Heine-Borel-Bolzano-Weierstrass Proof Clarification

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Proof. Let $x \in \mathbb{R}$. Choose an integer N such that $N > |x| + \epsilon$ for $\epsilon > 0$ (we can choose such an integer by the Archimedean property of $\mathbb{R}$). Then for every $n \geq N$, we have

$$|p_n| > n \geq N > |x| + \epsilon,$$

so

$$|p_n - x| \geq |p_n| - |x| > \epsilon.$$

by Corollary 2.1.4(b). Hence all but finitely many elements of the set $\{\, p_n \mid n \in \mathbb{N} \,\}$ lie outside the open interval $(x - \epsilon, x + \epsilon)$, which is the neighborhood of x with radius ϵ. That means no neighborhood of x contains infinitely many p_n; therefore by Theorem 2.2.15, x is not a limit point of $\{p_n\}$. □

Proof. Let V be a neighborhood of p with radius $\epsilon > 0$. Choose an $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. Then $n \geq N$ implies

$$|p_n - p| < \frac{1}{n} \leq \frac{1}{N} < \epsilon,$$

so $p_n \in V$. □

Theorem 2.4.3. This is the Bolzano-Weierstrass Theorem.

Proof. Let S be a bounded infinite subset of $\mathbb{R}$. Since S is bounded, choose numbers $a_1 < b_1$ with

$$S \subset I_1 \coloneqq [a_1, b_1].$$

Assume for some $n \geq 1$ we already have constructed a closed interval

$$I_n = [a_n, b_n] \qquad \text{with } a_n < b_n,$$

that contains infinitely many points of S. Define its midpoint m_n by

$$m_n = \frac{a_n + b_n}{2}.$$

Partition I_n into the two closed halves

$$I_{n,L} = [a_n, m_n], \qquad I_{n,R} = [m_n, b_n],$$

and observe that

$$I_{n,L} \cup I_{n,R} = I_n \quad \text{and} \quad I_{n,L} \cap I_{n,R} = \{m_n\}.$$

Because $S \cap I_n$ is infinite, we know that

$$S \cap I_n \not\subset \{m_n\}.$$

Consequently, $S \cap I_n$ must have points other than m_n. Write

$$S_L \coloneqq S \cap I_{n,L}, \qquad S_R \coloneqq S \cap I_{n,R}.$$

Then

$$S \cap I_n = S_L \cup S_R, \qquad S_L \cap S_R = S \cap \{m_n\} \subset \{m_n\}.$$

If both S_L and S_R are finite, then their union is finite. This contradicts the fact that $S \cap I_n$ is infinite. Therefore, at least at least one of S_L or S_R must be infinite, and hence at least one of $I_{n,L}$ or $I_{n,R}$ contains infinitely many points of S. Choose that half and call it I_{n+1}. Write

$$I_{n+1} = [a_{n+1}, b_{n+1}].$$

Then, by construction, $I_{n+1} \subset I_n$ for every n:

$$I_1 \supset I_2 \supset I_3 \supset \cdots.$$

We also have that the length of I_{n+1}, denoted $|I_{n+1}|$, is

$$|I_{n+1}| = \frac{1}{2}|I_n|.$$

Start with $|I_1|$ and apply this "half-length" rule repeatedly. Then we get

$$|I_2| = \frac{1}{2}|I_1|, \qquad |I_3| = \frac{1}{2}|I_2| = \left(\frac{1}{2}\right)^2 |I_1|, \qquad \ldots.$$

Thus, by induction, we have

$$|I_n| = \left(\frac{1}{2}\right)^{n-1} |I_1| = 2^{-n+1}|I_1|, \qquad \text{for } n \geq 1.$$

Then as $n \to \infty$, we have that $|I_n| \to 0$. So $\{I_n\}$ is a countable decreasing sequence of non-empty, closed, bounded intervals whose lengths satisfy

$$|I_n| = b_n - a_n = 2^{-n+1}(b_1 - a_1) \to 0 \quad \text{as} \quad n \to \infty.$$

Since the lengths of the intervals tend to 0, the Nested Intervals Property implies that

$$\bigcap_{n=1}^{\infty} I_n = \{x_0\}$$

for some point $x_0 \in \mathbb{R}$. Let $\epsilon > 0$ be given. Choose N large enough such that $|I_N| < \epsilon$. Since I_N contains infinitely many points of S, there exists a point $s \in S \cap I_N \subset S$ with $s \neq x_0$. Since the length of I_N is less than ϵ, $I_N \subset (x_0 - \epsilon, x_0 + \epsilon)$, so

$$0 < |s - x_0| < \epsilon.$$

Because this can be done for every $\epsilon > 0$, every neighborhood of x_0 contains a point of S other than x_0; hence x_0 is a limit point of S. □

Theorem 1. Let (a_n) be a convergent sequence in $\mathbb{R}$. Then (a_n) is Cauchy.

Proof. Let $a_n \to L \in \mathbb{R}$ and fix $\epsilon > 0$. Since (a_n) converges, there exists $N \in \mathbb{N}$ such that for $n > N$, we have

$$|a_n - L| < \frac{\epsilon}{2}.$$

Then, for all $m, n > N$, we have

$$\begin{aligned} |a_m - a_n| &= |a_m - L + L - a_n| \\ &\leq |a_m - L| + |L - a_n| \\ &= |a_m - L| + |a_n - L| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Therefore, (a_n) is Cauchy. □