Maths Dis

TP11

12/12

- fonction génératrice ordinaire de la suite $(a_n)_n \in \mathbb{N}$ est $\sum_{n \geq 0} a_n x^n$

- fonction génératrice exponentielle de la suite $(a_n)_n \in \mathbb{N}$ est $\sum_{n \geq 0} \frac{a_n}{n!} x^n$

~~où~~ et
- $\sum_{n \geq 0} x^n = \frac{1}{1-x}$
- $\sum_{n \geq 0} \frac{x^n}{n!} = e^x$.

et $\times$ $H_n = \sum_{k=1}^{n} \frac{1}{k}$; $(n \geq 1)$ et $H_0 = 0$. (nombres harmoniques)

1. $\sum_{n=0}^{\infty} H_n \frac{1}{10^n} = \cancel{\sum_{n=0}^{\infty} \left[\sum_{k=1}^{n} \frac{1}{k}\right] \frac{1}{1-\frac{1}{10}}}$

$\cancel{= \left(\sum_{n=0}^{\infty} H_n\right) \frac{10}{9}}$

$\cancel{= \frac{10}{9} \sum_{n=0}^{\infty} H_n}$

$\cancel{= \frac{10}{9} \sum_{n=0}^{\infty} \left(\sum_{k=1}^{n} \frac{1}{k}\right) = \frac{10}{9} (H_n)_n}$

$\cancel{= \frac{10}{9} \sum_{n \geq 0} H_n x^n}$

$\Rightarrow \sum H_n x^n = \frac{1}{1-x} \ln \frac{1}{1-x}$

$= \frac{1}{1-\frac{1}{10}} \ln \frac{1}{1-\frac{1}{10}}$

$\boxed{= \frac{10}{9} \ln \frac{10}{9}}$

2. 1. $\sum_{n=0}^{\infty} H_n \frac{1}{2^n} = \frac{1}{1-\frac{1}{2}} \ln \frac{1}{1-\frac{1}{2}}$

$\boxed{= 2 \ln 2}$

2. $\sum_{n=0}^{\infty} \binom{n}{2} \frac{1}{10^n} = \cancel{\sum_{n=0}^{\infty} \frac{n!}{2!(n-2)!} \frac{1}{10^n}}$ $\quad \cancel{\sum \binom{n}{k} a_k}$

$= \frac{\frac{1}{10}^2}{\left(1-\frac{1}{10}\right)^{2+1}}$ $\quad \sum_{n=0}^{\infty} \binom{n}{k} x^n = \frac{x^k}{(1-x)^{k+1}}$

$= \frac{\cancel{\frac{1}{100}}}{\left(\frac{9}{10}\right)^3}$

$\boxed{= \frac{10}{9^3}}$

3. 1. $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{1-\frac{1}{2}} - \underbrace{1}_{n=0} = 2 - 1 = \underline{1}$

2. $\sum_{n=1}^{\infty} \frac{n}{2^n} = \sum_{n=1} n \cdot \frac{1}{2^n}$ $\qquad \left(\frac{1}{1-x}\right)' = \frac{x}{(1-x)^2}$

$= \sum_{n=0}^{\infty} \frac{1}{2^n} \cdot n$

$= \frac{\frac{1}{2}}{\left(1-\frac{1}{2}\right)^2}$

$\boxed{= 2}$

3. $\sum_{n=1}^{\infty} \frac{1}{n 2^n} = \sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{2^n}$ $\qquad \cancel{\int \frac{1}{1-x} dx = \left[\ln\left(\frac{1}{1-x}\right)\right]}$

$\boxed{= -\ln \frac{1}{2}}$? Thm 11 (iii) $\int \frac{1}{1-x} dx = \cancel{\ln \frac{1}{1-x}}$

$= -\ln(1-x)$

4. $a_n = 2^n + 3^n$

$\Rightarrow$ FGO: $\sum_{n=0}^{\infty} (2^n + 3^n) x^n = \sum_{n=0}^{\infty} 2^n x^n + \sum_{n=}^{\infty} 3^n x^n$

$\boxed{= \frac{1}{1-2x} + \frac{1}{1-3x}}$

$\Rightarrow$ FGE: $\sum_{n=0}^{\infty} (2^n + 3^n) \frac{x^n}{n!} = \boxed{e^{2x} + e^{3x}}$

5. $a_n = 7a_{n-1} + 8a_{n-2} + 7n + 5/2$, $a_0 = 4$, $a_1 = 7$, $n \geq 2$

1. • $x^2 - 7x - 8 = 0$

$(x-8)(x+1) = 0$

$x = 8$ ou $x = -1$ $\qquad \Rightarrow A 8^n + B(-1)^n$

• $\tilde{a}_n = Cn + D$

$Cn + D = 7C(n-1) + 7D + 8C(n-2) + 8D + 7n + 5/2$

$Cn - 7Cn - 8Cn = 7n$ $\qquad | \quad D - 7D - 8D + 7C + 16C = \frac{5}{2}$

$-14C = 7$ $\qquad | \quad -14D + 23\left(-\frac{1}{2}\right) = \frac{5}{2}$

$\boxed{C = -\frac{1}{2}}$ $\qquad -14D = \frac{28}{2}$

$\boxed{D = -1}$

$\Rightarrow a_n = A 8^n + B(-1)^n - \frac{1}{2} n - 1$

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5\. 1. $a_n = \frac{3}{2} 8^n + \frac{7}{2}(-1)^n - \frac{n}{2} - 1.$

2\. $b_0 = 4$, $b_1 = 7$, $b_n = 7b_{n-1} + 8b_{n-2}$

$$f(x) = \sum_{n=0}^{\infty} b_n x^n$$

$$f(x) = \sum b_n x^n = 4 + 7x + \sum_{n=2}^{\infty} (7 b_{n-1} + 8 b_{n-2}) x^n$$

$$\sum_{n=2}^{\infty} 7 b_{n-1} x^{n-1} \times x$$

$$= 4 + 7x + 7x \underbrace{\sum_{n=2}^{\infty} b_{n-1} x^{n-1}}_{f(x) - 4} + 8x^2 \Big(\underbrace{\sum_{n=2}^{\infty} b_{n-2} x^{n-2}}_{f(x)}$$

$$= (4 - 21x) / (1 - 7x - 8x^2)$$

$$\boxed{= \frac{21x - 4}{8x^2 + 7x - 1}}$$

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3\. $8x^2 + 7x - 1 = (x+1)(8x-1)$

$$\frac{21x - 4}{8x^2 + 7x - 1} = \frac{A}{x+1} + \frac{B}{8x-1}$$

$$21x - 4 = A(8x-1) + B(x+1)$$

$$21 = 8A + B$$

$$-4 = -A + B \quad ; \quad \boxed{A = \frac{25}{9}, \; B = \frac{-11}{9}}$$

$$f(x) = \frac{21x - 4}{8x^2 + 7x - 1} = \frac{25}{9} \frac{1}{1+x} + \frac{11}{9} \frac{1}{1-8x} \quad \boxed{-\frac{11}{9} \frac{1}{8x-1}}$$

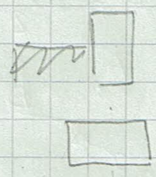
$$= \frac{25}{9} \sum_{n=0}^{\infty} (-x)^n + \frac{11}{9} \sum_{n=0}^{\infty} (+8x)^n$$

$$b_n = [x^n] f(x) = \boxed{\frac{25}{9} (-1)^n + \frac{11}{9} 8^n}$$

6. rect : $2 \times n$

vert : 2×1 — 4 eu.

hor : 1×2 — 1 eu.



~~pour chaque rect : 2 vert et n/2 hor~~

~~donc 4 × 2 et n/2 euro~~

~~=> 8 + n/2 par pavage~~

cas 1 : vertical → vertical / horizontal

cas 2 : horizontal → horizontal → vertical / horizontal

au tableau

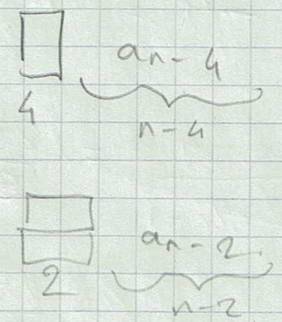
cas 1 : $\cancel{x}$ (vertical)

cas 2 : $2y$ (horizontal) } $4x + 2y$

a_n : nombre de pavages à n€

si n impair, $a_n = 0$ (pas de euro impair).

=> si n pair



$\Rightarrow a_n = a_{n-4} + a_{n-2}$

$a_n = \begin{cases} F_{n/2} & n \text{ pair} \\ 0 & n \text{ impair} \end{cases}$

a_0, a_1, a_2, a_3

$a_0 = 1$ (une seule manière de dépenser)

$a_1 = 0$

$a_2 = 1$

$a_3 = 0$

$$A(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 + a_1 x^1 + a_2 x^2 + a_3 x^3 + \sum_{n \geq 4} a_n x^n$$

$$= 1 + x^2 + \sum_{n \geq 4} (a_{n-2} + a_{n-4}) x^n$$

$$= 1 + x^2 + x^2 \sum_{n \geq 4} a_{n-2} x^{n-2} + x^4 \sum a_{n-4} x^{n-4}$$

terme qui manque $n \geq 4$

$$= 1 + x^2 + x^2 (A(x) - 1) + x^4 (A(x))$$

$\cancel{1 + x^2 (A}$

$$A(x) = \frac{1}{1 - x^2 - x^4}$$