Model Theory and Non-Classical Logic
Example Sheet 4 Solutions

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1. Suppose every subset of a Bishop-finite set is Bishop-finite. Let φ be any proposition—we want to show that $\varphi \vee \neg\varphi$. The set $[1]$ is clearly Bishop-finite. Define

$$A \coloneqq \{x \in [1] \mid \varphi\}.$$

Then there is a bijection $f\colon A \to [m]$ for some $m \in \mathbb{N}$. There is an algorithm such that given m it checks whether $[m]$ is inhabited (say by checking whether $0 \in [m]$). Hence we can decide whether $[m]$ is inhabited (if you are fussy, you can prove this by induction too).

If $[m]$ is inhabited then we get a proof of φ by considering f^{-1}, while if it is not inhabited then we get a proof of $\neg\varphi$ by first assuming that there is a proof of φ, from which it follows that A is inhabited, and then considering the image of f.

Conversely, suppose the law of the excluded middle holds. We let $A \subseteq B$ and suppose we can list $B = \{b_0, \ldots, b_{n-1}\}$. Then for all $b_i \in B$ we can ask whether it is in A or not, and if it is not we delete it. We get a subsequence which can be easily reindexed to enumerate A. Clearly this also works if we do not accept excluded middle but A is decidable in B.

2. (a) $\lambda q\colon (\phi \to \psi) \times (\chi \to \rho).\lambda r\colon \phi \to \chi.\lambda t\colon \phi.\langle \pi_1(q)t, \pi_2(q)rt\rangle$.
   (b) $\lambda p\colon (\phi \to \psi).\lambda q\colon \psi \to \chi.\lambda r\colon \phi.q(pr)$.
   (c) $\lambda p\colon \phi \to (\psi \to \chi).\lambda q\colon \phi \times \psi.(p\pi_1(q))\pi_2(q)$.
   (d) $\lambda p\colon \psi \times \phi \to \chi.\lambda q\colon \phi.\lambda r\colon \psi.p\langle r, q\rangle$.
   (e) $\lambda p\colon (\phi \to \chi) \times (\psi \to \chi).\lambda q\colon \phi + \psi.\mathrm{case}(q; x\colon \phi.\pi_1(p)x; y\colon \psi.\pi_2(p)y)$.

3. (a) This is pictorially obvious: if $X > M_1 > \cdots > M_k$ and $X > N_1 > \cdots > N_l$ then consider

an algorithm going as follows (it looked better on paper)

$$
\begin{tikzcd}
 & & & Y_3 & & & \\
 & Y_2^M \arrow[urr] & & Y_2 \arrow[ll] \arrow[rr] & & Y_2^N \arrow[ull] & \\
M_3 \arrow[ur] & & Y_1^M \arrow[ur] & & Y_1^N \arrow[ul] & & N_3 \arrow[ul] \\
 & M_2 \arrow[ul] \arrow[ur] & & Y_1 \arrow[ul] \arrow[ur] & & N_2 \arrow[ul] \arrow[ur] & \\
 & & M_1 \arrow[ul] \arrow[ur] & & N_1 \arrow[ul] \arrow[ur] & & \\
 & & & X \arrow[ul] \arrow[ur] & & &
\end{tikzcd}
$$

(b) Rather unenlightening checking

4. Suppose $\vdash_{IPC(\to)} \bot$. By the Curry-Howard correspondence, there is a λ-term M such that $\Vdash M : 0$. Using β-normalisation and subject reduction we can assume M is in β-normal form. As M cannot have any free variables, we must have M to be of the form $\lambda x : \sigma . P$. Hence the corresponding proof would be of the form $A \to B$, but this is not the form of $\bot$, contradiction.

5. Recall that we define $a \leq b$ to be the case iff $a \wedge b = a$. Then if $a \leq b$ we have, by the absorption laws that
$$a \vee b = (a \wedge b) \vee b = b.$$
Conversely, if $a \vee b = b$ then
$$a \wedge b = a \wedge (a \vee b) = a,$$
so $a \vee b = b$ is an equivalent definition for $a \leq b$.

We also note that from the definition of lattices we can deduce that $\wedge$ and $\vee$ are idempotent. Indeed, if a is an element of a lattice then, using both absorption laws,
$$a \vee a = a \vee (a \wedge (a \vee a)) = a,$$
and similarly,
$$a \wedge a = a \wedge (a \vee (a \wedge a)) = a.$$
Using these results we can prove that the operations $a \wedge -$ and $a \vee -$ are order preserving. Indeed, if $b \leq c$ then
$$(a \wedge b) \wedge (a \wedge c) = (a \wedge a) \wedge (b \wedge c) = a \wedge b,$$
and so $a \wedge b \leq a \wedge c$. Similarly, we have
$$(a \vee b) \vee (a \vee c) = (a \vee a) \vee (b \vee c) = a \vee c,$$

and so $a \vee b \leq a \vee c$.

Note as well that the absorption laws can be restated in terms of $\leq$ to say that $a \leq a \vee b$ and $a \wedge b \leq a$.

Finally I will prove the universal property of products for $\wedge$ and of coproducts for $\vee$. That is, if $c \leq a$ and $c \leq b$ we have that

$$(a \wedge b) \wedge c = a \wedge (b \wedge c) = a \wedge c = c,$$

so $c \leq a \wedge b$. Similarly, if we now assume that $a \leq c$ and $b \leq c$ then

$$(a \vee b) \vee c = a \vee (b \vee c) = a \vee c = c,$$

from which it follows that $a \vee b \leq c$.

(a) Suppose $b \leq c$. As $a \Rightarrow b \leq a \Rightarrow b$ we have that

$$(a \Rightarrow b) \wedge a \leq b \leq c.$$

Thus $a \Rightarrow b \leq a \Rightarrow c$. Now we do the other direction. As $- \wedge x$ preserves order we have that

$$b \wedge (c \Rightarrow a) \leq c \wedge (c \Rightarrow a).$$

We also have $c \Rightarrow a \leq c \Rightarrow a$, from which it follows that $c \wedge (c \Rightarrow a) \leq a$. Putting the inequalities together we get that

$$b \wedge (c \Rightarrow a) \leq a,$$

and hence $c \Rightarrow a \leq b \Rightarrow a$ as desired.

(b) By the absorption laws we have $a \leq a \vee b$. As $- \Rightarrow \perp$ is order-reversing by the previous item we have that $\neg(a \vee b) \leq \neg a$. By symmetry we also get $\neg(a \wedge b) \leq \neg b$. But then by universal product property of $\wedge$ we have $\neg(a \vee b) \leq \neg a \wedge \neg b$.

Also, from $x \wedge y \leq x$ we get that

$$\neg a \wedge \neg b \leq a \Rightarrow \perp,$$

and so

$$(\neg a \wedge \neg b) \wedge a \leq \perp,$$

and hence

$$a \leq (\neg a \wedge \neg b) \Rightarrow \perp.$$

By symmetry we also have

$$b \leq (\neg a \wedge \neg b) \Rightarrow \perp.$$

Using the universal coproduct property of $\vee$ we have that

$$a \vee b \leq (\neg a \wedge \neg b) \Rightarrow \perp.$$

Using the definition of $\Rightarrow$ now we get

$$(\neg a \wedge \neg b) \wedge (a \vee b) \leq \perp,$$

and hence $\neg a \wedge \neg b \leq \neg(a \vee b)$, so indeed we have $\neg(a \vee b) = \neg a \wedge \neg b$.

The other law has a typo and it should read $\neg a \vee \neg b \leq \neg(a \wedge b)$. Now we prove the other law. By the absorption laws we have $a \wedge b \leq a$. As $- \Rightarrow \perp$ is order-reversing we have that $\neg a \leq \neg(a \wedge b)$. By symmetry we also have $\neg b \leq \neg(a \wedge b)$. Hence by universal coproduct property of $\vee$ we get that $\neg a \vee \neg b \leq \neg(a \wedge b)$ as desired.

(c) First we show that $a \leq \neg\neg a$ (the reverse inequality isn't true in general). This follows from the fact that $a \Rightarrow \perp \leq a \Rightarrow \perp$ for then we have

$$a \wedge (a \Rightarrow \perp) \leq \perp,$$

and so $a \leq (a \Rightarrow \perp) \Rightarrow \perp = \neg\neg a$. As $- \Rightarrow \perp$ is order-reversing we have that $\neg\neg\neg a \leq \neg a$. In the other direction, note that as $\neg\neg a \leq \neg\neg a$ we get by definition of $\Rightarrow$ that

$$\neg a \wedge \neg\neg a \leq \perp,$$

and so $\neg a \leq \neg\neg a \Rightarrow \perp = \neg\neg\neg a$.

(d) By absorption law $a \wedge b \leq b$ and thus $b \leq a \Rightarrow b$. Using $b \wedge \top = b$ we have that $\top \wedge b \leq a \Rightarrow b$. It follows that $\top \leq b \Rightarrow (a \Rightarrow b)$. The reverse inequality is true as $\top$ is maximal so $\top = b \Rightarrow (a \Rightarrow b)$ as desired.

(e) By now we have repeatedly used the fact that $(x \Rightarrow y) \wedge x \leq y$ by observing that it follows from $x \Rightarrow y \leq x \Rightarrow y$. Now note that

$$\begin{aligned} a \wedge (a \Rightarrow b) \wedge (a \Rightarrow (b \Rightarrow c)) &= (a \wedge (a \Rightarrow b)) \wedge (a \wedge (a \Rightarrow (b \Rightarrow c))) \\ &\leq b \wedge (a \wedge (a \Rightarrow (b \Rightarrow c))) \\ &\leq b \wedge (b \Rightarrow c) \\ &\leq c \end{aligned}$$

where we have used idempotency of $\wedge$ at the start, and then the fact that $\wedge$ is order preserving. Now, by two applications of the definition of $\Rightarrow$ we get

$$a \Rightarrow (b \Rightarrow c) \leq (a \Rightarrow b) \Rightarrow (a \Rightarrow c).$$

By remembering that $x \wedge \top = \top$ we get that

$$\top \leq (a \Rightarrow (b \Rightarrow c)) \Rightarrow ((a \Rightarrow b) \Rightarrow (a \Rightarrow c)),$$

and the reverse inclusion follows since $\top$ is maximal.

(f) We have that

$$(a \Rightarrow \perp) \wedge a \leq \perp \leq b.$$

It follows that $\neg a \leq a \Rightarrow b$. As $- \Rightarrow \perp$ reverses inclusions we get that $\neg(a \Rightarrow b) \leq \neg\neg a$. Also, by absorption law we have that $a \wedge b \leq b$ and hence $b \leq a \Rightarrow b$. Again, by applying the order-reversing negation we get $\neg(a \Rightarrow b) \leq \neg b$. Using universal product property of $\wedge$ we get that

$$\neg(a \Rightarrow b) \leq \neg\neg a \wedge \neg b.$$

For the other direction, note that as $a \wedge (a \Rightarrow b) \leq b$ holds, we must have, as $\wedge$ preserves order, that

$$a \wedge (a \Rightarrow b) \wedge \neg b \leq b \wedge \neg b \leq \perp.$$

Thus, by definition of $\Rightarrow$ we get that

$$(a \Rightarrow b) \wedge \neg b \leq \neg a = \neg\neg a \Rightarrow \bot,$$

where we have used triple negation. Hence

$$\neg\neg a \wedge (a \Rightarrow b) \wedge \neg b \leq \bot,$$

and so

$$\neg\neg a \wedge \neg b \leq \neg(a \Rightarrow b).$$

This finishes the proof that $\neg(a \Rightarrow b) = \neg\neg a \wedge \neg b$.

(g) We use the previous result. Note that

$$\neg\neg(a \Rightarrow b) \wedge (\neg\neg a \wedge \neg b) = \neg\neg(a \Rightarrow b) \wedge \neg(a \Rightarrow b) \leq \bot.$$

It follows by two applications of the definition of $\Rightarrow$ that

$$\neg\neg(a \Rightarrow b) \leq \neg\neg a \Rightarrow \neg\neg b.$$

For the other direction, it follows since $\wedge$ preserves order, that

$$(\neg\neg a \Rightarrow \neg\neg b) \wedge \neg(a \Rightarrow b) = (\neg\neg a \Rightarrow \neg\neg b) \wedge (\neg\neg a \wedge \neg b) \leq \neg\neg b \wedge \neg b \leq \bot.$$

And hence it follows that

$$\neg\neg a \Rightarrow \neg\neg b \leq \neg\neg(a \Rightarrow b),$$

proving that $\neg\neg a \Rightarrow \neg\neg b = \neg\neg(a \Rightarrow b)$.

For the other identity we first use both De Morgan laws. The second one says that $\neg a \vee \neg b \leq \neg(a \wedge b)$. Applying negation reverses the order and so we get

$$\neg\neg(a \wedge b) \leq \neg(\neg a \vee \neg b) = \neg\neg a \wedge \neg\neg b,$$

where we used the other De Morgan law.

Now we prove the reverse inequality. Note that $\neg\top \leq \top$ and thus, as $\top$ is maximal and $\wedge$ preserves order, and we have the idempotent law, we have

$$\neg\top = \neg\top \wedge \neg\top \leq \neg\top \wedge \top \leq \bot.$$

By addition of superfluous hypothesis we have

$$\neg(b \Rightarrow (a \Rightarrow b)) \leq \bot.$$

By negation of implications this becomes

$$\neg\neg b \wedge \neg(a \Rightarrow b) \leq \bot,$$

and thus

$$\neg\neg b \leq \neg\neg(a \Rightarrow b).$$

Note that as $\wedge$ is order preserving we have that

$$((a \Rightarrow b) \wedge a) \wedge a \leq b \wedge a,$$

and so by the idempotent law and definition of $\Rightarrow$

$$a \Rightarrow b \leq a \Rightarrow a \wedge b.$$

Note that as $\neg$ is order-reversing it follows that $\neg\neg$ is order-preserving. Hence, going back to what we had,

$$\neg\neg b \leq \neg\neg(a \Rightarrow b) \leq \neg\neg(a \Rightarrow a \wedge b).$$

We already proved that double negation preserves $\Rightarrow$, hence

$$\neg\neg b \leq \neg\neg a \Rightarrow \neg\neg(a \wedge b),$$

which implies that

$$\neg\neg a \wedge \neg\neg b \leq \neg\neg(a \wedge b),$$

as desired.

6. Note that the inequality $(a \wedge b) \vee (a \wedge c) \leq a \wedge (b \vee c)$ holds in any lattice. Indeed, by absorption laws we have that $a \wedge b \leq a$ and $a \wedge c \leq a$, and so, by universal property of $\vee$ we have that

$$(a \wedge b) \vee (a \wedge c) \leq a.$$

Similarly, $a \wedge b \leq b \leq b \vee c$ and $a \wedge c \leq c \leq b \vee c$. Applying universal property of $\vee$ again yields

$$(a \wedge b) \vee (a \wedge c) \leq b \vee c.$$

We can put these together using the universal property of $\wedge$ to get

$$(a \wedge b) \vee (a \wedge c) \leq a \wedge (b \vee c).$$

For the reverse inequality, we observe that, by absorption law:

$$a \wedge b \leq (a \wedge b) \vee (a \wedge c).$$

It follows that

$$b \leq a \Rightarrow [(a \wedge b) \vee (a \wedge c)].$$

By symmetry we get

$$c \leq a \Rightarrow [(a \wedge b) \vee (a \wedge c)].$$

Applying the universal property of $\vee$ we have that

$$b \vee c \leq a \Rightarrow [(a \wedge b) \vee (a \wedge c)],$$

and thus we get the reverse inequality

$$a \wedge (b \vee c) \leq (a \wedge b) \vee (a \wedge c),$$

as desired. It follows that every Heyting algebra is distributive.

Now we show that every finite distributive lattice is a Heyting algebra. Firstly we show that the dual of the distributive law can be deduced, without any finiteness assumption. Indeed, using only the absorption laws and the fact that $\wedge$ distributes over $\vee$ we have that

$$\begin{aligned}
a \vee (b \wedge c) &= (a \vee (a \wedge b)) \vee (b \wedge c) \\
&= a \vee ((a \wedge b) \vee (b \wedge c)) \\
&= a \vee (b \wedge (a \vee c)) \\
&= (a \wedge (a \vee c)) \vee (b \wedge (a \vee c)) \\
&= (a \vee b) \wedge (a \vee c),
\end{aligned}$$

so $\vee$ distributes over $\wedge$ too. We will also be applying the distributive law over finite meets and joins, which can be easily proved to be valid by induction.

Let L be a finite distributive lattice. We first show that L is bounded. Indeed, we can define

$$\top \coloneqq \bigvee L,$$

as L is finite. Then by the absorption law and the idempotent law for any $a \in L$ we have

$$a \leq a \vee \left(\bigvee L\right) = \top.$$

Similarly, if we define

$$\bot \coloneqq \bigwedge L$$

then for any $a \in L$ we get

$$a \geq a \wedge \left(\bigwedge L\right) = \bot.$$

So L is bounded.

Now for $b, c \in L$ we define

$$S(b,c) \coloneqq \{x \in L \colon x \wedge b \leq c\}.$$

Note that $c \in S(b,c)$ so $S(b,c)$ is never empty. Set

$$b \Rightarrow c \coloneqq \bigvee S(b,c).$$

Now if $a \in L$ is such that $a \wedge b \leq c$ then $a \in S(b,c)$ and thus $a \leq b \Rightarrow c$. Conversely, suppose $a \leq b \Rightarrow c$. Then, as $\wedge$ preserves ordering, and we have a finite distributive law,

$$\begin{aligned}
a \wedge b &\leq (b \Rightarrow c) \wedge b \\
&= b \wedge \left(\bigvee_{x \in S(b,c)} x \right) \\
&= \bigvee_{x \in S(b,c)} (x \wedge b) \\
&\leq c,
\end{aligned}$$

where the last inequality follows from the fact that $x \wedge b \leq c$ for all $x \in S(b,c)$ and the universal property of finite applications of $\vee$. We hence see that L is a Heyting algebra, as desired.

7. Recall that for a topological space X, the lattice of its open subsets form a Heyting algebra by defining $U \Rightarrow V$ to be $((X \setminus U) \cup V)^\circ$. In particular, $\neg U$ is defined to be $(X \setminus U)^\circ$.

   (a) We work in the Heyting algebra given by the usual topology on $\mathbb{R}^2$. If we set $v(p) = \mathbb{R}^2 \setminus \{(0,0)\}$ and $v(q) = \emptyset$ then we get

$$\begin{aligned}
v(((p \to q) \to p) \to p) &= ((v(p) \Rightarrow v(q)) \Rightarrow v(p)) \Rightarrow v(p) \\
&= ((\{(0,0)\} \cup \emptyset)^\circ \Rightarrow v(p)) \Rightarrow v(p) \\
&= (\emptyset \Rightarrow v(p)) \Rightarrow v(p) \\
&= (\mathbb{R}^2 \cup (\mathbb{R}^2 \setminus \{(0,0)\})) \Rightarrow v(p) \\
&= \mathbb{R}^2 \Rightarrow v(p) \\
&= (\emptyset \cup (\mathbb{R}^2 \setminus \{(0,0)\}))^\circ \\
&= \mathbb{R}^2 \setminus \{(0,0)\} \\
&\neq \mathbb{R}^2 = \top.
\end{aligned}$$

   So this law cannot be valid.

   (b) It is enough to show that $\neg(p \wedge q) \to (\neg p \vee \neg q)$. We work in the space $\mathbb{R}$ with the usual topology. We let $v(p) = (-\infty, 0)$ and $v(q) = (0, \infty)$. Then,

$$\begin{aligned}
v(\neg(p \wedge q) \to (\neg p \vee \neg q)) &= v(\neg(p \wedge q)) \Rightarrow v(\neg p \vee \neg q) \\
&= (\mathbb{R} \setminus v(p \wedge q))^\circ \Rightarrow [(\mathbb{R} \setminus v(p))^\circ \cup (\mathbb{R} \setminus v(q))^\circ] \\
&= (\mathbb{R} \setminus (v(p) \cap v(q)))^\circ \Rightarrow [(0, \infty) \cup (-\infty, 0)] \\
&= \mathbb{R} \Rightarrow \mathbb{R}/\{0\} \\
&= (\mathbb{R} \setminus \mathbb{R} \cup \mathbb{R} \setminus \{0\})^\circ \\
&= \mathbb{R} \setminus \{0\} \\
&\neq \mathbb{R} = \top.
\end{aligned}$$

   Thus the law is not valid.

   (c) Again we work with $\mathbb{R}$ with the usual topology, and let $v(p) = (0, \infty)$ and $v(q) = \mathbb{R} \setminus \{0\}$. Then

$$\begin{aligned}
v((\neg q \to \neg p) \to (p \to q)) &= v(\neg q \to \neg p) \Rightarrow v(p \to q) \\
&= (v(\neg q) \Rightarrow v(\neg p)) \Rightarrow (v(p) \Rightarrow v(q)) \\
&= (\emptyset \Rightarrow (-\infty, 0)) \Rightarrow ((0, \infty) \Rightarrow \mathbb{R} \setminus \{0\}) \\
&= \mathbb{R} \Rightarrow \mathbb{R} \setminus \{0\} \\
&= \mathbb{R} \setminus \{0\} \\
&\neq \mathbb{R} = \top.
\end{aligned}$$

(d) Again we work with $\mathbb{R}$ with the usual topology. Let $v(p) = \mathbb{R}$ and $v(q) = \mathbb{R} \setminus \{0\}$. Then,

$$\begin{aligned} v((\neg\neg p \to \neg\neg q) \to (p \to q)) &= v(\neg\neg p \to \neg\neg q) \Rightarrow v(p \to q) \\ &= [\neg\neg\mathbb{R} \Rightarrow \neg\neg(\mathbb{R} \setminus \{0\})] \Rightarrow (\mathbb{R} \Rightarrow \mathbb{R} \setminus \{0\}) \\ &= (\mathbb{R} \Rightarrow \neg\emptyset) \Rightarrow \mathbb{R} \setminus \{0\} \\ &= \mathbb{R} \Rightarrow \mathbb{R} \setminus \{0\} \\ &= \mathbb{R} \setminus \{0\} \\ &\neq \mathbb{R} = \top. \end{aligned}$$

(e) Again we work with $\mathbb{R}$ with the usual topology. Let $v(p) = \mathbb{R} \setminus \{0\}$ and $v(q) = (0, \infty)$ and $v(r) = (-\infty, 0)$. Then

$$v((p \to (q \vee r) \to (p \to q) \vee (p \to r))) = [v(p) \Rightarrow v(q) \cup v(r)] \Rightarrow [(v(p) \Rightarrow v(q)) \cup (v(p) \Rightarrow v(r))]$$

which equals

$$[\mathbb{R} \setminus \{0\} \Rightarrow \mathbb{R} \setminus \{0\}] \Rightarrow [(\mathbb{R} \setminus \{0\} \Rightarrow (0, \infty)) \cup (\mathbb{R} \setminus \{0\} \Rightarrow (-\infty, 0))],$$

which is

$$\mathbb{R} \Rightarrow (0, \infty) \cup (-\infty, 0) = \mathbb{R} \Rightarrow \mathbb{R} \setminus \{0\},$$

which is easily seen to equal $\mathbb{R} \setminus \{0\}$, which is not $\top$.

8. We prove the contrapositive. Suppose $\nvdash_{IPC} \phi$ and $\nvdash_{IPC} \psi$. Then there must be Kripke models $(S, \leq, \Vdash)$ and $(T, \leq, \Vdash)$ such that there is $s \in S$ and $t \in T$ with $s \nVdash \phi$ and $t \nVdash \psi$. We can assume S and T are disjoint as sets. Now we construct a new Kripke model

$$(S \cup T \cup \{u\}, \leq, \Vdash),$$

by specifying that $s \leq u$ and $t \leq u$ and that u forces everything forced by s and everything forced by t. But then u does not force ψ nor ϕ and thus certainly not $\phi \vee \psi$. This shows that $\nvdash_{IPC} \phi \vee \psi$.

9. TODO

10. (a) Say a proposition ϕ is *stable* if $\vdash_{IPC} \neg\neg\phi \to \phi$ (note that $\phi \to \neg\neg\phi$ is easily provable in intuitionistic logic).

Claim 1. *For all quantifier-free sentences ϕ we have that ϕ^g is stable.*

Proof. By induction on the structure of ϕ. We freely use facts derived in Q5.

If ϕ is atomic then ϕ^g is $\neg\neg\phi$. We have to show that

$$\neg\neg\neg\neg\phi \to \neg\neg\phi.$$

By the triple negation law, this reduces to $\neg\neg\phi \to \neg\neg\phi$, which is clearly provable.

Suppose ψ^g and ϕ^g are stable. As double negation preserves $\wedge$ and $\to$, it is readily seen that $(\psi \wedge \phi)^g$ and $(\psi \to \phi)^g$ are stable. The fact that $(\psi \vee \phi)^g$ is stable follows from triple negation. $\square$

Now we show that if ϕ is a classical tautology then ϕ^g is an intuitionistic one. We prove the more general statement: if $\Gamma \vdash_{CPC} \phi$ then $\Gamma^g \vdash_{IPC} \phi^g$ where Γ is a context and $\Gamma^g := \{\psi^g \mid \psi \in \Gamma\}$.
Recall that we obtain classical logic by adjoining to intuitionistic logic the law of the excluded middle. Hence we can do induction on the classical derivation of ϕ.
If $\phi = \psi \vee \neg\psi$ and we have the derivation

$$\frac{}{\Gamma \vdash \psi \vee \neg\psi}\ \text{LEM}$$

using the law of excluded middle then we need to prove that

$$\begin{aligned}\phi^g &= \neg(\neg\psi^g \wedge \neg(\neg\psi)^g)\\ &= \neg(\neg\psi^g \wedge \neg\neg\psi^g)\end{aligned}$$

can be deduced in intuitionistic logic. But if we assume $\neg\psi^g \wedge \neg\neg\psi^g$ then we are immediately led to contradiction so this is true.
The case for $\bot$-elimination is trivial. If ϕ was derived by assumption

$$\frac{}{\Gamma, \phi \vdash \phi}\ \text{(Ax)}$$

Then clearly

$$\frac{}{\Gamma^g, \phi^g \vdash \phi^g}\ \text{(Ax)}$$

Now suppose $\phi = \psi_1 \to \psi_2$ was derived using the introduction rule for implication

$$\frac{\Gamma, \psi_1 \vdash \psi_2}{\Gamma \vdash \psi_1 \to \psi_2}\ (\to\text{-I})$$

By inductive hypothesis we can assume $\Gamma^g, \psi_1^g \vdash \psi_2^g$. It follows that $\Gamma^g \vdash \psi_1^g \to \psi_2^g$, and now we are done by definition of $(\psi_1 \to \psi_2)^g$.
Suppose we used the elimination rule for implication.

$$\frac{\Gamma \vdash \psi \to \phi \qquad \Gamma \vdash \psi}{\Gamma \vdash \phi}\ (\to\text{-E})$$

By induction hypothesis we assume that $\Gamma^g \vdash \psi^g$ and $\Gamma^g \vdash (\psi \to \phi)^g$. By definition we have $\Gamma^g \vdash \psi^g \to \phi^g$ and thus $\Gamma^g \vdash \phi^g$ as desired.
Suppose $\phi = \psi_1 \wedge \psi_2$ and we used the introduction rule for $\wedge$.

$$\frac{\Gamma \vdash \psi_1 \qquad \Gamma \vdash \psi_2}{\Gamma \vdash \psi_1 \wedge \psi_2}\ (\wedge\text{-I})$$

Then by induction $\Gamma^g \vdash \psi_1^g$ and $\Gamma^g \vdash \psi_2^g$ and thus $\Gamma^g \vdash \psi_1^g \wedge \psi_2^g$. Now we are done by definition of $(\psi_1 \wedge \psi_2)^g$.
Suppose we used one of the elimination rules of $\wedge$. Without loss of generality assume

$$\frac{\Gamma \vdash \psi \wedge \phi}{\Gamma \vdash \phi}\ (\wedge\text{-E})$$

Then $\Gamma^g \vdash \psi^g \wedge \phi^g$ and thus $\Gamma^g \vdash \phi^g$ as desired.

Now suppose we use one of the introduction rules for $\vee$. Without loss of generality assume $\phi = \psi_1 \vee \psi_2$ and

$$\frac{\Gamma \vdash \psi_1}{\Gamma \vdash \psi_1 \vee \psi_2}\ (\vee\text{-I})$$

We assume that $\Gamma^g \vdash \psi_1^g$. We need to show that $\Gamma^g \vdash \neg(\neg\psi_1^g \wedge \neg\psi_2^g)$. This is shown by the following proof tree (here W denotes the weakening rule, which was mentioned in lectures).

$$\dfrac{\dfrac{\dfrac{}{\Gamma^g, \neg\psi_1^g \wedge \neg\psi_2^g \vdash \neg\psi_1^g \wedge \neg\psi_2^g}\ (\text{Ax})}{\Gamma^g, \neg\psi_1^g \wedge \neg\psi_2^g \vdash \neg\psi_1^g}\ (\wedge\text{-E}) \qquad \dfrac{\Gamma^g \vdash \psi_1^g}{\Gamma^g, \neg\psi_1^g \wedge \neg\psi_2^g \vdash \psi_1^g}\ (\text{W})}{\dfrac{\Gamma^g, \neg\psi_1^g \wedge \neg\psi_2^g \vdash \bot}{\Gamma^g \vdash \neg(\neg\psi_1^g \wedge \neg\psi_2^g)}\ (\to\text{-I})}\ (\to\text{-E})$$

So we are done with this case.

Now suppose that we used the elimination rule for $\vee$ as follows.

$$\frac{\Gamma, \psi_1 \vdash \phi \qquad \Gamma, \psi_2 \vdash \phi \qquad \Gamma \vdash \psi_1 \vee \psi_2}{\Gamma \vdash \phi}\ (\vee\text{-E})$$

By induction we can assume that $\Gamma^g, \psi_1^g \vdash \phi^g$, that $\Gamma^g, \psi_2^g \vdash \phi^g$, and $\Gamma^g \vdash \neg(\neg\psi_1^g \wedge \neg\psi_2^g)$. We need to show that $\Gamma^g \vdash \phi^g$. But this is hard to do given the assumptions. Luckily, we proved that ϕ^g is stable, so it suffices to show $\Gamma^g \vdash \neg\neg\phi^g$. We show this using the proof tree at the end of this document. This settles all the cases and completes the induction.

As a corollary we see that if $\vdash_{CPC} \phi$ then $\vdash_{IPC} \phi^g$. The converse is easy to see: if $\vdash_{IPC} \phi^g$ then $\vdash_{CPC} \phi^g$ since intuitionistic proofs work in a classical setting, and a short induction shows that $\phi \leftrightarrow \phi^g$ is provable in classical logic. Hence we have shown that $\vdash_{CPC} \phi$ iff $\vdash_{IPC} \phi^g$. Notice that $\bot^g = \bot$, from which it follows that CPC is inconsistent, iff IPC is.

(b) We consider the sentence $\forall x.\varphi(x) \vee \neg\varphi(x)$. We construct a Kripke model as follows. Let $W = \{w_0, w_1, \ldots\}$ and the structures are defined

$$M_{w_n} \coloneqq \{1, \ldots, n\}$$

and we postulate that M_n models $\varphi(1), \ldots, \varphi(n-1)$. Note that for all n we have $w_n \nVdash \varphi(n)$ since $M \nvDash \varphi(n)$. But for $m > n$ we do have $M_m \vDash \varphi(n)$, so $w_n \nVdash \neg\varphi(n)$. It follows that, $w_n \nVdash \varphi(n) \vee \neg\varphi(n)$, and thus $w_n \nVdash \forall x.(\varphi(x) \vee \neg\varphi(x))$ for all n.

For we to have $w_n \Vdash \neg\neg\forall x.(\varphi(x) \vee \neg\varphi(x))$ we must have that for all $m \geq n$ there is some $m' \geq m$ so that $w_{m'} \Vdash \forall x.(\varphi(x) \vee \neg\varphi(x))$, and this is clearly not the case in this model. Thus $\neg\neg\forall x.(\varphi(x) \vee \neg\varphi(x))$ cannot be an intuitionistic theorem.

11. Don't know how to do the first part... If a proposition is intuitionistically valid then it is H-valid for any finite Heyting algebra H. For the converse, suppose ϕ is not intuitionistically valid. It suffices to find a finite Heyting algebra in which ϕ is not valid. We know that ϕ is not valid in A, the Lindebaum-Tarski algebra (for the empty theory). This means that there is some valuation $v\colon P \to A$ such that $v(\phi) \neq \top$.

However this Heyting algebra A is, in general, infinite. Nevertheless, as we are only interested in the valuation of ϕ, we can take $P' \subseteq P$ to be the finite set of primitive propositions appearing in ϕ, and then consider

$$A' := \langle \operatorname{im} v|_{P'} \rangle$$

the distributive sublattice generated by $\operatorname{im} v|_{P'}$ in A. As A' is finitely generated we have that it is a finite lattice. Moreover, by Q6, A' is a Heyting algebra. Let $v' \colon P \to A'$ be any valuation extending $v|_{P'}$. As the valuation of ϕ only depends on the valuations of P', we see that $v'(\phi) \neq \top$, finishing the proof.

$$
\begin{prooftree}
\AxiomC{$\Gamma^g, \psi_1^g \vdash \phi^g$}
\RightLabel{(W)}
\UnaryInfC{$\Gamma^g, \neg\phi^g, \psi_1^g \vdash \phi^g$}
\AxiomC{}
\RightLabel{(Ax)}
\UnaryInfC{$\Gamma^g, \neg\phi^g, \psi_1^g \vdash \neg\phi^g$}
\RightLabel{($\to$-E)}
\BinaryInfC{$\Gamma^g, \neg\phi^g, \psi_1^g \vdash \bot$}
\RightLabel{($\to$-I)}
\UnaryInfC{$\Gamma^g, \neg\phi^g \vdash \neg\psi_1^g$}
\AxiomC{$\Gamma^g, \psi_2^g \vdash \phi^g$}
\RightLabel{(W)}
\UnaryInfC{$\Gamma^g, \neg\phi^g, \psi_2^g \vdash \phi^g$}
\AxiomC{}
\RightLabel{(Ax)}
\UnaryInfC{$\Gamma^g, \neg\phi^g, \psi_2^g \vdash \neg\phi^g$}
\RightLabel{($\to$-E)}
\BinaryInfC{$\Gamma^g, \neg\phi^g, \psi_2^g \vdash \bot$}
\RightLabel{($\to$-I)}
\UnaryInfC{$\Gamma^g, \neg\phi^g \vdash \neg\psi_2^g$}
\RightLabel{($\wedge$-I)}
\BinaryInfC{$\Gamma^g, \neg\phi^g \vdash \neg\psi_1^g \wedge \neg\psi_2^g$}
\AxiomC{$\Gamma^g \vdash \neg(\neg\psi_1^g \wedge \neg\psi_2^g)$}
\RightLabel{($\to$-I)}
\UnaryInfC{$\Gamma^g, \neg\phi^g \vdash \neg(\neg\psi_1^g \wedge \neg\psi_2^g)$}
\RightLabel{($\to$-E)}
\BinaryInfC{$\Gamma^g, \neg\phi^g \vdash \bot$}
\RightLabel{($\to$-I)}
\UnaryInfC{$\Gamma^g \vdash \neg\neg\phi^g$}
\end{prooftree}
$$