Lecture 3

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Definition 3.1. The symbol $\vee$, read "or", is a logical operator. The statement $p \vee q$, read "p or q" has the following truth values:

1. $p \vee q \equiv T$ if $p \equiv T$,
2. $p \vee q \equiv T$ if $q \equiv T$,
3. $p \vee q \equiv F$ if and only if $p \equiv F$ and $q \equiv F$.

Note that $p \vee q \equiv T$ only requires that just one of the statements is true.

Remark 3.2. Note that truth values 1 and 2 of Definition 3.1 may be rewritten as

$p \vee q \equiv T$ if and only if $p \equiv T$ or $q \equiv T$.

The truth values of Definition 3.1 are written symbolically as

1. $(p \equiv T) \implies (p \vee q \equiv T)$,
2. $(q \equiv T) \implies (p \vee q \equiv T)$,
3. $((p \vee q) \equiv F) \iff ((p \equiv F) \wedge (q \equiv F))$.

Also note that the parenthesis aren't necessary, but shown for clarity.

Definition 3.3. The symbol $\wedge$, read "and", is a logical operator. The statement $p \wedge q$, read "p and q" has the following truth values:

1. $p \wedge q \equiv F$ if $p \equiv F$,
2. $p \wedge q \equiv F$ if $q \equiv F$,
3. $p \wedge q \equiv T$ if and only if $p \equiv T$ and $q \equiv T$.

Note that $p \wedge q \equiv F$ only requires that just one of the statements is false.

Remark 3.4. Note that truth values 1 and 2 of Definition 3.3 may be rewritten as

$p \wedge q \equiv F$ if and only if $p \equiv F$ or $q \equiv F$.

The truth values of Definition 3.3 are written symbolically as

1. $(p \equiv F) \implies ((p \wedge q) \equiv F)$,
2. $(q \equiv F) \implies ((p \wedge q) \equiv F)$,
3. $((p \wedge q) \equiv T) \iff ((p \equiv T) \wedge (q \equiv T))$.

Also note that the parenthesis aren't necessary, but shown for clarity.

Definition 3.5. The symbol $\oplus$, read "either or", is a logical operator. The statement $p \oplus q$, read "p either or q", has the following truth values:

1. $p \oplus q \equiv T$ if $p \equiv T$ and $q \equiv F$,
2. $p \oplus q \equiv T$ if $p \equiv F$ and $q \equiv T$,
3. $p \oplus q \equiv F$ if $p \equiv T$ and $q \equiv T$,
4. $p \oplus q \equiv F$ if $p \equiv F$ and $q \equiv F$.

Note that $p \oplus q \equiv (p \wedge \neg q) \vee (\neg p \wedge q)$. Thus for $p \oplus q$ to be true, exactly one of the statements must be true.

Remark 3.6. Note that truth values 1 and 2 of Definition 3.5 may be rewritten as

$p \oplus q \equiv T$ if and only if $p \equiv T$ and $q \equiv F$ or $p \equiv F$ and $q \equiv T$.

Similarly, the truth values 3 and 4 of Definition 3.5 may be rewritten as

$p \oplus q \equiv F$ if and only if $p \equiv T$ and $q \equiv T$ or $p \equiv F$ and $q \equiv F$.

Definition 3.7. The symbol $\downarrow$, read "neither nor", is a logical operator. The statement $p \downarrow q$, read "p neither nor q", has the following truth values:

1. $p \downarrow q \equiv F$ if $p \equiv T$,
2. $p \downarrow q \equiv F$ if $q \equiv T$,
3. $p \downarrow q \equiv T$ if and only if $p \equiv F$ and $q \equiv F$.

Note that $p \downarrow q \equiv \neg p \wedge \neg q$. Thus for $p \downarrow q$ to be true, both of the statements must be false. Likewise, for $p \downarrow q$ to be false, at least one of the statements must true. Also note that there are multiple widely used symbols for neither nor.

Remark 3.8. Note that the truth values 1 and 2 of Definition 3.7 may be rewritten as

$p \downarrow q \equiv F$ if and only if $p \equiv T$ or $q \equiv T$.

Definition 3.9. The Rules of Inference are the valid forms of making logical arguments. Let p, q, and r be statements. Then the Rules of Inference are defined as follows

1. Modus Ponens

$$\begin{array}{rl} & p \implies q \\ & p \\ \therefore & q \end{array}$$

2. Modus Tollens

$$\begin{array}{rl} & p \implies q \\ & \neg q \\ \therefore & \neg p \end{array}$$

3. Generalization

a. p **b.** q

$\therefore\ p \vee q$ $\therefore\ p \vee q$

4. Specialization

a. $p \wedge q$ **b.** $p \wedge q$

$\therefore\ p$ $\therefore\ q$

5. Conjunction

$$\begin{aligned} & p \\ & q \\ \therefore\ & p \wedge q \end{aligned}$$

6. Elimination

a. $\begin{aligned} & p \vee q \\ & \neg q \\ \therefore\ & p \end{aligned}$ **b.** $\begin{aligned} & p \vee q \\ & \neg p \\ \therefore\ & q \end{aligned}$

7. Transitivity

$$\begin{aligned} & p \implies q \\ & q \implies r \\ \therefore\ & p \implies r \end{aligned}$$

8. Proof by Division into Cases

$$\begin{aligned} & p \vee q \\ & p \implies r \\ & q \implies r \\ \therefore\ & r \end{aligned}$$

9. Contradiction Rule

$$\begin{aligned} & \neg p \implies F \\ \therefore\ & p \end{aligned}$$

Note that F means a contradiction and that each statement before the "therefore" statement is supposed to be true. For example, in Conjunction, both p and q are supposed to be true, otherwise the argument is trivial.

Remark 3.10. The 5th rule of Inference (Conjunction) is written in English as "if p is true and q is true, then p and q is true." Informally, one could write this as "if we know (are given or have proven) that p is true and q is true, then we know (can infer) that p and q is true."

Definition 3.11. An argument is said to be valid if it can be associated with one of the valid forms defined in Definition 3.9. Informally, this means if an argument can be rewritten or reordered (without changing its meaning) so that it matches one of the valid forms defined in Definition 3.9, then it is considered valid.

Definition 3.12. An argument is said to be invalid if it is not valid. Note that one may not include any invalid argument(s) in a mathematical/valid proof/argument.

Definition 3.13. Let U be the universe under consideration. Let A and B be sets such that $A \subset U$ and $B \subset U$. Then the set $A \cup B$, read "A union B" is defined as follows

$$A \cup B \coloneqq \{\, x \in U \mid x \in A \vee x \in B \,\}.$$

Note that here we assume that $U \neq \emptyset$. If $U = \emptyset$, then $A \cup B = \emptyset$ and $A = \emptyset$ and $B = \emptyset$.

Remark 3.14. Using the definitions in Definition 3.13. To determine the elements of the universe that are in $A \cup B$, one would look at each element of the universe and check whether that element is in A or is in B. Since every element in A is in U and every element in B is in U, a common method shown to students is to only consider the elements in A and B. However, by Definition 3.13, union technically considers all elements in the universe, not just the ones in A and B. It just so happens that the only elements that end up in $A \cup B$ are the elements in A or in B.

Theorem 3.15. Let U be the universe under consideration. Let A and B be sets such that $A \subset U$ and $B \subset U$. Then

1. $A \cup B = A$ if $B = \emptyset$,
2. $A \cup B = B$ if $A = \emptyset$,
3. $A \cup B = \emptyset$ if and only if $A = \emptyset = B$.

We leave the proof of this theorem as an exercise to the reader.

Definition 3.16. Let U be the universe under consideration. Let A and B be sets such that $A \subset U$ and $B \subset U$. Then the set $A \cap B$, read "A intersect B", is defined as follows

$$A \cap B \coloneqq \{ x \in U \mid x \in A \wedge x \in B \}.$$

Note that here we assume that $U \neq \emptyset$. If $U = \emptyset$, then $A \cap B = \emptyset$ and $A = \emptyset$ and $B = \emptyset$.

Theorem 3.17. Let U be the universe under consideration. Let A and B be sets such that $A \subset U$ and $B \subset U$. Then

$$A \cap B = \emptyset \text{ if } A = \emptyset \text{ or } B = \emptyset.$$

We leave the proof of this theorem as an exercise to the reader.

Definition 3.18. Let U be the universe under consideration. Let A be a set such that $A \subset U$. Then the set $A^{\complement}$, read "A complement" or "the complementation of A", is defined as follows

$$A^{\complement} \coloneqq \{ x \in U \mid x \notin A \}.$$

It follows immediately from the above definition that $\emptyset^{\complement} = U$ and that $U^{\complement} = \emptyset$ (as one may prove).

Definition 3.19. Let U be the universe under consideration. Let A and B be sets such that $A \subset U$ and $B \subset U$. Then the set $A \setminus B$, read "A set difference B" or "A set minus B", is defined as follows

$$A \setminus B \coloneqq \{ x \in U \mid x \in A \wedge x \notin B \}.$$

Thus $A \setminus B$ is the set of all the elements in A that are not in B.