Green's function in many body physics, using contour formalism

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This is largely based on the excellent textbook: *Nonequilibrium Many-body Theory of Quantum Systems: A Modern Introduction*, by Gianluca Stefanucci and Robert van Leeuwen.

Contents

1 Time-dependent Schrödinger equation and evolution operators

We aim to find a solution for the time-dependent Schrödinger equation.

$$\mathrm{i}\frac{\partial}{\partial t}|\Psi(t)\rangle = \hat{H}(t)\,|\Psi(t)\rangle$$

For time-independent Hamitonian $\hat{H}(t) = \hat{H}(t_0)$ (for all t), we have

$$|\Psi(t)\rangle = e^{-\mathrm{i}\hat{H}(t_0)(t-t_0)}|\Psi(t_0)\rangle$$

What we are actually looking for, is the evolution operator.

- The evolution operator $\hat{U}(t, t_0)$ is defined as:

  $$|\Psi(t)\rangle = \hat{U}(t, t_0)\,|\Psi(t_0)\rangle$$

  where $\hat{U}(t, t_0)$ is unitary.

- If $t > t_0$, $\hat{H}$ is piecewise constant, i.e. for $t_0 < t_1 < \cdots < t_n < t_{n+1} = t$

  $$\hat{H}(\tau) = \hat{H}(t_p) \text{ for } t_p < \tau < t_{p+1},$$

  Then we have

  $$\begin{aligned}|\Psi(t)\rangle &= \mathrm{e}^{-\mathrm{i}\hat{H}(t_n)(t-t_n)}\,|\Psi(t_n)\rangle = \mathrm{e}^{-\mathrm{i}\hat{H}(t_n)(t-t_n)}\mathrm{e}^{-\mathrm{i}\hat{H}(t_{n-1})(t_n-t_{n-1})}\,|\Psi(t_{n-1})\rangle \\ &= \mathrm{e}^{-\mathrm{i}\hat{H}(t_n)(t-t_n)}\mathrm{e}^{-\mathrm{i}\hat{H}(t_{n-1})(t_n-t_{n-1})}\cdots\mathrm{e}^{-\mathrm{i}\hat{H}(t_0)(t_1-t_0)}\,|\Psi(t_0)\rangle\end{aligned}$$

  For simplicity, let's assume for now all $t_{p+1} - t_p = \Delta t$.

- Now we are ready to introduce time-ordering operator $\hat{T}$. $\hat{T}$ acts on the product of m Hamitonians of different time in the following way:

  $$\hat{T}\left(\hat{H}\left(t_1\right)\cdots\hat{H}\left(t_m\right)\right) = \hat{H}\left(t_{P(1)}\right)\cdots\hat{H}\left(t_{P(m)}\right)$$

  where $t_1, \cdots, t_m$ are rearranged in the increasing order: $t_{P(1)} \geq \cdots \geq t_{P(m)}$.

- Why do we introduce the time-ordering operator? It is because we want to write

  $$\mathrm{e}^{-\mathrm{i}\hat{H}(t_n)\Delta t}\mathrm{e}^{-\mathrm{i}\hat{H}(t_{n-1})\Delta t)}\cdots\mathrm{e}^{-\mathrm{i}\hat{H}(t_0)\Delta t}$$

in a more compact way. However, $\hat{H}(t_i)$ and $\hat{H}(t_j)$ probably doesn't commute, therefore

$$\mathrm{e}^{-\mathrm{i}\hat{H}(t_n)\Delta t}\mathrm{e}^{-\mathrm{i}\hat{H}(t_{n-1})\Delta t)}\cdots\mathrm{e}^{-\mathrm{i}\hat{H}(t_0)\Delta t}\neq\mathrm{e}^{-\mathrm{i}\Delta t\left(\hat{H}(t_n)+\hat{H}(t_{n-1})+\cdots+\hat{H}(t_0)\right)}$$

But with time ordering operators, we actually have:

$$\mathrm{e}^{-\mathrm{i}\hat{H}(t_n)\Delta t}\mathrm{e}^{-\mathrm{i}\hat{H}(t_{n-1})\Delta t)}\cdots\mathrm{e}^{-\mathrm{i}\hat{H}(t_0)\Delta t}=\hat{T}\left(\mathrm{e}^{-\mathrm{i}\Delta t\left(\hat{H}(t_n)+\hat{H}(t_{n-1})+\cdots+\hat{H}(t_0)\right)}\right)$$

You can verify this through Taylor expansion, but another straightforward way to see this is that under $\hat{T}$, $\hat{H}(t_i)$ and $\hat{H}(t_j)$ does commute.

- In one word, for pieceswise constant Hamiltonian, we have

$$|\Psi(t)\rangle=\hat{T}\left(\mathrm{e}^{-\mathrm{i}\Delta t\left(\sum_p\hat{H}(t_p)\right)}\right)|\Psi(t_0)\rangle$$

- In the spirit of Riemann integral, for general time-dependent Hamiltonian, we should have

$$\begin{aligned}|\Psi(t)\rangle&=\lim_{\Delta t\to 0}\hat{T}\left(\mathrm{e}^{-\mathrm{i}\Delta t\left(\sum_p\hat{H}(t_p)\right)}\right)|\Psi(t_0)\rangle\\&=\hat{T}\left(\mathrm{e}^{-\mathrm{i}\int_{t_0}^{t}\mathrm{d}\tau\hat{H}(\tau)}\right)|\Psi(t_0)\rangle\end{aligned}$$

The time evolution operator is, for $t>t_0$

$$\hat{U}(t,t_0)=\hat{T}\left(\mathrm{e}^{-\mathrm{i}\int_{t_0}^{t}\mathrm{d}\tau\hat{H}(\tau)}\right)$$

- The time evolution operator satisfies the time-dependent Schrödinger equation itself:

$$\mathrm{i}\frac{\partial}{\partial t}\hat{U}(t,t_0)=\hat{H}(t)\hat{U}(t,t_0),\quad\hat{U}(t_0,t_0)=\mathbf{1}$$

From this we can also derive the formalism for time evolution opeartor above. We have

$$\hat{U}(t,t_0)=\mathbf{1}-\mathrm{i}\int_{t_0}^{t}\mathrm{d}t_1\hat{H}(t_1)\hat{U}(t_1,t_0)$$

We can substitute $\hat{U}$ back into itself again and again, and get:

$$\begin{aligned}\hat{U}(t,t_0)&=\mathbf{1}-\mathrm{i}\int_{t_0}^{t}\mathrm{d}t_1\hat{H}(t_1)+(-\mathrm{i})^2\int_{t_0}^{t}\mathrm{d}t_1\int_{t_0}^{t_1}\mathrm{d}t_2\hat{H}(t_1)\hat{H}(t_2)\hat{U}(t_2,t_0)\\&=\sum_{k=0}^{\infty}(-\mathrm{i})^k\int_{t_0}^{t}\mathrm{d}t_1\int_{t_0}^{t_1}\mathrm{d}t_2\cdots\int_{t_0}^{t_{k-1}}\mathrm{d}t_k\hat{H}(t_1)\hat{H}(t_2)\cdots\hat{H}(t_k)\end{aligned}$$

Note that

$$\int_{t_0}^{t}\mathrm{d}t_1\int_{t_0}^{t_1}\mathrm{d}t_2\cdots\int_{t_0}^{t_{k-1}}\mathrm{d}t_k\hat{H}(t_1)\hat{H}(t_2)\cdots\hat{H}(t_k)=\frac{1}{k!}\int_{t_0}^{t}\mathrm{d}t_1\int_{t_0}^{t}\mathrm{d}t_2\cdots\int_{t_0}^{t}\mathrm{d}t_k\hat{T}\left(\hat{H}(t_1)\hat{H}(t_2)\cdots\hat{H}(t_k)\right)$$

Therefore

$$\begin{aligned}\hat{U}(t,t_0)&=\sum_{k=0}^{\infty}\frac{(-\mathrm{i})^k}{k!}\int_{t_0}^{t}\mathrm{d}t_1\int_{t_0}^{t}\mathrm{d}t_2\cdots\int_{t_0}^{t}\mathrm{d}t_k\hat{T}\left(\hat{H}(t_1)\hat{H}(t_2)\cdots\hat{H}(t_k)\right)\\&=\hat{T}\left(\mathrm{e}^{-\mathrm{i}\int_{t_0}^{t}\mathrm{d}\tau\hat{H}(\tau)}\right)\end{aligned}$$

- Similarly, we can prove, for $t < t_0$, we have

$$\hat{U}(t, t_0) = \bar{T}\left(\mathrm{e}^{\mathrm{i}\int_t^{t_0}\mathrm{d}\tau \hat{H}(\tau)}\right)$$

where $\bar{T}$ is the anti-chronological time ordering operator, which reorders the operators with time increasing from the left to the right.

2 Operators in the Heisenberg picture

- With the evolution operator $\hat{U}(t, t_0)$,

$$|\Psi(t)\rangle = \hat{U}(t, t_0)|\Psi_0\rangle$$

- For operator $\hat{O}(t)$, the observation expection is

$$\left\langle \Psi(t) \left| \hat{O}(t) \right| \Psi(t) \right\rangle = \left\langle \Psi_0 \left| \hat{U}(t_0, t)\hat{O}(t)\hat{U}(t, t_0) \right| \Psi_0 \right\rangle$$

Therefore we define the following operator $\hat{O}_H$, which is called operator in the Heisenberg representation:

$$\hat{O}_H(t) = \hat{U}(t_0, t)\hat{O}(t)\hat{U}(t, t_0)$$

- For example, for density operator $\hat{n}(\boldsymbol{x}) = \hat{\psi}^\dagger(\boldsymbol{x})\hat{\psi}(\boldsymbol{x})$, we have $\hat{n}_H(\boldsymbol{x}, t)$ as followed:

$$\begin{aligned}\hat{n}_H(\boldsymbol{x}, t) &= \hat{U}(t_0, t)\hat{\psi}^\dagger(\boldsymbol{x})\hat{\psi}(\boldsymbol{x})\hat{U}(t, t_0) = \hat{U}(t_0, t)\hat{\psi}^\dagger(\boldsymbol{x})\hat{U}(t, t_0)\hat{U}(t_0, t)\hat{\psi}(\boldsymbol{x})\hat{U}(t, t_0) \\ &= \hat{\psi}_H^\dagger(\boldsymbol{x})\hat{\psi}_H(\boldsymbol{x})\end{aligned}$$

- Basic property: if $C = AB$, then $C_H = A_H B_H$; $[A_H, B_H] = ([A, B])_H$.
- (Anti-)commutation relation:

$$\left[\hat{\psi}(\boldsymbol{x}, t), \hat{\psi}^\dagger(\boldsymbol{x}', t)\right] = \delta(\boldsymbol{x} - \boldsymbol{x}')$$

- With some calculation, we have the equation of motion:

$$\mathrm{i}\frac{\partial}{\partial t}\hat{O}_H(t) = [\hat{O}_H(t), \hat{H}_H(t)]_- + \mathrm{i}\left(\frac{\partial \hat{O}}{\partial t}\right)_H(t)$$

- We consider the following Hamiltonian $\hat{H}(t)$

$$\begin{aligned}\hat{H}(t) &= \int \mathrm{d}\boldsymbol{x}\hat{\psi}^\dagger(\boldsymbol{x})\hat{h}(\boldsymbol{x}, t)\hat{\psi}(\boldsymbol{x}) + \frac{1}{2}\int \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{x}'\hat{\psi}^\dagger(\boldsymbol{x})\hat{\psi}^\dagger(\boldsymbol{x}')v(\boldsymbol{x}, \boldsymbol{x}')\hat{\psi}^\dagger(\boldsymbol{x}')\hat{\psi}^\dagger(\boldsymbol{x}) \\ &= H_0 + H_{\mathrm{int}}\end{aligned}$$

- With this Hamiltonian, we can figure out the equation of motion for field oprator. We need to calculate:

$$
\begin{aligned}
\left[\hat{\psi}(\boldsymbol{x}), \hat{\psi}^\dagger(\boldsymbol{x}')\hat{\psi}(\boldsymbol{x}')\right] &= \hat{\psi}(\boldsymbol{x})\hat{\psi}^\dagger(\boldsymbol{x}')\hat{\psi}(\boldsymbol{x}') - \hat{\psi}^\dagger(\boldsymbol{x}')\hat{\psi}(\boldsymbol{x}')\hat{\psi}(\boldsymbol{x}) \\
&= \hat{\psi}(\boldsymbol{x})\hat{\psi}^\dagger(\boldsymbol{x}')\hat{\psi}(\boldsymbol{x}') \mp \hat{\psi}^\dagger(\boldsymbol{x}')\hat{\psi}(\boldsymbol{x})\hat{\psi}(\boldsymbol{x}') = \hat{\psi}(\boldsymbol{x}')\delta(\boldsymbol{x}-\boldsymbol{x}')
\end{aligned}
$$

$$
\begin{aligned}
\left[\hat{\psi}^\dagger(\boldsymbol{x}), \hat{\psi}^\dagger(\boldsymbol{x}')\hat{\psi}(\boldsymbol{x}')\right] &= \hat{\psi}^\dagger(\boldsymbol{x})\hat{\psi}^\dagger(\boldsymbol{x}')\hat{\psi}(\boldsymbol{x}') - \hat{\psi}^\dagger(\boldsymbol{x}')\hat{\psi}(\boldsymbol{x}')\hat{\psi}^\dagger(\boldsymbol{x}) \\
&= \pm\hat{\psi}^\dagger(\boldsymbol{x}')\hat{\psi}^\dagger(\boldsymbol{x})\hat{\psi}(\boldsymbol{x}') - \hat{\psi}^\dagger(\boldsymbol{x}')\hat{\psi}(\boldsymbol{x}')\hat{\psi}^\dagger(\boldsymbol{x}) = -\hat{\psi}^\dagger(\boldsymbol{x}')\delta(\boldsymbol{x}'-\boldsymbol{x})
\end{aligned}
$$

$$
\begin{aligned}
\left[\hat{\psi}(\boldsymbol{x}), \hat{\psi}^\dagger(\boldsymbol{x}')\hat{\psi}^\dagger(\boldsymbol{x}'')\hat{\psi}(\boldsymbol{x}'')\hat{\psi}(\boldsymbol{x}')\right] &= \hat{\psi}(\boldsymbol{x})\hat{\psi}^\dagger(\boldsymbol{x}')\hat{\psi}^\dagger(\boldsymbol{x}'')\hat{\psi}(\boldsymbol{x}'')\hat{\psi}(\boldsymbol{x}') - \hat{\psi}^\dagger(\boldsymbol{x}')\hat{\psi}^\dagger(\boldsymbol{x}'')\hat{\psi}(\boldsymbol{x}'')\hat{\psi}(\boldsymbol{x}')\hat{\psi}(\boldsymbol{x}) \\
&= \hat{\psi}(\boldsymbol{x})\hat{\psi}^\dagger(\boldsymbol{x}')\hat{\psi}^\dagger(\boldsymbol{x}'')\hat{\psi}(\boldsymbol{x}'')\hat{\psi}(\boldsymbol{x}') - \hat{\psi}^\dagger(\boldsymbol{x}')\hat{\psi}^\dagger(\boldsymbol{x}'')\hat{\psi}(\boldsymbol{x})\hat{\psi}(\boldsymbol{x}'')\hat{\psi}(\boldsymbol{x}') \\
&= \left(\hat{\psi}(\boldsymbol{x})\hat{\psi}^\dagger(\boldsymbol{x}')\hat{\psi}^\dagger(\boldsymbol{x}'') - \hat{\psi}^\dagger(\boldsymbol{x}')\hat{\psi}^\dagger(\boldsymbol{x}'')\hat{\psi}(\boldsymbol{x})\right)\hat{\psi}(\boldsymbol{x}'')\hat{\psi}(\boldsymbol{x}') \\
&= \left([\hat{\psi}(\boldsymbol{x}), \hat{\psi}^\dagger(\boldsymbol{x}')]_\mp\hat{\psi}^\dagger(\boldsymbol{x}'') \pm \hat{\psi}^\dagger(\boldsymbol{x}')[\hat{\psi}(\boldsymbol{x}), \hat{\psi}^\dagger(\boldsymbol{x}'')]_\mp\right)\hat{\psi}(\boldsymbol{x}'')\hat{\psi}(\boldsymbol{x}') \\
&= \left(\delta(\boldsymbol{x}-\boldsymbol{x}')\hat{\psi}^\dagger(\boldsymbol{x}'') \pm \hat{\psi}^\dagger(\boldsymbol{x}')\delta(\boldsymbol{x}-\boldsymbol{x}'')]\right)\hat{\psi}(\boldsymbol{x}'')\hat{\psi}(\boldsymbol{x}')
\end{aligned}
$$

  Then we have:

$$
\begin{aligned}
\mathrm{i}\frac{\partial}{\partial t}\hat{\psi}_H(\boldsymbol{x},t) &= \hat{h}(\boldsymbol{x},t)\hat{\psi}_H(\boldsymbol{x},t) + \int \mathrm{d}\boldsymbol{x}' v(\boldsymbol{x},\boldsymbol{x}')\hat{\psi}_H^\dagger(\boldsymbol{x}',t)\hat{\psi}_H(\boldsymbol{x}',t)\hat{\psi}_H(\boldsymbol{x},t) \\
\mathrm{i}\frac{\partial}{\partial t}\hat{\psi}_H^\dagger(\boldsymbol{x},t) &= -\hat{h}(\boldsymbol{x},t)\hat{\psi}_H^\dagger(\boldsymbol{x},t) - \int \mathrm{d}\boldsymbol{x}' v(\boldsymbol{x},\boldsymbol{x}')\hat{\psi}_H^\dagger(\boldsymbol{x},t)\hat{\psi}_H^\dagger(\boldsymbol{x}',t)\hat{\psi}_H(\boldsymbol{x}',t)
\end{aligned}
$$

3 Schwinger-Keldysh contour

We start by considering how to calculate the time-dependent quantum average of an operator $\hat{O}(t)$. Let $|\Psi(t_0)\rangle = |\Psi_0\rangle$, for $t > t_0$, we have:

$$\langle\hat{O}\rangle(t) = \langle\Psi(t)|\hat{O}(t)|\Psi(t)\rangle = \langle\Psi_0|\hat{U}(t_0,t)|\hat{O}(t)|\hat{U}(t,t_0)|\Psi_0\rangle$$

With the formula derived in the previous section, we have:

$$\langle\hat{O}\rangle(t) = \left\langle\Psi_0\middle|\bar{T}\left(\mathrm{e}^{-\mathrm{i}\int_t^{t_0}\mathrm{d}\tau\hat{H}(\tau)}\right)\middle|\hat{O}(t)\middle|T\left(\mathrm{e}^{-\mathrm{i}\int_{t_0}^{t}\mathrm{d}\tau\hat{H}(\tau)}\right)\middle|\Psi_0\right\rangle$$

If we do Taylor expansions, our job will be reduced to evaluate the integrals of the following forms:

$$\bar{T}\left(\hat{H}(t_1)\cdots\hat{H}(t_n)\right)\hat{O}(t)\hat{T}\left(\hat{H}(t_1')\cdots\hat{H}(t_m')\right)$$

in which t_i, t_i' is between t_0 and t.

There are two separate time-ordering operators in this formula. We want to use a general time-ordering operator $\mathcal{T}$ to simplify this formulation.

- We start by defining an oriented contour:

$$\gamma := (t_0, t) \oplus (t, t_0) = \gamma_- \oplus \gamma_+$$

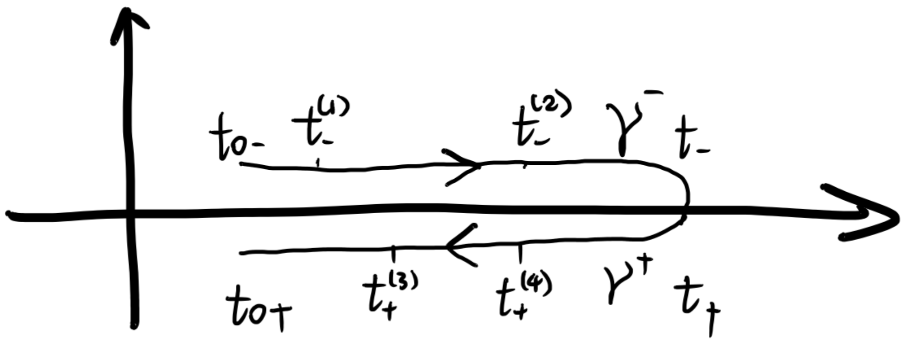

Figure 1: The oriented contour $\gamma = \gamma_- \oplus \gamma_+$

- γ_- is the forward branch. γ_+ is the backward branch.
- For a time $z \in \gamma$, if it's time t and it's on forward branch γ_-, we call it $z = t_-$, meaning that it's time t and it's used on the forward branch.
- For a time $z \in \gamma$, if it's time t and it's on backward branch γ_+, we call it $z = t_+$, meaning that it's time t and it's used on the backward branch.
- We have
$$\hat{H}(t_+) = \hat{H}(t_-) = \hat{H}(t), \quad \hat{O}(t_+) = \hat{O}(t_-) = \hat{O}(t)$$
- The oriented contour defines an ordering of time. For $t_-^{(1)}, t_-^{(2)}, t_+^{(3)}, t_+^{(4)}$, we have:
$$t_{0+} > t_+^{(3)} > t_+^{(4)} > t_+ = t_- > t_-^{(2)} > t_-^{(1)} > t_{0-}$$
- With this well-defined ordering, we can define a generic time-ordering operator $\mathcal{T}$, called the contour ordering operator, which operates as followed: for $z_1, \cdots, z_m \in \gamma$
$$\mathcal{T}\left(\hat{A}_1(z_1)\cdots\hat{A}_m(z_m)\right) = \hat{A}_{P(1)}(z_{P(1)})\cdots\hat{A}_{P(m)}(z_{P(m)})$$
where $z_{P(1)} > \cdots > z_{P(m)}$, which is the ordering defined by the contour.
- Therefore, we have
$$\begin{aligned}&\bar{T}\left(\hat{H}(t_1)\cdots\hat{H}(t_n)\right)\hat{O}(t)\hat{T}\left(\hat{H}(t_1')\cdots\hat{H}(t_m')\right)\\ =&\mathcal{T}\left(\hat{H}(t_{1+})\cdots\hat{H}(t_{n+})\hat{O}(t_\pm)\hat{H}(t'_{1-})\cdots\hat{H}(t'_{m-})\right)\end{aligned}$$
- Now we are ready to rewrite the quantum average in contour integral. We have
$$\begin{aligned}&\int_t^{t_0}\mathrm{d}t_1\cdots\int_t^{t_0}\mathrm{d}t_n\int_{t_0}^t\mathrm{d}t_1'\cdots\int_{t_0}^t\mathrm{d}t_m'\bar{T}\left(\hat{H}(t_1)\cdots\hat{H}(t_n)\right)\hat{O}(t)\hat{T}\left(\hat{H}(t_1')\cdots\hat{H}(t_m')\right)\\ =&\int_{\gamma_+}\mathrm{d}z_1\cdots\int_{\gamma_+}\mathrm{d}z_n\int_{\gamma_-}\mathrm{d}z_1'\cdots\int_{\gamma_-}\mathrm{d}z_m'\mathcal{T}\left(\hat{H}(z_1)\cdots\hat{H}(z_n)\hat{O}(t_\pm)\hat{H}(z_1')\cdots\hat{H}(z_m')\right)\end{aligned}$$

- Then, the original quantum average could be written in the form of

$$\langle O\rangle(t)=\left\langle \Psi_0 \middle| \mathcal{T}\left(\mathrm{e}^{-\mathrm{i}\int_{\gamma_+}\mathrm{d}z\hat{H}(z)}\hat{O}(t_\pm)\mathrm{e}^{-\mathrm{i}\int_{\gamma_-}\mathrm{d}z\hat{H}(z)}\right)\middle| \Psi_0\right\rangle$$

Note that all these operators in $\mathcal{T}$ commutes, therefore we have

$$\langle O\rangle(t)=\left\langle \Psi_0 \middle| \mathcal{T}\left(\mathrm{e}^{-\mathrm{i}\int_{\gamma}\mathrm{d}z\hat{H}(z)}\hat{O}(t_\pm)\right)\middle| \Psi_0\right\rangle$$

In fact, there is more that we can do. As γ here actually rely on t, we can replace γ with a new contour that doesn't rely on t. Let our new $\tilde{\gamma}$ be

$$\tilde{\gamma}=(t_0,+\infty)\oplus(+\infty,t_0)$$

Then, with this γ, we have

$$\mathcal{T}\left(\mathrm{e}^{-\mathrm{i}\int_{\tilde{\gamma}}\mathrm{d}z\hat{H}(z)}O(t_-)\right)=\hat{U}(t_0,\infty)\hat{U}(\infty,t)\hat{O}(t)\hat{U}(t,t_0)=\hat{U}(t_0,t)\hat{O}(t)\hat{U}(t,t_0)$$

$$\mathcal{T}\left(\mathrm{e}^{-\mathrm{i}\int_{\tilde{\gamma}}\mathrm{d}z\hat{H}(z)}O(t_+)\right)=\hat{U}(t_0,t)\hat{O}(t)\hat{U}(t,\infty)\hat{U}(\infty,t_0)=\hat{U}(t_0,t)\hat{O}(t)\hat{U}(t,t_0)$$

Then, we have the final result, which is very elegant:

$$\langle O\rangle(t)=\left\langle \Psi_0 \middle| \mathcal{T}\left(\mathrm{e}^{-\mathrm{i}\int_{\tilde{\gamma}}\mathrm{d}z\hat{H}(z)}\hat{O}(t_\pm)\right)\middle| \Psi_0\right\rangle$$

This $\tilde{\gamma}$ is refered to as Schwinger-Keldysh contour.

4 Konstantinov-Perel contour for ensemble averages

Now we turn to the ensemble averages, which is probably a mixed state, rather than a pure state in the previous section.

The setup of the problem is as followed:

- For Hamiltonian $\hat{H}^M$, (M is credited to Matsubra), let's assume its eigenpairs being $(E_k^M, |\Psi_k\rangle)$. The Hamiltonian could be reconstructed as

$$\hat{H}=\sum_k E_k^M\left|\Psi_k\right\rangle\left\langle\Psi_k\right|$$

- The density matrix, $\hat{\rho}$, is defined as

$$\hat{\rho}=\frac{1}{Z}\mathrm{e}^{-\beta H}=\frac{1}{Z}\sum_k \mathrm{e}^{-\beta E_k^M}\left|\Psi_k\right\rangle\left\langle\Psi_k\right|$$

where Z is set to be $\mathrm{Tr}(\mathrm{e}^{-\beta H})=\sum_k \mathrm{e}^{-\beta E_k^M}$ to make sure $\mathrm{Tr}(\hat{\rho})=1$. Here β is

$$\beta=\frac{1}{k_B T}$$

- With density matrix $\hat{\rho}$, the ensemble average of an operator $\hat{O}$ is

$$\langle \hat{O} \rangle = \mathrm{Tr}(\hat{\rho}\hat{O})$$

- For an ensemble in thermodynamic equilibrium, described by temperature T and chemical potential μ, the Hamlitonian $\hat{H}^M$ is

$$\hat{H}^M = \hat{H} - \mu \hat{N}$$

- In the previous section, we have the result

$$\hat{U}(t_0, t)\hat{O}(t)\hat{U}(t, t_0) = \mathcal{T}\left(\mathrm{e}^{-\mathrm{i}\int_{\tilde{\gamma}} \mathrm{d}z \hat{H}(z)} \hat{O}(t_\pm)\right)$$

Then the ensemble average is

$$\begin{aligned}\langle \hat{O} \rangle(t) &= \left\langle \mathrm{Tr}(\hat{U}(t_0, t)\hat{O}(t)\hat{U}(t, t_0)) \right\rangle = \mathrm{Tr}\left(\hat{\rho}\mathcal{T}\left(\mathrm{e}^{-\mathrm{i}\int_{\tilde{\gamma}} \mathrm{d}z \hat{H}(z)} \hat{O}(t_\pm)\right)\right) \\ &= \frac{\mathrm{Tr}\left(\mathrm{e}^{-\beta H^M}\mathcal{T}\left(\mathrm{e}^{-\mathrm{i}\int_{\tilde{\gamma}} \mathrm{d}z \hat{H}(z)} \hat{O}(t_\pm)\right)\right)}{\mathrm{Tr}\left(\mathrm{e}^{-\beta H^M}\right)}\end{aligned}$$

This is partly written using the contour formalism.

- In fact, we can try to express the above formula only using contour integrals. Let z_a, z_b be the points in complex plane, such that

$$z_b - z_a = -\mathrm{i}\beta$$

- Let γ^M be (any of) the contour starting from z_a ending in z_b, then we have

$$\mathrm{e}^{-\beta \hat{H}^M} = \mathrm{e}^{-\mathrm{i}\int_{\gamma^M} \mathrm{d}z \hat{H}^M}$$

For $z \in \gamma^M$, we can also define the value of Hamiltonian $H^M(z)$ on γ^M. We just choose $H^M(z) = H(t_0)$.

- We also have

$$\mathcal{T}\left(\mathrm{e}^{-\mathrm{i}\int_{\tilde{\gamma}} \mathrm{d}z \hat{H}(z)}\right) = \hat{U}(t_0, \infty)\hat{U}(\infty, t_0) = \mathbf{1}$$

- Then, combined those two equations, we have:

$$\langle \hat{O} \rangle(t) = \frac{\mathrm{Tr}\left(\mathrm{e}^{-\mathrm{i}\int_{\gamma^M} \mathrm{d}z \hat{H}^M}\mathcal{T}\left(\mathrm{e}^{-\mathrm{i}\int_{\tilde{\gamma}} \mathrm{d}z \hat{H}(z)} \hat{O}(t_\pm)\right)\right)}{\mathrm{Tr}\left(\mathrm{e}^{-\mathrm{i}\int_{\gamma^M} \mathrm{d}z \hat{H}^M}\mathcal{T}\left(\mathrm{e}^{-\mathrm{i}\int_{\tilde{\gamma}} \mathrm{d}z \hat{H}(z)}\right)\right)}$$

- In this case, $\hat{H}(z) = \hat{H}^M$ for all z.
- We haven't decided what is z_a, z_b and γ^M yet, just saying that $z_b - z_a = -\mathrm{i}\beta$. As shown in figure 2, we do have different choices.

Using γ^M and the original γ_-, γ_+, we can construct new oriented contour γ:

$$\gamma = \gamma_- \oplus \gamma_+ \oplus \gamma^M = \tilde{\gamma} \oplus \gamma^M$$

and define the ordering using this oriented contour γ.

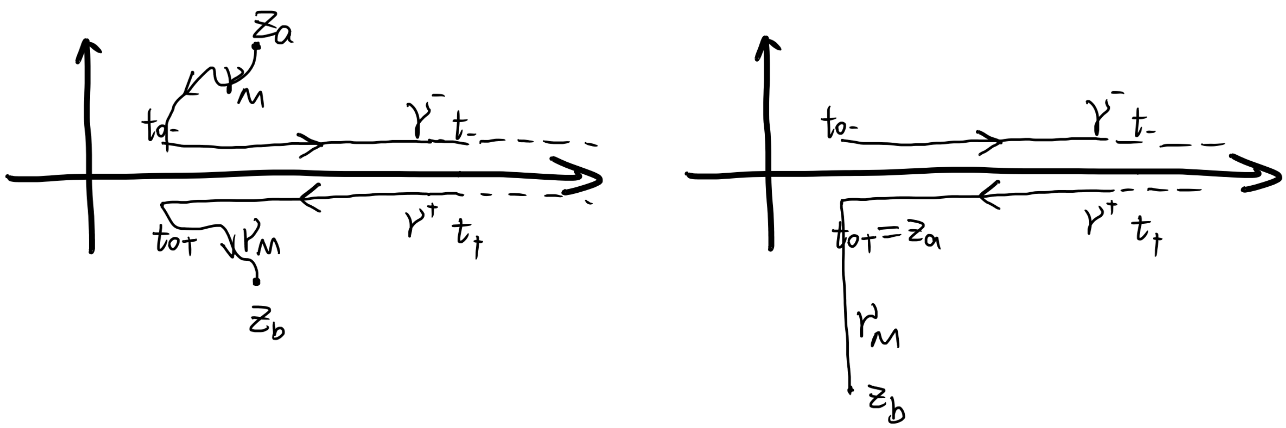

Figure 2: Different choices of z_a, z_b

- Now, for example, for the left choice of figure 2, we have:

$$\begin{aligned}\mathrm{Tr}\left(\mathrm{e}^{-\mathrm{i}\int_{\gamma^M}\mathrm{d}z\hat{H}^M}\mathcal{T}\left(\mathrm{e}^{-\mathrm{i}\int_{\tilde{\gamma}}\mathrm{d}z\hat{H}(z)}\hat{O}(t_\pm)\right)\right) &= \mathrm{Tr}\left(\mathrm{e}^{-\mathrm{i}\int_{z_a}^{t_0-}\mathrm{d}z\hat{H}^M}\mathcal{T}\left(\mathrm{e}^{-\mathrm{i}\int_{\tilde{\gamma}}\mathrm{d}z\hat{H}(z)}\hat{O}(t_\pm)\right)\mathrm{e}^{-\mathrm{i}\int_{t_0+}^{z_b}\mathrm{d}z\hat{H}^M}\right)\\ &= \mathrm{Tr}\left(\mathcal{T}\left(\mathrm{e}^{-\mathrm{i}\int_{\tilde{\gamma}\oplus\gamma^M}\mathrm{d}z\hat{H}(z)}\hat{O}(t_\pm)\right)\right)\end{aligned}$$

- In general, we have

$$\langle\hat{O}\rangle(t) = \frac{\mathrm{Tr}\left(\mathcal{T}\left(\mathrm{e}^{-\mathrm{i}\int_{\tilde{\gamma}\oplus\gamma^M}\mathrm{d}z\hat{H}(z)}\hat{O}(t_\pm)\right)\right)}{\mathrm{Tr}\left(\mathcal{T}\left(\mathrm{e}^{-\mathrm{i}\int_{\tilde{\gamma}\oplus\gamma^M}\mathrm{d}z\hat{H}(z)}\right)\right)}$$

In fact, this value could also be generalized to any $z \in \gamma$, not only real-time t, but also imaginary time z on γ^M:

$$\langle\hat{O}\rangle(z) = \frac{\mathrm{Tr}\left(\mathcal{T}\left(\mathrm{e}^{-\mathrm{i}\int_{\tilde{\gamma}\oplus\gamma^M}\mathrm{d}z\hat{H}(z)}\hat{O}(z)\right)\right)}{\mathrm{Tr}\left(\mathcal{T}\left(\mathrm{e}^{-\mathrm{i}\int_{\tilde{\gamma}\oplus\gamma^M}\mathrm{d}z\hat{H}(z)}\right)\right)}$$

5 Equation of motion on the contour

- As an analogy to the evolution operator for quantum system, we define the following evolution operator on the contour:

$$\hat{U}(z_2, z_1) = \begin{cases}\mathcal{T}\left(e^{-\mathrm{i}\int_{z_1}^{z_2}\mathrm{d}z\hat{H}(z)}\right), & z_2 \text{ later than } z_1\\ \bar{\mathcal{T}}\left(e^{+\mathrm{i}\int_{z_2}^{z_1}\mathrm{d}z\hat{H}(z)}\right), & z_1 \text{ later than } z_2\end{cases}$$

The integral is taken along the contour.

- This evolution operator has the following property:

$$\hat{U}(z,z) = \mathbf{1}, \quad \hat{U}(z_3, z_2)\hat{U}(z_2, z_1) = \hat{U}(z_3, z_1)$$

- Now let's calculate the derivative. If z is later than z_0, we have

$$\mathrm{i}\frac{\partial}{\partial z}\hat{U}(z,z_0)=\mathcal{T}\left(\mathrm{i}\frac{\partial}{\partial z}e^{-\mathrm{i}\int_{z_0}^{z}\mathrm{d}\bar{z}\hat{H}(\bar{z})}\right)=\mathcal{T}\left(\hat{H}(z)e^{-\mathrm{i}\int_{z_0}^{z}\mathrm{d}\bar{z}\hat{H}(\bar{z})}\right)=\hat{H}(z)\hat{U}(z,z_0)$$

$$\mathrm{i}\frac{\partial}{\partial z}\hat{U}(z_0,z)=\bar{\mathcal{T}}\left(\mathrm{i}\frac{\partial}{\partial z}e^{\mathrm{i}\int_{z_0}^{z}\mathrm{d}\bar{z}\hat{H}(\bar{z})}\right)=-\bar{\mathcal{T}}\left(\hat{H}(z)e^{\mathrm{i}\int_{z_0}^{z}\mathrm{d}\bar{z}\hat{H}(\bar{z})}\right)=-\hat{U}(z_0,z)\hat{H}(z)$$

- We have the following relation between evolution operator on contour $\hat{U}(z_2,z_1)$ and the ordinary evolution operator $\hat{U}(t_2,t_1)$:

$$\hat{U}(t_2,t_1)=\hat{U}(t_{2-},t_{1-})=\hat{U}(t_{2+},t_{1+})$$

- Let's denote by z_i the initial point of the contour, by z_f the final point of the contour. Then:

$$O(z)=\frac{\mathrm{Tr}\left(\hat{U}(z_f,z_i)\hat{U}(z_i,z)\hat{O}(z)\hat{U}(z,z_i)\right)}{\mathrm{Tr}\left(\hat{U}(z_f,z_i)\right)}$$

- It seems natural to define the contour Heisenberg picture:

$$\hat{O}_H(z)=\hat{U}(z_i,z)\hat{O}(z)\hat{U}(z,z_i)$$

- If z lies on the horizontal branches, the contour Heisenberg picture and the standard Heisenberg picture is:

$$\hat{O}_H(t_1)=\hat{O}_H(t_2)=\hat{O}_H(t)$$

- Equation of motion:

$$\begin{aligned}\mathrm{i}\frac{\partial}{\partial z}\hat{O}_H(z)=&\hat{U}(z_i,z)[\hat{O}(z),\hat{H}(z)]\hat{U}(z,z_i)+\mathrm{i}\left(\frac{\partial\hat{O}}{\partial z}\right)_H(z)\\=&[\hat{O}_H(z),\hat{H}_H(z)]+\mathrm{i}\left(\frac{\partial\hat{O}}{\partial z}\right)_H(z)\end{aligned}$$

- For example, field operators $\hat{\psi}_H(\boldsymbol{x},t),\hat{\psi}_H^\dagger(\boldsymbol{x},t)$ also have the contour analog $\hat{\psi}_H(\boldsymbol{x},z),\hat{\psi}_H^\dagger(\boldsymbol{x},z)$.
- For $\hat{\psi}(\boldsymbol{x},z),\hat{\psi}^\dagger(\boldsymbol{x},z)$, we only need to specify the definition on the γ^M:

$$\hat{\psi}(\boldsymbol{x},z\in\gamma^M)=\hat{\psi}(\boldsymbol{x}),\quad \hat{\psi}^\dagger(\boldsymbol{x},z\in\gamma^M)=\hat{\psi}^\dagger(\boldsymbol{x})$$

  In other words, the field operators are constant over the entire contour. Generally, we only consider $\hat{H}^M$ of the form:

$$\begin{aligned}\hat{H}^M&=\int\mathrm{d}\boldsymbol{x}\hat{\psi}^\dagger(\boldsymbol{x})\hat{h}(\boldsymbol{x},z)\hat{\psi}(\boldsymbol{x})+\frac{1}{2}\int\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{x}'\hat{\psi}^\dagger(\boldsymbol{x})\hat{\psi}^\dagger(\boldsymbol{x}')v(\boldsymbol{x},\boldsymbol{x}')\hat{\psi}(\boldsymbol{x}')\hat{\psi}(\boldsymbol{x})\\&=\hat{H}_0^M+\hat{H}_{\mathrm{int}}^M\end{aligned}$$

- Neglecting spins, we have the following equation of motion:

$$\mathrm{i}\frac{\partial}{\partial z}\hat{\psi}_H(\boldsymbol{x},z)=\hat{h}(\boldsymbol{x},z)\hat{\psi}_H(\boldsymbol{x},z)+\int\mathrm{d}\boldsymbol{x}'v(\boldsymbol{x},\boldsymbol{x}',z)\hat{\psi}_H^\dagger(\boldsymbol{x}',z)\hat{\psi}_H(\boldsymbol{x}',z)\hat{\psi}_H(\boldsymbol{x},z)$$

$$\mathrm{i}\frac{\partial}{\partial z}\hat{\psi}_H^\dagger(\boldsymbol{x},z)=-\hat{h}(\boldsymbol{x},z)\hat{\psi}_H^\dagger(\boldsymbol{x},z)-\int\mathrm{d}\boldsymbol{x}'v(\boldsymbol{x},\boldsymbol{x}',z)\hat{\psi}_H^\dagger(\boldsymbol{x},z)\hat{\psi}_H^\dagger(\boldsymbol{x}',z)\hat{\psi}_H(\boldsymbol{x}',z)$$

6 Operators correlators on the contour

- It has been shown that, in order to calculate $\langle O\rangle(t)$, after taylor expansion, the quantity that we are concerned about are traces of the following forms:

$$\hat{k}(z_1, \cdots, z_n) = \mathcal{T}\left(\hat{O}_1(z_1)\cdots\hat{O}_n(z_n)\right)$$

These are what we call operator correlators.

- Define $\theta(z_1, z_2)$:

$$\theta(z_1, z_2) = \begin{cases} 1 & z_1 > z_2 \\ 0 & z_1 \leq z_2 \end{cases}$$

its derivative is the delta function:

$$\frac{\mathrm{d}}{\mathrm{d}z_1}\theta(z_1, z_2) = \delta(z_1, z_2), \quad \frac{\mathrm{d}}{\mathrm{d}z_2}\theta(z_1, z_2) = -\delta(z_1, z_2)$$

- For two operator case $\mathcal{T}\left(\hat{O}_1(z_1)\hat{O}_2(z_2)\right)$, we have:

$$\mathcal{T}\left(\hat{O}_1(z_1)\hat{O}_2(z_2)\right) = \theta(z_1, z_2)\hat{O}_1(z_1)\hat{O}_2(z_2) + \theta(z_2, z_1)\hat{O}_2(z_2)\hat{O}_1(z_1)$$

Then the derivative is:

$$\frac{\mathrm{d}}{\mathrm{d}z_1}\mathcal{T}\left(\hat{O}_1(z_1)\hat{O}_2(z_2)\right) = \delta(z_1, z_2)\left[\hat{O}_1(z_1), \hat{O}_2(z_2)\right] + \mathcal{T}\left(\left(\frac{\mathrm{d}}{\mathrm{d}z_1}\hat{O}_1(z_1)\right)\hat{O}_2(z_2)\right)$$

- The commutator is actual for bosons. For fermions, we prefer anti-commutators. Therefore, if $\hat{O}_1, \hat{O}_2$ are fermionic operators, we modify our definition of time-ordering operator:

$$\mathcal{T}\left(\hat{O}_1(z_1)\hat{O}_2(z_2)\right) = \theta(z_1, z_2)\hat{O}_1(z_1)\hat{O}_2(z_2) - \theta(z_2, z_1)\hat{O}_2(z_2)\hat{O}_1(z_1)$$

Then, the derivative is:

$$\frac{\mathrm{d}}{\mathrm{d}z_1}\mathcal{T}\left(\hat{O}_1(z_1)\hat{O}_2(z_2)\right) = \delta(z_1, z_2)\left\{\hat{O}_1(z_1), \hat{O}_2(z_2)\right\} + \mathcal{T}\left(\left(\frac{\mathrm{d}}{\mathrm{d}z_1}\hat{O}_1(z_1)\right)\hat{O}_2(z_2)\right)$$

- Generally, for n operators, we define

$$\mathcal{T}\left(\hat{O}_1(z_1)\cdots\hat{O}_n(z_n)\right) = \sum_P (\pm)^P \theta(z_{P(1)}, \cdots, z_{P(n)})\hat{O}_{P(1)}(z_{P(1)})\cdots\hat{O}_{P(n)}(z_{P(n)})$$

where $\theta(z_1, \cdots, z_n)$ is defined as

$$\theta(z_1, \cdots, z_n) = \begin{cases} 1 & z_1 > z_2 > \cdots > z_n \\ 0 & \text{otherwise} \end{cases}$$

we have

$$\mathcal{T}\left(\hat{O}_1(z_1)\cdots\hat{O}_n(z_n)\right) = (\pm)^P \mathcal{T}\left(\hat{O}_{P(1)}(z_{P(1)})\cdots\hat{O}_{P(n)}(z_{P(n)})\right)$$

The plus sign is for Bosonic operator, and the minus sign is for Fermionic operator. Hamiltonian is treated as Bosonic operator.

- In other words, we have:

$$\frac{\mathrm{d}}{\mathrm{d}z_1}\mathcal{T}\left(\hat{O}_1(z_1)\hat{O}_2(z_2)\right) = \delta(z_1,z_2)\left[\hat{O}_1(z_1),\hat{O}_2(z_2)\right]_{\mp} + \mathcal{T}\left(\left(\frac{\mathrm{d}}{\mathrm{d}z_1}\hat{O}_1(z_1)\right)\hat{O}_2(z_2)\right)$$

- For M equal time operators, it acts like a bosonic/fermionic operator under $\mathcal{T}$, for M being even/odd.

- For n operators, we have:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}z_k}\mathcal{T}\left(\hat{O}_1(z_1)\cdots\hat{O}_n(z_n)\right) &= \mathcal{T}\left(\hat{O}_1(z_1)\cdots\hat{O}_{k-1}(z_{k-1})\left(\frac{\mathrm{d}}{\mathrm{d}z_k}\hat{O}_k(z_k)\right)\hat{O}_{k+1}(z_{k+1})\cdots\hat{O}_n(z_n)\right)\\ &\quad+\sum_P(\pm)^P\left(\frac{\mathrm{d}}{\mathrm{d}z_k}\theta(z_{P(1)},\cdots,z_{P(n)})\right)\hat{O}_{P(1)}(z_{P(1)})\cdots\hat{O}_{P(n)}(z_{P(n)})\end{aligned}$$

- Now we dedicate to calculate the second term. When moving z_k along the contour, only of when it passes along another z_l, will create a term like $\delta(z_k-z_l)$ in the derivative. Consider z_l is the term closest to z_k, there is nothing between them. We have:

  - if $l>k$,

$$\begin{aligned}\mathcal{T}\left(\hat{O}_1\cdots\hat{O}_n\right) &= (\pm)^{l-k-1}\mathcal{T}\left(\hat{O}_1\cdots\hat{O}_k\hat{O}_l\hat{O}_{k+1}\cdots\hat{O}_{l-1}\hat{O}_{l+1}\cdots\hat{O}_n\right)\\ &= (\pm)^{l-k-1}(\pm)^Q\left(\theta(z_k-z_l)\hat{O}_{Q(1)}\cdots\hat{O}_k\hat{O}_l\hat{O}_{Q(n)}\pm\theta(z_l-z_k)\hat{O}_{Q(1)}\cdots\hat{O}_l\hat{O}_k\hat{O}_{Q(n)}\right)\end{aligned}$$

  We see the above second term of the derivative in this case becomes:

$$\begin{aligned}&(\pm)^{l-k-1}(\pm)^Q\left(\delta(z_k-z_l)\hat{O}_{Q(1)}\cdots\left[\hat{O}_k,\hat{O}_l\right]_{\mp}\hat{O}_{Q(n)}\right)\\ =&(\pm)^{l-k-1}\delta(z_k,z_l)\mathcal{T}\left(\hat{O}_1\cdots\left[\hat{O}_k,\hat{O}_l\right]_{\mp}\hat{O}_{k+1}\cdots\hat{O}_{l-1}\hat{O}_{l+1}\cdots\hat{O}_n\right)\end{aligned}$$

  - if $k>l$, similarly we have the derivative as

$$(\pm)^{k-l}\delta(z_k,z_l)\mathcal{T}\left(\hat{O}_1\cdots\hat{O}_{l-1}\hat{O}_{l+1}\cdots\left[\hat{O}_k,\hat{O}_l\right]_{\mp}\hat{O}_{k+1}\cdots\hat{O}_n\right)$$

- Put them together, we have

$$\begin{aligned}&\sum_{l=k+1}^{n}(\pm)^{l-k-1}\delta(z_k,z_l)\mathcal{T}\left(\hat{O}_1\cdots\left[\hat{O}_k,\hat{O}_l\right]_{\mp}\hat{O}_{k+1}\cdots\hat{O}_{l-1}\hat{O}_{l+1}\cdots\hat{O}_n\right)\\ +&\sum_{l=1}^{k-1}(\pm)^{k-l}\delta(z_k,z_l)\mathcal{T}\left(\hat{O}_1\cdots\hat{O}_{l-1}\hat{O}_{l+1}\cdots\left[\hat{O}_k,\hat{O}_l\right]_{\mp}\hat{O}_{k+1}\cdots\hat{O}_n\right)\end{aligned}$$

- We will be working with field operators, which has a nice property that the (anti)commutator is a scalar function. Therefore the above formalism could be simplified further:

$$\begin{aligned}&\sum_{l=k+1}^{n}(\pm)^{l-k-1}\delta(z_k,z_l)\left[\hat{O}_k,\hat{O}_l\right]_{\mp}\mathcal{T}\left(\hat{O}_1\cdots\tilde{\hat{O}}_k\cdots\tilde{\hat{O}}_l\cdots\hat{O}_n\right)\\ +&\sum_{l=1}^{k-1}(\pm)^{k-l}\delta(z_k,z_l)\left[\hat{O}_k,\hat{O}_l\right]_{\mp}\mathcal{T}\left(\hat{O}_1\cdots\tilde{\hat{O}}_l\cdots\tilde{\hat{O}}_k\cdots\hat{O}_n\right)\end{aligned}$$

where $\tilde{\hat{O}}_k$ means exclude this term from the multiplication.

- In order to deal with the field operators, we introduce the following notation:

$$i = \boldsymbol{x}_i, z_i, \quad j = \boldsymbol{x}_j, z_j, \cdots$$
$$\delta(j;k) = \delta(z_j, z_k)\delta(\boldsymbol{x}_j - \boldsymbol{x}_k)$$

- As an example, we have

$$\frac{\mathrm{d}}{\mathrm{d}z_2}\mathcal{T}\left\{\hat{\psi}_H(1)\hat{\psi}_H(2)\hat{\psi}_H^\dagger(3)\hat{\psi}_H^\dagger(4)\right\} = \mathcal{T}\left\{\hat{\psi}_H(1)\left(\frac{\mathrm{d}}{\mathrm{d}z_2}\hat{\psi}_H(2)\right)\hat{\psi}_H^\dagger(3)\hat{\psi}_H^\dagger(4)\right\}$$
$$+ \delta(2;3)\mathcal{T}\left\{\hat{\psi}_H(1)\hat{\psi}_H^\dagger(4)\right\} \pm \delta(2;4)\mathcal{T}\left\{\hat{\psi}_H(1)\hat{\psi}_H^\dagger(3)\right\}$$

- We are specifically interested in the following correlators:

$$\hat{G}_n(1,\cdots,n;1',\cdots,n') = \frac{1}{\mathrm{i}^n}\mathcal{T}\left(\hat{\psi}_H(1)\cdots\hat{\psi}_H(n)\psi_H^\dagger(n')\cdots\psi_H^\dagger(1')\right)$$

For $n=0$, $\hat{G}_0 = \mathbf{1}$

- Therefore we have:

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}z_k}\hat{G}_n(1,\cdots,n;1',\cdots,n') = \frac{1}{\mathrm{i}^n}\mathcal{T}\left(\hat{\psi}_H(1)\cdots\left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}z_k}\hat{\psi}_H(k)\right)\cdots\hat{\psi}_H(n)\psi_H^\dagger(n')\cdots\psi_H^\dagger(1')\right)$$
$$+\sum_{j=1}^{n}(\pm)^{k+j}\delta(k;j')\hat{G}_{n-1}(1,\cdots,\tilde{k},\cdots,n;1',\cdots,\tilde{j}',\cdots,n')$$

Similarly:

$$-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}z_k'}\hat{G}_n(1,\cdots,n;1',\cdots,n') = \frac{1}{\mathrm{i}^n}\mathcal{T}\left(\hat{\psi}_H(1)\cdots\left(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}z_k'}\hat{\psi}_H(k)\right)\cdots\hat{\psi}_H(n)\psi_H^\dagger(n')\cdots\psi_H^\dagger(1')\right)$$
$$+\sum_{j=1}^{n}(\pm)^{k+j}\delta(j;k')\hat{G}_{n-1}(1,\cdots,\tilde{j},\cdots,n;1',\cdots,\tilde{k}',\cdots,n')$$

- Recall the equation of motion for field operators:

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}z_k}\hat{\psi}_H(k) = \hat{h}(k)\hat{\psi}_H(k) + \int \mathrm{d}\bar{1}v(k,\bar{1})\hat{\psi}_H^\dagger(\bar{1})\hat{\psi}_H(\bar{1})\hat{\psi}_H(k)$$
$$-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}z_k}\hat{\psi}_H^\dagger(k') = \hat{\psi}_H^\dagger(k')\hat{h}(k') + \int \mathrm{d}\bar{1}v(k',\bar{1})\hat{\psi}_H^\dagger(k')\hat{\psi}_H^\dagger(\bar{1})\hat{\psi}_H(\bar{1})$$

- If we insert the derivative of field operator back into the derivative of $\hat{G}_n$, we have something like $\hat{G}_{n+1}$. Note that: In order to ensure $\hat{\psi}_H^\dagger(1)$ is in front of $\hat{\psi}_H(1)$, we can introduce 1+:

$$\mathcal{T}\left(\cdots\hat{n}_H(\bar{1})\hat{\psi}_H(k)\cdots\right) = \mathcal{T}\left(\cdots\hat{\psi}_H^\dagger(\bar{1}^+)\hat{\psi}_H(\bar{1})\hat{\psi}_H(k)\cdots\right) = \pm\mathcal{T}\left(\cdots\hat{\psi}_H(k)\hat{\psi}_H(\bar{1})\hat{\psi}_H^\dagger(\bar{1}^+)\cdots\right)$$

Then we have:

$$\frac{1}{\mathrm{i}^n}\mathcal{T}\left\{\hat{\psi}_H(1)\ldots\left(\mathrm{i}\frac{d}{dz_k}\hat{\psi}_H(k)\right)\ldots\hat{\psi}_H(n)\hat{\psi}_H^\dagger\left(n'\right)\ldots\hat{\psi}_H^\dagger\left(1'\right)\right\}$$
$$= h(k)\hat{G}_n\left(1,\ldots,n;1',\ldots,n'\right) \pm \frac{1}{\mathrm{i}^n}\int d\bar{1}v(k;\bar{1})\mathcal{T}\left\{\hat{\psi}_H(1)\ldots\hat{\psi}_H(n)\hat{\psi}_H(\bar{1})\hat{\psi}_H^\dagger\left(\bar{1}^+\right)\hat{\psi}_H^\dagger\left(n'\right)\ldots\hat{\psi}_H^\dagger\left(1'\right)\right\}$$
$$= h(k)\hat{G}_n\left(1,\ldots,n;1',\ldots,n'\right) \pm \mathrm{i}\int d\bar{1}v(k;\bar{1})\hat{G}_{n+1}\left(1,\ldots,n,\bar{1};1,\ldots,n',\bar{1}^+\right)$$

Therefore we have:

$$\left[\mathrm{i}\frac{d}{dz_k} - h(k)\right] \hat{G}_n\left(1,\ldots,n;1',\ldots,n'\right)$$

$$= \pm\,\mathrm{i}\int d\bar{1} v(k;\bar{1}) \hat{G}_{n+1}\left(1,\ldots,n,\bar{1};1',\ldots,n',\bar{1}^{+}\right) + \sum_{j=1}^{n}(\pm)^{k+j}\delta\left(k;j'\right)\hat{G}_{n-1}\left(1,\ldots\tilde{k}\ldots,n;1',\ldots\tilde{j}'\ldots,n'\right)$$

Similarly, we have:

$$\hat{G}_n\left(1,\ldots,n;1',\ldots,n'\right)\left[-\mathrm{i}\frac{\overleftarrow{d}}{dz'_k} - h\left(k'\right)\right]$$

$$= \pm\,\mathrm{i}\int d\bar{1} v\left(k';\bar{1}\right) \hat{G}_{n+1}\left(1,\ldots,n,\bar{1}^{-};1',\ldots,n',\bar{1}\right) + \sum_{j=1}^{n}(\pm)^{k+j}\delta\left(j;k'\right)\hat{G}_{n-1}\left(1,\ldots\tilde{j}\ldots,n;1',\ldots\tilde{k}'\ldots,n'\right)$$

7 Martin-Schwinger hierarchy

- The $n-$particle Green's function G_n, is defined using $\hat{G}_n$:

$$\begin{aligned} G_n(1,\cdots,n;1,\cdots,n') &= \frac{\mathrm{Tr}\left(e^{-\beta H^M}\hat{G}_n(1,\cdots,n;1,\cdots,n'))\right)}{\mathrm{Tr}\left(e^{-\beta H^M}\right)} \\ &= \frac{1}{\mathrm{i}^n}\frac{\mathrm{Tr}\left(\mathcal{T}\left(e^{-\mathrm{i}\int_\gamma \mathrm{d}\bar{z}\hat{H}(\bar{z})}\hat{\psi}(1)\cdots\hat{\psi}(n)\psi^\dagger(n')\cdots\psi^\dagger(1')\right)\right)}{\mathrm{Tr}\left(\mathcal{T}\left(e^{-\mathrm{i}\int_\gamma \mathrm{d}\bar{z}\hat{H}(\bar{z})}\right)\right)} \end{aligned}$$

  This requires derivation!

- For G_1, by choosing $z_1 = z$, $z'_1 = z^+$, we have the time dependent ensemble average for $\hat{\psi}^\dagger(\boldsymbol{x}'_1)\hat{\psi}(\boldsymbol{x}_1)$, which could be used to calculate the ensemble average of any one-body operator. For example, the density operator:

$$n(\boldsymbol{x},z) = \frac{\mathrm{Tr}\left(e^{-\beta H^M}\hat{\psi}_H^\dagger(\boldsymbol{x},z)\hat{\psi}_H(\boldsymbol{x},z)\right)}{\mathrm{Tr}\left(e^{-\beta H^M}\right)} = \pm iG_1(\boldsymbol{x},z;\boldsymbol{x},z^+)$$

  and the current density operator:

$$\begin{aligned} \boldsymbol{j}(\boldsymbol{x},z) &= \frac{1}{2m\mathrm{i}}\frac{\mathrm{Tr}\left(e^{-\beta H^M}\left(\hat{\psi}_H^\dagger(\boldsymbol{x},z)\left(\nabla\hat{\psi}_H(\boldsymbol{x},z)\right) - \left(\nabla\hat{\psi}_H^\dagger(\boldsymbol{x},z)\right)\hat{\psi}_H(\boldsymbol{x},z)\right)\right)}{\mathrm{Tr}\left(e^{-\beta H^M}\right)} \\ &= \pm\frac{\nabla-\nabla'}{2m}G_1(\boldsymbol{x},z;\boldsymbol{x}',z^+)|_{\boldsymbol{x}=\boldsymbol{x}'} \end{aligned}$$

  For G_n it's similar. For example, the interaction energy is:

$$E_{\mathrm{int}}(z) = -\frac{1}{2}\int \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{x}' v(\boldsymbol{x},\boldsymbol{x}',z)G_2(\boldsymbol{x}',z,\boldsymbol{x},z;\boldsymbol{x}',z^+,\boldsymbol{x},z^+)$$

- We have the differential equation for G_n:

$$\left[\mathrm{i}\frac{d}{dz_k}-h(k)\right]G_n\left(1,\ldots,n;1',\ldots,n'\right)$$
$$=\pm\,\mathrm{i}\int d\bar{1}v(k;\bar{1})G_{n+1}\left(1,\ldots,n,\bar{1};1',\ldots,n',\bar{1}^+\right)+\sum_{j=1}^{n}(\pm)^{k+j}\delta\left(k;j'\right)G_{n-1}\left(1,\ldots\tilde{k}\ldots,n;1',\ldots\tilde{j}'\ldots,n'\right)$$

and

$$G_n\left(1,\ldots,n;1',\ldots,n'\right)\left[-\mathrm{i}\frac{\overleftarrow{d}}{dz_k'}-h\left(k'\right)\right]$$
$$=\pm\,\mathrm{i}\int d\bar{1}v\left(k';\bar{1}\right)G_{n+1}\left(1,\ldots,n,\bar{1}^-;1',\ldots,n',\bar{1}\right)+\sum_{j=1}^{n}(\pm)^{k+j}\delta\left(j;k'\right)G_{n-1}\left(1,\ldots\tilde{j}\ldots,n;1',\ldots\tilde{k}'\ldots,n'\right)$$

This is called the Martin-Schwinger hierarchy.

- We also have the following boundary condition for Green's function:

$$G_n(1,\cdots,\boldsymbol{x}_k,z_i,\cdots,n;1',\cdots,n')=\pm G_n(1,\cdots,\boldsymbol{x}_k,z_f,\cdots,n;1',\cdots,n')$$
$$G_n(1,\cdots,n;1'\cdots,\boldsymbol{x}_k',z_i,\cdots,n')=\pm G_n(1,\cdots,n;1'\cdots,\boldsymbol{x}_k',z_f,\cdots,n')$$

This is called the Kubo-Martin-Schwinger(KMS) relations.

We'll now discuss an example of the truncation of hierarchy, i.e. the approximation of Green's function.

- For $G(1,1')$, we have:

$$\left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}z_1}-h(1)\right)G(1;1')=\delta(1;1')\pm\mathrm{i}\int\mathrm{d}2v(1,2)G_2(1,2;1',2^+)$$
$$G(1;1')\left(-\mathrm{i}\frac{\overleftarrow{\mathrm{d}}}{\mathrm{d}z_1}-h(1')\right)=\delta(1;1')\pm\mathrm{i}\int\mathrm{d}2v(1',2)G_2(1,2^-;1',2)$$

- For $G_2(1,2;1',2')$, we have:

$$\left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}z_1}-h(1)\right)G_2(1,2;1',2')=\delta(1;1')G(2,2')\pm\delta(1;2')G(2;1')\pm\mathrm{i}\int\mathrm{d}3v(1,3)G_3(1,2,3;1',2',3^+)$$
$$\left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}z_2}-h(2)\right)G_2(1,2;1',2')=\pm\delta(2;1')G(1,2')+\delta(2;2')G(1;1')\pm\mathrm{i}\int\mathrm{d}3v(2,3)G_3(1,2,3;1',2',3^+)$$
$$G_2(1,2;1',2')\left(-\mathrm{i}\frac{\overleftarrow{\mathrm{d}}}{\mathrm{d}z_1}-h(1')\right)=\delta(1;1')G(2,2')\pm\delta(2;1')G(1;2')\pm\mathrm{i}\int\mathrm{d}3v(1',3)G_3(1,2,3^-;1',2',3)$$
$$G_2(1,2;1',2')\left(-\mathrm{i}\frac{\overleftarrow{\mathrm{d}}}{\mathrm{d}z_2}-h(2')\right)=\pm\delta(1;2')G(2,1')+\delta(2;2')G(1;1')\pm\mathrm{i}\int\mathrm{d}3v(2',3)G_3(1,2,3^-;1',2',3)$$

- This inspires us to write $G_2(1,2;1',2')$ into the following formulation:

$$G_2(1,2;1',2') = G(1;1')G(2;2') \pm G(1;2')G(2;1') + \Upsilon(1,2;1',2')$$

$\Upsilon(1,2;1',2')$ is called the correlation function. For $v=0$, we can verify that $G_2(1,2;1',2') = G(1;1')G(2;2') \pm G(1;2')G(2;1')$ satisfy the exact differential equation for G_2.

- The formulation

$$G_2(1,2;1',2') = G(1;1')G(2;2') \pm G(1;2')G(2;1')$$

is called the Hartree-Fock approximation. If we insert it back to the differential equation for $G(1;1')$, we have:

$$\left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}z_1} - h(1)\right) G(1;1') = \delta(1;1') \pm \mathrm{i}\int \mathrm{d}2 v(1,2)\left(G(1;1')G(2;2^+) \pm G(1;2^+)G(2;1')\right)$$

$$G(1;1')\left(-\mathrm{i}\frac{\overleftarrow{\mathrm{d}}}{\mathrm{d}z_1} - h(1')\right) = \delta(1;1') \pm \mathrm{i}\int \mathrm{d}2 v(1',2)\left(G(1;1')G(2^-;2) \pm G(1;2')G(2^-;1')\right)$$

- If we define Σ as:

$$\Sigma(1,2) = \delta(1;2)V_H(1) + \mathrm{i}v(1;2)G(1;2^+)$$

where

$$V_H(1) = \pm\mathrm{i}\int \mathrm{d}3 v(1;3)G(3,3^+) = \int \mathrm{d}\boldsymbol{x}_3 v(\boldsymbol{x}_1,\boldsymbol{x}_3,z_1)n(\boldsymbol{x}_3,z_1)$$

is the well-known Hartree potential. Then we have

$$\left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}z_1} - h(1)\right) G(1;1') = \delta(1;1') + \mathrm{i}\int \mathrm{d}2\Sigma(1,2)G(2;1')$$

$$G(1;1')\left(-\mathrm{i}\frac{\overleftarrow{\mathrm{d}}}{\mathrm{d}z_1} - h(1')\right) = \delta(1;1') + \mathrm{i}\int \mathrm{d}2 G(1;2)\Sigma(2;1')$$

This is the self-consistent equation.

- Now we discuss the derivation of Dyson's equation. Using integral by part and KMS boundary condition, we have:

$$\begin{aligned}
&\int \mathrm{d}1 G_0(2;1)\left[\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}z_1} - h(1)\right] G\left(1;1'\right) \\
&= \int \mathrm{d}1 G_0(2;1)\left[-\mathrm{i}\frac{\overleftarrow{\mathrm{d}}}{\mathrm{d}z_1} - h(1)\right] G\left(1;1'\right) + \mathrm{i}\int \mathrm{d}\boldsymbol{x}_1 G_0\left(2;\boldsymbol{x}_1,z_1\right) G\left(\boldsymbol{x}_1,z_1;1'\right)\Big|_{z_1=z_1}^{z_1=z_t} \\
&= \int d1\delta(2;1)G\left(1;1'\right) = G(2,1')
\end{aligned}$$

With this we can do integral on the self-consistent equation for $G(1;1')$, and we have:

$$G(1;2) = G_0(1;2) + \int \mathrm{d}3\mathrm{d}4 G_0(1;3)\Sigma(3;4)G(4;2)$$

Similarly, we can also get:

$$G(1;2) = G_0(1;2) + \int \mathrm{d}3\mathrm{d}4 G_(1;3)\Sigma(3;4)G_0(4;2)$$

Dyson's equation is also correct for the general case, not only for the Hartree-Fock approximation here.

We also can discuss the approximation beyond Hartree-Fock.

- For
$$G_2(1,2;1',2') = G(1;1')G(2;2') \pm G(1;2')G(2;1') + \Upsilon(1,2;1',2')$$
with
$$\left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}z_1} - h(1)\right) G(1;1') = \delta(1;1') \pm \mathrm{i}\int \mathrm{d}2 v(1,2) G_2(1,2;1',2^+)$$
$$\left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}z_1} - h(1)\right) G(1;2') = \delta(1;2') \pm \mathrm{i}\int \mathrm{d}2 v(1,2) G_2(1,2;2',2^+)$$
$$\left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}z_1} - h(1)\right) G_2(1,2;1',2') = \delta(1;1')G(2,2') \pm \delta(1;2')G(2;1') \pm \mathrm{i}\int \mathrm{d}3 v(1,3) G_3(1,2,3;1',2',3^+)$$
- Therefore we have
$$\begin{aligned}&\left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}z_1} - h(1)\right) \Upsilon(1,2;1',2')\\ &= \pm\,\mathrm{i}\int \mathrm{d}3 v(1,3) \left(G_3(1,2,3;1',2',3^+) - G_2(1,3;1',3^+)G(2;2') \mp G_2(1,3;2',3^+)G(2;1')\right)\end{aligned}$$
- Because
$$\begin{aligned}\left[\mathrm{i}\frac{d}{dz_2} - h(2)\right] G_3\left(1,2,3;1',2',3'\right) &= \pm\delta\left(2;1'\right) G_2\left(1,3;2,3'\right) + \delta\left(2;2'\right) G_2\left(1,3;1',3'\right)\\ &\pm\delta\left(2;3'\right) G_2\left(1,3;1',2'\right) \pm \mathrm{i}\int d4 v(2;4) G_4\left(1,2,3,4;1',2',3',4^+\right)\end{aligned}$$
- Therefore
$$\begin{aligned}&\left[\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}z_1} - h(1)\right]\left[\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}z_2} - h(2)\right] \Upsilon\left(1,2;1',2'\right) = \mathrm{i}v(1;2) G_2\left(1,2;1',2'\right)\\ &-\int d4 v(1;3) v(2;4) \left[G_4\left(1,2,3,4;1',2',3^+,4^+\right) - G_2\left(1,3;1',3^+\right) G_2\left(2,4;2',4^+\right)\right.\\ &\left.\mp G_2\left(1,3;2',3^+\right) G_2\left(2,4;1',4^+\right)\right].\end{aligned}$$
- If we neglect the higher order term, we have:
$$\left[\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}z_1} - h(1)\right]\left[\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}z_2} - h(2)\right] \Upsilon\left(1,2;1',2'\right) = \mathrm{i}v(1;2) G_2\left(1,2;1',2'\right)$$
Therefore
$$\Upsilon\left(1,2;1',2'\right) = \mathrm{i}\int \mathrm{d}3\mathrm{d}4 G_0(1;3) G_0(2;4) v(3;4) G_2\left(3,4;1',2'\right)$$

8 Exact solution of Martin-Schwinger's Hierarchy from Wick's theorem

- We'll first deal with the non-interacting case $G_{0,n}$, where $v=0$. We have the following result:

$$G_{0,n}\left(1,\ldots,n;1',\ldots,n'\right)=\left|\begin{array}{ccc} G_0\left(1;1'\right) & \ldots & G_0\left(1;n'\right) \\ \vdots & & \vdots \\ G_0\left(n;1'\right) & \ldots & G_0\left(n;n'\right) \end{array}\right|_{\pm}$$

This follows from Martin-Schwinger hierarchy if $v=0$. This is the so-called Wick's theorem.

- This inspires us to calculate G_n in a brute-force way. From the definition, we have:

$$G_n=\frac{1}{\mathrm{i}^n}\frac{\operatorname{Tr}\left[\mathcal{T}\left\{\mathrm{e}^{-\mathrm{i}\int_\gamma \mathrm{d}\bar{z}\hat{H}_0(\bar{z})}\mathrm{e}^{-\mathrm{i}\int_\gamma \mathrm{d}\bar{z}\hat{H}_{\mathrm{int}}(\bar{z})}\hat{\psi}(1)\ldots\hat{\psi}^{\dagger}\left(1'\right)\right\}\right]}{\operatorname{Tr}\left[\mathcal{T}\left\{e^{-\mathrm{i}\int_\gamma \mathrm{d}\bar{z}\hat{H}_0(\bar{z})}e^{-\mathrm{i}\int_\gamma \mathrm{d}\bar{z}\hat{H}_{\mathrm{int}}(\bar{z})}\right\}\right]}$$

where $\hat{H}_0$ and $\hat{H}_{\mathrm{int}}$ are respectively one-body and two-body terms for the Hamiltonian.

- If we introduce the notation:

$$\operatorname{Tr}\left[\mathcal{T}\left\{e^{-\mathrm{i}\int_\gamma d\bar{z}\hat{H}_0(\bar{z})}\ldots\right\}\right]=\langle\mathcal{T}\{\ldots\}\rangle_0$$

which could be physically interpreted as taking average over the non-interacting system. Then our interacting Green's function G_n, using Taylor's expansion, is

$$G_n=\frac{1}{\mathrm{i}^n}\frac{\sum_{k=0}^{\infty}\frac{(-\mathrm{i})^k}{k!}\int_\gamma d\bar{z}_1\ldots d\bar{z}_k\left\langle\mathcal{T}\left\{\hat{H}_{\text{int }}\left(\bar{z}_1\right)\ldots\hat{H}_{\text{int }}\left(\bar{z}_k\right)\hat{\psi}(1)\ldots\hat{\psi}^{\dagger}\left(1'\right)\right\}\right\rangle_0}{\sum_{k=0}^{\infty}\frac{(-\mathrm{i})^k}{k!}\int_\gamma d\bar{z}_1\ldots d\bar{z}_k\left\langle\mathcal{T}\left\{\hat{H}_{\text{int }}\left(\bar{z}_1\right)\ldots\hat{H}_{\text{int }}\left(\bar{z}_k\right)\right\}\right\rangle_0}$$

where $\hat{H}_{\mathrm{int}}$ is

$$\hat{H}_{\mathrm{int}}(z)=\frac{1}{2}\int\mathrm{d}z'\int\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{x}'v\left(\boldsymbol{x},z;\boldsymbol{x}',z'\right)\hat{\psi}^{\dagger}\left(\boldsymbol{x},z^{+}\right)\hat{\psi}^{\dagger}\left(\boldsymbol{x}',z^{+}\right)\hat{\psi}\left(\boldsymbol{x}',z'\right)\hat{\psi}(\boldsymbol{x},z)$$

where

$$v\left(\boldsymbol{x},z;\boldsymbol{x}',z'\right)=\delta\left(z,z'\right)\left\{\begin{array}{ll} v\left(\boldsymbol{x},\boldsymbol{x}',t\right) & \text{if } z=t_{\pm}\text{is on the horizontal branches of } \gamma \\ v^{\mathrm{M}}\left(\boldsymbol{x},\boldsymbol{x}'\right) & \text{if } z \text{ is on the vertical track of } \gamma \end{array}\right.$$

- Now we look at the case of one-particle Green's function. Let $a=\left(\boldsymbol{x}_a,z_a\right),b=\left(\boldsymbol{x}_b,z_b\right)$, we have

$$G(a;b)=\frac{1}{i}\frac{\sum_{k=0}^{\infty}\frac{(-\mathrm{i})^k}{k!}\int_\gamma dz_1\ldots dz_k\left\langle\mathcal{T}\left\{\hat{H}_{\text{int }}\left(z_1\right)\ldots\hat{H}_{\text{int }}\left(z_k\right)\hat{\psi}(a)\hat{\psi}^{\dagger}(b)\right\}\right\rangle_0}{\sum_{k=0}^{\infty}\frac{(-\mathrm{i})^k}{k!}\int_\gamma dz_1\ldots dz_k\left\langle\mathcal{T}\left\{\hat{H}_{\text{int }}\left(z_1\right)\ldots\hat{H}_{\text{int }}\left(z_k\right)\right\}\right\rangle_0}$$

The numerator could be rewritten as

$$\begin{aligned}&\sum_{k=0}^{\infty}\tfrac{1}{k!}\left(-\tfrac{\mathrm{i}}{2}\right)^k\int d1\ldots dkd1'\ldots dk'v\left(1;1'\right)\ldots v\left(k;k'\right)\\ &\quad\times\left\langle\mathcal{T}\left\{\hat{\psi}^{\dagger}\left(1^{+}\right)\hat{\psi}^{\dagger}\left(1'^{+}\right)\hat{\psi}\left(1'\right)\hat{\psi}(1)\ldots\hat{\psi}^{\dagger}\left(k^{+}\right)\hat{\psi}^{\dagger}\left(k'^{+}\right)\hat{\psi}\left(k'\right)\hat{\psi}(k)\hat{\psi}(a)\hat{\psi}^{\dagger}(b)\right\rangle_0\end{aligned}$$

The quantity inside time-ordering operator could be rewritten as

$$\left\langle\mathcal{T}\left\{\hat{\psi}(a)\hat{\psi}(1)\hat{\psi}\left(1'\right)\ldots\hat{\psi}(k)\hat{\psi}\left(k'\right)\hat{\psi}^{\dagger}\left(k^{+}\right)\hat{\psi}^{\dagger}\left(k^{+}\right)\ldots\hat{\psi}^{\dagger}\left(1'^{+}\right)\hat{\psi}^{\dagger}\left(1^{+}\right)\hat{\psi}^{\dagger}(b)\right\}\right\rangle_0$$

which is exactly

$$\mathrm{i}^{2k+1}Z_0G_{0,2k+1}\left(a,1,1',\ldots,k,k';b,1^{+},1'^{+},\ldots k^{+},k'^{+}\right)$$

Similarly, the k-th order term of the denominator is

$$\mathrm{i}^{2k}Z_0G_{0,2k}\left(1,1',\ldots,k,k';1^{+},1'^{+},\ldots k^{+},k'^{+}\right)$$

Therefore we have

$$G(a;b)=\frac{\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{\mathrm{i}}{2}\right)^k\int v\left(1;1'\right)\ldots v\left(k;k'\right)G_{0,2k+1}\left(a,1,1',\ldots;b,1^{+},1'^{+},\ldots\right)}{\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{\mathrm{i}}{2}\right)^k\int v\left(1;1'\right)\ldots v\left(k;k'\right)G_{0,2k}\left(1,1',\ldots;1^{+},1'^{+},\ldots\right)}$$

which could be further written using Wick's theorem

$$G(a,b)=\frac{\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{\mathrm{i}}{2}\right)^k\int v\left(1;1'\right)..v\left(k;k'\right)\begin{vmatrix}G_0(a;b) & G_0\left(a;1^{+}\right) & \ldots & G_0\left(a;k'^{+}\right)\\ G_0(1;b) & G_0\left(1;1^{+}\right) & \ldots & G_0\left(1;k'^{+}\right)\\ \vdots & \vdots & \ddots & \vdots\\ G_0\left(k';b\right) & G_0\left(k';1^{+}\right) & \ldots & G_0\left(k';k'^{+}\right)\end{vmatrix}_{\pm}}{\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{\mathrm{i}}{2}\right)^k\int v\left(1;1'\right)..v\left(k;k'\right)\begin{vmatrix}G_0\left(1;1^{+}\right) & G_0\left(1;1'^{+}\right) & \ldots & G_0\left(1;k'^{+}\right)\\ G_0\left(1';1^{+}\right) & G_0\left(1';1'^{+}\right) & \ldots & G_0\left(1';k'^{+}\right)\\ \vdots & \vdots & \ddots & \vdots\\ G_0\left(k';1^{+}\right) & G_0\left(k';1'^{+}\right) & \ldots & G_0\left(k';k'^{+}\right)\end{vmatrix}_{\pm}}$$

where the integrals are over $1,1',\cdots k,k'$.

- Similarly, we have

$$G_2(a,b;c,d)=\frac{\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{\mathrm{i}}{2}\right)^k\int v\left(1;1'\right)\ldots v\left(k;k'\right)\begin{vmatrix}G_0(a;c) & G_0(a;d) & \ldots & G_0\left(a;k'^{+}\right)\\ G_0(b;c) & G_0(b;d) & \ldots & G_0\left(b;k'^{+}\right)\\ \vdots & \vdots & \ddots & \vdots\\ G_0\left(k';c\right) & G_0\left(k';d\right) & \ldots & G_0\left(k';k'^{+}\right)\end{vmatrix}_{\pm}}{\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{\mathrm{i}}{2}\right)^k\int v\left(1;1'\right)\ldots v\left(k;k'\right)\begin{vmatrix}G_0\left(1;1^{+}\right) & G_0\left(1;1'^{+}\right) & \ldots & G_0\left(1;k'^{+}\right)\\ G_0\left(1';1^{+}\right) & G_0\left(1';1'^{+}\right) & \ldots & G_0\left(1';k'^{+}\right)\\ \vdots & \vdots & \ddots & \vdots\\ G_0\left(k';1^{+}\right) & G_0\left(k';1'^{+}\right) & \ldots & G_0\left(k';k'^{+}\right)\end{vmatrix}_{\pm}}$$

We also have Z/Z_0, which is just the denominator above

$$\frac{Z}{Z_0}=\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{\mathrm{i}}{2}\right)^k\int v\left(1;1'\right)\ldots v\left(k;k'\right)\begin{vmatrix}G_0\left(1;1^{+}\right) & G_0\left(1;1'^{+}\right) & \ldots & G_0\left(1;k'^{+}\right)\\ G_0\left(1';1^{+}\right) & G_0\left(1';1'^{+}\right) & \ldots & G_0\left(1';k'^{+}\right)\\ \vdots & \vdots & \ddots & \vdots\\ G_0\left(k';1^{+}\right) & G_0\left(k';1'^{+}\right) & \ldots & G_0\left(k';k'^{+}\right)\end{vmatrix}_{\pm}$$

This is the full mathematical description of many body perturbation theory.

9 Analytic continuity rules

Here we consider the analytic property of two-point correlators $k(z,z')$, such as one-particle Green's function.

$$k\left(z,z'\right)=\operatorname{Tr}\left[\hat{\rho}\hat{k}\left(z,z'\right)\right]=\operatorname{Tr}\left[\hat{\rho}\mathcal{T}\left\{\hat{O}_1(z)\hat{O}_2\left(z'\right)\right\}\right]$$

and the discussion here based on $\hat{O}(t_+)=\hat{O}(t_-)$. We only consider the contour as being the right side of figure 2.

- We have

  $$k\left(z,z'\right)=\theta\left(z,z'\right)k^>\left(z,z'\right)+\theta\left(z',z\right)k^<\left(z,z'\right)$$

  where $k^>$ and $k^<$ are:

  $$k^>\left(z,z'\right)=\operatorname{Tr}\left[\hat{\rho}\hat{O}_1(z)\hat{O}_2\left(z'\right)\right],\quad k^<\left(z,z'\right)=\pm\operatorname{Tr}\left[\hat{\rho}\hat{O}_2\left(z'\right)\hat{O}_1(z)\right]$$

- $k(z,z')$ is the same value for z,z' being on forward/backward branch:

  $$k^{\lessgtr}\left(t_+,z'\right)=k^{\lessgtr}\left(t_-,z'\right),\quad k^{\lessgtr}\left(z,t'_+\right)=k^{\lessgtr}\left(z,t'_-\right)$$

- A function is said to belong to Keldysh space, if

  $$k\left(z,z'\right)=k^{\delta}(z)\delta\left(z,z'\right)+\theta\left(z,z'\right)k^>\left(z,z'\right)+\theta\left(z',z\right)k^<\left(z,z'\right)$$

  Here $k^{\delta}(t)=k^{\delta}(t_-)=k^{\delta}(t_+)$.

- On the real time axis, we can define the greater and lesser Keldysh components as followed:

  $$\begin{aligned}k^>\left(t,t'\right)&\equiv k\left(t_+,t'_-\right)\\ k^<\left(t,t'\right)&\equiv k\left(t_-,t'_+\right)\end{aligned}$$

- We also define the left and right Keldysh components from $k(z,z')$ with one real time t and one imaginary time $t_0-\mathrm{i}\tau$:

  $$\begin{aligned}k^{\lceil}(\tau,t)&\equiv k\left(t_0-\mathrm{i}\tau,t_{\pm}\right)\\ k^{\rceil}(t,\tau)&\equiv k\left(t_{\pm},t_0-\mathrm{i}\tau\right)\end{aligned}$$

- We can also define the Matsubara component $k^M(\tau,\tau')$ with both contour arguments on the vertical track:

  $$\begin{aligned}k^{\mathrm{M}}\left(\tau,\tau'\right)&\equiv k\left(t_0-\mathrm{i}\tau,t_0-\mathrm{i}\tau'\right)\\ &=\delta\left(t_0-\mathrm{i}\tau,t_0-\mathrm{i}\tau'\right)k^{\delta}\left(t_0-\mathrm{i}\tau\right)+k_r^{\mathrm{M}}\left(\tau,\tau'\right)\end{aligned}$$

  where

  $$k_r^{\mathrm{M}}\left(\tau,\tau'\right)=\theta\left(\tau-\tau'\right)k^>\left(t_0-\mathrm{i}\tau,t_0-\mathrm{i}\tau'\right)+\theta\left(\tau'-\tau\right)k^<\left(t_0-\mathrm{i}\tau,t_0-\mathrm{i}\tau'\right)$$

  Because $\delta\left(t_0-\mathrm{i}\tau,t_0-\mathrm{i}\tau'\right)=\mathrm{i}\delta\left(\tau-\tau'\right)$, therefore

  $$k^{\mathrm{M}}\left(\tau,\tau'\right)=\mathrm{i}\delta\left(\tau-\tau'\right)k^{\delta}(\tau)+k_r^{\mathrm{M}}\left(\tau,\tau'\right)$$

- We also define the retarded and advanced component:

$$k^{\mathrm{R}}\left(t, t^{\prime}\right) \equiv k^{\delta}(t) \delta\left(t-t^{\prime}\right)+\theta\left(t-t^{\prime}\right)\left[k^{>}\left(t, t^{\prime}\right)-k^{<}\left(t, t^{\prime}\right)\right]$$
$$k^{\mathrm{A}}\left(t, t^{\prime}\right) \equiv k^{\delta}(t) \delta\left(t-t^{\prime}\right)-\theta\left(t^{\prime}-t\right)\left[k^{>}\left(t, t^{\prime}\right)-k^{<}\left(t, t^{\prime}\right)\right]$$

  and the time-ordered and anti time-ordered component:

$$k^{\mathrm{T}}\left(t, t^{\prime}\right) \equiv k\left(t_{-}, t_{-}^{\prime}\right)=k^{\delta}(t) \delta\left(t-t^{\prime}\right)+\theta\left(t-t^{\prime}\right) k^{>}\left(t, t^{\prime}\right)+\theta\left(t^{\prime}-t\right) k^{<}\left(t, t^{\prime}\right)$$
$$k^{\overline{\mathrm{T}}}\left(t, t^{\prime}\right) \equiv k\left(t_{+}, t_{+}^{\prime}\right)=-k^{\delta}(t) \delta\left(t-t^{\prime}\right)+\theta\left(t^{\prime}-t\right) k^{>}\left(t, t^{\prime}\right)+\theta\left(t-t^{\prime}\right) k^{<}\left(t, t^{\prime}\right)$$

  There are a lot to be filled in this section.....

10 Physics of one-particle Green's function

We begin by discussing some other formalism of Green's function.

- The Green's function could be regarded as a first quantization operator:

$$\hat{\mathcal{G}}\left(z_{1}, z_{2}\right)=\int d\boldsymbol{x}_{1} d\boldsymbol{x}_{2}\left|\boldsymbol{x}_{1}\right\rangle G(1 ; 2)\left\langle\boldsymbol{x}_{2}\right|$$

  whose matrix element is

$$\left\langle\boldsymbol{x}_{1}\left|\hat{\mathcal{G}}\left(z_{1}, z_{2}\right)\right| \boldsymbol{x}_{2}\right\rangle=G(1 ; 2)$$

- On the other hand, recall that

$$\hat{\psi}(\boldsymbol{x})=\sum_{i} \varphi_{i}(\boldsymbol{x}) \hat{d}_{i}, \quad \hat{\psi}^{\dagger}(\boldsymbol{x})=\sum_{i} \varphi_{i}^{*}(\boldsymbol{x}) \hat{d}_{i}^{\dagger}$$

  we can also define the following matrix Green's function:

$$G_{ji}\left(z_{1}, z_{2}\right)=\frac{1}{\mathrm{i}} \frac{\operatorname{Tr}\left[e^{-\beta \hat{H}^{\mathrm{M}}} \mathcal{T}\left\{\hat{d}_{j, H}\left(z_{1}\right) \hat{d}_{i, H}^{\dagger}\left(z_{2}\right)\right\}\right]}{\operatorname{Tr}\left[e^{-\beta \hat{H}^{\mathrm{M}}}\right]}$$

- Their relations are:

$$G(1 ; 2)=\sum_{ji} \varphi_{j}\left(\boldsymbol{x}_{1}\right) G_{ji}\left(z_{1}, z_{2}\right) \varphi_{i}^{*}\left(\boldsymbol{x}_{2}\right)=\sum_{ji}\left\langle\boldsymbol{x}_{1} \mid j\right\rangle G_{ji}\left(z_{1}, z_{2}\right)\left\langle i \mid \boldsymbol{x}_{2}\right\rangle$$
$$\hat{\mathcal{G}}\left(z_{1}, z_{2}\right)=\sum_{ji}|j\rangle G_{ji}\left(z_{1}, z_{2}\right)\langle i|$$

- For non-interacting Green's function, we have:

$$\left(\mathrm{i} \frac{\mathrm{d}}{\mathrm{d} z_{1}}-\hat{h}\left(z_{1}\right)\right) \hat{\mathcal{G}}_{0}\left(z_{1}, z_{2}\right)=\delta\left(z_{1}, z_{2}\right)$$

  For interactimg case, we have

$$\left[\mathrm{i} \frac{d}{d z_{1}}-\hat{h}\left(z_{1}\right)\right] \hat{\mathcal{G}}\left(z_{1}, z_{2}\right)-\int_{\gamma} d \bar{z} \hat{\Sigma}\left(z_{1}, \bar{z}\right) \hat{\mathcal{G}}\left(\bar{z}, z_{2}\right)=\delta\left(z_{1}, z_{2}\right)$$

Now we talk about Matsubara Green's function.

- Let $z_1 = t_0 - \mathrm{i}\tau_1$ and $z_2 = t_0 - \mathrm{i}\tau_2$. The definition of Matsubara Green's function is

$$G_{ji}^{\mathrm{M}}(\tau_1, \tau_2) = \frac{1}{\mathrm{i}}\left\{\theta(\tau_1 - \tau_2)\frac{\mathrm{Tr}\left[e^{(\tau_1-\tau_2-\beta)\hat{H}^{\mathrm{M}}}\hat{d}_j e^{(\tau_2-\tau_1)\hat{H}^{\mathrm{M}}}\hat{d}_i^\dagger\right]}{\mathrm{Tr}\left[e^{-\beta\hat{H}^{\mathrm{M}}}\right]} \pm \theta(\tau_2 - \tau_1)\frac{\mathrm{Tr}\left[e^{(\tau_2-\tau_1-\beta)\hat{H}^{\mathrm{M}}}\hat{d}_i^\dagger e^{(\tau_1-\tau_2)\hat{H}^{\mathrm{M}}}\hat{d}_j\right]}{\mathrm{Tr}\left[e^{-\beta\hat{H}^{\mathrm{M}}}\right]}\right\}$$

 Here $\hat{H}^M = \hat{H}(t_0) - \mu\hat{N}$.

- For any one-body operator $\hat{O}$

$$\hat{O} = \sum_{ij} O_{ij}\hat{d}_i^\dagger\hat{d}_j$$

 we have

$$\langle O\rangle = \frac{\mathrm{Tr}\left[e^{-\beta\hat{H}^M}\hat{O}\right]}{\mathrm{Tr}\left[e^{-\beta\hat{H}^M}\right]} = \sum_{ij} O_{ij}\frac{\mathrm{Tr}\left[e^{-\beta\hat{H}^M}\hat{d}_i^\dagger\hat{d}_j\right]}{\mathrm{Tr}\left[e^{-\beta\hat{H}^M}\right]} = \pm\mathrm{i}\sum_{ij} O_{ij}G_{ji}^{\mathrm{M}}\left(\tau, \tau^+\right)$$

- $G_{ji}^M(\tau_1, \tau_2)$ only depends on $\tau_1 - \tau_2$.

- KMS relation:

$$\hat{\mathcal{G}}^{\mathrm{M}}(0,\tau) = \pm\hat{\mathcal{G}}^{\mathrm{M}}(\beta,\tau), \quad \hat{\mathcal{G}}^{\mathrm{M}}(\tau,0) = \pm\hat{\mathcal{G}}^{\mathrm{M}}(\tau,\beta)$$

- Therefore the Matsubara Green's function could be expended using Fourier series:

$$\hat{\mathcal{G}}^{\mathrm{M}}(\tau_1, \tau_2) = \frac{1}{-\mathrm{i}\beta}\sum_{m=-\infty}^{\infty} e^{-\omega_m(\tau_1-\tau_2)}\hat{\mathcal{G}}^{\mathrm{M}}(\omega_m)$$

 where the Matsubara frequencies are

$$\omega_m = \begin{cases} \frac{\mathrm{i}2m\pi}{\beta} & \text{for bosons} \\ \frac{\mathrm{i}(2m+1)\pi}{\beta} & \text{for fermions} \end{cases}$$

- Let's look at example. For

$$\hat{H}^{\mathrm{M}} = \sum_{ij} h_{ij}^{\mathrm{M}}\hat{d}_i^\dagger\hat{d}_j$$

 with the equation of motion, we have

$$\left[-\frac{d}{d\tau_1} - \hat{h}^{\mathrm{M}}\right]\hat{\mathcal{G}}^{\mathrm{M}}(\tau_1, \tau_2) = \delta(-\mathrm{i}\tau_1 + \mathrm{i}\tau_2) = \mathrm{i}\delta(\tau_1 - \tau_2)$$

 with the help of

$$\delta(\tau) = \frac{1}{\beta}\sum_{m=-\infty}^{\infty}\begin{cases} e^{-\mathrm{i}\frac{2m\pi}{\beta}\tau} \\ e^{-\mathrm{i}\frac{(2m+1)\pi}{\beta}\tau}\tau \end{cases} = \frac{1}{\beta}\sum_{m=-\infty}^{\infty} e^{-\omega_m\tau}$$

 we get

$$\hat{\mathcal{G}}^{\mathrm{M}}(\omega_m) = \frac{1}{\omega_m - \hat{h}^{\mathrm{M}}}$$

Now we discuss lesser/greater Green's function.

- The definition is

$$G^<_{ji}(t,t') = \mp\mathrm{i}\frac{\mathrm{Tr}\left[e^{-\beta\hat{H}^M}\hat{d}^\dagger_{i,H}(t')\,\hat{d}_{j,H}(t)\right]}{\mathrm{Tr}\left[e^{-\beta\hat{H}^M}\right]} = \mp\mathrm{i}\sum_k \rho_k \left\langle \Psi_k \left| \hat{d}^\dagger_{i,H}(t')\,\hat{d}_{j,H}(t) \right| \Psi_k \right\rangle$$

$$G^>_{ji}(t,t') = -\mathrm{i}\frac{\mathrm{Tr}\left[e^{-\beta\hat{H}^\mathrm{M}}\hat{d}_{j,H}(t)\hat{d}^\dagger_{i,H}(t')\right]}{\mathrm{Tr}\left[e^{-\beta\hat{H}^\mathrm{M}}\right]} = -\mathrm{i}\sum_k \rho_k \left\langle \Psi_k \left| \hat{d}_{j,H}(t)\hat{d}^\dagger_{i,H}(t') \right| \Psi_k \right\rangle$$

- We have:

$$\left[G^>_{ji}(t,t')\right]^* = -G^>_{ij}(t',t)\,, \quad \left[G^<_{ji}(t,t')\right]^* = -G^<_{ij}(t',t)$$

  and consequently

$$\hat{\mathcal{G}}^\mathrm{R}(t,t') = \theta(t-t')\left[\hat{\mathcal{G}}^>(t,t') - \hat{\mathcal{G}}^<(t,t')\right] = \left[\hat{\mathcal{G}}^\mathrm{A}(t',t)\right]^\dagger$$

- The lesser Green's function could be used to calculate one-body operator. For example:

$$n_i(t) = \pm\mathrm{i}\sum_{ij} O_{ij}G^<_{ji}(t,t)$$

- For $t = t_0$, we have for all τ:

$$\hat{\mathcal{G}}(t_0,t_0) = \hat{\mathcal{G}}^\mathrm{M}(\tau,\tau^+)$$

- If $\hat{H}(t) = \hat{H}$ for $t > t_0$, we have

$$G^<_{ji}(t,t') = \mp\mathrm{i}\sum_k \rho_k \left\langle \Psi_k \left| e^{\mathrm{i}\hat{H}(t'-t_0)}\hat{d}^\dagger_i e^{-\mathrm{i}\hat{H}(t'-t)}\hat{d}_j e^{-\mathrm{i}\hat{H}(t-t_0)} \right| \Psi_k \right\rangle$$

- In general, $G^<_{ij}$ is not a function of $t'-t$, because Ψ_k is not necessarily the eigenstates of driving Hamiltonian $\hat{H}$.

11 Non-interacting Green's function

We are talking about non-interacting case, where the Hamiltonian is

$$\hat{H}(z) = \sum_{ij} h_{ij}(z)\hat{d}^\dagger_i\hat{d}_j = \sum_{ij}\langle i|\hat{h}(z)|j\rangle\hat{d}^\dagger_i\hat{d}_j$$

- We write $\hat{\mathcal{G}}$ as:

$$\hat{\mathcal{G}}(z_1,z_2) = \hat{\mathcal{U}}_L(z_1)\hat{\mathcal{F}}(z_1,z_2)\hat{\mathcal{U}}_R(z_2)$$

  Here $\hat{\mathcal{U}}_{L/R}(z)$ fulfill:

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}z}\hat{\mathcal{U}}_L(z) = \hat{h}(z)\hat{\mathcal{U}}_L(z), \quad \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}z}\hat{\mathcal{U}}_R(z) = -\hat{\mathcal{U}}_R(z)\hat{h}(z)$$

  with $\hat{\mathcal{U}}_L(t_{0-}) = \hat{\mathcal{U}}_R(t_{0-}) = \mathbf{1}$.

- The solution of $\hat{\mathcal{U}}_L$ and $\hat{\mathcal{U}}_R$ is

$$\hat{\mathcal{U}}_L(z) = \mathcal{T}\left\{e^{-\mathrm{i}\int_{t_0}^{z} \mathrm{d}\bar{z}\hat{h}(\bar{z})}\right\}, \quad \hat{\mathcal{U}}_R(z) = \overline{\mathcal{T}}\left\{e^{+\mathrm{i}\int_{t_0}^{z} \mathrm{d}\bar{z}\hat{h}(\bar{z})}\right\}$$

- Therefore we have the differential equation for $\hat{\mathcal{F}}$:

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}z_1}\hat{\mathcal{F}}\left(z_1, z_2\right) = \delta\left(z_1, z_2\right), \quad -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}z_2}\hat{\mathcal{F}}\left(z_1, z_2\right) = \delta\left(z_1, z_2\right)$$

- We have

$$\hat{\mathcal{F}}\left(z_1, z_2\right) = \theta\left(z_1, z_2\right)\hat{\mathcal{F}}^{>} + \theta\left(z_2, z_1\right)\hat{\mathcal{F}}^{<}$$

  where

$$\hat{\mathcal{F}}^{>} - \hat{\mathcal{F}}^{<} = -\mathrm{i}\hat{\mathbf{1}}$$

- With KMS relations:

$$\hat{\mathcal{G}}\left(t_{0-}, z'\right) = \pm\hat{\mathcal{G}}\left(t_0 - \mathrm{i}\beta, z'\right) \quad \begin{cases} + \text{ for bosons} \\ - \text{ for fermions} \end{cases}$$

  we have

$$\hat{\mathcal{F}}^{<} = \pm\hat{\mathcal{U}}_L\left(t_0 - \mathrm{i}\beta\right)\hat{\mathcal{F}}^{>} = \pm e^{-\beta\hat{h}^{\mathrm{M}}}\hat{\mathcal{F}}^{>} \quad \begin{cases} + \text{ for bosons} \\ - \text{ for fermions} \end{cases}$$

  Then

$$\hat{\mathcal{F}}^{<} = \mp\mathrm{i}\frac{1}{e^{\beta\hat{h}^{M}} \mp \hat{\mathbf{1}}} = \mp\mathrm{i}f\left(\hat{h}^{\mathrm{M}}\right)$$

  and

$$\hat{\mathcal{F}}^{>} = \pm\mathrm{i}\frac{1}{e^{-\beta\hat{h}^{\mathrm{M}}} \mp \hat{\mathbf{1}}} = -\mathrm{i}\bar{f}\left(\hat{h}^{\mathrm{M}}\right)$$

  where

$$f(\omega) = 1/\left[e^{\beta\omega} \mp 1\right], \quad \bar{f}(\omega) = 1 \pm f(\omega) = e^{\beta\omega}f(\omega)$$

Now we are talking about the Matsubara component:

- Let $z_1 = t_0 - \mathrm{i}\tau_1$, $z_2 = t_0 - \mathrm{i}\tau_2$, we have

$$\hat{\mathcal{U}}_L\left(t_0 - \mathrm{i}\tau\right) = e^{-\tau\hat{h}^{\mathrm{M}}}, \quad \hat{\mathcal{U}}_R\left(t_0 - \mathrm{i}\tau\right) = e^{\tau\hat{h}^{\mathrm{M}}}$$

  Then we have

$$\hat{\mathcal{G}}^{\mathrm{M}}\left(\tau_1, \tau_2\right) = -\mathrm{i}\left[\theta\left(\tau_1 - \tau_2\right)\bar{f}\left(\hat{h}^{\mathrm{M}}\right) \pm \theta\left(\tau_2 - \tau_1\right)f\left(\hat{h}^{\mathrm{M}}\right)\right]e^{-\left(\tau_1 - \tau_2\right)\hat{h}^{\mathrm{M}}}$$

- Another way to write this, using result from previous section, is that

$$\hat{\mathcal{G}}^{\mathrm{M}}\left(\tau_1, \tau_2\right) = \frac{1}{-\mathrm{i}\beta}\sum_{m=-\infty}^{\infty}\frac{e^{-\omega_m\left(\tau_1 - \tau_2\right)}}{\omega_m - \hat{h}^{\mathrm{M}}}$$

- We can prove they are equivalent.

Now let's talk about lesser and greater component. Inspired by

$$\hat{\mathcal{G}}(z_1, z_2) = -\mathrm{i}\hat{\mathcal{U}}_L(z_1)\left(\theta(z_1 - z_2)\bar{f}(\hat{h}^M) \pm \theta(z_2 - z_1)f(\hat{h}^M)\right)\hat{\mathcal{U}}_R(z_2)$$

and with

$$\hat{U}_L(t_\pm) = T\left(e^{-\mathrm{i}\int_{t_0}^{t}\mathrm{d}\bar{t}\hat{h}(\bar{t})}\right) = \hat{\mathcal{U}}(t), \quad \hat{\mathcal{U}}_R(t_\pm) = \hat{\mathcal{U}}^\dagger(t)$$

we have:

- The lesser component:
$$\hat{\mathcal{G}}^<(t_1, t_2) = \mp\mathrm{i}\hat{\mathcal{U}}(t_1)f(\hat{h}^M)\hat{\mathcal{U}}^\dagger(t_2)$$
- The greater component:
$$\hat{\mathcal{G}}^>(t_1, t_2) = -\mathrm{i}\hat{\mathcal{U}}(t_1)\bar{f}(\hat{h}^M)\hat{\mathcal{U}}^\dagger(t_2)$$
- Furthermore, we have:
$$\begin{aligned}\hat{\mathcal{G}}^<(t_1, t_2) &= \mp\mathrm{i}\sum_\lambda f(\epsilon_\lambda^M)\hat{\mathcal{U}}(t_1)|\lambda^M\rangle\langle\lambda^M|\hat{\mathcal{U}}^\dagger(t_2)\\ &= \mp\mathrm{i}\sum_\lambda f(\epsilon_\lambda^M)|\lambda^M(t_1)\rangle\langle\lambda^M(t_2)|\end{aligned}$$
- The density is
$$n(\boldsymbol{x}, t) = \pm\mathrm{i}G^<(\boldsymbol{x}, t; \boldsymbol{x}, t)$$
- If the Hamiltonian is constant, we have
$$\hat{\mathcal{G}}^<(t_1, t_2) = \mp\mathrm{i}f(\hat{h}^M)e^{-\mathrm{i}\hat{h}(t_1 - t_2)}$$
$$\hat{\mathcal{G}}^>(t_1, t_2) = -\mathrm{i}\bar{f}(\hat{h}^M)e^{-\mathrm{i}\hat{h}(t_1 - t_2)}$$
There is no dependence on t_0! Only dependence on $t_1 - t_2$.
- We can define the Fourier transform:
$$\hat{\mathcal{G}}^<(t_1, t_2) = \int\frac{\mathrm{d}\omega}{2\pi}e^{-\mathrm{i}\omega(t_1 - t_2)}\hat{\mathcal{G}}^<(\omega)$$
$$\hat{\mathcal{G}}^>(t_1, t_2) = \int\frac{\mathrm{d}\omega}{2\pi}e^{-\mathrm{i}\omega(t_1 - t_2)}\hat{\mathcal{G}}^>(\omega)$$
We can directly see that
$$\hat{\mathcal{G}}^<(\omega) = \mp2\pi\mathrm{i}f(\hat{h}^M)\delta(\omega - \hat{h})$$
$$\hat{\mathcal{G}}^>(\omega) = -2\pi\mathrm{i}\bar{f}(\hat{h}^M)\delta(\omega - \hat{h}) = \pm e^{\beta h^M}\hat{\mathcal{G}}^<(\omega)$$

Now let's discuss other components.

- The retarded Green's function is
$$\begin{aligned}\hat{\mathcal{G}}^R(t_1, t_2) &= \theta(t_1 - t_2)\left(\hat{\mathcal{G}}^>(t_1, t_2) - \hat{\mathcal{G}}^<(t_1, t_2)\right)\\ &= -\mathrm{i}\theta(t_1 - t_2)\hat{\mathcal{U}}(t_1)\hat{\mathcal{U}}^\dagger(t_2) = -\mathrm{i}\theta(t_1 - t_2)T\left(e^{-\mathrm{i}\int_{t_2}^{t_1}\mathrm{d}\bar{t}\hat{h}(\bar{t})}\right)\end{aligned}$$

- The advanced Green's function is

$$\hat{\mathcal{G}}^{A}(t_1,t_2)=\mathrm{i}\theta(t_2-t_1)\bar{T}\left(e^{\mathrm{i}\int_{t_2}^{t_1}\mathrm{d}\bar{t}\hat{h}(\bar{t})}\right)=\left(\hat{\mathcal{G}}^{R}(t_2,t_1)\right)^{\dagger}$$

  This doesn't depend on the original density matrix, only depends on the Hamiltonian.

- For constant Hamiltonian $\hat{h}(t)=\hat{h}$, we have

$$\hat{\mathcal{G}}^{R}(t_1,t_2)=-\mathrm{i}\theta(t_1-t_2)e^{-\mathrm{i}\hat{h}(t_1-t_2)}$$

$$\hat{\mathcal{G}}^{A}(t_1,t_2)=\mathrm{i}\theta(t_2-t_1)e^{\mathrm{i}\hat{h}(t_2-t_1)}$$

- The Fourier transform is defined as

$$\hat{\mathcal{G}}^{R,A}(t_1,t_2)=\int\frac{\mathrm{d}\omega}{2\pi}e^{-\mathrm{i}\omega(t_1-t_2)}\hat{\mathcal{G}}^{R,A}(\omega)$$

- We have the integration form of step function:

$$\theta\left(t_1-t_2\right)=\mathrm{i}\int\frac{d\omega}{2\pi}\frac{e^{-\mathrm{i}\omega(t_1-t_2)}}{\omega+\mathrm{i}\eta}$$

- Therefore

$$\hat{\mathcal{G}}^{\mathrm{R}}\left(t_1,t_2\right)=\int\frac{d\omega}{2\pi}\frac{e^{-\mathrm{i}(\omega+\hat{h})(t_1-t_2)}}{\omega+\mathrm{i}\eta}$$

  Changing variables from $\omega\to\omega-\hat{h}$, we have

$$\hat{\mathcal{G}}^{\mathrm{R}}(\omega)=\frac{1}{\omega-\hat{h}+\mathrm{i}\eta}=\sum_{\lambda}\frac{|\lambda\rangle\langle\lambda|}{\omega-\epsilon_{\lambda}+\mathrm{i}\eta}$$

  And because

$$\hat{\mathcal{G}}^{\mathrm{A}}(\omega)=\left[\hat{\mathcal{G}}^{\mathrm{R}}(\omega)\right]^{\dagger}$$

  we have

$$\hat{\mathcal{G}}^{\mathrm{A}}(\omega)=\frac{1}{\omega-\hat{h}-\mathrm{i}\eta}=\sum_{\lambda}\frac{|\lambda\rangle\langle\lambda|}{\omega-\epsilon_{\lambda}-\mathrm{i}\eta}$$

- Given that

$$\hat{\mathcal{G}}^{\mathrm{R}}-\hat{\mathcal{G}}^{\mathrm{A}}=\hat{\mathcal{G}}^{>}-\hat{\mathcal{G}}^{<}$$

  we have

$$\hat{\mathcal{G}}^{<}(\omega)=\pm f\left(\hat{h}^{\mathrm{M}}\right)\left[\hat{\mathcal{G}}^{\mathrm{R}}(\omega)-\hat{\mathcal{G}}^{\mathrm{A}}(\omega)\right]$$

  and

$$\hat{\mathcal{G}}^{>}(\omega)=\bar{f}\left(\hat{h}^{\mathrm{M}}\right)\left[\hat{\mathcal{G}}^{\mathrm{R}}(\omega)-\hat{\mathcal{G}}^{\mathrm{A}}(\omega)\right]$$

- In the case $\hat{h}^{\mathrm{M}}=\hat{h}-\mu$, we have

$$\hat{\mathcal{G}}^{<}(\omega)=\pm f(\omega-\mu)\left[\hat{\mathcal{G}}^{\mathrm{R}}(\omega)-\hat{\mathcal{G}}^{\mathrm{A}}(\omega)\right],\quad\hat{\mathcal{G}}^{>}(\omega)=\bar{f}(\omega-\mu)\left[\hat{\mathcal{G}}^{\mathrm{R}}(\omega)-\hat{\mathcal{G}}^{\mathrm{A}}(\omega)\right]$$

- Left and right component:

$$\hat{\mathcal{G}}^{\rceil}(t,\tau) = \mp\mathrm{i}\hat{\mathcal{U}}(t) f\left(\hat{h}^{\mathrm{M}}\right) e^{\tau \hat{h}^{\mathrm{M}}} = \mathrm{i}\hat{\mathcal{G}}^{\mathrm{R}}\left(t,t_0\right)\hat{\mathcal{G}}^{\mathrm{M}}(0,\tau)$$

$$\hat{\mathcal{G}}^{\lceil}(\tau,t) = -\mathrm{i}e^{-\tau\hat{h}^{\mathrm{M}}}\bar{f}\left(\hat{h}^{\mathrm{M}}\right)\hat{\mathcal{U}}^{\dagger}(t) = -\mathrm{i}\hat{\mathcal{G}}^{\mathrm{M}}(\tau,0)\hat{\mathcal{G}}^{\mathrm{A}}\left(t_0,t\right)$$

- We have time-ordered Green's function

$$\hat{\mathcal{G}}^{\mathrm{T}}\left(t_1,t_2\right) = -\mathrm{i}\hat{\mathcal{U}}\left(t_1\right)\left[\theta\left(t_1-t_2\right)\bar{f}\left(\hat{h}^{\mathrm{M}}\right) \pm \theta\left(t_2-t_1\right) f\left(\hat{h}^{\mathrm{M}}\right)\right]\hat{\mathcal{U}}^{\dagger}\left(t_2\right)$$

and we have

$$\hat{\mathcal{G}}^{\mathrm{T}}\left(t_1,t_2\right) = \int\frac{d\omega}{2\pi}e^{-\mathrm{i}\omega(t_1-t_2)}\underbrace{\left[\frac{\bar{f}\left(\hat{h}^{\mathrm{M}}\right)}{\omega-\hat{h}+\mathrm{i}\eta} \mp \frac{f\left(\hat{h}^{\mathrm{M}}\right)}{\omega-\hat{h}-\mathrm{i}\eta}\right]}_{\hat{\mathcal{G}}^{\mathrm{T}}(\omega)}$$

- For fermions in zero temperature, $\hat{h}^{\mathrm{M}} = \hat{h} - \epsilon_{\mathrm{F}}$, ϵ_{F} is the Fermi energy, and we have $f\left(\epsilon_\lambda - \mu\right) = \theta\left(\epsilon_{\mathrm{F}} - \epsilon_\lambda\right)$. Then we have

$$\hat{\mathcal{G}}^{\mathrm{T}}(\omega) = \sum_{\epsilon_\lambda > \epsilon_{\mathrm{F}}}\frac{|\lambda\rangle\langle\lambda|}{\omega-\epsilon_\lambda+\mathrm{i}\eta} + \sum_{\epsilon_\lambda < \epsilon_{\mathrm{F}}}\frac{|\lambda\rangle\langle\lambda|}{\omega-\epsilon_\lambda-\mathrm{i}\eta}$$

12 Interacting Green's function

Consider Hamiltonian $\hat{H}$ and corresponding $\hat{H}^M = \hat{H} - \mu\hat{N}$. Consider the eigenstates of $|\Psi_k\rangle$ of $\hat{H}^M$.

- We have

$$G_{ji}^{<}\left(t,t'\right) = \mp\mathrm{i}\sum\rho_k\left\langle\Psi_k\left|\hat{d}_i^{\dagger}e^{-\mathrm{i}\left(\hat{H}-E_k\right)(t'-t)}\hat{d}_j\right|\Psi_k\right\rangle$$
$$G_{ji}^{>}\left(t,t'\right) = -\mathrm{i}\sum_k\rho_k\left\langle\Psi_k\left|\hat{d}_je^{-\mathrm{i}\left(\hat{H}-E_k\right)(t-t')}\hat{d}_i^{\dagger}\right|\Psi_k\right\rangle$$

- Lehmann representation:

$$G_{ji}^{<}\left(t,t'\right) = \mp\mathrm{i}\sum_{pk}\rho_k\Phi_{pk}^*(i)\Phi_{pk}(j)e^{-\mathrm{i}(E_p-E_k)(t'-t)}$$

$$G_{ji}^{>}\left(t,t'\right) = -\mathrm{i}\sum_{pk}\rho_k\Phi_{kp}(j)\Phi_{kp}^*(i)e^{-\mathrm{i}(E_p-E_k)(t-t')}$$

where

$$\Phi_{kp}(i) = \left\langle\Psi_k\left|\hat{d}_i\right|\Psi_p\right\rangle$$

For this term to be nonzero, $|\Psi_p\rangle$ must contain one more particle than $|\Psi_k\rangle$.

- Then we get the Fourier transform:

$$G_{ji}^{<}(\omega)=\mp 2\pi\mathrm{i}\sum_{pk}\rho_k\Phi_{pk}^{*}(i)\Phi_{pk}(j)\delta\left(\omega-E_k+E_p\right)$$

and

$$G_{ji}^{>}(\omega)=-2\pi\mathrm{i}\sum_{pk}\rho_k\Phi_{kp}(j)\Phi_{kp}^{*}(i)\delta\left(\omega-E_p+E_k\right)$$

and we have the property

$$\mathrm{i}G_{jj}^{>}(\omega)\geq 0$$

$$\mathrm{i}G_{jj}^{<}(\omega)\geq 0 \quad \text{for bosons ;} \quad \mathrm{i}G_{jj}^{<}(\omega)\leq 0 \quad \text{for fermions}$$

- For retarded and advanced function, we have

$$\hat{\mathcal{G}}^{\mathrm{R}}\left(t,t'\right)=\mathrm{i}\int\frac{d\omega}{2\pi}\frac{e^{-\mathrm{i}\omega(t-t')}}{\omega+\mathrm{i}\eta}\int\frac{d\omega'}{2\pi}e^{-\mathrm{i}\omega'(t-t')}\left[\hat{\mathcal{G}}^{>}\left(\omega'\right)-\hat{\mathcal{G}}^{<}\left(\omega'\right)\right]$$

therefore

$$\hat{\mathcal{G}}^{\mathrm{R}}(\omega)=\mathrm{i}\int\frac{d\omega'}{2\pi}\frac{\hat{\mathcal{G}}^{>}\left(\omega'\right)-\hat{\mathcal{G}}^{<}\left(\omega'\right)}{\omega-\omega'+\mathrm{i}\eta}$$

with

$$\hat{\mathcal{G}}^{\mathrm{A}}(\omega)=\left[\hat{\mathcal{G}}^{\mathrm{R}}(\omega)\right]^{\dagger}$$

then

$$\hat{\mathcal{G}}^{\mathrm{A}}(\omega)=\mathrm{i}\int\frac{d\omega'}{2\pi}\frac{\hat{\mathcal{G}}^{>}\left(\omega'\right)-\hat{\mathcal{G}}^{<}\left(\omega'\right)}{\omega-\omega'-\mathrm{i}\eta}$$

therefore

$$G_{ji}^{\mathrm{R/A}}(\omega)=\sum_{pk}\frac{\Phi_{kp}(j)\Phi_{kp}^{*}(i)}{\omega-E_p+E_k\pm\mathrm{i}\eta}\left[\rho_k\mp\rho_p\right]$$

- For thermodynamic equilibrium,

$$\hat{H}^{\mathrm{M}}=\hat{H}-\mu\hat{N}$$

we have

$$\rho_k=\frac{e^{-\beta(E_k-\mu N_k)}}{\mathrm{Tr}\left[e^{-\beta(\hat{H}-\mu\hat{N})}\right]}=e^{-\beta(E_k-E_p)+\beta\mu(N_k-N_p)}\rho_p$$

- We can prove that

$$\hat{\mathcal{G}}^{>}(\omega)=\pm e^{\beta(\omega-\mu)}\hat{\mathcal{G}}^{<}(\omega)$$

- We have

$$\hat{\mathcal{G}}^{>}(\omega)-\hat{\mathcal{G}}^{<}(\omega)=\hat{\mathcal{G}}^{\mathrm{R}}(\omega)-\hat{\mathcal{G}}^{\mathrm{A}}(\omega)$$

Then

$$\hat{\mathcal{G}}^{<}(\omega)=\pm f(\omega-\mu)\left[\hat{\mathcal{G}}^{\mathrm{R}}(\omega)-\hat{\mathcal{G}}^{\mathrm{A}}(\omega)\right]$$

$$\hat{\mathcal{G}}^{>}(\omega)=\bar{f}(\omega-\mu)\left[\hat{\mathcal{G}}^{\mathrm{R}}(\omega)-\hat{\mathcal{G}}^{\mathrm{A}}(\omega)\right]$$

- For left component, we have

$$G_{ji}^{\Gamma}\left(\tau,t'\right)=-\mathrm{i}\sum_k \rho_k\left\langle \Psi_k\right| \underbrace{e^{\mathrm{i}(\hat{H}-\mu\hat{N})(-\mathrm{i}\tau)}\hat{d}_j e^{-\mathrm{i}(\hat{H}-\mu\hat{N})(-\mathrm{i}\tau)}}_{\hat{d}_{j,H}(t_0-\mathrm{i}\tau)}\underbrace{e^{\mathrm{i}\hat{H}(t'-t_0)}\hat{d}_i^{\dagger}e^{-\mathrm{i}\hat{H}(t'-t_0)}}_{d_{i,H}^{\dagger}(t')}\left|\Psi_k\right\rangle$$

$$=-\mathrm{i}e^{\mu\tau}\sum_k \rho_k\left\langle \Psi_k\left|\hat{d}_j e^{-\mathrm{i}\left(\hat{H}-E_k\right)(t_0-\mathrm{i}\tau-t')}\hat{d}_i^{\dagger}\right|\Psi_k\right\rangle$$

then

$$\hat{\mathcal{G}}^{\lceil}\left(\tau,t'\right)=e^{\mu\tau}\hat{\mathcal{G}}^{>}\left(t_0-\mathrm{i}\tau,t'\right)$$

Similarly

$$\hat{\mathcal{G}}^{\rceil}(t,\tau)=e^{-\mu\tau}\hat{\mathcal{G}}^{<}\left(t,t_0-\mathrm{i}\tau\right)$$

- For system in thermodynamic equilibrium, we have

$$G_{ji}^{\mathrm{M}}\left(\tau_1,\tau_2\right)=\frac{1}{\mathrm{i}}\sum_{kn}\rho_p\left[\theta\left(\tau_1-\tau_2\right)e^{\beta\left(E_p^{\mathrm{M}}-E_k^{\mathrm{M}}\right)}\pm\theta\left(\tau_2-\tau_1\right)\right]e^{-(\tau_1-\tau_2)\left(E_p^{\mathrm{M}}-E_k^{\mathrm{M}}\right)}\Phi_{kp}(j)\Phi_{kp}^*(i)$$

where

$$E_p^{\mathrm{M}}=E_p-\mu N_p$$

Recall that we proved

$$\frac{1}{-\mathrm{i}\beta}\sum_{m=-\infty}^{\infty}\frac{e^{-\omega_m\tau}}{\omega_m-E}=\frac{1}{\mathrm{i}}\left[\theta(\tau)e^{\beta E}\pm\theta(-\tau)\right]f(E)e^{-\tau E}$$

Then we have

$$G_{ji}^{\mathrm{M}}\left(\tau_1,\tau_2\right)=\frac{1}{-\mathrm{i}\beta}\sum_{m=-\infty}^{\infty}e^{-\omega_m(\tau_1-\tau_2)}\underbrace{\sum_{kp}\frac{\rho_p}{f\left(E_p^{\mathrm{M}}-E_k^{\mathrm{M}}\right)}\frac{\Phi_{kp}(j)\Phi_{kp}^*(i)}{\omega_m-E_p^{\mathrm{M}}+E_k^{\mathrm{M}}}}_{G_{ji}^{\mathrm{M}}(\omega_m)}.$$

we have

$$\frac{\rho_p}{f\left(E_p^{\mathrm{M}}-E_k^{\mathrm{M}}\right)}=\rho_p\left(e^{\beta\left(E_p^{\mathrm{M}}-E_k^{\mathrm{M}}\right)}\mp 1\right)=\rho_k\mp\rho_p$$

and

$$E_p^{\mathrm{M}}-E_k^{\mathrm{M}}=E_p-E_k-\mu\left(N_p-N_k\right)=E_p-E_k-\mu$$

therefore we have

$$G_{ji}^{\mathrm{M}}\left(\omega_m\right)=\sum_{kp}\frac{\Phi_{kp}(j)\Phi_{kp}^*(i)}{\omega_m+\mu-E_p+E_k}\left[\rho_k\mp\rho_p\right]$$

therefore

$$\hat{\mathcal{G}}^{\mathrm{M}}(\zeta)=\begin{cases}\hat{\mathcal{G}}^{\mathrm{R}}(\zeta+\mu) & \text{for }\ \mathrm{Im}[\zeta]>0\\ \hat{\mathcal{G}}^{\mathrm{A}}(\zeta+\mu) & \text{for }\ \mathrm{Im}[\zeta]<0\end{cases}$$

Thus $\mathcal{G}^M(\zeta)$is analytic everywhere except along the real axis where it can have poles or branch points.

- As a result,

$$\hat{\mathcal{G}}^{\mathrm{M}}(\omega\pm\mathrm{i}\eta)=\hat{\mathcal{G}}^{\mathrm{R/A}}(\omega+\mu)$$

which means that $\mathcal{G}^M$ has a discontinuity given by the difference $\mathcal{G}^R-\mathcal{G}^A$ when the complex frequency crosses the real axis.

13 Spectral function

- For systems that are initially in a pure state $\hat{\rho}=|\Psi_{N,0}\rangle\langle\Psi_{N,0}|$, We denote by $|\Psi_{N\pm1,m}\rangle$ the eigenstates of $\hat{H}$ with $N\pm1$ particles and define the quasi-particle wavefunctions P_m and the quasi-hole wavefunctions Q_m according to

$$P_m(i)=\left\langle\Psi_{N,0}\left|\hat{d}_i\right|\Psi_{N+1,m}\right\rangle,\quad Q_m(i)=\left\langle\Psi_{N-1,m}\left|\hat{d}_i\right|\Psi_{N,0}\right\rangle$$

- Then, the lesser and greater Green's functions become

$$G_{ji}^{<}\left(t,t'\right)=\mp\mathrm{i}\sum_m Q_m(j)Q_m^*(i)e^{-\mathrm{i}\left(E_{N-1,m}-E_{N,0}\right)\left(t'-t\right)}$$

$$G_{ji}^{>}\left(t,t'\right)=-\mathrm{i}\sum P_m(j)P_m^*(i)e^{-\mathrm{i}\left(E_{N+1,m}-E_{N,0}\right)\left(t-t'\right)}$$

- The Fourier transform of these functions are

$$G_{ji}^{<}(\omega)=\mp2\pi\mathrm{i}\sum Q_m(j)Q_m^*(i)\delta\left(\omega-\left[E_{N,0}-E_{N-1,m}\right]\right)$$

$$G_{ji}^{>}(\omega)=-2\pi\mathrm{i}\sum P_m(j)P_m^*(i)\delta\left(\omega-\left[E_{N+1,m}-E_{N,0}\right]\right)$$

  $G^<$ is peaked at the removal energies, and $G^>$ is peaked at the addition energies.

- We define

$$\hat{\mathcal{A}}(\omega)=\mathrm{i}\left[\hat{\mathcal{G}}^{>}(\omega)-\hat{\mathcal{G}}^{<}(\omega)\right]=\mathrm{i}\left[\hat{\mathcal{G}}^{\mathrm{R}}(\omega)-\hat{\mathcal{G}}^{\mathrm{A}}(\omega)\right]$$

- We have

$$A_{jj}(\omega)=2\pi\left[\sum_m\left|P_m(j)\right|^2\delta\left(\omega-\left[E_{N+1,m}-E_{N,0}\right]\right)\mp\sum_m\left|Q_m(j)\right|^2\delta\left(\omega-\left[E_{N,0}-E_{N-1,m}\right]\right)\right]$$

- It is not difficult to verify that:

$$\int\frac{d\omega}{2\pi}A_{ji}(\omega)=\delta_{ji}$$

- We have

$$\hat{\mathcal{G}}^{\mathrm{R}}(\omega)=\int\frac{d\omega'}{2\pi}\frac{\hat{\mathcal{A}}\left(\omega'\right)}{\omega-\omega'+\mathrm{i}\eta},\quad\hat{\mathcal{G}}^{\mathrm{A}}(\omega)=\int\frac{d\omega'}{2\pi}\frac{\hat{\mathcal{A}}\left(\omega'\right)}{\omega-\omega'-\mathrm{i}\eta}$$

- We have

$$\hat{\mathcal{G}}^{<}(\omega)=\mp\mathrm{i}f(\omega-\mu)\hat{\mathcal{A}}(\omega),\quad\hat{\mathcal{G}}^{>}(\omega)=-\mathrm{i}\bar{f}(\omega-\mu)\hat{\mathcal{A}}(\omega)$$

14 Galitskii–Migdal formula

- For systems of interacting identical particles with Hamiltonian $\hat{H}(t) = \hat{H}_0(t) + \hat{H}_{\text{int}}$, the time-dependent energy $E_S(t_1)$ is considered as the ensemble average of $\hat{H}_{\mathrm{S}}(t_1)$:

$$\hat{H}_{\mathrm{S}}(t_1) \equiv \hat{H}(t_1) - q\int \mathrm{d}\boldsymbol{x}_1 \hat{n}(\boldsymbol{x}_1)\,\delta V(1), \quad \delta V(1) = V(\boldsymbol{r}_1, t_1) - V(\boldsymbol{r}_1)$$

- Then we have

$$E_{\mathrm{S}}(z_1) = \sum_k \rho_k \langle \Psi_k | \hat{U}(t_{0-}, z_1) \left[\int \mathrm{d}\boldsymbol{x}_1 \mathrm{d}\boldsymbol{x}_2 \hat{\psi}^\dagger(\boldsymbol{x}_1) \left\langle \boldsymbol{x}_1 \left| \hat{h}_{\mathrm{S}}(z_1) \right| \boldsymbol{x}_2 \right\rangle \hat{\psi}(\boldsymbol{x}_2) + \frac{1}{2}\int \mathrm{d}\boldsymbol{x}_1 \mathrm{d}\boldsymbol{x}_2 v(\boldsymbol{x}_1, \boldsymbol{x}_2) \hat{\psi}^\dagger(\boldsymbol{x}_1) \hat{\psi}^\dagger(\boldsymbol{x}_2) \hat{\psi}(\boldsymbol{x}_2) \hat{\psi}(\boldsymbol{x}_1) \right] \hat{U}(z_1, t_{0-}) |\Psi_k\rangle$$

  where $\hat{h}_{\mathrm{S}}(z_1) = \hat{h}(z_1) - q\delta V(\hat{\boldsymbol{r}}_1, z_1)$.

- In contour formalism, we have

$$E_{\mathrm{S}}(z_1) = \pm \mathrm{i}\int \mathrm{d}\boldsymbol{x}_1 \mathrm{d}2 h_{\mathrm{S}}(1;2) G(2;1^+) - \frac{1}{2}\int \mathrm{d}\boldsymbol{x}_1 \mathrm{d}2 v(1;2) G_2(1,2;1^+,2^+)$$

- Since we have

$$\left[\left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}z_1} - \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}z_2}\right) G(1;2)\right]_{2=1^+} - \int \mathrm{d}3\left[h(1;3)G(3;1^+) + G(1;3^+)h(3;1)\right] = \pm 2\mathrm{i}\int \mathrm{d}3 v(1;3) G_2(1,3;1^+,3^+)$$

- Then, we have

$$E_{\mathrm{S}}(z_1) = \pm \mathrm{i}\int \mathrm{d}\boldsymbol{x}_1 \left\langle \boldsymbol{x}_1 \left| \left[\hat{h}_{\mathrm{S}}(z_1) - \frac{1}{2}\hat{h}(z_1)\right] \hat{\mathcal{G}}(z_1, z_1^+) \right| \boldsymbol{x}_1 \right\rangle \pm \frac{\mathrm{i}}{4}\int \mathrm{d}\boldsymbol{x}_1 \left[\left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}z_1} - \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}z_2}\right) \left\langle \boldsymbol{x}_1 \left| \hat{\mathcal{G}}(z_1, z_2) \right| \boldsymbol{x}_1 \right\rangle \right]_{z_2 = z_1^+}$$

- For $z_1 = t_0 - \mathrm{i}\tau_1$, $h_S(z_1) = h(z_1) = h^M$, then the initial energy E_S^M is

$$E_{\mathrm{S}}^{\mathrm{M}} = \pm\frac{\mathrm{i}}{2}\frac{1}{-\mathrm{i}\beta}\sum_m e^{\eta\omega_m}\int \mathrm{d}\boldsymbol{x} \left\langle \boldsymbol{x} \left| \left(\omega_m + \hat{h}^{\mathrm{M}}\right) \hat{\mathcal{G}}^{\mathrm{M}}(\omega_m) \right| \boldsymbol{x} \right\rangle$$

- For $z_1 = t_\pm$, we have $\hat{h}_{\mathrm{S}}(t) = \hat{h}(t) - q\delta V(\hat{\boldsymbol{r}}, t)$, then

$$E_{\mathrm{S}}(t) = \pm\frac{\mathrm{i}}{4}\int \mathrm{d}\boldsymbol{x} \left\langle \boldsymbol{x} \left| \left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} - \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t'} + 2\hat{h}(t)\right) \hat{\mathcal{G}}^<(t,t') \right| \boldsymbol{x} \right\rangle \Bigg|_{t'=t} - q\int \mathrm{d}\boldsymbol{x} n(\boldsymbol{x}, t)\delta V(\mathrm{r}, t)$$

- If $\hat{H}(t) = \hat{H}$, then

$$E_{\mathrm{S}} = \pm\frac{\mathrm{i}}{2}\int\frac{\mathrm{d}\omega}{2\pi}\int \mathrm{d}\boldsymbol{x} \left\langle \boldsymbol{x} \left| (\omega + \hat{h})\hat{\mathcal{G}}^<(\omega) \right| \boldsymbol{x} \right\rangle$$

  This is called the Galitskii–Migdal formula.

- For noninteracting system, the above formula becomes

$$E_{\mathrm{S}} = \sum_\lambda f\left(\epsilon_\lambda^{\mathrm{M}}\right)\epsilon_\lambda$$