1 Lecture 1

2 Lecture 2

Theorem 2.1 (Projection theorem)**.** *Let $C \subseteq \mathbb{R}^n$ be a closed convex set. For all points $y \in \mathbb{R}^n$ there is a unique $p_C(y) \in C$ so that $||y - x|| \geq ||y - p_C(y)||$ for all $x \in C$.*

Proof. This is a special case of Hilbert's projection theorem, which has an elementary proof on Wikipedia. $\square$

Proposition 2.2 (Obtuse angle criterion)**.** *Let $C \subseteq \mathbb{R}^n$ be a closed convex set and $y \in \mathbb{R}^n$. Then, for all $x \in C$,*

$$\langle y - p_C(y), x - p_C(y)\rangle \leq 0.$$

Proof. Let $\lambda \in (0, 1)$. As x and $p_C(y)$ are in C, so is any convex combination. By definition of $p_C(y)$ we must have

$$\begin{aligned} ||p_C(y) - y||^2 &\leq ||\lambda x + (1-\lambda)p_C(y) - y||^2 \\ &= ||\lambda(x - p_C(y)) - (y - p_C(y))||^2 \\ &= \lambda^2||x - p_C(y)||^2 - 2\lambda\langle x - p_C(y), y - p_C(y)\rangle + ||p_C(y) - y||^2. \end{aligned}$$

It follows by cancelling and rearranging that

$$\langle y - p_C(y), x - p_C(y)\rangle \leq \frac{\lambda}{2}||x - p_C(y)||^2.$$

As λ can be made arbitrarily small, we are done. $\square$

Next we prove the Separating hyperplane theorem. First, a lemma.

Lemma 2.3. *Let $C \subseteq \mathbb{R}^n$ be a convex set. The function $y \mapsto p_C(y)$ is continuous.*

Proof. Let $\varepsilon > 0$ be arbitrary. Suppose $y, y' \in \mathbb{R}^n$ are such that $||y - y'|| \leq \delta$ where $\delta := \varepsilon$. Then, by the Obtuse angle criterion,

$$\begin{aligned} 0 &\geq \langle y - p_C(y), p_C(y') - p_C(y)\rangle \\ &= \langle (y - p_C(y')) + (p_C(y') - p_C(y)), p_C(y') - p_C(y)\rangle \\ &= \langle y - p_C(y'), p_C(y') - p_C(y)\rangle + ||p_C(y') - p_C(y)||^2. \end{aligned}$$

It follows that

$$\begin{aligned} ||p_C(y') - p_C(y)||^2 &\leq \langle p_C(y') - y, p_C(y') - p_C(y)\rangle \\ &= \langle (p_C(y') - y') + (y' - y), p_C(y') - p_C(y)\rangle \\ &= \langle p_C(y') - y', p_C(y') - p_C(y)\rangle + \langle y' - y, p_C(y') - p_C(y)\rangle \\ &\leq \langle y' - y, p_C(y') - p_C(y)\rangle, \end{aligned}$$

where we have used the Obtuse angle criterion in the last inequality. By the Cauchy-Schwarz inequality, we finally obtain

$$||p_C(y') - p_C(y)||^2 \leq ||y' - y|| \cdot ||p_C(y') - p_C(y)||,$$

from which it follows that $||p_C(y') - p_C(y)|| \leq \varepsilon$ as desired. $\square$

Theorem 2.4 (Separating hyperplane theorem)**.** *Let $C \subseteq \mathbb{R}^n$ be a convex set and let $y \notin C$. Then there is $a \in \mathbb{R}^n \setminus \{0\}$ and $b \in \mathbb{R}$ such that for all $x \in C$ we have*

$$\langle a, x \rangle \leq b \quad \textit{and} \quad \langle a, y \rangle \geq b.$$

Proof. The proof for C closed was given in lectures. If C is not closed, take the closure $\bar{C}$ of C and obtain a and b as required. The only case where this does not work is if $y \in \partial C$, so we assume $y \in \partial C$.

Recall that $p \in \partial C$ if and only if every neighbourhood of p contains a point in C and a point not in C. $\square$