Report for exercise 4 from group E

Tasks addressed:	5
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The work on tasks was divided in the following way:

Çağatay Gültekin (03775999)	Task 1	100%
	Task 3	100%
Vatsal Sharma (03784922)	Task 2	100%
Zhitao Xu (03750803) **Project lead**	Task 4	66.7%
Gaurav Vaibhav (0366416)	Task 5	80%
Minxuan He (03764584)	Task 4	33.3%
	Task 5	20%
	Task	bonus

Report on task 1, Vector fields, orbits, and visualization

1.1 Matrices, Parameters, and Phase Portraits

In this task, we investigated the effect of varying the parameters on the 2D linear system's dynamic behavior and worked on phase portraits of dynamical systems.

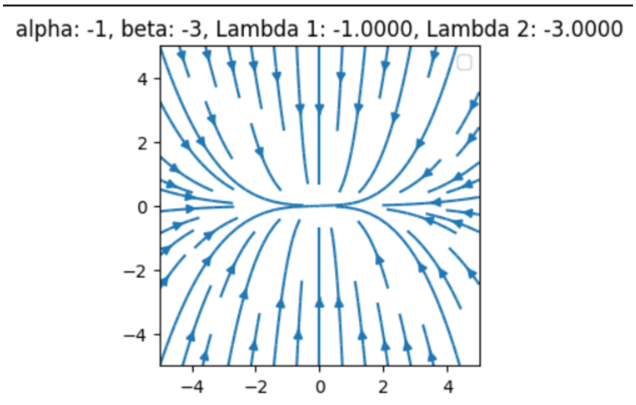
1.1.1 Nodes and Saddle

While drawing the "Stable Node", "Unstable Node" and "Unstable Saddle", we used this matrix with two parameters named "alpha" and "beta" 1:

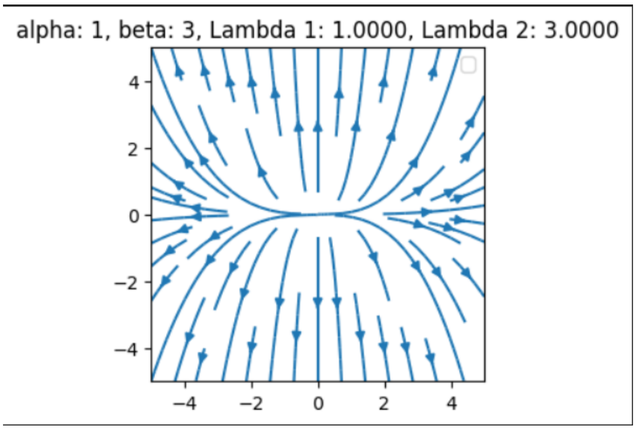
$$\begin{bmatrix} alpha & 0 \\ 0 & beta \end{bmatrix}$$

Figure 1: Node and Saddle Matrix

We needed all eigenvalues in real numbers in nodes and saddle. In a stable node, two negative eigenvalues; in an unstable node, two positive eigenvalues; in an unstable saddle, one positive and one negative eigenvalue. That is why we used a diagonal matrix(values other than 0 in the diagonal), our alpha value became our first eigenvalue, and beta values became our second eigenvalues. You can see the used parameters and phase portrait drawings in these images 2 and 3:



(a) Phase Portrait of the Stable Node



(b) Phase Portrait of the Unstable Node

Figure 2: Phase Portrait of the Stable and Unstable Node: As you can see from these images 2, in stable nodes, they are attracting into the steady state (which is a fixed point where $\frac{dx}{dt} = 0$) with straight lines. In unstable nodes, they are repelling from the steady state with straight lines.

1.1.2 Focuses

Then, we used different matrix for "Stable Focus" and "Unstable Focus" because they have imaginary values in their eigenvalues and we are only using real-valued parameter values. For the "Stable Focus" and "Unstable Focus, our matrix is 4:

You can see the used parameters and phase portrait drawings in these images 5:

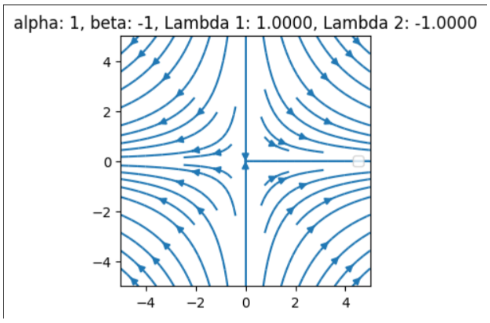


Figure 3: Phase Portrait of the Unstable Saddle: In unstable saddle 3, the lines are all unstable(repelling from the steady state) except in two stable(attracting through the steady state) directions with all linear lines.

$$\begin{bmatrix} alpha & beta \\ -1/4 & 0 \end{bmatrix}$$

Figure 4: Focus Matrix

1.2 Topologically equivalence

Even though we used the same matrix with different parameters in "Stable Node", "Unstable Node" and "Unstable Saddle" and the same matrices with different parameters in "Stable Focus" and "Unstable Focus", they are not topologically equivalent because of topological equivalency denotes the existence of a continuous one-to-one mapping that converts one system's orbit and behavior into another system's orbit and behavior. However, these five systems differ greatly in terms of stability and whether the orbitals are repulsive or attractive, and thus cannot be interconverted by continuous one-to-one mapping.

Report on task 2, Common bifurcations in nonlinear systems

This task is focussed on finding bifurcations in dynamical systems. In definitive terms, bifurcation theory deals with the sudden split observed in the behavior of systems, when the controlling parameters are slowly and smoothly changed. Here we focus on examining the bifurcations of two dynamical systems given by:-

$$\dot{x} = \alpha - x^2 \textbf{ (System - 1)}$$

and

$$\dot{x} = \alpha - 2x^2 - 3 \textbf{ (System - 2)}$$

2.1 Analysis of Steady States

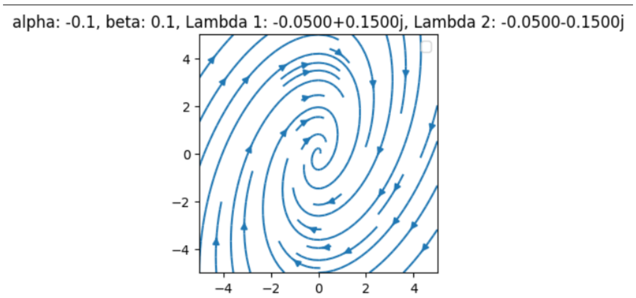
Here x represents the state variable of the system, and α is the parameter, which can be modified .
We perform an analysis for three different types of α. For $\alpha > 0$:

SYSTEM - 1
We perform an analysis for three different types of α.

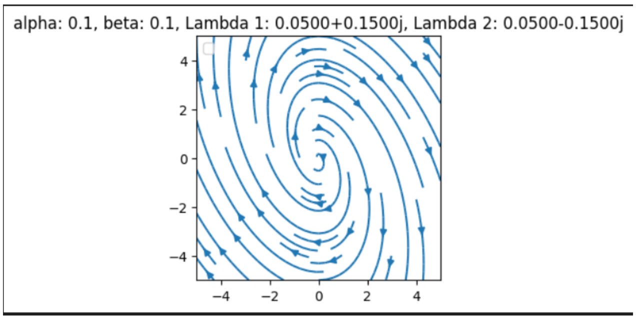
For $\alpha > 0$:
Here two steady states for x = $\pm\sqrt{\alpha}$ are achieved by equating $\dot{x} = 0$. Out of these, x = $\sqrt{\alpha}$ represents the stable equilibrium state, whereas x = -$\sqrt{\alpha}$ represents the unstable equilibrium state.

For $\alpha = 0$:
A half-stable point of equilibrium (saddle point) is achieved at x = 0.

For $\alpha < 0$:



(a) Phase Portrait of the Stable Focus



(b) Phase Portrait of the Unstable Focus

Figure 5: Phase Portrait of the Stable and Unstable Focus: As you can see from these images 5, in stable focuses, lines are attracted into the steady state with oscillatory lines. In unstable focuses, lines are repelling from the steady state with oscillatory lines.

No real states of equilibrium are achieved.

Note that the stability test can be done by observing the change of $\dot{x}$ at the specified point or by calculating the double derivative at that specific point. A positive double derivative indicates the presence of unstable equilibrium whereas a negative double derivative represents stable equilibrium at the point.

SYSTEM - 2
Here, by an intuitive examination, one can clearly predict a split in behavior at $\alpha = 3$. Hence here we analyse the situation for different values of alpha.

For $\alpha > 3$:
Here two steady states for x = $\pm\sqrt{(}\alpha$ - 3)/2 are achieved by equating $\dot{x} = 0$. Out of these, x = $\sqrt{(}\alpha$ - 3)/2 represents the stable equilibrium state, whereas x = -$\sqrt{(}\alpha$ - 3)/2 represents the unstable equilibrium state.

For $\alpha = 3$:
A half-stable point of equilibrium (saddle point) is achieved at x = 0.

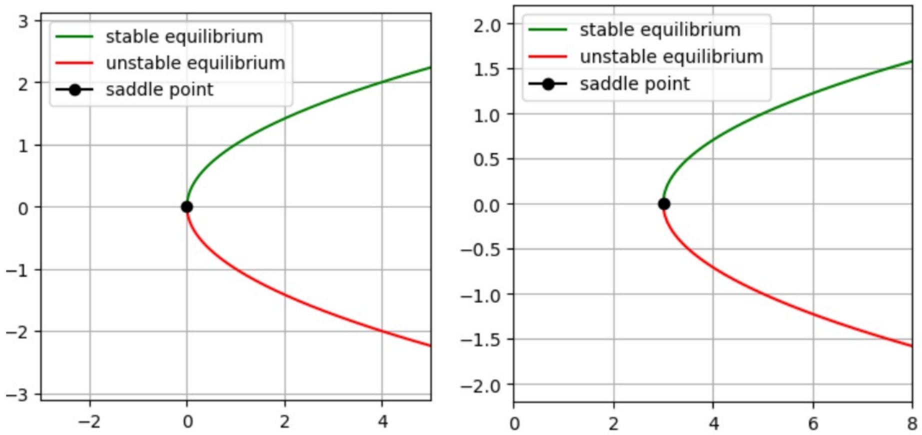
For $\alpha < 3$:
No real states of equilibrium are achieved.
See fig 6 for the bifurcation diagrams.

The type of bifurcation is **Saddle-Node Bifurcation**, as in both the systems we can clearly see how two fixed points collide and later annihilate each other, thus reducing the number of system's steady states to zero.

2.2 Topological Equivalence

Two systems of differential equations (as our systems) are considered to be **topologically equivalent** if there exists a homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$, such that $y = h(x)$, which maps orbits of the first system into orbits of the second system preserving the direction of time. And a **homeomorphism** is defined as a bijective and continuous function between topological spaces that has a continuous inverse function.
At $\boldsymbol{\alpha = 1}$, from our previous analysis, we can say that System-1 has 2 steady states, and System-2 has no real steady state, hence they **cannot be topologically equivalent**. The intuition behind this reasoning arises from the fact, that if one system has zero or one steady state and another has two steady states, they should have **fundamentally different qualitative behaviors** and therefore cannot be equated or mapped.
At $\boldsymbol{\alpha = -1}$, both the systems have 0 steady states, hence we could possibly have a relation that could map the respective orbits. By observing the bifurcation systems, one can safely conclude that the two systems are very similar. Both the differential equations share a similar kind of normal form too of the form

$$\dot{x} = A + Bx^2$$



(a) System - 1 (b) System - 2

Figure 6: Bifurcation Diagrams: State Variable x versus α

with A and B as constants and the highest order derivative on one side of the equation. In addition to it both the systems share the same nature, just that the fixed point of second system is shifted by 3 units and the parabolic curve squeezed by a factor of 2, which again proves the similarity structurally. Now on integrating these differential equations we get **tangential trajectories** for both the systems. Considering the initial state to be x = 0 at t = 0, the final trajectories come out as -

$$x = \tan(-t) \text{ (System - 1)}$$

$$x = \frac{\tan(-2\sqrt{2}t)}{\sqrt{2}} \text{ (System - 2)}$$

Hence in order to map the trajectories we need to rescale the x (state variable) of first system to $\sqrt{2}$x and we also need to scale the time variable from t to 2 * $\sqrt{2}$t, assumed that two independent dynamical systems can work on two separate time scales.

So at alpha = -1, both systems can be said to be equivalent in nature.

Report on task 3, Bifurcations in higher dimensions

3.1 Three Phase of Andronov-Hopf Bifurcation

In this task, we explored Andronov-Hopf bifurcation. It will be a bifurcation because when our parameter alpha changes, it won't be topologically equivalent to the old version of itself. Andronov-Hopf Bifurcation has two special coordinates which are x1 and x2. The only parameter it has is alpha.

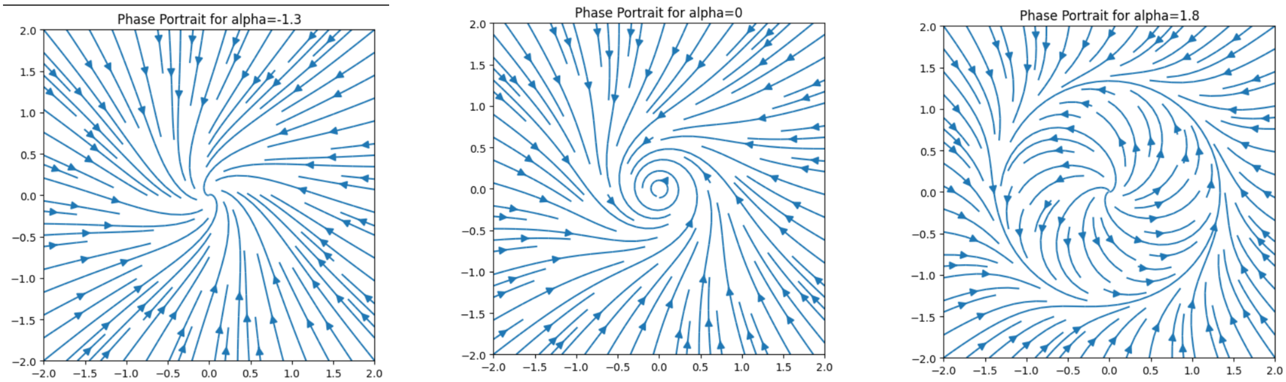
We used three different alpha values which are the same as we used in the class. These are the phase portraits 7:

3.2 Two orbits of the System

In this part, we used the same system in the 3.1. We compute two orbits of the system, one of them starting at (2,0), and the other one starting at (0.5,0). They have the same alpha value so the general portrait would be the same and we can compare these orbits. These are the orbits in the figure 8:

3.3.1 The Cusp Bifurcation

In this task, we worked with the Cusp Bifurcation. We had two parameters which are alpha1 and alpha2. In addition, we had a steady state which is x. In this task, it asked to have a 3D plot of the Cusp Bifurcation



(a) (b) (c)

Figure 7: Phase Portrait of the system with different alpha values: As you can see from the left image, it has an attractive steady state when it has a negative value of alpha and there is no limit cycle. In the middle image, by alpha becoming 0, a limit cycle exists out of a steady state. As you can see from the right image, when the alpha value becomes positive our steady state becomes a repelling steady state, and all the lines go into the limit cycle. Consequently, stable oscillatory behavior develops from a stable steady state.

when $\dot{x}$ is equal to 0. So, to do this, first, we generate a uniform distribution of alpha 2 and x. Then, we used $alpha1 = -alpha2 * x + x^3$ to calculate alpha1 samples because $\dot{x}$ is equal to 0. Then, we drawed this 3D plot of the Cusp Bifurcation 9:

3.3.2 Why is it called cusp bifurcation?

A particular kind of mathematical occurrence that arises in the study of bifurcations and dynamical systems theory is called a cusp bifurcation. Bifurcation theory addresses how a system's qualitative behavior varies with a change in a parameter. In cusp bifurcation, the term "cusp" describes a particular shape that appears in a system's bifurcation diagram. A cusp bifurcation occurs when a parameter crosses a critical value, causing an abrupt change or transition in the behavior of a dynamical system. The equilibrium or steady-state solutions of the system undergo a qualitative shift at this critical value.

Report on task 4, Chaotic dynamics

In this task, we will study two dynamical systems, i.e. logistic map and Lorenz attractor, both exhibiting chaotic behavior. Although both models are based on deterministic equations, their behavior is unpredictable. At certain parameter values, both systems can generate complex and irregular dynamics, which are highly sensitive to initial conditions.

4.1 Logistic Map

The logistic map is a discrete map defined by the following equation:

$$x_{n+1} = rx_n(1 - x_n), \quad n \in \mathbb{N}, \tag{1}$$

with the parameter $r \in (0, 4]$ and $x \in [0, 1]$. To analyze the behavior of this system when the parameter r takes different values, we first do some mathematical analysis, and then perform some simulation with Python under different settings of x and r, and finally give the bifurcation diagram.

4.1.1 Mathematical Analysis of Logistic Map

First we try to find out the fixed points of this system. As we know, at fixed points, the state value of the system will remain the same, i.e. $x_{n+1} = x_n$. We can directly apply this to the system equation as follows:

$$x = rx(1 - x). \tag{2}$$

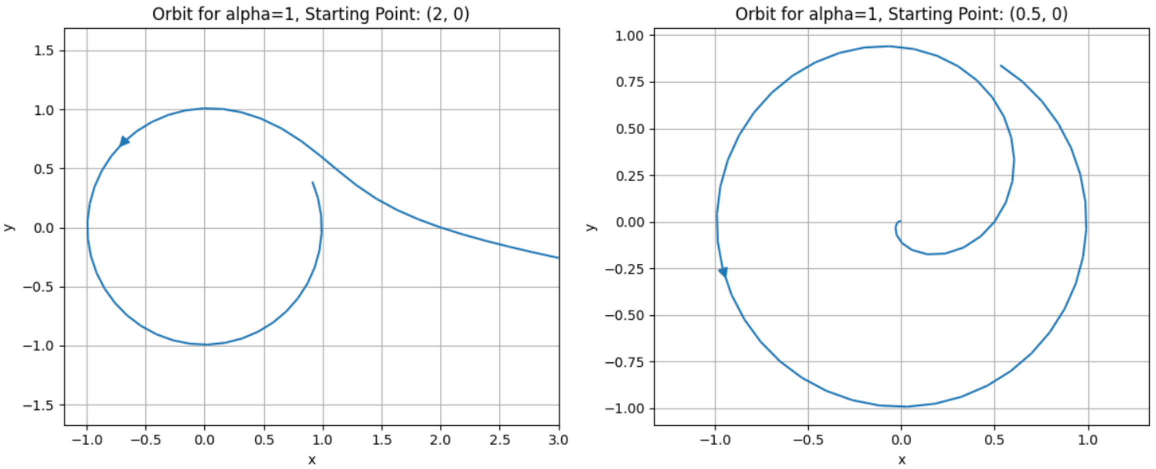


Figure 8: Orbits of the system: As you can see from both images, they are both attracting into the limit cycle that is caused by alpha having a positive value. After alpha has a positive value, the steady state becomes a repelling and limit cycle attracting every line everywhere. Then, they are stuck in the limit cycle. The main difference is an orbit with a start point of 0.5 starts the journey inside the limit cycle so it repels from a steady state and attracts to the limit cycle. However, orbit with start point 2 starts the journey outside of the limit cycle and starts attracting by limit cycle so it is going closer to the steady state than where it started.

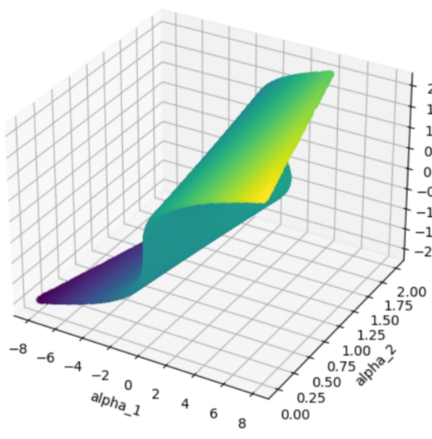


Figure 9: The bifurcation surface of the cusp bifurcation in a 3D plot.

We can find that, this equation can always be satisfied when $x = 0$, which means $x^* = 0$ is always a fixed point (**but not necessarily stable**), so we can only consider $x \neq 0$. Then we can find another fixed point $x^* = 1 - \frac{1}{r}$.

If x^* is a fixed point, then $x^* = f(x^*)$. We consider a point x_n close to x^*, and we want to know the position of x_{n+1} relative to x^*. This can be approximated using the Taylor expansion of $f(x)$ around x^*:

$$f(x_n) \approx f(x^*) + f'(x^*)(x_n - x^*). \tag{3}$$

Since $x^* = f(x^*)$, we have

$$x_{n+1} \approx x^* + f'(x^*)(x_n - x^*). \tag{4}$$

Now let's analyze the impact of $f'(x^*)$:

1. If $|f'(x^*)| < 1$, this implies that for any point x_n close to x^*, x_{n+1} will be closer to x^*. As iterations proceed, the point x_n will converge to x^*, indicating that the fixed point is stable.

2. If $|f'(x^*)| > 1$, this implies that x_{n+1} will move away from x^*, meaning that x_n will not converge to x^*, indicating that the fixed point is unstable.

3. If $|f'(x^*)| = 1$, this is a borderline case, and we need higher-order derivatives to determine the fixed point is stable or not.

For logistic map, we have 2 fixed points: $x_1^* = 0$ and $x_2^* = 1 - \frac{1}{r}$. The derivative of the map is calculated as follows:

$$f'(x) = \frac{d}{dx}(rx(1-x)) = r - 2rx. \tag{5}$$

Now let's analyze the stability of the two fixed points:

1. For $x_1^* = 0$, $f'(x_1^*) = r - 2r \cdot 0 = r$. So when $r < 1$, then $|f'(x_1^*)| < 1$, and the fixed point x_1^* is stable. If $r > 1$, then $|f'(x_1^*)| > 1$, and the fixed point x_1^* is unstable.

2. For $x_2^* = 1 - \frac{1}{r}$, $f'(x_2^*) = r - 2r(1 - \frac{1}{r}) = 2 - r$. When $1 < r < 3$, we have $|f'(x_2^*)| < 1$, indicating x_2^* is stable. But when $r > 3$, $|f'(x_2^*)| > 1$, resulting in the loss of stability at x_2^*. When $0 < r < 1$, x_2^* is negative and out of the range $[0, 1]$, so we don't need to consider this scenario.

4.1.2 Varying `r` from 0 to 2

In order to get an intuition of how this dynamical system behaves when `r` ranges from 0 to 2, we simulate the system by iterating the equation for multiple times with some initial state $x \in [0, 1]$ to see if it will converge to some steady state. We define a function `simulate` in `logistic.py`, which achieves 3 arguments, and they are a float number for `r`, an array for initial `x` values, and an integer for the number of iterations. We iteratively apply the logistic map, and preserve the result of each iteration for visualization. To make the initial state relatively comprehensive, we let `x` take 10 different values, i.e. $0, 0.1, \ldots, 0.9$. We make the plot to illustrate how these values will change during the iteration, and here we iterate 100 times. We show the plots with $r = 0.5, 1.0, 1.5, 2.0$ in Fig. 10.

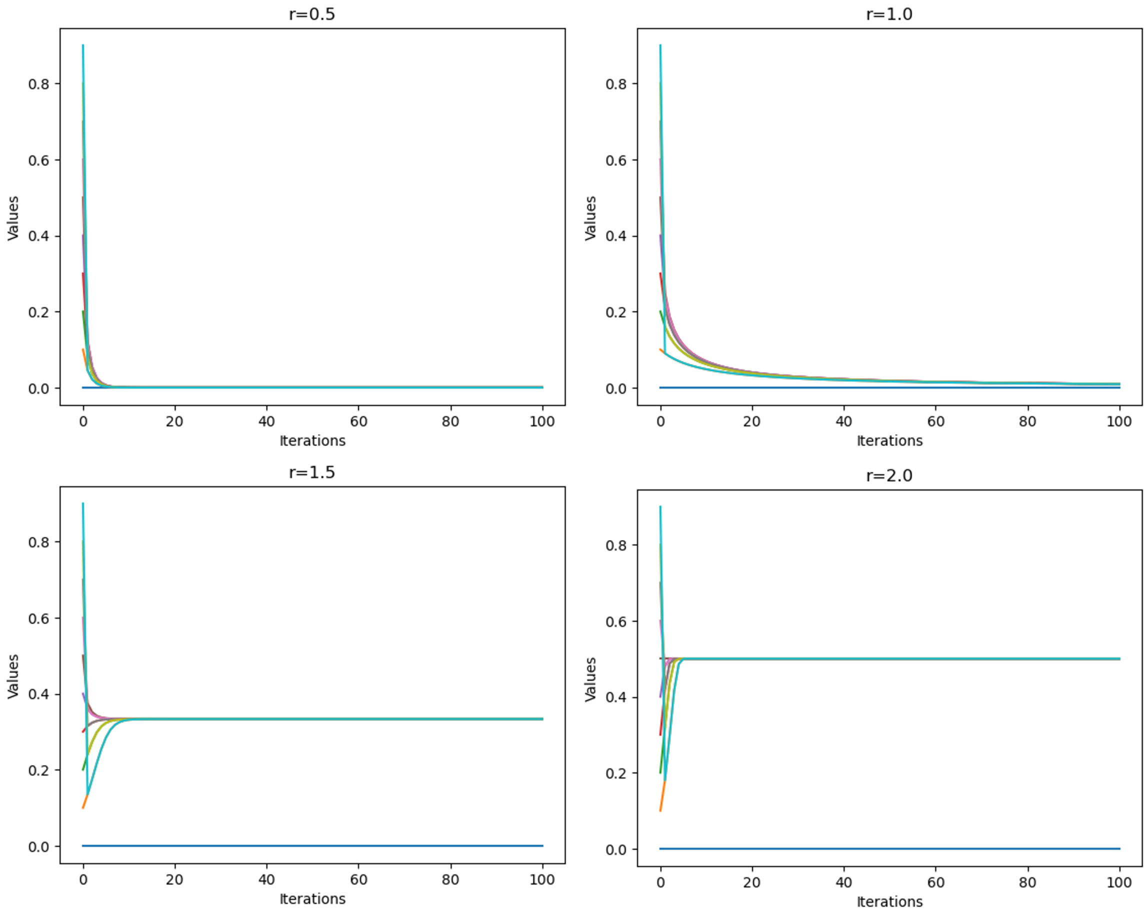


Figure 10: The change of values during simulation with $r = 0.5, 1.0, 1.5, 2.0$.

We can find out that, all the results coincide with the analysis above. When $0 < r < 1$, all the values converge to 0, indicating that 0 is the only stable point of the system. When $r = 1$, the speed of the convergence is much slower, and as we mentioned above, $x = 0$ begins to lose the stability. When $r > 1$, we can see that, the system has a new stable point. When $r = 1.5$, x converges to $\frac{1}{3}$, and when $r = 2.0$, x converges to 0.5. Both results reflect the analysis above that new stable point locates at $x_2^* = 1 - \frac{1}{r}$. As a result, varying `r` from 0 to 2 will lead to a bifurcation at $x = 1$, which is called fold bifurcation or saddle-node bifurcation. Before this bifurcation, the system has the only stable point $x = 0$, while after this bifurcation, the system has a new stable fixed point, and the original one is unstable.

4.1.3 Varying `r` from 2 to 4

According to the analysis above, when $2 < r < 3$, the system will also converge to $1 - \frac{1}{r}$. However, when $3 < r < 4$, the fixed point becomes unstable, and it's hard to mathematically analyze how the system will behave. So we simulate the system with Python to see how will it behave. Similarly, we use the `simulate` function, and set `r` to $2.5, 3.0, 3.5, 4.0$, respectively. Also we simulate for 100 iterations. Furthermore, we define a function `simulate_animation` to create an animation to dynamically illustrate how the behavior of the system changes while varying the value of `r`. We save the animation as a gif file in the repository. But here we only illustrate them statically and the results are shown in Fig. 11.

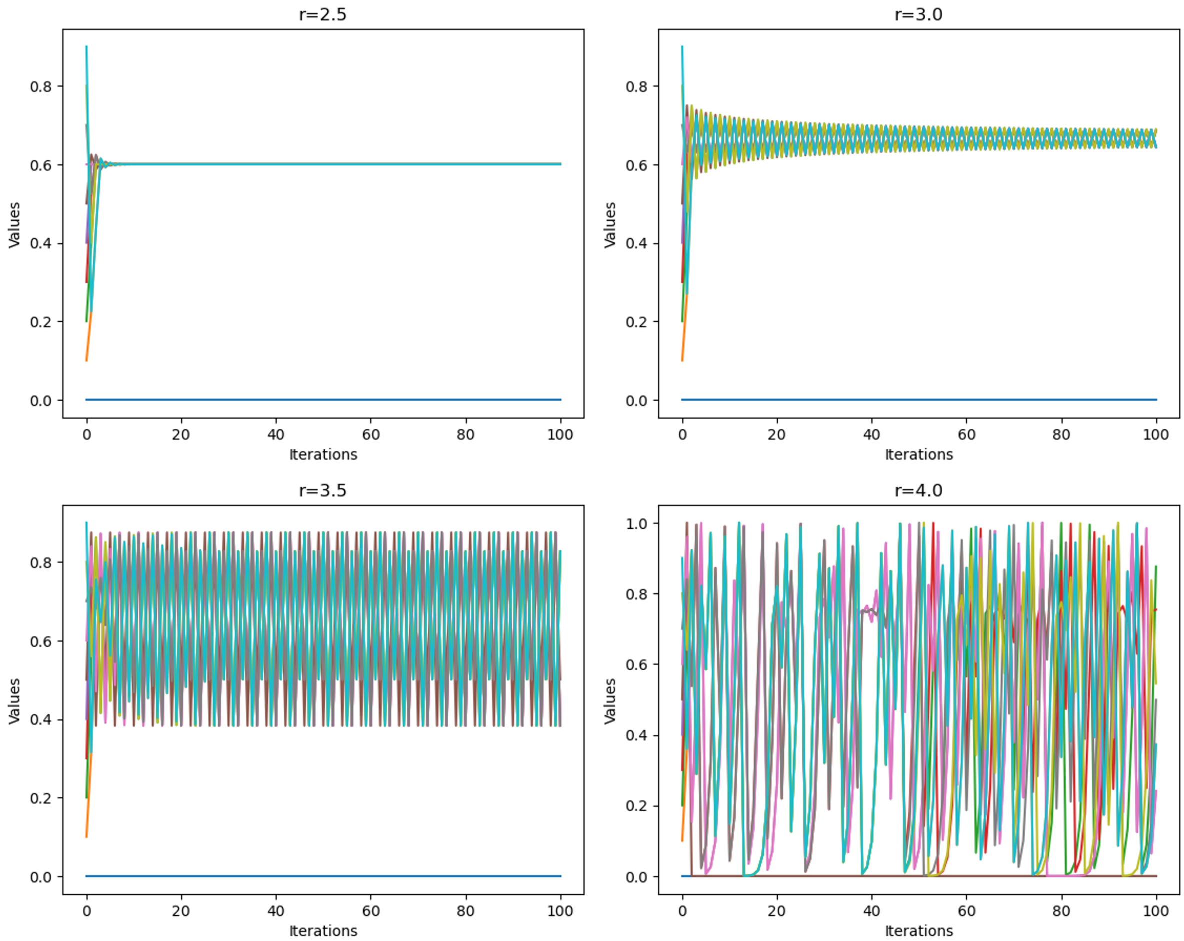


Figure 11: The change of values during simulation with $r = 2.5, 3.0, 3.5, 4.0$.

As we can see, when $r = 2.5$, the system will converge to $x = 1 - \frac{1}{2.5} = 0.6$. When $r = 3.0$, the convergence is very slow, so the fixed point loses the stability. But we can observe that the system still behaves in some deterministic pattern. It seems that the system oscillates between two values. When $r = 3.5$, the system still behaves deterministically after some iterations, but it's harder to tell from these plots in which pattern the system behaves. When $r = 4.0$, we can observe that there is no obvious pattern, and the system seems to run into chaos. To further investigate the behavior of the system when $3 < r \leq 4$, we managed to create the bifurcation diagram of this system, and the details are discussed in the following.

4.1.4 Bifurcation diagram

We define a function `logistic_bifurcation` to create the bifurcation diagram of this system. In this function, we sample 401 points at equal intervals from 0 to 4 for `r`, i.e. from $0.00, 0.01, \ldots$ up to $\ldots, 3.99, 4.00$. Each `r` value corresponds to 1000 `x` values from 0 to 0.999. We create an animation to dynamically illustrate the process of the simulation, during which we show the scatter plot at each frame with the horizontal axis to be `r` and the vertical axis to be `x`. As iteration proceeds, the sample points on the scatter plot will gather to form the bifurcation diagram. We also save the scatter plot at the last frame as the final bifurcation diagram. The gif file and the bifurcation diagram are saved in the repository. Here we only show the bifurcation diagram in Fig. 12.

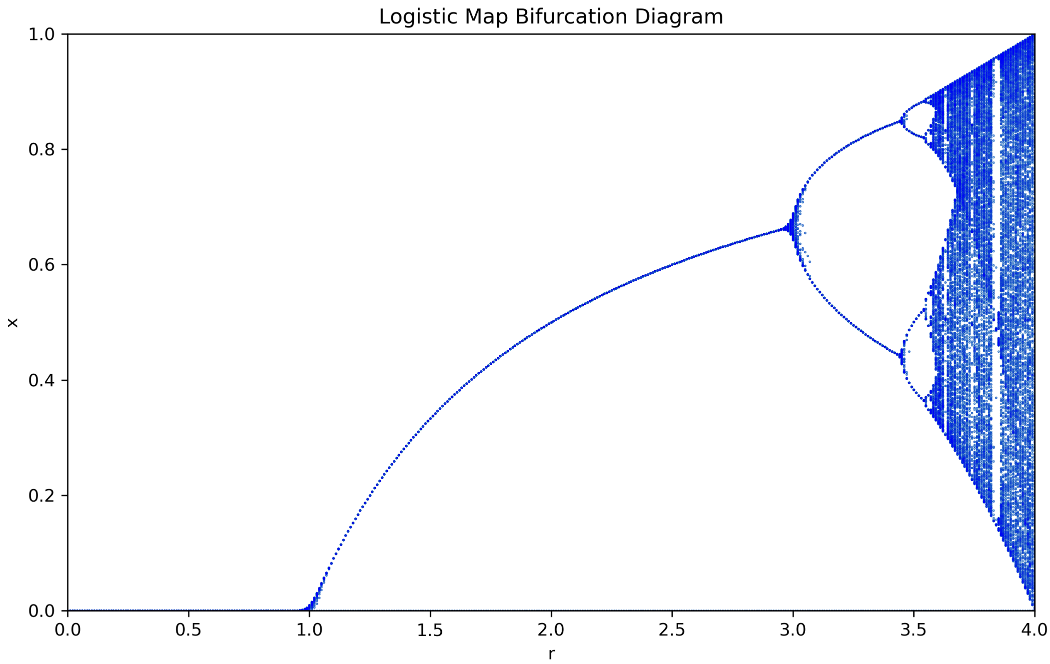


Figure 12: The bifurcation diagram of the logistic map.

From the figure, we can observe that, the system behaves exactly the same as what we analyzed above when $0 < r < 3$. When $r = 3$, we observe a bifurcation called period-doubling bifurcation. The previously stable fixed point $x = 1 - \frac{1}{r}$ becomes unstable, and the system starts oscillating between two values, entering a behavior with period = 2. We have a test with some specific values to prove this result. We set `r` to $3.1, 3.2, 3.3$ respectively and set `x` to $0.3, 0.5$ to see if the system will all initial values between 0 and 1 will end up with oscillating between two fixed values. The results are shown in Table 1.

r	x	iteration=97	iteration=98	iteration=99	iteration=100
3.1	0.3	0.55801413	0.76456652	0.55801413	0.76456652
	0.5	0.76456652	0.55801413	0.76456652	0.55801413
3.2	0.3	0.51304451	0.79945549	0.51304451	0.79945549
	0.5	0.79945549	0.51304451	0.79945549	0.51304451
3.3	0.3	0.47942702	0.82360328	0.47942702	0.82360328
	0.5	0.82360328	0.47942702	0.82360328	0.47942702

Table 1: Numerical Results of some Examples.

As we can see, if r is fixed and satisfies $3 < r \lesssim 3.5$, then starting from any initial state $0 < x < 1$, the system will always end up with oscillating between 2 fixed values, exhibiting a periodical pattern with period equal to 2. When $r \approx 3.5$, we observe another period-doubling bifurcation. The period of the system becomes 4, meaning that the system's state cycles between four values. As the value of r continues to increase, a new period-doubling bifurcation occurs, leading to a period of 8, which we can also observe in Fig. 12, although it is not very clear. Furthermore, we observe that as the period increases, the interval length of r corresponding to the same period is gradually decreasing. Therefore, we can reasonably speculate that new bifurcations will occur, leading to periods of 16, 32, etc., but these are not observable in the graph because the intervals of r are too narrow. As the value of r approaches 4, we can see that the system's behavior becomes unpredictable, and the system gradually enters a chaotic state.

4.2 Lorenz Attractor

Lorenz attractor is a nonlinear dynamical system in 3D space, and it is another example of chaotic behavior in deterministic systems. It is an important model to study chaotic dynamics and weather. The Lorenz system is a system of three differential equations:

$$\frac{dx}{dt} = \sigma(y - x), \tag{6}$$

$$\frac{dy}{dt} = x(\rho - z) - y, \tag{7}$$

$$\frac{dz}{dt} = xy - \beta z, \tag{8}$$

where x, y and z represent the states of the system with respect to time, and σ, ρ and β are the parameters of the system. The paths traced by these solutions in the x, y, z space form the so-called Lorenz attractor.

4.2.1 Chaotic Behavior of Lorenz Attractor

We first set the parameter values $\sigma = 10, \beta = 8/3, \rho = 28$, and set the initial point at $x_0 = (10, 10, 10)$ and the end time $T_{end} = 1000$. We want to visualize the trajectory of the Lorenz attractor under these settings. In order to do this, we define a function `lorenz_simulate`, in which we have a inner function to give the equations of the lorenz system. Then we call the function `odeint` from `scipy.integrate` and specify the system equations, the initial state, and the time steps. This function can numerically solve the system's equations and return the 3D system state at each time step. After we get the trajectory data, we plot it in 3D space with `matplotlib`. We want to find out that how a small perturbation in the initial state will affect the system, so we also plot the trajectory the Lorenz attractor with the initial point $\hat{x}_0 = (10 + 10^{-8}, 10, 10)$ while keeping the parameters unchanged. In addition, we also define a function `lorenz_compare` to plot the trajectories together for a more intuitive comparison, and compute the L2 norm of the difference of the two trajectories. We then plot the difference data against time, and add a red dash line representing 1 to see at what time is the difference larger than 1. Finally, we print the first time step at which the difference is larger than 1. The results of the original and the perturbed trajectories are shown in Fig. 13, and the comparison figure and the difference against time are shown in Fig. 14.

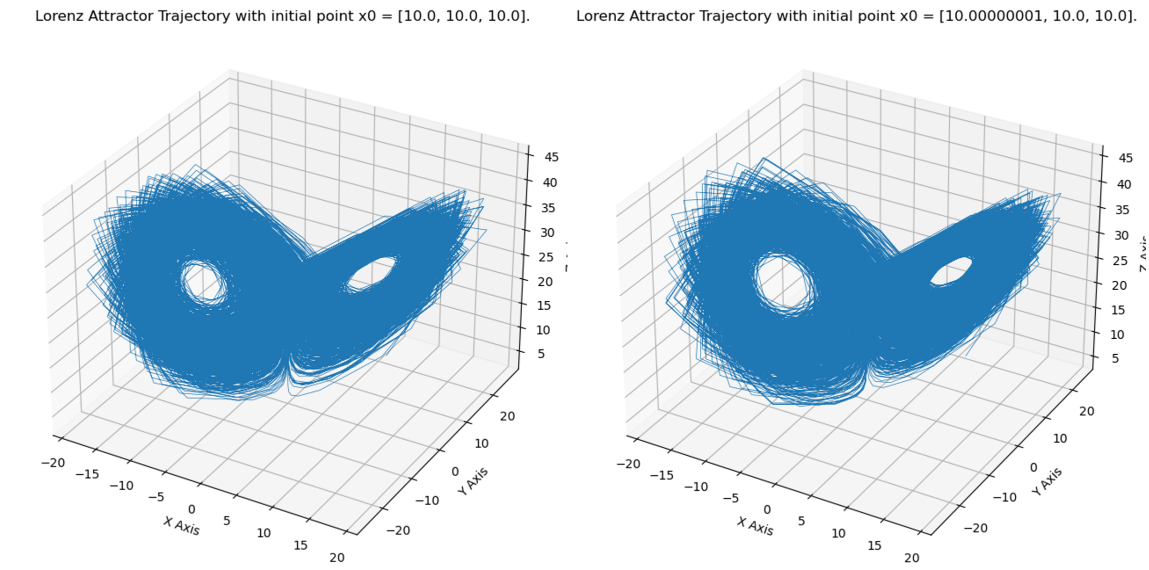


Figure 13: The original and perturbed trajectories of the Lorenz attractor with $\rho = 28$.

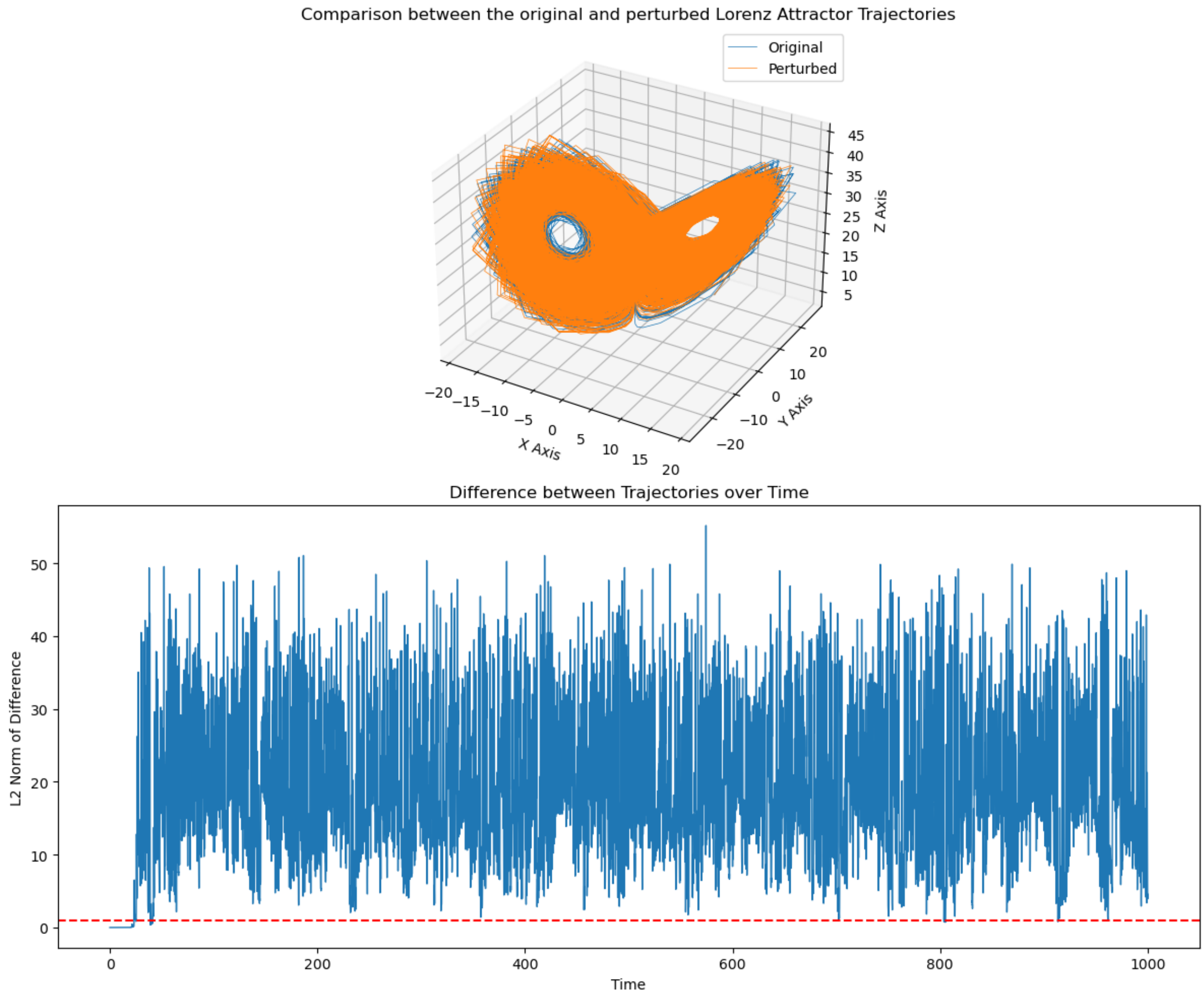


Figure 14: The comparison of the two trajectories and the difference against time with $\rho = 28$.

The shapes of the two trajectories are similar to the shape of ∞. From Fig. 13 and the top figure of Fig. 14, we can see that, we plot the 3D trajectories in 2D figures, so it's hard to tell the difference of the two trajectories visually. However, from the bottom figure of Fig. 14, we can see that the two trajectories are indeed quite different. At $T = 23.3$, the L2 norm of the difference begins to be larger than 1. After that, the difference at most of the time steps keeps being larger than 1, except only a few negligible time points. The difference shows no obvious pattern, and it can be up to even 50. These facts prove that, under the current settings of the parameters, the Lorenz attractor exhibits chaotic behavior. Although the equations of the system are deterministic, a small perturbation in the initial state will still cause significant difference in the results.

4.2.2 Deterministic Behavior of Lorenz Attractor

We have seen that the Lorenz attractor can show a chaotic pattern, however, it also can behave deterministically under some circumstances. Now we change the system parameter ρ to 0.5, and keep the other parameters unchanged. Similarly, we plot two trajectories starting from $x_0 = (10, 10, 10)$ and $\hat{x}_0 = (10 + 10^{-8}, 10, 10)$ respectively to see if there is any difference on trajectories. The trajectories are shown in Fig. 15, and the comparison figure and the difference against time are shown in Fig. 16.

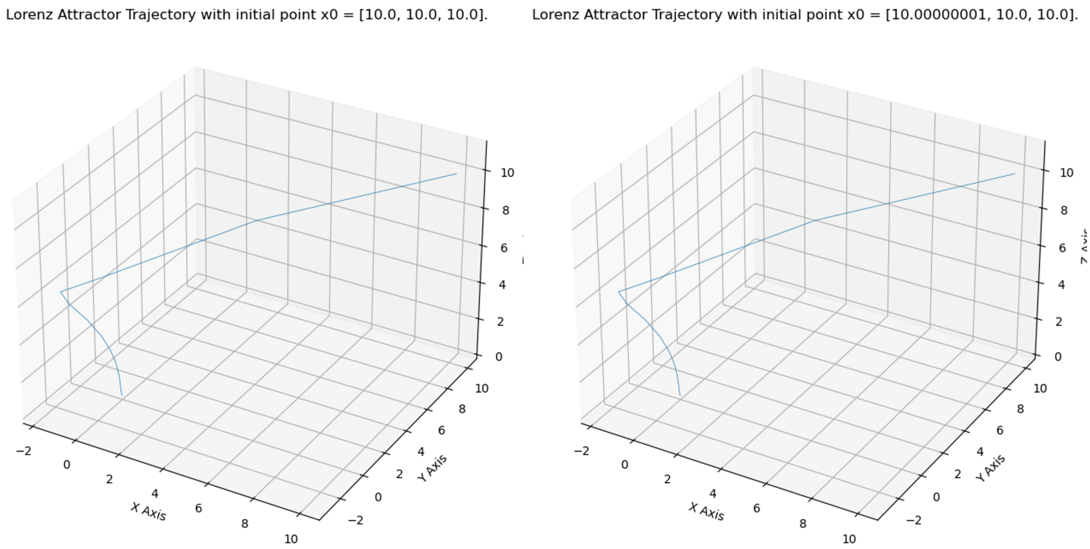


Figure 15: The original and perturbed trajectories of the Lorenz attractor with $\rho = 0.5$.

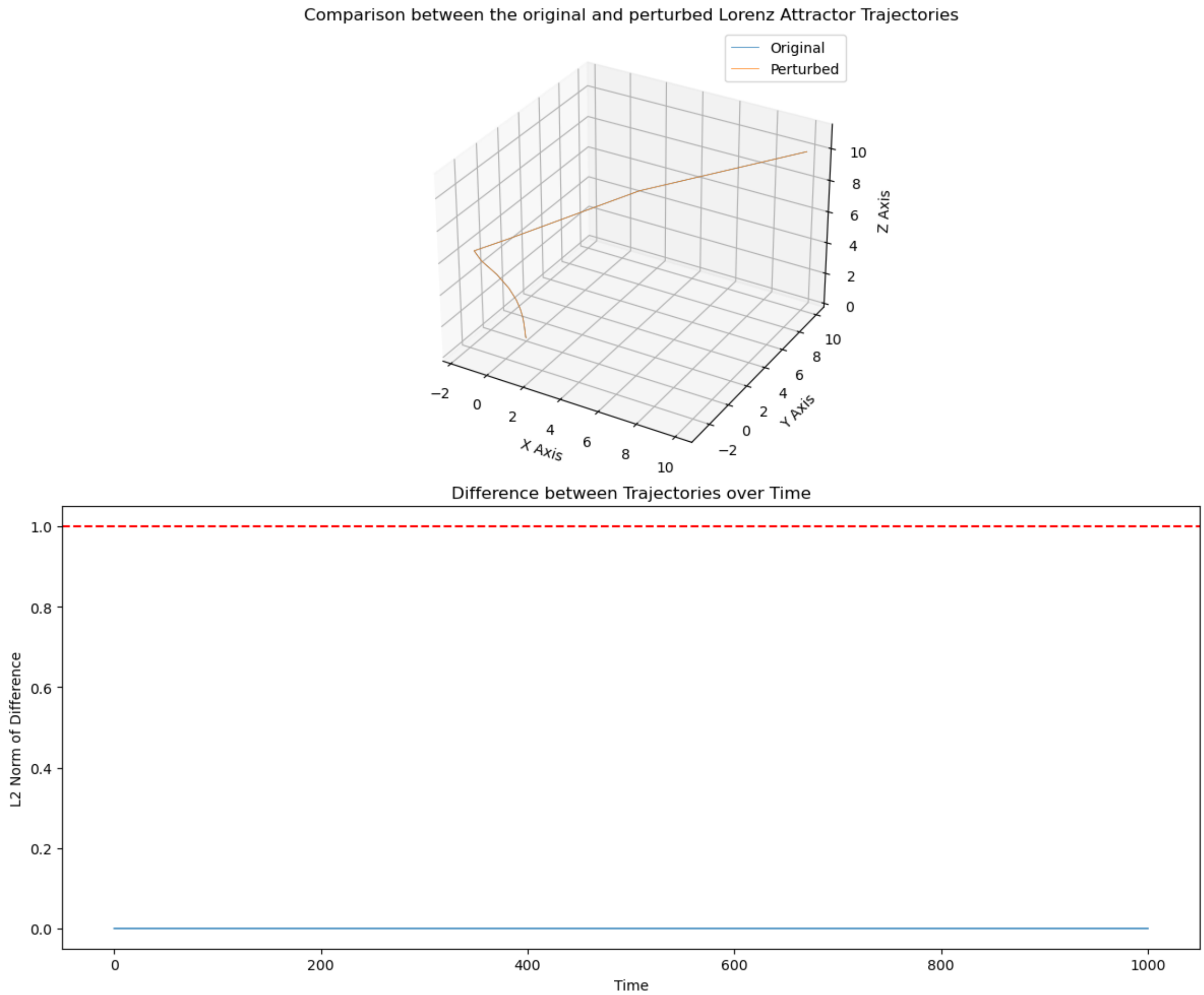


Figure 16: The comparison of the two trajectories and the difference against time with $\rho = 0.5$.

From these figures, we can see that, under the current settings of parameters, the Lorenz attractor shows a extremely simple and clear behavior, and the both trajectories are similar to the shape of a capital "L". From Fig. 16, we can find that, the two trajectories exactly coincide with each other, and the L2 norm of the difference keeps being 0 throughout the whole simulation time. This indicates that, when the parameter ρ is small, the system will show a deterministic pattern.

4.2.3 Short Discussion

From the facts above, we can see that, when the parameter $\rho = 28$ is large, the Lorenz attractor shows chaotic, complicated, and unpredictable behavior. While when the parameter $\rho = 0.5$ is small, the system shows simple and deterministic behavior. Consequently, if we vary the parameter ρ from 0.5 to 28, the Lorenz attractor goes from deterministic pattern to chaotic pattern, meaning that there definitely exists a bifurcation (or even more) between $\rho = 0.5$ and $\rho = 28$.

Report on task 5, Bifurcations in crowd dynamics

The incomplete code was completed by writing equations of Dynamical systems of SIR model which are given below:

$$\frac{dS}{dt} = A - \delta S - \frac{\beta SI}{S+I+R}$$

$$\frac{dI}{dt} = -(\delta + \nu)I - \mu(b, I)I + \frac{\beta SI}{S+I+R}$$

$$\frac{dR}{dt} = \mu(b, I)I - \delta R$$

The value of b was incremented slowly by 0.001 till it was found that at b=0.023, the trajectories of SIR model exhibit bifurcation.

For three different initial conditions, S,I,R trajectory values were obtained for sufficient time length after performing numerical integration. These were plotted and analysed for three different values of b 0.010, 0.022, 0.030. The plots are shown in fig [17]. The column titles shows the increasing values of b. The row titles correspond to different initial conditions.

As we can see in the 1st column subplots for b = 0.010, the trajectory curves for all three initial conditions exhibit the same behaviour i.e. tendency to go towards attracting node. When the value of b is slightly increased to 0.023, the three different curves exhibit different behaviours. The trajectory for initial condition $[195.3, 0.052, 4.4]$ spirals inwards to stable focus point (red dot) [see (1,2) subplot]. The second trajectory starting at $[195.7, 0.03, 3.92]$ spiral outwards to the stable limit cycle (red curve) [see (2,2) subplot]. While the third trajectory starting at $[193, 0.08, 6.21]$ spiral inwards to the stable limit cycle (red curve) [see (3,2) subplot]. Since at this value of b =0.023, the phase portrait trajectories have become topologically nonequivalent, this point is a bifurcation point. At b=0.030, all the trajectories (see last column subplots) are approaching the disease free stable equilibrium point $A/d, 0, 0$ i.e. $200, 0, 0$.

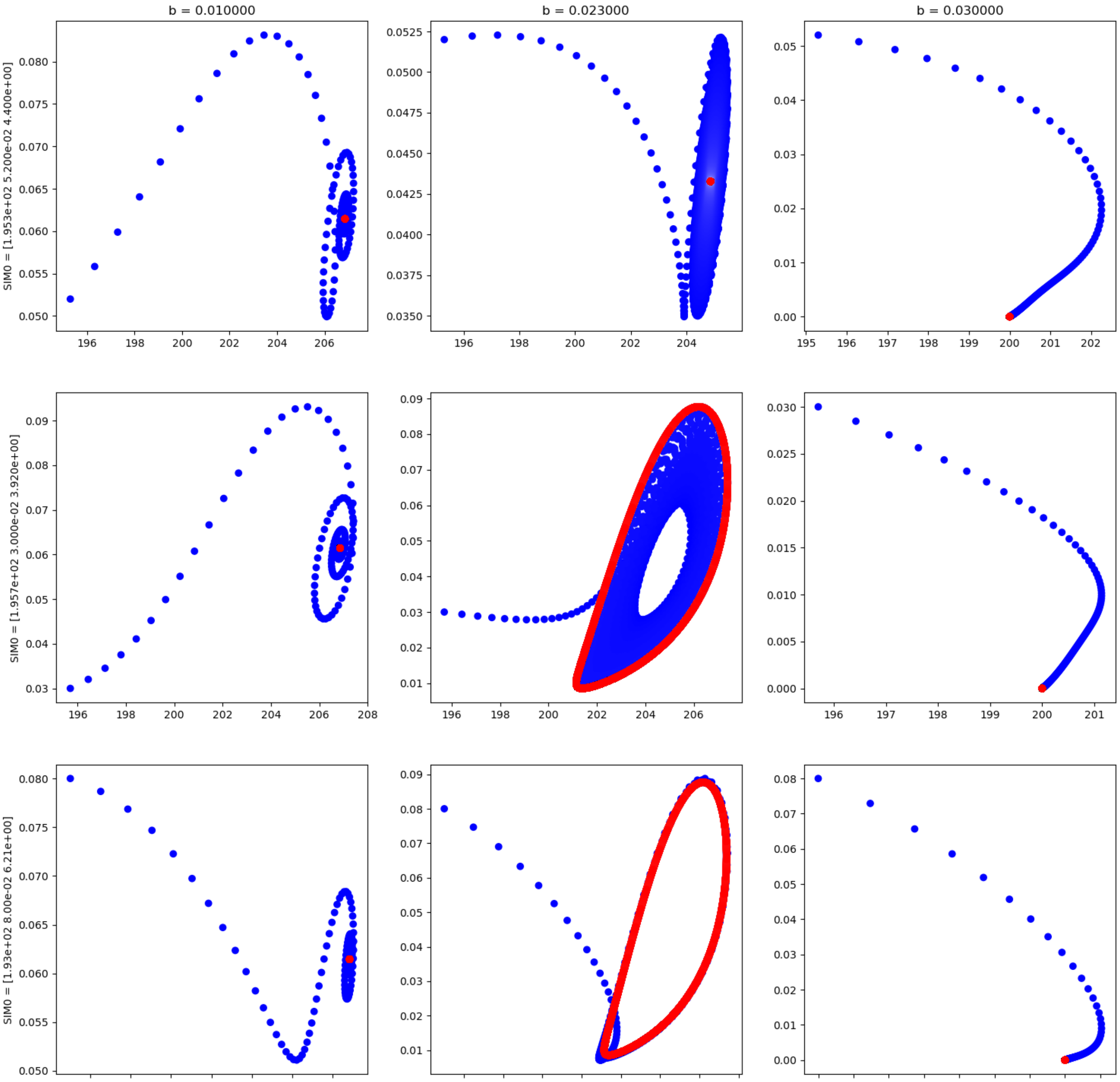


Figure 17: SI trajectories plots vs different values of b

The bifurcation phase portrait depicting all the trajectories together for b=0.023 is shown in fig [18].

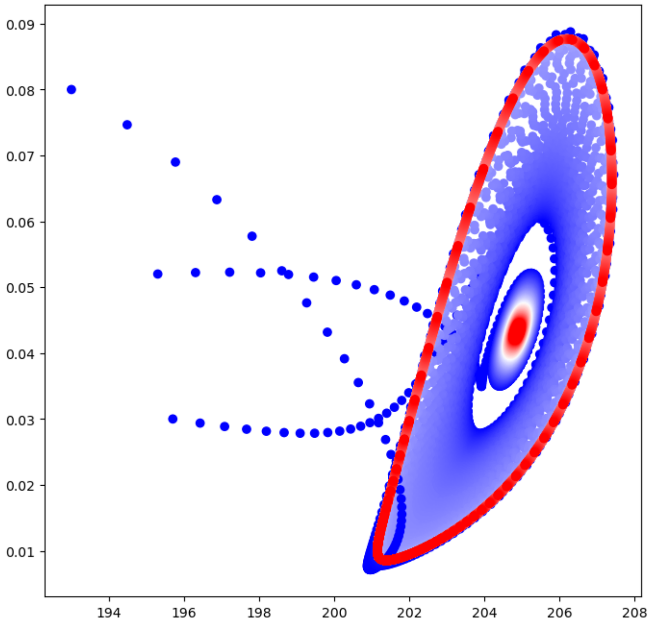


Figure 18: Bifurcation at b=0.023

The bifurcation type is Hopf due to the emergence of oscillations and stable limit cycles (periodic orbits) at b=0.023.

The variables used in computing reproduction rate are β (average number of adequate contacts per unit time with infectious individuals), d (per capita natural death rate), ν (per capita disease-induced death rate) and μ_0 (minimum recovery rate). As the β increases, the number of infected persons also increases as it leads to higher transmission rate. So on average, each infectious individual infects more susceptible individuals, leading to a more rapid spread of the disease. This is also clearly apparent from the figures 15 and 16.

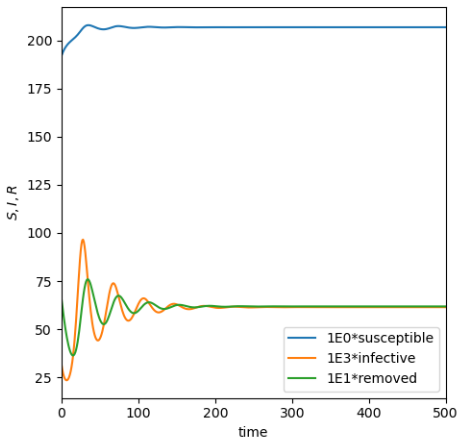


Figure 19: SIR plot for $beta = 11.5$

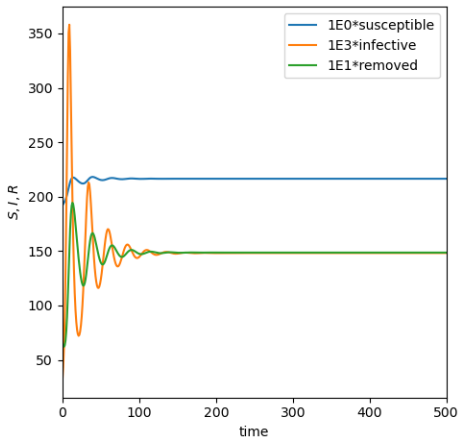


Figure 20: SIR plot for $beta = 11.9$

The disease free equilibrium $E_0 = (A/d, 0, 0)$ at $R_0 < 1$ is an attractive node implies that it is a stable equilibrium point of this dynamical system where system variables S, I, R don't change over time. As this point possess stability characteristics, the trajectories emerging near around this E_0 point have tendency to converge towards this point. This is illustrated in fig [21]. In other words, when slightly perturbed at this point, trajectory will return to this point over time.

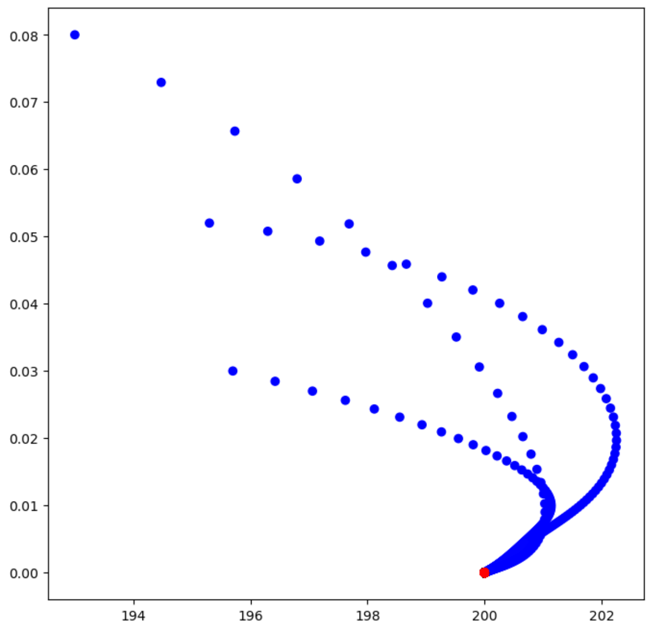


Figure 21: Attracting node E_0

Bouns

In [1], the backward bifurcation, saddle-node bifurcation, Hopf bifurcation and cusp type of Bogdanov–Takens bifurcation of the SIR model are proved. Task 5 has already analyzed Hopf bifurcation, and here I will add the analysis of backward bifurcation.

It can be obtained from Theorem 4.1 of [1]. For system (2.2), consider $\mathbb{R}_0$ as the bifurcation parameter. When $\mathbb{R}_0$=1, if b>A$(\mu_1-\mu_0)/\beta(\beta-\upsilon)$, then system (2.2) will undergo forward bifurcation. ; If b<A$(\mu_1-\mu_0)/\beta(\beta-\upsilon)$, then system (2.2) undergoes backward bifurcation; if b = A$(\mu_1-\mu_0)/\beta(\beta-\upsilon)$, then system (2.2) experiences pitchfork bifurcation.

Through the equation **??**, we can get that when μ_1=10.4, it makes $\mathbb{R}_0$=1. We start b from 0.02 and gradually

increase it to 0.05. We can see the trajectory change in the figure 22. The red and blue curves correspond to [195.3, 0.052, 4.4] and [195.7, 0.03, 3.92] respectively. Observing the figure 22, we found that it is similar to the Hopf bifurcation analysis of task 5.3, except that the b value is around 0.029.
The lower β, the system reaches a stable disease-free equilibrium state. However, when Bβ increases beyond a certain threshold, a backward bifurcation occurs, that is, a stable local equilibrium with a positive number of infected individuals. This persistence of infection at lower rates complicates disease control measures because it means the disease may persist in the population despite efforts to reduce transmission.

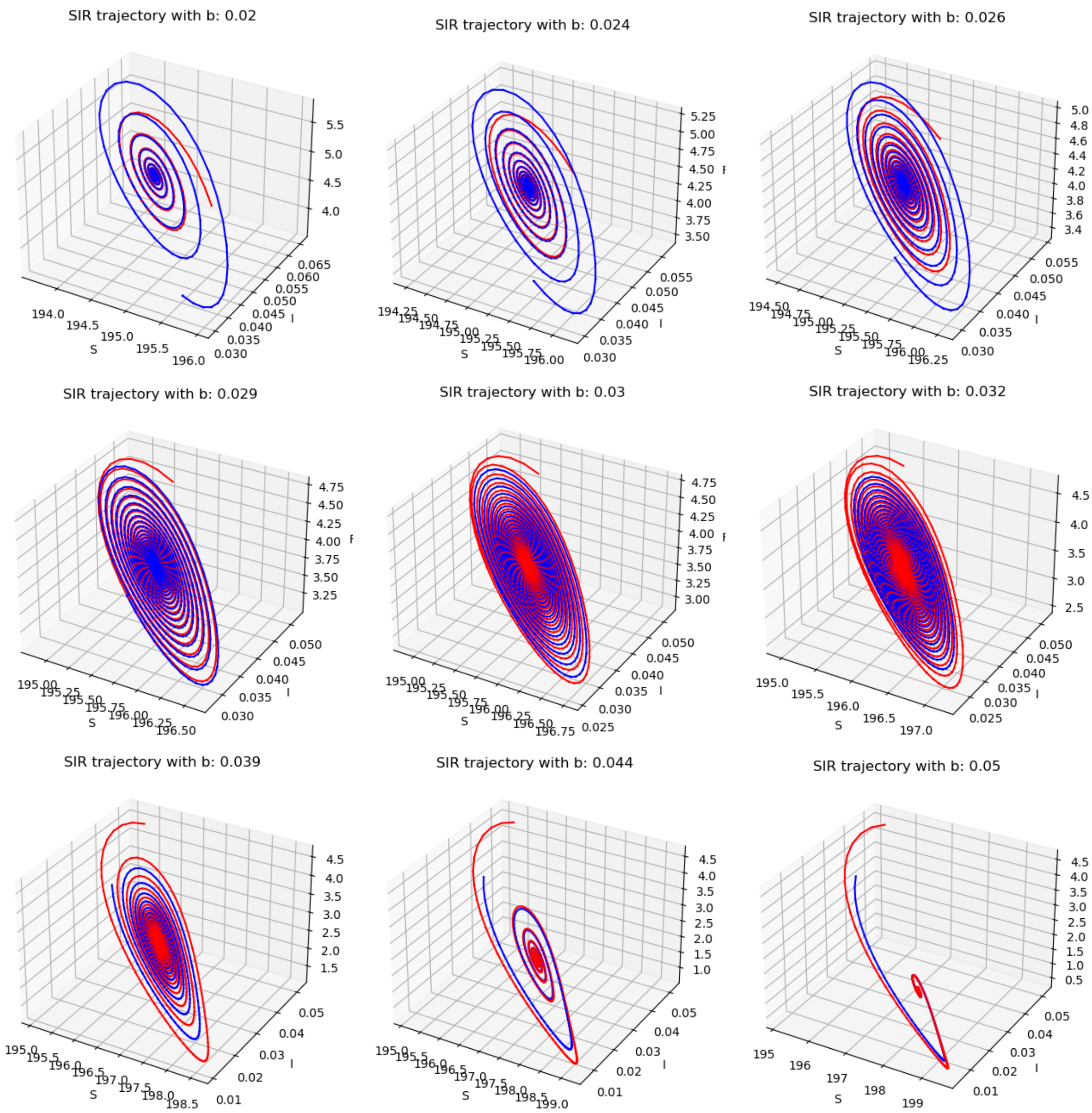


Figure 22: Trajectory of SIR variables by changing b from left to right and up to down

References

[1] Chunhua Shan and Huaiping Zhu. Bifurcations and complex dynamics of an SIR model with the impact of the number of hospital beds. Journal of Differential Equations, 257(5):1662–1688, September 2014.