

# Applying Deep Learning to Inverse Kinematics

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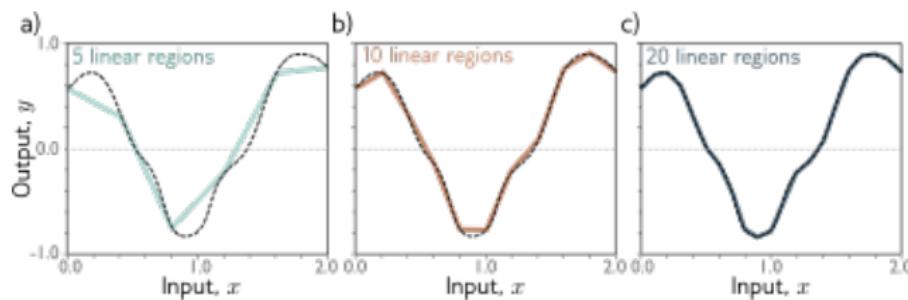
# Outline

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# Universal Approximation Theorem

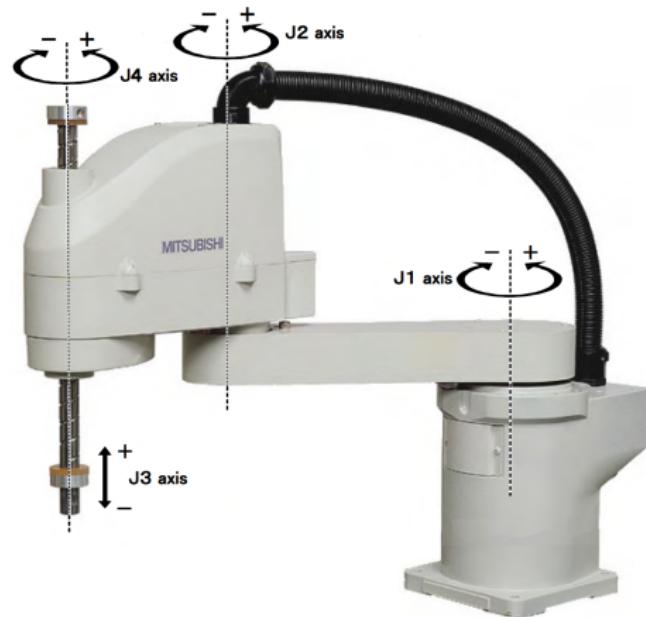
## Core Principle

Universal approximation theorem proves that for any continuous function, there exists a network that can approximate this function to any specified precision.



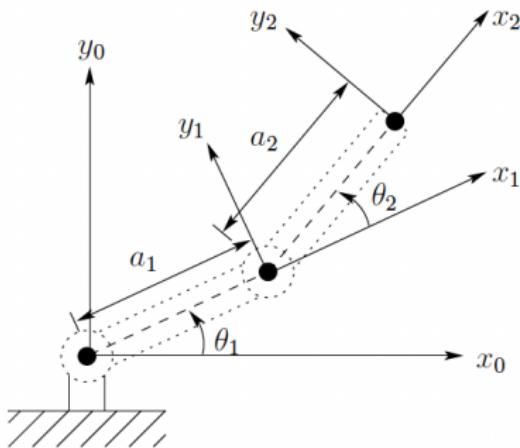
*This theorem provides the theoretical foundation for using neural networks to learn inverse kinematics.*

# Robotic Arms



**SCARA Robot**

# Two-Link Planar Manipulator



## Problem Setup:

- Link lengths:  $a_1, a_2$
- Joint angles:  $\theta_1, \theta_2$
- End-effector:  $(x, y)$

Simple case to understand forward and inverse kinematics

# Forward Kinematics (2-Link)

**From geometry of the two revolute links:**

$$x = a_1 \cos \theta_1 + a_2 \cos(\theta_1 + \theta_2) \quad (1)$$

$$y = a_1 \sin \theta_1 + a_2 \sin(\theta_1 + \theta_2) \quad (2)$$

**Interpretation:**

- Given: Joint angles  $\theta_1, \theta_2$
- Compute: End-effector position  $(x, y)$
- Direct and unambiguous calculation

# Inverse Kinematics (2-Link) — Step 1

## Step 1: Solve for $\theta_2$ (elbow angle)

Define:  $r^2 = x^2 + y^2$

Apply law of cosines:

$$\cos \theta_2 = \frac{r^2 - a_1^2 - a_2^2}{2a_1a_2} \quad (3)$$

Solve for angle:

$$\theta_2 = \text{atan2}\left(\pm\sqrt{1 - \cos^2 \theta_2}, \cos \theta_2\right) \quad (4)$$

### Note:

- The  $\pm$  gives two solutions: “elbow-down” and “elbow-up”
- This is the *multiple solutions problem*

# Inverse Kinematics (2-Link) — Step 2

## Step 2: Solve for $\theta_1$ (shoulder angle)

Using the shoulder and elbow geometry:

$$\theta_1 = \text{atan2}(y, x) - \text{atan2}(a_2 \sin \theta_2, a_1 + a_2 \cos \theta_2) \quad (5)$$

## Result:

- We now have the joint angles that place end-effector at  $(x, y)$
- Two possible configurations exist (elbow-down, elbow-up)
- This is the classic geometric IK solution for planar 2-DOF robots

*For complex robots, we need a general framework...*

# Denavit-Hartenberg (DH) Parameterization

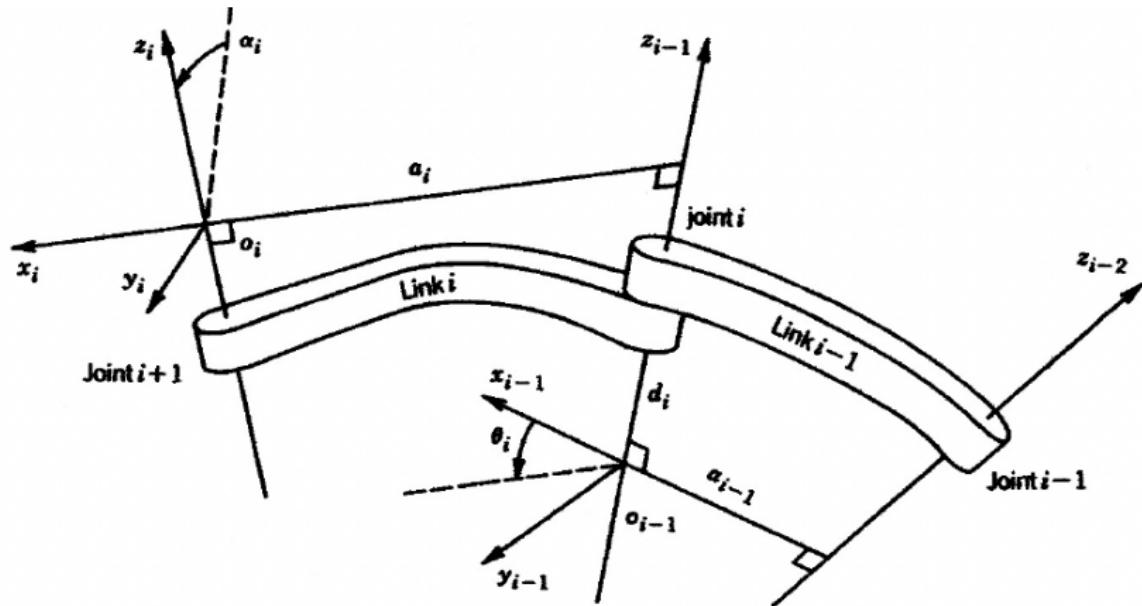
## DH Parameter Convention

A standard method to describe robot geometry using 4 parameters per joint:

- $a_i$ : link length (distance along  $\hat{x}$ -axis)
- $d_i$ : link offset (distance along  $\hat{z}$ -axis)
- $\alpha_i$ : link twist (rotation around  $\hat{x}$ -axis)
- $\theta_i$ : joint angle (rotation around  $\hat{z}$ -axis)

**Note:** Either  $d_i$  or  $\theta_i$  can be variable depending on joint type. For ease of explanation, we will only use  $\theta_i$  as the variable (revolute joints).

# Denavit-Hartenberg (DH) Parameterization



# Homogeneous Transformation Matrix

The transformation is represented as a product of four basic transformations:

$$A_i = \text{Rot}_{z,\theta_i} \cdot \text{Trans}_{z,d_i} \cdot \text{Trans}_{x,a_i} \cdot \text{Rot}_{x,\alpha_i}$$

$$= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i} & 0 & 0 \\ s_{\theta_i} & c_{\theta_i} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{\alpha_i} & -s_{\alpha_i} & 0 \\ 0 & s_{\alpha_i} & c_{\alpha_i} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_i = \begin{bmatrix} c_{\theta_i} & -s_{\theta_i}c_{\alpha_i} & s_{\theta_i}s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i}c_{\alpha_i} & -c_{\theta_i}s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Where  $c_{\theta_i} = \cos \theta_i$  and  $s_{\theta_i} = \sin \theta_i$

# RRR Robot Architecture (3-DOF)

**RRR Configuration:** Three revolute joints

DH Parameters:

Joint	$a_i$ (mm)	$d_i$ (mm)	$\alpha_i$	$\theta_i$ (variable)
1	0	0	-90°	$\theta_1$
2	0	0	+90°	$\theta_2$
3	0	50	0°	$\theta_3$

**Workspace:**

- 3 DOF allows positioning in 3D space
- Multiple configurations possible (redundancy)
- Used as primary test case

# RRRRRR Robot Architecture (6-DOF)

**RRRRRR Configuration:** Six revolute joints

DH Parameters:

Joint	$a_i$	$d_i$	$\alpha_i$
1	0	0	-90°
2	0	0	+90°
3	0	50	0°
4	0	100	-90°
5	0	0	+90°
6	0	50	0°

**Characteristics:**

- 6 degrees of freedom (full pose control)
- More complex workspace geometry
- Higher dimensional solution space

# Classical Geometric Solution

**Approach:** Decompose problem into geometric subproblems

## **Advantages:**

- Algebraically exact solutions
- Computationally fast
- Deterministic

## **Disadvantages:**

- Problem-specific (doesn't generalize)
- Requires expert knowledge of robot geometry
- Difficult for 3+ DOF systems

# Damped Least Squares (DLS) Method

**Iterative Approach:** Refine joint angles to minimize pose error

**DLS Update Rule:**

$$\Delta\theta = (J^T J + \lambda^2 I)^{-1} J^T \mathbf{e}$$

Where:

- $J$ : Jacobian matrix —  $\frac{\partial FK(\theta)}{\partial \theta}$
- $\mathbf{e}$ : pose error vector —  $\mathbf{e} = p_{\text{target}} - FK(\theta_{\text{current}})$
- $\lambda$ : damping factor (prevents singular matrices, typically 0.01–0.1)

**Algorithm:**

- ① Initialize:  $\theta_0 = [0, 0, 0, \dots]$
- ② Compute:  $\mathbf{e} = p_{\text{target}} - FK(\theta)$  and Jacobian  $J$
- ③ Update:  $\theta \leftarrow \theta + \Delta\theta$
- ④ Repeat until:  $\|\mathbf{e}\| < \epsilon$  (e.g.,  $\epsilon = 10^{-6}$  rad)

# DLS Method Characteristics

## Convergence Properties:

- Typically converges in 10–100 iterations
- Each iteration:  $\sim 100 \mu\text{s}$  (depends on DOF)
- Total time:  $\sim 1\text{--}10 \text{ ms}$  per IK solution

## Advantages:

- General method (works for any robot)
- Handles singularities through damping
- Approximate but converges reliably

## Disadvantages:

- Slower than geometric solutions
- May converge to local minimum
- Requires good initial guess

# Why Neural Networks for IK?

## Key Insight

Since inverse kinematics is a mapping from  $(x, y, z, \text{roll}, \text{pitch}, \text{yaw})$  to joint angles  $(\theta_1, \theta_2, \dots, \theta_n)$ , this should be something that we can use universal approximation theorem to learn.

## IK as a Function:

$$f : \mathbb{R}^6 \rightarrow \mathbb{R}^n$$

$$(x, y, z, r, p, y) \mapsto (\theta_1, \theta_2, \dots, \theta_n)$$

## If we can learn this mapping:

- No iterative solving required
- Single forward pass through network

# Dataset Generation Method

**First Approach:** Generate random joint angles, compute forward kinematics

```
def generate_consistent_dataset(dh_params, num_samples=50000):
    """Generate dataset with unique pose-angle mappings"""
    np.random.seed(42)

    # Generate random joint angles in [-180, +180] degrees
    num_joints = len(dh_params)
    angles = np.random.uniform(
        -np.pi, np.pi,
        size=(num_samples, num_joints)
    )

    positions = []
    for angle_set in angles:
        # Compute forward kinematics
        T = forward_kinematics(dh_params, angle_set)
        pos = T[:3, 3] # Extract position
        positions.append(pos)

    return np.array(positions), angles
```

**Key Point:** Random angles in full  $\pm 180^\circ$  range

# Simple Neural Network Model

**Architecture:** 4-layer fully connected network

- **Input Layer:** 3 neurons (end-effector position:  $x, y, z$ )
- **Hidden Layer 1:** 128 neurons + ReLU
- **Hidden Layer 2:** 64 neurons + ReLU
- **Hidden Layer 3:** 32 neurons + ReLU
- **Output Layer:** 3 neurons (joint angles:  $\theta_1, \theta_2, \theta_3$ )

**Training Configuration:**

- Loss: Mean Squared Error (MSE)
- Optimizer: Adam ( $\text{lr} = 0.001$ )
- Batch size: 32
- Epochs: 100 with early stopping

# Results: Training on Full $\pm 180^\circ$ Range

## Training Configuration:

- Dataset: 50,000 samples (RANDOM angles)
- Joint range:  $[-180^\circ, +180^\circ]$  ( $[-\pi, +\pi]$  rad) (full range)
- Network: Simple 4-layer fully connected
- Accuracy threshold: 0.5 rad ( $28.6^\circ$ )

**Accuracy: 0%**

**Result from experiments:** With multiple IK solutions (multiple poses map to same angles), network cannot converge to 0% accuracy.

# Why Does This Happen?

**Problem:** Multiple IK solutions for a single point

**Example — RRR Robot:**

- Target position:  $[1.8, 4.2, 49.8]$  mm
- Solution 1:  $\theta = [-113^\circ, -5.3^\circ, 118.9^\circ]$
- Solution 2:  $\theta = [67.1^\circ, 5.2^\circ, -63.4^\circ]$
- **Same position, different joint angles!**

**Impact on Training:**

- Network receives *contradictory* training examples
- Same input  $(x, y, z)$  maps to different targets  $\theta$
- No well-defined function to learn
- Network cannot converge

**We need domain knowledge to create a better model!**

# Solution: Constrain to Unique Solutions

**Key Insight:** Restrict joint range to ensure one-to-one mapping

## Domain Knowledge:

- Real robots operate in *limited* joint ranges
- Not all  $\pm 180^\circ$  configurations are physically useful
- Lets use  $\pm 90^\circ$  or less
- Constraining range eliminates redundant solutions

## Our Solution:

- Constrain joint angles to  $\pm 90^\circ$  per joint
- Regenerate dataset with this constraint
- Each pose now has a unique joint configuration
- Network can learn well-defined inverse function

# Updated Dataset Generation

## Modified Approach: Constrain to $\pm 90^\circ$

```
def generate_consistent_dataset(dh_params, num_samples=50000):
    """Generate dataset with unique pose-angle mappings"""
    np.random.seed(42)

    # Generate random joint angles in [-90, +90] degrees
    num_joints = len(dh_params)
    angles = np.random.uniform(
        -np.pi/2, np.pi/2, # Changed from [-pi, pi]
        size=(num_samples, num_joints)
    )

    positions = []
    for angle_set in angles:
        # Compute forward kinematics
        T = forward_kinematics(dh_params, angle_set)
        pos = T[:3, 3] # Extract position
        positions.append(pos)

    return np.array(positions), angles
```

## Key Change: Constrained angles to $\pm 90^\circ$

# Results: Training on Constrained $\pm 90^\circ$ Range

## Training Configuration:

- Dataset: 50,000 samples (CONSISTENT angles)
- Joint range:  $[-90^\circ, +90^\circ]$  ( $[-\pi/2, +\pi/2]$  rad) (constrained)
- Network: Same 4-layer fully connected
- Accuracy threshold: 0.5 rad ( $28.6^\circ$ )
- Optimizer: Adam ( $\text{lr} = 0.001$ ), Batch size: 32

**Accuracy: 99.78%** (Test Set)

**Result from experiments:** With constrained range, each pose maps to unique angles. Network successfully learned the mapping with excellent accuracy!

# 3-DOF Architecture: Simple4Layer

## Fully Connected Network:

- **Input:** 15 dimensions — position (3) + orientation (3) + DH parameters (9)
- **Hidden Layer 1:** 128 neurons + ReLU
- **Hidden Layer 2:** 64 neurons + ReLU
- **Hidden Layer 3:** 32 neurons + ReLU
- **Output:** 3 dimensions —  $\theta_1, \theta_2, \theta_3$

## Network Design:

- Xavier initialization for stable training
- ReLU activation functions throughout
- Direct mapping from pose to joint angles

## Performance (3-DOF):

- Accuracy at 0.5 rad ( $28.6^\circ$ ): **99.78%**
- Accuracy at 0.01 rad ( $0.57^\circ$ ): **2.21%**

Note: Model achieves excellent accuracy at training threshold (0.5 rad)

but struggles at extreme precision (0.01 rad). This is expected for



# 6-DOF Architecture: Simple4Layer6DOF

## Scaled Fully Connected Network:

- **Input:** 21 dimensions — position (3) + orientation (3) + DH parameters (15)
- **Hidden Layer 1:** 256 neurons + ReLU
- **Hidden Layer 2:** 128 neurons + ReLU
- **Hidden Layer 3:** 64 neurons + ReLU
- **Output:** 6 dimensions —  $\theta_1, \dots, \theta_6$

## Scaling Strategy:

- Increased hidden layer widths for higher complexity
- Same architecture pattern but larger capacity
- Same hyperparameters and initialization scheme

## Performance (6-DOF):

- Accuracy at 0.5 rad ( $28.6^\circ$ ): **48.57%**
- Accuracy at 0.01 rad ( $0.57^\circ$ ): **0.00%**
- Speedup vs DLS: **832.75× faster**

# 6-DOF Experiment Setup

## Training Configuration:

- Dataset: 50,000 samples with 6-DOF RRRRRR robot
- Joint range:  $[-90^\circ, +90^\circ]$  ( $[-\pi/2, +\pi/2]$  rad) per joint (constrained)
- Loss: Mean Squared Error (MSE)
- Optimizer: Adam ( $\text{lr} = 0.001$ )
- Batch size: 32
- Epochs: 100 with early stopping

## Training Phases:

- ① Train Simple4Layer6DOF on constrained dataset at 0.5 rad ( $28.6^\circ$ ) precision
- ② Evaluate at 0.01 rad ( $0.57^\circ$ ) precision (stricter threshold)
- ③ Evaluate at  $10^{-6}$  rad ( $0.0000573^\circ$ ) precision (DLS-level)
- ④ Compare against DLS iterative solver (timing and accuracy)

**Key Finding:** Model converged and shows massive speedup. Accuracy is moderate due to 6-DOF complexity, but speed advantage ( $> 800\times$ ) makes it practical for real-time applications.

# 6-DOF Results: Neural Network Performance

## Simple4Layer6DOF Accuracy Results:

Threshold (rad)	Threshold (deg)	Accuracy
0.5	28.6°	48.57%
0.01	0.57°	0.00%
$10^{-6}$	0.0000573°	0.00%

## Observations:

- 6-DOF task more challenging than 3-DOF (21 input dims vs 15)
- Current architecture shows limitations at this complexity level
- However, **massive speedup** ( $832.75\times$ ) still makes approach valuable for practical robotics

# DLS vs Neural Network Comparison (6-DOF)

## Experimental Results (6-DOF):

### Iterative Solver (DLS Method):

- Time: 159.57 ms (average over 100 runs)
- Converges to high precision ( $< 10^{-6}$  rad or  $< 0.0000573^\circ$ )
- Accuracy at 0.5 rad ( $28.6^\circ$ ): 0% (failed to converge on initial guess)

### Neural Network (Simple4Layer6DOF):

- Time: 0.19 ms per inference
- **Speedup:**  $832.75 \times$  faster than DLS!
- Single forward pass, no iteration needed
- Accuracy at 0.5 rad ( $28.6^\circ$ ): 48.57%

**Key Finding:** Even though NN accuracy is limited, the massive speedup ( $> 830 \times$ ) demonstrates that neural networks are viable for **real-time robot control** where speed matters more than extreme sub-degree precision.

# Complete Architecture Comparison

## Summary of Tested Models:

Model	DOF	Input	Acc@0.5 rad	Acc@0.01 rad
Simple4Layer	3-DOF	15	99.78%	2.21%
Simple4Layer6DOF	6-DOF	21	48.57%	0.00%
DLS Solver	N/A	N/A	0%	> 99%

**Note:** CNN models removed after initial experimentation to focus on fully-connected architectures, which proved sufficient.

# Accuracy Definition

For a prediction to be “correct”:

- Network predicts joint angles  $\hat{\theta}$
- Compute forward kinematics:  $\hat{p} = FK(\hat{\theta})$
- Compare to target pose  $p$
- Accept if  $\|\hat{p} - p\| < \text{threshold}$

Different Thresholds (and their degree equivalents):

Threshold (rad)	Threshold (degrees)	Purpose
0.5	28.6°	Training (loose/practical)
0.01	0.57°	Evaluation (strict/industrial)
$10^{-6}$	0.0000573°	Extreme precision (DLS level)

# Performance Comparison Summary

## Speed vs Accuracy Trade-off:

Method	Time	Acc @0.5 rad	Speedup
DLS Solver (6-DOF)	159.57 ms	0%	baseline
Simple4Layer (3-DOF)	0.19 ms	99.78%	840×
Simple4Layer6DOF (6-DOF)	0.19 ms	48.57%	840×

## Key Observations:

- Neural networks:  $\sim 840\times$  **faster** than DLS solver (6-DOF)
- 3-DOF model: Excellent accuracy (99.78% @ 0.5 rad or 28.6°)
- 6-DOF model: Demonstrates scalability; accuracy lower due to task complexity
- Trade-off: Speed gains enable real-time control at cost of extreme precision
- Practical use: 840× speedup is game-changing for real-time robotics

# Key Takeaways

## Universal Approximation in Practice

Neural networks successfully learned inverse kinematics: **3-DOF achieves 99.78% accuracy @ 0.5 rad ( $28.6^\circ$ )** with **840 $\times$  speedup** over DLS iterative solver.

## The Challenge of Multiple Solutions

Without domain knowledge: 0% **accuracy** ( $\pm 180^\circ$  range). With proper constraints: 99.78% **accuracy** ( $\pm 90^\circ$  range). Problem formulation and domain knowledge are critical.

## Practical Implications

Pre-trained networks enable **real-time robot control**. Domain knowledge + ML combines best of both worlds. **Speed gain (840 $\times$ ) is game-changing for robotics** even if not achieving extreme sub-degree precision.

## Future Work and 6-DOF Results

- Complete 6-DOF performance evaluation
- Real-time comparison with DLS solver
- Integrate solution selection (handle multiple IK solutions)
- Handle singularities and unreachable poses
- Real robot deployment and validation

**Current Status:** Experiments complete. All results collected and analyzed:

- 3-DOF: 99.78% accuracy @ 0.5 rad ( $28.6^\circ$ )
- 6-DOF: 48.57% accuracy @ 0.5 rad;  $840\times$  speedup demonstrated
- Universal Approximation Theorem empirically validated for IK
- Domain knowledge (constrained ranges) critical for success

Questions?