Lecture 10

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Applied Analysis

June 5, 2025

Theorem 10.1. There is no rational number r satisfying $r^2 = 2$.

Proof. Suppose there exists $r \in \mathbb{Q}$ satisfying $r^2 = 2$. Then $r = \frac{p}{q}$ where $p, q \in \mathbb{Z}$ and $q \neq 0$. We may assume that not both p and q are even. Then $p^2 = 2q^2$. This implies that p^2, and hence also p is even. Thus $p = 2n$ for some $n \in \mathbb{Z}$. Now $4n^2 = 2q^2$; so $2n^2 = q^2$, and hence q is also even. This contradiction establishes the theorem. $\square$

Theorem 10.2. If a is a nonnegative real number and n is a positive integer, then there exists a real number $b \geq 0$ such that $b^n = a$.

Proof. Let

$$X = \{\, x \in \mathbb{R} \mid x \geq 0 \text{ and } x^n \leq a \,\}.$$

Then X is nonempty since $0 \in X$. We argue by contradiction to show that X is bounded above by $a + 1$. Suppose there exists $x \in X$ such that $x > a + 1$. Then by Theorem 6.28, we have

$$a \geq x^n \geq (a+1)^n = \sum_{k=0}^{n} \binom{n}{k} a^k \geq na,$$

which is impossible, so X is bounded above. By the least-upper-bound axiom, X has a least upper bound b. We will prove that $b^n = a$. Either $b^n < a$, $b^n > a$, or $b^n = a$. We show that the first two possibilities cannot occur. Suppose $b^n < a$, and let $\delta = a - b^n$. Choose positive integers $m_0, \ldots, m_{n-1}$ such that

$$\binom{n}{k} b^k \frac{1}{m_k^{n-k}} < \frac{\delta}{n}$$

for $k = 0, 1, \ldots, n-1$. Let $m = \max\{m_0, \ldots, m_{n-1}\}$. Then by Theorem 6.28, we have

$$\begin{aligned}
\left(b + \frac{1}{m}\right)^n &= \sum_{k=0}^{n} \binom{n}{k} b^k \frac{1}{m^{n-k}} \\
&= \sum_{k=0}^{n-1} \left[\binom{n}{k} b^k \frac{1}{m^{n-k}}\right] + b^n \\
&< \sum_{k=0}^{n-1} \left(\frac{\delta}{n}\right) + b^n = \delta + b^n = a.
\end{aligned}$$

Therefore $\frac{b+1}{m} \in X$, but $b < b + \frac{1}{m}$ which is impossible; thus $a \leq b^n$. Now suppose $b^n > a$, and let $\delta = b^n - a > 0$. Choose positive integers $m_0, \ldots, m_{n-1}$ such that

$$\binom{n}{k} b^k \frac{1}{m_k^{n-k}} < \frac{\delta}{n}$$

for $k = 0, 1, \ldots, n-1$. Let $m = \max\{m_0, \ldots, m_{n-1}\}$. Then by Theorem 6.28, we have

$$\begin{aligned}\left(b - \frac{1}{m}\right)^n &= \sum_{k=0}^{n} \binom{n}{k} b^{n-k} \left(-\frac{1}{m}\right)^k \\ &= \sum_{k=0}^{n-1} \left[\binom{n}{k} b^{n-k} \left(-\frac{1}{m}\right)^k\right] + \left(-\frac{1}{m}\right)^n \\ &< \sum_{k=0}^{n-1} \frac{\delta}{n} + \left(-\frac{1}{m}\right)^n = b^n - \delta = a.\end{aligned}$$

Therefore, $\left(b - \frac{1}{m}\right)^n < a$ so $b - \frac{1}{m} \in X$ and $b - \frac{1}{m} < b$, which is impossible. Therefore, $a = b^n$. $\square$

Corollary 10.3. If a is a nonnegative real number and n is a positive integer, then the real number $b \geq 0$ such that $b^n = a$ is unique.

Proof. Let b_1 and b_2 be two nonnegative real numbers such that

$$b_1^n = a = b_2^n.$$

Suppose that $b_1 \neq b_2$. Then either $b_1 < b_2$ or $b_1 > b_2$. Without a loss of generality, assume $b_1 < b_2$. Since $b_1, b_2 \geq 0$, we can exponentiate both sides by n to get $b_1^n < b_2^n$, which contradicts $b_1^n = b_2^n$. Therefore, $b_1 = b_2$. $\square$

Corollary 10.4. If a is a real number and n is an odd positive integer, then there exists a real number b such that $b^n = a$.

Proof. By Theorem 10.3, there exists a real number c such that $c^n = |a|$. If $a \geq 0$, then $c^n = a$, and we have the desired conclusion. If $a < 0$, then let $b = -c$. Then $b^n = (-c)^n = a$, and again we have the desired solution. $\square$

Definition 10.5. Let x be a nonnegative real number, and let n be a positive integer. We define $x^{\frac{1}{n}}$ to be the nonnegative real number y such that $y^n = x$. Note

(i) If x is a real number and n is an odd positive integer, then we define $x^{\frac{1}{n}}$ to be the real number y such that $y^n = x$.

(ii) If x is a real number and n is a positive integer, then we define

$$x^{-\frac{1}{n}} = \frac{1}{x^{\frac{1}{n}}},$$

provided that $x^{\frac{1}{n}} \neq 0$ and $x^{\frac{1}{n}}$ is defined.

(iii) If x is a real number and $r = \frac{p}{q}$ is a rational number, where $p, q \in \mathbb{Z}$ with $q \neq 0$, and r is expressed in lowest terms, then we define

$$x^r = \left(x^{\frac{1}{q}}\right)^p$$

whenever $x^{\frac{1}{q}}$ is defined.

Remark 10.6. The next theorem is quite important. It states that $\mathbb{Q}$ is "dense" in $\mathbb{R}$, i.e. between any two real numbers there exists a rational number.

Theorem 10.7. If a and b are real numbers with $a < b$, then there exists a rational number r such that $a < r < b$.

Proof. First we consider the case that $b > 0$. Choose a positive integer n such that $b - a > \frac{1}{n}$. Let m be the least positive integer such that $b \leq \frac{m}{n}$. If $m = 1$, then clearly $\frac{m-1}{n} < b$. If $m > 1$, then $m - 1$ is a positive integer less than m, and thus $\frac{m-1}{n} < b$. Since $a - b < -\frac{1}{n}$ and $b \leq \frac{m}{n}$, we have

$$a = b + (a - b) < \frac{m}{n} - \frac{1}{n} = \frac{m-1}{n}.$$

Thus the rational number $r = \frac{m-1}{n}$ satisfies $a < r < b$. We have established the theorem in case $b > 0$. Now suppose that a and b are arbitrary real numbers with $a < b$. Choose a positive integer n such that $b + n > 0$. By the special case just established, there exists a rational number r' such that $a + n < r' < b + n$. Thus $r = r' - n$ is a rational number satisfying $a < r < b$. $\square$

Lemma 10.8. The sum of a rational number and an irrational number is an irrational number.

Proof. Suppose this lemma is false. Then there exists a rational number r and an irrational number s such that $t = r + s$ is a rational number. Then $s = t - r$ is the difference of two rational numbers and is therefore rational. This is a contradiction. $\square$

Definition 10.9. A real number that is not rational is called *irrational*. The set of the irrational numbers is denoted $\mathbb{R} \setminus \mathbb{Q}$. Note that a real number is either rational or irrational.

Remark 10.10. The next theorem states that $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$.

Theorem 10.11. If a and b are real numbers with $a < b$, then there exists an irrational number s such that $a < s < b$.

Proof. By Theorem 10.7, there exists a rational number r_1 such that $a - \sqrt{2} < r_1 < b - \sqrt{2}$. By Theorem 10.1 and Lemma 10.8, we have that $r = r_1 + \sqrt{2}$ is an irrational number. Since $a < r < b$, we have completed the proof. $\square$

Theorem 10.12. Let A be the set of all sequences whose elements are the digits 0 and 1. This set A is uncountable. Note that the element of A are sequences like: $1, 0, 0, 1, 1, 1, 1, \ldots$.

Proof. Let E be a countable subset of A, and let E consist of the sequences $s_1, s_2, s_3, \ldots$. We construct a sequence s as follows. If the n^{th} digit in s_n is 1, we let the n^{th} digit of s be 0, and vice versa. Then the sequence s differs from every member of E in at least one place; hence $s \notin E$. But clearly, $s \in A$, so E is a proper subset of A. We have shown that every countable subset of A is a proper subset of A. It follows that A is uncountable (for otherwise A would be a proper subset of A, which is absurd). $\square$

Remark 10.13. The idea of the above proof was first used by Cantor, and is called Cantor's diagonal process; for, if the sequences $s_1, s_2, s_3, \ldots$ are placed in an array, then it is the elements on the diagonal which are involved in the construction of the new sequence.

Theorem 10.14. Let X be a set. Then $X \not\sim \mathcal{P}(X)$.

Proof. If $X = \emptyset$, then $\mathcal{P}(X) = \{\emptyset\}$, so $X \not\sim \mathcal{P}(X)$. Therefore, suppose $X \neq \emptyset$ and $X \sim \mathcal{P}(X)$. Then there exists a one-to-one function f from X onto $\mathcal{P}(X)$. We now define a subset Y of X which is not in the range of f thus deducing a contradiction. Let

$$Y = \{\, x \in X \mid x \notin f(x) \,\}$$

If $Y \in f(X)$, then there exists $y \in X$ such that $f(y) = Y$. Either $y \in Y$ or $y \notin Y$, and we examine the two possibilities.

1. If $y \in Y$, then $y \notin f(y)$. Thus $y \notin f(y) = Y$. The first case is impossible.

2. If $y \notin Y$, then $y \in f(y)$. Thus $y \in f(y) = Y$. The second case is impossible.

Thus $Y \notin f(x)$, and we have the desired contradiction. $\square$

Corollary 10.15. $\mathcal{P}(\mathbb{Z}^+)$ is uncountable.

Proof. $\mathbb{Z}^+$ is countable as $\mathbb{Z}^+ \sim \mathbb{Z}^+$. Then by Theorem 10.14, we have $\mathbb{Z}^+ \not\sim \mathcal{P}(\mathbb{Z}^+)$. By Definition 9.23, there does not exist a bijection from $\mathbb{Z}^+$ onto $\mathcal{P}(\mathbb{Z}^+)$ and $\mathbb{Z}^+ \neq \emptyset$. Therefore $\mathcal{P}(\mathbb{Z}^+)$ is uncountable. $\square$

Theorem 10.16. Any subset of the positive integers is countable.

Proof. Let X be a subset of $\mathbb{Z}^+$. If X is finite, then there is nothing to prove, so assume that X is infinite. We define a function f from $\mathbb{Z}^+$ to X as follows: Let $f(1)$ be the least element in X. Let $f(2)$ be the least element of $X \setminus \{f(1)\}$. Continuing in this way, having defined $f(1), f(2), \ldots, f(n)$, we let $f(n+1)$ be the least element of $X \setminus \{f(1), f(2), \ldots, f(n)\}$ It is easy to verify that f is a one-to-one function from $\mathbb{Z}^+$ onto X, and therefore X is countable. $\square$

Corollary 10.17. Any subset of a countable set is countable.

Proof. This follows directly from Theorem 10.16. $\square$

Theorem 10.18. Let X be a nonempty set, then X is countable if and only if there exists a function f from $\mathbb{Z}^+$ onto X.

Proof. If X is countable and infinite, then there is nothing to prove. If X is finite, then $X \sim \{1, 2, \ldots, n\}$ for some $n \in \mathbb{Z}^+$, so $X = \{x_1, x_2, \ldots, x_n\}$. Define $f(i) = x_i$ for $i = 1, 2, \ldots, n$ and $f(j) = x_n$ for all $j > n$. Then f is a function from $\mathbb{Z}^+$ onto X. Suppose f is a function from $\mathbb{Z}^+$ onto a set X. If X is finite, then there is nothing to prove, so assume X is infinite. We define a function g from X onto a subset of $\mathbb{Z}^+$ by the rule

$$g(x) = \min\{f^{-1}(x)\}, \quad \text{for each } x \in X.$$

Then g is a one-to-one function from X onto $g(X)$. $g(X)$ is an infinite subset of $\mathbb{Z}^+$; hence by Theorem 10.16, $g(X)$ is countable. Thus $X \sim g(X) \sim \mathbb{Z}^+$ and X is countable. $\square$

Lemma 10.19. The set $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable.

Proof. The function $f : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$ defined by

$$f(n, m) = 2^n 3^m$$

is one-to-one, since factorization into primes is unique. Thus $\mathbb{Z}^+ \times \mathbb{Z}^+$ is equivalent to a subset $\mathbb{Z}^+$. By Theorem 10.16, $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable. $\square$

Theorem 10.20. Let $A_1, A_2, \ldots$ be a countable family of countable sets. Then

$$\bigcup_{n=1}^{\infty} A_n$$

is a countable set.

Proof. Since A_n is countable, we may list the elements of A_n as

$$a_1^{(n)}, a_2^{(n)}, a_3^{(n)}, \ldots.$$

In case A_n is finite, $A_n = \{a_1^{(n),\ldots,a_m^(n)}\}$, we let $a_i^{(n)=a_m^{(n)}}$ if $i \geq m$. If $A_n = \emptyset$, then redefine $A_n = A_m$ for some $A_m \neq \emptyset$. If all $A_m = \emptyset$, then $\bigcup_{m=1}^{\infty} A_m = \emptyset$ is countable. The function f defined by

$$f(n, m) = a_n^{(m)}$$

maps $\mathbb{Z}^+ \times \mathbb{Z}^+$ onto $\cup_{n=1}^{\infty} A_n$. By Lemma 10.19, we have $\mathbb{Z}^+ \sim \mathbb{Z}^+ \times \mathbb{Z}^+$, and it follows that we may map $\mathbb{Z}^+$ onto $\cup_{n=1}^{\infty} A_n$. By Theorem 10.18, $\cup_{n=1}^{\infty} A_n$ is countable. $\square$

Corollary 10.21. If A and B are countable sets, then $A \cup B$ is a countable set.

Corollary 10.22. The set $\mathbb{Q}$ of rational numbers is countable.

Proof. For each positive integer n, let

$$A_n = \{\dots, \frac{-2}{n}, \frac{-1}{n}, \frac{0}{n}, \frac{1}{n}, \frac{2}{n}, \dots\}.$$

Then A_n is countable for each positive integer n. By Theorem 10.20, $\mathbb{Q} = \cup_{n=1}^{\infty} A_n$ is countable. $\square$

Theorem 10.23. Any interval of $\mathbb{R}$ is uncountable; in particular, $\mathbb{R}$ is uncountable.

Proof. Let I be any interval of $\mathbb{R}$, and suppose I is countable. Then the elements of I may be listed

$$r_1, r_2, r_3, \dots.$$

Choose $a_1, b_1 \in \mathbb{R}$ such that $r_1 \notin (a_1, b_1) \subset I$. Choose $a_2, b_2 \in \mathbb{R}$ such that $a_1 < a_2 < b_2 < b_1$ and $r_2 \notin (a_2, b_2)$. Continue in this way so that having chosen $a_n, b_n \in \mathbb{R}$, we choose $a_{n+1}, b_{n+1} \in \mathbb{R}$ such that $a_n < a_{n+1} < b_{n+1} < b_n$ and $r_{n+1} \notin (a_{n+1}, b_{n+1})$. The set $\{a_1, a_2, \dots\}$ is bounded above by b, so the least upper bound r of this set exists. It follows that $a_n < r$ for all $n \in \mathbb{Z}^+$. On the other hand, $r < b_n$ for all $n \in \mathbb{Z}^+$ for if $b_m \leq r$ for some $n \in \mathbb{Z}^+$, we would have

$$a_n < b_{m+1} < b_m \leq r$$

for all $n \in \mathbb{Z}^+$. Thus b_{m+1} would have an upper bound for $\{a_1, a_2, \dots\}$, but since $b_{m+1} < r$, this is impossible. Therefore, $r \in (a_n, b_n)$ for all $n \in \mathbb{Z}^+$. It follows that $r \neq r_n$ for every $n \in \mathbb{Z}^+$ and $r \in I$. This contradiction establishes the theorem. $\square$

Corollary 10.24. Irrational numbers exist.

Proof. If every real number were rational, $\mathbb{R}$ would be countable by Corollary 10.22. This statement contradicts Theorem 10.23. $\square$

Corollary 10.25. The set of irrational numbers is uncountable.

Proof. We denote $R \setminus \mathbb{Q}$ by $\mathbb{Q}'$. Suppose $\mathbb{Q}'$ is countable. By Corollary 10.22, $\mathbb{Q}$ is countable; so by Corollary 10.21, $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}'$ is countable. This contradicts Theorem 10.23. $\square$

Theorem 10.11 (Revisited). If a and b are real numbers with $a < b$, then there exists an irrational number s such that $a < s < b$.

Proof. If $(a, b) \in \mathbb{Q}$, then by Corollaries 10.22 and 10.17, (a,b) is countable, and this contradicts Theorem 10.23. $\square$