Set Intersection with Minimal Support
The SIMS-Problem

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1. Problem definition

Given a number $n \in \mathbb{N}$, find the minimal number $k \in \mathbb{N}$ such that there are n sets $A_1, ..., A_n$ containing numbers in $[k]$, i.e $A_i \subseteq \{1, ..., k\}$ satisfying:

$$|A_i \cap A_j| = |i - j| \text{ for all } 1 \leq i < j \leq n$$

For example, for $n = 4$, the answer would be $k = 5$, with which we could pick the 4 sets as:

$$\begin{aligned} A_1 &= \{1, 2, 3, 4\} \\ A_2 &= \{1, 5\} \\ A_3 &= \{1, 2\} \\ A_4 &= \{1, 3, 4, 5\} \end{aligned}$$

or a more visual alternative:

```
A1:  1  2  3  4
A2:  1           5
A3:  1  2
A4:  1     3  4  5
```

You can try to find sets which only use the numbers 1 to 4 but will hopefully be convinced that $k = 5$ is optimal.

2. Best known bounds

In the following table, we record our best known values for k.

n	**optimal value with combinatorial solver**	**optimal value with LP solver**
0	0	0
1	0	0
2	1	1
3	2	2
4	5	5
5	9	9
6	16	16
7	24	24
8		36
9		50
10		70
11		91
12		120
13		150
14		189
15		231
16		280
17		336
18		398
19		468
20		547
21		630
22		728
23		≤ 827
24		≤ 944
25		≤ 1064
26		≤ 1198
27		≤ 1341
28		≤ 1493
29		≤ 1661
30		≤ 1838
31		≤ 2027
32		≤ 2232
33		≤ 2442
34		≤ 2680
35		≤ 2918
36		≤ 3179

Our strategy in solving this problem combinatorically will be explained in Section 5. Our formulation of this problem as an (I-)LP will be explained in Section 6

3. Upper Bounds

Our current best upper bound comes from an explicit construction. This bound is not tight, there are more optimal solutions starting from $n = 6$, but we have not yet been able to derive a pattern out of those.

Our construction works as follows:

For each set-distance $i \in \{1, ..., n - 1\}$, and for each coset representative $a \in \{0, ..., i - 1\}$, if there are atleast 2 two set indices $b, c \in \{1, ..., n\}$ with $b \bmod i = c \bmod i = a$, i.e. they lie in the same coset, then define $\varphi(i)$ new unused numbers to add to all sets with indices in that coset and increase k by $\varphi(i)$, the number of newly added and used numbers.

For example, for $n = 6$, this constructions yields the following:

```
     i=1| i=2   |         i=3         |        i=4        |        i=5
A₁:   1 | 2     | 4  5                | 10  11            | 14  15  16  17
A₂:   1 |     3 |       6  7          |         12  13    |
A₃:   1 | 2     |             8  9    |                   |
A₄:   1 |     3 | 4  5                |                   |
A₅:   1 | 2     |       6  7          | 10  11            |
A₆:   1 |     3 |             8  9    |         12  13    | 14  15  16  17
```

From this we can also see that this is not optimal, since for $n = 6$, there is a solution with $k = 16$.

<u>EXPLICITCONSTRUCTION</u>(n):

$A_1, ..., A_n = \{\}$
$k = 0$
For $i \in \{1, ..., n - 1\}$
    For $a \in \{1, ..., \min(i, n - i)\}$ // cosets with atleast two set indices
        For $b \in \{1, ..., \varphi(i)\}$ // number of elements to add
            For $j \in \{0, 1, ..., n \div i\}$ // set indices to add to
                $A_{a+ji} = A_{a+ji} \cup \{k + b\}$
            End
        End
        $k += \varphi(i)$
    End
End
return $A_1, ..., A_n$

euler identity for $\sum_{d|n} \varphi(d) = n$

4. Lower Bounds

5. Combinatorial approach

6. Linear Programming approach