Model Theory and Non-Classical Logic
Example Sheet 3 Solutions

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A partial function $f\colon M \to N$ is called *partial elementary* if for all $\varphi(\bar{x}) \in \mathcal{L}$ and $\bar{d} \in \operatorname{dom}(f)$

$$M \models \varphi(\bar{d}) \iff N \models \varphi(f(\bar{d})).$$

In particular, these functions are injective (consider $\varphi(x, y)$ to be the formula $(x = y)$). If $f\colon M \to N$ is partial elementary and $A \coloneqq \operatorname{dom} f$ then we call the map $f\colon A \to N$ just elementary.

We also use the following strengthening of elementary amalgamation (which is proved in the same way).

Proposition 1. *Let M be a structure. Suppose A is a subset of M and $f\colon A \to M$ is an elementary map. Then there exists an elementary extension N of M and an elementary embedding $g\colon M \to N$ such that the following diagram commutes.*

$$\begin{tikzcd}
& N & \\
M \arrow[ur, "g"] & & M \arrow[ul, hook'] \\
& A \arrow[ul, hook] \arrow[ur, "f"'] &
\end{tikzcd}$$

Proof. □

1. Let p be a complete 1-type. Then for all $a \in \mathbb{Q}$ exactly one of $x < a$, $x = a$, and $x > a$ is in p.

   Suppose that $(x = a) \in p$ for some $a \in \mathbb{Q}$. We claim that $p = \operatorname{tp}^{\mathbb{Q}}(a/\mathbb{Q})$. By Corollary 1.4.7, there is an elementary extension M of $\mathbb{Q}$ such that $p = \operatorname{tp}^{M}(a/\mathbb{Q})$, where we used the fact that $(x = a) \in p$. But $\operatorname{tp}^{M}(a/\mathbb{Q}) = \operatorname{tp}^{\mathbb{Q}}(a/\mathbb{Q})$ as the extension is elementary, so the claim follows.

   Now assume that $(x = a) \notin p$ for all $a \in \mathbb{Q}$. Then the sets

   $$U \coloneqq \{a \in \mathbb{Q} \mid (x < a) \in p\} \quad \text{and} \quad L \coloneqq \{b \in \mathbb{Q} \mid (b < x) \in p\}$$

   partition $\mathbb{Q}$. Note that if $a \in U$ and $b \in L$, then $b < a$. Indeed, the sets are disjoint, so the only alternative is that $a < b$. But from this it follows that in any realization of the type p, say by an element c, we must have $c < a < b < c$ and thus $c < c$. As $\mathbb{Q} \models \forall y.\neg(y < y)$ this is a contradiction. Hence the complete type p gives rise to a partition U, L of $\mathbb{Q}$ such that $L < U$ in the above sense.

Conversely, any such partition can be extended to a complete type by the Ultrafilter Principle, and it is clear that this correspondence is bijective.

2. We prove a more general result. Let M be a structure and S a subset of M together with an elementary map $f\colon S \to M$. Then there is an elementary extension N of M and an automorphism of N extending f.

   To see that the result immediately solves the problem, observe the following. As $\mathrm{tp}^M(\bar{a}/A) = \mathrm{tp}^M(\bar{b}/B)$ it follows that the map $f\colon A \cup \{\bar{a}\} \to M$ given by fixing A and sending $\bar{a} \mapsto \bar{b}$ is elementary.

   Now we prove the result. We construct a chain of structures and maps $(M_i, f_i)_{i<\omega}$ satisfying the following properties for all $i \geq 0$.

   - $M_i \preccurlyeq M_{i+1}$.
   - $\operatorname{im} f_i \subseteq \operatorname{dom} f_{i+1}$ and f_{i+1} extends f_i.
   - If i is even $\operatorname{dom}(f_{i+1}) = M_i$.
   - If i is odd $\operatorname{im}(f_{i+1}) = M_i$.

   Furthermore we will have $M_0 \coloneqq M$ and $f_0 = f$.

   Suppose, for a moment, that we have finished the construction. Then let $N \coloneqq \bigcup_{i<\omega} M_i$, which is an elementary extension of M. Also let $\sigma \coloneqq \bigcup_{i<\omega} f_i\colon N \to N$, which is clearly an automorphism since $\operatorname{dom}(\sigma) = \operatorname{im}(\sigma) = N$. Also σ extends $f_0 = f$ so we will be done.

   We continue with the construction. Use elementary amalgamation to have the following diagram.

$$
\begin{tikzcd}
 & M_1 & \\
M_0 \arrow[ur, "f_1"] & & M_0 \arrow[ul, hook] \\
 & S \arrow[ul, hook] \arrow[ur, "f=f_0"'] &
\end{tikzcd}
$$

   It is easy to check that this satisfies the requirements. We can apply amalgamation to f_1^{-1} to obtain the following diagram

$$\begin{tikzcd}
 & M_2 & \\
M_1 \arrow[ur, hook] & & M_1 \arrow[ul, "g"'] \\
M_0 \arrow[u, hook] & & \\
 & \mathrm{im}(f_1) \arrow[ul, "f_1^{-1}"] \arrow[uur, hook] &
\end{tikzcd}$$

Define $f_2\colon \operatorname{im} g \to M_1$ to be g^{-1}. It follows from the diagram that $g \circ f_1(x) = x$ for $x \in M_0$. Hence we have $f_1(x) = f_2(x)$ for all $x \in M_0$ and so f_2 extends f_1. Furthermore it is easy to see that f_2 is an isomorphism $\operatorname{im} g \to M_1$ so in particular it is surjective and so $\mathrm{im}(f_2) = M_1$. Hence all conditions are satisfied. Keep going on in this way (apply amalgamation to f_i when i is even and to f_i^{-1} when i is odd).

3. First we show that, given a finite number of complete types $p_1, \ldots, p_k \in S_n^M(M)$ there is an elementary extension N of M realizing all of them. For $k = 0$ this is trivial. Now suppose there is an elementarily extension N' of M realizing $p_1, \ldots, p_{k-1}$. Note that $S_n^M(M) = S_n^{N'}(M)$ since the extension is elementary, so in particular $p_k \in S_n^{N'}(M)$ and by Proposition 1.4.6 there is an elementary extension N of N' realizing p_k. Obviously N is an elementary extension of M realizing $p_1, \ldots, p_k$ so we are done by induction.

   Back to the main problem. To the language $\mathcal{L}$ we add a constant for each element of M and we add n constants $c_1^p, \ldots, c_n^p$ for every $p \in S_n^M(M)$. In the expanded language, consider the theory

$$\left(\bigcup S_n^M(M)\right) \cup \mathrm{Diag}_{\mathrm{el}}(M)$$

   where each $\varphi(\bar{x}) \in p \in S_n^N$ is replaced by $\varphi(\bar{c}^p)$. Clearly if this theory is consistent then we are done. But every finite subset of this theory is satisfied by an elementary extension of M that has to realize only finitely many types, so we are done by our previous result.

4.

   (a) Let p, q be distinct types. Without loss of generality, we assume that there is a formula $\varphi(\bar{x})$ such that $\varphi \in p$ but $\varphi \notin q$. Then $[\![\varphi]\!]$ is a clopen set containing p but not q. This shows that $S_n^M(A)$ is totally disconnected.

   For the second part, we need a claim.

   Claim 1. *Let F be a set of $\mathcal{L}_A$ formulae with n variables. Add n new constants $\bar{c}$ to the language. Then the set $\mathcal{C} := \{\, [\![\varphi]\!] \mid \varphi(\bar{x}) \in F \,\}$ covers*

$S_n^M(A)$ *if and only if the theory*

$$\mathcal{T} := \mathrm{Th}_A(M) \cup \{\, \neg\varphi(\bar{c}) \mid \varphi(\bar{x}) \in F \,\}$$

is inconsistent.

Proof. Suppose $\mathcal{T}$ were consistent. Then $\neg F$ is an n-type, which, by the Ultrafilter Principle, can be extended to a complete n-type $q \in S_n^M(A)$. For all $\varphi \in F$ we must have $\neg\varphi \in q$, which means $\varphi \notin q$; thus $\mathcal{C}$ does not cover q.

Conversely, suppose that there is some $q \in S_n^M(A)$ such that $\varphi \notin q$ for all $\varphi \in F$. That means that $\neg\varphi \in q$ for all $\varphi \in F$ since q is complete. By definition of type, we have that $\mathrm{Th}_A(M) \cup q$ is consistent when we replace the variables $\bar{x}$ in q by the constants $\bar{c}$. It follows that $\mathcal{T}$ is consistent. $\square$

Back to the problem, let $\mathcal{C}$ be an open cover of $S_n^M(A)$. As open sets are unions of basis elements, we can assume that $\mathcal{C}$ is of the form $\{\, [\![\varphi]\!] \mid \varphi(\bar{x}) \in F \,\}$ for some set of $\mathcal{L}_A$-formulae F.

Now we know that $\mathcal{T}$ is inconsistent, where $\mathcal{T}$ is as in Claim 1. By the Compactness Theorem (for first-order logic) there is a finite subset $\mathcal{T}'$ of $\mathcal{T}$ that is inconsistent. Hence there is a finite subset F' of F such that $\mathrm{Th}_A(M) \cup \{\neg\varphi(\bar{c}) \mid \varphi(\bar{x}) \in F'\}$ is inconsistent. Again by Claim 1, the set $\mathcal{C}' := \{\, [\![\varphi]\!] \mid \varphi(\bar{x}) \in F' \,\}$, which is a finite subset of $\mathcal{C}$, covers $S_n^M(A)$.

(b) I think that he meant to define $f^*(p) := \{\phi(\bar{x}, f(\bar{a})) \mid \phi(\bar{x}, \bar{a}) \in p\}$, and that we need to show that $f^*(p) \in S_n^N(f(A))$.

To show that $f^*(p) \in S_n^N(f(A))$ first we need to show that

$$\mathrm{Th}_{f(A)}(N) \cup f^*(p)$$

is satisfiable. By assumption p is an n-type, so there is an elementary extension X of M and a tuple $\bar{r} \in X$ with $\phi(\bar{r}, \bar{a})$ for all $\phi(\bar{x}, \bar{a}) \in p$. Note that X can also be interpreted as an $\mathcal{L}_{f(A)}$-structure. As f is elementary it is clear that $X \models \mathrm{Th}_{f(A)}(N)$ and is immediate $X \models \phi(\bar{r}, f(\bar{a}))$. This all shows that $f^*(p)$ is an n-type; and it is complete since p is complete.

Now we show that f^* is continuous. By general topology, it suffices to show that for each basis element $[\![\varphi]\!] \subseteq S_n^N(f(A))$ the set $(f^*)^{-1}([\![\varphi]\!])$ is open in $S_n^M(A)$.

So, let $\varphi(\bar{x}, f(\bar{a}))$ be an $\mathcal{L}_{f(A)}$-formula. Then

$$\begin{aligned}
(f^*)^{-1}([\![\varphi]\!]) &= \{p \in S_n^M \mid f^*(p) \in [\![\varphi]\!]\} \\
&= \{p \in S_n^M \mid \varphi \in f^*(p)\} \\
&= \{p \in S_n^M \mid \varphi(\bar{x}, f(\bar{a})) = \psi(\bar{x}, f(\bar{a})) \text{ for some } \psi(\bar{x}, \bar{a}) \in p\}.
\end{aligned}$$

But notice that, as f is injective, $\varphi(\bar{x}, f(\bar{a})) = \psi(\bar{x}, f(\bar{a}))$ implies that $\psi = \phi$. Thus,

$$(f^*)^{-1}(\llbracket\varphi\rrbracket) = \{p \in S_n^M \mid \varphi(\bar{x}, \bar{a}) \in p\} = \llbracket\varphi(\bar{x}, \bar{a})\rrbracket$$

which is open in $S_n^M(A)$.

5. In the proof of Theorem 1.4.11 we make a small modification. Let $f\colon \omega \to \omega \times \omega$ be a bijection. When we define θ_s for odd $s = 2i+1$ then if $f(i) = (j, k)$ instead of taking $\bar{d}_i$ we take $\bar{d}_j$, run the same process to get ψ and we let φ be a formula in p_k that is not implied by ψ. The rest of the proof is the same, except that at the very end we notice that if $\bar{c} \in C^n$ then $\bar{c} = \bar{d}_j$ for some j and that if $k < \omega$ then we can define $i := f^{-1}(j, k)$; it then follows that θ_{2i+2} implies that $\bar{c}$ does not realize p_k. As $\bar{c}$ and k were arbitrary, we are done.

6. Let $\mathcal{L}$ be the language which has as signature $\aleph_2$ constants, say $\{c_\alpha\}_{\alpha<\aleph_2}$. Let $\mathcal{T}$ be the theory of uncountable sets, i.e.

$$\mathcal{T} := \{c_\alpha \neq c_{\alpha'} \mid \alpha < \alpha' < \aleph_1\}.$$

Define

$$p := \{x \neq c_\alpha \mid \alpha < \aleph_2\}.$$

It is easy to check that this is a 1-type of $\mathcal{T}$ (any set of cardinality bigger than $\aleph_2$ is a model of $\mathcal{T}$ realizing p).

For the sake of contradiction, suppose this type is isolated by a formula $\varphi(x)$. This means that $\mathcal{T} \cup \{\varphi(x)\}$ is satisfiable and

$$\mathcal{T} \models \forall x.(\varphi(x) \to x \neq c_\alpha)$$

for all $\alpha < \aleph_2$. Choose α such that c_α does not appear in $\mathcal{T}$ nor in $\varphi(x)$; this is possible because $\mathcal{T}$ mentions only $\aleph_1$-many constants and $\varphi(x)$ only finitely many. Hence by generalization we have

$$\mathcal{T} \models \forall x, y.(\varphi(x) \to x \neq y).$$

In particular,

$$\mathcal{T} \models \forall x.(\varphi(x) \to x \neq x).$$

contradicting the fact that $\mathcal{T} \cup \varphi(x)$ is satisfiable. Thus p is not isolated. However there can be no countable model of $\mathcal{T}$ ommiting p since $\mathcal{T}$ has no countable models!

7. Let $\mathcal{M}$ be a countable model of PA. If $\mathcal{L}$ denotes the language of arithmetic, we add a constant c to the language and let $\mathcal{L}^+ := \mathcal{L}_{\{c\}\cup\mathcal{M}}$. We define the $\mathcal{L}^+$-theory

$$\mathcal{T} := \mathrm{Diag}_{\mathrm{el}}(M) \cup \{c > m \mid m \in \mathcal{M}\}.$$

This theory is clearly consistent by Compactness.

Say an element $m \in \mathcal{M}$ is *natural* if the interval $[0, m]$ is finite. If m is natural then there are $m_1, \dots, m_k \in \mathcal{M}$ such that

$$\mathcal{M} \models \forall x. \left(\bigwedge_{i=1}^{k} x \neq m_i \to x > m \right)$$

If m_0 is not natural we say it is *unnatural*. For every unnatural $m_0 \in \mathcal{M}$ we define

$$p_{m_0} \coloneqq \{x \neq m \mid m \in \mathcal{M}\} \cup \{x < m_0\}.$$

We claim that p_{m_0} is a non-isolated type over $\mathcal{T}$ for all unnatural $m_0 \in \mathcal{M}$. Firstly, if $p \subseteq p_{m_0}$ is finite then $\mathcal{T} \cup p$ is satisfiable precisely because $[0, m_0]$ is infinite, so p_{m_0} is a 1-type over $\mathcal{T}$.

Suppose, for the sake of contradiction, that p_{m_0} is isolated by an $\mathcal{L}^+$-formula $\varphi(x)$. Write $\varphi(x) = \psi(x, c)$ where ψ is an $\mathcal{L}_{\mathcal{M}}$-formula. As φ is an isolating formula we have that $\mathcal{T} \cup \{\varphi(x)\}$ is satisfiable. So let $\mathcal{M}'$ be a satisfying structure. Clearly $\mathcal{M}'$ is an elementary extension of $\mathcal{M}$ that includes two constants $c^{\mathcal{M}'}, d^{\mathcal{M}'}$ such that $\mathcal{M}' \models c^{\mathcal{M}'} > m$ for all $m \in \mathcal{M}$ and $\mathcal{M}' \models \psi(d^{\mathcal{M}'}, c^{\mathcal{M}'})$. As $\varphi(x)$ isolates p_{m_0} we have in particular that $\psi(x, c)$ implies that $x < m_0$. Thus

$$\mathcal{M}' \models \psi(d^{\mathcal{M}'}, c^{\mathcal{M}'}) \wedge d^{\mathcal{M}'} < m_0.$$

It follows that

$$\mathcal{M}' \models \exists z < m_0 . \psi(z, c^{\mathcal{M}'}).$$

Furthermore, for all $m \in \mathcal{M}$ we have that

$$\mathcal{M}' \models \exists y > m \exists z < m_0 . \psi(z, y).$$

Thus, as $\mathcal{M}$ is an elementary substructure of $\mathcal{M}'$ we have that

$$\mathcal{M} \models \exists y > m . \exists z < m_0 . \psi(z, y),$$

for all $m \in \mathcal{M}$. Hence,

$$\mathcal{M} \models \forall x . \exists y > x . \exists z < m_0 . \psi(z, y).$$

Now, PA proves the pigeonhole principle. Think of z as the pigeonholes, y as the pigeons, and $\psi(z, y)$ as the statement "pigeon y is assigned pigeonhole z"; then it is clear that, as we are trying to fit infinitely many pigeons in finitely many pigeonholes there is a pigeonhole with arbitrarily many pigeoholes (of course $[0, m_0]$ is not actually finite but PA proves the pigeonhole principle nevertheless). Thus,

$$\mathcal{M} \models \exists z < m_0 . \forall x . \exists y > x . \psi(z, y).$$

In particular, there is some $m_1 \in \mathcal{M}$ such that $m_1 < m_0$ and

$$\mathcal{M} \models \forall x . \exists y > x . \psi(m_1, y).$$

We claim that $\mathcal{T} \cup \{\psi(m_1, c)\}$ is consistent. If it were inconsistent then, by Compactness, there is $n \in \mathcal{M}$ such that

$$\mathrm{Diag}_{\mathrm{el}}(M) \cup \{c > n\} \models \neg\psi(m_1, c),$$

and thus $\mathrm{Diag}_{\mathrm{el}}(M) \models c > n \to \neg\psi(m_1, c)$. As $\mathrm{Diag}_{\mathrm{el}}(M)$ does not mention c we have, by generalization,

$$\mathrm{Diag}_{\mathrm{el}}(M) \models \forall y > n. \neg\psi(m_1, c),$$

contradicting the defining property of m_1. We conclude that $\mathcal{T} \cup \{\psi(m_1, c)\}$ is consistent. As $\varphi(x)$ isolates p_{m_0} we also must have

$$\mathcal{T} \models \psi(m_1, c) \to m_1 \neg m_1,$$

a contradiction. Thus none of the types p_{m_0} is isolated. By Q5, and the fact that $\mathcal{L}^+$ is countable, there is a countable model $\mathcal{N}$ of $\mathcal{T}$ omitting all p_{m_0} for $m_0 \in \mathcal{M}$ unnatural.

We claim that $\mathcal{N}$ is a proper end extension. It is clearly an (elementary) extension, and it is proper since $c^{\mathcal{N}}$ is greater than all elements of $\mathcal{M}$. If it weren't an end extension then there must be some $n \in \mathcal{N} \setminus \mathcal{M}$ and $m_0 \in \mathcal{M}$ such that $n < m_0$. Clearly this cannot happen for m_0 unnatural since $\mathcal{N}$ omits p_{m_0}. So suppose m_0 is natural. But then there are $m_1, \ldots, m_k \in \mathcal{M}$ such that

$$\mathcal{M} \models \forall x. \left(\bigwedge_{i=1}^{k} x \neq m_i \to x > m_0 \right).$$

As $\mathcal{N}$ is an elementary extension, we have that $\mathcal{N}$ also models this sentence, and thus $n > m_0$ as well as $n < m_0$, a contradiction. Thus $\mathcal{N}$ is a proper end extension of $\mathcal{M}$.

8. Let $M \coloneqq \{m_1, m_2, \ldots\}$ and $N \coloneqq \{n_1, n_2, \ldots\}$ be two countable ω-saturated, elementarily equivalent $\mathcal{L}$-structures. We construct a sequence $f_0, f_1, \ldots$ such that for all $i \in \mathbb{N}$:

   - f_i is an elementary partial function $M \to N$;
   - f_{i+1} extends f_i;
   - $\mathrm{dom}(f_i)$ (and hence $\mathrm{cod}(f_i)$) is finite;
   - $\{m_1, \ldots, m_i\} \subseteq \mathrm{dom}(f_i)$ and $\{n_1, \ldots, n_i\} \subseteq \mathrm{cod}(f_i)$.

   Define f_0 to be the empty function, which is elementary since M and N are elementarily equivalent. Suppose f_i has been defined, and let $D \coloneqq \mathrm{dom}(f_i)$ and $C \coloneqq \mathrm{cod}(f_i)$ be finite. Consider the complete 1-type

   $$\mathrm{tp}^M(m_{i+1}/D).$$

   Using the notation of Question 4 (b), we note that $f_i^*(p)$ is a complete 1-type of N by basically the same argument as in Q4 and the fact that f_i

is elementary. Since N is ω-saturated, it follows that there is some $n \in N$ realizing this type. Let $g\colon D \cup \{m_{i+1}\} \to C \cup \{n\}$ be the extension of f_i that sends $m_{i+1} \mapsto n$ (if $m_{i+1} \in D$ then just let $g = f_i$). Thus g is elementary by construction.

Similarly, consider the complete 1-type $p \coloneqq \mathrm{tp}^N(n_{i+1}/C \cup \{n\})$. As g is elementary it follows that

$$\{\varphi(x, \bar{d})\colon \varphi(x, g(\bar{d})) \in p \text{ for some } \bar{d} \in D \cup \{m_{i+1}\}\}$$

is a 1-type for M so it has a realization $m \in M$. Finally, we let $f_{i+1}\colon D \cup \{m, m_{i+1}\} \to C \cup \{n, n_{i+1}\}$ be the extension of g mapping $m \mapsto n_{i+1}$. For the same reasons as before, f_{i+1} is elementary. This finishes the construction.

Now let $f\colon M \to N$ be the union of all f_i. By construction, f is defined everywhere, elementary, and surjective. It is also injective since $N \models f(m) = f(m')$ will imply $M \models m = m'$. The fact that f is a homomorphism can be similarly verified.

9. We assume $\mathcal{T}$ is, in addition, consistent, since otherwise the result is trivial.

   Suppose there are finitely many equivalence classes for formulae (of a given arity). We claim that countable $\mathcal{T}$-models are ω-saturated. Indeed, suppose $M \models \mathcal{T}$ is a model and let $\bar{a} \subseteq M$ be a finite tuple. Let $p \in S_n^M(\bar{a})$; we have to show that M realizes p.

   By assumption we can consider p as a finite set of formulae to be satisfied, since $M \models \mathcal{T}$ and $\mathcal{T}$ has finitely many equivalence classes for formulae of a given arity, and the arity of formulae in p is bounded by $n+|\bar{a}|$. By taking the conjunction of all these formulae, we are left with a single formulae $\varphi(\bar{x}, \bar{a})$ and we have to show that $M \models \exists\bar{x}.\varphi(\bar{x}, \bar{a})$. But as p is a type, we know it is realized in an elementary extension N of M, and so $N \models \exists\bar{x}.\varphi(\bar{x}, \bar{a})$. As the extension is elementary we get that M realizes p. This shows that all models of $\mathcal{T}$ are ω-saturated.

   Now, any two countable models of $\mathcal{T}$ are elementarily equivalent (since $\mathcal{T}$ is complete) and ω-saturated, thus isomorphic by Q8. Hence $\mathcal{T}$ is $\aleph_0$-categorical, as desired.

   Conversely, suppose there is some n such that are infinitely many $\mathcal{T}$-equivalence classes of formulae with n variables $\bar{x} = (x_1, \ldots, x_n)$. We claim the following.

   Claim 2. *All models of $\mathcal{T}$ are infinite.*

   Proof. Suppose $\mathcal{T}$ has a finite model M. Then M thinks there are only finitely many equivalence classes of formulae with n-variables. Indeed, the equivalence class of a formulae $\varphi(\bar{x})$ is determined by its truth value on its inputs, i.e. we can see the formula as a Boolean function $M^n \to \{0, 1\}$, and there are only finitely many of those since M is finite. Hence there are formulae $\varphi_1(\bar{x}), \ldots, \varphi_k(\bar{x})$ such that for all formulae $\varphi(\bar{x})$ there is some $i \leq k$ with

   $$M \models \forall x.(\varphi(\bar{x}) \leftrightarrow \varphi_i(\bar{x})).$$

As $\mathcal{T}$ is complete we see that the same is true if we replace M by $\mathcal{T}$ in the above, and thus there are only finitely many $\mathcal{T}$-equivalence classes of formulae with n variables, a contradiction. $\square$

Now, the condition on the equivalence classes can be equivalently stated to say that the Lindenbaum-Tarski algebra $\mathcal{B}_n(\mathcal{T})$ is infinite. Hence the Fréchet filter on $\mathcal{B}_n(\mathcal{T})$ is proper and can be extended to a free ultrafilter on $\mathcal{B}_n(\mathcal{T})$ (this is by ES2 Q5(b)). This ultrafilter corresponds to a complete type p, and the fact that is free means that p is not isolated (!).

By the ommitting types theorem (here we use the countability of the language), there is a countable model $N \models \mathcal{T}$ that omits p. By definition of types there is a model $M \models \mathcal{T}$ that realizes p, say with a tuple $\bar{m} \in M$. By our claim M is infinite. Using the downwards Löwenheim–Skolem theorem (again using $|\mathcal{L}| = \aleph_0$) we can find an elementary substructure M' of M such that M' is countable and $\bar{m} \in M'$. As the substructure is elementary, we have $M' \models \mathcal{T}$ and that $\bar{m}$ realizes p in M'.

But then M' and N are countable models of $\mathcal{T}$, and one of the realizes p while the other ones omits p. We conclude that $\mathcal{T}$ is *not* $\aleph_0$-categorical.

Consider a theory $\mathcal{S}$ with quantifier elimination in a language whose signature has finitely many relation symbols (of arity at least 1) and no function symbols. The language clearly is countable. We claim that $\mathcal{S}$ is complete. Indeed, it suffices to decide quantifier-free sentences, but these are only Boolean combinations of $\perp$ by the signature, so this is trivial.

Again, by quantifier-elimination, each formula is equivalent to a quantifier-free formula, and using the disjunctive normal form and by analysing all the possible literals we can reduce everything to finitely many formulae.

10. Let ε be an infinite linear order contained in an infinite model of $\mathcal{T}$. It follows from the Ehrenfeucht–Mostowski theorem (by Skolemizing the language first) that there is an Ehrenfeucht–Mostowski functor F such that $\mathrm{Th}(F)$ expands $\mathrm{Th}(M, \eta)$. As a particular case of this, the sentences satisfied in M are also satisfied in $\mathrm{Th}(F)$. More particularly, F takes values in $\mathcal{T}$-models.

    Note that there is a homomorphism $G \to F(\eta)$ given by $g \mapsto F(g)$; that this is a homomorphism follows from functoriality of F. But if $F(g) = F(h)$ then, as F extends maps, we have that $g = h$, and so this homomorphism identifies G with a subgroup of $\mathrm{Aut}(F(\eta))$.

11. Let T be the set of all closed terms in $\mathcal{L}$, and define an equivalence relation on T by saying that for all $s, t \in T$

    $$s \sim t \iff s = t \in \Gamma.$$

    We claim that this is an equivalence relation. It is reflexive by the first condition of $=$-closed. Suppose $s = t \in \Gamma$. If we define $\varphi(x) := (x = s)$ then, as $\varphi(s) \in \Gamma$ we have $\varphi(t) = (t = s) \in \Gamma$; hence the relation is symmetric.

Finally, suppose $s = t$ and $t = u$ are in Γ. Let $\psi(x) \coloneqq x = u$. Then as $s = t$ and $\psi(t) \in \Gamma$ it follows that $\psi(s) \in \Gamma$, and so $s = u$ is in Γ; this shows that the relation is transitive.

Define $M \coloneqq T/\sim$, and if $t \in T$ we denote by $[t]$ the corresponding equivalence class. We make M into an $\mathcal{L}$-structure as follows. For a constant c we define $c^M \coloneqq [c]$. For a function symbol f of arity n we inductively define

$$f([t_1], \ldots, [t_n]) = [f(t_1, \ldots, t_n)].$$

We need to check that this assignment is well-defined. Suppose $t_1, \ldots, t_n, t'_1, \ldots, t'_n$ are such that $t_i \sim t'_i$ for all i. It can be proved by induction on n that

$$f(t_1, \ldots, t_n) \sim f(t'_1, \ldots, t'_n).$$

For relation symbols R we define

$$([t_1], \ldots, [t_n]) \in R^M \iff R(t_1, \ldots, t_n) \in T.$$

It can be checked that this is well-defined and thus M becomes an $\mathcal{L}$-structure. The rest can be easily shown by induction over the structure of formulae.

12.

(a) Clearly the formula $x = x$ is contained in $\mathrm{Th}(M, \omega)$ so $S(\eta)$ contains $t = t$ for every closed $\mathcal{L}_\eta$-term. Suppose $S(\eta)$ contains $\phi(s(\bar{c}), \bar{c})$ and $s(\bar{c}) = t(\bar{c})$, where s, t are $\mathcal{L}$-terms, ϕ is an atomic $\mathcal{L}$-formula, and $\bar{c} \in [\eta]^k$. It follows that if $\bar{d} \in [\omega]^k$ then

$$M \models \phi(s(\bar{d}), \bar{d}) \wedge s(\bar{d}) = t(\bar{d}),$$

so $M \models \phi(t(\bar{d}), \bar{d})$ and hence $\phi(t(\bar{x}), \bar{x})$ is in $\mathrm{Th}(M, \omega)$, from which the claim follows.

(b) Use Q11 on $S(\eta)$ to obtain an $\mathcal{L}$-structure $F(\eta)$ such that the set of atomic $\mathcal{L}_\eta$sentences true in $F(\eta)$ is exactly $S(\eta)$, and whose every element is the interpretation of some closed term in $\mathcal{L}_\eta$.

It follows that $F(\eta)$ contains a copy of η by its interpretation of constants, and this is a faithful copy since $x = y$ is obviously not in $\mathrm{Th}(M, \omega)$. Furthermore for every closed $\mathcal{L}$-term t there is an $\mathcal{L}$-term s and some $\bar{d} \in [\omega]^k$ such that

$$M \models t = s(\bar{d})$$

since M is generated by ω; it follows readily that $F(\eta)$ must be generated by η.

If $g \colon \eta \to \varepsilon$ is an order embedding then for every atomic formula ϕ (and hence for every quantifier free formula by induction) if $\bar{c} \in [\eta]^k$ then $F(\eta) \models \phi(\bar{c})$ implies $\phi(\bar{x}) \in \mathrm{Th}(M, \omega)$ which in turn implies that $F(\varepsilon) \models \phi(g(\bar{c}))$ where we crucially use the fact that g respects ordering. Using the method of diagrams we get some embedding $F(\eta) \to F(\varepsilon)$ which extends (and indeed it is determined by) g. This makes sure that the assignment is functorial so this is indeed an EM functor. Furthermore, $F(\omega)$ is generated by ω, from which it follows that $F(\omega) = M$.

(c) By the Lemma right after the definition of $\mathrm{Th}(F)$, we have that all quantifier-free sentences in $\mathrm{Th}(M, \omega)$ are contained in both $\mathrm{Th}(F)$ and $\mathrm{Th}(G)$. But then for every quantifier-free formula $\varphi(\bar{x})$ and $\bar{c} \in [\eta]^k$ we get that

$$F(\eta) \models \varphi(\bar{a}) \iff G(\eta) \models \varphi(\bar{a}).$$

As η is a generating set, we get again by the method of diagrams an isomorphism $F(\eta) \to G(\eta)$ fixing η.