{ for all } c \in A.$$

For $c \in A$ let $\alpha(c)$ be the sequence above, i.e. a but the m-th value replaced by c. Similarly, let $\alpha'(c)$ be the sequence a' but replacing the m-th value with c. Finally, let $\alpha^*(c)$ be the sequence $\alpha(c)$ but with the n-th value replaced by $t_A(\alpha(c))$ Then we can reformulate our statement as so:

$$\alpha(c) \in (p[t/x_n])_A \text{ for all } c \in A.$$

By the inductive hypothesis, this new statement is equivalent to

$$\alpha^*(c) \in p_A \text{ for all } c \in A.$$

Note that, for all c, we have that $\alpha^*(c)$ and $\alpha'(c)$ agree on all values (including the m-th) except possibly on the n-th value, where we have $t_A(\alpha(c))$ and $t_A(a)$ fro $\alpha^*(c)$ and $\alpha'(c)$ respectively. We claim that in fact *they do* agree on the n-th value, i.e. $t_A(\alpha(c)) = t_A(a)$.

Indeed, by definition, a and $\alpha(c)$ agree on all values except possibly on the m-th value. However, we assumed (and this is the first and only time we use the assumption when $A \neq \emptyset$) that the free variables of t do not appear bound in w. Clearly x_m is bound in w (recall that we insist that variables that are being bound appear in the formula). Thus x_m cannot be a free variable of t, which implies that a and $\alpha(c)$ agree on free variables; hence $t_A(\alpha(c)) = t_A(a)$ by variable redundancy. Thus $\alpha^*(c)$ is the same sequence as $\alpha'(c)$. Therefore we can, once again, reformulate our statement:

$$\alpha'(c) \in p_A \text{ for all } c \in A.$$

This is manifestly equivalent to $a' \in ((\forall x_m)p)_A$, as desired. □

We now postulate our axioms to be substitution instances of these propositions.

(a) $(p \Rightarrow (q \Rightarrow p))$

(b) $((p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r)))$

(c) $(\neg\neg p \Rightarrow p)$
(here p, q, r may be any formulae of $\mathcal{L}$)

(d) $((\forall x)p \Rightarrow p[t/x])$
(here p is any formula with $x \in \mathrm{FV}(p)$, t any term whose free variables don't occur bound in p)

(e) $((\forall x)(p \Rightarrow q) \Rightarrow (p \Rightarrow (\forall x)q))$
(p, q formulae, $x \notin \mathrm{FV}(p)$)

(f) $(\forall x)(x = x)$

(g) $(\forall x, y)((x = y) \Rightarrow (p \Rightarrow p[y/x]))$
(p any formula with $x \in \mathrm{FV}(p)$, y not bound in p and distinct from x)

Proposition 0.4.3. *All the axioms above are tautologies.*

Proof. Let p, q, r be formulae in $\mathcal{L}$ and let A be an $\mathcal{L}$-structure.

(a) First suppose that $A = \emptyset$. Then, if there are free variables in p or q then it is clear that $\emptyset \models (p \Rightarrow (q \Rightarrow p))$. Otherwise, p and q are sentences and so they have a truth value. Case-by-case analysis reveals that $\emptyset \models (p \Rightarrow (q \Rightarrow p))$. Now assume that A is nonempty.

Note that for all $a \in A^{\mathbb{N}}$ we have

$$(p \Rightarrow (q \Rightarrow p))_A(a) = (\Rightarrow_2)(p_A(a), (\Rightarrow_2)(p_A(a), q_A(a)))$$

as elements of $2 = \{0, 1\}$. Plugging in the possible values for $p_A(a)$ and $q_A(a)$ we conclude that in all cases $(p \Rightarrow (q \Rightarrow p))_A(a) = 1$.

(b) Similar to (a).

(c) Similar to (a).

(d) Suppose $x \in \mathrm{FV}(p)$ and t is any term whose free variables don't occur bound in p. It is easy to see that the axiom is never a sentence, so $\emptyset$ models it. Assume now that $A \neq \emptyset$.

Let $a = (a_1, a_2, \ldots) \in A^{\mathbb{N}}$ and consider

$$(\Rightarrow_2)(((\forall x)p)_A(a), p[t/x]_A(a))$$

If $((\forall x)p)_A(a) = 0$ then the above equals 1, clearly. Now suppose $((\forall x)p)_A(a) = 1$ and let $x = x_n$ for some n. This means that, for all $a'_n \in A$ we have

$$(a_1, \ldots, a_{n-1}, a'_n, a_{n+1}, \ldots) \in p_A.$$

Set $a'_n \coloneqq t_A(a)$. By Lemma 0.4.2, the above implies that $p[t/x]_A(a) = 1$, as desired.

(e) Let $x \notin \mathrm{FV}(p)$. If the axiom is not a sentence then it has $\emptyset$ as a model. Suppose now that the axiom is a sentence; this is easily seen to imply that q has x as its only free variable. Clearly $(\forall x)(p \Rightarrow q)$ is satisfied in $\emptyset$. Note that $(\forall x)q$ is also a satisfied sentence in $\emptyset$. Therefore the whole axiom is seen to be satisfied in $\emptyset$. Now assume $A \neq \emptyset$.

Let $a = (a_1, a_2, \ldots) \in A^{\mathbb{N}}$. If $((\forall x)(p \Rightarrow q))_A(a) = 0$ then the formula is true for a. So, assume that $a \in ((\forall x)(p \Rightarrow q))_A$. Let $x = x_n$ for some n. We have that, for all $a'_n \in A$:

$$(a_1, \ldots, a_{n-1}, a'_n, a_{n+1}, \ldots) \in (p \Rightarrow q)_A.$$

In other words, for all $a'_n \in A$:

$$(\Rightarrow_2)(p_A, q_A)(a_1, \ldots, a_{n-1}, a'_n, a_{n+1}, \ldots) = 1$$

But, as $x_n \notin \mathrm{FV}(p)$, the value $p_A(a_1, \ldots, a_{n-1}, a'_n, a_{n+1}, \ldots)$ does not depend on a'_n by variable redundancy. Thus we conclude that for all $a'_n \in A$.

$$(\Rightarrow_2)(p_A(a), q_A((a_1, \ldots, a_{n-1}, a'_n, a_{n+1}, \ldots))) = 1.$$

From this, it is easy to deduce that $a \in (p \Rightarrow (\forall x)q)_A$, as desired.

(f) The empty set is easily seen to model this axiom. Let $a = (a_1, a_2, \ldots) \in A^{\mathbb{N}}$ and let $x = x_n$l Then $a \in ((\forall x)(x = x))_A$ iff for all $a'_n \in A$ we have

$$(a_1, \ldots, a_{n-1}, a'_n, a_{n+1}, \ldots) \in (x = x)_A.$$

This happens iff for all a'_n we have

$$x_A(a_1, \ldots, a_{n-1}, a'_n, a_{n+1}, \ldots) = x_A(a_1, \ldots, a_{n-1}, a'_n, a_{n+1}, \ldots),$$

which is manifestly true.

(g) The empty set is easily seen to model this axiom. Let $x = x_n \in \mathrm{FV}(p)$ and $y = x_m$ be not bound in p with $n \neq m$. Have some $a = (a_1, a_2, \ldots) \in A^{\mathbb{N}}$. We need to show that

$$a \in ((\forall x_n)(\forall x_m)((x_n = x_m) \Rightarrow (p \Rightarrow p[x_m/x_n])))_A$$

For $c_n, c_m \in A$ define $\alpha(c_n, c_m)$ to be the sequence a but with the i-th value replaced by c_i for $i \in \{n, m\}$ (recall that $n \neq m$). Then the above proposition is equivalent to

$$\alpha(c_n, c_m) \in ((x_n = x_m) \Rightarrow (p \Rightarrow p[x_m/x_n]))_A \text{ for all } c_n, c_m \in A$$

We need to prove the above. To that end, let $c_n, c_m \in A$ be arbitrary. If $\alpha(c_n, c_m) \notin (x_n = x_m)_A$ then we do have the inclusion above. So, assume

$\alpha(c_n, c_m) \in (x_n = x_m)_A$; this clearly implies that $c \coloneqq c_n = c_m$. Now, we need to show that

$$\alpha(c,c) \in (p \Rightarrow p[x_m/x_n])_A.$$

If $\alpha(c,c) \notin p_A$ then the above is true. Therefore we can suppose $\alpha(c,c) \in p_A$. We want to prove that $\alpha(c,c) \in (p[x_m/x_n])_A$. As x_m is not bound in p we can apply Lemma 0.4.2 which tells us that it suffices to show that $\alpha(c,c)' \in p_A$, where $\alpha(c,c)'$ denotes the sequence $\alpha(c,c)$ but replacing the n-th value by $(x_m)_A(\alpha(c,c)) = c$. Clearly $\alpha(c,c)' = \alpha(c,c)$ and we supposed at the start that $\alpha(c,c) \in A$. Thus we are done.

□

To our deductive system we add the following rules of inference.

(MP) From p and $(p \Rightarrow q)$, we may infer q, *provided* either q has a free variable or p is a sentence.

(Gen) From p we may infer $(\forall x)p$, *provided* x does not occur free in any premiss which has been used in the proof of p (but is a free variable of p).

Formally, we define our concept of deduction as follows.

Definition 0.4.4 (Deduction sequence)**.** Let S be a set of formulae. A *deduction sequence* on S is a finite sequence on the set of formulae of $\mathcal{L}$, defined inductively below.

(i) The empty sequence is a deduction sequence.

(ii) If $(p_1, \ldots, p_n)$ is a deduction sequence and p is an axiom or an element of S, then $(p_1, \ldots, p_n, p)$ is a deduction sequence.

(iii) Let $(p_1, \ldots, p_n)$ be a deduction sequence. Suppose there are $1 \leq i, j \leq n$ so that p_j is the formula $(p_i \Rightarrow p)$ for some p, and, in addition either p_i is a sentence or p has a free variable. Then $(p_1, \ldots, p_n, p)$ is a deduction sequence.

(iv) Suppose $(p_1, \ldots, p_n)$ is a deduction sequence so that p_n has a free variable x but x is not a free variable of p_i for $i < n$, and $p_n \notin S$. Then, if $p = (\forall x)p_n$, we have that $(p_1, \ldots, p_n, p)$ is a deduction sequence.

(v) That is all.

Definition 0.4.5 (Syntactic entailment)**.** Let S be a set of formulae and p a formula. We say that S *syntactically entails* p, written as $S \vdash p$, if there is a deduction sequence terminating at p.

0.5 Properties of first-order languages

Again, we fix a language $\mathcal{L}$ with a set of variables $X = \{x_1, x_2, \ldots\}$

0.6 Completeness

The aim of this section is to prove the Completeness theorem. Before that, we need to prove the Soundness theorem. First, a couple of lemmata.

Lemma 0.6.1. *Let S be a set of formulae and let p and q be formulae so that either q has a free variable or p is a sentence. If $S \models p$ and $S \models (p \Rightarrow q)$, then $S \models q$.*

Proof. Let A be an $\mathcal{L}$-structure. We want to show that one of the two following statements holds:

(I) A is empty and there is a non-sentence in $S \cup \{q\}$.

(II) $\bigcap_{r \in S} r_A \subseteq q_A$.

We know that one of these two statements holds:

(a) A is empty and there is a non-sentence in $S \cup \{p\}$.

(b) $\bigcap_{r \in S} r_A \subseteq p_A$.

Suppose (a) holds. If there is a non-sentence in S then there is a non-sentence in $S \cup \{q\}$ and A is empty, i.e. (I) holds. Otherwise, if p is a non-sentence, then, by the premiss of the statement q is a non-sentence and again (I) holds. So, from now on, assume (b) holds

Similarly, we also know that one of the two statements below holds:

(a') A is empty and there is a non-sentence in $S \cup \{p, q\}$.

(b') $\bigcap_{r \in S} r_A \subseteq (p \Rightarrow q)_A$.

Suppose (a') holds. If q is a non-sentence then (I) holds, and if instead the non-sentence is in $S \cup \{p\}$ we have reduced to case (a). So, we can assume (b') holds. But (b) and (b') are easily seen to imply (II), even when A is empty. $\square$

Lemma 0.6.2. *Let S be a set of formulae, p a formula, and x a variable so that x does not occur free in any formulae of S. If $S \models p$ then $S \models (\forall x)p$.*

Proof. Let A be an $\mathcal{L}$-structure. As before, we want to show that one of the two following statements holds:

(I) A is empty and there is a non-sentence in $S \cup \{(\forall x)p\}$.

(II) $\bigcap_{r \in S} r_A \subseteq ((\forall x)p)_A$.

The hypothesis is that one of the two following statements holds. (We know that p is a non-sentence already)

(a) A is empty

(b) $\bigcap_{r \in S} r_A \subseteq p_A$.

Suppose (a) is true. If there is a non-sentence in $S \cup \{(\forall x)p\}$ then (I) holds, so assume that S is a set of sentences, and that $(\forall x)p$ is a sentence. But then, as A is empty, we have that the indicator of $((\forall x)p)_A$ is constant with value 1, implying that (II) holds.

Now suppose (a) is not true. Then A is nonempty and (b) holds. Let $a = (a_1, a_2, \ldots) \in r_A \subseteq A^{\mathbb{N}}$ for all $r \in S$. We would like to show that $a \in ((\forall x)p)_A$, which, if $x = x_n$, is equivalent to the statement

$$(a_1, \ldots, a_{n-1}, a'_n, a_{n+1}, \ldots) \in p_A \text{ for all } a'_n \in A.$$

So, fix some $a'_n \in A$ and set $a' \coloneqq (a_1, \ldots, a_{n-1}, a'_n, a_{n+1}, \ldots)$. By assumption x_n is not a free variable of r for all $r \in S$. By variable redundancy, we conclude that $a' \in r_A$ for all $r \in S$. Finally, (b) implies that $a' \in p_A$, as desired. $\square$

Proposition 0.6.3 (the Soundness Theorem)**.** *Let S be a set of formulae and p a formula. If $S \vdash p$ then $S \models p$.*

Proof. It is enough to show that, for all deduction sequences σ, all formulae of σ are semantically entailed by S. We use induction on the set of deduction sequences.

The claim is vacuously true when σ is the empty sequence. Suppose $\sigma = (p_1, \ldots, p_n, p)$, where $S \models p_i$ for all i, and p is an axiom or an element of S. It easily follows that $S \models p$ (recall axioms are tautologies).

Now suppose that $\sigma = (p_1, \ldots, p_n, p)$, where $S \models p_i$ for all i, and there are $1 \leq i, j, \leq n$ so that p_j is the formula $(p_i \Rightarrow p)$, and, in addition, either p_i is a sentence or p has a free variable. Then Lemma 0.6.1 says that $S \models p$.

Finally, suppose that $\sigma = (p_1, \ldots, p_n, p)$, where $S \models p_i$ for all i, and that $p_n \notin S$ has a free variable x but x is not a free variable of p_i for $i < n$. In addition, we suppose $p = (\forall x)p_n$. Let $S' = \{p_1, \ldots, p_{n-1}\}$. We claim that $S' \models p_n$ $\square$

Chapter 1

Model Theory

1.1 Substructures and diagrams

Definition 1.1.1 ($\mathcal{L}$-homomorphism)**.** Let M and N be $\mathcal{L}$-structures. An $\mathcal{L}$-*homomorphism* is a map $\eta\colon M \to N$ such that given $\bar{a} = (a_1, \dots, a_n) \in M^n$:

- for all function symbols f of arity n we have that

  $$\eta(f^M(\bar{a})) = f^N(\eta^n(\bar{a})),$$

  in other words the diagram

  $$\begin{tikzcd} M^n \arrow[r, "\eta^n"] \arrow[d, "f^M"'] & N^n \arrow[d, "f^N"] \\ M \arrow[r, "\eta"'] & N \end{tikzcd}$$

  commutes;

- for all relation symbols R of arity n we have that

  $$\bar{a} \in R^M \text{ if and only if } \eta^n(\bar{a}) \in R^N.$$

An injective $\mathcal{L}$-homomorphism is an $\mathcal{L}$-*embedding* and an invertible one is an $\mathcal{L}$-*isomorphism*. If $M \subseteq N$ and the inclusion map is an $\mathcal{L}$-homomorphism we say that M is a *substructure* of N, and N is an *extension* of M.

We are going to stop writing $\bar{m} \in M^n$ where n is the length of $\bar{m}$ and just write $\bar{m} \in M$ when n can be inferred or its unimportant.

Examples 1.1.2.

(a) Let $\mathcal{L}$ be the language of groups. Then $(\mathbb{N}, +, 0)$ is a subset of the the integers $(\mathbb{Z}, +, 0)$, but it is not a substructure.

(b) If M is an $\mathcal{L}$-structure and $X \subseteq M$ then X is the domain of a substructure of M iff it is closed under the interpretation of all function symbols.

Indeed, the inclusion $\iota \colon X \to M$ clearly preserves relations. But if it is not closed under some function f^M then there is no way to interpret f^X.

(c) It follows from the previous point that the intersection of a family of substructures is a substructure: indeed, applying a function f^M to anything in the intersection will land on all substructures (since these are closed under function symbols) and thus in the intersection.

The substructure generated by $X \subseteq M$ is defined to be the intersection of all substructures of M containing X; it is denoted by $\langle X \rangle_M$. Again, by the previous point, $\langle X \rangle_M$ is also the intersection of all subsets of M that are closed under function symbols.

Hence

$$\langle X \rangle_M = X \cup \{t^M(\bar{m}) \mid t \text{ a term and } \bar{m} \in X\}.$$

Indeed, the RHS is obviously closed under function symbols and no strict subset of it could possibly be. Therefore $|\langle X \rangle_M| \leq |X| + |\mathcal{L}|$.