

# Mathematics Short Sum Up

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Convert  $d$ -dimensional potential summations into a universal integral summation by Laplace transform

$$G(\mathbf{L}, \mathbf{v}) = D \int_0^\infty t^{C-1} (-\delta(\mathbf{v}, B) + \sum_{\mathbf{k} \in \mathbb{Z}^d} e^{-A\pi \|\mathbf{Lk} + \mathbf{v}\|^2 t - \frac{B}{t}}) dt, \quad (1)$$

with different constants  $A, B, C, D$ , then use integral split up

$$G(\mathbf{L}, \mathbf{v}) = \int_0^1 \sum \cdot dt + \int_1^\infty \sum \cdot dt = G_{Fourier}(\mathbf{L}, \mathbf{v}) + G_{direct}(\mathbf{L}, \mathbf{v}), \quad (2)$$

where  $G_{Fourier}$  denotes long-range term and  $G_{direct}$  denotes short-range term. Since convergence with  $t$  in  $G_{Fourier}$  is slow, we do

$$t \rightarrow \frac{1}{t} \quad (3)$$

and Poisson summation

$$\sum_{\mathbf{k} \in \mathbb{Z}^d} e^{-2\pi i \mathbf{w} \cdot \mathbf{Lk} - \pi t \|\mathbf{Lk} + \mathbf{v}\|^2} = \frac{t^{-\frac{d}{2}} e^{2\pi i \mathbf{w} \cdot \mathbf{v}}}{\det \mathbf{L}} \sum_{\mathbf{k} \in \mathbb{Z}^d} e^{2\pi i \mathbf{L}' \mathbf{k} \cdot \mathbf{v} - \frac{\pi}{t} \|\mathbf{L}' \mathbf{k} + \mathbf{w}\|^2}. \quad (4)$$

We obtain

$$\begin{aligned} G_{Fourier}(\mathbf{L}, \mathbf{v}) &= -\frac{D\delta(\mathbf{v}, B)}{C} + \frac{\delta(B)}{\det \mathbf{L}} \frac{D}{(C - \frac{d}{2})A^{\frac{d}{2}}} \\ &+ \frac{1}{\det \mathbf{L}} \frac{D}{A^{\frac{d}{2}}} \sum_{\mathbf{k} \in \mathbb{Z}^d, \pi \|\mathbf{L}' \mathbf{k}\|^2 / A + B > 0} e^{2\pi i \mathbf{L}' \mathbf{k} \cdot \mathbf{v}} K_{C-\frac{d}{2}}\left(\frac{\pi \|\mathbf{L}' \mathbf{k}\|^2}{A} + B, 0\right) \end{aligned} \quad (5)$$

and

$$G_{direct}(\mathbf{L}, \mathbf{v}) = D\delta(\mathbf{v})B^C(\Gamma(-C) - \Gamma(-C, B)) + D \sum_{\mathbf{k} \in \mathbb{Z}^d, \|\mathbf{Lk} + \mathbf{v}\| \neq 0} K_{-C}(A\pi \|\mathbf{Lk} + \mathbf{v}\|^2, B). \quad (6)$$

with incomplete Bessel function

$$K_\nu(x, y) = \int_1^\infty t^{-\nu-1} e^{-xt-y/t} dt. \quad (7)$$

We show that summation of incomplete Bessel functions

$$\sum_{\mathbf{k} \in \mathbb{Z}^d, \pi\alpha \|\mathbf{Lk} + \mathbf{v}\|^2 + \gamma > 0} K_\nu(\pi\alpha \|\mathbf{Lk} + \mathbf{v}\|^2 + \gamma, \beta), \quad (8)$$

with  $\alpha > 0, \beta \geq 0, \gamma \geq 0$ , is bound by the Gaussian Lattice Sum

$$E(\mathbf{L}, \mathbf{v}, \alpha) = \sum_{\mathbf{k} \in \mathbb{Z}^d} e^{-\pi\alpha \|\mathbf{Lk} + \mathbf{v}\|^2} < \frac{d}{2} \left(\frac{2}{\rho}\right)^d \Gamma\left(\frac{d}{2}, \left(\frac{\rho}{2}\right)^2\right). \quad (9)$$

where  $\rho = \min\{\sqrt{\pi\alpha} \|\mathbf{Lk}\| \mid \mathbf{k} \in \mathbb{Z}^d, \mathbf{k} \neq \mathbf{0}\}$ .