Model Theory and Non-Classical Logic
Example Sheet 3 Solutions

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1. Let p be a complete 1-type. Then for all $a \in \mathbb{Q}$ exactly one of $x < a$, $x = a$, and $x > a$ is in p.

   Suppose that $(x = a) \in p$ for some $a \in \mathbb{Q}$. We claim that $p = \mathrm{tp}^{\mathbb{Q}}(a/\mathbb{Q})$. By Corollary 1.4.7, there is an elementary extension M of $\mathbb{Q}$ such that $p = \mathrm{tp}^{M}(a/\mathbb{Q})$, where we used the fact that $(x = a) \in p$. But $\mathrm{tp}^{M}(a/\mathbb{Q}) = \mathrm{tp}^{\mathbb{Q}}(a/\mathbb{Q})$ as the extension is elementary, so the claim follows.

   Now assume that $(x = a) \notin p$ for all $a \in \mathbb{Q}$. Then the sets

   $$U \coloneqq \{a \in \mathbb{Q} \mid (x < a) \in p\} \quad \text{and} \quad L \coloneqq \{b \in \mathbb{Q} \mid (b < x) \in p\}$$

   partition $\mathbb{Q}$. Note that if $a \in U$ and $b \in L$, then $b < a$. Indeed, the sets are disjoint, so the only alternative is that $a < b$. But from this it follows that in any realization of the type p, say by an element c, we must have $c < a < b < c$ and thus $c < c$. As $\mathbb{Q} \models \forall y.\neg(y < y)$ this is a contradiction. Hence the complete type p gives rise to a partition U, L of $\mathbb{Q}$ such that $L < U$ in the above sense.

   Conversely, suppose we had a partition

2.

3. First we show that, given a finite number of complete types $p_1, \dots, p_k \in S_n^M(M)$ there is an elementary extension N of M realizing all of them. For $k = 0$ this is trivial. Now suppose there is an elementarily extension N' of M realizing $p_1, \dots, p_{k-1}$. Note that $S_n^M(M) = S_n^M(N')$ since the extension is elementary, so in particular $p_k \in S_n^M(N')$ and by Proposition 1.4.6 there is an elementary extension N of N' realizing p_k. Obviously N is an elementary extension of M realizing $p_1, \dots, p_k$ so we are done by induction.

   Back to the main problem. To the language $\mathcal{L}$ we add a constant for each element of M and we add n constants $c_1^p, \dots, c_n^p$ for every $p \in S_n^M(M)$. In the expanded language, consider the theory

   $$\left(\bigcup S_n^M(M)\right) \cup \mathrm{Diag}_{\mathrm{el}}(M)$$

where each $\varphi(\bar{x}) \in p \in S_n^N$ is replaced by $\varphi(\bar{c}^p)$. Clearly if this theory is consistent then we are done. But every finite subset of this theory is satisfied by an elementary extension of M that has to realize only finitely many types, so we are done by our previous result.

4.

(a) Let p, q be distinct types. Without loss of generality, we assume that there is a formula $\varphi(\bar{x})$ such that $\varphi \in p$ but $\varphi \notin q$. Then $[\![\varphi]\!]$ is a clopen set containing p but not q. This shows that $S_n^M(A)$ is totally disconnected.

For the second part, we need a claim.

Claim 1. *Let F be a set of $\mathcal{L}_A$ formulae with n variables. Add n new constants $\bar{c}$ to the language. Then the set $\mathcal{C} \coloneqq \{ [\![\varphi]\!] \mid \varphi(\bar{x}) \in F \}$ covers $S_n^M(A)$ if and only if the theory*

$$\mathcal{T} \coloneqq \mathrm{Th}_A(M) \cup \{ \neg\varphi(\bar{c}) \mid \varphi(\bar{x}) \in F \}$$

is inconsistent.

Proof. Suppose $\mathcal{T}$ were consistent. Then $\neg F$ is an n-type, which, by the Ultrafilter Principle, can be extended to a complete n-type $q \in S_n^M(A)$. For all $\varphi \in F$ we must have $\neg\varphi \in q$, which means $\varphi \notin q$; thus $\mathcal{C}$ does not cover q.

Conversely, suppose that there is some $q \in S_n^M(A)$ such that $\varphi \notin q$ for all $\varphi \in F$. That means that $\neg\varphi \in q$ for all $\varphi \in F$ since q is complete. By definition of type, we have that $\mathrm{Th}_A(M) \cup q$ is consistent when we replace the variables $\bar{x}$ in q by the constants $\bar{c}$. It follows that $\mathcal{T}$ is consistent. □

Back to the problem, let $\mathcal{C}$ be an open cover of $S_n^M(A)$. As open sets are unions of basis elements, we can assume that $\mathcal{C}$ is of the form $\{ [\![\varphi]\!] \mid \varphi(\bar{x}) \in F \}$ for some set of $\mathcal{L}_A$-formulae F.

Now we know that $\mathcal{T}$ is inconsistent, where $\mathcal{T}$ is as in Claim 1. By the Compactness Theorem (for first-order logic) there is a finite subset $\mathcal{T}'$ of $\mathcal{T}$ that is inconsistent. Hence there is a finite subset F' of F such that $\mathrm{Th}_A(M) \cup \{\neg\varphi(\bar{c}) \mid \varphi(\bar{x}) \in F'\}$ is inconsistent. Again by Claim 1, the set $\mathcal{C}' \coloneqq \{ [\![\varphi]\!] \mid \varphi(\bar{x}) \in F' \}$, which is a finite subset of $\mathcal{C}$, covers $S_n^M(A)$.

(b) I think that he meant to define $f^*(p) \coloneqq \{\phi(\bar{x}, f(\bar{a})) \mid \phi(\bar{x}, \bar{a}) \in p\}$, and that we need to show that $f^*(p) \in S_n^N(f(A))$.

To show that $f^*(p) \in S_n^N(f(A))$ first we need to show that

$$\mathrm{Th}_{f(A)}(N) \cup f^*(p)$$

is satisfiable. By assumption p is an n-type, so there is an elementary extension X of M and a tuple $\bar{r} \in X$ with $\phi(\bar{r}, \bar{a})$ for all $\phi(\bar{x}, \bar{a}) \in p$. Note that X can also be interpreted as an $\mathcal{L}_{f(A)}$-structure. As f is elementary it is clear that $X \models \mathrm{Th}_{f(A)}(N)$ and is immediate $X \models \phi(\bar{r}, f(\bar{a}))$. This all shows that $f^*(p)$ is an n-type; and it is complete since p is complete.

Now we show that f^* is continuous. By general topology, it suffices to show that for each basis element $[\![\varphi]\!] \subseteq S_n^N(f(A))$ the set $(f^*)^{-1}([\![\varphi]\!])$ is open in $S_n^M(A)$.

So, let $\varphi(\bar{x}, f(\bar{a}))$ be an $\mathcal{L}_{f(A)}$-formula. Then

$$\begin{aligned}
(f^*)^{-1}([\![\varphi]\!]) &= \{p \in S_n^M \mid f^*(p) \in [\![\varphi]\!]\} \\
&= \{p \in S_n^M \mid \varphi \in f^*(p)\} \\
&= \{p \in S_n^M \mid \varphi(\bar{x}, f(\bar{a})) = \psi(\bar{x}, f(\bar{a})) \text{ for some } \psi(\bar{x}, \bar{a}) \in p\}.
\end{aligned}$$

But notice that, as f is injective, $\varphi(\bar{x}, f(\bar{a})) = \psi(\bar{x}, f(\bar{a}))$ implies that $\psi = \phi$. Thus,

$$(f^*)^{-1}([\![\varphi]\!]) = \{p \in S_n^M \mid \varphi(\bar{x}, \bar{a}) \in p\} = [\![\varphi(\bar{x}, \bar{a})]\!]$$

which is open in $S_n^M(A)$.