PDE notes: Sobolev space, taught by Sungjin Oh

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1 Sobolev spaces

We start by introducing the Sobolev norms. We require the readers to have knowledge about the distribution space $\mathcal{D}'(U)$. If you don't, just think of $\mathcal{D}'(U)$ for now as the function space that is big enough to include all the functions of our interest.

Definition 1.1 (Sobolev norm). *Let $u \in \mathcal{D}'(U)$, U is an open subset in $\mathbb{R}^d$. The k-th order L^p-based Sobolev norm of u is defined as*

$$\|u\|_{W^{k,p}(U)} = \sum_{\alpha:|\alpha|\leq k} \|D^\alpha u\|_{L^p} \tag{1}$$

Here α is the multi-index, $D^\alpha u$ is distributional derivatives (or, derivatives in the weak sense).

For the Sobolev norm to be finite, we need $D^\alpha u \in L^p$. The definition of Sobolev spaces follow naturally.

Definition 1.2 (Sobolev spaces). *Sobolev spaces $W^{k,p}(U)$ is defined as*

$$W^{k,P}(u) = \{u \in \mathcal{D}'(0) : \|u\|_{\omega^{k,P}}(0) < +\infty\} \tag{2}$$

and is called the L^P-Sobolev space of order K on U.

Let $C_c^\infty(U)$ be the space of compactly supported smooth functions. It is a subset of $W^{k,p}(U)$. We define its closure in $W^{k,p}(U)$ as

Definition 1.3 ($W_0^{k,p}(U)$). *$W_0^{k,p}(U)$ is defined as the closure of $C_c^\infty(U)$ (with respect to $\|\cdot\|_{W^{k,p}}$)*

$$W_0^{k_1 p}(u) = \overline{C_c^\infty(u)} \subseteq W^{k,p}(u) \tag{3}$$

This is the set of $u \in W^{k,p}(U)$ that vanishes (to appropriate orders) on ∂U.

For $p = 2$, we introduce the following notation:

$$H^k(U) = W^{k,2}(U), \quad H_0^k(U) = W_0^{k,2}(U) \tag{4}$$

Now we are ready to discuss some preliminary properties of Sobolev spaces.

Theorem 1.1 (Sobolve spaces are Banach space). *For any $k \in \mathbb{Z}_+$, $1 \leq p \leq +\infty$,*

$$\begin{aligned} &\left(W^{k,p}(U), \|\cdot\|_{W^{k,p}}\right) \text{ is a Banach space.} \\ &\left(W_0^{k,p}(U), \|\cdot\|_{W^{k,p}}\right) \text{ is a Banach space.} \end{aligned}$$

The proof of this is first to verify that $\|u\|_{W^{k,p}(U)}$ is a norm. Then, assert that for a Cauchy sequence $\{u_m\}$ in $W^{k,p}(U)$, $\{u_m\}$ and $D^\alpha u_m$ have limit in L^p, denoted as u and u_α. Finally, verify that $D^\alpha u = u_\alpha$.

It is straightforward to have:

Theorem 1.2 ($H^k(U), H_0^k(U)$ is Hilbert space). *For any $k \in \mathbb{Z}_+$, $1 \leq p \leq +\infty$,*

$$\begin{aligned} &\left(H^k(U), \langle\cdot,\cdot\rangle\right) \text{ is a Hilbert space.} \\ &\left(H_0^k(U), \langle\cdot,\cdot\rangle\right) \text{ is a Hilbert space.} \end{aligned}$$

We then have the Fourier analytic characterization of $H^k(U)$:

Theorem 1.3 (Fourier analytic characterization of $H^k(U)$). *For $u \in H^k(U)$, we have*

$$\|u\|_{H^k} \simeq \|\hat{u}\|_{L^2} + \left\||\xi|^k \hat{u}\right\|_{L^2} \simeq \left\|\left(1 + \|\,\xi|^2\right)^{\frac{k}{2}} \hat{u}\right\|_{L^2}. \tag{5}$$

Here we use the notation that $A \simeq B \Leftrightarrow \exists c_1, c_2 > 0, A \leq c_1 B \& B \leq c_2 A.$ [1]

[1] The proof of this should be verified.

Now we define the negative order Sobolev spaces:

Definition 1.4 (negative order Sobolev spaces)**.** *For $k \in \mathbb{Z}_+$, $1 < p < +\infty$, we define the negative order Sobolev norm:*

$$\|u\|_{W^{-k,p}(U)} = \inf \left\{ \sum_{\alpha:|\alpha|\leq k} \|g_\alpha\|_{L^p} : u = \sum_{\alpha:|\alpha|\leq k} D^\alpha g_\alpha \right\} \tag{6}$$

and the negative order Sobolev Spaces:

$$W^{-k,p}(U) = \{u \in \mathcal{D}'(U) : u = \sum_{\alpha:|\alpha|\leq k} D^\alpha g_\alpha, g_\alpha \in L^p(U)\}. \tag{7}$$

If $g \in L^p$, then $D_{x^1} g \in W^{-1,p}(U)$. If $u \in W^{k,p}(U)$, then $D_{x^j} g \in W^{k-1,p}(U)$. [2]

Acutually, the negative order Sobolev spaces serves as some kind of dual space:

Theorem 1.4. *For $k \in \mathbb{Z}_+$, $1 < p < \infty$, $\frac{1}{p'} + \frac{1}{p} = 1$, we have*

$$\left(W_0^{k,p}(U)\right)^* = W^{-k,p'}(U)$$

Proof. (1) We first prove $W^{-k,p'}(U) \subseteq \left(W_0^{k,p}(U)\right)^*$. For $v \in W^{-k,p'}(U)$, there exits g_α such that $v = \sum_{\alpha:|\alpha|\leq k} D^\alpha g_\alpha$. Then for $u \in W_0(^{k,p}(U)$, we have:

$$\langle v, u\rangle = \int v u \mathrm{d}x = \sum_{\alpha:|\alpha|\leq k} \int D^\alpha g_\alpha u \mathrm{d}x = \sum_{\alpha:|\alpha|\leq k} \int (-1)^{|\alpha|} g_\alpha D^\alpha u \mathrm{d}x \leq \sum_{\alpha:|\alpha|\leq k} \|g_\alpha\|_{L^{p'}} \|D^\alpha u\|_{L^p} \tag{8}$$

We pick the g_α such as $\sum_{\alpha:|\alpha|\leq k} \|g_\alpha\|_{L^{p'}} \leq 2\|v\|_{W^{-k,p}(U)}$, then $\langle v, u\rangle \leq C\|v\|_{W^{-k,p'}}\|u\|_{W^{k,p}}$. This means v as a linear operator is bounded.

(2) Now we prove $\left(W_0^{k,p}(U)\right)^* \subseteq W^{-k,p'}(U)$. The idea is Hahn-Banach theorem, which states that for a normed vector space X, $Y \subseteq X$, if $\ell : Y \to \mathbb{R}$ is a linear bounded functional on Y, i.e $|\ell(u)| \leq c\|u\|$, then there exists an extension $\widetilde{\ell} : X \to \mathbb{R}$ such that $\widetilde{\ell}(u) \leq c\|u\|$ and $\widetilde{\ell}|_Y = \ell$.

Let $\ell : W_0^{k,p}(U) \to \mathbb{R}$ be a bounded linear functional on $W_0^{k,p}(U)$.

Define $T : C_c^\infty(U) \to L^p(U)^{\oplus K(k)}$ as $u \mapsto (u, D_{x^1}u, \cdots, D_{x^d}u, \cdots, D^\alpha u)$. Then $\|Tu\| \leq C\|u\|_{W^{k,p}}$, and T is injective. Moreover, $T : (C_c^\infty(U), \|\cdot\|_{W^{k,p}}) \to (T(C_c^\infty(U)), \|\cdot\|)$ is an isomorphism. Then we can define $\widetilde{\ell} : T(C_c^\infty(U)) \to \mathbb{R}$, where:

$$\widetilde{\ell}(Tu) = \ell(u)$$

us also bounded. Then according to Hahn-Banach, $\widetilde{\ell}$ could be extended to $\widetilde{\widetilde{\ell}} : L^p(U)^{\oplus K} \to \mathbb{R}$, which is also bounded. $\widetilde{\widetilde{\ell}} \in \left(L^p(U)^{\oplus K}\right)^*$, which means that there exists $\tilde{g}_\alpha \in L^{p'}(U)$, such that

$$\widetilde{\widetilde{\ell}}(\tilde{u}) = \sum_\alpha \langle \tilde{g}_\alpha, \tilde{u}_\alpha\rangle$$

Then for $u \in C_c^\infty(U)$, $\ell(u) = \widetilde{\ell}(Tu) = \widetilde{\widetilde{\ell}}(Tu) = \sum_\alpha \langle \widetilde{g}_\alpha, (Tu)_\alpha\rangle = \sum_\alpha \langle \widetilde{g}_\alpha, D^\alpha u\rangle$. Therefore let $g_\alpha = (-1)^{|\alpha|}\widetilde{g}_\alpha$, we have

$$\ell(u) = \langle v, u\rangle = \sum_{\alpha:|\alpha|\leq k} (-1)^{|\alpha|} \langle g_\alpha, D^\alpha u\rangle = \sum_{\alpha:|\alpha|\leq k} \langle D^\alpha g_\alpha, u\rangle$$

which completes the proof. □

[2] One needs to verify that $W^{-k,p}(U)$ is a Banach space.

2 A digression on functional analysis

Let X, Y be Banach spaces. $P : X \to Y$. We are often concerned with the following two problems:

- Existence: For $f \in Y$, is there $u \in X$, such that $Pu = f$?
- Uniqueness: If $u, u' \in X$, $Pu = Pu'$, is $u = u'$? This is equivalent to if $Pu = 0$, is $u = 0$?

P would be a linear differential operator, such as $-\Delta$ or $\Box$. We often prove the a-priori estimate for a PDE, i.e. if $u \in X$, $Pu = f$, then $\|u\|_X \le c\|f\|_Y$.

These two problems (existence and uniqueness) are related to each other by duality.

Theorem 2.1. *Let X, Y be Banach spaces, $P : X \to Y$ is bounded linear operator. Denote $P^* : Y^* \to X^*$ as the adjoint of P , i.e.*

$$\langle v, Pu\rangle = \langle P^* v, u\rangle, \quad \forall u \in X, \forall v \in Y^*$$

Suppose $\exists C > 0$ s.t.

$$\|u\|_X \le C\|Pu\|_Y, \forall u \in X$$

then

- *(Uniqueness for $Pu = f$) If $u \in X$, $Pu = 0$, then $u = 0$.*
- *(Existence for $P^*v = g$) $\forall g \in X^*$, $\exists v \in Y^*$, s.t. $P^*v = g$ and $\|v\|_{Y^*} \le C\|g\|_{X^*}$. (C is the same constant as above.)*

Proof. The proof for the first part is straightforward. We focus on the second part.

Our goal is to find $v \in Y^*$ such that $P^*v = g$, which means $\langle g, u\rangle = \langle P^*v, u\rangle = \langle v, Pu\rangle$ for any u. This motivates us to define $\ell : P(X) \to \mathbb{R}$ by

$$\ell(Pu) = \langle g, u\rangle$$

This is well-defined since P is injective. We want to prove ℓ is bounded on $P(X)$, and then we can use Hahn-Banach theorem.

Actually, $|\ell(Pu)| = |\langle g, u\rangle| \le \|g\|_{X^*}\|u\|_X \le C\|g\|_{X^*}\|Pu\|_Y$. Then according to Hahn-Banach theorem, $\exists v \in Y^*$ as an extension of ℓ, s.t. $\langle v, Pu\rangle = \langle g, u\rangle$ and $\|v\|_{Y^*} \le c\|g\|_{X^*}$ ☐

Now, how about existence theorem for $Pu = f$? Let's find an easy way out, and assume that X is reflexive. (i.e. $X \to (X^*)^* : u \mapsto (u' \mapsto \langle u^*, u'\rangle)$ is an isomorphism.)

Theorem 2.2. *Let X, Y be Banach spaces, $P : X \to Y$ is bounded linear operator. X is reflexive. Suppose*

$$\|v\|_{Y^*} \le C\|P^*v\|_{X^*}$$

then

- *(Uniqueness for $P^*v = g$) If $v \in Y^*$, $P^*v = 0$, then $v = 0$.*
- *(Existence for $Pu = f$) $\forall f \in Y$, $\exists u \in X$, s.t. $Pu = f$ and $\|u\|_X \le C\|f\|_Y$. (C is the same constant as above.)*

The proof is similar as above. Note that all sobolev spaces $W_0^{k,p}(U), 1 < p < \infty$ are reflexive. [3] Also note that

$$(\operatorname{Ran} P)^\perp = \ker P^* \quad \ker P = {}^\perp (\operatorname{Ran} P^*)$$

Here for $U \subseteq Y$, we define $U^\perp = \{v \in Y^* : \langle v, f\rangle = 0 \quad \forall f \in U\}$. And for $V \subseteq X^*$, we define ${}^\perp V = \{u \in X : \langle g, u\rangle = 0 \quad \forall g \in V\}$. As a consequence, if $\ker P^* = 0$, then [4]

$$(\operatorname{Ran} P)^\perp = \{0\} \Longleftrightarrow \overline{\operatorname{Ran} P} = Y$$

[3] This needs to be proved.
[4] This needs tp be proved.

In finite dimensions, $\overline{\operatorname{Ran} P} = \operatorname{Ran} P$. In infinite dimensions, $\overline{\operatorname{Ran} P} = \operatorname{Ran} P$ not necessarily holds.

Now we demonstrate that $\|v\|_{Y^*} \leq C\|P^*v\|_{X^*}$ is not too much to ask.

Theorem 2.3. *X, Y are Banach spaces, $P : X \to Y$ is bounded linear operator. If $P(X) = Y$, then there exists some $C > 0$ such that*

$$\|v\|_{Y^*} \leq C\|P^*v\|_{X^*}$$

Proof. By the open mapping theorem, $P(B_x)$ is open and $0 \in P(B_x)$. Then there exists $c > 0$, s.t. $cB_Y \subseteq P(B_x)$. Then we have

$$\|P^*v\|_{X^*} = \sup_{u:\|u\|_X \leq 1} |\langle P^*v, u\rangle| = \sup_{f \in P(\overline{B_x})} |\langle v, f\rangle| \geq \sup_{f \in cB_Y} |\langle v, f\rangle| = c\|v\|_{Y^*}$$

□

We use the above results on an example. Consider

$$\begin{cases} -u'' = f & \text{in} \quad (0,1) \\ u = 0 & \text{at} \quad x = 0, 1 \end{cases}$$

We want to discuss its solvability in $H_0^1((0,1))$.

- Recall that $\|u\|_{H^1}^2 = \|u\|_{L^2}^2 + \|u'\|_{L^2}^2$. and $\left(H_0^1((0,1))\right)^* = H^{-1}(0,1)$. In this case $X = H_0^1((0,1))$, $Y = H^{-1}(0,1)$, $Pu = -u''$. Therefore we want to prove $\|u\|_{H^1} \leq c\|f\|_{H^{-1}}$ where $f = -u''$, $u \in H_0^1((0,1))$.

  Proof. We only need to consider $u \in C_c^\infty((0,1))$, according to the approximation theorem that we will introduce later. We have

  $$\int -u''u\mathrm{d}x = \int fu\mathrm{d}x$$

  and $\int -u''u\mathrm{d}x = \int (u')^2\mathrm{d}x$. Note that there is no boundary term since $u \in C_c^\infty((0,1))$.

  For $\|u\|_{L^2}$, we have

  $$u(x) = \int_0^x u'(x')\mathrm{d}x'$$

  then $|u(x)| \leq \int_0^1 |u'(x')|\mathrm{d}x' \leq \|u'\|_{L^2}$. The second inequality is because of Cauchy-Schwartz inequality. Then $\int_0^1 |u(x)|^2\mathrm{d}x \leq \sup_{(0,1)} |u|^2 \leq \|u'\|_{L^2}^2$. Therefore

  $$\|u\|_{H^1}^2 \leq C\langle f, u\rangle \leq C\|f\|_{H^{-1}}\|u\|_{H^1}$$

  □

- With this section's first theorem, we have if $-u'' = 0$ and $u \in H_0^1((0,1))$m then $u = 0$.

- Let's compute P^*. $\langle P^*v, u\rangle = \langle v, Pu\rangle$ for all $v \in (H^{-1})^*, u \in H_0^1$. Note that by reflexibility of H_0^1, we have $(H^{-1})^* = H_0^1$. Then

  $$\langle v, Pu\rangle = \int_0^1 v(-u'')\mathrm{d}x = \int_0^1 v'u'\mathrm{d}x = \int_0^1 -v''u\mathrm{d}x$$

  Therefore $P^*v = -v''$, $Y^* = H_0^1((0,1))$, $X^* = H^{-1}((0,1))$. Then according to the second theorem, we have $\forall f \in H^{-1}$, $\exists u \in H_0^1$, s.t. $Pu = f$.

3 Approximation(density) theorems and extension theorem

The main tools of approximation theorems are convolution, mollifiers and smooth partition of unity.

Lemma 3.1 (Mollifiers). *Let φ be a smooth and compactly supported function. $\int \varphi \mathrm{d}x = 1$. Mollifiers are defined as*

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon^d}\varphi\left(\frac{x}{\varepsilon}\right)$$

Note that $\int \varphi_\varepsilon = 1$. For $u \in L^p(\mathbb{R}^d)$, $1 \le p < \infty$, we have

$$\|\varphi_\varepsilon * u - u\|_{L^p} \to 0 \text{ as } \varepsilon \to 0$$

where

$$\varphi_\varepsilon * u = \int \varphi_\varepsilon(x-y)u(y)dy$$

Here $\varphi_\varepsilon * u$ is C^∞. [5]

Proof. The key of the proof is continuity of translation on L^p, i.e. for $z \in \mathbb{R}^d$, $u \in L^p$, define translation τ_z as

$$\tau_z u(x) = u(x-z)$$

and we have $\lim_{|z|\to 0} \|\tau_z u - u\|_{L^p} = 0$ [6]. Then

$$\begin{aligned}
(\varphi_\varepsilon * u)(x) - u(x) &= \int u(x-y)\varphi_\varepsilon(y)\mathrm{d}y - u(x) = \int (u(x-y)-u(x))\,\varphi_\varepsilon(y)\mathrm{d}y \\
\|(\varphi_\varepsilon * u) - u\|_{L^p} &= \| \int (u(x-y)-u(x))\,\varphi_\varepsilon(y)\mathrm{d}y\|_{L^p} \\
&\le \int \|\,(u(\cdot - y) - u(\cdot))\,\|_{L^p}|\varphi_\varepsilon(y)|\mathrm{d}y = \int \|\,(u(\cdot - \varepsilon y) - u(\cdot))\,\|_{L^p}|\varphi(y)|\mathrm{d}y
\end{aligned}$$

Therefore as $\varepsilon \to 0$, $\|\,(u(\cdot - \varepsilon y) - u(\cdot))\,\|_{L^p}|\varphi(y)|$ goes to 0 (pointwise). Then according to Dominated Convergence theorem, the above integral goes to 0. □

Lemma 3.2 (Smooth partition of unity). [7] *$\{U_\alpha\}_{\alpha\in\mathcal{A}}$ is an open covering of U in $\mathbb{R}^d$. Then there exists a smooth partition of unity $\{\chi_\alpha\}$ on U subordinate to $\{U_\alpha\}_{\alpha\in\mathcal{A}}$, i.e.:*

- $\sum_\alpha \chi_\alpha(x) = 1$ *on U, and $\forall x \in U$, $\exists$ only finitely many nonzero $\chi_\alpha(x)$.*
- $\operatorname{supp}\chi_\alpha \subseteq U_\alpha$.
- χ_α *is smooth.*

Approximation theorems basically says that: given $u \in W^{k,p}(U)$, we want to approximate it with something *better*, for example smooth and/or has a nice support property.

We first state the approximation theorem in $\mathbb{R}^d$.

Theorem 3.1 (approximation theorem in $\mathbb{R}^d$). *For integer $k \ge 0$, $1 \le p < \infty$, we have*

- $C^\infty(\mathbb{R}^d)$ *is dense in* $W^{k,p}(\mathbb{R}^d)$;
- $C_c^\infty(\mathbb{R}^d)$ *is dense in* $W^{k,p}(\mathbb{R}^d)$

Proof. The proof of the first part is a straightforward application of mollification. The proof of the second part is done by approximating with $f\chi(\frac{\cdot}{R})$, $R \to \infty$.[8] □

[5] This needs proof.
[6] This needs proof!
[7] This needs to be proved.
[8] This needs to be elaborated.

Now we turn to the approximation in open subset U. A local approximation is obvious[9]. Let's look at a global approximation.

Theorem 3.2 (Global approximation theorem for open set U)**.** *For integer $k \geq 0$, $1 \leq p < \infty$, let U be any open subset of $\mathbb{R}^d$. Then $C^\infty(U)$ is dense in $W^{k,p}(U)$.*

Proof. For $u \in W^{k,p}(U)$, fix $\varepsilon > 0$. We want to find $v \in C^\infty(U)$ s.t.

$$\|u - v\|_{W^{k,p}} \leq C\varepsilon$$

where C is an independent constant.

Step 1: Define

$$U_j = \{x \in U : \mathrm{dist}(x, \partial U) > \frac{1}{j}\}, \quad V_j = U_{j+3} \backslash \bar{U}_{j+1}, \quad j \geq 1$$

Then we have $U \subseteq \bigcup_{j=0}^{\infty} V_j$. (with a good open set V_0) Then $\exists \chi_j$ being smooth parts of identity that is subordinate to V_j:

$$u = \sum_{j=1}^{\infty} u\chi_j = \sum_{j=1}^{\infty} u_j$$

With $\operatorname{supp} \chi_j \subseteq V_j$, then $\operatorname{supp} u_j = \operatorname{supp} u\chi_j \subseteq V_j$, therefore u_j is $C_c^\infty(\mathbb{R}^d)$.

Step 2: With a smooth $\varphi \in C_c^\infty(\mathbb{R}^d$, $\int \varphi = 1$, $\mathrm{supp}(\varphi) \subseteq B_1(0)$, then we have mollifier

$$\varphi_{\varepsilon_j} = \frac{1}{\varepsilon_j^d}\varphi(\frac{\cdot}{\varepsilon_j})$$

Here $\mathrm{supp}(\varphi_j) \subseteq B_{\varepsilon_j}(0)$. Then we define $v_j = \varphi_{\varepsilon_j} * u_j$, $v = \sum_{j=1}^{\infty} v_j$. ε_j is taken to satisfy

$$\|u_j - v_j\|_{W^{k,p}} \leq 2^{-j}\varepsilon, \quad \mathrm{supp}(v_j) \subseteq U_j \backslash \bar{U}_{j+3}$$

Then v is well-defined and smooth because for any open set $V \subset\subset U$ there are at most finitely many nonzero terms in the sum. And

$$\|v - u\|_{W^{k,p}} \leq \sum_{j=1}^{\infty} \|v_j - u_j\|_{W^{k,p}} \leq \sum_{j=1}^{\infty} 2^{-j}\varepsilon = \varepsilon$$

□

What if we want to approximate u with smooth functions on $\bar{U}$? The answer is yes we can as long as $\bar{U}$ has a good enough boundary. Let us first define what is $C^\infty(\bar{U})$:

Definition 3.1 ($C^\infty(\bar{U})$)**.** *$C^\infty(\bar{U})$ is defined as*

$$C^\infty(\bar{U}) = \{u : U \to \mathbb{R} : u \text{ is the restriction to } U \text{ of a smooth function } \tilde{u} \in C^\infty(\tilde{U}), U \subset \tilde{U}\}$$

and recall the definition of boundary being C^k:

Definition 3.2 (C^k boundary)**.** *∂U is of class C^k if $\forall x_0 \in \partial U$, $\exists \gamma = \gamma(x_0) > 0$, s.t. up to relabeling the variables,*

$$B_r(x_0) \cap U = \{x \in B_r(x_0) : x^d > \gamma(x^1, \cdots, x^{d-1})\}$$

for some C^k function $\gamma = \gamma(x^1, \cdots, x^{d-1})$ on $B_r(x_0) \cap \mathbb{R}^{d-1} \times \{x_0^d\}$.

Theorem 3.3 (Global approximation theorem for bounded open set U with $C^\infty(\bar{U})$)**.** *For integer $k \geq 0$, $1 \leq p < \infty$, let U be an open subset of $\mathbb{R}^d$ with boundary ∂U of class C^1, then $C^\infty(\bar{U})$ is dense in $W^{k,p}(U)$.*

[9] This needs proof

Proof. Step 1: By the definition of C^1-regularity of ∂U, ∂U can be covered by balls $\{U_k = B_{r_k}(x_k)\}_{k=1}^K$ in each of which U can be represented as the region above some C^1 graph. Since ∂U is compact (why?), then K is finite.

Let U_0 be an open set that contains $U \backslash \bigcup_{k=1}^K U_k$, then $\{U_0, U_1, \cdots, U_k\}$ is an open covering of U. Let $\{\chi_k\}_{k=0}^K$ be a smooth partition of identity subordinate to $\{U_k\}_{k=0}^K$, and with that , for $u \in W^{k,p}(U)$, we have

$$u = \sum_{k=0}^K u\chi_k =: u_0 + \sum_{k=1}^K u_k$$

u_0 is compactly supported, for which we can use mollification to approximate.

We will focus on a single u_k.

Step 2: Let $\varepsilon > 0$. We want to approximate u_k, whose support is $B_{r_k}(x_k)$. Without loss of generality, we assume $x_k = 0$. Denote u_k, U_k as u, U. We will use a two-step approximation:

$$\|u - v\| \leq \|u - w_\eta\| + \|w_\eta - v\|$$

Here $w_\eta(x) = u(x + \eta e_d)$, $e_d = (0, \cdots, 0, 1)^T$. For η small enough, we have (because of continuity of translation in L^p)

$$\|u - w_\eta\|_{W^{k,p}(U \cap B_{r_0}(0))} < \frac{1}{2}\varepsilon$$

And let $v = \varphi_\delta * w_\eta$. With δ small enough, v is well-defined on $V = B_{r_k/2}(0)$. More importantly $v \in C^\infty\overline{(V \cap \{x^d > \gamma(x_1, \cdots, x^{d-1})\})}$. Then take r_k in step 1 as $r_k/2$ here, we are done. □

Now we discuss extension theorem. Extension theorem is a tool to deal with $u \in W^{k,p}(U)$, where U is a bounded domain.

Theorem 3.4 (Extension)**.** *For $k \geq 0$ nonnegative integer, $1 \leq p < \infty$, U is a bounded domain with C^k boundary. V is an open set such that $\bar{U} \subset V$. Then $\exists \mathcal{E} : W^{k,p}(U) \to W^{k,p}(\mathbb{R}^d)$, such that*

- *(Extension) $\mathcal{E}u|_U = u$,*
- *(linear and bounded) $\mathcal{E}$ is linear and*

$$\|\mathcal{E}u\|_{W^{k,p}(\mathbb{R}^d)} \leq C\|u\|_{W^{k,p}(U)}$$

- *(support prescription)* $\operatorname{supp} \mathcal{E}u \subset V$.

Proof. It suffices to only consider $u \in C^\infty(\bar{U})$ (by density theorem and the linear and bounded property of $\mathcal{E}$).

Step 1: reduction to the half-ball case.

As in approximation theorem for bounded domain, we construct $U_0, U_1, \cdots, U_k$ and $\chi_0, \chi_1, \cdots, \chi_k$. Define $u_k = \chi_k u$. Note that u_0 is already in $W^{k,p}(\mathbb{R}^d)$ and $\operatorname{supp} u_0 \subset U_0 \subset V$. For $k > 1$, $\operatorname{supp} u_k \subset U_k \cap U$. Observe that if we define

$$\begin{cases} y^j = x^j - x_k^j, \quad j = 1, \cdots, d-1 \\ y^d = x^d - \gamma(x^1, \cdots, x^{d-1}) \end{cases}$$

Then $U_k \cap U$ gets mapped into $\widetilde{U} = \left\{y \in B_{\widetilde{r}}(0) : y^d > 0\right\}$. Note that $x \mapsto y$ is C^k, and u_k smooth, $u_k(y) = u_k(x(y))$ satisfies that

$$\|u_k(y)\|_{W_y^{k,p}}(\widetilde{U}) \leq C \|u_k(\lambda)\|_{W_x^{k,p}}$$

according to the chain rule.

Step 2: extension in the half-ball case. $U = B_r^+(0)$, $W = B_{r/2}^+(0)$, $\operatorname{supp} u \subset W$. The idea is higher order reflection method. Consider

$$\widetilde{u} = \begin{cases} u, & x^d > 0 \\ \sum_{i=0}^{K} \alpha_j u(x^1, \cdots, x^{d-1}, -\beta_i x^d), & x^d < 0 \end{cases}$$

where $0 < \beta_j < 1$ but not yet determined. We need to match the normal derivatives on $\{x^d = 0\}$ up to order k. Note that

$$\partial_{x^d}^j \left(u\left(x^1, \cdots, x^{d-1}, -\beta_i x^d\right)\right) = (-1)^j \beta_i^j \left(\partial_{x^d}^j u\right)\left(x^1 \cdots, x^{d-1}, -\beta_i x^d\right)$$

Therefore we have

$$\begin{cases} u\left(x^1, \cdots, x^{d-1}, 0+\right) = \sum_{j=0}^{k} \alpha_j u\left(x^1, \cdots, x^{d-1}, 0+\right) \\ \partial_{x^d} u\left(x^1, \cdots, x^{d-1}, 0+\right) = \sum_{j=0}^{k} \alpha_j \left(-\beta_j\right)\left(\partial_{x_d} u\right)\left(x^1, \cdots, x^{d-1}, 0+\right) \\ \cdots \\ \partial_{x^d}^k u\left(x^1, \cdots, x^{d-1}, 0+\right) = \sum_{j=0}^{k} \alpha_j \left(-\beta_j\right)^k \left(\partial_{x_d}^k u\right)\left(x^1, \cdots, x^{d-1}, 0+\right) \end{cases}$$

which means

$$\begin{cases} 1 = \sum_{j=0}^{k} \alpha_j \\ 1 = \sum_{j=0}^{k} \alpha_j \left(-\beta_j\right) \\ \vdots \\ 1 = \sum_{j=0}^{k} \alpha_j \left(-\beta_j\right)^k \end{cases}$$

and as long as all β_j are distinct, then there exists such α_j to satisfy this equation system. (Recall the property of Vandermonde matrix.)

Finally, use an appropriate smooth cutoff χ_V (s.t. $\chi_V = 1$ on U and $\operatorname{supp} \chi_V \subset V$) to define $\mathcal{E}u$, i.e. $\mathcal{E}u = \chi_v \widetilde{u}$. [10] $\square$

4 Trace theorem

For $u \in C(\bar{U})$, we can restrict it to ∂U and define $u|_{\partial U}$. We define such an operation as $\operatorname{tr}_{\partial U}$, where $\operatorname{tr}_{\partial U} u = u|_{\partial U}$. Note that $\operatorname{tr}_{\partial U} u = u|_{\partial U}$ is linear. We are concerned that whether we can have such an operation for general $u \in W^{1,p}(U)$.

Theorem 4.1 (trace theorem, non-sharp)**.** *U bounded open subset of $\mathbb{R}^d$. ∂U is C^1, $1 < p < \infty$.* [11] *Then for $u \in C^1(\bar{U})$, we have*

$$\|\operatorname{tr}_{\partial U} u\|_{\partial U} \leq C\|u\|_{W^{1,p}(U)}$$

Note that as a consequence, $\operatorname{tr}_{\partial U}$ is extended (uniquely) by continuity (and density theorem) to $\operatorname{tr}_{\partial U} : W^{1,p}(U) \to L^p(\partial U)$. We call $\operatorname{tr}_{\partial U} u$ the trace of u on ∂U.

[10] How did we enforce that this mapping is bounded?
[11] why $1 < p$?

Proof. Same as before, we reduce to the case of half ball $B(0,r)$. Let ξ be a cutoff function that is 1 on $B(0,\frac{r}{2})$ and supported on $B(0,r)$. Let $(x',0)=(x_1,\ldots,x_{n-1},0)\in\{x_n=0\}$,

$$\begin{aligned}\int_{\partial U\cap B(0,r/2)}|u|^p\mathrm{d}x' &\le \int_{\{x_n=0\}\cap B(0,r/2)}\xi|u|^p\mathrm{d}x' = -\int_{B_+(0,r)}(\xi|u|^p)_{x_n}\,\mathrm{d}x\\ &= -\int_{B_+(0,r)}|u|^p\xi_{x_n}+p|u|^{p-1}(\operatorname{sgn}u)u_{x_n}\xi\mathrm{d}x \le C\int_{B_+(0,r)}|u|^p+|Du|^p\mathrm{d}x\end{aligned}$$

□

A nontrivial result of the trace operator is that

Theorem 4.2. *For bounded domain U, ∂U being C^1, $u\in W^{1,p}(U)$, we have*

$$u\in W_0^{1,p}(U)\Leftrightarrow \operatorname{tr}_{\partial U}u=0 \text{ on } \partial U$$

The detailed proof of this could be found on Evans. We focus now on understanding the sharp trace theorem in a restricted setting. Let $p=2$, we will use Fourier transform:

$$\begin{cases} U=\mathbb{R}^d_+=\{x\in\mathbb{R}^d : x^d>0\}\\ \partial U=\{(x',0)\in\mathbb{R}^d\}\simeq\mathbb{R}^{d-1}\end{cases}$$

Recall that

$$\|u\|^2_{H^k}\simeq\|(1+|\xi|^2)^{\frac{k}{2}}\hat{u}\|^2_{L^2_\xi}, k\ge 0$$

where k could be any positive real number. In this way, we have defined the fractional (L^2-based) Sobolev spaces.

Theorem 4.3 (sharp trace theorem)**.** *For $u\in C^1(\bar{\mathbb{R}}^d_+)\cap H^1(\mathbb{R}^d_+)$, we have*

$$\|\operatorname{tr}_{\partial U}u\|_{H^{\frac{1}{2}}(\mathbb{R}^{d-1})}\le C\|u\|_{H^1(\mathbb{R}^d_+)}$$

Proof. Take $u\in C^1(\bar{\mathbb{R}}^d_+)\cap H^1(\mathbb{R}^d_+)$. Using the extension like before, $\exists\widetilde{u}\in C^1(\mathbb{R}^d)$,

$$\|\widetilde{u}\|_{H^1(\mathbb{R}^d)}\le c\|u\|_{H^1(\mathbb{R}^d_+)}$$

We have

$$\operatorname{tr}_{\partial U}u\,(x')=\widetilde{u}\,(x',0)=\int\mathcal{F}_{x^d}\widetilde{u}\,(x',\xi_d)\,\frac{\mathrm{d}\xi_d}{2\pi}$$

where $\mathcal{F}$ represents the Fourier transform. We have

$$\mathcal{F}_{x'}\operatorname{tr}_{\partial U}u\,(\xi')=\int\mathcal{F}\widetilde{u}(\xi',\xi_d)\frac{\mathrm{d}\xi_d}{2\pi}$$

Then

$$\begin{aligned}\|\operatorname{tr}_{\partial U}u\|_{H^s(\mathbb{R}^{d-1})} &\sim \left\|(1+|\xi'|^2)^{\frac{s}{2}}\mathcal{F}_{x'}\operatorname{tr}_{\partial U}\widetilde{u}(\xi')\right\|_{L^2_{\xi'}}\\ &= \left\|(1+|\xi'|^2)^{\frac{s}{2}}\int\mathcal{F}\widetilde{u}(\xi',\xi_d)\frac{\mathrm{d}\xi_d}{2\pi}\right\|_{L^2_{\xi'}}\\ &= \left\|\int\frac{(1+|\xi'|^2)^{\frac{s}{2}}}{(1+|\xi'|^2+\xi_d^2)^{\frac{1}{2}}}(1+|\xi'|^2+\xi_d^2)^{\frac{1}{2}}\mathcal{F}\widetilde{u}(\xi',\xi_d)\frac{\mathrm{d}\xi_d}{2\pi}\right\|_{L^2_{\xi'}}\\ &\le \left\|\left(\int\frac{(1+|\xi'|^2)^s}{(1+|\xi'|^2+\xi_d^2)}\mathrm{d}\xi_d\right)^{\frac{1}{2}}\|(1+|\xi'|^2+\xi_d^2)^{\frac{1}{2}}\mathcal{F}\widetilde{u}\|_{L^2_{\xi_d}}\right\|_{L^2_{\xi'}}\\ &\le \sup_{\xi'\in\mathbb{R}^{d-1}}\left(\int\frac{(1+|\xi'|^2)^s}{(1+|\xi'|^2+\xi_d^2)}\mathrm{d}\xi_d\right)^{\frac{1}{2}}\|(1+|\xi'|^2+\xi_d^2)^{\frac{1}{2}}\mathcal{F}\widetilde{u}\|_{L^2_\xi}\end{aligned}$$

When $s\ge 1/2$, the first term on the right hand side is bounded. □

It turns out that the image of $\mathrm{tr}_{\partial U}$ is exactly $H^{\frac{1}{2}}$.

Theorem 4.4 (Existence of extension from ∂U). *$\exists$ bounded linear map $\mathrm{ext}_{\partial u}: H^{\frac{1}{2}}\left(\mathbb{R}^{d-1}\right) \to H^1\left(\mathbb{R}^d_+\right)$, s.t. $\mathrm{tr}_{\partial U}\cdot \mathrm{ext}_{\partial U} = \mathrm{id}$.*

Proof. For $g \in \mathcal{S}(\mathbb{R}^{d-1})$, consider $\eta \in C_c^\infty(\mathbb{R})$, $\eta = 1$ for $|s| < 1$, $\eta = 0$ for $|s| > 2$. Define $u = \mathrm{ext}_{\partial u} g$ using the following way:

$$\mathcal{F}_{x'} u(\xi', x^d) = \eta(x^d)\mathrm{e}^{-x^d|\xi'|}\hat{g}(\xi')$$

we want to verify that $u \in H^1(\mathbb{R}_+)$. First consider

$$\begin{aligned}
&\|u\|_{L^2}^2 + \|\partial_1 u\|_{L^2}^2 + \cdots + \|\partial_{d-1} u\|_{L^2}^2 \\
=&\left\| \left(1+|\xi'|^2\right)^{\frac{1}{2}} \mathcal{F}_x u\left(\xi', x^d\right) \right\|_{L^2_{\xi'}, L^2_{x^d}} \\
=&\left\| \left(1+|\xi'|^2\right)^{\frac{1}{2}} \eta(x^d)\mathrm{e}^{-x^d|\xi'|}\hat{g}(\xi') \right\|_{L^2_{\xi'}, L^2_{x^d}} \\
=&\left\| \left(1+|\xi'|^2\right)^{\frac{1}{4}} \|\eta(x^d)\mathrm{e}^{-x^d|\xi'|}\|_{L^2_{x^d}} \left(1+|\xi'|^2\right)^{\frac{1}{4}} \hat{g}(\xi') \right\|_{L^2_{\xi'}}
\end{aligned}$$

$\left(1+|\xi'|^2\right)^{\frac{1}{4}} \hat{g}(\xi')$ is bounded since $g \in H^{1/2}$, and $\left(1+|\xi'|^2\right)^{\frac{1}{4}} \|\eta(x^d)\mathrm{e}^{-x^d|\xi'|}\|_{L^2_{x^d}}$ is bounded with some calculations. Now we only need to verify $\partial_x^d u \in L^2$. We have

$$\partial_{x^d} u = \partial_{x^d}\left(\eta\left(x^d\right)v\right) = \eta'(x^d)v + \eta\partial_{x^d} v, \quad \mathcal{F}_{x'} v = e^{-x^d|\xi'|}\hat{g}(\xi').$$

It is very natural to have $\|\eta'\left(x^d\right)v\|_{L^2} \le C\|v\|_{L^2(x^d \operatorname{supp}\eta)}$. Then let's look at the second term:

$$\begin{aligned}
\|\eta\partial_{x^d} v\|_{L^2_{x'}, L^2_{x^d}} &= \left\|\eta\partial_{x^d}\left(e^{-x^d|\xi'|}\hat{g}(\xi')\right)\right\|_{L^2_{\xi'}, L^2_{x^d}} \\
&= \left\|\eta\,|\xi'|\,e^{-x^d|\xi'|}\hat{g}(\xi')\right\|_{L^2_{\xi'}, L^2_{x^d}} = \||\xi'|\,\mathcal{F}_{x'} u\|_{L^2_{\xi'}, L^2_{x^d}} \le C\|g\|_{H^{\frac{1}{2}}}
\end{aligned}$$

Now we are done. $\square$

5 Sobolev-type inequalities (1)

Sobolev-type inequalities are quantitative generations of fundamental theorem of calculus, which use derivatives to control function. Below we will prove them for smooth functions, and then according to density theorems, these inequalities hold in the corresponding Sobolev spaces.

For now we consider $W^{1,p}(U)$, $U \in \mathbb{R}^n$. We will see that we have to talk about the following three cases seperately:

$$1 \le p < n, \quad n < p \le \infty, \quad p = n$$

For now we look at the case $1 \le p < n$. We want something like

$$\|u\|_{L^q(\mathbb{R}^n)} \le C\|Du\|_{L^p(\mathbb{R}^n)}$$

A natural question would be what's the relation between p and q. If the above equation holds, then consider $u_\lambda(x) = u(\lambda x)$, then

$$\|u_\lambda\|_{L^q} = \left(\int |u(\lambda x)|^q \mathrm{d}x\right)^{1/q} = \lambda^{-d/q}\|u\|_{L^q}, \quad \|Du_\lambda\|_{L^p} = \left(\int |\lambda Du(\lambda x)|^p \mathrm{d}x\right)^{1/p} = \lambda^{(p-d)/p}\|Du\|_{L^p}$$

Therefore we need $-d/q = (p-d)/p$. In other words,

$$\frac{1}{q} = \frac{1}{p} - \frac{1}{d}$$

From here we can also see that $p < d$ is necessary. Below for $1 \le p < d$, the Sobolev conjugate of p is always defined as

$$p^* = \frac{dp}{d-p}$$

Theorem 5.1 (Gagliardo-Nirenberg-Sobolev inequality)**.** *For $1 \le p < d$ (which makes $d \ge 2$), $u \in C_c^\infty(\mathbb{R}^d)$, there exists constant C depending only on d, p such that*

$$\|u\|_{L^{\frac{d}{d-1}}} \le C\|Du\|_{L^1}$$

We introduce the following notation:

$$(x^1, \cdots, \widehat{x^j}, \cdots, x^d) = (x^1, \cdots, x^{j-1}, x^{j+1}, \cdots, x^d)$$

To prove this theorem we need the following lemma

Lemma 5.1 (Loomis-Whitney inequality)**.** *For $d \ge 2$, $j = 1, \cdots, d$, and $f_j = f_j(x^1, \cdots, \widehat{x^j}, \cdots, x^d)$. Then we have*

$$\left\| \prod_{j=1}^d f_j \right\|_{L^1(\mathbb{R}^d)} \le \prod_{j=1}^d \|f_j\|_{L^{d-1}(\mathbb{R}^{d-1})}$$

We first prove the lemma.

Proof. Integrate in each variable, and then use Holder's inequality:

$$\int \left| \prod_{j=1}^d f_j \right| \mathrm{d}x^1 = |f_1| \int |f_2| \dots |f_d| \, \mathrm{d}x^1 \le |f_1| \, \|f_2\|_{L^{d-1}_{x^1}} \cdots \|f_d\|_{L^{d-1}_{x_1}}$$

$$\iint \left| \prod_{j=1}^d f_j \right| \mathrm{d}x^1 \mathrm{d}x^2 \le \int |f_1| \, \|f_2\|_{L^{d-1}_{x^1}} \cdots \|f_d\|_{L^{d-1}_{x^1}} \, \mathrm{d}x^2 = \|f_2\|_{L^{d-1}_{x^1}} \|f_1\|_{L^{d-1}_{x^2}} \|f_3\|_{L^{d-1}_{x^1x^2}} \cdots \|f_d\|_{L^{d-1}_{x^1x^2}}$$

$$\cdots$$

$$\int \cdots \int \left| \prod_{j=1}^d f_j \right| \mathrm{d}x^1 \cdots \mathrm{d}x^d \le \prod_{j=1}^d \|f_j\|_{L^{d-1}_{x^1, \cdots, \hat{x^j}, \cdots x^d}}$$

$\square$

Note that L-W inequality has the following geometric meaning. For $E \in \mathbb{R}^d$, let $\pi_j(E) = \{x \in \mathbb{R}^d : x^j = 0, \exists (x^j)', s.t. (x^1, \cdots, (x^j)', \cdots, x^d) \in E\}$. Can me bound the measure of E by $|\pi_j(E)|$? Yes!

$$|E| = \int 1_E \mathrm{d}x \le \int \prod_{j=1}^d i_{\pi_j(E)}(x^1, \cdots, \widehat{x^j}, \cdots, x^d) \mathrm{d}x \le \prod_{j=1}^d |\pi_j(E)|^{\frac{1}{d-1}}$$

The second inequality is the L-W inequality.

Now we are ready to prove the original theorem.

Proof. For $x \in \mathbb{R}^d$, we have

$$u(x) = \int_{-\infty}^{x^j} \partial_{x^j} u(x^1, \cdots, x^{j-1}, y, x^{j+1}, \cdots, x^d) \mathrm{d}y$$

$$|u(x)| \le \int_{-\infty}^{+\infty} |Du(x^1, \cdots, x^{j-1}, y, x^{j+1}, \cdots, x^d)| \mathrm{d}y := \widetilde{f}_j(x^1, \cdots, \widehat{x^j}, \cdots, x^d)$$

$$|u(x)| \le \left(\prod_{j=1}^d \widetilde{f}_j \right)^{1/d}, \quad |u(x)|^{\frac{d}{d-1}} \le \prod_{j=1}^d \widetilde{f}_j^{\frac{1}{d-1}} := \prod_{j=1}^d f_j$$

Then,

$$\|u\|_{L^{\frac{d}{d-1}}}^{\frac{d}{d-1}} = \int |u|^{\frac{d}{d-1}} \leq \int \prod_{j=1}^{d} f_j \leq \prod_{j=1}^{d} \|f_j\|_{L^{d-1}} = \prod_{j=1}^{d} \left(\int |f_j|^{d-1} \mathrm{d}x^1 \cdots \mathrm{d}\widehat{x^j} \cdots \mathrm{d}x^d \right)^{\frac{1}{d-1}}$$
$$\leq \left(\int |Du| \right)^{\frac{d}{d-1}} \leq \|Du\|_{L^1}^{\frac{d}{d-1}}$$

□

Now we naturally have the following theorem.

Theorem 5.2 (Sobolev inequalities for L^p-based spaces)**.** *For $1 \leq p < d$ (which makes $d \geq 2$), $u \in C_c^\infty(\mathbb{R}^d)$, there exists constant C depending only on d, p such that*

$$\|u\|_{L^{p^*}} \leq C\|Du\|_{L^p}$$

Proof. Let $v = |u|^q$, where $q\frac{d}{d-1} = p^*$. Then $|Dv| = q|u|^{q-1}|Du|$. [12] Then,

$$\|u\|_{L^{p^*}}^{p^*} = \int |u|^{p^*} \mathrm{d}x = \int |v|^{\frac{d}{d-1}} \mathrm{d}x \leq \left(\int |Dv| \mathrm{d}x \right)^{\frac{d}{d-1}} \simeq \left(\int |u|^{q-1}|Du| \mathrm{d}x \right)^{\frac{d}{d-1}} \leq \left(\||u|^{q-1}\|_{L^{p^*}} \|Du\|_{L^p} \right)^{\frac{d}{d-1}}$$

Here $\||u|^{q-1}\|_{L^{p'}} = (\int |u|^{(q-1)p'})^{1/p'} = (\int |u|^{p^*})^{1/p'}$, where $p' = p/(p-1)$. The rest is easy to check. □

Theorem 5.3 (Estimates for $W^{1,p}$)**.** *For $1 \leq p < d$ (which makes $d \geq 2$), we have:*

- *For $u \in W^{1,p}(\mathbb{R}^d)$, $\|u\|_{L^{p^*}} \leq C\|Du\|_{L^p}$,*
- *U is a bounded domain, with ∂U being C^1, for $u \in W^{1,p}(U)$, we have*

$$\|u\|_{L^{p^*}(U)} \leq C\|u\|_{W^{1,p}(U)}$$

Sketch of the proof. For the first part, we can use density theorem of C_c^∞ in $W^{1,p}(\mathbb{R}^d)$. For the second theorem, we need to first use extension theorem then use density theorem. Note that G-N-S inequality only works for C_c^∞ functions. □

We have another further result for $W_0^{1,p}(U)$:

Theorem 5.4 (Estimates for $W_0^{1,p}(U)$)**.** *For $1 \leq p < d$ (which makes $d \geq 2$), U is a bounded domain. For $u \in W_0^{1,p}(U)$, we have*

$$\|u\|_{L^q(U)} \leq C\|Du\|_{L^p(U)}, \quad \forall 1 \leq q \leq p^*$$

Note that the difference from the previous theorem is that the right hand side is $\|Du\|_{L^p(U)}$ instead of $\|u\|_{W^{1,p}(U)}$. The proof of this theorem requires the approximation of C_c^∞ directly, without extension. Particularly, we have

$$\|u\|_{L^p(U)} \leq C\|Du\|_{L^p(U)}$$

This is called Poincare-Fredrich inequality, one of the Poincare-type inequalities.

[12] Why?

6 Sobolev-type inequalities (2)

Now we discuss what happened if $p \geq d$. Consider $u \in C^\infty(U)$, we have

$$u(x)-u(y)=\int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}u\left(x+s(y-x)\right)\mathrm{d}s, \quad \frac{\mathrm{d}}{\mathrm{d}s}u\left(x+s(y-x)\right)=(y-x)\cdot Du(x+s(y-x))$$

For $r>0$,

$$\begin{aligned}\frac{1}{|B_r(x)|}\int_{B_r(x)}|u(x)-u(y)|\,\mathrm{d}y &\leq \frac{1}{|B_r(x)|}\int_{B_r(x)}\int_0^1\left|\frac{\mathrm{d}}{\mathrm{d}s}u\left(x+s(y-x)\right)\right|\mathrm{d}s\mathrm{d}y\\ &\leq C'\frac{1}{r^d}\int_{B_r(x)}\int_0^1|(y-x)|\,|Du(x+s(y-x))|\,\mathrm{d}s\mathrm{d}y\\ &\leq C'\frac{1}{r^d}\int_0^r\mathrm{d}\rho\int_{\mathbb{S}^{d-1}}\mathrm{d}\omega\int_0^1\mathrm{d}s\left(\rho\,|Du(x+s\rho\omega)|\,\rho^{d-1}\right)\end{aligned}$$

Note that

$$\begin{aligned}&\int_0^1\mathrm{d}s\int_0^r\mathrm{d}\rho\,|Du(x+s\rho\omega)|\,\rho^d \quad (t=s\rho)\\ =&\int_0^1\mathrm{d}s\int_0^{sr}\frac{\mathrm{d}t}{s}\,|Du(x+t\omega)|\left(\frac{t}{s}\right)^d\\ =&\int_0^r\mathrm{d}t\int_{t/r}^1\mathrm{d}s\frac{1}{s^{d+1}}t^d\,|Du(x+t\omega)|\leq c\int_0^r r^d\,|Du(x+t\omega)|\,\mathrm{d}t\end{aligned}$$

Then we can continue the previous inequality

$$\begin{aligned}\frac{1}{|B_r(x)|}\int_{B_r(x)}(u(x)-u(y))\,\mathrm{d}y &\leq C'\frac{1}{r^d}\int_0^r\mathrm{d}\rho\int_{\mathbb{S}^{d-1}}\mathrm{d}\omega\int_0^1\mathrm{d}s\left(\rho\,|Du(x+s\rho\omega)|\,\rho^{d-1}\right)\\ &\leq C\int_{\mathbb{S}^{d-1}}\mathrm{d}\omega\int_0^r|Du(x+t\omega)|\,\mathrm{d}t=C\int_{B_r(x)}\frac{|Du|}{|x-y|^{d-1}}\mathrm{d}y\end{aligned}$$

In other words, we have proved the following lemma:

Lemma 6.1. *For $u\in C^\infty(\mathbb{R}^d)$, $d\geq 2$[13], we have*

$$\frac{1}{|B_r(x)|}\int_{B_r(x)}|u(x)-u(y)|\,\mathrm{d}y\leq C\int_{B_r(x)}\frac{|Du|}{|x-z|^{d-1}}\mathrm{d}z$$

With this we have the following theorem for $p>d$:

Theorem 6.1. *For $u\in C^\infty(\mathbb{R}^d)$, $d\geq 2$, let $\alpha=1-\frac{d}{p}$, then we have*

$$|u(x)-u(y)|\leq C|x-y|^\alpha\|Du\|_{L^p}$$

Proof. Let $r=|x-y|$. Let $W=B(x,r)\cap B(y,r)$. Then

$$\begin{aligned}|u(x)-u(y)| &= \frac{1}{|W|}\int_W|u(x)-u(y)|\,\mathrm{d}z\\ &\leq \frac{1}{|W|}\int_W|u(x)-u(z)|\mathrm{d}z+\frac{1}{|W|}\int_W|u(y)-u(z)|\mathrm{d}z\\ &\leq \frac{1}{|W|}\int_{B_r(x)}|u(x)-u(z)|\mathrm{d}z+\frac{1}{|W|}\int_{B_r(y)}|u(y)-u(z)|\mathrm{d}z\\ &= \frac{|B_r(x)|}{|W|}\frac{1}{|B_r(x)|}\int_{B_r(x)}|u(x)-u(z)|\mathrm{d}z+\frac{|B_r(y)|}{|W|}\frac{1}{|B_r(y)|}\int_{B_r(y)}|u(y)-u(z)|\mathrm{d}z\\ &\leq C\int_{B_r(x)}\frac{|Du|}{|x-z|^{d-1}}\mathrm{d}z\leq C\|Du\|_{L^p(B_r(x))}\|\frac{1}{|x-z|^{d-1}}\|_{L^{p'}}\leq C'\|Du\|_{L^p}|x-y|^\alpha\end{aligned}$$

□

[13] Why $d\geq 2$?

In order to carry on, we introduce the definition of Holder space:

Definition 6.1 (Holder space). *The Holder semi-norm is defined as*

$$[u]_{C^\alpha(U)} = \sup_{x,y\in U, x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha}$$

$[u]_{C^\alpha(U)}$ *is a semi-norm since* $[u]_{C^\alpha(U)}$ *doesn't mean* $u = 0$*. Therefore, we define Holder norm* $\|u\|_{C^\alpha(U)}$ *as*

$$\|u\|_{C^{0,\alpha}(U)} = [u]_{C^{0,\alpha}(U)} + \|u\|_{L^\infty}$$

and correspondingly, we can define the Holder space

$$C^{0,\alpha}(U) = \{u \in C(U), \|u\|_{C^{0,\alpha}} < \infty\}$$

This definition could be generalized to define Holder space $C^{k,\alpha}$*:*

$$C^{k,\alpha}(U) = \{u \in C^k(U), \|D^\beta u\|_{C^{0,\alpha}} < \infty, \forall\beta, |\beta| = k\}$$

Now we are ready to prove the following inequality:

Theorem 6.2 (Morrey's inequality). $d \geq 2$*. For* U *bounded domain in* $\mathbb{R}^d$ *with* C^1 *boundary,* $p > d$*, if* $u \in W^{1,p}(U)$*, then* $u \in C^{0,\alpha}(U)$ *with* $\alpha = 1 - \frac{d}{p}$*, and we have*

$$\|u\|_{C^{0,\alpha}(U)} \leq C\|u\|_{W^{1,p}(U)}$$

Proof. By extension and density theorem, it suffices to consider $u \in C^\infty(\mathbb{R}^d)$, $\operatorname{supp} u \subset V$, V is a bounded set such that $\bar{U} \subset V$. According to the previous theorem, we have $[u]_{C^\alpha(V)} \leq C\|Du\|_{L^p}$. It remains to show

$$\|u\|_{L^\infty} \leq C\|u\|_{W^{1,p}}$$

Since

$$\left|u(x) - \frac{1}{|B_r|}\int_{B_r(x)} u(z)\mathrm{d}z\right| \leq \frac{1}{|B_r|}\int_{B_r(x)} |u(x)-u(z)|\,\mathrm{d}z \leq C\int_{B_r(x)} \frac{|Du|}{|x-z|^{d-1}}\mathrm{d}z \leq Cr^\alpha\|Du\|_{L^p(B_r(x))}$$

Take $r = 1$, we have

$$|u(x)| \leq c\int_{B_1(x)} |u(z)|\mathrm{d}z + C\|Du\|_{L^p(B_r(x))} \leq C'\|u\|_{W^{1,p}(U)}$$

□

For $p = d$, the above theorem break down since $W^{1,d}(U)$ is not a subset of $L^\infty(U)$. A counterexample is that $d = 2$, $U = B_1(0)$, and

$$u = \log(\log(10 + \frac{1}{|x|}))$$

In other words, $L^\infty(U)$ is not big enough. One remedy is to consider the bounded mean oscillation (BMO) space.

Definition 6.2 (BMO). *For* $u \in L^1_{\text{loc}}(U)$*, we define*

$$[u]_{BMO} = \sup_{B_r(x_0)\subset U} \frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} \left|u(z) - \frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} u(y)\mathrm{d}y\right| \mathrm{d}z$$

and recall the famous Hardy-Littlewood theorem, which we'll use later:

Theorem 6.3 (Hardy-Littlewood)**.** *For $u \in L^1_{\text{loc}}(U)$, $Mu(x)$ is defined as*

$$Mu(x) = \sup_{r>0} \frac{1}{B_r(x)} \int_{B_r(x)} |u|$$

It is obvious that $|Mu| \le \|u\|_{L^\infty}$. Moreover, for all $1 < p < \infty$, we have

$$\|Mu\|_{L^p} \le C\|u\|_{L^p}$$

Then we have the following theorem:

Theorem 6.4. *$U = \mathbb{R}^d$, $u = L^1_{\text{loc}}(\mathbb{R}^d)$, $d \ge 2$, then $[u]_{BMO} < \infty$ [14] and*

$$[u]_{BMO} \le C\|u\|_{W^{1,d}}$$

Proof. We'll prove

$$[u]_{BMO} \le C\|Du\|_{L^d}$$

Fix $B_r(x)$, we want to prove $\exists c$ independent of $B_r(x)$ and u, such that

$$\frac{1}{|B_r(x)|} \int_{B_r(x)} \left| u(z) - \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y)\mathrm{d}y \right| \mathrm{d}z \le C\|Du\|_{L^d}$$

we have:

$$\begin{aligned} LHS &= \frac{1}{|B_r(x)|} \int_{B_r(x)} \left| \frac{1}{|B_r(z)|} \int_{B_r(z)} u(z)\mathrm{d}y - \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y)\mathrm{d}y \right| \mathrm{d}z \\ &\le \frac{1}{|B_r(x)|^2} \int_{B_r(x)} \int_{B_{2r}(y)} |u(z) - u(y)|\, \mathrm{d}y\mathrm{d}z \\ &\le C' \frac{1}{|B_r(x)|} \int_{B_r(x)} \int_{B_{2r}(y)} \frac{|Du(z)|}{|z-y|^{d-1}} \mathrm{d}y\mathrm{d}z \end{aligned}$$

One might think of using Young's inequality:

$$\|f * g\|_{L^r} \le \|f\|_{L^p}\|g\|_{L^q}, \quad 1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$$

However, this fails since $\frac{1}{|z-x|^{d-1}} \notin L^q$. We will need the famous dyadic decomposition and Hardy-Littlewood theorem. We have

$$\begin{aligned} \int_{B_{2r}(y)} \frac{|Du(z)|}{|z-y|^{d-1}} \mathrm{d}z &\le \sum_{2^k \le 2r} \int_{A_k} \frac{1}{(2^k)^{d-1}} |Du(z)|\mathrm{d}z, \quad A_k = \{2^k \le |z-y| \le 2^{k+1}\} \\ &\le \sum_{2^k \le 2r} \frac{1}{(2^k)^{d-1}} \int_{B_{2^k}(y)} |Du(z)|\mathrm{d}z \le C_2 \sum_{2^k \le 2r} 2^k M(|Du|)(y) \end{aligned}$$

where $M(|Du|)$ is the Hardy-littlewood function of Du. Then

$$\begin{aligned} &\frac{1}{|B_r(x)|} \int_{B_r(x)} \int_{B_{2r}(y)} \frac{|Du(z)|}{|z-y|^{d-1}} \mathrm{d}y\mathrm{d}z \\ \le & C_2 \frac{1}{|B_r(x)|} \int_{B_r(x)} \mathrm{d}y \sum_{2^k \le 2r} 2^k M(|Du|)(y) \\ \le & C_3 \frac{1}{|B_r(x)|} \int_{B_r(x)} rM(|Du|)(y)\mathrm{d}y = C'''r \frac{1}{|B_r(x)|} \int_{B_r(x)} M(|Du|)(y)\mathrm{d}y \\ \le & C_3 \frac{1}{|B_r(x)|} r\|M(|Du|)\|_{L^d(B_r(x))} \|1\|_{L^{\frac{d}{d-1}}(B_r(x))} = C_4 \|M(|Du|)\|_{L^d(B_r(x))} \le C\||Du|\|_{L^d} \end{aligned}$$

where we use the Hardy-Littlewood inequality in the last step. □

[14] (why?)

7 Compactness of Sobolev embedding

Definition 7.1 (Compact operator)**.** *$T : X \to Y$, X, Y are Banach spaces. T is a compact operator, if $T(B_x)$ is precompact, or equivalently, $\forall$ bounded sequence $\{x_n\} \subset X$, $\{Tx_n\}$ has a convergent subsequence.*

Definition 7.2 (Compact embedding)**.** *Consider an embedding $l : X \to Y$, where $X \subset Y$ and l is a bounded linear injective mapping. We say that X is compactly embedded in Y if and only if l is compact as a mapping, i.e. $\forall$ bounded sequence $\{x_n\} \subset X$, it has a convergent subsequence with respect to $\|\cdot\|_Y$.*

Since we have $W^{1,p}(U) \subset L^p(U)$, it is natural to ask whether such embedding is compact.

A natural starting point to discuss compactness is the Arzela-Ascoli theorem:

Theorem 7.1 (Arzela-Ascoli)**.** *K is compact set, $F \subset C(K)$, if*

1. *(Local boundedness) $\forall x \in K$, $\exists M_x > 0$, s.t. $\forall f \in K$, $|f(x)| \leq M_x$ holds.*[15]
2. *(Equicontinuity) $\forall \varepsilon > 0$, $\exists \delta > 0$, $\forall f \in F$, $|f(x) - f(y)| < \varepsilon$ if $|x - y| < \delta$.*

Then F has a convergent subsequence. (uniformly)

Let us first discuss the compactness of the embedding of Holder spaces.

Theorem 7.2 (Compactness of embedding $C^{0,\alpha}(U) \subset C^{0,\alpha'}(U)$, $0 < \alpha' < \alpha < 1$.)**.** *U is a bounded open subset of $\mathbb{R}^d$, $0 < \alpha' < \alpha < 1$. Then the embedding $C^{0,\alpha}(U) \subset C^{0,\alpha'}(U)$ is compact.*

Proof. Step 1: Using Arzela-Ascoli, we know that $C^{0,\alpha}(U) \subset C(U)$ is a compact embedding.

Step 2: according to step 1, if $\{u_n\} \subset C^{0,\alpha}(U)$, s.t. $\|u_n\|_{C^{0,\alpha}} \leq M$, $\exists$ subsequence u_{n_j} s.t. $\{u_{n_j}\} \subset C(U)$, $u_{n_j} \to u_\infty$ in $C(U)$. We claim that in fact $\|u_{n_j} - u_\infty\|_{C^{0,\alpha'}} \to 0$ when $j \to \infty$.

Since

$$\frac{|v(x) - v(y)|}{|x - y|^{\alpha'}} \leq \left(\frac{|v(x) - v(y)|}{|x - y|^{\alpha}}\right)^{\frac{\alpha'}{\alpha}} (|v(x)| + |v(y)|)^{1 - \frac{\alpha'}{\alpha}}$$

therefore $\forall v \in C^{0,\alpha'}(U)$,

$$[v]_{C^{0,\alpha'}} \leq c\|v\|_{L^\infty}^{1-\frac{\alpha'}{\alpha}} [v]_{C^{0,\alpha}}^{\frac{\alpha'}{\alpha}}$$

then

$$[u_{n_j} - u_\infty]_{C^{0,\alpha'}} \leq c\|u_{n_j} - u_\infty\|_{L^\infty}^{1-\frac{\alpha'}{\alpha}} [u_{n_j} - u_\infty]_{C^{0,\alpha}}^{\frac{\alpha'}{\alpha}}$$

On the right hand side, the first term goes to 0, the second term is bounded, and we are done. ☐

Our main theorem would be proving that for $1 \leq p < d$, the embedding $W^{1,p}(U) \subset L^q(U)$ is compact, where $1 \leq q < p^*$. We'll need the following lemma:

Lemma 7.1 (Mollifier)**.** *Recall that if $v \in L^p(\mathbb{R}^d)$, $\varphi \in C_c^\infty(\mathbb{R}^d)$, $\int \varphi = 1$, $\varphi_\varepsilon * v \to v (\varepsilon \to 0)$ in L^p. Now consider $v \in W^{k,p}(\mathbb{R}^d)$, and for φ assume that $\int x^\alpha \varphi \mathrm{d}x = 0$ for $1 \leq |\alpha| \leq k$. Then we have*

$$\|\varphi_\varepsilon * v - v\|_{L^p} \leq C\varepsilon^k \|\partial^{(k)} v\|_{L^p}$$

Proof. We'll only prove this in the case of $k = 1$. Without loss of generality, let us assume that $\operatorname{supp} \varphi \subset B_1(0)$. Then we have

$$\begin{aligned}
(\varphi_\varepsilon * v)(x) - v(x) &= \frac{1}{\varepsilon^d} \int_{B(x,\varepsilon)} \eta\left(\frac{x - z}{\varepsilon}\right) (v(z) - v(x))\mathrm{d}z \\
&= \int_{B(0,1)} \varphi(y) (v(x - \varepsilon y) - v(x))\mathrm{d}y \\
&= \int_{B(0,1)} \varphi(y) \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} u(x - \varepsilon t y)\mathrm{d}t\mathrm{d}y \\
&= -\varepsilon \int_{B(0,1)} \varphi(y) \int_0^1 Du(x - \varepsilon t y) \cdot y \mathrm{d}t\mathrm{d}y
\end{aligned}$$

[15] Doesn't this upper bound need to be uniform for all x?

Then

$$\begin{aligned}\int |(\varphi_\varepsilon * v)(x) - v(x)|^p \, \mathrm{d}x &= \varepsilon^p \int \left(\int_{B(0,1)} \varphi(y) \int_0^1 Du(x - \varepsilon t y) \cdot y \mathrm{d}t \mathrm{d}y \right)^p \mathrm{d}x \\ &\le \varepsilon^p \int \left(\int_0^1 \int_{B(0,1)} \varphi(y) \, |Dv(x - \varepsilon t y)| \, \mathrm{d}y \mathrm{d}t \right)^p \mathrm{d}x \\ &\le C_1 \varepsilon^p \left(\int_0^1 \int \int_{B(0,1)} \varphi(y)^p \, |Dv(x - \varepsilon t y)|^p \, \mathrm{d}y \mathrm{d}x \mathrm{d}t \right) \\ &\le C_2 \varepsilon^p \int |Dv|^p \mathrm{d}x = C_2 \varepsilon^p \|Dv\|_{L^p}^p\end{aligned}$$

Therefore

$$\|\varphi_\varepsilon * v - v\|_{L^p} \le C\varepsilon \|Dv\|_{L^p}$$

□

Theorem 7.3 (Rellich-Kondrachov compactness theorem). *For $d \ge 2$, U bounded in $\mathbb{R}^d$ with C^1 boundary, the embedding $W^{1,p}(U) \subset L^q(U)$ is compact, for $1 \le q < p^*$.*

(Recall that according to Sobolev inequality, the embedding of $W^{1,p}(U) \subset L^{p^*}(U)$ is compact.)

Proof. Step 1: reduce to compactness of $W^{1,p}(U) \to L^p(U)$:

- Case 1: $p \ge q \ge 1$, with U being bounded, we have $\|v\|_{L^q(U)} \le |U|^{1/q - 1/p} \|v\|_{L^p}$.
- Case 2: $p < q < p^*$, we have

  $$\|v\|_{L^q(U)} \le \|v\|_{L^p(U)}^\theta \|v\|_{L^{p^*}(U)}^{1-\theta}, \quad \frac{d}{q} = \frac{d}{p}\theta + \frac{d}{p^*}(1-\theta)$$

  where on the right hand side, the second term is bounded by the Sobolev inequality and the first term is bounded by the compactness result of $W^{1,p}(U) \to L^p(U)$.

Step 2: prove compactness of $W^{1,p}(U) \to L^p(U)$.

Given $\{u_n\} \subset W^{1,p}(U)$, $\|u_n\|_{W^{1,p}(U)} \le M < \infty$. By extension, $\exists \widetilde{u}_n$, an extension of u_n, such that

$$\|\widetilde{u}_n\|_{W^{1,p}(\mathbb{R}^d)} \le C\|u_n\|_{W^{1,p}(U)} \le CM$$

and $\operatorname{supp} \widetilde{u}_n \subset V$ (V is bounded open, $U \subset V$). Introduce $\varphi \in C_c^\infty(\mathbb{R}^d)$, $\int \varphi = 1$, and we have

$$\widetilde{u}_n = \varphi_\varepsilon * \widetilde{u}_n + (\widetilde{u}_n - \varphi_\varepsilon * \widetilde{u}_n)$$

By lemma, we know that $\|(\widetilde{u}_n - \varphi_\varepsilon * \widetilde{u}_n)\|_{L^p} \le C\varepsilon M$, independent of n. For $v_{n,\varepsilon} = \varphi_\varepsilon * \widetilde{u}_n$ for each fixed ε, we have

$$\|v_{n,\varepsilon}\|_{L^\infty} + \|\nabla v_{n,\varepsilon}\|_{L^\infty} \le C_\varepsilon M$$

because of Holder inequality

$$\left| \int \varphi_\varepsilon(x-y) \widetilde{u}_n(y) \mathrm{d}y \right| \le \|\widetilde{u}_n\|_{L^p} \|\varphi_\varepsilon\|_{L^{p'}} = C_\varepsilon \|\widetilde{u}_n\|_{L^p}$$

and also $v_{n,\varepsilon}$ for fixed ε satisfy the condition of using Arzela-Ascoli theorem.

Therefore any l, $\exists$ a subsequence $\{\widetilde{u}_{n_k^l}\}_{k=1}^\infty$ and a ε_l, such that

- $\|\widetilde{u}_{n_k^l} - \varphi_{\varepsilon_l} * \widetilde{u}_{n_k^l}\|_{L^p} < 2^{-l}$;
- $\|v_{n_{k'}^l, \varepsilon_l} - v_{n_{k''}^l, \varepsilon_l}\| < 2^{-k}$, for all $k', k'' \ge k$. (This follows from Arzela-Ascoli theorem.)

Then use a diagonal argument, we have constructed a sequence $\{\widetilde{u}_{n_l^l}\}$ such that it is a Cauchy sequence in L^p. □

A direct result is that

Theorem 7.4. *The embedding of $W^{1,p}(U) \to L(U)$ is compact for any p for bounded U with C^1 boundary.* [16] *The embedding of $W_0^{1,p}(U) \to L(U)$ is compact for any p for bounded U.*

8 Poincare-type inequality

Poincare-type inequality refers to the type of inequality that uses the regularity of Du to control u, with some additional conditions to fix the inequality.

Theorem 8.1 (Poincare)**.** *$1 \le p < \infty$, let $U \subset \mathbb{R}^d$ be a bounded domain with C^1 boundary, then for $u \in W^{1,p}(U)$ with $\int u \mathrm{d}x = 0$, we have*

$$\|u\|_{L^p(U)} \le C_U \|Du\|_{L^p(U)}$$

Proof using compactness. We argue by contradiction.

Assume that for n, $\exists u_n$, $\|u_n\|_{L^p} \ge n\|\nabla u_n\|_{L^p}$, and $u_n \in W^{1,p}$, $\int u_n = 0$. By normalization, without loss of generality, let $\|u_n\|_{L^p} = 1$, then $\|\nabla u_n\|_{L^p} \le \frac{1}{n}$. Therefore the $W^{1,p}$ norm of $\|u_n\|$ is bounded, by Rellich-Kondrachov theorem (and something else), there exists a subsequence, still denoted by $\{u_n\}$, such that

- $u_n \to u_\infty$ in L^p. This implies that $\|u_n\|_{L^p} \to \|u\|_{L^p} = 1$.
- $Du_n \rightharpoonup Du$ weakly. Since $\|Du_n\|_{L^p} \to 0$, this implies that $Du = 0$, then u is a constant. Recall that $u_n \to u_\infty$ in L^p, therefore $\int u_n \mathrm{d}x \to \int u \mathrm{d}x$, and we knew that $\int u \mathrm{d}x = 0$. Therefore $u = 0$.

Contradiction is achieved. □

Using this, we can prove that

$$\left\| u - \frac{1}{|B_r(x)|} \int_{B_r(x)} u \mathrm{d}x \right\|_{L^p(B(x,r))} \le Cr\|Du\|_{L^p(B(x,r))}$$

where C is independent of x, r. This implies that for $u \in W^{1,d}\left(\mathbb{R}^d\right) \cap L^1\left(\mathbb{R}^d\right)$, we have $u \in BMO$.

We have covered Poincare-Fredrich inequality. Another type of Poincare inequality is the Hardy's inequality:

Theorem 8.2. *For $u \in C_c^\infty(\mathbb{R}^d)$, $d > 2$, we have*

$$\|\frac{1}{|x|}u\|_{L^2} \le C\|Du\|_{L^2}$$

Proof. Using polar coordinate, we only need to prove

$$\int_0^\infty \frac{1}{r^2} u^2 r^{d-1} \mathrm{d}r \le C \int |\partial_r u|^2 r^{d-1} \mathrm{d}r$$

Consider that

$$\begin{aligned}
0 \le \int_0^\infty \left(\partial_r u + \frac{\alpha}{r} u\right)^2 r^{d-1} \mathrm{d}r &= \int_0^\infty \left((\partial_r u)^2 + \frac{2\alpha}{r} u \partial_r u + \frac{\alpha^2}{r^2} u^2\right) r^{d-1} \mathrm{d}r \\
&= \int_0^\infty (\partial_r u)^2 \mathrm{d}r + \alpha^2 \int_0^\infty \frac{1}{r^2} u^2 r^{d-1} \mathrm{d}r + \alpha \int_0^\infty \partial_r(u^2) r^{d-2} \mathrm{d}r \\
&= \int_0^\infty (\partial_r u)^2 \mathrm{d}r - \left((d-2)\alpha - \alpha^2\right) \int_0^\infty \frac{1}{r^2} u^2 r^{d-1} \mathrm{d}r
\end{aligned}$$

[16] When $p > d$, use Morrey's inequality and Arzela-Ascoli inequality. What about $p = d$?

Plug in $\alpha = \frac{d-2}{2}$, we have

$$\int_0^\infty \frac{1}{r^2} u^2 r^{d-1} \mathrm{d}r \leq \left(\frac{2}{d-2}\right)^2 \int |\partial_r u|^2 r^{d-1} \mathrm{d}r$$

$\square$

[17]

References

[17] How to prove the general form of Hardy's inequality?