Combinatorics

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Chapter 1

Introduction and Basic Results

This is a set of lecture notes taken by me from the Part III course “Combinatorics”, lectured by Professor Béla Bollobás in Michaelmas, 2023. I take full responsibility for any mistakes in these notes.

1 Chains, Antichains and Scattered Sets of Vectors

The following are two results we will be able to prove later in the course.

Theorem 1.1 (Littlewood and Clifford, 1943). *Let $z_1, \ldots, z_n \in \mathbb{C}$ where $|z_k| \geq 1$ for all k. Pick some $r > 0$. Then there exists some c depending only on r such that, if $\epsilon_k = \pm 1$ then, of the 2^n possible sums of the form $\sum_{k=1}^n \epsilon_k z_k$, at most $\frac{c 2^n \log n}{\sqrt{n}}$ of these fall into a circle of radius r.*

Paul Erdős improved this result in the special case where the points are real numbers.

Theorem 1.2 (Erdős, 1945). *Let $x_1, \ldots, x_n \in \mathbb{R}$ where $x_k \geq 1$ for all k. If $\epsilon_k = \pm 1$, then, out of the 2^n sums of the form $\sum_{k=1}^n \epsilon_k x_k$ at most $\binom{n}{\lfloor n/2 \rfloor}$ fall in the interior of an interval of length 2.*

We can see that this bound is the best possible. Take n to be even (the odd case is similar) and $x_k = 1$ for all k. Then the sum is 0 exactly $\binom{n}{n/2}$ of the time (same number of positive and negative ϵs) and otherwise the absolute value of the sum is greater than or equal to 2.

1.1 Hall's Marriage Theorem and Consequences

By $G = (U, W; E)$ we usually mean that G is a bipartite graph with bipartition $U \sqcup W$, and with edge-set E. In this case, we say that G is a *complete matching* from U into W if there is a subgraph H of G that contains all vertices of G and, for all $u \in U$ and $w \in U$,

$$d_H(u) = 1 \text{ and } d_H(w) \leq 1.$$

(Of course, $d_H(\cdot)$ refers to the degree of a vertex in H.)

This is a bit too formal. What we really mean by a complete matching from U to W is that there is a way to pair every element of U with an element of W it is adjacent to so that no element of U has to "share". Usually one takes U to be a set of women, V to be a set of men, and edges whenever a woman likes a man. Finding a complete matching is then the problem of marrying each women to a man (this is old-fashioned, I know) she likes. Hall's Marriage Theorem gives a necessary and sufficient condition under which this problem is solvable.

If $A \subseteq G$ we denote by $\Gamma(A)$ the set of neighbours of A in G (i.e. the set of vertices in G adjacent to at least one vertex of A). First suppose that G has a complete matching. Clearly, if $A \subseteq U$ then $|A| \leq |\Gamma(A)|$ since otherwise we wouldn't have enough men to marry all the women of A. What is surprising is that this trivial condition is not only necessary but actually sufficient for G to have a complete matching.

Theorem 1.3 (Hall's Marriage Theorem)**.** *Let $G = (U, W; E)$ be a bipartite graph. Then G has a complete matching if and only if for all $A \subseteq U$ we have $|A| \leq |\Gamma(A)|$.*

Proof. TODO □

Hall's Theorem is very useful, especially in situations where it seems like you don't need it. Let $\mathcal{F}$ be a *set system*, i.e. a sequence $(\mathcal{F}_1, \ldots, \mathcal{F}_m)$ of sets where each $\mathcal{F}_i$ is a finite set. A *set of distinct representatives* of $\mathcal{F}$ is a sequence $(a_i)_{i=1}^m$ with $a_i \in \mathcal{F}_i$ and $a_i \neq a_j$ for all i, j with $i \neq j$.

When does $\mathcal{F}$ has a set of distinct representatives? If $\mathcal{F}$ has one then clearly for all $I \subset [m]$ we would have

$$\left| \bigcup_{i \in I} \mathcal{F}_i \right| \geq |I|.$$

This leads us to an equivalent formulation of Hall's Theorem.

Theorem 1.4. *A set system $\mathcal{F}$ has a set of distinct representatives if and only if for all $I \subset [m]$ we have*

$$\left| \bigcup_{i \in I} \mathcal{F}_i \right| \geq |I|.$$

Proof. One implication is obvious. For the other one, define a bipartite graph $([m], \bigcup_{i=0}^m \mathcal{F}_i)$ where the set of neighbours of $i \in [m]$ is just all elements of $\mathcal{F}_\rangle$. Then the hypothesis is exactly Hall's condition and so, by Hall's theorem, there is a complete matching from $[m]$ to $\bigcup_{i=0}^m \mathcal{F}_i$, that of course gives us a set of distinct representatives of $\mathcal{F}$. □

Next, we explore some simple applications of Hall's theorem.

Corollary 1.5. *Let $G = (U, W)$ be a bipartite graph with at least one edge such that $d(u) \geq d(w)$ for all $u \in U$ and $w \in W$. Then there is a complete matching from U to W.*

Proof. We use Hall's theorem. Let $A \subseteq U$ and let d be an integer such that

$$d(w) \leq d \leq d(u)$$

for all $u \in U$ and $w \in W$—such an integer exists because of the hypothesis (take, e.g., $d = \min_{u \in U} d(u)$). Consider the number of edges between A and $\Gamma(A)$, call it e. Each vertex in A contributes at least d edges, so $d|A| \leq e$. Similarly, each vertex in $\Gamma(A)$ contributes at most d edges, so $e \leq d|\Gamma(A)|||$. Putting this together we see that $d|A| \leq d|\Gamma(A)|$. Note that $d \neq 0$ since G has at least one edge. Thus, we can conclude that $|A| \leq |\Gamma(A)|$. As A was arbitrary Hall's theorem applies and we are done. □

Definition 1.6 (Weight)**.** Let (U, W) be a bipartite graph and let $A \subseteq U$ and $B \subseteq W$. Define the *weight* of A and the weight of B, denoted by $w(A)$ and $w(B)$ respectively, as

$$w(A) = \frac{|A|}{|U|}$$

$$w(B) = \frac{|B|}{|W|}.$$

Definition 1.7 (Biregularity)**.** A bipartite graph (U, W) is *biregular* with biregularity (k, l) if $d(u) = k$ and $d(w) = l$ for all $u \in U$ and $w \in W$.

It turns out biregular graphs have complete matchings. Before showing this we will prove a lemma.

Lemma 1.8. *For all biregular graphs (U, W) and all $A \subseteq U$ we have*

$$w(A) \leq w(\Gamma(A)).$$

Proof. Note that we have that the number of edges of G is both $k|U|$ and $l|W|$. Also, if $A \subseteq U$, we have that the number of edges from A to $\Gamma(A)$ is $k|A|$ and this number is at most the edges coming out of $\Gamma(A)$, i.e. $l|\Gamma(A)|$. Therefore

$$w(A) = \frac{k|A|}{k|U|} \leq \frac{l|\Gamma(A)|}{k|U|} = \frac{|\Gamma(A)|}{|W|} = w(\Gamma(A)).$$ □

Corollary 1.9. *Let (U, W) be a biregular graph. If $|U| \leq |W|$ then there is a complete matching from $|U|$ to $|W|$ and vice versa.*

Proof. Without loss of generality suppose $|U| \leq |W|$. Hall's condition is immediately satisfied since by Lemma 1.8 we have, for all $A \subseteq U$:

$$|A| = |U|w(A) \leq |W|w(\Gamma(A)) = |\Gamma(A)|.$$ □

2 Sperner's Theorem

Now we would like to study the graph with vertex set $\mathcal{P}([n])$ with an edge between two subsets of $[n]$ if one of them contains the other. Define an *antichain* in this graph to be a set $\mathcal{A}$ of subsets of $[n]$ so that for all distinct $A, B \in \mathcal{A}$ we have $A \nsubseteq B$ and $B \nsubseteq A$. How big can an antichain be? First, a lemma.

Corollary 2.1. *Let $s, r, n \in \mathbb{N}$ be with $0 \leq r < s \leq n$. Define $X = [n]$. If $|s - \frac{n}{2}| \leq |r - \frac{n}{2}|$, then there exists some injective function $f \colon X^{(r)} \to X^{(s)}$ such that for all $A \in X^{(r)}$ we have $A \subseteq f(A)$. Similarly, if $|s - \frac{n}{2}| \geq |r - \frac{n}{2}|$, then there exists some injection $g \colon X^{(s)} \to X^{(r)}$ such that for all $B \in X^{(s)}$ we have $B \supseteq g(B)$.*

Proof. Now we construct a biregular graph that we will often come back to. Let G be a bipartite graph with bipartition $(X^{(r)}, X^{(s)})$ with an edge joining $A \in X^{(r)}$ and $B \in [n]^{(s)}$ iff $A \subseteq B$. We claim G is biregular with biregularity

$$\left(\binom{n-r}{s-r}, \binom{s}{r}\right).$$

Indeed, if $A \in X^{(r)}$ then, to construct some $B \in X^{(s)}$ with $A \subseteq B$ we add $s - r$ elements from the set $X \setminus A$ which has size $n - r$. Similarly, if $B \in X^{(s)}$ and we want to construct some $A \in X^{(r)}$ with $A \subseteq B$ then we just choose r elements from B (which has size s) and make it into our A.

If $|s - \frac{n}{2}| \leq |r - \frac{n}{2}|$ then $\binom{n}{r} \leq \binom{n}{s}$ and so, by Corollary 1.9 there is a complete matching from $X^{(r)}$ to $X^{(s)}$: this tells us how to define our f. Similarly, if $|s - \frac{n}{2}| \geq |r - \frac{n}{2}|$ then $\binom{n}{r} \geq \binom{n}{s}$ and we do the same thing to define g. $\square$

Theorem 2.2 (Sperner, 1928)**.** *Let $\mathcal{A} \subseteq \mathcal{P}([n])$ be an antichain for some $n > 1$. Then $|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}$.*

Proof. The idea is to prove that $\mathcal{P}([n])$ can be covered by $\binom{n}{\lfloor n/2 \rfloor}$ chains, i.e. subsets $\mathcal{B} \subset \mathcal{P}([n])$ such that for all $A, B \in \mathcal{B}$ we have $A \subseteq B$ or $B \subseteq A$. Note that any antichain can intersect a chain at most once, thus the claim follows.

For s with $n/2 < s \leq n$ let $g_s \colon X^{(s)} \to X^{(s-1)}$ be an injection with $B \supseteq g(B)$, which we know exists by Corollary 2.1. Similarly, for r with $0 \leq r < n/2$ let $f \colon X^{(r)} \to X^{(r+1)}$ be an injection with the property that $A \subseteq f(A)$ for all $A \in X^{(r)}$.

Let $m = \lfloor n/2 \rfloor$. For $A \in X^{(r)}$ with $0 \leq r < n/2$ we construct the chain generated by A, denoted as $\mathcal{C}_A$, with the following algorithm.

1. $A \in \mathcal{C}_A$
2. Let $M \in X^{(t)}$ be the unique greatest element of $\mathcal{C}_A$.
   (a) If $0 \leq t < n/2$ declare $f_t(M) \in \mathcal{C}_A$
   (b) If $m \leq t < n$ and there is some $C \in X^{(t+1)}$ so that $g_{t+1}(C) = M$ we declare $C \in \mathcal{C}_A$.

(c) Else, terminate.

3. Repeat (b).

Similarly, if $B \in X^{(s)}$ with $n/2 < s \leq n$ we construct the chain generated by B:

1. $B \in \mathcal{C}_B$
2. Let $M \in X^{(t)}$ be the unique least element of $\mathcal{C}_B$.
   (a) If $n/2 < t \leq n$ declare $g_t(M) \in \mathcal{C}_B$
   (b) If $0 < t \leq m$ and there is some $C \in X^{(t-1)}$ so that $f_{t-1}(C) = M$ we declare $C \in \mathcal{C}_{\mathcal{B}}$.
   (c) Else, terminate.
3. Repeat (b).

Finally, define a set $\mathcal{C}$ of chains of $\mathcal{P}(X)$ in stages as follows.

1. Repeat (a) for $k = 0, 1, \ldots, n$ in that order.
   (a) If there is an element $A \in X^{(k)}$ so that A does not appear in any chain of $\mathcal{C}$ declare $\mathcal{C}_A \in \mathcal{C}$.

It is clear that the chains in $\mathcal{C}$ cover $\mathcal{P}(X)$. Furthermore, the injectivity of f and g guarantee that the chains are disjoint. Finally, each chain contains an element of $X^{(m)}$ by definition of $\mathcal{C}_A$. Therefore $|\mathcal{C}| = \binom{n}{m}$ as desired. □

Using Sperner's Theorem, Erdős proved Theorem 1.2 as follows.

Proof of Theorem 1.2. Let I be an interval of length 2. For $\epsilon = (\epsilon_i)_{i=1}^n$ set $x_\epsilon := \sum_{i=1}^n \epsilon_i x_i$ and

$$F_\epsilon := \{i \mid \epsilon_i = 1\}.$$

Let $\mathcal{F} = \{F_\epsilon \mid x_\epsilon \text{ is in the interior of } I\}$. Then $\mathcal{F}$ is an antichain, for if $F_\epsilon \subsetneq F_\delta$ we would have that x_δ contains one more "plus sign" than x_ϵ and thus $x_\delta - x_\epsilon \geq 2$, contradicting that x_δ and x_ϵ are in the interior of I. Thus, by Sperner's Theorem, we have that $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$. □

3 More general posets

There are more general settings in which the previous results hold. We assume the definition of a poset: the definitions of chains and antichains generalize to the obvious ones.

There is one more definition that we will need. Let $(S, \leq)$ be a poset and suppose $x, y \in S$. Then y *covers* x if $x < y$ and there is no $z \in S$ with $x < z < y$.

3.1 Regular graded posets

Now we begin by generalizing properties of $\mathcal{P}([n])$. Say a poset $(S, \le)$ is *graded* if S has a partition $S = \bigsqcup_{i=0}^{m} S_i$ such that each S_i is an antichain and if $x < y$ there are elements $x_i, x_{i+1}, \ldots, x_j \in S$ such that $x_k \in S_k$ for all k and

$$x = x_i < x_{i+1} < \cdots < x_j = y.$$

(Furthermore we assume that for all k with $0 < k < m$ and $x \in S_k$ there is some $x' \in S_{k-1}$ and $x'' \in S_{k+1}$ such that $x' < x < x''$ to rule out e.g., trivial posets. All of our posets are connected). Clearly the definition implies that if y covers x then $x \in S_k$ and $y \in S_{k+1}$ for some k.

Furthermore, the graded poset $(S, \le)$ is said to be *regular* if for all k with $0 \le k \le m$ there exists some integers r_k and s_k so that for all $x \in S_k$ we have that x covers exactly r_k elements of S_{k-1} (if $k > 0$) and is covered by exactly s_k elements of S_{k+1} (if $k < m$).

Just in case we haven't emphasised this enough: for us, the perennial example of a regular graded poset is $\mathcal{P}([m])$. However, soon we will meet other regular graded posets which could use with results analogous to Sperner's Theorem, for example.

For a graded poset $(S, \le)$ and $A \subseteq S$ we define the *weight* of A as follows.

(a) If $A \subseteq S_i$ for some i we define $w(A) \coloneqq \frac{|A|}{|S_i|}$.

(b) For any $A \subseteq S$ we define

$$w(A) \coloneqq \sum_{i=0}^{m} w(A \cap S_i).$$

Theorem 3.1. *Let A be an antichain in a regular graded poset $(S, \le)$. Then $w(A) \le 1$.*

First proof. First we fix some notation. From now on, we abbreviate $A \cap S_i$ by A_i. If $B \subseteq S_i$ for some i denote by $\Gamma(B)$ the set of elements of S_{i+1} that cover an element of B.

We claim that for all $0 \le k \le m$ we have

$$\sum_{i=0}^{k} w(A_i) \le w\left(\bigcup_{i=0}^{k} \Gamma^{k-i}(A_i)\right)$$

Note that the argument of w in the right-hand side is a subset of S_k so this is well-defined. Use induction on k. For $k = 0$ this reduces to $w(A_0) = w(A_0)$.

Now suppose $k > 0$. As A is an antichain, A_k does not contain elements from $\Gamma(A_{k-1})$ nor of $\Gamma^2(A_{k-2})$, and so on. Therefore,

$$\begin{aligned} w\left(\bigcup_{i=0}^{k} \Gamma^{k-i}(A_i)\right) &= w(A_k) + w\left(\bigcup_{i=0}^{k-1} \Gamma^{k-i}(A_i)\right) \\ &= w(A_k) + w\left(\Gamma\left(\bigcup_{i=0}^{k-1} \Gamma^{k-i-1}(A_i)\right)\right) \\ &\geq w(A_k) + w\left(\bigcup_{i=0}^{k-1} \Gamma^{k-i-1}(A_i)\right), \end{aligned}$$

where the last inequality is a consequence of Lemma 1.8 by noting that S_t and S_{t+1} form a biregular graph for all t with the covering relation (that is what it means for the graded poset to be regular) and that the weight and Γ defined for bipartite graphs and for graded posets coincide in this case. Therefore, by the inductive hypothesis, we have

$$w\left(\bigcup_{i=0}^{k} \Gamma^{k-i}(A_i)\right) \geq w(A_k) + \sum_{i=0}^{k-1} w(A_i).$$

as desired. This closes the induction. Now specialize to the case where $k = m$ to get

$$w(A) = \sum_{i=0}^{m} w(A_i) \leq w\left(\bigcup_{i=0}^{m} \Gamma^{m-i}(A_i)\right) \leq 1,$$

where the last inequality follows from the fact that $w(S_m) = 1$ and subsets of S_m cannot exceed this weight. $\square$

Second proof. Suppose $A \neq \emptyset$ for otherwise the theorem is trivial. We define the *span* of A to be the maximum integer k so that there exists some i with both A_i and A_{i+k} being nonempty.

We use induction on the span k of A. If k is zero then $A \subseteq S_i$ for some i and clearly $w(A) = w(A_i) \leq 1$. Assume $k \geq 1$. Let h be such that $A_h \neq \emptyset$ and $A_j = \emptyset$ for all $j > h$. Let A'_{h-1} be the set of elements in S_{h-1} that are covered by some element of A_h; clearly A and A'_{h-1} are disjoint. By Lemma 1.8 we have $w(A'_{h-1}) \geq w(A_h)$. Then, if we replace A_h by A'_{h-1} in A we cannot decrease the weight. That is, if we define

$$A' \coloneqq (A \setminus A_h) \cup A'_{h-1},$$

then $w(A') \geq w(A)$. But notice that the span of A' is less than k. The inductive hypothesis implies $w(A') \leq 1$ and we are done. $\square$

We will also give a third proof but we need some preliminaries. Note that a regular graded poset $(S, \leq)$ has various parameters, say $(r_i, s_i)_{i=0}^{m}$ where if $x \in S_k$ it covers r_k elements in S_{k-1} and is covered by s_k elements in S_{k+1}. We assume $r_0 = 0$ and $s_m = 0$ but otherwise $r_k, s_k \geq 1$—this is in order to rule out disconnected posets.

As we are disregarding disconnected posets, it is clear that every maximal chain in S has length $m+1$. For $x \in S$ denote by $\mu(x)$ the number of maximal chains containing x. Let us calculate $\mu(x)$.

If $x \in S_k$ for some k we can construct a maximal chain by choosing one element from each of $S_0, S_1, \ldots, S_{k-1}$ these elements are comparable to x, and similarly for $S_{k+1}, \ldots, S_m$. We know how many choices there are for S_{k-1} and S_{k+1}: these are r_k and s_k respectively. And once we've made those choices we know how many choices there are for S_{k-2} and S_{k+2}: these are r_{k-1} and s_{k+1}. Continuing on we conclude that if $x \in S_k$

$$\mu(x) = \left(\prod_{i=1}^{k} r_i\right)\left(\prod_{i=k}^{m-1} s_i\right).$$

Note that this number doesn't depend on the specific element x, only on its level k. Let M denote the number of maximal chains in S. As every maximal chain must pass through all levels, we conclude that for all $1 \leq k \leq m$ and for all $x \in S_k$ we have

$$M = |S_k|\mu(x).$$

Third proof. Note that maximal chains passing through different elements of A must be distinct since two distinct elements of the antichain A cannot form part of a chain. It follows that

$$\begin{aligned} M &\geq \sum_{x \in A} \mu(x) \\ &= \sum_{k=0}^{m} |A_h| \frac{M}{S_h} \\ &= M \sum_{k=0}^{m} \frac{|A_h|}{|S_h|} = Mw(A). \end{aligned}$$

As our posets are connected there is at least one maximal chain, i.e., $M > 0$. Divide by M and we are done. □

3.2 Back to $\mathcal{P}(n)$

We can apply the previous theorem to the specific case of $\mathcal{P}([n])$ (we abbreviate this as $\mathcal{P}(n)$ now).

Theorem 3.2. *If $\mathcal{A} \subseteq \mathcal{P}(n)$ is an antichain then $\sum_{A \in \mathcal{A}} \binom{n}{|A|}^{-1} \leq 1$. Equivalently, if we define $f_k \coloneqq |\mathcal{A} \cap X^{(k)}|$ then $\sum_{k=0}^{n} f_k \binom{n}{k}^{-1} \leq 1$.*

First proof. Clearly the second statement, in the language of graded posets, is just saying that the weight of A is less than or equal to 1, which is just Theorem 3.1. The first statement is also referring to the weight, but the sum runs over individual elements of $\mathcal{A}$ rather than the intersections over different levels. □

Second proof. This proof is due to Lubell. We say that $A \in \mathcal{P}(n)$ is contained in a permutation $\pi = x_1 x_2 \cdots x_n$ of $[n]$ if $A = \{x_1, \ldots, x_k\}$ where $k = |A|$.

Every permutation π contains at most one element of $\mathcal{A}$. This is because, if it contained two, one of them would be a subset of the other and $\mathcal{A}$ is an antichain. Also, given a set $A \in \mathcal{P}(n)$ with $|A|= k$ we see that A is contained in exactly $k!\,(n-k)!$ permutations. Trivially, the number of permutations containing an element of $\mathcal{A}$ is at most the number of permutations of $\mathcal{P}(n)$. In other words,

$$\sum_{A \in \mathcal{A}} |A|!\,(n-|A|)! \leq n! \qquad \square$$

4 Symmetric Chains

Definition 4.1. A chain in $\mathcal{P}(n)$ is called symmetric if it is of the form $C_i \subseteq C_{i+1} \subseteq \cdots \subseteq C_{n-i}$ and $|C_j|= j$ for all j.

Examples 4.2. In $\mathcal{P}(6)$ the following is a symmetric chain.

$$\{1\} \subseteq \{1,4\} \subseteq \{1,3,4\} \subseteq \{1,3,4,6\} \subseteq \{1,3,4,5,6\}.$$

The one-element chain $\{2,4,5\}$ is also symmetric in $\mathcal{P}(6)$ (but not in any other $\mathcal{P}(n)$). Similarly, the chain

$$\{2,5,6\} \subseteq \{2,4,5,6\}$$

is symmetric in $\mathcal{P}(7)$ but not symmetric anywhere else.

Theorem 4.3. *Every powerset has a partition into symmetric chains. Furthermore, such a partition of $\mathcal{P}(n)$ has size exactly $\binom{n}{\lfloor n/2 \rfloor}$.*

Proof. First, suppose that such a partition P existed. Symmetric chains always contain an element of $[n]^{(\lfloor n/2 \rfloor)}$ but they can contain at most one since any two are not comparable. Therefore there is a function $f\colon P \to [n]^{(\lfloor n/2 \rfloor)}$ sending each chain of the partition to the unique element of $[n]^{(\lfloor n/2 \rfloor)}$ it contains. Clearly f is injective (since elements of P are disjoint) and surjective (since P covers all of $\mathcal{P}(n)$). Hence f is a bijection and thus $|P|= \binom{n}{\lfloor n/2 \rfloor}$.

It remains to show that such partitions exist. We use induction on n. For $n = 0$ the single one-element chain $\{\emptyset\}$ works. Let $n \geq 2$ and assume that the result holds for $n-1$. Then, let P be a partition of $\mathcal{P}(n-1)$ into symmetric chains let $\mathcal{C} = \{C_i, C_{i+1}, \ldots, C_j\}$ be a chain in P.

If $i < j$ define two more chains

$$\begin{aligned} \mathcal{C}' &\coloneqq \{C_i, \ldots, C_j, C_j \cup \{n\}\} \\ \mathcal{C}'' &\coloneqq \{C_i \cup \{n\}, \ldots, C_{j-1} \cup \{n\}\}. \end{aligned}$$

If $i = j$ then just $\mathcal{C}'$ as above and omit $\mathcal{C}''$. Clearly both of these are symmetric chains in $\mathcal{P}(n)$. If we let P' be the collection of all such $\mathcal{C}'$ and $\mathcal{C}''$ then P' partitions $\mathcal{P}(n)$ into symmetric chains. $\square$

Write $l(n,i)$ for the number of symmetric chains of length $n+1-2i$ in a partition of $\mathcal{P}(n)$. Then

$$l(n,i) = \binom{n}{i} - \binom{n}{i-1}.$$

Indeed, the number of symmetric chains starting at level i would be $\binom{n}{i}$ but there are $\binom{n}{i-i}$ symmetric chains that start at a lower level and pass through level i.

Theorem 4.4. *Let $x_1, \ldots, x_n$ be vectors of norm at least 1 in some normed space. For a set $A \subseteq X \coloneqq [n]$, set $x_A \coloneqq \sum_{i \in A} x_i$. Let $\mathcal{A} \subseteq \mathcal{P}(n)$ be such that $A, B \in \mathcal{A}$ then $||x_A - x_B|| < 1$. Then $|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}$.*

To prove this, we need another result. Call a set $\mathcal{A} \subseteq \mathcal{P}(n)$ *sparse* or *scattered* if for all distinct $A, B \in \mathcal{A}$ we have $||x_A - x_B|| \geq 1$.

Let a partition P of $\mathcal{P}(n)$ be such that for all i, the number of components of size $n+1-2i$ in P is precisely $l(n,i)$. We call such a partition a *symmetric decomposition into sparse sets* abbreviated SDSS. Clearly Theorem 4.4 is a simple corollary of Theorem 4.3 and the following.

Theorem 4.5 (Kleitman)**.** *For all n, $\mathcal{P}(n)$ has a symmetric decomposition into sparse sets.*