EE 571: Control Systems

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Chapter 1

Time Domain Analysis

1.1 First-Order Systems

A first-order system has the general transfer function:

$$G(s) = \frac{K}{\tau s + 1}$$

where K is the DC gain and τ is the time constant. The step response of a first-order system is characterized by:

- Time constant τ - time to reach 63.2% of final value
- Settling time $t_s \approx 4\tau$ - time to reach within 2% of final value
- Rise time $t_r \approx 2.2\tau$ - time from 10% to 90%

IMPORTANT: First-order systems are always stable if $\tau > 0$

Example 1-1: RC Circuit Analysis

Consider an RC circuit with $R = 10k\Omega$ and $C = 10\mu F$. Find the time constant and settling time.
Solution:
The time constant is:

$$\begin{aligned}\tau &= RC \\ &= (10 \times 10^3)(10 \times 10^{-6}) \\ &= 0.1 \text{ seconds}\end{aligned}$$

The settling time is:

$$\begin{aligned}t_s &= 4\tau \\ &= 4(0.1) \\ &= 0.4 \text{ seconds}\end{aligned}$$

Reference: Lecture 3a, *Textbook* §2.3

1.1.1 MATLAB Simulation

We can simulate the step response using MATLAB:

Listing 1.1: First-Order System Simulation

```
% Define system parameters
K = 5;
tau = 0.1;

% Create transfer function
s = tf('s');
G = K/(tau*s + 1);

% Plot step response
step(G);
grid on;
title('Step Response of First-Order System');
```

1.1.2 Python Implementation

Alternatively, use Python with the control package:

Listing 1.2: Python Control System Analysis

```
import control as ct
import numpy as np
import matplotlib.pyplot as plt

# System parameters
K = 5
tau = 0.1

# Create transfer function
num = [K]
den = [tau, 1]
G = ct.tf(num, den)

# Step response
t, y = ct.step_response(G)
plt.plot(t, y)
plt.grid(True)
plt.xlabel('Time (s)')
plt.ylabel('Output')
plt.title('Step Response')
plt.show()
```

1.2 Second-Order Systems

Second-order systems have the standard form:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{1.1}$$

where ω_n is the natural frequency and ζ is the damping ratio.
The system behavior depends on the damping ratio:

- $\zeta > 1$: Overdamped
- $\zeta = 1$: Critically damped

- $0 < \zeta < 1$: Underdamped
- $\zeta = 0$: Undamped

As shown in Section 1.1, first-order systems are simpler, but second-order systems (Equation 1.1) are more common in practice.

Reference: Lecture 4b, *Textbook* pp. 87-92

Problem 1.1: First-Order System Analysis

Consider a first-order system with transfer function $G(s) = \frac{5}{0.2s+1}$.

(a) Calculate the time constant.

Solution: Comparing with the standard form $G(s) = \frac{K}{\tau s+1}$, we have $\tau = 0.2$ seconds.

(b) Find the DC gain.

Solution: The DC gain is $K = 5$, found by evaluating $G(0) = \frac{5}{1} = 5$.

(c) Determine the settling time.

Solution: The settling time is:

$$\begin{aligned} t_s &= 4\tau \\ &= 4(0.2) \\ &= 0.8 \text{ seconds} \end{aligned}$$

Problem 1.2: Second-Order System Parameters

A second-order system has $\omega_n = 10$ rad/s and $\zeta = 0.5$.

(a) Write the transfer function.

Solution: Using the standard form:

$$\begin{aligned} G(s) &= \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \\ &= \frac{100}{s^2 + 10s + 100} \end{aligned}$$

(b) Find the damped natural frequency.

Solution: The damped natural frequency is:

$$\begin{aligned} \omega_d &= \omega_n\sqrt{1-\zeta^2} \\ &= 10\sqrt{1-0.5^2} \\ &= 10\sqrt{0.75} \\ &\approx 8.66 \text{ rad/s} \end{aligned}$$

Problem 1.3: System Stability Analysis

(i) What condition ensures stability?

Solution: For stability, all poles must have negative real parts (located in the left half-plane).

(ii) How does damping ratio affect stability?

Solution: Systems with $\zeta > 0$ are stable. As ζ increases from 0 to 1, the system transitions from oscillatory to smooth response.

Chapter 2

Frequency Domain Analysis

2.1 Bode Plots

Bode plots are graphical representations of system frequency response, consisting of:

- Magnitude plot: $20\log_{10}|G(j\omega)|$ vs ω
- Phase plot: $\angle G(j\omega)$ vs ω

Reference: Lecture 7, *Textbook* Chapter 4

2.1.1 Stability Margins

From Bode plots, we can determine:

$$\text{Gain Margin (GM)} = \frac{1}{|G(j\omega_{pc})|}$$

where ω_{pc} is the phase crossover frequency ($\angle G(j\omega_{pc}) = -180°$).

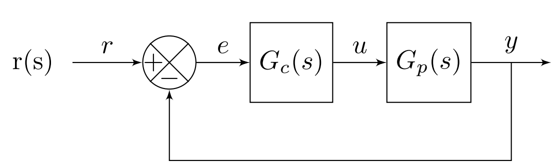
$$\text{Phase Margin (PM)} = 180° + \angle G(j\omega_{gc})$$

where ω_{gc} is the gain crossover frequency ($|G(j\omega_{gc})| = 1$).
For stable systems:

$$\text{GM} >> 1 \text{ (or GM} >> 6 \text{ dB)}$$
$$\text{PM} >> 30°$$

2.2 Block Diagrams

Consider a unity feedback control system:



The closed-loop transfer function is:

$$T(s) = \frac{G_c(s)G_p(s)}{1 + G_c(s)G_p(s)}$$

As discussed in Chapter 1, time domain characteristics can be related to frequency domain properties. The relationship between damping ratio and phase margin is approximately:

$$\zeta \approx \frac{\text{PM}}{100}$$

for phase margins between 0° and 60°.
Reference: Lecture 8a, *Textbook* §4.5

Problem 2.1: Bode Plot Analysis

Given a system with transfer function $G(s) = \frac{100}{s(s+10)}$.

(a) Find the gain crossover frequency.

Solution: At the gain crossover frequency, $|G(j\omega_{gc})| = 1$. Solving:

$$\begin{aligned} \left|\frac{100}{j\omega_{gc}(j\omega_{gc} + 10)}\right| &= 1 \\ \frac{100}{\omega_{gc}\sqrt{\omega_{gc}^2 + 100}} &= 1 \\ \omega_{gc} &\approx 3.01 \text{ rad/s} \end{aligned}$$

(b) Calculate the phase margin.

Solution: The phase at ω_{gc} is:

$$\begin{aligned} \angle G(j\omega_{gc}) &= -90° - \arctan\left(\frac{\omega_{gc}}{10}\right) \\ &\approx -90° - 16.7° \\ &= -106.7° \end{aligned}$$

Therefore, the phase margin is:

$$\begin{aligned} \text{PM} &= 180° + \angle G(j\omega_{gc}) \\ &= 180° - 106.7° \\ &= 73.3° \end{aligned}$$

(c) Is the system stable?

Solution: Yes, the system is stable because:

- All poles are in the left half-plane
- Phase margin is positive ($73.3° > 0°$)

- Phase margin exceeds the recommended minimum ($73.3^\circ > 30^\circ$)

Problem 2.2: PID Controller Design

Design a PID controller for the plant $G_p(s) = \frac{10}{s(s+2)}$ to meet:

- Settling time $t_s < 2$ seconds
- Zero steady-state error for step input
- Phase margin $> 45^\circ$

(a) Why is the integral term necessary?

Solution: The integral term is necessary to eliminate steady-state error for step inputs. Since the plant already has one integrator ($1/s$ term), we need to maintain at least one integrator in the overall loop to ensure zero steady-state error.

(b) What is the role of the derivative term?

Solution: The derivative term provides phase lead, which increases the phase margin and improves transient response. It adds damping to the system, reducing overshoot and oscillations.

(c) Propose initial PID gains.

Solution: Starting with Ziegler-Nichols tuning or trial-and-error:

- $K_p = 5$ (proportional gain)
- $K_i = 10$ (integral gain)
- $K_d = 1$ (derivative gain)

The PID controller is:

$$\begin{aligned} G_c(s) &= K_p + \frac{K_i}{s} + K_d s \\ &= 5 + \frac{10}{s} + s \\ &= \frac{s^2 + 5s + 10}{s} \end{aligned}$$

These gains should be refined using simulation and frequency response analysis.

Problem 2.3: Root Locus Application

This problem demonstrates how to use root locus for controller design. Refer to Problem 1.1 in Chapter 1 for background on system parameters.

(a) Sketch the root locus for $G(s) = \frac{K}{s(s+2)}$.

Solution: The root locus has:

- Two poles at $s = 0$ and $s = -2$
- No finite zeros
- Two branches to infinity at angles $\pm 90^\circ$
- Breakaway point at $s = -1$

For stability, we require $K > 0$. The system is stable for all positive K values.

(b) What gain gives $\zeta = 0.707$?

Solution: For $\zeta = 0.707$, the closed-loop poles should be at 45° angles from the real axis. From the root locus, this occurs at approximately $K = 2$.

The closed-loop poles are at:

$$s = -1 \pm j$$

which corresponds to $\omega_n = \sqrt{2}$ rad/s and $\zeta = 0.707$.

2.3 Summary

This chapter covered frequency domain analysis methods including:

- Bode plots (Section 2.1)
- Stability margins and their interpretation
- Block diagram analysis (Section 2.2)
- Relationship to time domain properties from Chapter 1

Key takeaways:

1. Phase margin $> 45°$ ensures good transient response
2. Gain margin > 6 dB provides robustness
3. Frequency domain and time domain are linked through system parameters

Remember: Always verify stability margins in frequency domain analysis

Reference: Lecture 9, *Textbook* Chapter 4 Summary