Lecture 7

Alex Hassett
Applied Analysis

November 4, 2025

Remark 7.1. Note that we have not discussed the real numbers prior to this. The expectation is that the reader has a good understanding of the fundamental properties of addition and multiplication (the two fundamental operations on the real numbers). In this lecture we will discuss sequences of the real numbers.

Definition 7.2. Let X be a set. A *sequence* of elements of X is a function from the set of positive integers ($\mathbb{Z}^+$) into X.

Definition 7.3. A *sequence of real numbers* (or *real sequence*) is a function from $\mathbb{Z}^+$ into $\mathbb{R}$. The usual notation for a real sequence is $\{a_n\}_{n=1}^{\infty}$, where a denotes the function from $\mathbb{Z}^+$ into $\mathbb{R}$ and a_n is the value of the function at the positive integer n. The notation $\{a_n\}$ is also used to denote a real sequence. The number a_n is called the *nth term* of the sequence $\{a_n\}_{n=1}^{\infty}$.

Example 7.4. A sequence may be defined by giving an explicit formula for the nth term. For example, the formula

$$a_n = \frac{1}{n}$$

defines the sequence whose value at the positive integer n is $\frac{1}{n}$. The first three terms of this sequence are

$$a_1 = 1, \quad a_2 = \frac{1}{2}, \quad a_3 = \frac{1}{3}.$$

A sequence may also be defined inductively. For example,

$$a_1 = 0, \quad a_2 = 1, \quad a_{n+2} = \frac{a_n + a_{n+1}}{2}, \quad n = 1, 2, \ldots$$

define the sequence whose first six terms are

$$0, 1, \frac{1}{2}, \frac{3}{4}, \frac{5}{8}, \frac{11}{16}.$$

Definition 7.5. Given a sequence $\{a_n\}$, we use the notation

$$\sum_{n=p}^{q} a_n \quad (p \leq q)$$

to denote the sum $a_p + a_{p+1} \cdots + a_q$. With $\{a_n\}$ we associate a sequence $\{s_n\}$, where

$$s_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k$$

for $n \geq k$. The numbers s_n are called the partial sums of the series, where s_n is called the *nth partial sum* of the series. For $\{s_n\}$ we also use the symbolic expression

$$a_1 + a_2 + a_3 + \cdots$$

or, more concisely,

$$\sum_{n=1}^{\infty} a_n. \tag{1}$$

The symbol (1) is called an *infinite series*, or just a *series*. Sometimes, for convenience of notation, we shall consider series of the form

$$\sum_{n=0}^{\infty} a_n. \tag{2}$$

And frequently, when there is no possible ambiguity, or when the distinction is immaterial, we shall simply write $\sum a_n$ in place of (1) or (2).

Remark 7.6. Note that a *finite series* is called a *summation*. Furthermore, for readers who are familiar with limits and convergence we make the following addition to Definition 7.5: If $\{s_n\}$ converges to s, we say that the series *converges*, and write

$$\sum_{n=1}^{\infty} a_n = s.$$

The number s is called the *sum* of the series, but it should be clearly understood that s is the limit of a sequence of sums, and is not obtained simply by addition. If $\{s_n\}$ diverges, then the series is said to diverge.

Definition 7.7. An *arithmetic sequence* $\{a_n\}$ is defined by the formula

$$a_n = a_1 + (n-1)d$$

where the $d = a_2 - a_1$. Also note that a_1 is the first term of the sequence. The formula for the nth partial sum of the $\{a_n\}$ is

$$s_n = n\left(\frac{a_1 + a_n}{2}\right).$$

Note that there is no general formula for the corresponding sequence (series) $\{s_n\}$.

Definition 7.8. A *geometric sequence* $\{a_n\}$ is defined by the formula

$$a_n = a_1 r^{n-1}$$

where $r = \frac{a_2}{a_1} \neq 1$ and a_1 is the first term of the sequence. The nth partial sum s_n of the corresponding sequence $\{s_n\}$ is given by the formula

$$s_n = \frac{a_1(1-r^n)}{1-r} = \frac{a_1(r^n - 1)}{r-1}.$$

The formula for the series is given by

$$s = \sum_{n=1}^{\infty} a_n = \frac{a_1}{1-r}$$

where $-1 < r < 1$, i.e. $|r| < 1$.

Remark 7.9. In a standard Real Analysis course, we generally discuss the convergence/divergence of series. However, this will be omitted from this lecture as that will be covered in the future. The following theorems will be discussing the algebraic properties of series.

Definition 7.10. The absolute value function from $\mathbb{R}$ into $\mathbb{R}$, denoted $|\cdot|$, is defined by

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

where $x \in \mathbb{R}$.

Theorem 7.11. Let $\{a_m\}$ and $\{b_m\}$ be real sequences. Then the following statements are true for any integer $n \geq m$:

(i) $\sum_{k=m}^{n} a_k = \sum_{k=m}^{n-1} a_k + a_n$,

(ii) $\sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k = \sum_{k=m}^{n} (a_k + b_k)$,

(iii) $c \sum_{k=m}^{n} a_k = \sum_{k=m}^{n} c a_k$ for $c \in \mathbb{R}$.

The above statements are three of field axioms ($\mathbb{R}$, by definition, is a field) and as such we cannot prove them (and this is more of a definition than a theorem).

Remark 7.12. Observe that

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \cdots + n^2,$$

and that

$$\sum_{j=1}^{n} j^2 = 1^2 + 2^2 + \cdots + n^2.$$

Hence,

$$\sum_{k=1}^{n} k^2 = \sum_{j=1}^{n} j^2.$$

The variables k and j for the above summations are often referred to as *local variables*. Clearly, for any sequence $\{a_n\}$ the following is true

$$\sum_{k=p}^{q} a_k = \sum_{n=p}^{q} a_n,$$

where $q \geq p$ (note that q can be infinity, but if $q = \infty$ then $q > p$). Thus, one may always interchange local variables so long as the change is "consistent" (the value of the summation does not change).

Example 7.13. In this example, we provide some examples where we interchange local variables while maintaining "consistency." Let $\{a_n\}$ be a real sequence. Then we provide a common example of such an interchange: Let $j = n - 1$ and define the sequence $\{b_j\}$ by $b_j = a_{j+1}$, then

$$\sum_{n=1}^{m} a_n = \sum_{j=0}^{m-1} b_j.$$

The above interchange is consistent as the value of the summation does not change. Also note that since local variables are only defined within the scope of the summation, we could (and often do) write

$$\sum_{n=1}^{m} a_n = \sum_{n=0}^{m-1} b_n.$$

For the last example, consider the real sequence $\{\frac{k}{n+k}\}$. A summation of this sequence can be written as

$$\sum_{k=1}^{n+1} \frac{k}{n+k}$$

We make the following exchange of local variables: Let $j = k - 1$. Notice that $j = k - 1 \implies k = j + 1$. Thus, substituting $j + 1$ for k, in the above summation, we have

$$\sum_{k=1}^{n+1} \frac{k}{n+k} = \sum_{j=0}^{n} \frac{j+1}{n+(j+1)}.$$

Note that the bounds for the above summation come from the following statements:

Lower bound: $k = 1 \implies j = 1 - 1 = 0$

Upper bound: $k = n + 1 \implies j = (n + 1) - 1 = n + 1 - 1 = n.$

Finally, as is standard, we rewrite the summation using the original local variable:

$$\sum_{k=1}^{n+1} \frac{k}{n+k} = \sum_{k=0}^{n} \frac{k+1}{n+(k+1)}.$$

Definition 7.14. A *field* is a set F with two operations, called *addition* and *multiplication*, which satisfy the following so-called "field axioms" (A), (M), and (D):

(A) Axioms for addition

- (A1) If $x \in F$ and $y \in F$, then their sum $x + y$ is in F.
- (A2) Addition is commutative: $x + y = y + x$ for all $x, y \in F$.
- (A3) Addition is associative: $(x + y) + z = x + (y + z)$ for all $x, y, z \in F$.
- (A4) F contains an element 0 such that $0 + x = x$ for every $x \in F$.
- (A5) To every $x \in F$ corresponds an element $-x \in F$ such that

$$x + (-x) = 0.$$

(M) Axioms for multiplication

- (M1) If $x \in F$ and $y \in F$, then their product xy is in F.
- (M2) Multiplication is commutative: $xy = yx$ for all $x, y \in F$.
- (M3) Multiplication is associative: $(xy)z = x(yz)$ for all $x, y, z \in F$.
- (M4) F contains an element $1 \neq 0$ such that $1x = x$ for every $x \in F$.
- (M5) If $x \in F$ and $x \neq 0$, then there exists an element $\frac{1}{x} \in F$ such that

$$x \cdot (\frac{1}{x}) = 1.$$

(D) The distributive law

$$x(y + z) = xy + xz$$

holds for all $x, y, z \in F$.

Definition 7.15. Let S be a set. An *order* on S is a binary relation, denoted by $<$, with the following two properties:

(i) If $x \in S$ and $y \in S$, then one and only one (exactly one) of the following statements

$$x < y, \quad x = y, \quad y < x$$

is true.

(ii) Let $x, y, z \in S$. If $x < y$ and $y < z$, then $x < z$.

Note that the condition of statement (ii) is read "If x is less than y and y is less than z, then x is less than z."

Definition 7.16. An *ordered set* is a set S in which an order is defined. For example, $\mathbb{Q}$ is an ordered set if $r < s$ is defined to mean that $s - r$ is a positive rational number.

Theorem 7.17. If n is a positive integer, then $n \geq 1$.

Proof. Let $S(n)$ be the statement "$n \geq 1$." Since $1 \geq 1$, $S(1)$ is true. If $n \geq 1$, then $n+1 > n \geq 1$; so if $S(n)$ is true, then $S(n+1)$ is true. Therefore $S(n)$ is true for every positive integer n by Theorem 4.24. $\square$

Theorem 7.18. If $m, n \in \mathbb{Z}^+$, then $m+n \in \mathbb{Z}^+$.

Proof. Let $S(m)$ be the statement "$m+n \in \mathbb{Z}^+$ for all $n \in \mathbb{Z}^+$." The term $1+n \in \mathbb{Z}^+$ for all $n \in \mathbb{Z}^+$ since $\mathbb{Z}^+$ is a successor set, and so $S(1)$ is true. Next suppose that $m+n \in \mathbb{Z}^+$ for all $n \in \mathbb{Z}^+$. Then $(m+1)+n = (m+n)+1 \in \mathbb{Z}^+$ for all $n \in \mathbb{Z}^+$ since $\mathbb{Z}^+$ is a successor set ($m+n \in \mathbb{Z}^+$ implies $(m+n)+1 \in \mathbb{Z}^+$). Thus if $S(m)$ is true, then $S(m+1)$ is true, so by Theorem 4.24, we have that $S(m)$ is true for all $m \in \mathbb{Z}^+$. $\square$

Lemma 7.19. If $n \in \mathbb{Z}^+$, then either $n-1=0$ or $n-1 \in \mathbb{Z}^+$.

Proof. Let
$$G = \{\, n \in \mathbb{Z}^+ \mid n-1=0 \text{ or } n-1 \in \mathbb{Z}^+ \,\}.$$
We show that G is a successor set. Clearly $1 \in G$. Suppose $n \in G$. Then $(n+1)-1 = n \in \mathbb{Z}^+$, so $n+1 \in G$. Thus G is a successor set by Definition 4.20. Therefore $G = \mathbb{Z}^+$ because $\mathbb{Z}^+ \subset G$ (by Lemma 4.23) and $G \subset \mathbb{Z}^+$. $\square$

Lemma 7.20. If $m, n \in \mathbb{Z}^+$ and $m < n$, then $n-m \in \mathbb{Z}^+$.

Proof. We proceed by induction on m letting $S(m)$ be the statement of the lemma. Suppose $1 < n$ for $n \in \mathbb{Z}^+$. By Lemma 7.19, either $n-1=0$ or $n-1 \in \mathbb{Z}^+$. Since $n \neq 1$, we have that $n-1 \in \mathbb{Z}^+$. Now suppose that if $m, n \in \mathbb{Z}^+$ and $m < n$, then $n-m \in \mathbb{Z}^+$. Suppose $m+1 < n$ and $m, n \in \mathbb{Z}^+$. By Lemma 7.19 and Theorem 7.17, we have $n-1 \in \mathbb{Z}^+$. Since $m < n-1$, by induction we see that $(n-1)-m \in \mathbb{Z}^+$. Thus $n-(m+1) \in Z^+$, and we have completed the inductive step. $\square$

Lemma 7.21. Let n be a positive integer. Then no positive integer m satisfies the inequality $n < m < n+1$.

Proof. Suppose that there exists $m \in \mathbb{Z}^+$ such that $n < m < n+1$ holds. By Lemma 7.20, we have that $m-n \in \mathbb{Z}^+$. On the other hand, since $m < n+1$, we have that $m-n < 1$, which contradicts Theorem 7.17. $\square$

Theorem 7.22. The Well-Ordering Theorem states that if X is a non-empty subset of the positive integers, then X contains a least element; that is, there exists $a \in X$ such that $a \leq x$ for all $x \in X$.

Proof. To prove this theorem we use induction on n, letting $S(n)$ be the statement: "If $n \in X$, then X contains a least element." If $1 \in X$, then 1 is the least element of X by Theorem 7.17. Now suppose that $S(n)$ is true and suppose $n+1 \in X$. Since $S(n)$ is true, $X \cup \{n\}$ contains a least element m. If $m \in X$, then m is the least element of X. If $m \notin X$, then $m = n$ and $n \leq x$ for all $x \in X$. Since $n \notin X$, we have $n+1 \leq x$ for all $x \in X$ by Lemma 7.21. In this case, $n+1$ is the least element of X. $\square$