Lecture 4

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Definition 4.1. Let p and q be predicates. Then the contrapositive of the statement $p \implies q$ is $\neg q \implies \neg p$.

Theorem 4.2. Let p and q be predicates. Then $p \implies q \equiv \neg q \implies \neg p$, i.e. a statement is logically equivalent to its contrapositive.

Proof. By Theorem 1.1, we have

$$\begin{aligned} p \implies q &\equiv \neg p \vee q \\ &\equiv q \vee \neg p \\ &\equiv \neg(\neg q) \vee \neg p \\ p \implies q &\equiv \neg q \implies \neg p, \end{aligned}$$

as desired. □

Definition 4.3. Let p and q be predicates. Then the converse of the statement $p \implies q$ is $q \implies p$.

Theorem 4.4. Let p and q be predicates. Then $p \implies q \not\equiv q \implies p$, i.e. a statement is not necessarily logically equivalent to its converse. Note that it is possible for a statement to be logically equivalent to its converse, but it is not generally true.

Proof. Let p be true and let q be false. Then $p \implies q \equiv \bot$ and $q \implies p \equiv \top$. Therefore, $p \implies q \not\equiv q \implies p$. □

Remark 4.5. Note that for all future lectures and theorems, $\top$ will denote true (tautology) and $\bot$ will denote false (contradiction).

Definition 4.6. Let p and q be predicates. Then the inverse of the statement $p \implies q$ is $\neg p \implies \neg q$.

Theorem 4.7. Let p and q be predicates. Then $p \implies q \not\equiv \neg p \implies \neg q$, i.e. a statement is not necessarily logically equivalent to its inverse. Note that it is possible for a statement to be logically equivalent to its inverse, but it generally won't be.

Proof. Suppose $p \equiv \bot$ and $q \equiv \top$. Then $p \implies q \equiv \top$, but $\neg p \implies \neg q \equiv \bot$. Therefore, $p \implies q \not\equiv \neg p \implies \neg q$. □

Remark 4.8. The term universe of discourse (universe), usually denoted $\mathbb{D}$, is another term for domain of discourse (see Definition 2.7). We will use the term universe for all future lectures. In previous lectures, we used the letter U to denote the universe, but for future lectures we will now use $\mathbb{D}$ to denote the universe under consideration.

Definition 4.9. The empty set, denoted $\emptyset$, is the unique set having no elements. Note that any set that is not the empty set is said to be "non-empty".

Definition 4.10. A statement S is said to be "vacuously true" if it resembles a statement of the form $p \implies q$ (this is called a material conditional statement) where the antecedent (p) is known to be false. Note that q is called the consequent.

Remark 4.11. Many properties regarding the empty set are vacuously true; as we will demonstrate in Example 4.12.

Example 4.12. By the definition of subset, the empty set is a subset of any set A. That is, every element x of $\emptyset$ belongs to A. Indeed, if it were not true that every element of $\emptyset$ is in A, then there would be at least one element of $\emptyset$ that is not present in A. Since there are no elements of $\emptyset$ at all, there is no element of $\emptyset$ that is not in A. Any statement that begins with "for every element of $\emptyset$" is not making a substantive claim; it is a vacuous truth. This is often paraphrased as "everything is true of the elements of the empty set."

Remark 4.13. To prove that for any set A, $\emptyset \subset A$ one could rewrite this as a material conditional statement: for all $x \in \mathbb{D}$, if $x \in \emptyset$, then $x \in A$. Symbolically, the previous material conditional statement can be written as $\forall x \in \mathbb{D}(x \in \emptyset \implies x \in A)$. Since there are no elements in the empty set, the antecedent $x \in \emptyset$ is always false. Therefore, by Definition 4.10, this statement is vacuously true.

Definition 4.14. A family (or collection) of sets is defined to be a collection of sets, i.e. a family of sets is a set of sets. A collection F of subsets of a given set S is called a family of sets over S. Note that a family of sets must be a set.

Definition 4.15. In set theory, a class is defined to be a collection of sets (or sometimes other mathematical objects) that can be unambiguously defined by a property that all its members share. Note that classes need not be sets themselves. For instance, the class of all sets ($\mathcal{C} \coloneqq \{\, A \mid A \text{ is a set} \,\}$) is not a set itself as it is too large to be a set (the explanation of this is well beyond the scope of these lectures).

Definition 4.16. If $\mathcal{A}$ is a collection of sets, we define

$$\cup\mathcal{A} \coloneqq \{\, x \mid x \in A \text{ for some (any) } A \in \mathcal{A} \,\}.$$

In case the family $\mathcal{A}$ is indexed by the positive integers, i.e.

$$\mathcal{A} = \{A_1, A_2, A_3, \dots\}$$

we write

$$\cup\mathcal{A} = \cup_{n=1}^{\infty} A_n.$$

Note that $\cup\mathcal{A}$ is read "the union of $\mathcal{A}$" or just "union $\mathcal{A}$".

Definition 4.17. If $\mathcal{A}$ is a collection of sets, we define

$$\cap\mathcal{A} \coloneqq \{\, x \mid x \in A \text{ for every (all) } A \in \mathcal{A} \,\}.$$

In case the family $\mathcal{A}$ is indexed by the positive integers, i.e.

$$\mathcal{A} = \{A_1, A_2, A_3, \dots\}$$

we write

$$\cap\mathcal{A} = \cap_{n=1}^{\infty} A_n.$$

Note that $\cap\mathcal{A}$ is read "the intersection of $\mathcal{A}$".

Remark 4.18. Another, equivalent, definition of $\cup\mathcal{A}$ is

$$\cup\mathcal{A} \coloneqq \{\, x \mid x \in A \text{ for at least one } A \in \mathcal{A} \,\}.$$

This means that x only needs to be in at least one of the sets in $\mathcal{A}$ to be in $\cup\mathcal{A}$. Furthermore, for x to be in $\cap\mathcal{A}$, x needs to be in every single set in $\mathcal{A}$. This idea is demonstrated in Example 4.19.

Example 4.19. Let $\mathcal{A}$ be a family of sets defined by

$$\mathcal{A} \coloneqq \{A_1, A_2, A_3, \dots\}$$

where $A_1 = \{1\}$, $A_2 = \{1, 2\}$, $\dots$, $A_n = \{1, 2, \dots, n\}$, $\dots$. Then

$$\cup\mathcal{A} = \cup_{n=1}^{\infty} A_n = \mathbb{Z}^+,$$

and

$$\cap\mathcal{A} = \cap_{n=1}^{\infty} A_n = \{1\}.$$

Definition 4.20. A subset X of $\mathbb{R}$ is said to be a successor set if

1. $1 \in X$,
2. $n \in X \implies n + 1 \in X$.

Note that since $\mathbb{R}$ is a successor set, successor sets exist.

Lemma 4.21. If $\mathcal{A}$ is any non-empty collection of successor sets, then $\cap\mathcal{A}$ is a successor set.

Proof. By Definition 4.20, we have that $1 \in A$ for every $A \in \mathcal{A}$, so $1 \in \cap\mathcal{A}$ by Definition 4.17. Suppose that $n \in \cap\mathcal{A}$. Then $n \in A$ for every $A \in \mathcal{A}$ by Definition 4.17. Since every set A in $\mathcal{A}$ is a successor set, by Definition 4.20, we have that $n + 1 \in A$ for every $A \in \mathcal{A}$. Thus $n + 1 \in \cap\mathcal{A}$ by Definition 4.17. Since $n \in A$ for every $A \in \mathcal{A}$ by Definition 4.17 and $A \subset \mathbb{R}$ by Definition 4.20, we have that $\cap\mathcal{A} \subset \mathbb{R}$. Therefore, by Definition 4.20, $\cap\mathcal{A}$ is a successor set. $\square$

Definition 4.22. The set of all positive integers ($\mathbb{Z}^+$) is the intersection of the family of all successor sets. Formally,

$$\mathbb{Z}^+ \coloneqq \cap\{\, X \mid X \text{ is a successor set} \,\}.$$

Lemma 4.23. If a set X is a successor set, then $\mathbb{Z}^+ \subset X$, i.e. $\mathbb{Z}^+$ is the "smallest" successor set.

Proof. By Lemma 4.21 and Definition 4.22, $\mathbb{Z}^+$ is a successor set. Let $x \in \mathbb{Z}^+$. Then x is in every successor set by Definition 4.22. In particular $x \in X$. Therefore, $\mathbb{Z}^+ \subset X$. $\square$

Theorem 4.24. Suppose that for each positive integer n, we have a statement $S(n)$. Also suppose that

1. $S(1)$ is true,
2. if $S(n)$ is true, then $S(n + 1)$ is true.

Then $S(n)$ is true for every positive integer n. Note that this is often referred to as the Theorem of Mathematical Induction.

Proof. Let $G = \{\, n \in \mathbb{Z}^+ \mid S(n) \text{ is true} \,\}$. Then $G \subset \mathbb{Z}^+$. On the other hand, we have that $1 \in G$ by supposition 1. Furthermore, if $n \in G$, then $n + 1 \in G$ by supposition 2. Thus G is a successor set by Definition 4.20. By Lemma 4.23, we have that $\mathbb{Z}^+ \subset G$. Therefore, $G = \mathbb{Z}^+$. $\square$

Corollary 4.25. The set of all positive integers equals the set

$$\mathbb{Z}^+ = \{1, 2, 3, \dots\}.$$

Proof. Let $\mathcal{F}$ denote the family of all successor sets. Define

$$\mathbb{Z}^+ \coloneqq \bigcap_{S \in \mathcal{F}} S.$$

We will prove that $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$. First, we prove that $\{1, 2, 3, \ldots\} \subseteq \mathbb{Z}^+$. Let $n \in \{1, 2, 3, \ldots\}$. We will show that $n \in S$ for every successor set S, which implies $n \in \mathbb{Z}^+$. We proceed by induction on n.

Base case. For $n = 1$, by Definition 4.20, $1 \in S$ for every successor set S. Hence $1 \in \mathbb{Z}^+$.

Inductive step. Suppose that for some $k \geq 1$, we have $k \in S$ for every successor set S. By Definition 4.20, it follows that $k + 1 \in S$ for every successor set S. Thus, $k + 1 \in \mathbb{Z}^+$. By Theorem 4.24, it follows that $n \in \mathbb{Z}^+$ for all $n \in \{1, 2, 3, \ldots\}$. Therefore,

$$\{1, 2, 3, \ldots\} \subseteq \mathbb{Z}^+.$$

Next, we prove that $\mathbb{Z}^+ \subseteq \{1, 2, 3, \ldots\}$. Consider the set $T = \{1, 2, 3, \ldots\}$. We claim that T is a successor set. Indeed, $1 \in T$, and for every $n \in T$, we have $n + 1 \in T$ by construction of T. Hence $T \in \mathcal{F}$. Since $\mathbb{Z}^+$ is the intersection of all successor sets and T is one of them, it follows that

$$\mathbb{Z}^+ \subseteq T = \{1, 2, 3, \ldots\}.$$

Combining the two inclusions, we conclude that

$$\mathbb{Z}^+ = \{1, 2, 3, \ldots\}.$$

$\square$

Definition 4.26. If X is a set, then we let $\mathcal{P}(X)$ denote the collection/set of all subsets of X. $\mathcal{P}(X)$ is called the power set of X. Formally,

$$\mathcal{P}(X) \coloneqq \{\, A \mid A \subset X \,\}.$$