Lecture 9

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Definition 9.1. Let x and y be real numbers. Then

(i) x is *negative* if $-x$ is positive.

(ii) $x > y$ means $x - y$ is positive.

(iii) $x \geq y$ means $x > y$ or $x = y$.

(iv) $x < y$ means $y > x$.

(v) $x \leq y$ means $y \geq x$.

The inequality $x > y$ ($x < y$) is read "x is greater (less) than y" and the inequality $x \geq y$ ($x \leq y$) is read "x is greater (less) than or equal to y."

Definition 9.2. A non-empty subset X of $\mathbb{R}$ is said to be *bounded above* if there exists a real number a such that $x \leq a$ for all $x \in X$. The number a is called an *upper bound* for X.

Definition 9.3. A non-empty subset X of $\mathbb{R}$ is said to be *bounded below* if there exists a real number a such that $x \geq a$ for all $x \in X$. The number a is called a *lower bound* for X.

Definition 9.4. Let X be a non-empty subset of $\mathbb{R}$. A number a in $\mathbb{R}$ is said to be a *least upper bound* (*supremum*) for X if

(i) a is an upper bound for X.

(ii) If b is an upper bound for X, then $a \leq b$.

Note that part (ii) can be equivalently stated as follows: if $b < a$, then b is not an upper bound for X. By Definition 9.2, this last statement is equivalent to

(ii′) If $b < a$, then there exists $x \in X$ such that $b < x$.

If a is the supremum of a set X, then we let $a = \sup X$. If X is a finite set, then we also denote $\sup X$ by $\max X$.

Definition 9.5. Let X be a non-empty subset of $\mathbb{R}$. A number a in $\mathbb{R}$ is said to be a *greatest lower bound* (*infimum*) for X if

(i) a is a lower bound for X.

(ii) If b is a lower bound for X, then $b \leq a$.

Note that part (ii) can be equivalently stated as follows: if $b > a$, then b is not a lower bound for X. By Definition 5.1, this last statement is equivalent to

(ii′) If $b > a$, then there exists $x \in X$ such that $b > x$.

If a is the infimum of a set X, then we let $a = \inf X$. If X is a finite set, then we also denote $\inf X$ by $\min X$.

Theorem 9.6. Let X be a subset of $\mathbb{R}$. If a and b are least upper bounds for X, then $a = b$. In other words, the supremum is unique.

Proof. By Definition 9.4, since a is a least upper bound for X and b is an upper bound for X, we have $a \leq b$. Similarly, $b \leq a$, and thus $a = b$. $\square$

Theorem 9.7. Let X be a subset of $\mathbb{R}$. If a and b are greatest lower bounds for X, then $a = b$. In other words, the infimum is unique.

Proof. By Definition 9.5, since a is a greatest lower bound for X and b is a lower bound for X, we have $a \geq b$. Similarly, $b \geq a$, and thus $a = b$. $\square$

Definition 9.8. The *real numbers* $\mathbb{R}$ is a set of objects satisfying Axioms 1 to 13 as listed in the following:

Axiom 1. There is a binary operation called *addition* and denoted $+$ such that if x and y are real numbers, then $x + y$ is a real number.

Axiom 2. Addition is associative.

$$(x + y) + z = x + (y + z)$$

for all $x, y, z \in \mathbb{R}$.

Axiom 3. Addition is commutative.

$$x + y = y + x$$

for all $x, y \in \mathbb{R}$.

Axiom 4. An additive identity exists. There exists a real number denoted 0 which satisfies

$$x + 0 = x = 0 + x$$

for all $x \in \mathbb{R}$.

Axiom 5. Additive inverses exist. For each $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that

$$x + y = 0 = y + x.$$

Note that the number y of Axiom 5 is denoted as $-x$.

Axiom 6. There is a binary operation called *multiplication* and denoted $\cdot$ such that if x and y are real numbers, then $x \cdot y$ (or xy) is a real number.

Axiom 7. Multiplication is associative.

$$(xy)z = x(yz)$$

for all $x, y, z \in \mathbb{R}$.

Axiom 8. Multiplication is commutative.

$$xy = yx$$

for all $x, y \in \mathbb{R}$.

Axiom 9. A multiplicative identity exists. There exists a real number, different from 0, denoted 1 which satisfies

$$x \cdot 1 = x = 1 \cdot x$$

for all $x \in \mathbb{R}$.

Axiom 10. Multiplicative inverses exist for non-zero real numbers. For any $x \in \mathbb{R}$ with $x \neq 0$, there exists $y \in \mathbb{R}$ such that

$$xy = 1 = yx.$$

Note that the number y of Axiom 10 is denoted x^{-1} or $\frac{1}{x}$.

Axiom 11. Multiplication distributes over addition.

$$x(y+z) = xy + xz \quad \text{and} \quad (y+z)x = yx + zx$$

for all $x, y, z \in \mathbb{R}$.

Axiom 12. There exists a subset $\mathbb{R}^+$ of $\mathbb{R}$ called the *positive real numbers* satisfying

(i) If x and y are in $\mathbb{R}^+$, then $x + y$ and xy are in $\mathbb{R}^+$.

(ii) If x is in $\mathbb{R}$, then exactly one of the following statements is true

$$x \in \mathbb{R}^+ \quad \text{or} \quad x = 0 \quad \text{or} \quad -x \in \mathbb{R}^+.$$

Note that this is also called the *order axiom* of the real numbers.

Axiom 13. A non-empty subset of real numbers which is bounded above has a least upper bound. Note that this is called the *least upper bound axiom* of the real numbers or the *completeness axiom* of the real numbers.

Remark 9.9. For Axiom 13 of Definition 9.8, the real numbers are complete in the sense that there are no "holes" in the real line ($\mathbb{R}$). Informally, if there were a hole in the real line, then the set of numbers to the left of the hole would have no least upper bound.

Theorem 9.10. The additive identity of axiom 4 is unique, that is, if there exists $0' \in \mathbb{R}$ such that that $x + 0' = x$ for all $x \in \mathbb{R}$, then $0 = 0'$.

Proof. Suppose that there exists $0' \in \mathbb{R}$ such that $x + 0' = x$ for all $x \in \mathbb{R}$. Then $0 + 0' = 0$. On the other hand, from axiom 4, we have $0' + 0 = 0'$. Since addition is commutative (axiom 3),

$$0 = 0 + 0' = 0' + 0 = 0'.$$

□

Theorem 9.11. The additive inverse of axiom 5 is unique, that is, if there exists $y \in \mathbb{R}$ such that $x + y = 0$ for all $x \in \mathbb{R}$, then $-x = y$.

Proof. Suppose that there exists $y \in \mathbb{R}$ such that $x + y = 0$. By Theorem 9.10 we have that 0 is unique. By axiom 5, there exists $-x \in \mathbb{R}$ such that $x + (-x) = 0$. By axiom 4, we have

$$-x = -x + 0 = -x + (x + y) = (-x + x) + y = 0 + y = y.$$

□

Theorem 9.12. The multiplicative identity of axiom 9 is unique, that is, if there exists $1'$ in $\mathbb{R}$ such that $x \cdot 1' = x = 1' \cdot x$ for all $x \in \mathbb{R}$, then $1 = 1'$.

Proof. Suppose 1 and $1'$ are both multiplicative identities in $\mathbb{R}$. By axiom 9, we have

$$1 \cdot 1' = 1.$$

Similarly, by axiom 9 we also have

$$1' \cdot 1 = 1'.$$

But, by axiom 8, multiplication in $\mathbb{R}$ is commutative, so

$$1 \cdot 1' = 1' \cdot 1.$$

Thus

$$1 = 1'.$$

Therefore, the multiplicative identity is unique. □

Theorem 9.13. For any non-zero real number $x \in \mathbb{R}$, the multiplicative inverse of x is unique, that is, if there exists y in $\mathbb{R}$ such that $xy = 1 = yx$, then $x^{-1} = y$.

Proof. Suppose y and x^{-1} are both multiplicative inverses of x. Then

$$x \cdot y = 1, \quad x \cdot x^{-1} = 1.$$

Now compute

$$y = y \cdot 1 = y \cdot (x \cdot x^{-1}) = (y \cdot x) \cdot x^{-1} = 1 \cdot x^{-1} = x^{-1}.$$

Thus $y = x^{-1}$. Therefore, the multiplicative inverse of x is unique. □

Theorem 9.14. $x \cdot 0 = 0$ for all x in $\mathbb{R}$.

Proof. By axiom 4, we have $x \cdot (0 + 0)$. On the other hand, $x \cdot (0 + 0) = x \cdot 0 + x \cdot 0$ by axiom 11. Thus $x \cdot 0 = x \cdot 0 + x \cdot 0$. Now

$$x \cdot 0 + [-(x \cdot 0)] = (x \cdot 0 + x \cdot 0) + [-(x \cdot 0)].$$

Again using axiom 5 and axiom 2, we have

$$0 = x \cdot 0 + 0,$$

and by axiom 4, we have $0 = x \cdot 0$. □

Theorem 9.15. For all real numbers x, the additive inverse of the additive inverse of x is x. That is, for all $x \in \mathbb{R}$,

$$-(-x) = x.$$

Proof. Let $x \in \mathbb{R}$. By axiom 5, there exists a unique $-x \in \mathbb{R}$ such that $x + (-x) = 0$. Now consider $-(-x) \in \mathbb{R}$ such that $-x + (-(-x)) = 0$. Adding x to both sides of the previous equation (see Theorem 9.16 and Theorem 9.17), we have

$$x + [-x + (-(-x))] = x + 0$$

Using axioms 2, 4, and 5 we have

$$\begin{aligned} (x + (-x)) + (-(-x)) &= x \\ 0 + (-(-x)) &= x \\ -(-x) &= x. \end{aligned}$$

□

Theorem 9.16. The set of real numbers $\mathbb{R}$, equipped with the usual operations of addition and multiplication, forms a ring.

Proof. By Definition 9.8, the set $\mathbb{R}$ satisfies the following

- $(\mathbb{R}, +)$ is an abelian group by Axioms 1, 2, 3, 4, and 5.
- $(\mathbb{R}, \cdot)$ is a semigroup by Axioms 6 and 7.
- Multiplication distributes over addition by Axiom 11.

Therefore, $\mathbb{R}$ is a ring by Definition 8.30. □

Theorem 9.17. The *cancellation property of addition* is stated as follows: let $x, y, z \in \mathbb{R}$. Then

$$x = y \iff x + z = y + z.$$

Proof. (⇒) Suppose $x = y$. Then by the axioms of a ring (associativity and commutativity of addition), we have
$$x + z = y + z.$$
(⇐) Conversely, suppose $x + z = y + z$. Then by the existence of additive inverses, we may add $-z$ to both sides
$$(x + z) + (-z) = (y + z) + (-z).$$
By associativity,
$$x + (z + (-z)) = y + (z + (-z)).$$
Since $z + (-z) = 0$,
$$x + 0 = y + 0.$$
Finally, by the additive identity property,
$$x = y.$$
Thus, the equivalence holds. □

Theorem 9.18.

(i) $1 > 0$.

(ii) If $x > y$ and $y > z$, then $x > z$, $x, y, z \in \mathbb{R}$.

(iii) If $x > y$, then $x + z > y + z$, $x, y, z \in \mathbb{R}$.

(iv) If $x > y$ and $z > 0$, then $xz > yz$, $x, y, z \in \mathbb{R}$.

(v) If $x > y$ and $z < 0$, then $xz < yz$, $x, y, z \in \mathbb{R}$.

Proof. By Axiom 12, exactly one of the following statements is true:
$$1 \in \mathbb{R}^+ \quad \text{or} \quad 1 = 0 \quad \text{or} \quad -1 \in \mathbb{R}^+.$$
By Axiom 9, $1 \neq 0$. Suppose $-1 \in \mathbb{R}^+$. Then by Axiom 12, $(-1)(-1) = 1 \in \mathbb{R}^+$ (by Theorem 9.15), and hence both $1 \in \mathbb{R}^+$ and $-1 \in \mathbb{R}^+$, which contradicts Axiom 12. Therefore $1 \in \mathbb{R}^+$, and (i) holds. For (ii), if $x > y$ and $y > z$, then $x - y, y - z \in \mathbb{R}^+$ by Definition 9.1. By Axiom 12, we have
$$x - z = (x - y) + (y - z) \in \mathbb{R}^+,$$
and thus $x > z$. For (iii), if $x > y$, then $x - y \in \mathbb{R}^+$. Then
$$(x + z) - (y + z) = (x - y) \in \mathbb{R}^+,$$
and hence $x + z > y + z$. For (iv), if $x > y$ and $z > 0$, then $x - y \in \mathbb{R}^+$ and $z \in \mathbb{R}^+$. By Axiom 12, we have
$$(x - y)z = xz - yz \in \mathbb{R}^+,$$
so $xz > yz$. For (v), if $x > y$ and $z < 0$, then $x - y \in \mathbb{R}^+$ and $-z \in \mathbb{R}^+$. Then
$$(x - y)(-z) = -(xz - yz) \in \mathbb{R}^+,$$
which implies $xz - yz \notin \mathbb{R}^+$ and $xz < yz$. □

Definition 9.19. Let $a, b \in \mathbb{R}$ with $a < b$. Then we define

(i) $(a, b) = \{\, x \in \mathbb{R} \mid a < x < b \,\}$,

(ii) $[a, b] = \{\, x \in \mathbb{R} \mid a \le x \le b \,\}$,

(iii) $[a, b) = \{\, x \in \mathbb{R} \mid a \le x < b \,\}$,

(iv) $(a, b] = \{ x \in \mathbb{R} \mid a < x \leq b \}$,

(v) $(-\infty, a) = \{ x \in \mathbb{R} \mid x < a \}$,

(vi) $(-\infty, a] = \{ x \in \mathbb{R} \mid x \leq a \}$,

(vii) $[a, \infty) = \{ x \in \mathbb{R} \mid a \leq x \}$,

(viii) $(a, \infty) = \{ x \in \mathbb{R} \mid a < x \}$,

(ix) $(-\infty, \infty) = \mathbb{R}$.

We call (c, d) an *open interval*; $[c, d]$ a *closed interval*; and either $[c, d)$ or $(c, d]$ a *half-open interval*, where c or d are possibly $\pm\infty$.

Definition 9.20. An *ordered field* is a *field* F which is also an *ordered set*, such that

(i) $x + y < x + z$ if $x, y, z \in F$ and $y < z$,

(ii) $xy > 0$ if $x \in F, y \in F$, $x > 0$, and $y > 0$.

If $x > 0$, we call x *positive*.

Theorem 9.21. There exists an ordered field $\mathbb{R}$ which satisfies Axiom 13. Moreover, $\mathbb{R}$ contains $\mathbb{Q}$ as a subfield. The members (elements) of $\mathbb{R}$ are called *real numbers*.

Remark 9.22. The statement that $\mathbb{R}$ contains $\mathbb{Q}$ as a subfield means that $\mathbb{Q} \subset \mathbb{R}$ and that the operations of addition and multiplication in $\mathbb{R}$, when applied to members of $\mathbb{Q}$, coincide with the usual operations on rational numbers; also the positive rational numbers are positive elements of $\mathbb{R}$. Note that Lecture 12 will be the construction of $\mathbb{Q}$ and the proof of Theorem 9.21. For now, we state the theorem without proof.

Definition 9.23. Let X and Y be non-empty sets. We say that X is *equivalent* to Y and write $X \sim Y$ if there exists a one-to-one function from X onto Y. Note that $\sim$ is an equivalence relation.

Definition 9.24. For any positive integer n, let J_n be the set

$$J_n \coloneqq \{ x \in \mathbb{Z}^+ \mid x \leq n \}.$$

In other words, let J_n be the set of positive integers $1, 2, \ldots, n$. Then for any set A we say

(i) A is *finite* if $A \sim J_n$ for some n (the empty set is also considered to be finite).

(ii) A is *infinite* if A is not finite.

(iii) A is *countable* if $A \sim \mathbb{Z}^+$.

(iv) A is *uncountable* if A is neither finite nor countable.

(v) A is *at most countable* if A is finite or countable.

For two finite sets A and B, we have $A \sim B$ if and only if A and B contain the same number of elements.

Theorem 9.25.

(i) Let $\varepsilon > 0$. Then $|x| < \varepsilon$ if and only if $-\varepsilon < x < \varepsilon$ and $|x| \leq \varepsilon$ if and only if $-\varepsilon \leq x \leq \varepsilon$.

(ii) $x \leq |x|$ for all $x \in \mathbb{R}$.

(iii) $|xy| = |x||y|$ for all $x, y \in \mathbb{R}$.

(iv) $|x + y| \leq |x| + |y|$ for all $x, y \in \mathbb{R}$.

Note that (iii) is called the *triangle inequality*.

Proof. To prove part (i),

($\Rightarrow$) Suppose $|x| < \varepsilon$. If $x \geq 0$, then $|x| = x < \varepsilon$, so $-\varepsilon < x < \varepsilon$. If $x < 0$, then $|x| = -x < \varepsilon$, so $-\varepsilon < x < \varepsilon$.

($\Leftarrow$) Suppose $-\varepsilon < x < \varepsilon$. Then if $x \geq 0$, $|x| = x < \varepsilon$. If $x < 0$, $|x| = -x < \varepsilon$.

The proof for $|x| \leq \varepsilon$ is similar. To prove part (ii),

Case 1. If $x \geq 0$, then $x = |x| \geq x$.

Case 2. If $x < 0$, then $|x| = -x > x$.

In either case, $x \leq |x|$. To prove part (iii),

Case 1. If $x = 0$ or $y = 0$, then $|xy| = 0 = |x||y|$.

Case 2. Suppose $x > 0$, $y > 0$: then $xy > 0$, so $|xy| = xy = |x||y|$.

Case 3. Suppose $x > 0$, $y < 0$: then $xy < 0$, so $|xy| = -(xy) = x(-y) = |x||y|$.

The other cases follow similarly. To prove part (iv),

Case 1. If $x + y \geq 0$, then $|x + y| = x + y \leq |x| + |y|$ by part (ii).

Case 2. If $x + y < 0$, then

$$|x + y| = -(x + y) = -x - y \leq |x| + |y|$$

by part (ii).

This completes the proof. $\square$

Theorem 9.26. For all $x, y \in \mathbb{R}$, we have

$$|x + y| = |x| + |y| \iff xy \geq 0.$$

In other words, the Triangle Inequality for real numbers is an equality if and only if $xy \geq 0$ or $x = 0$ or $y = 0$.

Proof. ($\Rightarrow$) Suppose $|x + y| = |x| + |y|$. We will argue by cases.

Case 1. If $x + y \geq 0$, then $|x + y| = x + y$. Thus, we have

$$x + y = |x| + |y|.$$

But since $|x| \geq x$ and $|y| \geq y$, this is possible only if $x, y \geq 0$, so $xy \geq 0$.

Case 2. If $x + y < 0$, then $|x + y| = -(x + y)$. Thus, we have

$$-(x + y) = |x| + |y|.$$

Again, this forces $x, y \leq 0$, so $xy \geq 0$.

($\Leftarrow$) Now suppose $xy \geq 0$. Again we argue by cases.

Case 1. If $x, y \geq 0$, then

$$|x + y| = x + y = |x| + |y|.$$

Case 2. If $x, y \leq 0$, then

$$|x + y| = -(x + y) = |x| + |y|.$$

In both cases the equality holds. Thus, $|x + y| = |x| + |y|$ if and only if $xy \geq 0$. $\square$

Definition 9.27. Let X and Y be sets. A function f from X into Y is a rule that assigns to each element $x \in X$ an element $f(x) \in Y$. We may describe f by the notation

$$x \mapsto f(x).$$

This is read as "x is mapped to $f(x)$." When defining the full function, we often write: define a function $f : X \to Y$ by

$$x \mapsto f(x).$$

Definition 9.28. In mathematical writing, the symbols $\Rightarrow$ and $\Leftarrow$ denote logical implication.

- The symbol $\Rightarrow$ is read as "implies." That is, for statements P and Q, the expression

  $$P \Rightarrow Q$$

  means "if P is true, then Q is true." This is often called the "if" (or "forward") direction of an "if and only if" statement (proof).

- The symbol $\Leftarrow$ is read as "is implied by." That is,

  $$P \Leftarrow Q$$

  means "if Q is true, then P is true." This is often called the "only if" (or "forward") direction of an "if and only if" statement (proof).

- The symbol $\iff$ is read as "if and only if" or "is equivalent to." That is,

  $$P \iff Q$$

  means "P is true if and only if Q is true." It asserts that both $P \Rightarrow Q$ and $Q \Rightarrow P$ hold.

Example 9.29. In the proof of Theorem 9.26, we wrote "($\Leftarrow$) Suppose $|x + y| = |x| + |y|$." This could be restated as "We first prove the if (or forward) direction. Suppose $|x + y| = |x| + |y|$." In the same proof, we also wrote "($\Rightarrow$) Now suppose $xy \geq 0$." This could be restated as "Now we prove the only if (or reverse) direction. Suppose $xy \geq 0$."

Theorem 9.30. Let S be a set of cases, and suppose there is an equivalence relation $\sim$ on S such that

(i) The property $P(x)$ to be proven is preserved under $\sim$ (that is, if $x \sim y$, then $P(x) \Leftrightarrow P(y)$).

(ii) Every element of S belongs to some equivalence class under $\sim$.

Then it suffices to prove $P(x)$ for one representative of each equivalence class. In particular, if all cases are symmetric under $\sim$, one may choose any single case to prove, and this is justified by writing "*without loss of generality.*" Note that the acronym for this is *WLOG*.

Proof. Let x_0 be a representative of some equivalence class $[x_0]$ under $\sim$. Suppose we prove $P(x_0)$. Now let $x \in [x_0]$ be arbitrary. Since $x \sim x_0$ and P is preserved under $\sim$, it follows that $P(x) \iff P(x_0)$. Therefore $P(x)$ holds for all $x \in [x_0]$. Since S is the union of its equivalence classes, repeating this argument for each equivalence class establishes $P(x)$ for all $x \in S$. In particular, if all cases are symmetric (i.e., there is only one equivalence class), proving $P(x)$ for any one x suffices. □

Example 9.31. Let $a, b \in \mathbb{R}$. We prove that

$$\max(a, b) \geq a \quad \text{and} \quad \max(a, b) \geq b.$$

Proof. By definition of $\max(a, b)$, we have

$$\max(a, b) = \begin{cases} a, & \text{if } a \geq b, \\ b, & \text{if } b > a. \end{cases}$$

This implies that we have two cases,

Case 1. Suppose $a \geq b$. Then

$$\max(a, b) = a.$$

Clearly,

$$\max(a, b) = a \geq a \quad \text{and} \quad \max(a, b) = a \geq b.$$

Case 2. Suppose $b > a$. *Without loss of generality*, we may assume $a \leq b$ and proceed as in Case 1 with b in place of a. Then we have

$$\max(a, b) = b \geq a \quad \text{and} \quad \max(a, b) = b \geq b.$$

Since the two cases are symmetric under swapping a and b, the argument for one case applies equally to the other, and "without loss of generality" we may consider only the case $a \leq b$. Thus, we have

$$\max(a, b) \geq a \quad \text{and} \quad \max(a, b) \geq b.$$

□

Theorem 9.32. A non-empty subset of real numbers which is bounded below has a greatest lower bound.

Proof. Let X be a non-empty subset of real numbers which is bounded below, and let Y be the set of lower bounds for X. Let $c \in X$. Then $y \leq c$ for $y \in Y$. Thus Y is bounded above, and by the least-upper-bound axiom Y has a least upper bound a. We will show that a is the greatest lower bound of X. Let $x \in X$. Then $y \leq x$ for all $y \in Y$, and thus x is an upper bound for Y. Since a is the least upper bound of Y, we have $a \leq x$. Therefore, a is a lower bound for X. Let b be any lower bound for X. Then $b \in Y$, and hence $b \leq a$. By Definition 9.5, a is the greatest lower bound of X. □