Lecture 11

Alex Hassett
Applied Analysis

June 12, 2025

Remark 11.1. We advise the reader completely ignore Theorem 11.2. It is not relevant to the content of these lectures and was simply added for completeness. It is not relevant to anything in any past (or future) lectures and the content of it was not taught.

Theorem 11.2. Let $(F, +, \cdot, \leq_F)$ and $(G, +, \cdot, \leq_G)$ be ordered fields, each of which satisfies the least–upper–bound property. Then there exists a unique order–field isomorphism

$$\varphi : F \longrightarrow G.$$

Proof. Throughout, $\mathbb{Q}$ denotes the usual field of rationals.

1. Embedding of $\mathbb{Q}$. Define injections
$$\iota_F(q) = q \cdot 1_F \in F, \qquad \iota_G(q) = q \cdot 1_G \in G.$$
Thus $\mathbb{Q} \subset F$ and $\mathbb{Q} \subset G$ as ordered subfields; below we identify $q \in \mathbb{Q}$ with its image in either field.

2. Density of $\mathbb{Q}$. Because each field has the least–upper–bound property, it is *Archimedean*[1]. Consequently, between any $a < b$ in F (resp. G) there is a $q \in \mathbb{Q}$ with $a < q < b$; hence $\mathbb{Q}$ is order–dense in both fields.

3. Definition of φ. For $x \in F$ choose a rational sequence (q_n) with $q_n \to x$ in F (density ensures such a choice, e.g. q_n satisfying $|x - q_n| < 1/n$). Because $\mathbb{Q}$ is a Cauchy sequence in G and G is complete, the sequence (q_n) converges in G; denote its limit by
$$\varphi(x) \; := \; \lim_G q_n.$$

4. Well–definedness. If (r_n) is a second rational sequence with $r_n \to x$ in F, then $q_n - r_n \to 0$ in F. Since $|q_n - r_n|$ is the same real number in either field, $q_n - r_n \to 0$ also in G, so both sequences have the same limit. Hence φ is well defined.

5. Field homomorphism property. Let $x, y \in F$ with rational approximations (q_n) and (r_n). Continuity of addition and multiplication in an ordered field gives
$$\varphi(x + y) = \lim_G (q_n + r_n) = \big(\lim_G q_n\big) + \big(\lim_G r_n\big) = \varphi(x) + \varphi(y),$$
$$\varphi(xy) = \lim_G (q_n r_n) = \big(\lim_G q_n\big)\big(\lim_G r_n\big) = \varphi(x)\varphi(y), \qquad \varphi(1_F) = 1_G.$$
Thus φ is a field homomorphism.

6. Order preservation. If $x <_F y$, pick rationals $q_n < r_n$ with $q_n \to x$, $r_n \to y$. Then $q_n < r_n$ in G for every n, hence $\varphi(x) = \lim_G q_n < \lim_G r_n = \varphi(y)$. Therefore φ is order preserving and in particular injective.

[1] If not, the set $\{n \in \mathbb{Z}^+\}$ would be bounded above but would lack a least upper bound, contradicting completeness.

7. Surjectivity. Apply the same construction with the roles of F and G reversed to obtain a homomorphism $\psi : G \to F$. The compositions satisfy

$$\psi \circ \varphi = \mathrm{id}_F, \qquad \varphi \circ \psi = \mathrm{id}_G,$$

so φ is bijective.

8. Uniqueness. Let $\chi : F \to G$ be any order–field homomorphism fixing each $q \in \mathbb{Q}$. For $x \in F$ choose rational $q_n \to x$ in F. Then by continuity,

$$\chi(x) = \chi\big(\lim_F q_n\big) = \lim_G \chi(q_n) = \lim_G q_n = \varphi(x),$$

so $\chi = \varphi$. Hence there exists exactly one order–field isomorphism between F and G.

□

Definition 11.3. A *cut* is, by definition, any set $\alpha \subset \mathbb{Q}$ with the following three properties.

(i) α is not empty, and $\alpha \neq \mathbb{Q}$.

(ii) If $p \in \alpha, q \in \mathbb{Q}$, and $q < p$, then $q \in \alpha$.

(iii) If $p \in \alpha$, then $p < r$ for some $r \in \alpha$.

Note that (iii) says that α has no largest member; (ii) implies two facts which can be used freely:

If $p \in \alpha$ and $q \notin \alpha$, then $p < q$.

If $r \notin \alpha$ and $r < s$, then $s \notin \alpha$.

Definition 11.4. A *complex number* is an ordered pair (a, b) of real numbers. In other words, we define the set of the complex numbers, $\mathbb{C}$ by

$$\mathbb{C} = \{\, (a, b) \mid a \in \mathbb{R} \text{ and } b \in \mathbb{R} \,\}.$$

Let $x = (a, b)$ and $y = (c, d)$ be two complex numbers. We define

$$x + y = (a + c, b + d),$$

$$xy = (ac - bd, ad + bc).$$

Theorem 11.5. Addition and multiplication as defined in Definition 11.4 turn the set of all complex numbers into a field, with $(0, 0)$ and $(1, 0)$ in the role of 0 and 1.

Proof. Let $x = (a, b)$, $y = (c, d)$, and $z = (e, f)$.

(A1) is clear.

(A2) $x + y = (a + c, b + d) = (c + a, d + b) = y + x$.

(A3)

$$\begin{aligned}(x + y) + z &= (a + c, b + d) + (e, f)\\ &= (a + c + e, b + d + f)\\ &= (a, b) + (c + e, d + f)\\ &= x + (y + z).\end{aligned}$$

(A4) $x + 0 = (a, b) + (0, 0) = (a, b) = x$.

(A5) Put $-x = (-a, -b)$. Then $x + (-x) = (0, 0) = 0$.

(M1) is clear.

(M2) $xy = (ac - bd, ad + bc) = (ca - db, da + cb) = yx$.

(M3)

$$\begin{aligned}(xy)z &= (ac - bd, ad + bc)(e, f)\\ &= (ace - bde - adf - bcf, acf - bdf + ade + bce)\\ &= (a, b)(ce - cdf, cf + de)\\ &= x(yz).\end{aligned}$$

(M4) $1x = (1, 0)(a, b) = (a, b) = x$.

(M5) If $x \neq 0$, then $(a, b) \neq (0, 0)$, which means that at least one of the real numbers a, b is different from 0. Hence $a^2 + b^2 > 0$, and we can define

$$\frac{1}{x} = \left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2}\right).$$

Then

$$\begin{aligned}x \cdot \frac{1}{x} &= (a, b)\left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2}\right)\\ &= (1, 0)\\ &= 1.\end{aligned}$$

(D)

$$\begin{aligned}x(y + z) &= (a, b)(c + e, d + f)\\ &= (ac + ae - bd - bf, ad + af + bc + be)\\ &= (ac - bd, ad + bc) + (ae - bf, af + be)\\ &= xy + xz.\end{aligned}$$

□

Theorem 11.6. For any real numbers a and b we have

$$(a, 0) + (b, 0) = (a + b, 0), \quad (a, 0)(b, 0) = (ab, 0).$$

The proof is trivial.

Remark 11.7. Theorem 11.6 shows that the complex numbers of the form $(a, 0)$ have the same arithmetic properties as the corresponding real numbers a. We can therefore identify $(a, 0)$ with a (notice the similarity with the integers). This identification gives us the real field as a subfield of the complex field.

Definition 11.8. The square root of -1, denoted i, is defined as $i = (0, 1)$.

Theorem 11.9. $i^2 = -1$.

Proof.

$$i^2 = (0, 1)(0, 1) = (-1, 0) = -1.$$

□

Theorem 11.10. If a and b are real, then $(a, b) = a + bi$.

Proof.

$$\begin{aligned} a+bi &= (a,0)+(b,0)(0,1) \\ &= (a,0)+(0,b) \\ &= (a,b). \end{aligned}$$

□

Definition 11.11. If a, b are real and $z = a+bi$, then the complex number $\bar{z} = a-bi$ is called the *conjugate* of z. The numbers a and b are the *real part* and the *imaginary part* of z, respectively. We shall occasionally write

$$a = \mathrm{Re}(z), \qquad b = \mathrm{Im}(z).$$

Theorem 11.12. If z and w are complex, then

(a) $\overline{z+w} = \bar{z}+\bar{w}$,

(b) $\overline{zw} = \bar{z}\cdot\bar{w}$,

(c) $z+\bar{z} = 2\mathrm{Re}(z)$, $z-\bar{z} = 2i\ \mathrm{Im}(z)$,

(d) $z\bar{z}$ is real and positive (except when $z=0$).

Proof. (a), (b), and (c) are quite trivial. To prove (d), write $z = a+bi$, and note that $z\bar{z} = a^2+b^2$. □

Definition 11.13. If z is a complex number, its *absolute value* $|z|$ is the nonnegative square root of $z\bar{z}$; that is, $|z| = (z\bar{z})^{\frac{1}{2}}$. Note that the existence (and uniqueness) of $|z|$ follows from Theorem 10.2 and part (d) of Theorem 11.2. Note that when x is real, then $\bar{x} = x$, hence $|x| = \sqrt{x^2}$. Thus $|x| = x$, if $x \geq 0$, then $|x| = -x$ if $x < 0$.

Theorem 11.14. Let z and w be complex numbers. Then

(a) $|z| > 0$ unless $z = 0$, then $|0| = 0$,

(b) $|\bar{z}| = |z|$,

(c) $|zw| = |z||w|$,

(d) $|\mathrm{Re}\, z| \leq |z|$,

(e) $|z+w| \leq |z|+|w|$.

Proof. (a) and (b) are trivial. Put $z = a+bi$, $w = c+di$, with $a,b,c,d \in \mathbb{R}$. Then

$$|zw|^2 = (ac-bd)^2+(ad+bc)^2 = (a^2+b^2)(c^2+d^2) = |z|^2|w|^2$$

or $|zw|^2 = (|z||w|)^2$. Now (c) follows from the uniqueness assertion of Corollary 10.3. To prove (d), note that $\bar{z}w$ is the conjugate of $z\bar{w}$, so that $z\bar{w}+\bar{z}w = 2\mathrm{Re}(z\bar{w})$. Hence

$$\begin{aligned} |z+w|^2 = (z+w)(\bar{z}+\bar{w}) &= z\bar{z}+z\bar{w}+\bar{z}w+w\bar{w} \\ &= |z|^2+2\mathrm{Re}(z\bar{w})+|w|^2 \\ &\leq |z|^2+2|z\bar{w}|+|w|^2 \\ &= |z|^2+2|z||w|+|w|^2 = (|z|+|w|)^2. \end{aligned}$$

Now (e) follows by taking square roots. □

Notation 11.15. If $x_1, \ldots, x_n$ are complex numbers, then we write

$$x_1+x_2+\cdots+x_n = \sum_{j=1}^{n} x_j.$$

Theorem 11.16. If $a_1, \dots, a_n$ and $b_1, \dots, b_n$ are complex numbers, then

$$\left| \sum_{j=1}^{n} a_j \bar{b}_j \right|^2 \leq \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2.$$

This inequality is usually known as the *Schwarz inequality.*

Proof. Put $A = \sum |a_j|^2$, $B = \sum |b_j|^2$, $C = \sum a_j \bar{b}_j$ (in all sums in this proof, j runs over the values $1, \dots, n$). If $B = 0$, then $b_1 = \cdots = b_n = 0$, and the conclusion is trivial. Assume therefore that $B > 0$. By Theorem 11.12, we have

$$\begin{aligned}\sum |Ba_j - Cb_j|^2 &= \sum (Ba_j - Cb_j)(B\bar{a}_j - \overline{Cb}_j) \\ &= B^2 \sum |a_j|^2 - B\bar{C} \sum a_j \bar{b}_j - BC \sum \bar{a}_j b_j + |C|^2 \sum |b_j|^2 \\ &= B^2 A - B|C|^2 \\ &= B(AB - |C|^2).\end{aligned}$$

Since each term in the first sum is nonnegative, we see that

$$B(AB - |C|^2) \geq 0.$$

Since $B > 0$, it follows that $AB - |C|^2 \geq 0$. This is the desired inequality. □

Definition 11.17. For each positive integer k, let $\mathbb{R}^k$ be the set of all ordered k-tuples

$$\mathbf{x} = (x_1, x_2, \dots, x_k),$$

where $x_1, \dots, x_k$ are real numbers, called the *coordinates* of $\mathbf{x}$. The elements of $\mathbb{R}^k$ are called points, or vectors, especially when $k > 1$. We shall denote vectors by boldfaced letters. If $\mathbf{y} = (y_1, \dots, y_k)$ and if α is a real number, put

$$\begin{aligned}\mathbf{x} + \mathbf{y} &= (x_1 + y_1, \dots, x_k + y_k), \\ &= \alpha\mathbf{x} = (\alpha x_1, \dots, \alpha x_k).\end{aligned}$$

so that $\mathbf{x} + \mathbf{y} \in \mathbb{R}^k$ and $\alpha\mathbf{x} \in \mathbb{R}^k$. This defines addition of vectors, as well as multiplication of a vector by a real number (a scalar). These two operations satisfy the commutative, associative, and distributive laws (the proof is trivial, in view of the analogous laws for the real numbers) and make $\mathbb{R}^k$ into a *vector space over the real field.* The zero element of $\mathbb{R}^k$ (sometimes called the *origin* or the *null vector*) is the point $\mathbf{0}$, all of whose coordinates are 0.

Definition 11.18. The "*inner product*" of $\mathbf{x} \in \mathbb{R}^k$ and $\mathbf{y} \in \mathbb{R}^k$ by

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{k} x_i y_i$$

and the *norm* of $\mathbf{x}$ by

$$\|\mathbf{x}\| = (\mathbf{x} \cdot \mathbf{x})^{\frac{1}{2}} = \left(\sum_{i=1}^{k} x_i^2 \right)^{\frac{1}{2}}.$$

The structure defined in Definitions 11.17 and 11.18 (the vector space $\mathbb{R}^k$ with the above inner product and norm) is called *euclidean k-space.*

Theorem 11.19. Suppose $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z} \in \mathbb{R}^k$, and α is real. Then

(a) $\|\mathbf{x}\| \geq 0$;

(b) $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{0}$;

(c) $\|\alpha\mathbf{x}\| = |\alpha|\,\|\mathbf{x}\|$;

(d) $|\mathbf{x}\cdot\mathbf{y}| \le \|\mathbf{x}\|\,\|\mathbf{y}\|$;

(e) $\|\mathbf{x}+\mathbf{y}\| \le \|x\| + \|y\|$;

(f) $\|\mathbf{x}-\mathbf{z}\| \le \|\mathbf{x}-\mathbf{y}\| + \|\mathbf{y}-\mathbf{z}\|$.

Proof. (a), (b), and (c) are obvious, and (d) is an immediate consequence of the Schwarz inequality. By (d) we have

$$\begin{aligned}\|\mathbf{x}+\mathbf{y}\|^2 &= (\mathbf{x}+\mathbf{y})\cdot(\mathbf{x}+\mathbf{y})\\ &= \mathbf{x}\cdot\mathbf{x} + 2\mathbf{x}\cdot\mathbf{y} + \mathbf{y}\cdot\mathbf{y}\\ &\le \|\mathbf{x}\|^2 + 2\,\|\mathbf{x}\|\,\|\mathbf{y}\| + \|\mathbf{y}\|^2\\ &= (\|\mathbf{x}\| + \|\mathbf{y}\|)^2,\end{aligned}$$

so that (e) is proved. Finally, (f) follows from (e) if we replace $\mathbf{x}$ by $\mathbf{x}-\mathbf{y}$ and $\mathbf{y}$ by $\mathbf{y}-\mathbf{z}$. □

Definition 11.20. The extended real number system, sometimes denoted $\overline{\mathbb{R}}$, consists of the real field $\mathbb{R}$ and two symbols, $+\infty$ (or ∞) and $-\infty$. We preserve the original order in $\mathbb{R}$, and define

$$-\infty < x < +\infty$$

for every $x \in \mathbb{R}$. The extended real number system does not form a field, but it is customary to make the following conventions:

(a) If x is real then
$$x + \infty = +\infty, \qquad x - \infty = -\infty, \qquad \frac{x}{+\infty} = \frac{x}{-\infty} = 0.$$

(b) If $x > 0$, then $x\cdot(+\infty) = +\infty$ and $x\cdot(-\infty) = -\infty$.

(c) If $x < 0$, then $x\cdot(+\infty) = -\infty$ and $x\cdot(-\infty) = +\infty$.

When it is desired to make the distinction between real numbers and the symbols $+\infty$ and $-\infty$ we call the real numbers *finite*.

Remark 11.21. From Definition 11.20, it is clear that $+\infty$ is an upper bound of every subset of the extended real number system, and that every nonempty subset has a least upper bound. If, for example, E is a nonempty set of real numbers which is not bounded above in $\mathbb{R}$, then $\sup E = +\infty$ in the extended real number system. Exactly the same remarks apply to lower bounds.