MATH-F307 - Mathématiques discrètes

Séance 1 (19 septembre 2018)

Exercice 1. Soient A et B deux ensembles finis avec $|A| = a$ et $|B| = b$ $(a, b \in \mathbb{N})$. Que valent:

1. $|A \times B|$,
2. $|B^A|$ où $B^A := \{f : A \to B\}$,
3. $|\{f : A \to B : f \text{ est une injection de } A \text{ dans } B\}|$,
4. $|\mathrm{S}(A)|$ où $\mathrm{S}(A)$ est l'ensemble de toutes les permutations de A.

Exercice 2. Quels sont les ensembles F non vides ayant la propriété suivante:

1. pour tout ensemble X, $|F^X| = 1$?
2. pour tout ensemble Y, $|Y^F| = 1$?

Exercice 3. Soient $f : A \to B$ et $g : B \to C$ deux fonctions. Démontrer:

1. $g \circ f$ injective $\Rightarrow$ f injective;
2. $g \circ f$ surjective $\Rightarrow$ g surjective;
3. $g \circ f$ bijective $\Rightarrow$ (f injective et g surjective).

Exercice 4.
Donner une bijection explicite entre l'ensemble des bijections de $[n]$ vers $[n]$ et l'ensemble $[n] \times [n-1] \times \cdots \times [1]$.

Exercice 5. Montrer que les ensembles $\mathbb{N}$ et $\mathbb{Z}$ ont la même cardinalité. Donner une bijection explicite entre les deux ensembles.

Exercice 6. Montrer que si n est un entier, alors $n^3 - n$ est pair.

Exercice 7. Montrer que $\sqrt{3}$ n'est pas rationel.

Exercice 8. Montrer qu'il n'existe pas d'entiers a et b telle que: $18a + 6b = 1$.

Exercice 9. Just for fun:

Stacy and Sam Smyth were known for throwing a heck of a good party. At one of their wild gatherings, five couples were present (this included the Smyth's, of course). The attendees were cordial, and some even shook hands with other guests. Although we have no idea who shook hands with whom, we do know that no one shook hands with themselves and no one shook hands with his or her spouse. Given these facts, a guest might not shake anyone's hand or might shake as many as eight other people's hands. At midnight, Sam Smyth gathered the crowd and asked the nine other people how many hands each of them had shaken. Much to Sam's amazement, each person gave a different answer. That is, someone didn't shake any hands, someone else shook one hand, someone else shook two hands, someone else shook three hands, and so forth, down to the last person, who shook eight hands. Given this outcome, determine the exact number of hands that Stacy Smyth shook.