

# Category Theory

Hernán Ibarra Mejía

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This is a set of lecture notes taken by me from the Part III course “Category Theory”, lectured by Professor Peter Johnstone in Michaelmas, 2023. I take full responsibility for any mistakes in these notes. Since Professor Johnstone provides his own notes, I will be writing ‘skeletal lecture notes’, i.e. only definitions, theorems, and proofs with little by way of motivation and almost no examples. I might expand on the proofs and statements given without rigorous proofs more than Professor Johnstone does, but that’s my only contribution in terms of content.

## 1 Definitions and Examples

**Definition 1.1** (Categories). A *category*  $\mathcal{C}$  consists of the following.

1. A collection  $\text{ob } \mathcal{C}$  of objects  $A, B, C, \dots$
2. A collection  $\text{mor } \mathcal{C}$  of morphisms  $f, g, h, \dots$
3. Two operations  $\text{dom}, \text{cod}: \text{mor } \mathcal{C} \rightarrow \text{ob } \mathcal{C}$ . We write  $f: A \rightarrow B$  to mean  $\text{dom } f = A$  and  $\text{cod } f = B$ .
4. An operation  $\text{id}: \text{ob } \mathcal{C} \rightarrow \text{mor } \mathcal{C}$  so that  $\text{dom}(\text{id } A) = \text{cod}(\text{id } A) = A$ . We write  $1_A$  for  $\text{id } A$ .
5. A partial binary operation on  $\text{mor } \mathcal{C}$  called composition. We usually denote it by concatenation i.e.,  $(f, g) \mapsto fg$ .

Categories are also required to satisfy the following axioms.

1. The composition  $fg$  is defined if and only if  $\text{dom } f = \text{cod } g$ .
2. We have that  $\text{dom } fg = \text{dom } g$  and  $\text{cod } fg = \text{cod } f$ .
3. If the composites are defined, we have  $f1_A = f$  and  $1_Ag = g$ .
4. Whenever  $fg$  and  $gh$  are defined we have  $f(gh) = (fg)h$ .

For any category  $\mathcal{C}$  there is the *opposite category*  $\mathcal{C}^{\text{op}}$  with the same objects as  $\mathcal{C}$  and morphisms  $f: A \rightarrow B$  in  $\mathcal{C}^{\text{op}}$  are just morphisms  $f: B \rightarrow A$  in  $\mathcal{C}$ . It is easily verified that this gives a category.

**Definition 1.2** (Subcategories). Let  $\mathcal{C}$  be a category. We say  $\mathcal{D}$  is a *subcategory* of  $\mathcal{C}$  if  $\mathcal{D}$  is a category with  $\text{ob } \mathcal{D} \subseteq \text{ob } \mathcal{C}$  and  $\text{mor } \mathcal{D} \subseteq \text{mor } \mathcal{C}$  so that the operations of domain, codomain, identity, and composition in  $\mathcal{D}$  are just the restriction of those of  $\mathcal{C}$ .

**Definition 1.3** (Isomorphisms). Let  $\mathcal{C}$  be a category. An *isomorphism* is a morphism  $f: A \rightarrow B$  in  $\mathcal{C}$  so that there exists some other morphism  $g: B \rightarrow A$  with the property that

$$gf = 1_A \quad \text{and} \quad fg = 1_B.$$

In this case we call  $g$  the *inverse* of  $f$ . If there is an isomorphism  $A \rightarrow B$  we say that  $A$  and  $B$  are *isomorphic* and write  $A \cong B$ .

**Proposition 1.4.** *Inverses of isomorphisms are unique.*

*Proof.* Suppose  $f: A \rightarrow B$  is an isomorphism in some category, and let  $g, g': B \rightarrow A$  be inverses of  $f$ . Then,

$$g = g1_B = g(fg') = (gf)g' = 1_A g' = g'. \quad \square$$

Note that in the above proof we have used all axioms for categories. Given that inverses are unique, we will denote the unique inverse of an isomorphism  $f$  by  $f^{-1}$ .

**Proposition 1.5.** *In all categories  $\cong$  is an equivalence relation.*

*Proof.* For the rest of the proof, fix a category  $\mathcal{C}$  and let  $A, B, C \in \text{ob } \mathcal{C}$  be arbitrary. Note that  $A \cong A$  since the identity morphism  $1_A: A \rightarrow A$  is an isomorphism (this is because  $1_A 1_A = 1_A$ ).

Suppose  $A \cong B$  and let  $f: A \rightarrow B$  be an isomorphism. Then we have

$$f^{-1}f = 1_A \quad \text{and} \quad ff^{-1} = 1_B,$$

which shows that  $f^{-1}: B \rightarrow A$  is an isomorphism and hence  $B \cong A$ .

Finally, suppose that  $A \cong B$  and  $B \cong C$ . Let  $f: A \rightarrow B$  and  $g: B \rightarrow C$  be isomorphisms. We claim that  $gf$  is an isomorphism. Indeed,

$$(gf)(f^{-1}g^{-1}) = g((ff^{-1})g^{-1}) = g(1_B g^{-1}) = gg^{-1} = 1_C,$$

and similarly  $(f^{-1}g^{-1})(gf) = 1_A$ , showing that  $f^{-1}g^{-1}$  is an inverse for  $gf$ . Hence  $A \cong C$ .  $\square$

**Definition 1.6** (Functors). Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories. A functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  consists of mappings  $F: \text{ob } \mathcal{C} \rightarrow \text{ob } \mathcal{D}$  and  $F: \text{mor } \mathcal{C} \rightarrow \text{mor } \mathcal{D}$  satisfying the following.

1. Compatibility with domains and codomains, i.e.,  $F(\text{dom } f) = \text{dom } Ff$  and  $F(\text{cod } f) = \text{cod } Ff$ .
2. Compatibility with identities, i.e.,  $F(1_A) = 1_{FA}$ .
3. Compatibility with composition, i.e.,  $F(fg) = (Ff)(Fg)$ .

When there is a functor  $\mathcal{C}^{\text{op}} \rightarrow \mathcal{D}$  or  $\mathcal{C} \rightarrow \mathcal{D}^{\text{op}}$  (these two are equivalent) we sometimes call it a *contravariant* functor  $\mathcal{C} \rightarrow \mathcal{D}$ . We use the term *covariant* to mean functors in the original sense.

Note that functors can be composed. That is, if we have categories  $\mathcal{C}, \mathcal{D}$  and  $\mathcal{E}$ , together with functors  $F: \mathcal{C} \rightarrow \mathcal{D}$  and  $G: \mathcal{D} \rightarrow \mathcal{E}$  we can define a functor  $GF: \mathcal{C} \rightarrow \mathcal{E}$ . This is done in the obvious way: if  $A, B \in \text{ob } \mathcal{C}$  and  $f: A \rightarrow B$  is a morphism in  $\mathcal{C}$  we let

$$GF(A) := G(F(A)) \quad \text{and} \quad GF(f) := G(F(f)).$$

That this is a functor is verified immediately. Note that composition is associative as well. Therefore, if we wanted to, we can define a category **Cat** whose objects are all (small) categories and its morphisms are functors. Proving that this is a category is easy though tedious.

**Definition 1.7** (Natural transformations). Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories and  $F, G: \mathcal{C} \rightarrow \mathcal{D}$  be functors. A *natural transformation*  $\alpha: F \rightarrow G$  is a collection of morphisms  $\alpha_A: FA \rightarrow GB$  of  $\text{mor } \mathcal{D}$  for all  $A \in \text{ob } \mathcal{C}$  so that if  $f: A \rightarrow B$  is a morphism of  $\mathcal{C}$  the following square (called the *naturality square*) commutes.

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ \alpha_A \downarrow & & \downarrow \alpha_B \\ GA & \xrightarrow{Gg} & GB \end{array}$$

Fix two categories  $\mathcal{C}$  and  $\mathcal{D}$ . We can build a category  $[\mathcal{C}, \mathcal{D}]$  whose objects are all functors  $\mathcal{C} \rightarrow \mathcal{D}$  and the morphisms are natural transformations with dom and cod defined in the obvious way.

If  $F: \mathcal{C} \rightarrow \mathcal{D}$  we define  $1_F: F \rightarrow F$  to be the natural transformation consisting of morphisms  $(1_F)_A: FA \rightarrow FA$  for each  $A \in \text{ob } \mathcal{C}$  by specifying  $(1_F)_A := 1_{FA}$ . Then  $1_F$  is a natural transformation since if  $f: A \rightarrow B$  is a morphism in  $\mathcal{C}$  the naturality square

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ (1_F)_A \downarrow & & \downarrow (1_F)_B \\ FA & \xrightarrow{Ff} & FB \end{array}$$

obviously commutes by the identity axiom of categories.

Note that if we have natural transformations  $\alpha: F \rightarrow G$  and  $\beta: G \rightarrow H$  for functors  $F, G, H: \mathcal{C} \rightarrow \mathcal{D}$  we can define the composite natural transformation  $\beta\alpha: F \rightarrow H$  by stating that for all  $A \in \mathcal{C}$  we have  $(\beta\alpha)_A = \beta_A\alpha_A$ . This is a natural transformation since if  $f: A \rightarrow B$  for  $A, B \in \text{ob } \mathcal{C}$  we have that in the following diagram

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ \alpha_A \downarrow & & \downarrow \alpha_B \\ GA & \xrightarrow{Gf} & GB \\ \beta_A \downarrow & & \downarrow \beta_B \\ HA & \xrightarrow{Hf} & HB \end{array}$$

both squares commute since both  $\alpha$  and  $\beta$  are natural transformations, so the outer rectangle, which is the naturality square for  $(\beta\alpha)_A$ , commutes. We only need to check associativity and the identity axiom (the other axioms are obvious from the construction).

Let  $F, G, H, I: \mathcal{C} \rightarrow \mathcal{D}$  be functors and let  $\alpha: F \rightarrow G$ ,  $\beta: G \rightarrow H$ , and  $\gamma: H \rightarrow I$  be natural transformations. Then for all  $A \in \text{ob } \mathcal{C}$  we have  $((\alpha\beta)\gamma)_A = (\alpha_A\beta_A)\gamma_A$  and  $(\alpha(\beta\gamma))_A = \alpha_A(\beta_A\gamma_A)$  by definitions; and these are equal by associativity in  $\text{mor } \mathcal{D}$ . Hence,  $(\alpha\beta)\gamma = \alpha(\beta\gamma)$  so associativity holds.

Finally, let  $F, G, H: \mathcal{C} \rightarrow \mathcal{D}$  be functors and let  $\alpha: F \rightarrow G$  and  $\beta: G \rightarrow H$  be natural transformations. Consider  $1_G\alpha$ . For all  $A \in \text{ob } \mathcal{C}$  we have

$$(1_G\alpha)_A = (1_G)_A\alpha_A = 1_{GA}\alpha_A = \alpha_A,$$

where the last equality is the axiom of identity in  $\mathcal{D}$ . Therefore  $1_G\alpha = \alpha$ . Similarly, for  $\beta 1_G$  we have for all  $A \in \text{ob } \mathcal{C}$ ,

$$(\beta 1_G)_A = \beta_A(1_G)_A = \beta_A 1_{GA} = \beta_A,$$

and so  $\beta 1_G = \beta$ . This verifies the last axiom so  $[C, D]$  is indeed a category. As you can see, verifying whether something that is obviously a category *is* a category is quite a lot of very boring and repetitive work so I promise not to do it ever ever ever ever ever ever ever ever again.

**Lemma 1.8.** *Let  $\alpha: F \rightarrow G$  be a natural transformation between functors  $F, G: \mathcal{C} \rightarrow \mathcal{G}$ . Then  $\alpha$  is an isomorphism in  $[\mathcal{C}, \mathcal{D}]$  if and only if  $\alpha_A$  is an isomorphism in  $\mathcal{D}$  for all  $A \in \text{ob } \mathcal{C}$ .*

*Proof.* Suppose  $\alpha$  is an isomorphism in  $[\mathcal{C}, \mathcal{D}]$ , and let  $\alpha^{-1}: G \rightarrow F$  be its inverse. If  $A \in \text{ob } \mathcal{C}$  we have that

$$1_{FA} = (1_F)_A = (\alpha^{-1}\alpha)_A = \alpha_A^{-1}\alpha_A,$$

and similarly  $1_{GA} = \alpha_A\alpha_A^{-1}$ . Thus  $\alpha_A$  is an isomorphism.

Conversely, suppose that  $\alpha_A$  is an isomorphism for all  $A \in \text{ob } \mathcal{C}$ . Then define  $\alpha^{-1}: \mathcal{G} \rightarrow \mathcal{F}$  as the collection of morphisms  $\alpha_A^{-1}$  for all  $A$ . We only need to check that this is a natural transformation. This is the case, since, if  $g: C \rightarrow D$  is a morphism in  $\mathcal{D}$  we have that the naturality square

$$\begin{array}{ccc} GC & \xrightarrow{Gg} & GD \\ \alpha_C^{-1} \downarrow & & \downarrow \alpha_D^{-1} \\ FC & \xrightarrow{Fg} & FD \end{array}$$

commutes. To see this, just calculate:

$$(Fg)(\alpha_C^{-1}) = (\alpha_D^{-1}\alpha_D)(Fg)(\alpha_C^{-1}) \stackrel{!}{=} \alpha_C^{-1}(Gg)(\alpha_C)\alpha_C^{-1} = \alpha_C^{-1}Gg.$$

Note that in  $\stackrel{!}{=}$  we used the naturality of  $\alpha$ . □

**Definition 1.9** (Equivalence of categories). Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories. By an *equivalence* between  $\mathcal{C}$  and  $\mathcal{D}$  we mean a pair of functors  $F: \mathcal{C} \rightarrow \mathcal{D}$  and  $G: \mathcal{D} \rightarrow \mathcal{C}$ , together with natural isomorphisms  $\alpha: 1_{\mathcal{C}} \rightarrow GF$  and  $\beta: FG \rightarrow 1_{\mathcal{D}}$ .

We write  $\mathcal{C} \simeq \mathcal{D}$  if there exists an equivalence between  $\mathcal{C}$  and  $\mathcal{D}$ .

For  $A, B$  objects of some category  $\mathcal{C}$  we define the notation  $\mathcal{C}(A, B)$  to mean the collection of morphisms  $f \in \text{mor } \mathcal{C}$  such that  $\text{dom } f = A$  and  $\text{cod } f = B$ .

**Definition 1.10.** Let  $F: \mathcal{C} \rightarrow \mathcal{D}$  be a functor.

- (a) We say  $F$  is *faithful* if, given  $f, g \in \text{mor } \mathcal{C}$  the equations  $\text{dom } f = \text{dom } g$ ,  $\text{cod } f = \text{cod } g$  and  $Ff = Fg$  imply  $f = g$ .

- (b) We say that  $F$  is *full* if, given  $g: FA \rightarrow FB$  there exists  $f: A \rightarrow B$  in  $\mathcal{C}$  so that  $Ff = g$ .
- (c) We say that  $F$  is *essentially surjective* if every  $B \in \text{ob } \mathcal{D}$  is isomorphic to  $FA$  for some  $A \in \mathcal{C}$ .

Equivalently, we say  $F$  is faithful (respectively full) if for all  $A, B \in \text{ob } \mathcal{C}$  the assignment  $\mathcal{C}(A, B) \rightarrow \mathcal{D}(F(A), F(B))$  given by  $f \mapsto Ff$  is one-to-one (resp. onto).

Note that a subcategory  $\mathcal{D}$  of  $\mathcal{C}$  induces a faithful functor  $F: \mathcal{D} \rightarrow \mathcal{C}$  which is given by inclusion. If  $F$  is also full, we say that  $\mathcal{D}$  is a *full subcategory* of  $\mathcal{C}$ .

**Proposition 1.11.** *Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories, and suppose  $F: \mathcal{C} \rightarrow \mathcal{D}$  is a functor. If  $A, B \in \text{ob } \mathcal{C}$  are isomorphic, then so are  $F(A)$  and  $F(B)$ . Furthermore, if  $F$  is full and faithful, then the converse holds.*

*Proof.* Suppose  $A, B \in \text{ob } \mathcal{C}$  are isomorphic, and let  $f: A \rightarrow B$  be an isomorphism. Then, by functoriality,

$$(Ff^{-1})(Ff) = F(f^{-1}f) = F1_A = 1_{FA},$$

and similarly  $(Ff)(Ff^{-1}) = 1_{FB}$ . It follows that  $Ff^{-1} = (Ff)^{-1}$  and  $Ff$  is an isomorphism  $FA \rightarrow FB$ .

Now suppose that  $F$  is full and faithful and let us prove the converse. Assume  $A, B \in \text{ob } \mathcal{C}$  are such that  $FA \cong FB$ . Let  $g: FA \rightarrow FB$  be an isomorphism. By fullness, there is a morphism  $f: A \rightarrow B$  so that  $Ff = g$ . Similarly, there is a morphism  $f^*: B \rightarrow A$  so that  $Ff^* = g^{-1}$ . Then, by functoriality,

$$F(f^*f) = (Ff^*)(Ff) = g^{-1}g = 1_{FA} = F(1_A).$$

By faithfulness,  $f^*f = 1_A$ . An analogous argument shows that  $ff^* = 1_B$  and thus  $f^* = f^{-1}$  and  $f$  is an isomorphism.  $\square$

**Lemma 1.12.** *Let  $F: \mathcal{C} \rightarrow \mathcal{D}$  be a functor. Then  $F$  is part of an equivalence  $\mathcal{C} \simeq \mathcal{D}$  if and only if  $F$  is full, faithful, and essentially surjective*

*Proof.* Suppose  $F$  is part of an equivalence  $\mathcal{C} \simeq \mathcal{D}$  and let  $G: \mathcal{D} \rightarrow \mathcal{C}$  be a functor so that  $\alpha: 1_{\mathcal{C}} \rightarrow GF$  and  $\beta: FG \rightarrow 1_{\mathcal{D}}$  are natural isomorphisms. That  $F$  is essentially surjective is immediate: if  $B \in \mathcal{D}$  we see that  $F(G(B))$  is isomorphic to  $B$  via the isomorphism  $\beta_B$ .

Let  $f, g \in \text{mor } \mathcal{C}$  with  $\text{dom } f = \text{dom } g$ ,  $\text{cod } f = \text{cod } g$  and  $Ff = Fg$ . The naturality square for  $g$  gives

$$\begin{array}{ccc} A & \xrightarrow{g} & B \\ \alpha_A \downarrow & & \downarrow \alpha_B \\ GFA & \xrightarrow{GFg} & GFB \end{array}.$$

By Lemma 1.8,  $\alpha_A$  and  $\alpha_B$  are isomorphisms, so  $g = \alpha_B^{-1}(GFg)\alpha_A$ . Similarly, from the naturality square for  $f$  we get  $f = \alpha_B^{-1}(GFf)\alpha_A$  (here we use that  $f$  and  $g$  have the same domain and codomain). But as  $Ff = Fg$  we clearly have  $f = g$ .

Now we prove that  $F$  is full, so let  $g: FA \rightarrow FB$  be a morphism in  $\mathcal{D}$ . We want to obtain a map in  $\mathcal{C}$  so it seems reasonable to apply  $G$  to get a map  $Gg: GFA \rightarrow GFB$ . As we want a map  $A \rightarrow B$  we can use the natural transformations.

$$\begin{array}{ccc} GFA & \xrightarrow{Gg} & GFB \\ \alpha_A \uparrow & & \uparrow \alpha_B \\ A & & B \end{array}$$

The only thing to do is to define  $f := \alpha_B^{-1}(Gg)\alpha_A$  and verify  $Ff = g$ . Consider the naturality square for  $f$ .

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \alpha_A \downarrow & & \downarrow \alpha_B \\ GFA & \xrightarrow{GFf} & GFB \end{array}$$

Plugging in the definition of  $f$  into this square yields the equality  $Gg = GFf$ . But we already proved that functors that are part of equivalences are faithful; in particular  $G$  is faithful so (since  $g$  and  $Ff$  obviously have matching domain and codomain) we have  $g = Ff$  as desired. We have finished with the first implication of lemma.

Conversely, assume  $F$  is a full, faithful, and essentially surjective. For all  $B \in \text{ob } \mathcal{D}$  pick some  $GB \in \text{ob } \mathcal{C}$  such that  $FGB$  is isomorphic to  $B$ ; this is possible by essential surjectivity. Furthermore, pick some isomorphism  $\beta_B: FGB \rightarrow B$ . For each morphism  $g: B \rightarrow B'$  in  $\mathcal{D}$  note that  $g^* = \beta_{B'}^{-1}(g)\beta_B: FGB \rightarrow FGB'$ . So, let  $Gg: GB \rightarrow GB'$  be a morphism so that  $FGg = g^*$ ; this is possible since  $F$  is full. We claim that this defines a functor  $G: \mathcal{D} \rightarrow \mathcal{C}$ . It is clearly an assignment  $\text{ob } \mathcal{D} \rightarrow \text{ob } \mathcal{C}$  and  $\text{mor } \mathcal{D} \rightarrow \text{mor } \mathcal{C}$  compatible with domains and codomains.

Let  $B \in \text{ob } \mathcal{D}$ . Note that

$$FG(1_B) = (1_B)^* = \beta_B^{-1}1_B\beta_B = 1_{FGB} = F(1_{GB}).$$

Thus, by faithfulness of  $F$ , we have that  $G(1_B) = 1_{GB}$ .

Now let  $g: B \rightarrow B'$  and  $h: B' \rightarrow B''$  be morphisms in  $\mathcal{D}$ . We perform a similar calculation.

$$\begin{aligned} F(G(h)G(g)) &= (FGh)(FGg) = h^*g^* = \beta_{B''}^{-1}(h)\beta_{B'}\beta_{B'}^{-1}(g)\beta_B \\ &= \beta_{B''}^{-1}(hg)\beta_B \\ &= (hg)^* = FG(hg). \end{aligned}$$

Again by faithfulness, we have  $G(h)G(g) = G(hg)$ . We have shown  $G$  is a functor.

Next, we define the natural transformations. We have already sneakily defined  $\beta: FG \rightarrow 1_{\mathcal{D}}$  as a collection of isomorphisms, so we need to check it is indeed a natural transformation. Let  $g: B \rightarrow B'$  be a morphism in  $\mathcal{D}$ . When we consider the square

$$\begin{array}{ccc} B & \xrightarrow{g} & B' \\ \beta_B \downarrow & & \downarrow \beta_{B'} \\ FGB & \xrightarrow{FGg} & FGB' \end{array}$$

we see that we defined  $Gg$  precisely so this holds! Hence  $\beta$  is a natural transformation. By Lemma 1.8, it is also a natural isomorphism.

Let  $A \in \text{ob } \mathcal{C}$ . Then  $\beta_{FA}^{-1}$  is a morphism  $FA \rightarrow FGFA$  in  $\mathcal{D}$ . By fullness of  $F$ , there is a morphism, say  $\alpha_A: A \rightarrow GFA$  such that  $F\alpha_A = \beta_{FA}^{-1}$ . We claim that this defines a natural isomorphism  $\alpha: 1_{\mathcal{C}} \rightarrow GF$ .

First, we check naturality. If  $f: A \rightarrow B$  is a morphism of  $\mathcal{C}$ , the relevant square is:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \alpha_A \downarrow & & \downarrow \alpha_B \\ GFA & \xrightarrow{GFf} & GFA \end{array}$$

Apply  $F$  to this square to get

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & B \\ F\alpha_A = \beta_{FA}^{-1} \downarrow & & \downarrow F\alpha_B = \beta_{FB}^{-1} \\ GFA & \xrightarrow{FGFf} & GFA \end{array}$$

which is just the naturality square for  $FA$  with respect to  $\beta$ ! Thus it commutes and, by faithfulness, the original square commutes too. It follows that  $\alpha$  is a natural transformation.

Finally, let's check that  $\alpha_A$  is an isomorphism for all  $A$ . Indeed,  $\beta_{FA}$  is a morphism  $FGFA \rightarrow FA$  in  $\mathcal{D}$  so, by fullness, there is a morphism  $\gamma_A: GFA \rightarrow A$  such that  $F\gamma_A = \beta_{FA}$ . Note that

$$F(\alpha_A \gamma_A) = (F\alpha_A)(F\gamma_A) = \beta_{FA}^{-1} \beta_{FA} = 1_{FGFA} = F(1_{GFA}).$$

By faithfulness,  $\alpha_A \gamma_A = 1_{GFA}$ . A similar calculation yields  $\gamma_A \alpha_A = 1_A$ . Hence  $\alpha_A$  is an isomorphism and  $\gamma_A = \alpha_A^{-1}$ . By Lemma 1.8  $\alpha$  is a natural isomorphism.  $\square$

**Definition 1.13** (Skeletal). Let  $\mathcal{C}$  be a category. We say  $\mathcal{C}$  is *skeletal* if every isomorphism class of  $\text{ob } \mathcal{C}$  has just one member, i.e. if isomorphic objects in  $\mathcal{C}$  are necessarily identical.

By a *skeleton* of  $\mathcal{C}$  we mean a full subcategory  $\mathcal{C}'$  containing just one object from each isomorphism class.

Note that any equivalence between skeletal categories is bijective on objects and hence an isomorphism of categories.



**Definition 1.14** (Monos and Epis). Let  $f: A \rightarrow B$  be a morphism of a category  $\mathcal{C}$ . We say  $f$  is a *monomorphism* (or *monic*) if for all  $g, h \in \text{mor } \mathcal{C}$  we have that  $fg = fh$  implies  $g = h$  whenever the composites are defined.

Dually,  $f$  is an *epimorphism* (or *epic*) if  $gf = hf$  implies  $g = h$ .

Note that isomorphisms are both monic and epic. We give a special name to categories where the converse holds.

**Definition 1.15** (Balanced). Let  $\mathcal{C}$  be a category. We say  $\mathcal{C}$  is balanced if all morphisms of  $\mathcal{C}$  that are both monic and epic are also isomorphisms.

## 2 The Yoneda Lemma

**Definition 2.1** (Locally small). We say a category  $\mathcal{C}$  is locally small if, for all  $A, B \in \text{ob } \mathcal{C}$  there is a set  $\mathcal{C}(A, B)$  so that the class of morphisms  $f \in \text{mor } \mathcal{C}$  with  $\text{dom } f = A$  and  $\text{cod } f = B$  is in one-to-one correspondence with  $\mathcal{C}(A, B)$ .

Given an object  $A$  of a locally small category  $\mathcal{C}$  we define a functor  $\mathcal{C}(A, -): \mathcal{C} \rightarrow \mathbf{Set}$ . For  $B \in \text{ob } \mathcal{C}$  we map  $B \mapsto \mathcal{C}(A, B)$ , and if  $f: B \rightarrow C$  is a morphism of  $\mathcal{C}$  we give a function  $f^*: \mathcal{C}(A, B) \rightarrow \mathcal{C}(A, C)$  defined by

$$f^*(g: A \rightarrow B) := (fg: A \rightarrow C)$$

We claim this is functorial. Compatibility with domains and codomains is immediate. If  $f': C \rightarrow D$  is a morphism in  $\mathcal{C}$  we have  $(f'f)g = f'(fg)$  for all  $g: A \rightarrow B$ , so we have compatibility with composition. Also, the identity  $1_B$  is sent to the map  $g \mapsto 1_B g = g$ , i.e.  $1_{\mathcal{C}(A, B)}$ , so we have compatibility with identities.

(From now on we use the notation  $f^*$  to mean the function  $g \mapsto fg$  and  $f_*$  to mean  $g \mapsto gf$ . This is awfully useful).

More is true, however. We have defined an assignment  $\text{ob } \mathcal{C} \rightarrow \text{ob}[\mathcal{C}, \mathbf{Set}]$  by sending  $A \mapsto \mathcal{C}(A, -)$ . We can extend this to a contravariant functor  $\mathcal{Y}$  from  $\mathcal{C}$  to  $[\mathcal{C}, \mathbf{Set}]$  by specifying where morphisms go.

Fix  $f: B \rightarrow A$ , a morphism in  $\mathcal{C}$  (which corresponds to a morphism  $A \rightarrow B$  in  $\mathcal{C}^{\text{op}}$ ); our task is to define a natural transformation  $\mathcal{Y}f: \mathcal{C}(A, -) \rightarrow \mathcal{C}(B, -)$ .

To do this, for all  $C \in \text{ob } \mathcal{C}$  we need to define  $(\mathcal{Y}f)_C: \mathcal{C}(A, C) \rightarrow \mathcal{C}(B, C)$ , a set-function. But this is easy:

$$f_*(g: A \rightarrow C) := (gf: B \rightarrow C).$$

Now we need to verify naturality. Let  $h: C \rightarrow D$  be a morphism in  $\mathcal{C}$ . The relevant square is the following.

$$\begin{array}{ccc} \mathcal{C}(A, C) & \xrightarrow{h^*} & \mathcal{C}(A, D) \\ f_* \downarrow & & \downarrow f_* \\ \mathcal{C}(B, C) & \xrightarrow{h^*} & \mathcal{C}(B, D) \end{array}$$

At a purely formal level, it is obvious that the square commutes:  $g: A \rightarrow B$  is sent to  $hgf$  along the diagonal of the square, no matter which path we take. Hence  $\mathcal{Y}f$  is really a natural transformation. We have defined  $\mathcal{Y}$  by an assignment  $\text{ob } \mathcal{C} \rightarrow \text{ob}[\mathcal{C} \rightarrow \mathbf{Set}]$  and a contravariant assignment  $\text{mor } \mathcal{C} \rightarrow \text{mor}[\mathcal{C} \rightarrow \mathbf{Set}]$ . It remains to show that  $\mathcal{Y}$  is really a functor. That it respects domains and codomains is clear from the construction.

Let  $A \in \text{ob } \mathcal{C}$ . We would like to show that  $\mathcal{Y}1_A$  is equal to the identity natural transformation  $1_{\mathcal{C}(A, -)}$ . To verify this, it suffices to show that each component is an identity morphism. So, let  $D \in \text{ob } \mathcal{C}$ . Note that  $(\mathcal{Y}1_A)_D$  sends  $h: A \rightarrow D$  to the morphism  $h1_A = h$ . Hence,  $\mathcal{Y}$  is compatible with identities.

Next, let  $f: A \rightarrow B$  and  $g: B \rightarrow C$  be morphisms in  $\mathcal{C}$ ; we aim to show  $\mathcal{Y}$  respects composition. We begin by the commutative diagram

$$\begin{array}{ccccc} & & gf & & \\ & \curvearrowright & & \curvearrowleft & \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C \end{array}$$

We ask whether this is still commutative under  $\mathcal{Y}$  (remember it is contravariant):

$$\begin{array}{ccccc} & & \mathcal{Y}(gf) & & \\ & \curvearrowright & & \curvearrowleft & \\ \mathcal{C}(C, -) & \xrightarrow{\mathcal{Y}_g} & \mathcal{C}(B, -) & \xrightarrow{\mathcal{Y}_f} & \mathcal{C}(A, -) \end{array} .$$

It suffices to verify whether it is “pointwise” commutative, i.e., if  $D \in \text{ob } \mathcal{C}$  that we have

$$\begin{array}{ccccc} & & (gf)_* & & \\ & \curvearrowright & & \curvearrowleft & \\ \mathcal{C}(C, D) & \xrightarrow{g_*} & \mathcal{C}(B, D) & \xrightarrow{f_*} & \mathcal{C}(A, D) \end{array} .$$

This is immediate: if  $h: C \rightarrow D$  is a morphism then  $(gf)_*h = h(gf)$  while

$$f_*g_*h = (hg)f,$$

and we are done by associativity.

Dually, for a fixed object  $B$  of  $\mathcal{C}$  we define the functor

$$\mathcal{C}(-, B) := \mathcal{C}^{\text{op}}(B, -): \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set},$$

and the same constructions works in this context.

**Lemma 2.2** (Yoneda Lemma). *Let  $\mathcal{C}$  be a locally small category,  $A \in \text{ob } \mathcal{C}$  and  $F: \mathcal{C} \rightarrow \mathbf{Set}$ . Then there is a one-to-one and onto correspondence between natural transformations  $\mathcal{C}(A, -) \rightarrow F$  and elements of  $FA$ . Moreover, this correspondence is natural in  $A$  and  $F$ .*

A few words on the statement of the lemma. Note that a priori the collection of natural transformations  $\mathcal{C}(A, -) \rightarrow F$  is not a set, so it doesn’t make sense to say that this collection is in bijection with  $FA$ . Of course, during the proof we will construct such a “bijection”, proving that this collection is indexed by a set (so it is “small”).

Now to explain naturality. It is helpful to rephrase the lemma in the following form.

**Lemma 2.3** (Yoneda Lemma’). *Let  $\mathcal{C}$  be a locally small category,  $A \in \text{ob } \mathcal{C}$  and  $F: \mathcal{C} \rightarrow \mathbf{Set}$ . Then  $[\mathcal{C}, \mathbf{Set}](\mathcal{C}(A, -), F)$  is a set and*

$$[\mathcal{C}, \mathbf{Set}](\mathcal{C}(A, -), F) \cong FA,$$

*naturally in  $A$  and  $F$ .*

This says exactly the same thing. Now, when we say a morphism  $FA \rightarrow GA$  is natural in  $A$  we just mean that the collection of morphisms  $\alpha_A: FA \rightarrow GA$  define a natural transformation  $F \rightarrow G$ . This is easy enough, but in the statement of Yoneda Lemma' we have a map (actually an isomorphism) between two expressions with *two* free variables,  $A$  and  $F$ , rather than one. How to interpret this?

We will give an informal description now, and formalize later. First, define a functor from the product category  $\mathcal{C} \times [\mathcal{C}, \mathbf{Set}]$  (defined in the obvious way, details later) to the category of sets  $\mathbf{Set}$  given on objects by

$$(A, F) \mapsto FA$$

(There is only one way to define the assignment on morphisms, details later). Yoneda Lemma says that there is another way to define a functor  $\mathcal{C} \times [\mathcal{C}, \mathbf{Set}] \rightarrow \mathbf{Set}$  given on objects by

$$(A, F) \mapsto [\mathcal{C}, \mathbf{Set}](\mathcal{C}(A, -), F)$$

and this is naturally isomorphic to the first functor. So, saying that the correspondence is natural in two variables is just stating that there is a natural isomorphism between two functors with domain a certain product category. Compare this with the following situation from linear algebra: we say a function is linear in two given variables if there is a bilinear map (with domain a product of vector spaces) representing the given function (we will come back to this analogy later, when we study functors on product categories). It is a fact that if a correspondence is natural in both variables then it is “jointly natural” (cf. a function  $f: \mathbb{R}^2 \rightarrow \mathbb{R}$  can be continuous in both variables without being continuous); don't worry if you don't know what this means since precise statement will be given.

## Digression on Products

(This subsection elaborates on some exercises from the book [Leinster2014].) Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories. The *product category*  $\mathcal{C} \times \mathcal{D}$  has objects ordered pairs  $(A, B)$  for  $A \in \mathcal{C}$  and  $B \in \mathcal{D}$  and morphisms  $(A, B) \rightarrow (A', B')$  ordered pairs  $(f, g)$  where  $f: A \rightarrow A'$  in  $\mathcal{C}$  and  $g: B \rightarrow B'$  in  $\mathcal{D}$ . For all  $(A, B)$  we define  $1_{(A, B)} = (1_A, 1_B)$ . Let  $(f, g): (A, B) \rightarrow (A', B')$  and  $(h, i): (A', B') \rightarrow (A'', B'')$ . Composition is defined by

$$(h, i)(f, g) := (hf, ig).$$

That is, business as usual.

**Proposition 2.4.** *For all categories  $\mathcal{C}$  and  $\mathcal{D}$ , the product category  $\mathcal{C} \times \mathcal{D}$  is a category*

*Proof.* We need to verify associativity for composition and the identity axiom (the other axioms are obviously satisfied). For the rest of the proof, let

$A_1, A_2, A_3, A_4$  be objects of  $\mathcal{C}$  and  $B_1, B_2, B_3, B_4$  be objects of  $\mathcal{D}$ . In addition, let  $(f, g): (A_1, B_1) \rightarrow (A_2, B_2)$  and  $(h, i): (A_2, B_2) \rightarrow (A_3, B_3)$  and  $(j, k): (A_3, B_3) \rightarrow (A_4, B_4)$  be morphisms in  $\mathcal{C} \times \mathcal{D}$ .

Note that

$$((j, k)(h, i))(f, g) = (jh, ki)(f, g) = ((jh)f, (ki)g),$$

and similarly,

$$(j, k)((h, i)(f, g)) = (j, k)(hf, ig) = (j(hf), k(ig)),$$

so the results are the same by associativity of  $\mathcal{C}$  and  $\mathcal{D}$ .

Finally, we can verify the identity axiom for  $1_{(A_2, B_2)}$ . We have that

$$(f, g)1_{(A_2, B_2)} = (f, g)(1_{A_2}, 1_{B_2}) = (f1_{A_2}, g1_{B_2}) = (f, g).$$

Similarly,  $1_{(A_2, B_2)}(h, i) = (h, i)$  and we are done.  $\square$

Now we would like to study functors  $\mathcal{C} \times \mathcal{D} \rightarrow \mathcal{E}$ ; these are sometimes called *bifunctors*. Such functors can be regarded as an interlocking pair of families of functors, much like in the bilinear/linear analogy.

Let  $F: \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{E}$  be a functor. For each  $A \in \text{ob } \mathcal{C}$  we define the functor  $F^A: \mathcal{D} \rightarrow \mathcal{E}$  as follows. For each  $B \in \text{ob } \mathcal{D}$  we define  $F^A B = F(A, B)$ .

- For each  $B \in \text{ob } \mathcal{D}$  we define  $F^A B := F(A, B)$ .
- For each  $g: B \rightarrow B'$  in  $\mathcal{D}$  we define  $F^A g := F(1_A, g)$ .

We claim that this is functorial. Indeed, for all  $B \in \text{ob } \mathcal{D}$  we have that

$$F^A(1_B) = F(1_A, 1_B) = F(1_{(A, B)}) = 1_{F(A, B)} = 1_{F^A B},$$

so  $F$  respects identities. Now let  $g: B \rightarrow B'$  and  $h: B' \rightarrow B''$  be maps in  $\mathcal{D}$ . We have that

$$(F^A h)(F^A g) = (F(1_A, h))(F(1_A, g)) = F((1_A, h)(1_A, g)) = F(1_A, hg) = F^A(hg),$$

so composition is preserved. Hence  $F^A$  is indeed a functor for each  $A \in \text{ob } \mathcal{C}$ .

Similarly we can define  $F_B$  for all  $B \in \mathcal{D}$ . Either go through the same construction again or note that there is a functor  $R: \mathcal{D} \times \mathcal{C} \rightarrow \mathcal{C} \times \mathcal{D}$  that reverses the order of the factors in the obvious way; verifying that this is a functor (and in fact an isomorphism of categories) is extremely easy. Then define  $F_B := (FR)^B$ .

(Technically we don't need  $F$  to be a functor to define  $F^A$  and  $F_B$ , so we define these even when  $F$  is just an assignment from objects to objects and morphisms to morphisms  $\mathcal{C} \times \mathcal{D} \rightarrow \mathcal{E}$ . We will need this to state the Bifunctor Criterion later.)

So, the functor  $F$  can be decomposed into two families of functors. Given the families, can we reconstruct  $F$ ? The answer is yes.

**Lemma 2.5.** *Let  $\mathcal{C}, \mathcal{D}$  and  $\mathcal{E}$  be categories. Suppose we have  $(F^A)_{A \in \text{ob } \mathcal{C}}$ , a family of functors  $\mathcal{D} \rightarrow \mathcal{E}$ , and  $(F_B)_{B \in \text{ob } \mathcal{D}}$ , a family of functors  $\mathcal{C} \rightarrow \mathcal{E}$ . Furthermore, assume that the following two conditions are satisfied:*

- (a) *If  $A \in \text{ob } \mathcal{C}$  and  $B \in \text{ob } \mathcal{D}$  then  $F^A(B) = F_B(A)$ .*
- (b) *If  $f: A \rightarrow A'$  is a morphism in  $\mathcal{C}$  and  $g: B \rightarrow B'$  is a morphism in  $\mathcal{D}$ , then*

$$(F^{A'}g)(F_Bf) = (F_{B'}f)(F^Ag).$$

*Then there is a unique functor  $F: \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{E}$  such that the functors  $F^A$  and  $F_B$  are precisely those defined earlier.*

**Remark 2.6.** Note that the converse is true. That is, if  $F: \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{E}$  is a functor then  $F^A(B) = F_B(A) = F(A, B)$ . In addition,

$$(F^{A'}g)(F_Bf) = (F(1_{A'}, g))(F(f, 1_B)) = F(1_{A'}f, g1_B) = F(f, g),$$

and similarly  $(F_{B'}f)(F^Ag) = F(f, g)$ .

*Proof.* Let us define the functor  $F$  by specifying  $F(A, B) := F^A(B) = F_B(A)$  for all  $(A, B) \in \text{ob}(\mathcal{C} \times \mathcal{D})$ . We also specify that if  $(f, g): (A, B) \rightarrow (A', B')$  is a morphism in  $\mathcal{C} \times \mathcal{D}$  we have that

$$F(f, g) := (F^{A'}g)(F_Bf) = (F_{B'}f)(F^Ag).$$

It takes a moment staring at the following diagrams to realize that the composites have the right domains and codomains.

$$F(A, B) \xrightarrow{F_Bf} F(A', B) \xrightarrow{F^{A'}g} F(A', B')$$

$$F(A, B) \xrightarrow{F^Ag} F(A, B') \xrightarrow{F_{B'}f} F(A', B')$$

By Remark 2.6 if  $F$  is a functor, then  $F^A$  and  $F_B$  are defined properly.

Let  $(A, B) \in \text{ob}(\mathcal{C} \times \mathcal{D})$  and note that

$$\begin{aligned} F(1_{(A, B)}) &= F(1_A, 1_B) = (F^A 1_B)(F_B 1_A) = (1_{F^A B})(1_{F_B A}) \\ &= (1_{F(A, B)})(1_{F(A, B)}) \\ &= 1_{F(A, B)}. \end{aligned}$$

Next, let there be maps in  $\mathcal{C} \times \mathcal{D}$  as follows

$$(A, B) \xrightarrow{(f, g)} (A', B') \xrightarrow{(h, i)} (A'', B'') .$$

Then we have

$$\begin{aligned}
F(h, i)F(f, g) &= (F^{A''}i)(F_{B'}h)(F^{A'}g)(F_Bf) \\
&= (F^{A''}i)(F_{B'}h)(F^{B'}g)(F_Af) \\
&= (F^{A''}i)(F_{B'}(hf))(F_Af) \\
&= (F_{B''}(hf))(F^A i)(F^A g) \\
&= F_{B''}(hf)F^A(ig) \\
&= F(hf, ig).
\end{aligned}$$

Hence  $F$  is really a functor. Now we need to show that it is unique. Let  $G: \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{E}$  be a functor so that  $G^A = F^A$  and  $G_B = F_B$  for all  $A \in \text{ob } \mathcal{D}$  and  $B \in \mathcal{D}$ . By Remark 2.6 it is clear that the values of  $G$  on objects and morphisms are already determined, so  $F = G$  as functors.  $\square$

For example, earlier we defined a functors  $\mathcal{C}(A, -): \mathcal{C} \rightarrow \mathbf{Set}$  for any locally small category  $\mathcal{C}$  and object  $A \in \text{ob } \mathcal{C}$ ; similarly we had a functor  $\mathcal{C}(-, B): \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$  for all objects  $B \in \text{ob } \mathcal{C}$ . It is clear that for all  $A, B \in \mathcal{C}$  we have

$$\mathcal{C}(A, -)B = \mathcal{C}(A, B) = \mathcal{C}(-, B)A.$$

Let  $f: A' \rightarrow A$  and  $g: B \rightarrow B'$  be morphisms in  $\mathcal{C}$  (note the direction of  $f$ ). Then for any  $h \in \mathcal{C}(A, B)$  we have

$$(\mathcal{C}(A', -)g)(\mathcal{C}(-, B)f)h = g^*(f_*h) = g(hf),$$

and

$$(\mathcal{C}(-, B')f)(\mathcal{C}(A, -)g)h = (gh)f,$$

so the expressions match by associativity. Then Lemma 2.5 applies and gives a functor

$$\mathcal{C}(-, -): \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathbf{Set}.$$

We don't need to check that this is a functor: this is guaranteed by the lemma!

Often, however, we are defining a functor  $\mathcal{C} \times \mathcal{D} \rightarrow \mathcal{E}$  directly, and we would like to check functoriality. This is where the Bifunctor Criterion applies.

**Corollary 2.7** (Bifunctor Criterion). *Let  $\mathcal{C}, \mathcal{D}$  and  $\mathcal{E}$  be categories, and let  $F: \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{E}$  be an assignment  $F: \text{ob}(\mathcal{C} \times \mathcal{D}) \rightarrow \text{ob}(\mathcal{E})$  and  $F: \text{mor}(\mathcal{C} \times \mathcal{D}) \rightarrow \text{mor}(\mathcal{E})$ . Then  $F$  is a functor if and only if  $F^A$  and  $F_B$  are functors for all  $A \in \text{ob } \mathcal{C}$  and  $B \in \text{ob } \mathcal{D}$  that satisfy the following condition*

- If  $f: A \rightarrow A'$  is a morphism in  $\mathcal{C}$  and  $g: B \rightarrow B'$  is a morphism in  $\mathcal{D}$ , then

$$(F^{A'}g)(F_Bf) = (F_{B'}f)(F^A g).$$

*Proof.* If  $F$  is a functor then  $F^A$  and  $F_B$  are functors: we verified this when we defined them (and they satisfy the condition by Remark 2.6). Conversely,

assume  $F^A$  and  $F_B$  are functors for all  $A \in \text{ob } \mathcal{C}$  and  $B \in \text{ob } \mathcal{D}$  satisfying the condition. By definition we have

$$F^A(B) = F(A, B) = F_B(A)$$

for all  $A, B$ . Thus Lemma 2.5 applies and we are done.  $\square$

Now we turn to study naturality between functors whose domain is a product category; this is the situation in the Yoneda Lemma. In particular, we will show that such transformations are natural in both variables simultaneously iff they are natural in each variable separately.

**Lemma 2.8.** *Let  $F, G: \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{E}$  be functors. Suppose we have  $\alpha$ , a family of morphisms  $\alpha_{A,B}: F(A, B) \rightarrow G(A, B)$  in  $\mathcal{E}$  for all  $(A, B) \in \text{ob}(\mathcal{C} \times \mathcal{D})$ . Then  $\alpha$  is a natural transformation  $F \rightarrow G$  if and only if it satisfies the following two conditions.*

- For each  $A \in \text{ob } \mathcal{C}$  the family the collection of morphisms  $\alpha_{A,B}: F^A(B) \rightarrow G^A(B)$  for  $B \in \text{ob } \mathcal{D}$  defines a natural transformation  $F^A \rightarrow G^A$ .
- For each  $B \in \text{ob } \mathcal{D}$  the family the collection of morphisms  $\alpha_{A,B}: F_B(A) \rightarrow G_B(A)$  for  $A \in \text{ob } \mathcal{C}$  defines a natural transformation  $F_B \rightarrow G_B$ .

*Proof.* First suppose  $\alpha$  is a natural transformation. Let  $A \in \text{ob } \mathcal{C}$  and let  $g: B \rightarrow B'$  be a morphism in  $\mathcal{D}$ . Consider the naturality square for  $\alpha$  with respect to  $(1_A, g)$ .

$$\begin{array}{ccc} F^A(B) & \xrightarrow{F(1_A, g)} & F^A(B') \\ \alpha_{A,B} \downarrow & & \downarrow \alpha_{A,B'} \\ G^A(B) & \xrightarrow{G(1_A, g)} & G^A(B') \end{array}$$

This is exactly the naturality square we needed to check, so the first condition is satisfied, and the second is checked similarly.

Conversely, suppose that the conditions are satisfied. We need to check that  $\alpha$  is a natural transformation. Let  $(f, g)$  be a morphism  $(A, B) \rightarrow (A', B')$  in  $\mathcal{C} \times \mathcal{D}$ . The square we need to check is

$$\begin{array}{ccc} F(A, B) & \xrightarrow{F(f, g)} & F(A', B') \\ \alpha_{A,B} \downarrow & & \downarrow \alpha_{A', B'} \\ G(A, B) & \xrightarrow{G(f, g)} & G(A', B') \end{array}$$

By Remark 2.6 we can decompose  $F(f, g)$  and  $G(f, g)$  as follows.

$$\begin{array}{ccccc} F(A, B) & \xrightarrow{F^A g} & F(A, B') & \xrightarrow{F_{B'} f} & F(A', B') \\ \alpha_{A,B} \downarrow & & \alpha_{A,B'} \downarrow & & \downarrow \alpha_{A', B'} \\ G(A, B) & \xrightarrow{G^A g} & G(A, B') & \xrightarrow{G_{B'} f} & G(A', B') \end{array}$$



The left square commutes by the first condition, and the right one commutes because of the second one. It follows that the whole rectangle commutes.  $\square$

## End of Digression

Let's go back to Yoneda. Let  $\mathcal{C}$  be a locally small category. We define a bifunctor called  $\text{ev}$  (for “evaluation”)

$$\text{ev}: \mathcal{C} \times [\mathcal{C}, \mathbf{Set}] \rightarrow \mathbf{Set}.$$

If  $A \in \text{ob } \mathcal{C}$  and  $F: \mathcal{C} \rightarrow \mathbf{Set}$  is a functor we define,

$$\text{ev}(A, F) := FA.$$

Now suppose  $f: A \rightarrow B$  is a morphism in  $\mathcal{C}$  and  $\alpha: F \rightarrow G$  is a natural transformation between functors  $F, G: \mathcal{C} \rightarrow \mathbf{Set}$ . We then have the naturality square for  $f$ :

$$\begin{array}{ccc} FA = \text{ev}(A, F) & \xrightarrow{Ff} & FB \\ \alpha_A \downarrow & & \downarrow \alpha_B \\ GA & \xrightarrow{Gf} & GB = \text{ev}(B, G) \end{array} .$$

We define  $\text{ev}(f, \alpha): \text{ev}(A, F) \rightarrow \text{ev}(B, G)$  to be the diagonal of this square, i.e.,

$$\text{ev}(f, \alpha) := \alpha_B(Ff) = (Gf)\alpha_A.$$

Next we verify that this construction is functorial. This could be done using the Bifunctor Criterion but we choose to do it by hand. Let  $A \in \text{ob } \mathcal{C}$  and  $F: \mathcal{C} \rightarrow \mathbf{Set}$  be a functor. We have that

$$\text{ev}(1_{(A, F)}) = \text{ev}(1_A, 1_F) = (1_F)_A(F1_A) = 1_{FA}1_{FA} = 1_{\text{ev}(A, F)},$$

so identities are preserved. Next, let  $f: A \rightarrow B$  and  $g: B \rightarrow C$  be morphisms in  $\mathcal{C}$ , and let  $\alpha: F \rightarrow G$  and  $\beta: G \rightarrow H$  be natural transformations. We will now give our first real diagram chase!

$$\begin{array}{ccccc} FA & \xrightarrow{Ff} & FB & \xrightarrow{Fg} & FC \\ \alpha_A \downarrow & & \downarrow \alpha_B & & \downarrow \alpha_C \\ GA & \xrightarrow{Gf} & GB & \xrightarrow{Gg} & GC \\ \beta_A \downarrow & & \downarrow \beta_B & & \downarrow \beta_C \\ HA & \xrightarrow{Hf} & HB & \xrightarrow{Hg} & HC \end{array}$$

Every small square is commutative by naturality of  $\alpha$  and  $\beta$ ; thus the big square is commutative too. Consider maps  $FA \rightarrow HC$ . Going along the diagonal of the big square (i.e. going along either of the sides of the big square) we get  $\text{ev}(gf, \beta\alpha)$ . On the other hand, if we go along the diagonal of the upper-left

small square and then along the diagonal of the lower-right small square we get  $\text{ev}(g, \beta) \text{ev}(f, \alpha)$ . Hence

$$\text{ev}((g, \beta)(f, \alpha)) = \text{ev}(gf, \beta\alpha) = \text{ev}(g, \beta) \text{ev}(f, \alpha),$$

and  $\text{ev}$  is indeed a functor. (Don't get discouraged if this goes too fast, just stare at the diagram for as long as you need to).

As we have said before, there is another functor  $\mathcal{C} \times [\mathcal{C}, \mathbf{Set}] \rightarrow \mathbf{Set}$ . The Yoneda Lemma says that this functor exists and is naturally isomorphic to  $\text{ev}$ . We might as well construct the functor before proving the Yoneda Lemma.

Part of the problem in the construction is that the assignment on objects is not a priori well-defined; we need to verify that the image of the functor indeed lies in  $\mathbf{Set}$ . Though this is part of the content of Yoneda Lemma, we will prove it separately.

**Proposition 2.9.** *Let  $\mathcal{C}$  be a locally small category,  $A \in \text{ob } \mathcal{C}$  and  $F: \mathcal{C} \rightarrow \mathbf{Set}$  a functor. Then the class*

$$[\mathcal{C}, \mathbf{Set}](\mathcal{C}(A, -), F)$$

*is a set isomorphic to  $FA$ .*

*Proof.* We define a class functions

$$\Phi_{(A, F)}: [\mathcal{C}, \mathbf{Set}](\mathcal{C}(A, -), F) \rightarrow FA$$

$$\Psi_{(A, F)}: FA \rightarrow [\mathcal{C}, \mathbf{Set}](\mathcal{C}(A, -), F)$$

as follows. (We drop the subscript of  $(A, F)$  for simplicity but we are going to need them in the proof of the Yoneda Lemma). For  $\alpha: \mathcal{C}(A, -) \rightarrow F$  a natural transformation define

$$\Phi(\alpha) := \alpha_A(1_A) \in F_A.$$

Fix  $x \in FA$ . Define, for all  $B \in \text{ob } \mathcal{C}$ , a set-function  $\Psi(x)_B: \mathcal{C}(A, B) \rightarrow FB$  by the rule

$$\Psi(x)_B(f: A \rightarrow B) := (Ff)x$$

We claim that the collection of all such functions form a natural transformation  $\Psi(x): \mathcal{C}(A, -) \rightarrow F$ . Indeed, let  $g: B \rightarrow C$  be a morphism in  $\mathcal{C}$ . We need to check the commutativity of the square

$$\begin{array}{ccc} \mathcal{C}(A, B) & \xrightarrow{g^*} & \mathcal{C}(A, C) \\ \Psi(x)_B \downarrow & & \downarrow \Psi(x)_C \\ FB & \xrightarrow{Fg} & FC \end{array}$$

Let  $f: A \rightarrow B$  be arbitrary. Then

$$\begin{aligned}
(Fg)(\Psi(x)_B)f &= (Fg)(Ffx) \\
&= (Fg)(Ff)x \\
&= (F(gf))x \\
&= \Psi(x)_C(gf) \\
&= \Psi(x)_C(g^*)f,
\end{aligned}$$

which shows that the diagram commutes. Hence  $\Psi(x)$  is a natural transformation for all  $x \in FA$ .

Now we show that  $\Phi$  and  $\Psi$  are inverses of each other. Let  $x \in FA$  be arbitrary. Then

$$\Phi(\Psi(x)) = \Psi(x)_A(1_A) = (F1_A)x = 1_{FA}x = x.$$

Now let  $\alpha: \mathcal{C}(A, -) \rightarrow F$  be a natural transformation. We have

$$\Psi(\Phi(\alpha)) = \Psi(\alpha_A(1_A)).$$

We would like to verify that this is the same natural transformation as  $\alpha$ . Check the components: for  $B \in \text{ob } \mathcal{C}$  and  $f: A \rightarrow B$  we see that

$$\Psi(\alpha_A(1_A))_B f = (Ff)(\alpha_A(1_A)).$$

By naturality of  $\alpha$  in  $f$  we get the following commutative square.

$$\begin{array}{ccc}
\mathcal{C}(A, A) & \xrightarrow{f^*} & \mathcal{C}(A, B) \\
\alpha_A \downarrow & & \downarrow \alpha_B \\
FA & \xrightarrow{Ff} & FB
\end{array}$$

Hence,

$$\Psi(\alpha_A(1_A))_B f = \alpha_B f^* 1_A = \alpha_B (f 1_A) = \alpha_B f.$$

Thus the natural transformations are equal and  $\Psi(\Phi(\alpha)) = \alpha$ .  $\square$

Thanks to Proposition 2.9 we can define, for any locally small category  $\mathcal{C}$ , a functor

$$\mathcal{F}: \mathcal{C} \times [\mathcal{C}, \mathbf{Set}] \rightarrow \mathbf{Set}$$

So that for  $A \in \text{ob } \mathcal{C}$  and  $\mathcal{F}: \mathcal{C} \rightarrow \mathbf{Set}$  we define

$$\mathcal{F}(A, F) := [\mathcal{C}, \mathbf{Set}](\mathcal{C}(A, -), F).$$

Now onto morphisms. Let  $f: A \rightarrow B$  be a morphism in  $\mathcal{C}$  and let  $\alpha: F \rightarrow G$  be a natural transformation of functors  $F, G: \mathcal{C} \rightarrow \mathbf{Set}$ . We define

$$\mathcal{F}(f, \alpha): [\mathcal{C}, \mathbf{Set}](\mathcal{C}(A, -), F) \rightarrow [\mathcal{C}, \mathbf{Set}](\mathcal{C}(B, -), G)$$

as follows. For  $\beta: \mathcal{C}(A, -) \rightarrow F$  a natural transformation we define  $(\mathcal{F}(f, \alpha))(\beta)$  to be the natural transformation  $\mathcal{C}(B, -) \rightarrow G$  whose  $C$ -component is, for all  $C \in \text{ob } \mathcal{C}$ , the composite

$$\mathcal{C}(B, C) \xrightarrow{f_*} \mathcal{C}(A, C) \xrightarrow{\beta_C} FC \xrightarrow{\alpha_C} GC .$$

We need to check that this is indeed a natural transformation. Let  $g: C \rightarrow D$  be a morphism in  $\mathcal{C}$ . Consider the diagram:

$$\begin{array}{ccc} \mathcal{C}(B, C) & \xrightarrow{g^*} & \mathcal{C}(B, D) \\ f_* \downarrow & & \downarrow f_* \\ \mathcal{C}(A, C) & \xrightarrow{g^*} & \mathcal{C}(A, D) \\ \beta_C \downarrow & & \downarrow \beta_C \\ FC & \xrightarrow{Fg} & FD \\ \alpha_C \downarrow & & \downarrow \alpha_D \\ GC & \xrightarrow{Gg} & GD \end{array} .$$

The outer rectangle is the naturality square for  $(\mathcal{F}(f, \alpha))(\beta)$  in  $g$ , so to check whether it is commutative it suffices to check commutativity of the three small squares. The top square is obviously commutative. The middle one is commutative by naturality of  $\beta$ , and the bottom one by naturality of  $\alpha$ . Thus  $(\mathcal{F}(f, \alpha))(\beta)$  is indeed a natural transformation.

Let's remember what we are doing. We have just defined the functor  $\mathcal{F}$  on objects and on morphisms, but we haven't checked functoriality yet. So, we continue.

We could use the Bifunctor Criterion but again we do it by hand.<sup>1</sup> Let  $A \in \text{ob } \mathcal{C}$  and  $F: \mathcal{C} \rightarrow \mathbf{Set}$  a functor. We would like to show that  $\mathcal{F}(1_A, 1_F) = \mathcal{F}(1_A, 1_F)$ , which is a set-function from  $[\mathcal{C}, \mathbf{Set}](\mathcal{C}(A, -), F)$  to itself, is just the identity on that set. Then let  $\beta: \mathcal{C}(A, -) \rightarrow F$  be a natural transformation; our task is to show that we have the equality of natural transformations  $\mathcal{F}(1_A, 1_F)\beta = \beta$ . It suffices to check this component-wise, so let  $C \in \text{ob } \mathcal{C}$  and note that  $(\mathcal{F}(1_A, 1_F)\beta)_C$  is the composite

$$\mathcal{C}(A, C) \xrightarrow{(1_A)_*} \mathcal{C}(A, C) \xrightarrow{\beta_C} FC \xrightarrow{(1_F)_C} FC ,$$

which is equal to  $\beta_C$ , which is what we wanted. Hence  $\mathcal{F}$  respects identities.

Now, let  $f: A \rightarrow B$  and  $g: B \rightarrow C$  be morphisms in  $\mathcal{C}$ . In addition, let  $\alpha: F \rightarrow G$  and  $\beta: G \rightarrow H$  be natural transformations between functors  $F, G, H: \mathcal{C} \rightarrow \mathbf{Set}$ . We need to check the equality of set-functions

$$\mathcal{F}(gf, \beta\alpha) = \mathcal{F}(g, \beta) \circ \mathcal{F}(f, \alpha).$$

<sup>1</sup>Then why did we prove the Bifunctor Criterion? I don't know, unfortunately.

Let  $\gamma: \mathcal{C}(A, -) \rightarrow F$  be a natural transformation and  $D \in \text{ob } \mathcal{C}$ . It suffices to calculate

$$(\mathcal{F}(g, \beta) \circ \mathcal{F}(f, \alpha)(\gamma))_D = ((\mathcal{F}(g, \beta))(\mathcal{F}(f, \alpha)(\gamma)))_D.$$

By definition, this is the composite

$$\mathcal{C}(C, D) \xrightarrow{g_*} \mathcal{C}(B, D) \xrightarrow{(F(f, \alpha)(\gamma))_D} GD \xrightarrow{\beta_D} HD$$

Expanding the map in the middle with the definition we get

$$\mathcal{C}(C, D) \xrightarrow{g_*} \mathcal{C}(B, D) \xrightarrow{f_*} \mathcal{C}(A, D) \xrightarrow{\gamma_D} FD \xrightarrow{\alpha_D} GD \xrightarrow{\beta_D} HD .$$

Clearly this is the same as the composite

$$\mathcal{C}(C, D) \xrightarrow{(gf)_*} \mathcal{C}(A, D) \xrightarrow{\gamma_D} FD \xrightarrow{(\beta\alpha)_D} HD ,$$

which is just  $(\mathcal{F}(gf, \beta\alpha)(\gamma))_D$ . As  $\gamma$  and  $D$  were arbitrary, we are done. We have shown that  $\mathcal{F}$  is a functor. Now we can state and prove the final version of the Yoneda Lemma.

**Lemma 2.10** (The Yoneda Lemma). *Let  $\mathcal{C}$  be a locally small category. Then  $\text{ev}$  and  $\mathcal{F}$  are naturally isomorphic as functors  $\mathcal{C} \times [\mathcal{C}, \mathbf{Set}] \rightarrow \mathbf{Set}$ .*

*Proof.* In the proof of Proposition 2.9 we already constructed the relevant isomorphisms,  $\Phi$  and  $\Psi$ , so all we need to check is that they are natural (we are using Lemma 1.8 once again to save us some work). It is only necessary to check that one of  $\Phi$  and  $\Psi$  is natural. We do it for  $\Psi: \text{ev} \rightarrow \mathcal{F}$ .

Let  $f: A \rightarrow B$  be a morphism in  $\mathcal{C}$  and  $\alpha: F \rightarrow G$  a natural transformation of functors  $F, G: \mathcal{C} \rightarrow \mathbf{Set}$ . We need to check the following naturality square.<sup>2</sup>

$$\begin{array}{ccc} FA & \xrightarrow{\text{ev}(f, \alpha)} & GB \\ \Psi_{(A, F)} \downarrow & & \downarrow \Psi_{(B, G)} \\ [\mathcal{C}, \mathbf{Set}](\mathcal{C}(A, -), F) & \xrightarrow{\mathcal{F}(f, \alpha)} & [\mathcal{C}, \mathbf{Set}](\mathcal{C}(B, -), G) \end{array}$$

Fix  $x \in FA$ . Checking commutativity for  $x$  is checking the equality of two natural transformations, so we should do so component-wise. Fix some  $C \in \text{ob } \mathcal{C}$ . It is enough to show that

$$(\mathcal{F}(f, \alpha)(\Psi_{(A, F)}(x)))_C = (\Psi_{(B, G)}(\text{ev}(f, \alpha)(x)))_C. \quad (1)$$

Start with the left-hand side. By definition,  $(\mathcal{F}(f, \alpha)(\Psi_{(A, F)}(x)))_C$  is the composite

$$\mathcal{C}(B, C) \xrightarrow{f_*} \mathcal{C}(A, C) \xrightarrow{(\Psi_{(A, F)}(x))_C} FC \xrightarrow{\alpha_C} GC .$$

---

<sup>2</sup>I know it doesn't look like a square.

Let  $g \in \mathcal{C}(B, C)$ . Then, following the diagram

$$g \longmapsto gf \longmapsto F(gf)(x) \longmapsto \alpha_C F(gf)(x) .$$

So we have just shown

$$(\mathcal{F}(f, \alpha)(\Psi_{(A, F)}(x)))_C(g) = \alpha_C F(gf)(x).$$

Now look at the left-hand side of equation (1). We can evaluate at  $g$  to get

$$\begin{aligned} (\Psi_{(B, G)}(\text{ev}(f, \alpha)(x)))_C(g) &= (Gg) \text{ev}(f, \alpha)(x) \\ &= (Gg)((Gf)\alpha_A)(x) \\ &= G(gf)\alpha_A(x) \\ &= \alpha_C F(gf)(x), \end{aligned}$$

where the last equality is just naturality of  $\alpha$  in  $gf$ . Thus we see that (1) holds.  $\square$

**Corollary 2.11.** *For any locally small category  $\mathcal{C}$  the Yoneda embedding  $\mathcal{Y}$ , which is a functor  $\mathcal{C}^{\text{op}} \rightarrow [\mathcal{C}, \mathbf{Set}]$ , is full and faithful.*

*Proof.* Let  $A, B \in \text{ob } \mathcal{C}$  be arbitrary. Let  $F = \mathcal{C}(B, -) = \mathcal{Y}(B)$  in the proof of Proposition 2.9. Then  $\Psi$  is a bijection

$$\mathcal{C}(B, A) = \mathcal{C}^{\text{op}}(A, B) \rightarrow [\mathcal{C}, \mathbf{Set}](\mathcal{C}(A, -), \mathcal{C}(B, -)).$$

But note that for all  $f \in \mathcal{C}(B, A)$ , all  $C \in \text{ob } \mathcal{C}$ , and all  $g: A \rightarrow C$  we have

$$\Psi(f)_C(g) = g^* f = gf = f^* g = \mathcal{Y}(f)_C(g),$$

so  $\Psi$  and  $\mathcal{Y}$  coincide in this case.  $\square$

This says that any locally small category is equivalent to a full subcategory of a functor category  $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$ .

**Definition 2.12** (Representables). If  $\mathcal{C}$  is a locally small category, we say a functor  $F: \mathcal{C} \rightarrow \mathbf{Set}$  is *representable* if it is naturally isomorphic to  $\mathcal{C}(A, -)$  for some  $A \in \mathcal{C}$ . By a *representation* of  $F$  we mean a pair  $(A, x)$  where  $A \in \text{ob } \mathcal{C}$  and  $x \in FA$  is such that  $\Psi(x)$  is a natural isomorphism  $\mathcal{C}(A, -) \rightarrow F$ .

**Corollary 2.13.** *Let  $\mathcal{C}$  be locally small. If  $(A, x)$  and  $(B, y)$  are both representations of a functor  $F: \mathcal{C} \rightarrow \mathbf{Set}$ , then there is a unique isomorphism  $f: A \rightarrow B$  such that  $Ff(x) = y$ .*

*Proof.* Note that  $\Psi(x)^{-1}\Psi(y)$  is an isomorphism  $\mathcal{C}(B, -) \rightarrow \mathcal{C}(A, -)$ . By Corollary 2.11, which says that  $\mathcal{Y}$  is full and faithful, there is a unique  $f: A \rightarrow B$  (necessarily an isomorphism) such that the following diagram commutes.

$$\begin{array}{ccc} \mathcal{C}(B, -) & \xrightarrow{\mathcal{Y}f} & \mathcal{C}(A, -) \\ & \searrow \Psi(y) & \swarrow \Psi(x) \\ & F & \end{array}$$

Specialize to the object  $B$ .

$$\begin{array}{ccc} \mathcal{C}(B, B) & \xrightarrow{f_*} & \mathcal{C}(A, B) \\ & \searrow \Psi(y)_B & \swarrow \Psi(x)_B \\ & FB & \end{array}$$

Further specialize to the identity  $1_B$  and note that one path gives

$$(\Psi(x)_B f_*) 1_B = \Psi(x)_B (1_B f) = \Psi(x)_B f = Ff(x)$$

while the other gives

$$\Psi(y)_B 1_B = F1_B(y) = 1_{FB}(y) = y.$$

Thus  $Ff(x) = y$ . Now let  $f': A \rightarrow B$  be an isomorphism such that  $Ff'(x) = y$ . Then for any  $C \in \text{ob } \mathcal{C}$  and  $g: B \rightarrow C$  we have

$$\begin{aligned} (\Psi(x)_C (f')_*) g &= \Psi(x)_C (gf') \\ &= F(gf')x \\ &= (Fg)(Ff')(x) \\ &= Fg(y) \\ &= \Psi(y)_C(g). \end{aligned}$$

This implies that the diagram we drew when we defined  $f$  commutes when we replace  $f$  by  $f'$ . Hence, by uniqueness of  $f$ , we have  $f = f'$ .  $\square$