Lecture 5

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Definition 5.1. Equality is a relationship between two expressions (quantities), stating that they represent (have) the same mathematical object (value). Equality between A and B is written $A = B$, and read "A equals B". In this equality, A and B are distinguished by calling them *left-hand side* (*LHS*), and *right-hand side* (*RHS*). The basic properties of equality are as follows:

1. Reflexivity (law of identity)

   For every a, one has $a = a$.

2. Symmetry

   For every a and b, if $a = b$, then $b = a$.

3. Transitivity

   For every a, b, and c, if $a = b$ and $b = c$, then $a = c$.

4. Substitution (substitution property)

   Formally, this is Definition 5.7. Informally, this means that if $a = b$, then a can replace b in any mathematical expression or formula without changing its meaning.

5. Operation application

   For every a and b, with some operation (function from a set into itself) $f(x)$, if $a = b$, then $f(a) = f(b)$.

Definition 5.2. Two objects that are not equal are said to be distinct.

Definition 5.3. The law (axiom) of identity is stated as follows: for every a, $a = a$. Informally, this is stated: "Whatever is, is." Symbolically,

$$\forall a(a = a).$$

Definition 5.4. The law (axiom) of non-contradiction is stated as follows: nothing can both be and not be. In other words, two or more contradictory statements cannot both be true in the same sense at the same time. Symbolically,

$$\neg(a \wedge \neg a),$$

where a is a statement.

Definition 5.5. The law (axiom) of excluded middle is stated as follows: everything must either be or not be. Symbolically,

$$p \vee \neg p,$$

where p is a statement.

Remark 5.6. Note that in all past lectures and all future lectures (including the current lecture), we assume that the Definitions 5.3-5.5 are true. Collectively these three laws are known as the laws of thought.

Definition 5.7. The substitution property (also referred to as "Leibniz's law") is stated as follows: if two things are equal, then any property of one must be a property of the other. This can be stated formally as: for every a and b, and any formula $\phi(x)$, with a free variable x, if $a = b$, then $\phi(a)$ implies $\phi(b)$. Symbolically,

$$(a = b) \implies [\phi(a) \implies \phi(b)].$$

Definition 5.8. Let X and Y be sets. X and Y are said to be disjoint if $X \cap Y = \emptyset$.

Definition 5.9. The axiom of extensionality is stated as follows: two sets are equal (are the same set) if they have the same elements. Symbolically,

$$\forall x \forall y [\forall z (z \in x \iff z \in y) \implies x = y]$$

Definition 5.10. The axiom of regularity is stated as follows: every non-empty set x contains an element y such that x and y are disjoint sets. Symbolically (in first-order logic),

$$\forall x (x \neq \emptyset \implies (\exists y \in x)(y \cap x = \emptyset)).$$

Remark 5.11. Note that in axiomatic set theory *everything* is a set, i.e. we don't work with other objects. So even if one were to denote some objects by $1, 2, 3, 4, 5$, they are in fact sets. For example, if we use the *standard construction* of positive integers in ZFC, then we have

$$\begin{aligned} 0 &= \emptyset, \\ 1 &= \{0\}, \\ 2 &= \{0, 1\}, \\ 3 &= \{0, 1, 2\}, \\ 4 &= \{0, 1, 2, 3\}, \\ 5 &= \{0, 1, 2, 3, 4\}. \end{aligned}$$

The axiom of regularity says that one of the elements of the set $A = \{1, 2, 3, 4, 5\}$ is a set which is disjoint with A. Indeed, 1 is such a set - the only element of 1 is $0 = \emptyset$, which is not an element of A; hence $1 \cap A = \emptyset$.

Definition 5.12. Subsets are commonly constructed using set builder notation. In general, a subset of a set z obeying a formula $\varphi(x)$ with one free variable x may be written as

$$\{ x \in z \mid \varphi(x) \}.$$

The axiom schema of specification states that this subset always exists (it is an axiom schema because there is one axiom for each φ). Formally, let φ be any formula in the language of ZFC with all free variables among $x, z, w_1, \ldots, w_n$ (y is not free in φ). Then,

$$\forall z \forall w_1 \forall w_2 \cdots \forall w_n \exists y \forall x [x \in y \iff ((x \in z) \wedge \varphi(x, w_1, w_2, \ldots, w_n, z))].$$

Note that the axiom schema of specification can only construct subsets and does not allow the construction of entities of the more general form:

$$\{ x \mid \varphi(x) \}.$$

Definition 5.13. The axiom of pairing is stated as follows: if x and y are sets, then there exists a set which contains x and y as elements. Symbolically,

$$\forall x \forall y \exists z ((x \in z) \wedge (y \in z)).$$

Definition 5.14. The axiom of union states that for any set X, there exists a set $\bigcup X$ which consists of just the elements of the elements of that set X. More formally: given any set X, there is a set Y such that, for any element u, u is a member of Y if and only if there is a set z such that u is a member of z and z is a member of X. Symbolically,

$$\forall X \exists Y \forall u (u \in Y \iff \exists z ((u \in z) \wedge (z \in X))).$$

Note that this formula doesn't directly assert the existence of $\bigcup X$, the set $\bigcup X$ can be constructed from Y in the above formula using the axiom schema of specification:

$$\bigcup X = \{ x \in Y \mid \exists u (x \in u \wedge u \in X) \}.$$

Definition 5.15. The axiom schema of replacement states that the image of a set under any definable function is also a set. Formally, let φ be any formula in the language of ZFC whose free variables are among $x, y, A, w_1, w_2, \ldots, w_n$, so that in particular B is not free in φ. Then,

$$\forall A \forall w_1 \forall w_2 \cdots \forall w_n [\forall x(x \in A \implies \exists! y \varphi) \implies \exists B \forall x(x \in A \implies \exists y(y \in B \wedge \varphi))].$$

In other words, if the relation φ represents a definable function f, A represents its domain, and $f(x)$ is a set for every $x \in A$, then the range of f is a subset of some set B.

Definition 5.16. The axiom of infinity as follows: Let $S(w)$ abbreviate $w \cup \{w\}$, where w is some set (we can see that $\{w\}$ is a valid set by applying the axiom of pairing with $x = y = w$ so that the set z is $\{w\}$). Then there exists a set X such that the empty set $\emptyset$, defined axiomatically, is an element of X and, whenever a set y is an element of X then $S(y)$ is also an element of X. Symbolically,

$$\exists X[\emptyset \in X \wedge \forall y(y \in X \implies S(y) \in X)].$$

Definition 5.17. By definition, a set z is a subset of a set x if and only if every element of z is also an element of x:

$$(z \subset x) \iff (\forall q(q \in z \implies q \in x)).$$

The axiom of power set states that for any set x, there is a set y that contains every subset of x:

$$\forall x \exists y \forall z(z \subset x \implies z \in y).$$

The axiom of schema of specification is then used to define the power set.

Definition 5.18. A choice function is a function f, defined on a collection X of non-empty sets, such that for every set A in X, $f(A)$ is an element of A. With this concept, the axiom of choice is stated as: for any set X of non-empty sets, there exists a choice function f that is defined on X and maps each set of X to an element of that set. Symbolically,

$$\forall X[\emptyset \notin X \implies \exists f : X \to \bigcup_{A \in X} A \text{ such that } \forall A \in X(f(A) \in A)].$$

Remark 5.19. Definitions 5.9 to 5.10 and Definitions 5.12 to 5.17 are known as the *Zaermelo-Fraenkel (ZF) set theory* axioms. Definitions 5.9 to 5.10 and Definitions 5.12 to 5.18 are known as the *Zaermelo-Fraenkel-Choice (ZFC) set theory* axioms. Note that the ZF and ZFC axioms are constructed/stated using first-order logic.

Definition 5.20. An ordered pair of elements a and b, written (a, b), is the set

$$(a, b) = \{\{a\}, \{a, b\}\}.$$

a is called the *first element* of (a, b) and b is called the *second element* of (a, b).

Theorem 5.21. Let (a, b) and (c, d) be ordered pairs. Then $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.

Proof. Suppose $a = c$ and $b = d$. Then $\{a\} = \{c\}$ and $\{a, b\} = \{c, d\}$. Therefore $(a, b) = (c, d)$. Conversely, suppose that $(a, b) = (c, d)$. Then

$$\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}. \tag{1}$$

First, we consider the case that $a = b$. Now

$$\begin{aligned} (a, b) &= \{\{a\}, \{a, b\}\} \\ &= \{\{a\}, \{a\}\} \\ &= \{\{a\}\}, \end{aligned}$$

and

$$\begin{aligned}(c,d) &= \{\{c\},\{c,d\}\}\\ &= \{\{a\}\}.\end{aligned}$$

Thus $\{a\} = \{c\} = \{c,d\}$, and therefore $a = b = c = d$. Now suppose that $a \neq b$. From equation (1) we see that either $\{c\} = \{a\}$ or $\{c\} = \{a,b\}$. Since $a \neq b$, we must have $\{c\} = \{a\}$, which implies that $a = c$. Again using equation (1), we have that either $\{a,b\} = \{c\}$ or $\{a,b\} = \{c,d\}$. If $\{a,b\} = \{c\}$, then $a = b = c$, which is not the case. Thus $\{a,b\} = \{c,d\}$. It follows that $b = c$ or $b = d$, but if $b = c$, we would have the contradiction $b = c = a$. Therefore, $b = d$, and we have established the theorem. $\square$

Definition 5.22. Let X and Y be sets. Then the *Cartesian product* of X and Y, denoted $X \times Y$, is the set

$$X \times Y := \{\, (x,y) \mid x \in X \text{ and } y \in Y \,\}.$$

Definition 5.23. Let X and Y be sets. A *function* f from X into Y is a subset of $X \times Y$ satisfying

- (i) If (x,y) and (x,y') belong to (are in) f, then $y = y'$.
- (ii) If $x \in X$, then $(x,y) \in f$ for some $y \in Y$.

If f is a function from X into Y, we write $f : X \to Y$. The crucial property of a function from X into Y is that with *each* [(ii)] element x in X, there is associated a *unique* [(i)] element $y \in Y$.

Definition 5.24. Let f be a function from X into Y. Let $A \subset X$ and $B \subset Y$.

- (i) X is called the *domain* of f.
- (ii) Y is called the *codomain* of f.
- (iii) If $(x,y) \in f$, we write $y = f(x)$ and call y the *(direct) image* of x under f.
- (iv) The *range* of f is the set
$$\{\, f(x) \in Y \mid x \in X \,\}.$$
- (v) The *image* of A under f is the set
$$f(A) = \{\, f(x) \in Y \mid x \in A \,\}.$$
- (vi) The *inverse image* of B under f is the set
$$f^{-1}(B) = \{\, x \in X \mid f(x) \in B \,\}.$$
- (vii) f is function from X *onto* (*surjective*) Y if $f(X) = Y$.
- $(viii)$ f is *one-to-one* (*injective*) if $f(x) = f(x') \implies x = x'$ for all $x, x' \in X$.

Definition 5.25. Let X and Y be sets. If f is a one-to-one function from X into Y, then the *inverse function* of f, denoted f^{-1}, from the range of f onto X is the set

$$f^{-1} = \{\, (y,x) \mid y \in \text{Range } f \text{ and } f(x) = y \,\}$$

such that for each $y \in \text{Range } f$,

$$(y,x) \in f^{-1} \iff (x,y) \in f.$$

Notice that the *function* f^{-1} is defined only if f is one-to-one, but that $f^{-1}(B)$ (inverse image of B under f) is defined for an arbitrary function f and for all sets $B \subset Y$.

Definition 5.26. Let X and Y be sets. Let f be a one-to-one function from X onto Y. Then the inverse function of f, denoted f^{-1}, from Y onto X is the set

$$f^{-1} = \{\, (y, x) \in Y \times X \mid f(x) = y \,\}$$

such that for each $y \in Y$,

$$x = f^{-1}(y) \iff f(x) = y.$$

Note that a function that is one-to-one and onto is called a *bijection*, i.e. a function that is both surjective and injective is bijective.

Definition 5.27. Let X, Y, and Z be sets. If $g : X \to Y$ and $f : Y \to Z$, we define the *composition* $f \circ g : X \to Z$, read "f circle g", by the rule

$$(f \circ g)(x) = f(g(x)) \text{ for each } x \in X.$$

Definition 5.28. Let X and Y be sets. If $f : X \to Y$ and $A \subset X$, we define the function

$$f\big|_A = \{\, (x, y) \in f \mid x \in A \,\}.$$

$f|_A$ is called the *restriction* of f to A, and f is said to be an *extension* of $f|_A$ to X.

Definition 5.29. A function that maps a set into itself is called an *operator*. Formally, let X be a set then the function $f : X \to X$ is an operator.