Model Theory and Non-Classical Logic
Example Sheet 3 Solutions

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December 6, 2023

1. Let p be a complete 1-type. Then for all $a \in \mathbb{Q}$ exactly one of $x < a$, $x = a$, and $x > a$ is in p.

   Suppose that $(x = a) \in p$ for some $a \in \mathbb{Q}$. We claim that $p = \mathrm{tp}^{\mathbb{Q}}(a/\mathbb{Q})$. By Corollary 1.4.7, there is an elementary extension M of $\mathbb{Q}$ such that $p = \mathrm{tp}^{M}(a/\mathbb{Q})$, where we used the fact that $(x = a) \in p$. But $\mathrm{tp}^{M}(a/\mathbb{Q}) = \mathrm{tp}^{\mathbb{Q}}(a/\mathbb{Q})$ as the extension is elementary, so the claim follows.

   Now assume that $(x = a) \notin p$ for all $a \in \mathbb{Q}$. Then the sets

   $$U \coloneqq \{a \in \mathbb{Q} \mid (x < a) \in p\} \quad \text{and} \quad L \coloneqq \{b \in \mathbb{Q} \mid (b < x) \in p\}$$

   partition $\mathbb{Q}$. Note that if $a \in U$ and $b \in L$, then $b < a$. Indeed, the sets are disjoint, so the only alternative is that $a < b$. But from this it follows that in any realization of the type p, say by an element c, we must have $c < a < b < c$ and thus $c < c$. As $\mathbb{Q} \models \forall y.\neg(y < y)$ this is a contradiction. Hence the complete type p gives rise to a partition U, L of $\mathbb{Q}$ such that $L < U$ in the above sense.

   Conversely, any such partition can be extended to a complete type by the Ultrafilter Principle, and it is clear that this correspondence is bijective.

2. TODO

3. First we show that, given a finite number of complete types $p_1, \ldots, p_k \in S_n^M(M)$ there is an elementary extension N of M realizing all of them. For $k = 0$ this is trivial. Now suppose there is an elementarily extension N' of M realizing $p_1, \ldots, p_{k-1}$. Note that $S_n^M(M) = S_n^M(N')$ since the extension is elementary, so in particular $p_k \in S_n^M(N')$ and by Proposition 1.4.6 there is an elementary extension N of N' realizing p_k. Obviously N is an elementary extension of M realizing $p_1, \ldots, p_k$ so we are done by induction.

   Back to the main problem. To the language $\mathcal{L}$ we add a constant for each element of M and we add n constants $c_1^p, \ldots, c_n^p$ for every $p \in S_n^M(M)$. In the expanded language, consider the theory

   $$\left(\bigcup S_n^M(M)\right) \cup \mathrm{Diag}_{\mathrm{el}}(M)$$

where each $\varphi(\bar{x}) \in p \in S_n^N$ is replaced by $\varphi(\bar{c}^p)$. Clearly if this theory is consistent then we are done. But every finite subset of this theory is satisfied by an elementary extension of M that has to realize only finitely many types, so we are done by our previous result.

4.

(a) Let p, q be distinct types. Without loss of generality, we assume that there is a formula $\varphi(\bar{x})$ such that $\varphi \in p$ but $\varphi \notin q$. Then $[\![\varphi]\!]$ is a clopen set containing p but not q. This shows that $S_n^M(A)$ is totally disconnected.

For the second part, we need a claim.

Claim 1. *Let F be a set of $\mathcal{L}_A$ formulae with n variables. Add n new constants $\bar{c}$ to the language. Then the set $\mathcal{C} := \{ [\![\varphi]\!] \mid \varphi(\bar{x}) \in F \}$ covers $S_n^M(A)$ if and only if the theory*

$$\mathcal{T} := \mathrm{Th}_A(M) \cup \{ \neg\varphi(\bar{c}) \mid \varphi(\bar{x}) \in F \}$$

is inconsistent.

Proof. Suppose $\mathcal{T}$ were consistent. Then $\neg F$ is an n-type, which, by the Ultrafilter Principle, can be extended to a complete n-type $q \in S_n^M(A)$. For all $\varphi \in F$ we must have $\neg\varphi \in q$, which means $\varphi \notin q$; thus $\mathcal{C}$ does not cover q.

Conversely, suppose that there is some $q \in S_n^M(A)$ such that $\varphi \notin q$ for all $\varphi \in F$. That means that $\neg\varphi \in q$ for all $\varphi \in F$ since q is complete. By definition of type, we have that $\mathrm{Th}_A(M) \cup q$ is consistent when we replace the variables $\bar{x}$ in q by the constants $\bar{c}$. It follows that $\mathcal{T}$ is consistent. □

Back to the problem, let $\mathcal{C}$ be an open cover of $S_n^M(A)$. As open sets are unions of basis elements, we can assume that $\mathcal{C}$ is of the form $\{ [\![\varphi]\!] \mid \varphi(\bar{x}) \in F \}$ for some set of $\mathcal{L}_A$-formulae F.

Now we know that $\mathcal{T}$ is inconsistent, where $\mathcal{T}$ is as in Claim 1. By the Compactness Theorem (for first-order logic) there is a finite subset $\mathcal{T}'$ of $\mathcal{T}$ that is inconsistent. Hence there is a finite subset F' of F such that $\mathrm{Th}_A(M) \cup \{\neg\varphi(\bar{c}) \mid \varphi(\bar{x}) \in F'\}$ is inconsistent. Again by Claim 1, the set $\mathcal{C}' := \{ [\![\varphi]\!] \mid \varphi(\bar{x}) \in F' \}$, which is a finite subset of $\mathcal{C}$, covers $S_n^M(A)$.

(b) I think that he meant to define $f^*(p) := \{\phi(\bar{x}, f(\bar{a})) \mid \phi(\bar{x}, \bar{a}) \in p\}$, and that we need to show that $f^*(p) \in S_n^N(f(A))$.

To show that $f^*(p) \in S_n^N(f(A))$ first we need to show that

$$\mathrm{Th}_{f(A)}(N) \cup f^*(p)$$

is satisfiable. By assumption p is an n-type, so there is an elementary extension X of M and a tuple $\bar{r} \in X$ with $\phi(\bar{r}, \bar{a})$ for all $\phi(\bar{x}, \bar{a}) \in p$. Note that X can also be interpreted as an $\mathcal{L}_{f(A)}$-structure. As f is elementary it is clear that $X \models \mathrm{Th}_{f(A)}(N)$ and is immediate $X \models \phi(\bar{r}, f(\bar{a}))$. This all shows that $f^*(p)$ is an n-type; and it is complete since p is complete.

Now we show that f^* is continuous. By general topology, it suffices to show that for each basis element $[\![\varphi]\!] \subseteq S_n^N(f(A))$ the set $(f^*)^{-1}([\![\varphi]\!])$ is open in $S_n^M(A)$.

So, let $\varphi(\bar{x}, f(\bar{a}))$ be an $\mathcal{L}_{f(A)}$-formula. Then

$$\begin{aligned}(f^*)^{-1}([\![\varphi]\!]) &= \{p \in S_n^M \mid f^*(p) \in [\![\varphi]\!]\}\\ &= \{p \in S_n^M \mid \varphi \in f^*(p)\}\\ &= \{p \in S_n^M \mid \varphi(\bar{x}, f(\bar{a})) = \psi(\bar{x}, f(\bar{a})) \text{ for some } \psi(\bar{x}, \bar{a}) \in p\}.\end{aligned}$$

But notice that, as f is injective, $\varphi(\bar{x}, f(\bar{a})) = \psi(\bar{x}, f(\bar{a}))$ implies that $\psi = \phi$. Thus,

$$(f^*)^{-1}([\![\varphi]\!]) = \{p \in S_n^M \mid \varphi(\bar{x}, \bar{a}) \in p\} = [\![\varphi(\bar{x}, \bar{a})]\!]$$

which is open in $S_n^M(A)$.

5. In the proof of Theorem 1.4.11 we make a small modification. Let $f\colon \omega \to \omega \times \omega$ be a bijection. When we define θ_s for odd $s = 2i+1$ then if $f(i) = (j, k)$ instead of taking $\bar{d}_i$ we take $\bar{d}_j$, run the same process to get ψ and we let φ be a formula in p_k that is not implied by ψ. The rest of the proof is the same, except that at the very end we notice that if $\bar{c} \in C^n$ then $\bar{c} = \bar{d}_j$ for some j and that if $k < \omega$ then we can define $i := f^{-1}(j, k)$; it then follows that θ_{2i+2} implies that $\bar{c}$ does not realize p_k. As $\bar{c}$ and k were arbitrary, we are done.

6. We need to find an uncountable language $\mathcal{L}$, an $\mathcal{L}$-theory $\mathcal{T}$, and a non-isolated n-type p of $\mathcal{T}$ such that p is realized in every countable model of $\mathcal{T}$. Hmm... not sure about this one. Feels like it super easy though.

7. Can't do it... I think it uses Omitting Types Theorem at some point.

8. Let $M := \{m_1, m_2, \ldots\}$ and $N := \{n_1, n_2, \ldots\}$ be two countable ω-saturated, elementarily equivalent $\mathcal{L}$-structures. A partial function $f\colon M \to N$ is called *elementary* if for all $\varphi(\bar{x}) \in \mathcal{L}$ and $\bar{d} \in \mathrm{dom}(f)$

$$M \models \varphi(\bar{d}) \iff N \models \varphi(f(\bar{d})).$$

We construct a sequence $f_0, f_1, \ldots$ such that for all $i \in \mathbb{N}$:

- f_i is an elementary partial function $M \to N$;
- f_{i+1} extends f_i;
- $\mathrm{dom}(f_i)$ (and hence $\mathrm{cod}(f_i)$) is finite;
- $\{m_1, \ldots, m_i\} \subseteq \mathrm{dom}(f_i)$ and $\{n_1, \ldots, n_i\} \subseteq \mathrm{cod}(f_i)$.

Define f_0 to be the empty function, which is elementary since M and N are elementarily equivalent. Suppose f_i has been defined, and let $D \coloneqq \mathrm{dom}(f_i)$ and $C \coloneqq \mathrm{cod}(f_i)$ be finite. Consider the complete 1-type

$$\mathrm{tp}^M(m_{i+1}/D).$$

Using the notation of Question 4 (b), we note that $f_i^*(p)$ is a complete 1-type of N by basically the same argument as in Q4 and the fact that f_i is elementary. Since N is ω-saturated, it follows that there is some $n \in N$ realizing this type. Let $g\colon D \cup \{m_{i+1}\} \to C \cup \{n\}$ be the extension of f_i that sends $m_{i+1} \mapsto n$ (if $m_{i+1} \in D$ then just let $g = f_i$). Thus g is elementary by construction.

Similarly, consider the complete 1-type $p \coloneqq \mathrm{tp}^N(n_{i+1}/C \cup \{n\})$. As g is elementary it follows that

$$\{\varphi(x, \bar{d})\colon \varphi(x, g(\bar{d})) \in p \text{ for some } \bar{d} \in D \cup \{m_{i+1}\}\}$$

is a 1-type for M so it has a realization $m \in M$. Finally, we let $f_{i+1}\colon D \cup \{m, m_{i+1}\} \to C \cup \{n, n_{i+1}\}$ be the extension of g mapping $m \mapsto n_{i+1}$. For the same reasons as before, f_{i+1} is elementary. This finishes the construction.

Now let $f\colon M \to N$ be the union of all f_i. By construction, f is defined everywhere, elementary, and surjective. It is also injective since $N \models f(m) = f(m')$ will imply $M \models m = m'$. The fact that f is a homomorphism can be similarly verified.