

Model Theory and Non-Classical Logic

Example Sheet 3 Solutions

Hernán Ibarra Mejia

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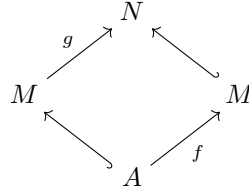
A partial function $f: M \rightarrow N$ is called *partial elementary* if for all $\varphi(\bar{x}) \in \mathcal{L}$ and $\bar{d} \in \text{dom}(f)$

$$M \models \varphi(\bar{d}) \iff N \models \varphi(f(\bar{d})).$$

In particular, these functions are injective (consider $\varphi(x, y)$ to be the formula $(x = y)$). If $f: M \rightarrow N$ is partial elementary and $A := \text{dom } f$ then we call the map $f: A \rightarrow N$ just elementary.

We also use the following strengthening of elementary amalgamation (which is proved in the same way).

Proposition 1. *Let M be a structure. Suppose A is a subset of M and $f: A \rightarrow M$ is an elementary map. Then there exists an elementary extension N of M and an elementary embedding $g: M \rightarrow N$ such that the following diagram commutes.*



Proof.

□

1. Let p be a complete 1-type. Then for all $a \in \mathbb{Q}$ exactly one of $x < a$, $x = a$, and $x > a$ is in p .

Suppose that $(x = a) \in p$ for some $a \in \mathbb{Q}$. We claim that $p = \text{tp}^{\mathbb{Q}}(a/\mathbb{Q})$. By Corollary 1.4.7, there is an elementary extension M of \mathbb{Q} such that $p = \text{tp}^M(a/\mathbb{Q})$, where we used the fact that $(x = a) \in p$. But $\text{tp}^M(a/\mathbb{Q}) = \text{tp}^{\mathbb{Q}}(a/\mathbb{Q})$ as the extension is elementary, so the claim follows.

Now assume that $(x = a) \notin p$ for all $a \in \mathbb{Q}$. Then the sets

$$U := \{a \in \mathbb{Q} \mid (x < a) \in p\} \quad \text{and} \quad L := \{b \in \mathbb{Q} \mid (b < x) \in p\}$$

partition \mathbb{Q} . Note that if $a \in U$ and $b \in L$, then $b < a$. Indeed, the sets are disjoint, so the only alternative is that $a < b$. But from this it follows

that in any realization of the type p , say by an element c , we must have $c < a < b < c$ and thus $c < c$. As $\mathbb{Q} \models \forall y. \neg(y < y)$ this is a contradiction. Hence the complete type p gives rise to a partition U, L of \mathbb{Q} such that $L < U$ in the above sense.

Conversely, any such partition can be extended to a complete type by the Ultrafilter Principle, and it is clear that this correspondence is bijective.

2. We prove a more general result. Let M be a structure and S a subset of M together with an elementary map $f: S \rightarrow M$. Then there is an elementary extension N of M and an automorphism of N extending f .

To see that the result immediately solves the problem, observe the following. As $\text{tp}^M(\bar{a}/A) = \text{tp}^M(\bar{b}/B)$ it follows that the map $f: A \cup \{\bar{a}\} \rightarrow M$ given by fixing A and sending $\bar{a} \mapsto \bar{b}$ is elementary.

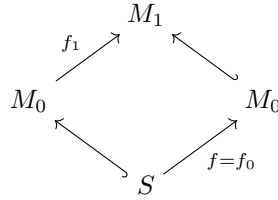
Now we prove the result. We construct a chain of structures and maps $(M_i, f_i)_{i < \omega}$ satisfying the following properties for all $i \geq 0$.

- $M_i \preceq M_{i+1}$.
- $\text{im } f_i \subseteq \text{dom } f_{i+1}$ and f_{i+1} extends f_i .
- If i is even $\text{dom}(f_{i+1}) = M_i$.
- If i is odd $\text{im}(f_{i+1}) = M_i$.

Furthermore we will have $M_0 := M$ and $f_0 = f$.

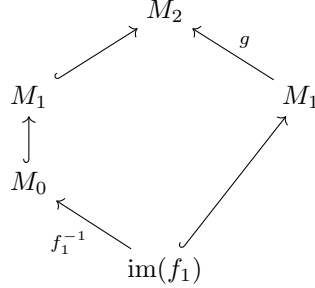
Suppose, for a moment, that we have finished the construction. Then let $N := \bigcup_{i < \omega} M_i$, which is an elementary extension of M . Also let $\sigma := \bigcup_{i < \omega} f_i: N \rightarrow N$, which is clearly an automorphism since $\text{dom}(\sigma) = \text{im}(\sigma) = N$. Also σ extends $f_0 = f$ so we will be done.

We continue with the construction. Use elementary amalgamation to have the following diagram.



It is easy to check that this satisfies the requirements. We can apply amal-

gamation to f_1^{-1} to obtain the following diagram



Define $f_2: \text{im } g \rightarrow M_1$ to be g^{-1} . It follows from the diagram that $g \circ f_1(x) = x$ for $x \in M_0$. Hence we have $f_1(x) = f_2(x)$ for all $x \in M_0$ and so f_2 extends f_1 . Furthermore it is easy to see that f_2 is an isomorphism $\text{im } g \rightarrow M_1$ so in particular it is surjective and so $\text{im}(f_2) = M_1$. Hence all conditions are satisfied. Keep going on in this way (apply amalgamation to f_i when i is even and to f_i^{-1} when i is odd).

3. First we show that, given a finite number of complete types $p_1, \dots, p_k \in S_n^M(M)$ there is an elementary extension N of M realizing all of them. For $k = 0$ this is trivial. Now suppose there is an elementary extension N' of M realizing p_1, \dots, p_{k-1} . Note that $S_n^M(M) = S_n^{N'}(M)$ since the extension is elementary, so in particular $p_k \in S_n^{N'}(M)$ and by Proposition 1.4.6 there is an elementary extension N of N' realizing p_k . Obviously N is an elementary extension of M realizing p_1, \dots, p_k so we are done by induction.

Back to the main problem. To the language \mathcal{L} we add a constant for each element of M and we add n constants c_1^p, \dots, c_n^p for every $p \in S_n^M(M)$. In the expanded language, consider the theory

$$\left(\bigcup S_n^M(M) \right) \cup \text{Diag}_{\text{el}}(M)$$

where each $\varphi(\bar{x}) \in p \in S_n^M(M)$ is replaced by $\varphi(\bar{c}^p)$. Clearly if this theory is consistent then we are done. But every finite subset of this theory is satisfied by an elementary extension of M that has to realize only finitely many types, so we are done by our previous result.

4.

- (a) Let p, q be distinct types. Without loss of generality, we assume that there is a formula $\varphi(\bar{x})$ such that $\varphi \in p$ but $\varphi \notin q$. Then $\llbracket \varphi \rrbracket$ is a clopen set containing p but not q . This shows that $S_n^M(A)$ is totally disconnected.

For the second part, we need a claim.

Claim 1. *Let F be a set of \mathcal{L}_A formulae with n variables. Add n new constants \bar{c} to the language. Then the set $\mathcal{C} := \{ \llbracket \varphi \rrbracket \mid \varphi(\bar{x}) \in F \}$ covers*

$S_n^M(A)$ if and only if the theory

$$\mathcal{T} := \text{Th}_A(M) \cup \{ \neg\varphi(\bar{c}) \mid \varphi(\bar{x}) \in F \}$$

is inconsistent.

Proof. Suppose \mathcal{T} were consistent. Then $\neg F$ is an n -type, which, by the Ultrafilter Principle, can be extended to a complete n -type $q \in S_n^M(A)$. For all $\varphi \in F$ we must have $\neg\varphi \in q$, which means $\varphi \notin q$; thus \mathcal{C} does not cover q .

Conversely, suppose that there is some $q \in S_n^M(A)$ such that $\varphi \notin q$ for all $\varphi \in F$. That means that $\neg\varphi \in q$ for all $\varphi \in F$ since q is complete. By definition of type, we have that $\text{Th}_A(M) \cup q$ is consistent when we replace the variables \bar{x} in q by the constants \bar{c} . It follows that \mathcal{T} is consistent. \square

Back to the problem, let \mathcal{C} be an open cover of $S_n^M(A)$. As open sets are unions of basis elements, we can assume that \mathcal{C} is of the form $\{ \llbracket \varphi \rrbracket \mid \varphi(\bar{x}) \in F \}$ for some set of \mathcal{L}_A -formulae F .

Now we know that \mathcal{T} is inconsistent, where \mathcal{T} is as in Claim 1. By the Compactness Theorem (for first-order logic) there is a finite subset \mathcal{T}' of \mathcal{T} that is inconsistent. Hence there is a finite subset F' of F such that $\text{Th}_A(M) \cup \{ \neg\varphi(\bar{c}) \mid \varphi(\bar{x}) \in F' \}$ is inconsistent. Again by Claim 1, the set $\mathcal{C}' := \{ \llbracket \varphi \rrbracket \mid \varphi(\bar{x}) \in F' \}$, which is a finite subset of \mathcal{C} , covers $S_n^M(A)$.

- (b) I think that he meant to define $f^*(p) := \{ \phi(\bar{x}, f(\bar{a})) \mid \phi(\bar{x}, \bar{a}) \in p \}$, and that we need to show that $f^*(p) \in S_n^N(f(A))$.

To show that $f^*(p) \in S_n^N(f(A))$ first we need to show that

$$\text{Th}_{f(A)}(N) \cup f^*(p)$$

is satisfiable. By assumption p is an n -type, so there is an elementary extension X of M and a tuple $\bar{r} \in X$ with $\phi(\bar{r}, \bar{a})$ for all $\phi(\bar{x}, \bar{a}) \in p$. Note that X can also be interpreted as an $\mathcal{L}_{f(A)}$ -structure. As f is elementary it is clear that $X \models \text{Th}_{f(A)}(N)$ and is immediate $X \models \phi(\bar{r}, f(\bar{a}))$. This all shows that $f^*(p)$ is an n -type; and it is complete since p is complete.

Now we show that f^* is continuous. By general topology, it suffices to show that for each basis element $\llbracket \varphi \rrbracket \subseteq S_n^N(f(A))$ the set $(f^*)^{-1}(\llbracket \varphi \rrbracket)$ is open in $S_n^M(A)$.

So, let $\varphi(\bar{x}, f(\bar{a}))$ be an $\mathcal{L}_{f(A)}$ -formula. Then

$$\begin{aligned} (f^*)^{-1}(\llbracket \varphi \rrbracket) &= \{ p \in S_n^M \mid f^*(p) \in \llbracket \varphi \rrbracket \} \\ &= \{ p \in S_n^M \mid \varphi \in f^*(p) \} \\ &= \{ p \in S_n^M \mid \varphi(\bar{x}, f(\bar{a})) = \psi(\bar{x}, f(\bar{a})) \text{ for some } \psi(\bar{x}, \bar{a}) \in p \}. \end{aligned}$$

But notice that, as f is injective, $\varphi(\bar{x}, f(\bar{a})) = \psi(\bar{x}, f(\bar{a}))$ implies that $\psi = \phi$. Thus,

$$(f^*)^{-1}(\llbracket \varphi \rrbracket) = \{p \in S_n^M \mid \varphi(\bar{x}, \bar{a}) \in p\} = \llbracket \varphi(\bar{x}, \bar{a}) \rrbracket$$

which is open in $S_n^M(A)$.

5. In the proof of Theorem 1.4.11 we make a small modification. Let $f: \omega \rightarrow \omega \times \omega$ be a bijection. When we define θ_s for odd $s = 2i+1$ then if $f(i) = (j, k)$ instead of taking \bar{d}_i we take \bar{d}_j , run the same process to get ψ and we let φ be a formula in p_k that is not implied by ψ . The rest of the proof is the same, except that at the very end we notice that if $\bar{c} \in C^n$ then $\bar{c} = \bar{d}_j$ for some j and that if $k < \omega$ then we can define $i := f^{-1}(j, k)$; it then follows that θ_{2i+2} implies that \bar{c} does not realize p_k . As \bar{c} and k were arbitrary, we are done.
6. Let \mathcal{L} be the language which has as signature \aleph_2 constants, say $\{c_\alpha\}_{\alpha < \aleph_2}$. Let \mathcal{T} be the theory of uncountable sets, i.e.

$$\mathcal{T} := \{c_\alpha \neq c_{\alpha'} \mid \alpha < \alpha' < \aleph_1\}.$$

Define

$$p := \{x \neq c_\alpha \mid \alpha < \aleph_2\}.$$

It is easy to check that this is a 1-type of \mathcal{T} (any set of cardinality bigger than \aleph_2 is a model of \mathcal{T} realizing p).

For the sake of contradiction, suppose this type is isolated by a formula $\varphi(x)$. This means that $\mathcal{T} \cup \{\varphi(x)\}$ is satisfiable and

$$\mathcal{T} \models \forall x.(\varphi(x) \rightarrow x \neq c_\alpha)$$

for all $\alpha < \aleph_2$. Choose α such that c_α does not appear in \mathcal{T} nor in $\varphi(x)$; this is possible because \mathcal{T} mentions only \aleph_1 -many constants and $\varphi(x)$ only finitely many. Hence by generalization we have

$$\mathcal{T} \models \forall x, y.(\varphi(x) \rightarrow x \neq y).$$

In particular,

$$\mathcal{T} \models \forall x.(\varphi(x) \rightarrow x \neq x).$$

contradicting the fact that $\mathcal{T} \cup \varphi(x)$ is satisfiable. Thus p is not isolated. However there can be no countable model of \mathcal{T} omitting p since \mathcal{T} has no countable models!

7. Let \mathcal{M} be a countable model of PA. If \mathcal{L} denotes the language of arithmetic, we add a constant c to the language and let $\mathcal{L}^+ := \mathcal{L}_{\{c\} \cup \mathcal{M}}$. We define the \mathcal{L}^+ -theory

$$\mathcal{T} := \text{Diag}_{\text{el}}(\mathcal{M}) \cup \{c > m \mid m \in \mathcal{M}\}.$$

This theory is clearly consistent by Compactness.

Say an element $m \in \mathcal{M}$ is *natural* if the interval $[0, m]$ is finite. If m is natural then there are $m_1, \dots, m_k \in \mathcal{M}$ such that

$$\mathcal{M} \models \forall x. \left(\bigwedge_{i=1}^k x \neq m_i \rightarrow x > m \right)$$

If m_0 is not natural we say it is *unnatural*. For every unnatural $m_0 \in \mathcal{M}$ we define

$$p_{m_0} := \{x \neq m \mid m \in \mathcal{M}\} \cup \{x < m_0\}.$$

We claim that p_{m_0} is a non-isolated type over \mathcal{T} for all unnatural $m_0 \in \mathcal{M}$. Firstly, if $p \subseteq p_{m_0}$ is finite then $\mathcal{T} \cup p$ is satisfiable precisely because $[0, m_0]$ is infinite, so p_{m_0} is a 1-type over \mathcal{T} .

Suppose, for the sake of contradiction, that p_{m_0} is isolated by an \mathcal{L}^+ -formula $\varphi(x)$. Write $\varphi(x) = \psi(x, c)$ where ψ is an $\mathcal{L}_{\mathcal{M}}$ -formula. As φ is an isolating formula we have that $\mathcal{T} \cup \{\varphi(x)\}$ is satisfiable. So let \mathcal{M}' be a satisfying structure. Clearly \mathcal{M}' is an elementary extension of \mathcal{M} that includes two constants $c^{\mathcal{M}'}, d^{\mathcal{M}'}$ such that $\mathcal{M}' \models c^{\mathcal{M}'} > m$ for all $m \in \mathcal{M}$ and $\mathcal{M}' \models \psi(d^{\mathcal{M}'}, c^{\mathcal{M}'})$. As $\varphi(x)$ isolates p_{m_0} we have in particular that $\psi(x, c)$ implies that $x < m_0$. Thus

$$\mathcal{M}' \models \psi(d^{\mathcal{M}'}, c^{\mathcal{M}'}) \wedge d^{\mathcal{M}'} < m_0.$$

It follows that

$$\mathcal{M}' \models \exists z < m_0. \psi(z, c^{\mathcal{M}'}).$$

Furthermore, for all $m \in \mathcal{M}$ we have that

$$\mathcal{M}' \models \exists y > m \exists z < m_0. \psi(z, y).$$

Thus, as \mathcal{M} is an elementary substructure of \mathcal{M}' we have that

$$\mathcal{M} \models \exists y > m. \exists z < m_0. \psi(z, y),$$

for all $m \in \mathcal{M}$. Hence,

$$\mathcal{M} \models \forall x. \exists y > x. \exists z < m_0. \psi(z, y).$$

Now, PA proves the pigeonhole principle. Think of z as the pigeonholes, y as the pigeons, and $\psi(z, y)$ as the statement “pigeon y is assigned pigeonhole z ”; then it is clear that, as we are trying to fit infinitely many pigeons in finitely many pigeonholes there is a pigeonhole with arbitrarily many pigeons (of course $[0, m_0]$ is not actually finite but PA proves the pigeonhole principle nevertheless). Thus,

$$\mathcal{M} \models \exists z < m_0. \forall x. \exists y > x. \psi(z, y).$$

In particular, there is some $m_1 \in \mathcal{M}$ such that $m_1 < m_0$ and

$$\mathcal{M} \models \forall x. \exists y > x. \psi(m_1, y).$$

We claim that $\mathcal{T} \cup \{\psi(m_1, c)\}$ is consistent. If it were inconsistent then, by Compactness, there is $n \in \mathcal{M}$ such that

$$\text{Diag}_{\text{el}}(M) \cup \{c > n\} \models \neg\psi(m_1, c),$$

and thus $\text{Diag}_{\text{el}}(M) \models c > n \rightarrow \neg\psi(m_1, c)$. As $\text{Diag}_{\text{el}}(M)$ does not mention c we have, by generalization,

$$\text{Diag}_{\text{el}}(M) \models \forall y > n. \neg\psi(m_1, c),$$

contradicting the defining property of m_1 . We conclude that $\mathcal{T} \cup \{\psi(m_1, c)\}$ is consistent. As $\varphi(x)$ isolates p_{m_0} we also must have

$$\mathcal{T} \models \psi(m_1, c) \rightarrow m_1 \neg m_1,$$

a contradiction. Thus none of the types p_{m_0} is isolated. By Q5, and the fact that \mathcal{L}^+ is countable, there is a countable model \mathcal{N} of \mathcal{T} omitting all p_{m_0} for $m_0 \in \mathcal{M}$ unnatural.

We claim that \mathcal{N} is a proper end extension. It is clearly an (elementary) extension, and it is proper since $c^{\mathcal{N}}$ is greater than all elements of \mathcal{M} . If it weren't an end extension then there must be some $n \in \mathcal{N} \setminus \mathcal{M}$ and $m_0 \in \mathcal{M}$ such that $n < m_0$. Clearly this cannot happen for m_0 unnatural since \mathcal{N} omits p_{m_0} . So suppose m_0 is natural. But then there are $m_1, \dots, m_k \in \mathcal{M}$ such that

$$\mathcal{M} \models \forall x. \left(\bigwedge_{i=1}^k x \neq m_i \rightarrow x > m_0 \right).$$

As \mathcal{N} is an elementary extension, we have that \mathcal{N} also models this sentence, and thus $n > m_0$ as well as $n < m_0$, a contradiction. Thus \mathcal{N} is a proper end extension of \mathcal{M} .

8. Let $M := \{m_1, m_2, \dots\}$ and $N := \{n_1, n_2, \dots\}$ be two countable ω -saturated, elementarily equivalent \mathcal{L} -structures. We construct a sequence f_0, f_1, \dots such that for all $i \in \mathbb{N}$:

- f_i is an elementary partial function $M \rightarrow N$;
- f_{i+1} extends f_i ;
- $\text{dom}(f_i)$ (and hence $\text{cod}(f_i)$) is finite;
- $\{m_1, \dots, m_i\} \subseteq \text{dom}(f_i)$ and $\{n_1, \dots, n_i\} \subseteq \text{cod}(f_i)$.

Define f_0 to be the empty function, which is elementary since M and N are elementarily equivalent. Suppose f_i has been defined, and let $D := \text{dom}(f_i)$ and $C := \text{cod}(f_i)$ be finite. Consider the complete 1-type

$$\text{tp}^M(m_{i+1}/D).$$

Using the notation of Question 4 (b), we note that $f_i^*(p)$ is a complete 1-type of N by basically the same argument as in Q4 and the fact that f_i

is elementary. Since N is ω -saturated, it follows that there is some $n \in N$ realizing this type. Let $g: D \cup \{m_{i+1}\} \rightarrow C \cup \{n\}$ be the extension of f_i that sends $m_{i+1} \mapsto n$ (if $m_{i+1} \in D$ then just let $g = f_i$). Thus g is elementary by construction.

Similarly, consider the complete 1-type $p := \text{tp}^N(n_{i+1}/C \cup \{n\})$. As g is elementary it follows that

$$\{\varphi(x, \bar{d}) : \varphi(x, g(\bar{d})) \in p \text{ for some } \bar{d} \in D \cup \{m_{i+1}\}\}$$

is a 1-type for M so it has a realization $m \in M$. Finally, we let $f_{i+1}: D \cup \{m, m_{i+1}\} \rightarrow C \cup \{n, n_{i+1}\}$ be the extension of g mapping $m \mapsto n_{i+1}$. For the same reasons as before, f_{i+1} is elementary. This finishes the construction.

Now let $f: M \rightarrow N$ be the union of all f_i . By construction, f is defined everywhere, elementary, and surjective. It is also injective since $N \models f(m) = f(m')$ will imply $M \models m = m'$. The fact that f is a homomorphism can be similarly verified.

9. We assume \mathcal{T} is, in addition, consistent, since otherwise the result is trivial.

Suppose there are finitely many equivalence classes for formulae (of a given arity). We claim that countable \mathcal{T} -models are ω -saturated. Indeed, suppose $M \models \mathcal{T}$ is a model and let $\bar{a} \subseteq M$ be a finite tuple. Let $p \in S_n^M(\bar{a})$; we have to show that M realizes p .

By assumption we can consider p as a finite set of formulae to be satisfied, since $M \models \mathcal{T}$ and \mathcal{T} has finitely many equivalence classes for formulae of a given arity, and the arity of formulae in p is bounded by $n + |\bar{a}|$. By taking the conjunction of all these formulae, we are left with a single formulae $\varphi(\bar{x}, \bar{a})$ and we have to show that $M \models \exists \bar{x}.\varphi(\bar{x}, \bar{a})$. But as p is a type, we know it is realized in an elementary extension N of M , and so $N \models \exists \bar{x}.\varphi(\bar{x}, \bar{a})$. As the extension is elementary we get that M realizes p . This shows that all models of \mathcal{T} are ω -saturated.

Now, any two countable models of \mathcal{T} are elementarily equivalent (since \mathcal{T} is complete) and ω -saturated, thus isomorphic by Q8. Hence \mathcal{T} is \aleph_0 -categorical, as desired.

Conversely, suppose there is some n such that there are infinitely many \mathcal{T} -equivalence classes of formulae with n variables $\bar{x} = (x_1, \dots, x_n)$. We claim the following.

Claim 2. *All models of \mathcal{T} are infinite.*

Proof. Suppose \mathcal{T} has a finite model M . Then M thinks there are only finitely many equivalence classes of formulae with n -variables. Indeed, the equivalence class of a formulae $\varphi(\bar{x})$ is determined by its truth value on its inputs, i.e. we can see the formula as a Boolean function $M^n \rightarrow \{0, 1\}$, and there are only finitely many of those since M is finite. Hence there are formulae $\varphi_1(\bar{x}), \dots, \varphi_k(\bar{x})$ such that for all formulae $\varphi(\bar{x})$ there is some $i \leq k$ with

$$M \models \forall \bar{x}.(\varphi(\bar{x}) \leftrightarrow \varphi_i(\bar{x})).$$

As \mathcal{T} is complete we see that the same is true if we replace M by \mathcal{T} in the above, and thus there are only finitely many \mathcal{T} -equivalence classes of formulae with n variables, a contradiction. \square

Now, the condition on the equivalence classes can be equivalently stated to say that the Lindenbaum-Tarski algebra $\mathcal{B}_n(\mathcal{T})$ is infinite. Hence the Fréchet filter on $\mathcal{B}_n(\mathcal{T})$ is proper and can be extended to a free ultrafilter on $\mathcal{B}_n(\mathcal{T})$ (this is by ES2 Q5(b)). This ultrafilter corresponds to a complete type p , and the fact that is free means that p is not isolated (!).

By the omitting types theorem (here we use the countability of the language), there is a countable model $N \models \mathcal{T}$ that omits p . By definition of types there is a model $M \models \mathcal{T}$ that realizes p , say with a tuple $\bar{m} \in M$. By our claim M is infinite. Using the downwards Löwenheim–Skolem theorem (again using $|\mathcal{L}| = \aleph_0$) we can find an elementary substructure M' of M such that M' is countable and $\bar{m} \in M'$. As the substructure is elementary, we have $M' \models \mathcal{T}$ and that \bar{m} realizes p in M' .

But then M' and N are countable models of \mathcal{T} , and one of the realizes p while the other ones omits p . We conclude that \mathcal{T} is *not* \aleph_0 -categorical.

Consider a theory \mathcal{S} with quantifier elimination in a language whose signature has finitely many relation symbols (of arity at least 1) and no function symbols. The language clearly is countable. We claim that \mathcal{S} is complete. Indeed, it suffices to decide quantifier-free sentences, but these are only Boolean combinations of \perp by the signature, so this is trivial.

Again, by quantifier-elimination, each formula is equivalent to a quantifier-free formula, and using the disjunctive normal form and by analysing all the possible literals we can reduce everything to finitely many formulae.

10. Let ε be an infinite linear order contained in an infinite model of \mathcal{T} . It follows from the Ehrenfeucht–Mostowski theorem (by Skolemizing the language first) that there is an Ehrenfeucht–Mostowski functor F such that $\text{Th}(F)$ expands $\text{Th}(M, \eta)$. As a particular case of this, the sentences satisfied in M are also satisfied in $\text{Th}(F)$. More particularly, F takes values in \mathcal{T} -models.

Note that there is a homomorphism $G \rightarrow F(\eta)$ given by $g \mapsto F(g)$; that this is a homomorphism follows from functoriality of F . But if $F(g) = F(h)$ then, as F extends maps, we have that $g = h$, and so this homomorphism identifies G with a subgroup of $\text{Aut}(F(\eta))$.

11. Let T be the set of all closed terms in \mathcal{L} , and define an equivalence relation on T by saying that for all $s, t \in T$

$$s \sim t \iff s = t \in \Gamma.$$

We claim that this is an equivalence relation. It is reflexive by the first condition of $=$ -closed. Suppose $s = t \in \Gamma$. If we define $\varphi(x) := (x = s)$ then, as $\varphi(s) \in \Gamma$ we have $\varphi(t) = (t = s) \in \Gamma$; hence the relation is symmetric.

Finally, suppose $s = t$ and $t = u$ are in Γ . Let $\psi(x) := x = u$. Then as $s = t$ and $\psi(t) \in \Gamma$ it follows that $\psi(s) \in \Gamma$, and so $s = u$ is in Γ ; this shows that the relation is transitive.

Define $M := T/\sim$, and if $t \in T$ we denote by $[t]$ the corresponding equivalence class. We make M into an \mathcal{L} -structure as follows. For a constant c we define $c^M := [c]$. For a function symbol f of arity n we inductively define

$$f([t_1], \dots, [t_n]) = [f(t_1, \dots, t_n)].$$

We need to check that this assignment is well-defined. Suppose $t_1, \dots, t_n, t'_1, \dots, t'_n$ are such that $t_i \sim t'_i$ for all i . It can be proved by induction on n that

$$f(t_1, \dots, t_n) \sim f(t'_1, \dots, t'_n).$$

For relation symbols R we define

$$([t_1], \dots, [t_n]) \in R^M \iff R(t_1, \dots, t_n) \in T.$$

It can be checked that this is well-defined and thus M becomes an \mathcal{L} -structure. The rest can be easily shown by induction over the structure of formulae.

12.

- (a) Clearly the formula $x = x$ is contained in $\text{Th}(M, \omega)$ so $S(\eta)$ contains $t = t$ for every closed \mathcal{L}_η -term. Suppose $S(\eta)$ contains $\phi(s(\bar{c}), \bar{c})$ and $s(\bar{c}) = t(\bar{c})$, where s, t are \mathcal{L} -terms, ϕ is an atomic \mathcal{L} -formula, and $\bar{c} \in [\eta]^k$. It follows that if $\bar{d} \in [\omega]^k$ then

$$M \models \phi(s(\bar{d}), \bar{d}) \wedge s(\bar{d}) = t(\bar{d}),$$

so $M \models \phi(t(\bar{d}), \bar{d})$ and hence $\phi(t(\bar{x}), \bar{x})$ is in $\text{Th}(M, \omega)$, from which the claim follows.

- (b) Use Q11 on $S(\eta)$ to obtain an \mathcal{L} -structure $F(\eta)$ such that the set of atomic \mathcal{L}_η -sentences true in $F(\eta)$ is exactly $S(\eta)$, and whose every element is the interpretation of some closed term in \mathcal{L}_η .

It follows that $F(\eta)$ contains a copy of η by its interpretation of constants, and this is a faithful copy since $x = y$ is obviously not in $\text{Th}(M, \omega)$. Furthermore for every closed \mathcal{L} -term t there is an \mathcal{L} -term s and some $\bar{d} \in [\omega]^k$ such that

$$M \models t = s(\bar{d})$$

since M is generated by ω ; it follows readily that $F(\eta)$ must be generated by η .

If $g: \eta \rightarrow \varepsilon$ is an order embedding then for every atomic formula ϕ (and hence for every quantifier free formula by induction) if $\bar{c} \in [\eta]^k$ then $F(\eta) \models \phi(\bar{c})$ implies $\phi(\bar{x}) \in \text{Th}(M, \omega)$ which in turn implies that $F(\varepsilon) \models \phi(g(\bar{c}))$ where we crucially use the fact that g respects ordering. Using

the method of diagrams we get some embedding $F(\eta) \rightarrow F(\varepsilon)$ which extends (and indeed it is determined by) g . This makes sure that the assignment is functorial so this is indeed an EM functor. Furthermore, $F(\omega)$ is generated by ω , from which it follows that $F(\omega) = M$.

- (c) By the Lemma right after the definition of $\text{Th}(F)$, we have that all quantifier-free sentences in $\text{Th}(M, \omega)$ are contained in both $\text{Th}(F)$ and $\text{Th}(G)$. But then for every quantifier-free formula $\varphi(\bar{x})$ and $\bar{c} \in [\eta]^k$ we get that

$$F(\eta) \models \varphi(\bar{a}) \iff G(\eta) \models \varphi(\bar{a}).$$

As η is a generating set, we get again by the method of diagrams an isomorphism $F(\eta) \rightarrow G(\eta)$ fixing η .

13. Let $M = F(\eta)$. It suffices to show that there are at most $|\mathcal{L}|$ many sets $\text{tp}^M(\bar{a}/M)$ as $\bar{a} \in [\eta]^n$ ranges. Since η generates M we have that any \bar{a} can be written as an n -tuple of \mathcal{L} -terms closed by some choice of elements of η . By indiscernibility, the complete type of \bar{a} does not depend on the choice of elements of η , only on the terms. And there are at most $n \times |\mathcal{L}| = |\mathcal{L}|$ of choosing these.