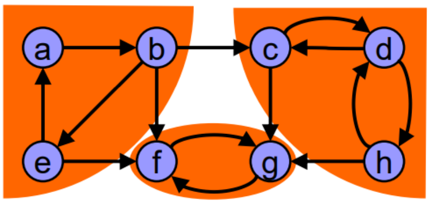
[3] Directed graphs, Strongly Connected Components, Euler trail

- **silně souvislé komponenty (strongly connected components)**
  - Orientovaný graf G=(V,E) se nazývá silně souvislý, existuje-li v každém směru mezi každými dvěma uzly v grafu cesta.
  - Silně souvislé komponenty grafu G jsou jeho maximální silně souvislé podgrafy.
  - **SCC(v)** = $\{u \in V \mid$ existuje cesta v G z u do v a cesta z v do $u\}$



Kosaraju-Sharirův algoritmus

input: graph $G = (V, E)$

output: set of strongly connected components (sets of vertices)

1. S = empty stack;
2. **while** S does not contain all vertices **do**
   Choose an arbitrary vertex v not in S;
   DFS-Walk'(v) and each time that DFS finishes expanding a vertex u, push u onto S;
3. Reverse the directions of all arcs to obtain the transpose graph;
4. **while** S is nonempty **do**
   v = pop(S);
   if v is UNVISITED **then** DFS-Walk(v);
   The set of visited vertices will give the strongly connected component containing v;

DFS-Walk

input: Graph G.

```
procedure DFS-Walk(Vertex u) {
    state[u] = OPEN; d[u] = ++time;
    for each Vertex v in succ(u)
        if (state[v] == UNVISITED) then {p[v] = u; DFS-Walk(v); }
    state[u] = CLOSED; f[u] = ++time;
}

procedure DFS-Walk'(Vertex u) {
    state[u] = OPEN; d[u] = ++time;
    for each Vertex v in succ(u)
        if (state[v] == UNVISITED) then {p[v] = u; DFS-Walk'(v); }
    state[u] = CLOSED; f[u] = ++time; push u to S;
}
```

output: array p pointing to predecessor vertex, array d with times of vertex opening and array f with time of vertex closing.

Optimalizovaný DFS-Walk

input: Graph G.

```
procedure DFS-Walk(Vertex u) {
    state[u] = OPEN;
    for each Vertex v in succ(u)
        if (state[v] == UNVISITED) then  DFS-Walk(v);
    state[u] = CLOSED;
}

procedure DFS-Walk'(Vertex u) {
    state[u] = OPEN;
    for each Vertex v in succ(u)
        if (state[v] == UNVISITED) then  DFS-Walk'(v);
    state[u] = CLOSED; push u to S;
}
```

- Kosaraju-Sharir uskuteční dva kompletní průchody grafem.
- Je-li graf reprezentován seznamem sousedů, pak má lineární složitost **$\Theta(|V|+|E|)$**.
- Je-li graf reprezentován maticí sousednosti, pak má složitost **$O(|V|^2)$**.

// Vizualizace algoritmů přikládám, ze šetrnostních důvodů, až na konec souboru.

Tarjanův algoritmus

input: graph $G = (V, E)$
output: set of strongly connected components

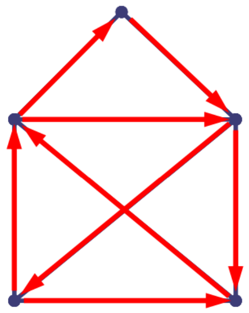
```
// every node has following fields:
// index: a unique number to ID node
// lowlink: ties node to others in SCC
// pred: pointer to stack predecessor
// instack: true if node is in stack

procedure push( v )
// stack may be null
  v.pred    = S;
  v.instack = true;
  S         = v;
end push;


function pop( v )
// val param v is stack copy
  S         = v.pred;
  v.pred    = null;
  v.instack = false;
  return v;
end pop;
```

```
procedure find_scc( v )
  v.index = v.lowlink = ++index;
  push( v );
  foreach node w in succ( v ) do
    if w.index = 0 then // not yet visited
      find_scc( w );
      v.lowlink = min( v.lowlink, w.lowlink );
    elsif w.instack then
      v.lowlink = min( v.lowlink, w.index );
    end if
  end foreach

  if v.lowlink = v.index then // v: head of SCC
    SCC++ // track how many SCCs found
    repeat
      x = pop( S );
      add x to current strongly connected component;
    until x = v;
    output the current strongly connected component;
  end if
end find_scc;

index = 0;    // unique node number > 0
S = null;     // pointer to node stack
SCC = 0;      // number of SCCs in G
foreach node v in V do
  if v.index = 0 then // yet unvisited
    find_scc( v );
  end if
end foreach;
```

- Tarjan uskuteční jediný kompletní průchod grafem.
- seznam sousedů → lineární složitost **Θ(|V|+|E|)**
- matice sousednosti → složitost **O(|V|²)**
- Tarjan poběží rychleji než Kosaraju-Sharir.

Eulerovský tah (Euler's trail)

- Existuje v (orientovaném či neorientovaném) grafu G tah, který navštíví každý uzel právě jednou?
- **tah** je podobný cestě, ale hrany se nemohou opakovat (uzly ano)



- [**Theorem**] Graf G obsahuje eulerovský tah, právě tehdy když je souvislý a má 0 či 2 uzly lichého stupně.

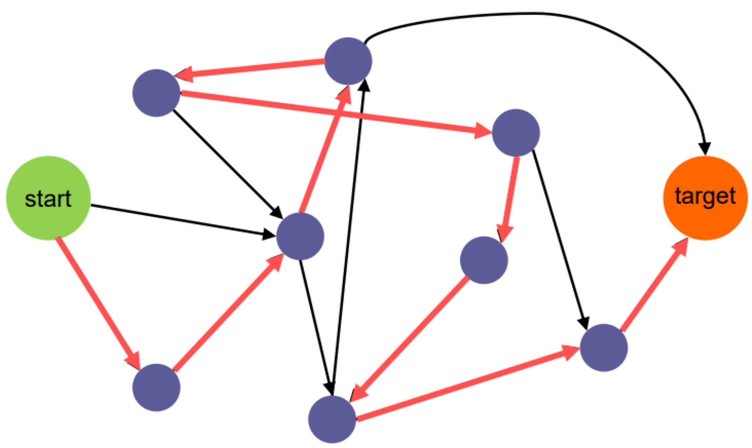
- lze rozlišit dva případy:
  - Eulerův tah začíná a **končí ve stejném uzlu** (Eulerian tour)
    - každý uzel musí mít sudý stupeň
  - začíná a končí **v různých uzlech**
    - oba uzly musí mít lichý stupeň a ostatní uzly musí mít sudý stupeň

- Euler uskuteční jen jeden kompletní průchod grafem.
- seznam sousedů → lineární složitost **Θ(|V|+|E|)**
- matice sousednosti → složitost **O(|V|²)**

```
input: graph G = (V, E)
output: trail (as a stack with edges)

procedure euler-trail(vertex v);
{
 foreach vertex u in succ(v) do {
   remove edge(v,u) from graph;
   euler-trail(u);
   push(edge(v,u));
  }
}
```

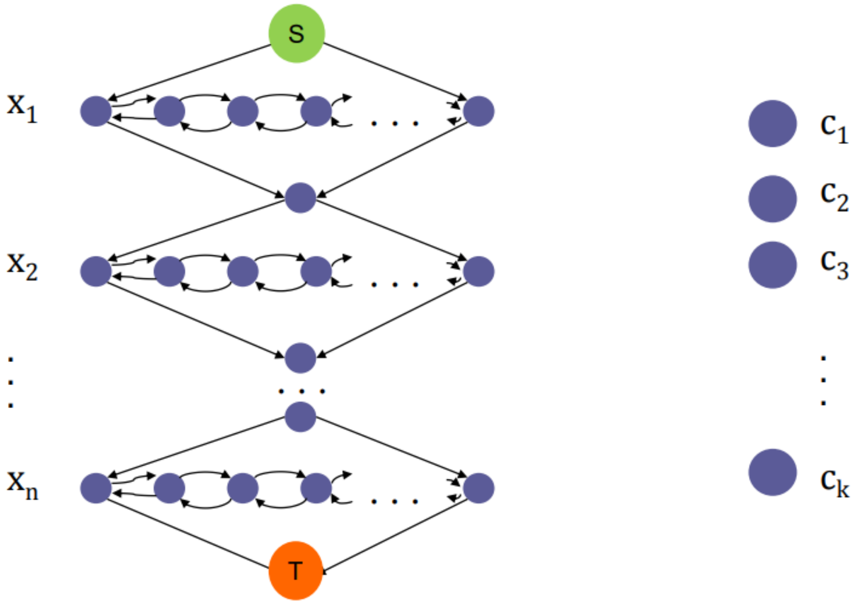
Hamiltonovská cesta (Hamiltonian Path)

- Existuje v (orientovaném či neorientovaném) grafu G cesta, která navštíví každý uzel právě jednou?

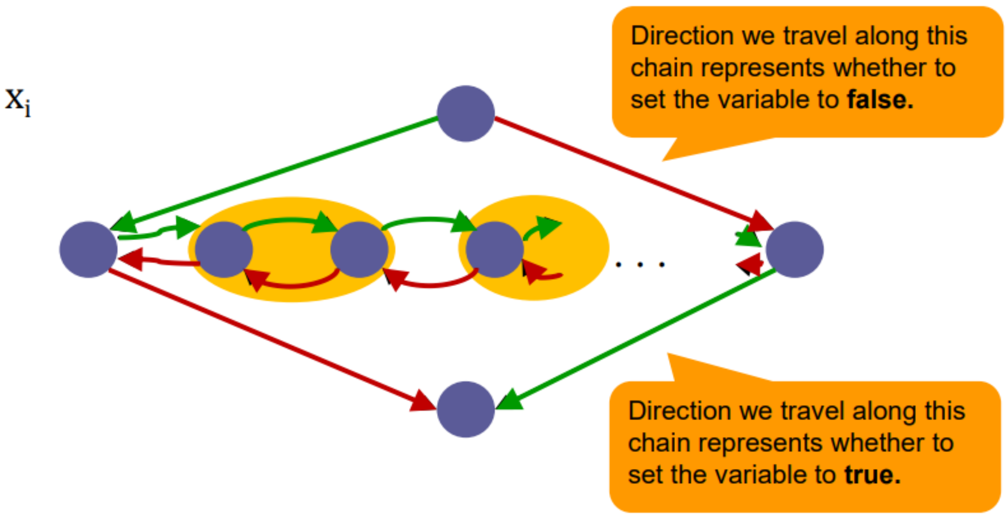


- velice složitý problém (**NP-úplný**)
- nápad na redukci:
  - Předpokládejme, že máme black box na řešení Hamiltonovské cesty.
  - Známe již NP-úplný problém splnitelnosti výrokových formulí (SAT – satisfiability)
  - Ideou je v polynomiálním čase převést vstup v podobě SAT instancí na instance vhodné pro náš black box, a to tak, že řešení z black boxu bude přímo reprezentovat SAT řešení daného vstupu.
  - Vyřešíme-li tedy black box v polynomiálním čase, tak můžeme v polynomiálním čase vyřešit i SAT.

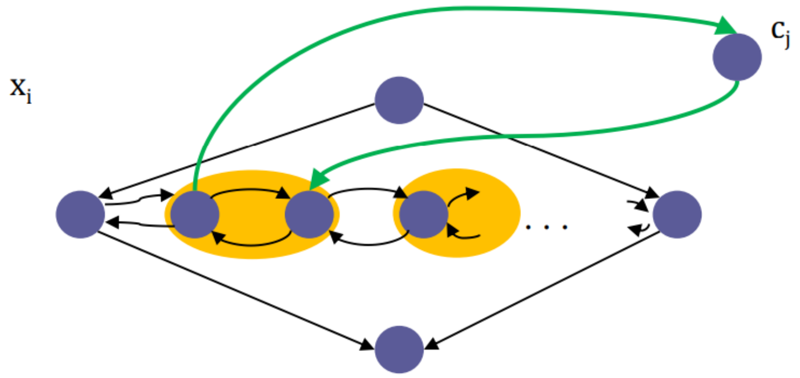
- **high-level struktura**:



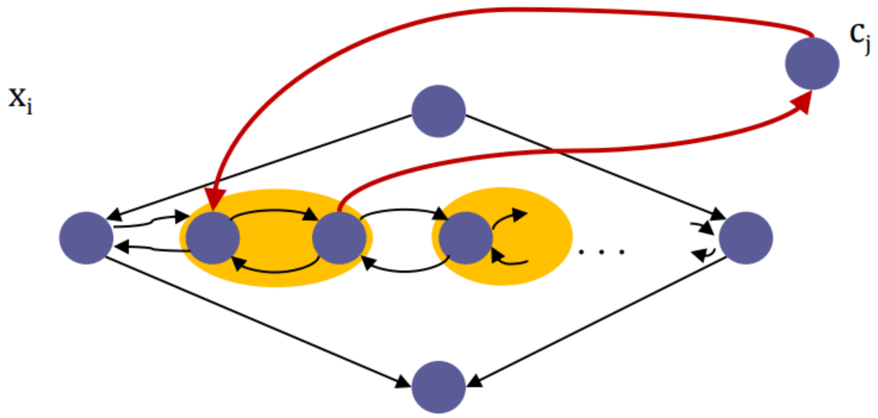
- **interní struktura proměnné x_i:**
  - počet výskytů proměnné x_i v celém SAT přesně koresponduje s počtem dvojic ve žlutých oválech



- pokud c_j obsahuje **pozitivní** literál: x_i



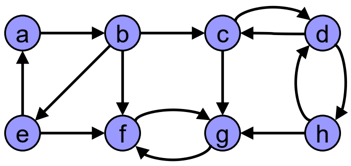
- pokud c_j obsahuje **negativní** literál: $\neg x_i$



Zdroj: https://cw.fel.cvut.cz/old/_media/courses/a4m33pal/2012pal03.pdf

// Zde dokument nekončí, následují vizualizace algoritmů.

Kosaraju-Sharir Algorithm



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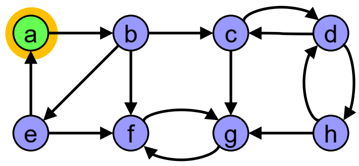

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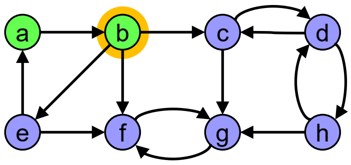

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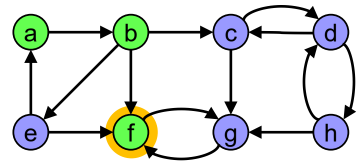

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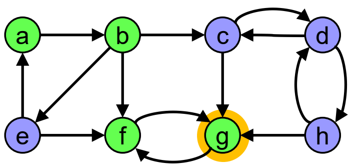

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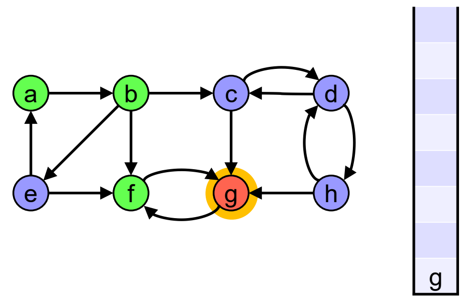

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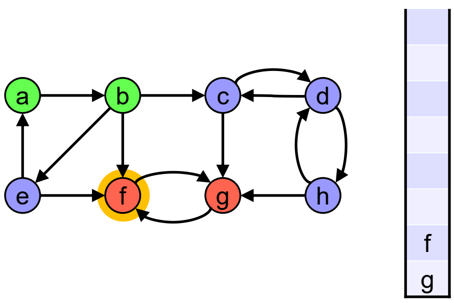

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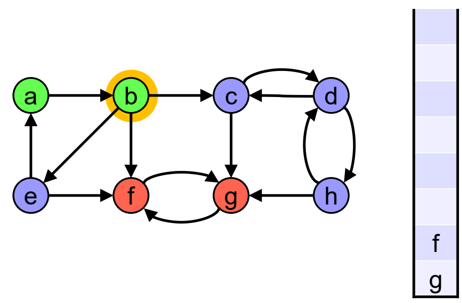

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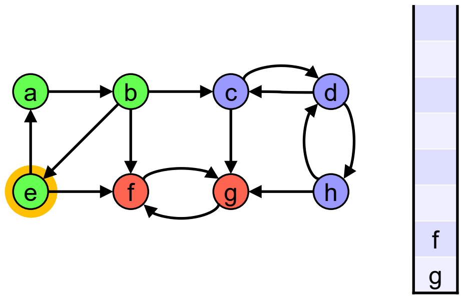

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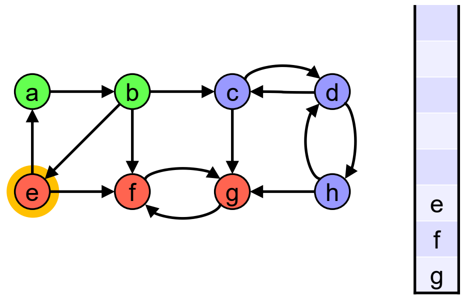

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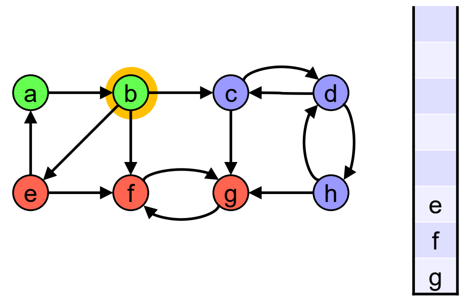

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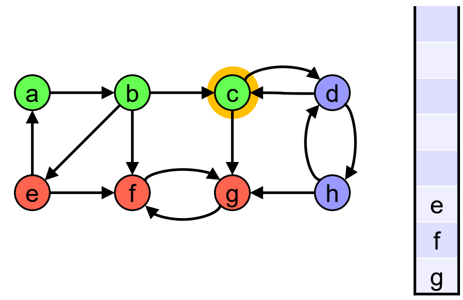

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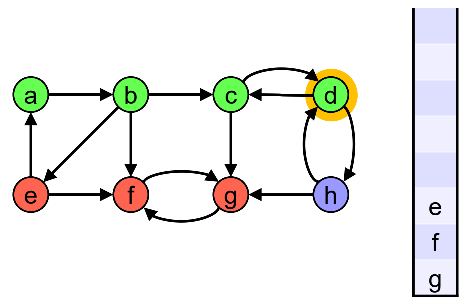

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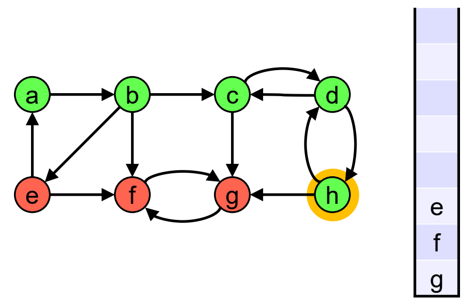

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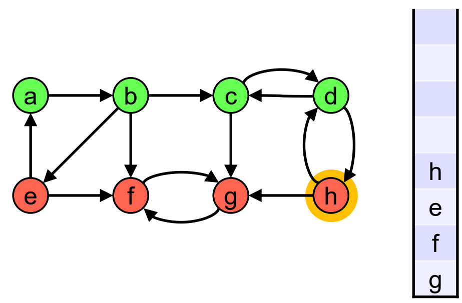

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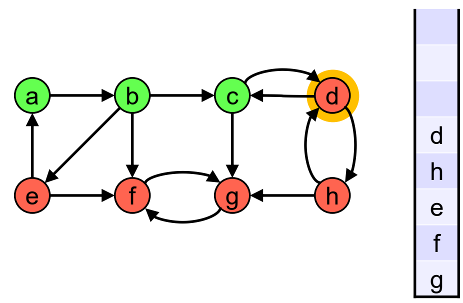

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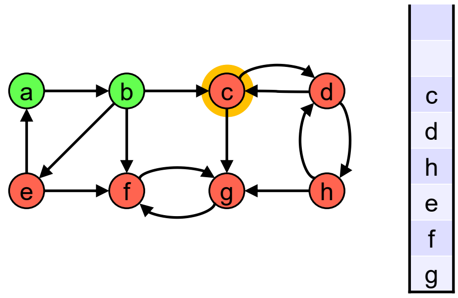

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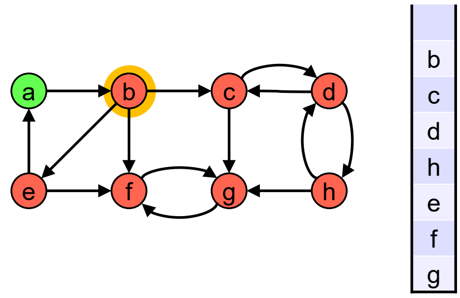

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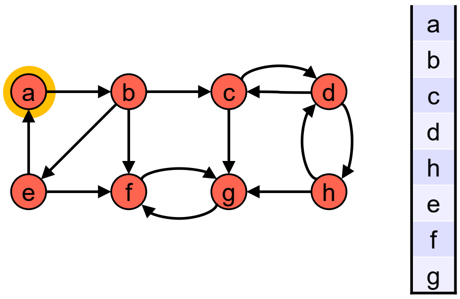

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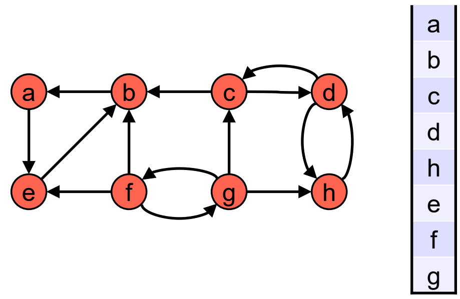

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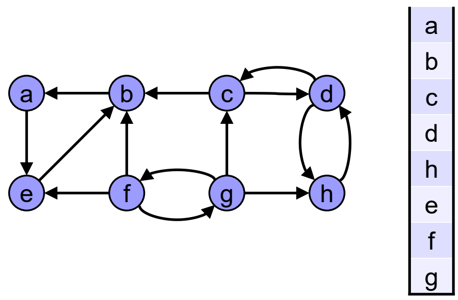

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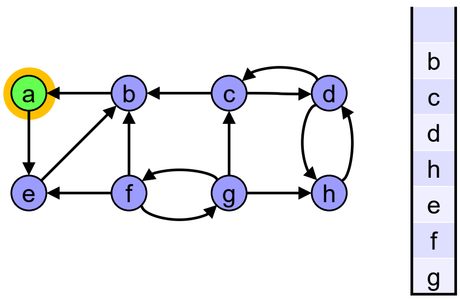

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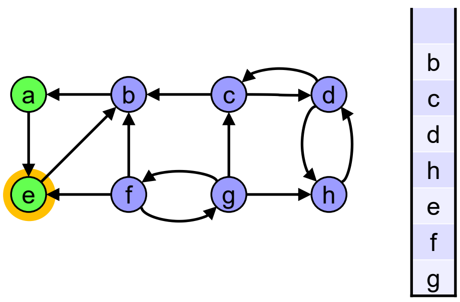

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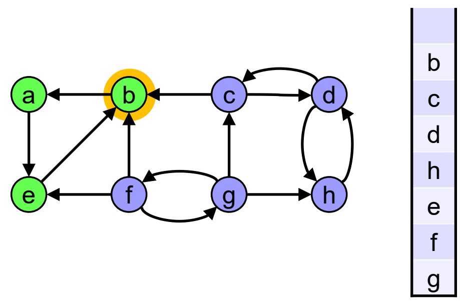

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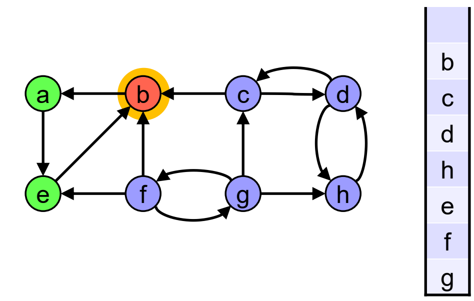

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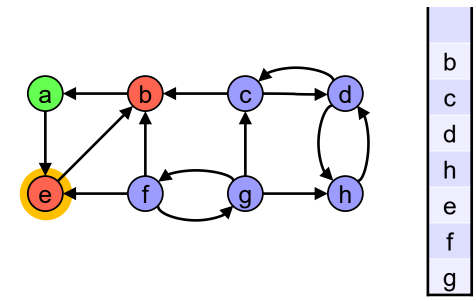

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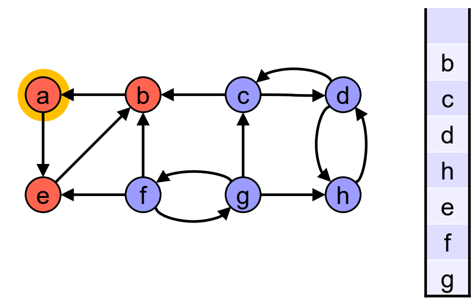

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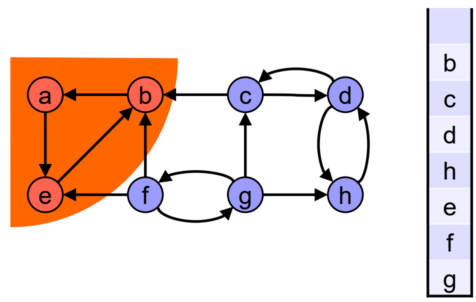

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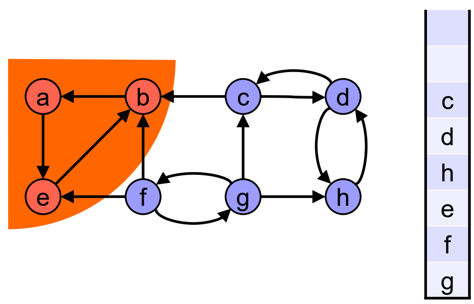

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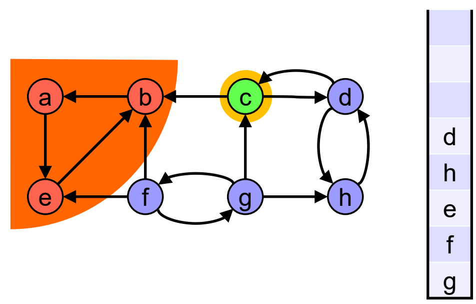

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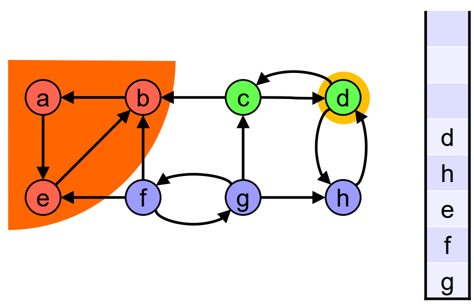

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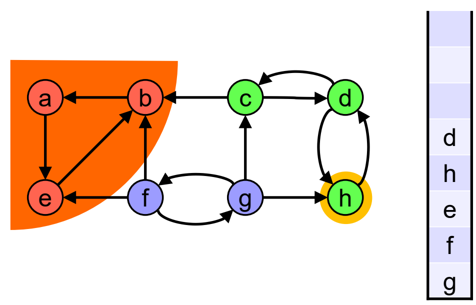

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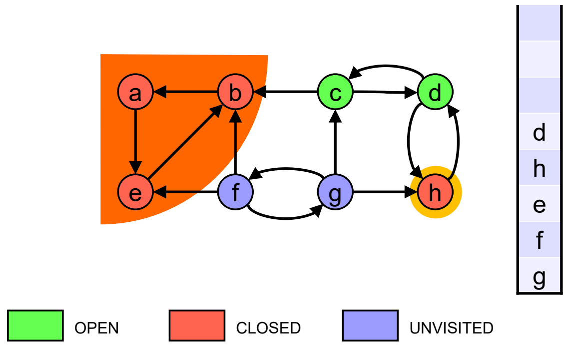

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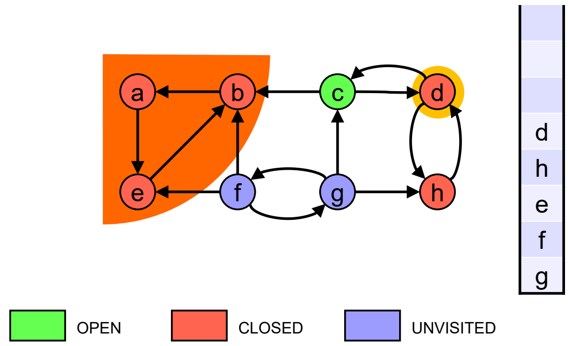

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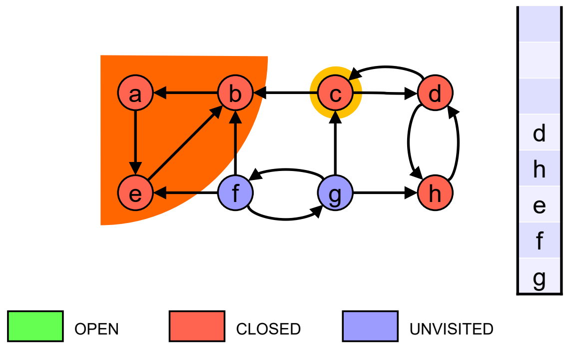

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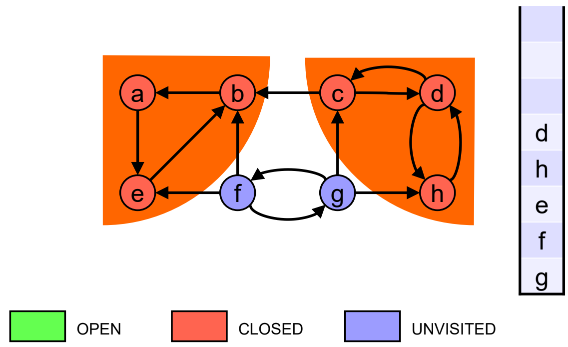

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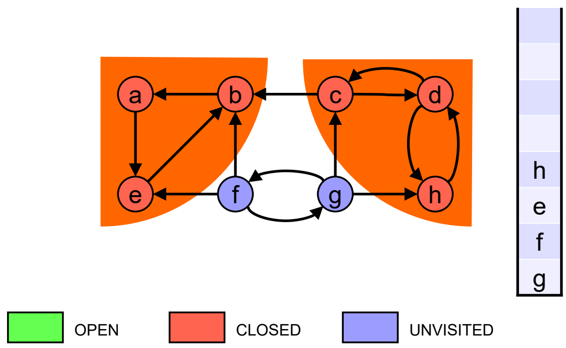

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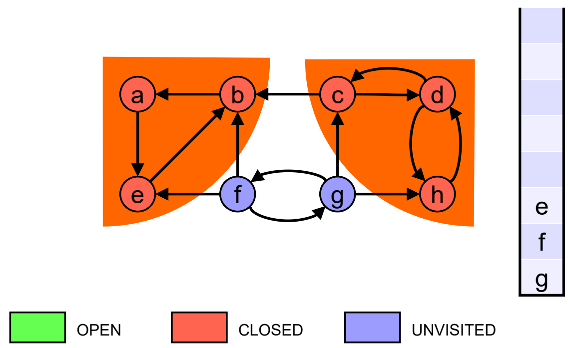

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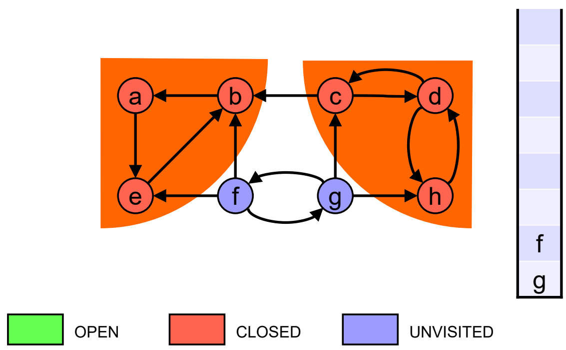

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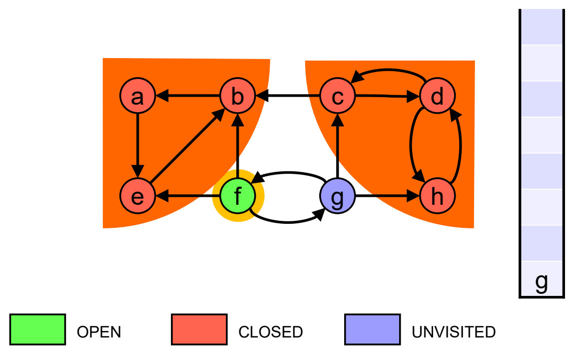

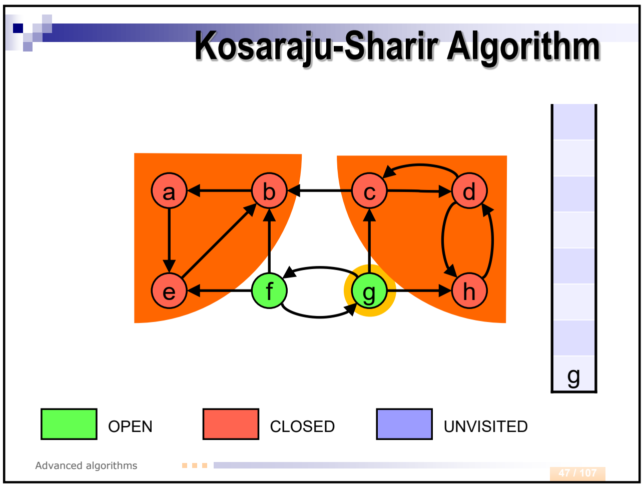
Kosaraju-Sharir Algorithm
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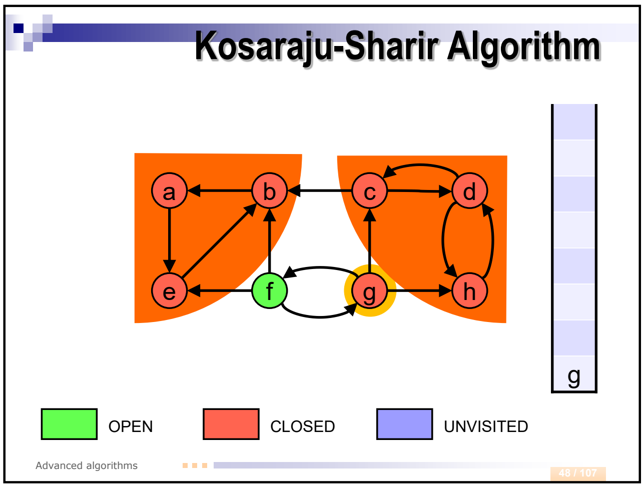
Kosaraju-Sharir Algorithm
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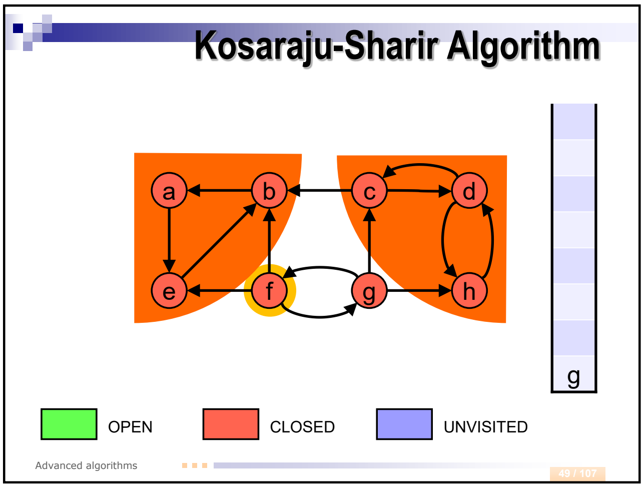
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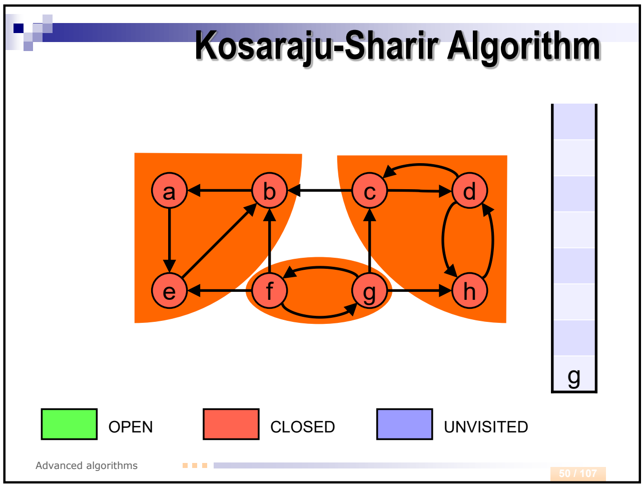
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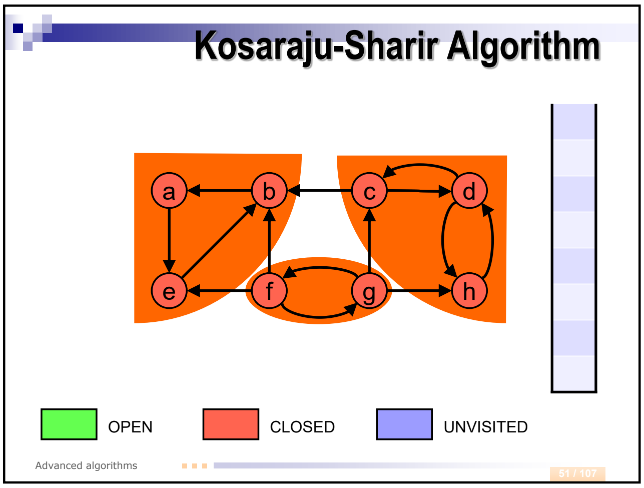
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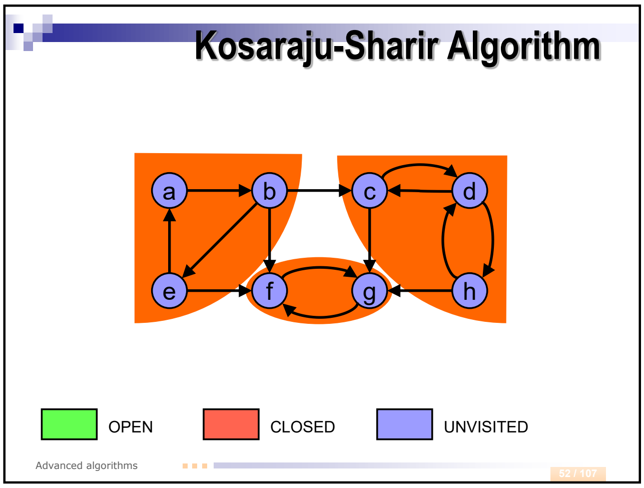
Kosaraju-Sharir Algorithm
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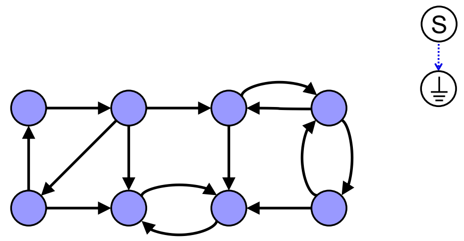
Tarjan's Algorithm



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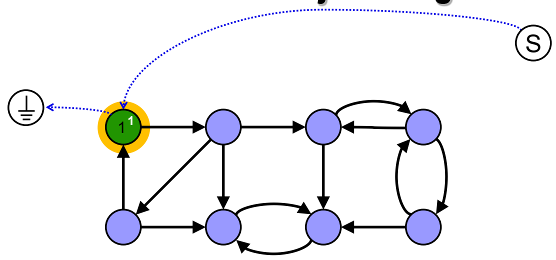

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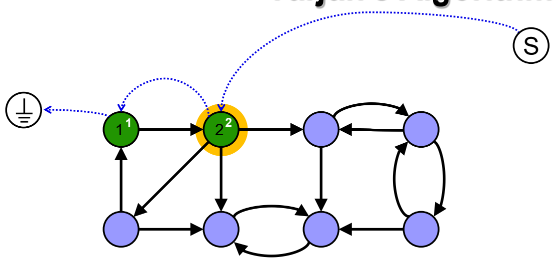

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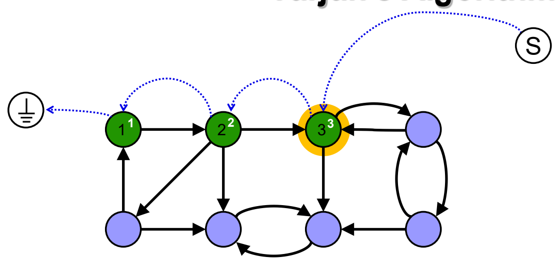

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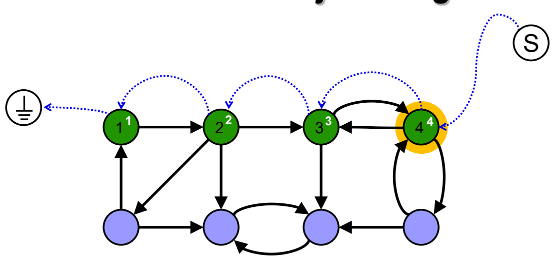

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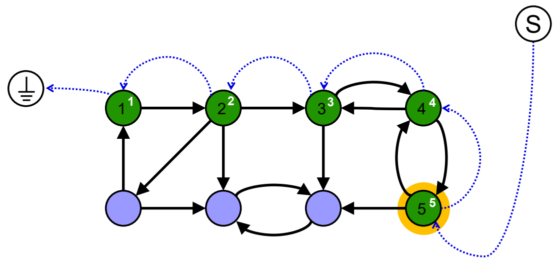

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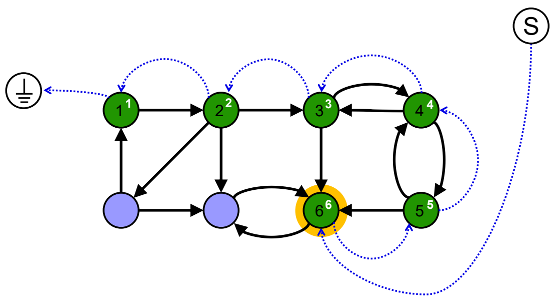

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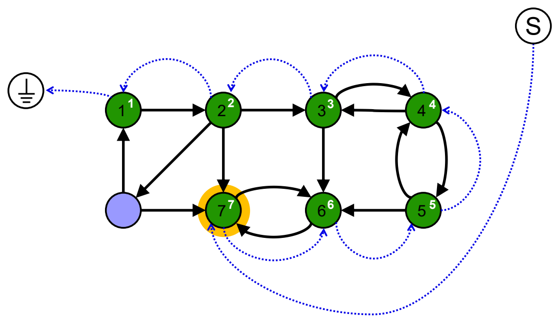

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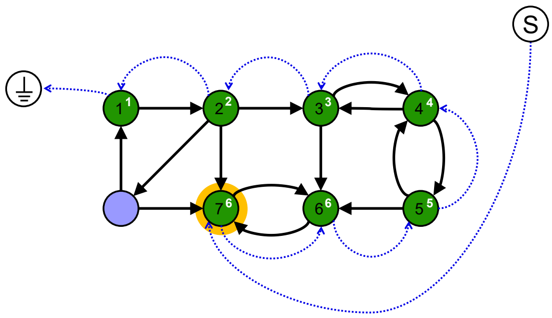

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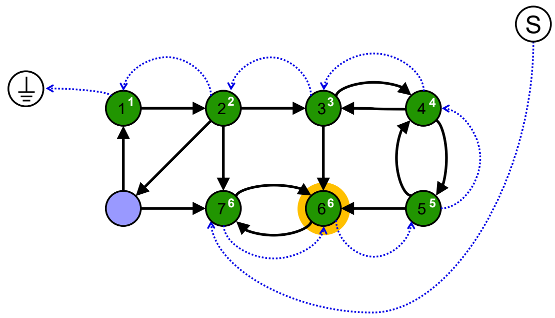

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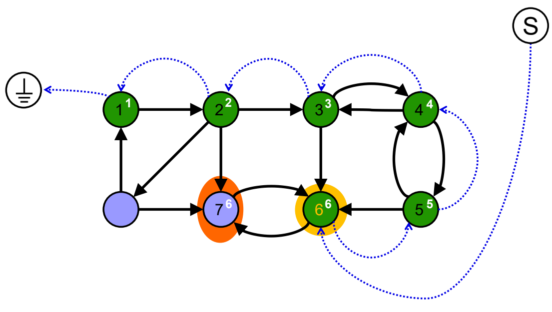

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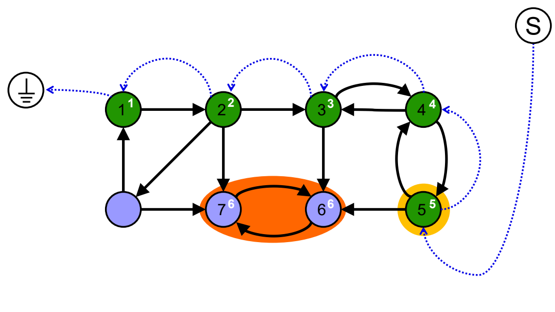

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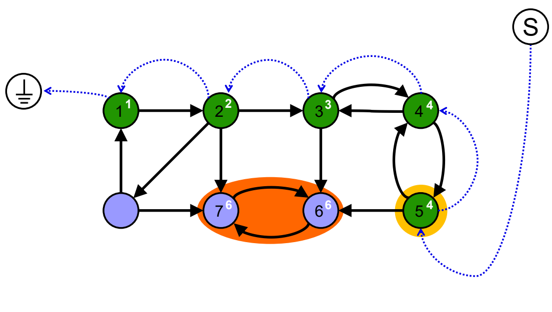

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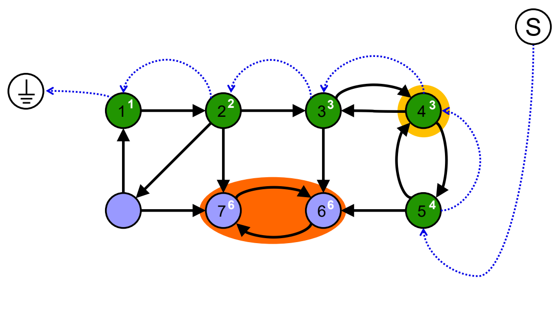

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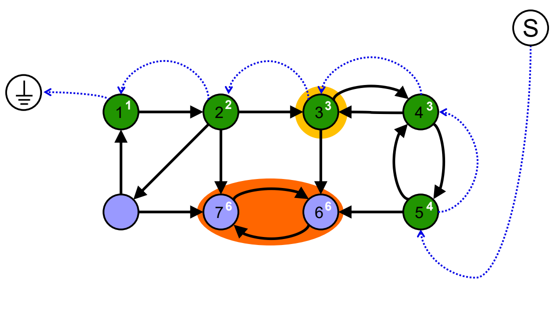

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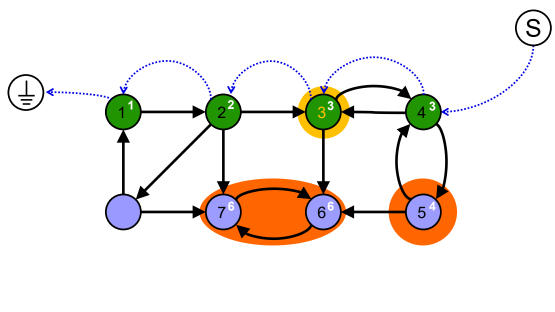

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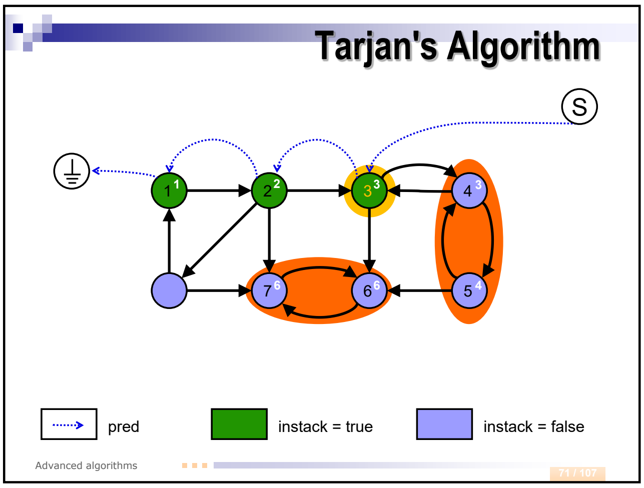
Tarjan's Algorithm
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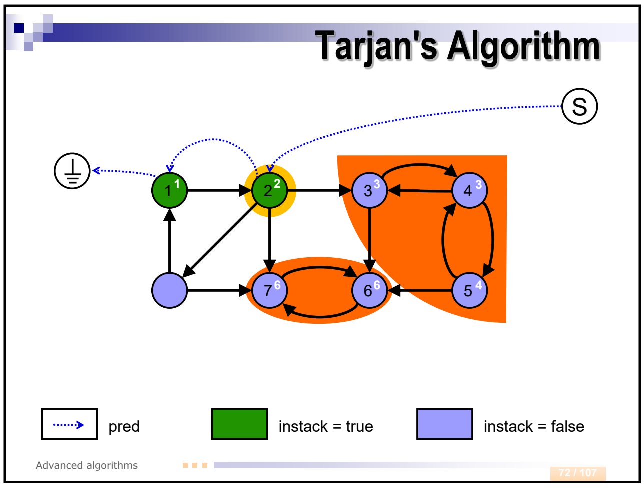
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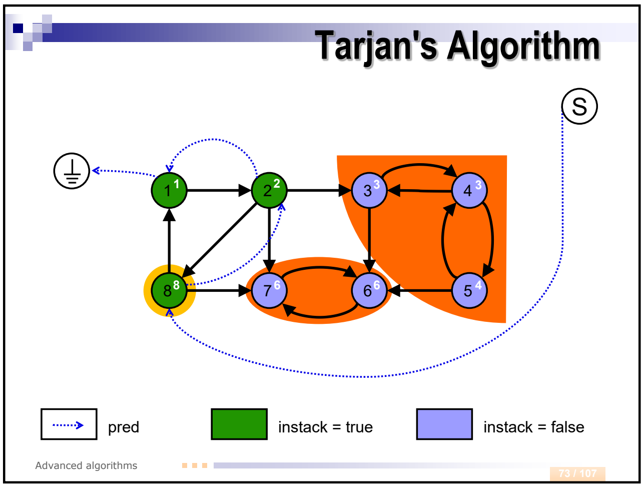
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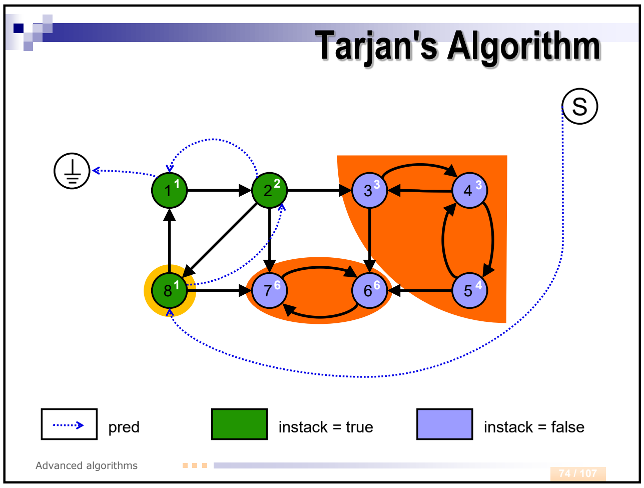
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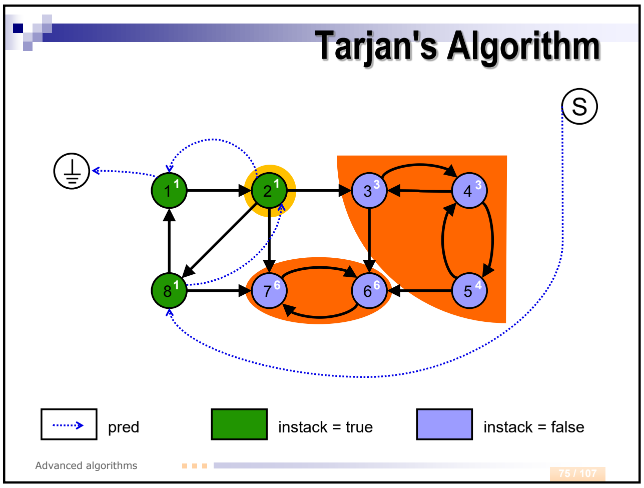
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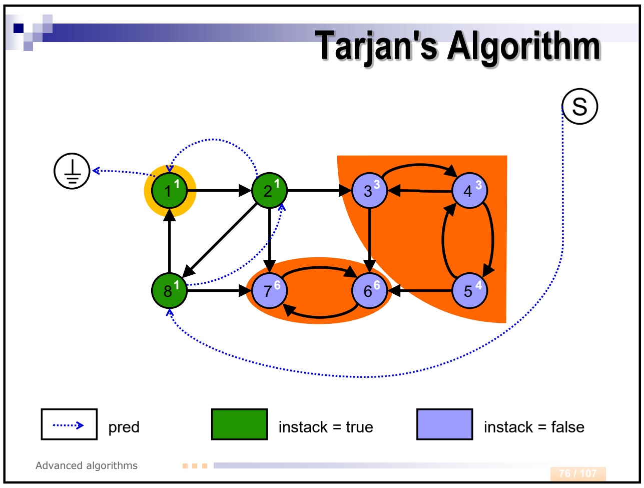
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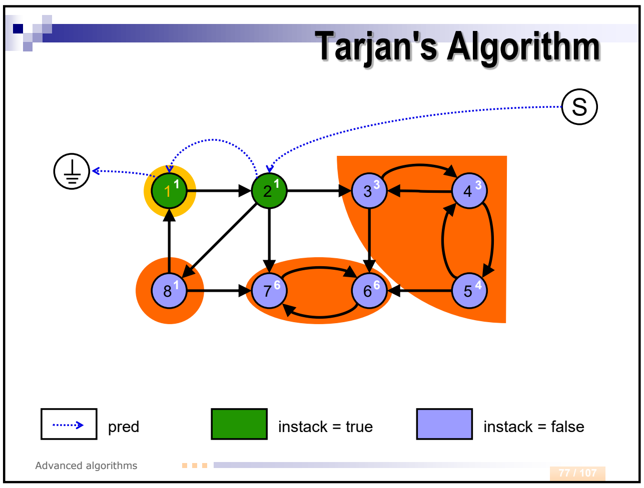
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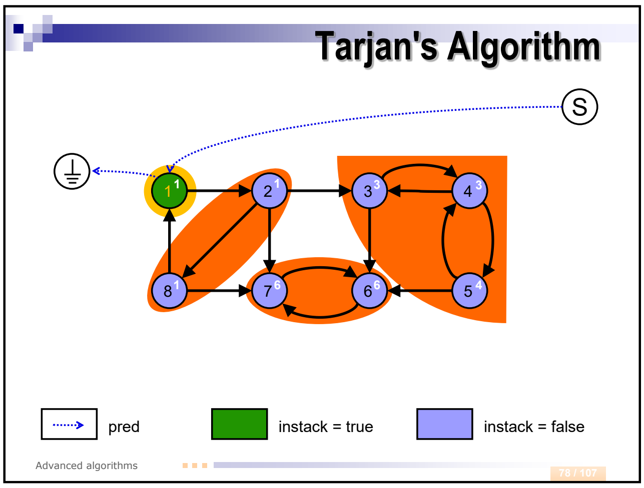
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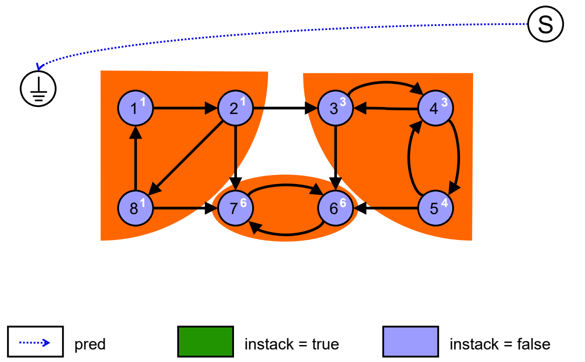
Tarjan's Algorithm



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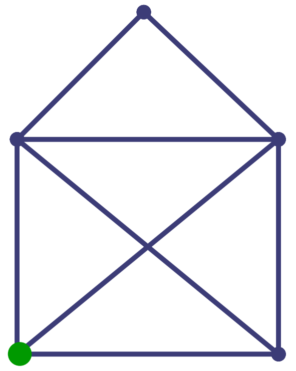

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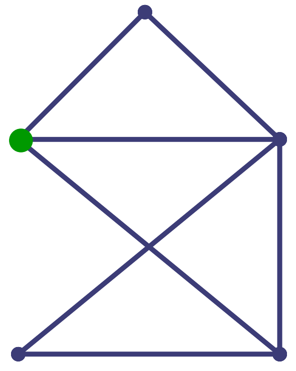

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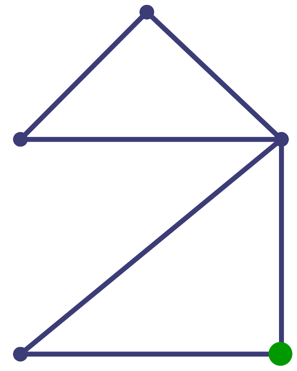

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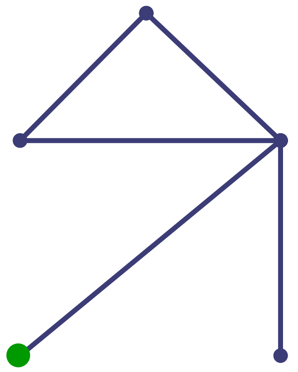

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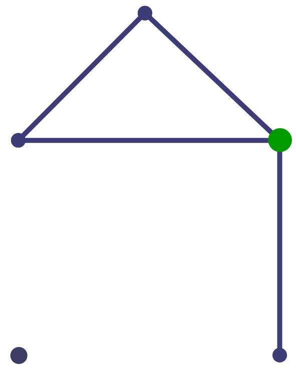

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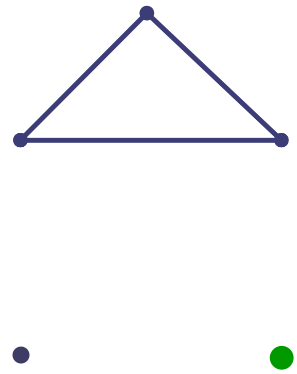

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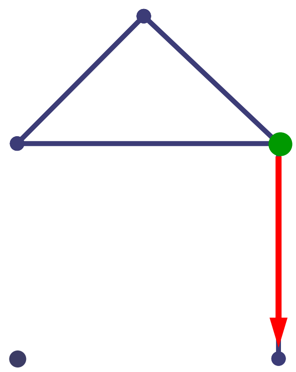

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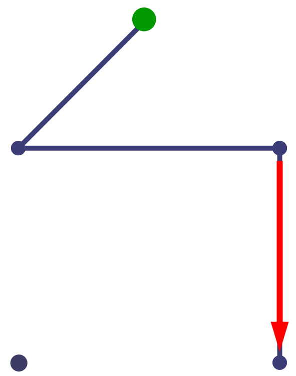

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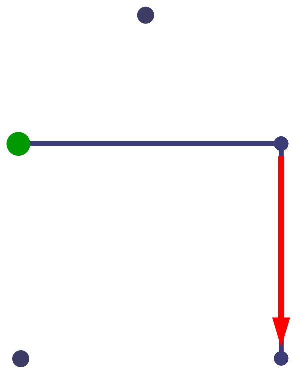

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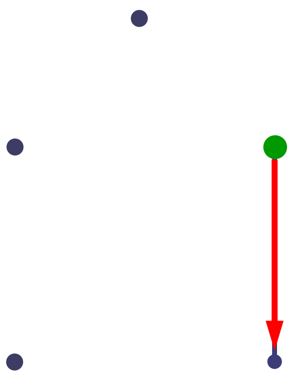

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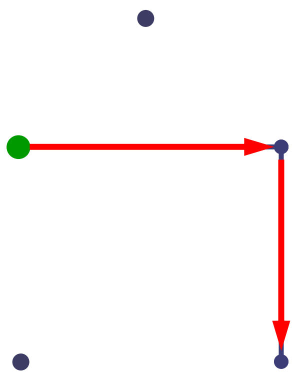

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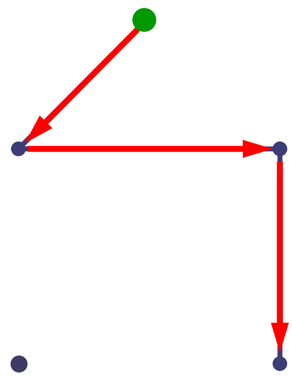

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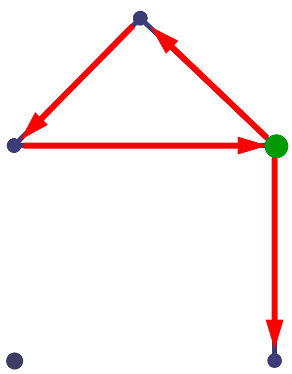

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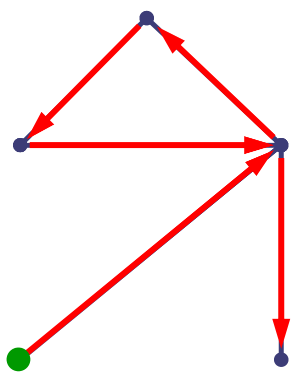

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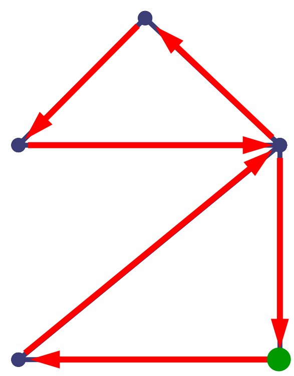

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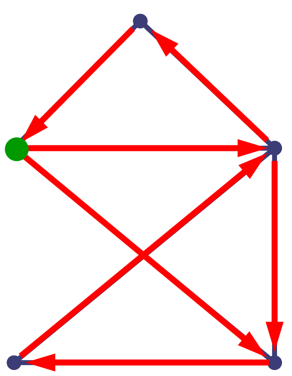

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