Geometric Group Theory
ES1 Solutions

1. Torus and Klein bottle respectively,
2. (a) It suffices to show that z commutes with the generators, but this is immediate.
   (b) The centre of A_5 is trivial so z must be send to the identity (here we use surjectivity). It remains to find elements of order 2 and 3 in A_5 such that their product has order 5 and they generate the whole group. The permutations $(1\,2)(3\,4)$ and $(1\,3\,5)$ satisfy these conditions.
3. Define $\varphi(k) \coloneqq ba^k b^{-1} a^{-2^k}$ for $k \geq 1$. Check by induction that

$$\varphi(k) = \prod_{r=0}^{k-1} a^{2^r}(bab^{-1}a^{-2})a^{-2^r}$$

in $F \coloneqq F(\{a,b\})$. It follows that $\varphi(k)$ is trivial in $BS(1,2)$ for all $k \geq 1$. Now define $\Phi(n,k) \coloneqq b^{n-1}\varphi(k)b^{-(n-1)}$ for all $n, k \geq 1$. Obviously $\Phi(n,k)$ becomes trivial in $BS(1,2)$. An induction on m shows that for all $1 \leq m \leq n$ we have the following identity in F

$$\prod_{r=0}^{m-1} \Phi(n-r, 2^r) = b^n ab^{-m} a^{-(2\uparrow\uparrow m)} b^{-(n-m)},$$

where $2\uparrow\uparrow m$ refers to the number $2^{2^{2^{\cdot^{\cdot^{\cdot}}}}}$ where 2 appears m times (this is Knuth's arrow notation). In particular, for $m = n$ we have

$$\Psi(n) \coloneqq \prod_{r=0}^{n-1} \Phi(n-r, 2^r) = b^n ab^{-n} a^{-(2\uparrow\uparrow n)}.$$

Hence

$$\begin{aligned}
\Phi(n)a\Phi^{-1}(n)a^{-1} &= b^n ab^{-n} a^{-(2\uparrow\uparrow n)} a(a^{2\uparrow\uparrow n} b^n a^{-1} b^{-n})a^{-1} \\
&= b^n ab^{-n}(a^{-(2\uparrow\uparrow n)} a a^{2\uparrow\uparrow n})(b^n a^{-1} b^{-n})a^{-1} \\
&= b^n ab^{-n} ab^n a^{-1} b^{-n} a^{-1} \\
&= w_n.
\end{aligned}$$

We can then calculate an upper bound for the area of w_n. Clearly the area of $\phi(k)$ is at most k, and hence the same goes for $\Phi(n,k)$, independently of n. It then follows that $\Psi(n)$ has area at most

$$1 + 2 + 4 + \ldots + 2^{n-1} = 2^n - 1.$$

Thus the area of w_n is at most $(2^n - 1) + (2^n - 1) = 2^{n+1} - 2$.

4. Let X be the presentation complex of $\mathbb{Z}^2 \cong \langle a, b \mid [a,b] \rangle$. Let $D \hookrightarrow X$ be a van-Kampen diagram for w_n of minimal area. The covering map $\tilde{X} \to X$ induces a map $\tilde{X} \to D \ldots$

5. (a) Isometries are affine maps with determinant ± 1. In $\mathbb{R}$, this means that all isometries are either translations or multiplication by -1 followed by translations. It immediately follows that B generates D_∞ and since $sr = t$ we get that A is also a generating set.

   (b) Omitted

6. Suppose $\langle A \mid R \rangle$ has a solvable word problem. Given some n, it is easy to find an algorithm that will construct the closed ball of radius n around 1 of $\mathrm{Cay}_A(F(A))$; call this graph G_n. As G_n is finite, we can check, using the algorithm for the word problem, which vertices should be identified in G_n according to R. The result will be B_n.

   Conversely, suppose we can construct B_n for any n. Given a word of length n, simply check whether the corresponding path in B_n ends at the identity or not.

7.