Lecture 1

Alex Hassett
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Theorem 1.1. Let p, q, and r be logical statements (meaning they can either be true (T) or false (F)). The logical equivalences are as follows

1. Identity laws
   (a) $p \wedge T \equiv p$
   (b) $p \vee F \equiv p$
2. Domination laws
   (a) $p \vee T \equiv T$
   (b) $p \wedge F \equiv F$
3. Idempotent laws
   (a) $p \vee p \equiv p$
   (b) $p \wedge p \equiv p$
4. Double negation law

   $\neg(\neg p) \equiv p$
5. Commutative laws
   (a) $p \vee q \equiv q \vee p$
   (b) $p \wedge q \equiv q \wedge p$
6. Associative laws
   (a) $(p \vee q) \vee r \equiv p \vee (q \vee r)$
   (b) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
7. Distributive laws
   (a) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee q)$
   (b) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
8. De Morgan's laws
   (a) $\neg(p \wedge q) \equiv \neg p \vee \neg q$
   (b) $\neg(p \vee q) \equiv \neg p \wedge \neg q$
9. Absorption laws
   (a) $p \vee (p \wedge q) \equiv p$
   (b) $p \wedge (p \vee q) \equiv p$

10. Negation laws
    (a) $p \vee \neg p \equiv T$
    (b) $p \wedge \neg p \equiv F$

Theorem 1.2. Let p and q be logical statements. Then $p \implies q$, read "p implies q", is the statement "if p, then q". Note that $p \implies q \equiv \neg p \vee q$.

Corollary 1.3. Let p and q be logical statements. Let $p \equiv r \implies t$ where r and t are logical statements. Then the following formal English statements are all logically equivalent to $p \implies q$.

1. If p, then q.
2. Suppose p. Then q.
3. p implies q.
4. Suppose p such that q.
5. Let p be such that q.
6. If r such that t, then q.

Note that the above equivalencies are only a few among the hundreds of equivalent ways to express the statement $p \implies q$ in formal English.

Definition 1.4. Let $S(p,q)$ be the statement $p \implies q$, i.e. let $S(p,q) \equiv p \implies q$. To non-trivially prove that $S(p,q)$ is true, $S(p,q) \equiv T$, one must suppose $p \equiv T$, then show that $q \equiv T$. In formal English, this would be written "suppose p is true, then show that q is also true."

Definition 1.5. A set is a collection of objects. For a set X, we say that x is an element of X by writing $x \in X$, read "x in X". We say x is not an element of X by writing $x \notin X$, read "x not in X".

Definition 1.6. A set Y is a subset of a set X, written $Y \subset X$, if for every $y \in Y$, we have that $y \in X$. We say that Y is a proper subset of X if $Y \subset X$ and $Y \neq X$.

Theorem 1.7. Let X be a set. Then $X \subset X$.

Proof. If $X = \emptyset$, then $X \subset X$ because $\emptyset \subset \emptyset$. Now suppose $X \neq \emptyset$ and let $x \in X$. Then $x \in X$. Therefore $X \subset X$. □

Definition 1.8. Let ζ, read "zeta", be a logical statement. Then the negation of ζ, written $\neg\zeta$ is the logical opposite of ζ, i.e. its negation. If $\zeta \equiv T$, then $\neg\zeta \equiv F$ and vice versa.

Definition 1.9. Let x and y be defined in a first-order language. Then x is said to be defined equivalently to y, written $x \approx y$, if the definition of x is logically equivalent to the definition of y. Some examples of this are

1. Let A be a set. Let $x \in A$ and let $y \in A$. Then $x \approx y$.
2. Let x be the statement "if p, then q". Let y be the statement "p implies q". Then $x \approx y$. Note that we can also say that $x \equiv y$, because x and y are propositional (logical) statements.

Corollary 1.10. Let x and y be defined equivalently. Then x is equivalent to y, but x does not necessarily equal y.