PDE notes: Elliptic PDE, taught by Sung-jin Oh

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It is claimed by some mathematician that

> If you do integration by part, you are doing PDE. If you do Holder's inequality, you are doing probability theory. If you do Young's inequality, you are doing mathematical analysis.

And as a corollary, by learning elliptic PDE, you are on the right path to master PDE, probability and mathematical analysis.

Part 4, Part 6 and Part 7 are not well-written now and needs development. Please avoid them for now.

The main references are Sung-jin's lecture and Evan's book Evans [2010]. For general boundary value problems (Part 6), we refer to Taylor [2011]. For unique continuation (part 7), we refer to Fragnelli [2021].

Contents

Introduction

We'll use Einstein's summation convention frequently.

Roughly speaking, elliptic PDEs are generalizations of $-\Delta u = f$. Recall that for $-\sum_j \partial_j^2$, it becomes $-\sum_j (\xi_j^2) = -|\xi|^2$ in the Fourier space, and is nonzero for $\xi \neq 0$. This is the general feature for elliptic equation.

Now we begin to describe the definition of elliptic PDEs.

Definition 0.1 (Elliptic PDE). *1. Consider positive definite operator P, if for $u : U \to \mathbb{R}^N$, (Pu) takes values in $\mathbb{R}^N$:*

$$(Pu)^I = \sum_{J,|\alpha|=K} A^I_{J,\alpha_1,\cdots,\alpha_d} \partial^\alpha u^J + \text{ lower order terms } (|\alpha| < K)$$

We define $\sigma_{prin}(P)$ as

$$\sigma_{prin}(P)^I_J = \mathrm{i}^k \sum_{J,|\alpha|=K} A^I_{\alpha_1,\cdots,\alpha_d} \xi_1^{\alpha_1} \cdots \xi_d^{\alpha_d}$$

therefore $\sigma_{prin}(P)$ could be viewed as a $N \times N$ matrix.

2. *We say P is elliptic if $\sigma_{\mathrm{prin}}(P)$ is invertible for all $x \in U$, $\xi \neq 0$.*

3. *For $N = 1$, i.e. the scalar case, we have*

$$Pu = \sum_{|\alpha|=K} a_\alpha(x) \partial^\alpha u + \text{ lower order terms } (|\alpha| < K)$$

The first nontrivial case is

$$Pu = a^{ij} \partial_i \partial_j u + b^i \partial_i u + cu, \quad a^{ij} = a^{ji}$$

Ellipticity means that $a^{ij} \xi_i \xi_j \neq 0, \forall \xi_i, \xi_j$, which means that a^{ij} is definite.

We take $a(x) > 0$, $\forall x \in U$.

Definition 0.2 (Uniform ellipticity). *$\exists \lambda > 0$, s.t. $a^{ij}(x)\xi_i\xi_j \geq \lambda$, $\forall x \in U$ and $|\xi| = 1$.*

Elliptic equations appear naturally in optimization problems due to calculus of variation, and it appears frequently in math and physics. It also often arises as a part of evolutionary problems. An example of that is in incompressible Euler equation, $u : \mathbb{R}^3 \to \mathbb{R}^3$, we have

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = 0 \\ \nabla \cdot u = 0 \end{cases}$$

If we take divergence of the first equation, we have

$$-\Delta p = \nabla(u \cdot \nabla u)$$

We'll cover the following topics.

- Boundary value problems for elliptic PDE, existness and uniqueness.
- Regularity properties: for $Pu = f$, where P is an elliptic operator of order K. If f has regularity of order k, we are expecting u to have regularity of order $k + K$.
- Maximum principle (mostly for the scalar case)

1 Boundary value problem

We consider the case where $d \geq 2$, $N = 1$, and we assume the uniform regularity of P. The regularity of a, b, c should be *nice*. U is a bounded domain in $\mathbb{R}^d$ with *nice* boundary. For now, we require $a, b, c \in L^\infty(U)$ and U has C^1 boundary. The boundary value problem is formulated as followed:

$$\begin{cases} Pu = f, & \text{in } U \\ u = g & \text{on } \partial U \end{cases} \quad \text{(Dirichlet)}, \quad \begin{cases} Pu = f, & \text{in } U \\ \dfrac{\partial u}{\partial n} = g & \text{on } \partial U \end{cases} \quad \text{(Neumann)}$$

We'll study the solvability for $u \in H^1(U)$. We'll focus on the Dirichlet boundary problem, which can be reduced to the case that $g = 0$: consider $\widetilde{g}$, for which g is the trace of $\widetilde{g}$. Consider $v = u - \widetilde{g}$, we have

$$\begin{cases} Pv = f - P\widehat{g} = \widetilde{f}, & \text{in } U \\ v = 0 & \text{on } \partial U \end{cases}$$

Definition 1.1 (Divergence form)**.** *P is divergence form, if*

$$\begin{aligned} Pu &= \partial_i(a^{ij}\partial_j u) + \partial_i(b^i u) + cu \\ &= a^{ij}\partial_i\partial_j u + (\partial_j a^{ij} + b^i)\partial_i u + (\partial_i b^i + c)u \end{aligned}$$

Note that as long as a^{ij} is C^1, P could always be written in the divergence form.

We focus on the following zero boundary value problem:

$$\begin{cases} Pu = f, & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases} \tag{1}$$

where

$$Pu = -\partial_j(a^{jk}\partial_k u) + b^j\partial_j u + cu$$

The notion of *weak solution*, is defined as followed:

Definition 1.2. *Motivated by integration by parts, it is natural to define the following bilinear form:*

$$B[u, v] := \int_U a^{ij}u_i v_j + b^i u_i v + cuv \mathrm{d}x, \quad u, v \in H_0^1(U)$$

$u \in H_0^1(U)$ is defined to be the solution of the boundary value problem (1) if and only if

$$B[u, v] = (f, v), \quad \forall v \in H_0^1(U)$$

Here $f \in H^{-1}$, and for $f = f^0 - \partial_i f^i$, $(f, v) = \int_U f^0 v + f^i v_i \mathrm{d}x$.

Our discussion of uniqueness and existence based on the following energy estimate, or a-priori estimate:[1]

Theorem 1.1.
- *Energy estimates: For the above bilinear form, there exists some constants $\alpha, \beta > 0$ and $\gamma \geq 0$ such that*

$$|B[u, v]| \leq \alpha\|u\|_{H_0^1(U)}\|v\|_{H_0^1(U)}$$

and

$$\beta\|u\|^2_{H_0^1(U)} \leq B[u, u] + \gamma\|u\|^2_{L^2(U)}$$

for all $u, v \in H_0^1(U)$.

[1] Does this proof use boundedness? What is the optimal regularity for b, c?

- *a-priori estimate: if $u \in H^1$, then $\exists c > 0, \gamma \geq 0$, s.t.*

$$\|u\|_{H^1(U)} \leq c\|Pu\|_{H^{-1}(U)} + \gamma\|u\|_{L^2(U)}$$

The proof of the energy estimates could be found as theorem 6.2 on Evans. The proof of the a-priori estimate simply take advantage of the fact that $B(u,u) = (f,u) \leq \|f\|_{H^{-1}}\|u\|_{H^1}$.

We'll also need the a-priori estimate for P^* later. Since P^* has the following form:

$$P^*u = \left(-\partial_j(a^{jk}\partial_k u)\right) - \partial_j(b^j u) + cu$$

Therefore the energy estimates also apply to P^*.

Now let us look at our operator P. Clearly P is $H^1(U) \to H^{-1}(U)$. However, to build the solvability theory for $u|_{\partial U} = 0$, we restrict it to be:

$$P : H_0^1(U) \to H^1(U)$$

We want to answer the question of solvability, which means

$$\text{solvability} \Leftrightarrow \text{uniqueness and existence} \Leftrightarrow \text{1-1 mapping and onto of } P$$

Using our functional analysis language, we let $X = H_0^1(U)$, $Y = H^{-1}(U)$.

1.1 Case 1: Both P and P^* obey good energy estimates

By *good*, we mean that the γ in Theorem 1.1 is 0. In this case, we have (from Theorem 1.1),

$$\|u\|_X^2 \leq cB[u,u], \quad \|u\|_X \leq c\|Pu\|_Y$$

Based on this, there are two ways in functional analysis to establish the uniqueness and existence. The first way is through the Lax-Milgram theorem:

Theorem 1.2 (Lax-Milgram theorem). *For a bilinear mapping $B : X \times X \to \mathbb{R}$, if $\exists \alpha, C$ such that*

$$|B[u,v]| \leq \alpha\|u\|_X\|v\|_X, \quad \|u\|_X^2 \leq cB[u,u]$$

then for any $f \in X^*$, $\exists u \in X$, such that

$$B[u,v] = \langle f, v\rangle$$

Note that if B is symmetric, this is an immediate result of Riesz representation theorem. We actually don't require B to be symmetric.

In this boundary value problem, we get to use Lax-Milgram theorem since Y coincides with X^*. We actually have another slightly generalized way in functional analysis to do this, which is a result that we mentioned earlier:

Theorem 1.3. *For Banach spaces X, Y, and P being a bounded linear operator, we have:*

1. *If $\|u\|_X \leq c\|Pu\|_Y$, then*
   - $\mathrm{Ker}(P) = \{0\}$
   - *$\forall g \in X^*$, $\exists v \in Y^*$ s.t. $P^*v = g$ ($\mathrm{ran}(P^*) = X^*$) and $\|v\|_{Y^*} \leq c\|g\|_{X^*}$.*
2. *If $\|v\|_{Y^*} \leq c'\|P^*v\|_{X^*}$, then*
   - $\mathrm{Ker}(P^*) = \{0\}$
   - *$\forall f \in Y$, $\exists u \in X$ s.t. $Pu = f$ ($\mathrm{ran}(P) = Y$) and $\|u\|_X \leq c'\|f\|_Y$.*

Note that previously when we prove this theorem, we cheated by assume X is reflexive in part two. Even though this is true in most cases, we actually don't need X to be reflexive. The proof of the second part would be using *closed range theorem* where we need to show $\mathrm{ran}(P)$ being close.[2]

With either of these approaches, we can prove existence theorem for $\gamma = 0$:

Theorem 1.4. *For cases with energy estimates that $\gamma = 0$, such as*

- $\tilde{P}u = -\partial_j(a^{jk}\partial_k u)$ *(the proof needs Poincare inequality, therefore needs U to be bounded);*
- *For general P, there exists μ such that, for $\lambda \geq \mu$, $\tilde{P} = P_\lambda = P + \lambda$ satisfy the above estimate.*

we have: $\forall f \in H^{-1}(U)$, there exists a unique $u \in H_0^1(U)$, s.t. $\tilde{P}u = f$ in U.

The proof of this can use either approaches we mentioned.

1.2 Case 2: For general P

Recall the definition of compact operators in functional analysis.

Definition 1.3 (Compact operator)**.** *For $K : X \to Y$ and ball $B_X = \{x \in X, \|x\| < 1\}$. K is called compact if and only if $K(\bar{B_x})$ is compact.*

Lemma 1.1 (Fredholm theory)**.** *For operator K, we have*

- *For $K : X \to Y$, if K is compact, then K^* is compact;*
- *Solvability of $(I + K)x = y$:*
  - $\mathrm{Ker}(I + K)$ *is finite dimensional.*
  - $\exists n_0 \geq 1$, *s.t.* $\mathrm{Ker}(I + K)^n = \mathrm{Ker}(I + K)^{n_0}$, *for* $n \geq n_0$.
  - $\mathrm{ran}(I + K)$ *is closed* $\Rightarrow \mathrm{ran}(I + K) =^{\perp} \mathrm{Ker}(I + K^*)$.
  - $\dim \mathrm{Ker}(I + K) = \dim \mathrm{Ker}(I + K^*)$

Evans proves this for $X = Y = H$ is a Hilbert space. We'll only use this to carry on. Why is this lemma useful? Consider $(P + \mu)$, for which by case 1, for μ sufficiently positive, $(P + \mu)^{-1} : H^{-1}(U) \to H_0^1(U)$ well defined. Then

$$(P + \mu)u - \mu u = Pu = f \Leftrightarrow (I - \mu(P + u)^{-1})u = (P + \mu)^{-1}f$$

and according to Rellich-Kondrachov, because U is bounded, then $l : H_0^1(U) \to L^2(U)$ is compact. Therefore, $(P + \mu)^{-1}$ as an operator from $L^2(U)$ to $L^2(U)$ is compact, since it is a composition of a compact operator and a linear and bounded operator.

$$(P + \mu)^{-1} : L^2(U) \subset H^{-1}(U) \to H_0^1(U) \to L^2(U)$$

And then we can use the above lemma and reach the following conclusion:

Theorem 1.5 (Uniqueness and existence using Fredholm alternative)**.** *P is as before, U is a bounded domain with C^1 boundary.*

1. *Using the above argument and Fredholm theory*

$$\textit{(solvability)} \quad \forall f \in L^2(U), \exists \text{ unique } u \in L^2(U), s.t. Pu = f$$

or

$$(\exists \textit{ nonzero solutions}) \quad \exists u \in L^2(U), u \neq 0, Pu = 0$$

[2] Please finish this detail by yourself.

2. *Using the above statement, we could show*

$$\text{(solvability)} \quad \begin{cases} \forall f \in H^{-1}(U), \exists \text{ unique } u \in H_0^1(U), s.t. Pu = f \\ \qquad \text{and } \exists c > 0 \text{ independent of } u, s.t. \|u\|_{H^1(U)} \le c\|f\|_{H^{-1}(U)} \end{cases}$$

or

$$(\exists \text{ nonzero solutions}) \quad \exists u \in H_0^1(U), u \neq 0, Pu = 0$$

3. *If the second scenario holds, we have* $\mathrm{Ker}(P) = \mathrm{Ker}(P^*) < +\infty$*. And given* $f \in H^{-1}(U)$*,* $\exists u \in H_0^1(U)$*, s.t.* $Pu = f$ *if and only if* $\langle f, v\rangle = 0$ *for* $\forall v \in \mathrm{Ker}(P^*)$*.*

Later, we'll use weak maximum principle to prove the uniqueness property, and then using Fredholm alternative theorem to prove the solvability.

We have some final remarks:

- There is a result for eigen problems using properties of compact operators.
- On separable Hilbert space, the set of compact operators are the closure set of all finite-dimensional range operator.

2 L^2-based interior regularity

Let us first look at a prototypical example. Consider

$$-\Delta u = f \text{ in } U$$

Intuitively, we are hoping that u is more regular than f by order 2.

- If $f \in H^k(U)$, we hope to have the following *interior regularity*: $\forall V \subset\subset U$, we have

$$\|u\|_{H^{k+2}}(V) \le C\|f\|_{H^k(U)} + C\|u\|_{L^2(U)}$$

  where C is dependent on V. This is the L^2-based regularity theory.

- If $f \in C^{k,\alpha}(U)$, we hope that

$$\|u\|_{C^{k+2},\alpha}(V) \le C\|f\|_{C^{k,\alpha}(U)} + C\|u\|_{L^\infty(U)}$$

  where C is dependent on V. This is the Schauder theory.

For now we focus on the L^2-based regularity theory. The idea is mainly *integration by parts* (energy method).

To demonstrate the main idea, let us first assume that we already know that $u \in H^{k+2}(V)$. Later, this allows us to commute the operator with derivatives.

Let us first look at how to obtain the H^1 bound for u

$$-\Delta u = f \Rightarrow \int -\Delta u u \xi^2 = \int f u \xi^2$$

where ξ is a smooth cutoff function:

$$\xi = \begin{cases} 1, & \text{in } V \\ 0, & \text{near } \partial U \end{cases}$$

Using integration by parts, we have

$$\int -\Delta u u \xi^2 = \sum_{j=1}^d \partial_j u \partial_j (u\xi^2) = \sum_{j=1}^d (\partial_j u)^2 \xi^2 + 2u\xi\partial_j u \partial_j \xi$$

Then

$$\int_U |Du|^2\xi^2 \leq \left|\int_U fu\xi^2 \mathrm{d}x\right| + 2\left|\int_U u\xi Du\cdot D\xi \mathrm{d}x\right|$$

$$2\left|\int_U u\xi Du\cdot D\xi\mathrm{d}x\right| \leq 2\left(\int_U |Du|^2\xi^2\right)^{\frac{1}{2}}\left(\int_U u^2|D\xi|^2\right)^{\frac{1}{2}} \leq \varepsilon\int_U |Du|^2\xi^2 + \frac{1}{\varepsilon}\int_U u^2|D\xi|^2$$

Choose $\varepsilon = \frac{1}{2}$, we have

$$\frac{1}{2}\int_V |Du|^2\mathrm{d}x \leq \frac{1}{2}\int_U |Du|^2\xi^2\mathrm{d}x \leq \left|\int_U fu\xi^2\mathrm{d}x\right| + 2\int_U u^2|D\xi|^2\mathrm{d}x$$

and similarly for $\left|\int_U fu\xi^2\mathrm{d}x\right|$, we have $\left|\int_U fu\xi^2\mathrm{d}x\right| \leq c(\|f\|_{L^2(U)} + \|u\|_{L^2(U)})$, and since $|D\xi|$ can be bounded, we are done.

Now let us look at the H^2 bound for u. Assuming that we know $u \in H^2(u)$. With $-\Delta u = f$, we have $-\Delta\partial_j u = \partial_j f$, and then all the above argument could be applied to $\partial_j u$ and $\partial_j f$. In other words, we have

$$\int_U |D\partial_j u|^2\xi^2\mathrm{d}x \leq 2\left|\int_U \partial_j f\partial_j u\xi^2\mathrm{d}x\right| + C_1\|\partial_j u\|_{L^2(U)}$$

where

$$\left|\int_U \partial_j f\partial_j u\xi^2\mathrm{d}x\right| = \left|\int_U f\partial_j\partial_j u\xi^2\mathrm{d}x\right| \leq \frac{1}{4\varepsilon}\int_U f^2\xi^2 + \varepsilon\int_U |\partial_j\partial_j u|^2\xi^2$$

which leads to

$$\int_U |D\partial_j u|^2\xi^2\mathrm{d}x \leq C_2\int_U f^2\mathrm{d}x + C_2\|Du\|_{L^2(U)}$$

Note that $V \subset\subset \exists W \subset\subset U$ on the right hand side, we can replace $C_2\|Du\|_{L^2(U)}$ by $C_2\|Du\|^2_{L^2(W)}$, and then use the H^1 bound $\|Du\|^2_{L^2(W)} \leq C_3\|f\|_{L^2(U)} + C_3\|u\|_{L^2(U)}$, we have

$$\int_V |D\partial_j u|^2\xi^2\mathrm{d}x \leq C\|f\|_{L^2(U)} + C\|u\|_{L^2(U)}, \quad \forall j$$

Along with the H^1 bound, we obtain what we want.

Now we look at the real case. Setting

$$Pu = -\partial_j(a^{jk}\partial_k u) + b^j\partial_j u + cu$$

where $u : U \to \mathbb{R}$, U is an open set[3], and $\exists\lambda > 0$ s.t. $a(x) \geq \lambda I$ for all $x \in U$. Let $a, b, c \in L^\infty(U)$, and for H^2 bound, we need $\partial a \in L^\infty(U)$.

Theorem 2.1 (Interior H^2-regularity). *Let $u \in H^1(U)$ be a weak solution to $Pu = f$ on U. Then $\forall V \subset\subset U$, $u \in H^2(V)$ and*

$$\|u\|_{H^2(V)} \leq C\left(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}\right)$$

The idea is roughly:

$$\begin{aligned}\int_U -\partial_j(a^{jk}\partial_k v)v\xi^2\mathrm{d}x &= \int_U a^{jk}\partial_k v\partial_j v\xi^2\mathrm{d}x + 2\int_U a^{jk}\partial_k vv\xi\partial_j\xi\mathrm{d}x \\ &\geq \lambda\int_U |Dv|^2\xi^2\mathrm{d}x - C_1\|a\|_{L^\infty}\int_U |Dv|\xi|v||D\xi|\mathrm{d}x\end{aligned}$$

where

$$\int_U |Dv|\xi|v||D\xi|\mathrm{d}x \leq \frac{\lambda}{2C_1\|a\|_{L^\infty}}\int |Dv|^2\xi^2\mathrm{d}x + \frac{C_1\|a\|_{L^\infty}}{\lambda}\int |D\xi|^2v^2\mathrm{d}x$$

[3] Do we need bounded?

Therefore

$$\int_U -\partial_j(a^{jk}\partial_k v)v\xi^2 \mathrm{d}x \geq \frac{\lambda}{2}\int_U |Dv|^2\xi^2\mathrm{d}x - \frac{C_1\|a\|_{L^\infty}}{\lambda}\int |D\xi|^2v^2\mathrm{d}x$$

Since we do not know a-priori that $u \in H^2(V)$, we need to modify to replace the derivative with difference quotients (i.e. finite differences):

$$D_j^h v(x) = \frac{v(x+he_j)-v(x)}{h}$$

Intuitively, $D_j^h v(x) \to \partial_j v$ as $h \to 0$. Rigorously, we have the following theorem:

Theorem 2.2 (D_j^h). *1. Suppose $1 \leq p < \infty$ and $u \in W^{1,p}(U)$. Then for each $V \subset\subset U$,*

$$\|D^h u\|_{L^p(V)} \leq C\|Du\|_{L^p(U)}$$

2. Assume $1 < p < \infty$, $u \in L^p(V)$, and there exists a constant C such that $\|D^h u\|_{L^p(V)} \leq C$ for all $\forall\ 0 < |h| < \frac{1}{2}\operatorname{dist}(V, \partial U)$. Then

$$u \in W^{1,p}(V), \quad \textit{with } \|Du\|_{L^p(V)} \leq C$$

The proof of this could be found in Evans.

Proof of H^2 regularity. Step 0: since $u \in H^1(U)$, and $Pu = f$ in $U \Leftrightarrow \langle Pu, \varphi\rangle = \langle f, \varphi\rangle$, $\forall\varphi \in C_c^\infty(U) \Leftrightarrow \langle Pu, \varphi\rangle = \langle f, \varphi\rangle$, $\forall \varphi \in H_0^1(U) = (H^{-1}(U))^* \Leftrightarrow$ the following

$$\int_U a^{jk}\partial_k u\partial_j\varphi + b^j\partial_j u\varphi + cu\varphi = \int_U f\varphi\mathrm{d}x, \quad \forall\varphi \in H_0^1(U)$$

let $\widetilde{f} = f - b^j\partial_j u - cu$, we have

$$\int_U a^{jk}\partial_k u\partial_j\varphi = \int_U \widetilde{f}\varphi\mathrm{d}x, \quad \forall\varphi \in H_0^1(U)$$

Step 1: commute the equation with D_j^h. Note that

$$D_j^h(u\cdot v)(x) = D_j^h u(x)v(x) + u^h(x)D_j^h v(x), \quad u^h(x) = u(x+h)$$

and

$$\int_U uD_j^{-h}v\mathrm{d}x = -\int_U vD_j^h u\mathrm{d}x$$

Consider $\varphi = -D_j^{-h}(\xi^2 D_j^h u) \in H_0^1(U)$. We have

$$\int_U a^{lk}\partial_k u\partial_l\left(-D_j^{-h}(\xi^2D_j^hu)\right) = \int_U \widetilde{f}\left(-D_j^{-h}(\xi^2D_j^hu)\right)\mathrm{d}x$$

For the left hand side, we have

$$\begin{aligned}\int_U a^{lk}\partial_k u\partial_l\left(-D_j^{-h}(\xi^2D_j^hu)\right)\mathrm{d}x &= -\int_U a^{lk}\partial_k uD_j^{-h}\left(\partial_l(\xi^2D_j^hu)\right)\mathrm{d}x\\ &= \int_U D_j^h\left(a^{lk}\partial_k u\right)\partial_l(\xi^2D_j^hu)\mathrm{d}x\\ &= \int_U a^{lk,h}\partial_k\left(D_j^hu\right)\partial_l(\xi^2D_j^hu)\mathrm{d}x + \int_U\left(D_j^ha^{lk}\right)\partial_k u\partial_l(\xi^2D_j^hu)\mathrm{d}x\end{aligned}$$

Using an argument like we showed in the motivating case, we can prove for the above left hand side:

$$LHS \geq \frac{\lambda}{2}\int_U \xi^2|D_k^hDu|^2\mathrm{d}x - C\int_U |Du|^2\mathrm{d}x$$

Step 2: give an upper bound for the right hand side:

$$RHS \le C \int (|f| + |Du| + |u|)|v|\mathrm{d}x$$

and

$$\begin{aligned}\int_U |v|^2 \mathrm{d}x \le C_1 \int_U \left|D(\xi^2 D_k^h u)\right|^2 \mathrm{d}x &\le C_2 \int_W \left|D_k^h u\right|^2 + \xi^2 \left|D_k^h Du\right|^2 \mathrm{d}x \\ &\le C \int_U |Du|^2 + \xi^2 \left|D_k^h Du\right|^2 \mathrm{d}x\end{aligned}$$

Then we have

$$RHS \le \frac{\lambda}{4} \int_U \xi^2 \left|D_k^h Du\right|^2 \mathrm{d}x + C \int_U f^2 + u^2 + |Du|^2 \mathrm{d}x$$

Combine step 1 and 2, we have

$$\int_V \left|D_k^h Du\right|^2 \mathrm{d}x \le \int_U \xi^2 \left|D_k^h Du\right|^2 \mathrm{d}x \le C \int_U f^2 + u^2 + |Du|^2 \mathrm{d}x$$

for $k = 1, \cdots, n$ and all sufficiently small $|h| \neq 0$. Then according to the previous theorem, we have

$$\|u\|_{H^2(V)} \le C \left(\|f\|_{L^2(U)} + \|u\|_{H^1(U)}\right)$$

and using $V \subset\subset W \subset\subset U$, we have

$$\|u\|_{H^2(V)} \le C \left(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}\right)$$

□

Now we turn to the result of higher order regularity:

Theorem 2.3 (H^k interior regularity)**.** *With same hypothesis as before, except*

- $|D^\alpha a| \le A,\quad$ *for* $|\alpha| \le K-1,\quad |D^\alpha b| + |D^\alpha c| \le A,\quad$ *for* $|\alpha| \le K-2$
- $f \in H^{k-2}(U)$

Then for any $V \subset\subset U$*,* $u \in H^k(U)$ *and*

$$\|u\|_{H^k(V)} \le C \left(\|f\|_{H^{k-2}(U)} + \|u\|_{L^2(U)}\right)$$

Sketch of the proof:

- The idea is to commute the equation with D^β, for $|\beta| \le K-1$ and look at the equation for $D^\beta u$:

$$\begin{aligned}D^\beta f = D^\beta(Pu) &= D^\beta\left(-\partial_j\left(a^{jk}\partial_k u\right) + b^j \partial_j u + cu\right) \\ &= -\partial_j\left(a^{jk}\partial_k D^\beta u\right) + D^\beta\left(b^j \partial_j u\right) + D^\beta(cu) - \sum_{\gamma \le \beta, \gamma \neq \beta} \binom{\beta}{\gamma} \partial_j (D^{\beta-\gamma} a^{jk} \partial_k D^\gamma u) c_\gamma\end{aligned}$$

- Using induction for m.

See details in Evans.

3 L^2-based boundary regularity

Theorem 3.1 (H^2 boundary regularity)**.** *Same hypothesis as in the H^2 interior regularity, except:*

- $u \in H_0^1(U)$ *(i.e. $u|_{\partial U} = 0$).*
- ∂U *is* C^2.

Then $u \in H^2(U)$ and

$$\|u\|_{H^2(U)} \leq C\left(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}\right)$$

Note that if $u \in H_0^1(U)$ is the unique solution, we further have

$$\|u\|_{H^2(U)} \leq C\|f\|_{L^2(U)}$$

We describe the basic ideas of the proof. For details, please check Evans Assume $u \in H^2(U)$ and omit the contribution of b and c.

$$f = -\partial_j\left(a^{jk}\partial_k u\right) + \cdots$$

For boundary regularity, only $(d-1)$-many directions (tangent to ∂U) are admissible.

$$\partial_l f = -\partial_l\left(\partial_j\left(a^{jk}\partial_k u\right)\right) + \cdots$$

For the sake of simplicity, take the special case when $\operatorname{supp} u \subset B_{1/2}(0) \cap \mathbb{R}_+^d$, $U = B_1(0) \cap \mathbb{R}_+^d$. In this case,

$$\partial_l f = -\left(\partial_j\left(\partial_l a^{jk}\partial_k u\right)\right) - \left(\partial_j\left(a^{jk}\partial_l\partial_k u\right)\right), \quad l = 1, \cdots, d-1$$

With localizer ζ, multiply by $\zeta^2\partial_l u$, we have

$$\|\zeta D\partial_l u\|_{L^2} \leq \|\zeta f\|_{L^2} + \|u\|_{L^2}$$

This uses integration by part and $u|_{\partial U} = 0$, $\partial_l u|_{\partial U} = 0$.

It remains to control

$$\|\zeta\partial_{x^d}\partial_{x^d} u\|$$

The equation allows to express this using everything else. Since $a^{jk} \geq \lambda I$, then $a^{dd} \geq \lambda$, then we have $\partial_d^2 u = -\frac{1}{a^{dd}}(\cdots)$

Now it remains to reduce the general case to the above easy case. Let us use the smooth partition of unity and boundary straightening to reduce to the half-ball case. For any $x \in \partial U$, $\exists B_r(x)$ s.t. $U \cap B_r(x) = \{x^d > \gamma(x^1, \cdots, x^{d-1})\}$ for some C^2 function $\gamma : \{(x^1)^2 + \cdots + (x^{d-1})^2 < r\} \to y^l = .x^l, \quad l = 1, \cdots, d-1, \quad y^d = x^d - \gamma(x^1, \cdots, x^d)$. Then boundary $\subset \{y^d = 0\}$. By compactness, $U \subset \bigcup_{k=1}^K U_k \bigcup U_0$, $\exists$ smooth partition of 1: $\{\chi_k\}_{k=0}^K$ subject to $\{U_k\}_{k=0}^K$

$$u = \chi_0 u + \sum_{k=1}^K \chi_k u$$

The first term has interior regularity. For each $\chi_k u$, when we change x to $y = y(x)$, we are reduced to the half-ball case which is already covered. We need to check:

- support properties and geometry
- Elliptic constant of the resulting equation is still $\sim \lambda$.
- $\partial\tilde{a}(y), \partial\tilde{b}(y)$ obeys the same bound.

$$a^{jk} = \frac{\partial x^j}{\partial y^{j'}}\tilde{a}^{j'k'}\frac{\partial x^k}{\partial y^{k'}}$$

From H^2 bound of $u\chi_k(y)$, come back to $u\chi_k(x)$.

We have the following remark:

- We have the corresponding theorem for H^k interior regularity, where we need$f \in H^{k-2}(U)$ and $\partial U \in C^k$.
- L^2-based theory (H^1 weak solution) is useful for deriving the existence of the solution. And we only need $a \in L^\infty$ to derive the H^1 bound. But it is wasteful in terms of regularity required for a for higher regularity. (Consider $-\partial_j(a^{jk}(u)\partial_k u) = f$)
- We need a regularity theory that works well for nonlinear equations: Schauder theory, i.e. elliptic regularity in $C^{k,\alpha}$. Holder spaces are closed under algebra, especially under product, which is what makes L^2 regularity problematic.

4 Schauder theory

Schauder theory is the ”Holder-based” elliptic regularity theory.

Theorem 4.1 (Schauder, interior, divergence form). *Consider*

$$Pu = -\partial_j(a^{jk}\partial_k u)$$

and U is an open subset of $\mathbb{R}^d$, $u \in C^{k,\alpha}(\bar{U})$, $a \geq \lambda I$, and $a \in C^{k-1,\alpha}$, $Pu = f$ in U, $f \in C^{k-2,\alpha}(\bar{U})$, $k \geq 1$. (if $f \in C^{-1,\alpha}$, $\exists f^0, f^j \in C^{0,\alpha}(U)$, we have $f = f^0 + \sum_{j=1}^d \partial_j f_j$.) Then we have, $\forall V \subset\subset U$, $\exists C = C_V$,

$$\|u\|_{C^{k,\alpha}(V)} \leq C\left(\|u\|_{C^0(U)} + \|f\|_{C^{k-2,\alpha}(U)}\right)$$

Theorem 4.2 (Schauder, interior, non-divergence form). *Consider*

$$Qu = -a^{jk}\partial_j\partial_k u$$

and U is an open subset of $\mathbb{R}^d$, $u \in C^{k,\alpha}(\bar{U})$, $a \geq \lambda I$, and $a \in C^{k-2,\alpha}$, $Pu = f$ in U, $f \in C^{k-2,\alpha}(\bar{U})$, $k \geq 2$. (if $f \in C^{-1,\alpha}$, $\exists f^0, f^j \in C^{0,\alpha}(U)$, we have $f = f^0 + \sum_{j=1}^d \partial_j f_j$.) Then we have, $\forall V \subset\subset U$, $\exists C = C_V$,

$$\|u\|_{C^{k,\alpha}(V)} \leq C\left(\|u\|_{C^0(U)} + \|f\|_{C^{k-2,\alpha}(U)}\right)$$

Definition 4.1 ($C^{k,\alpha}$ domains). *We say that ∂U is $C^{k,\alpha}$ if $\forall x \in \partial U$, $\exists r > 0$, s.t. (after possible rearranging the axes)*

$$U \cap B_r(x) = \{y \in B_r(x), y^d > \gamma(y^1, \cdots, y^{d-1}), \gamma \in C^{k,\alpha}\}$$

Theorem 4.3 (Schauder, boundary, divergence form). *Same as above, and ∂U is $C^{k,\alpha}$ and U is bounded. $Pu = f$ in U and $u|_{\partial U} = 0$. Then we have*

$$\|u\|_{C^{k,\alpha}(U)} \leq C\left(\|u\|_{C^0(U)} + \|f\|_{C^{k-2,\alpha}(U)}\right)$$

Theorem 4.4 (Schauder, boundary, non-divergence form). *Same as above, and ∂U is $C^{k,\alpha}$ and U is bounded. $Qu = f$ in U and $u|_{\partial U} = 0$. Then we have*

$$\|u\|_{C^{k,\alpha}(U)} \leq C\left(\|u\|_{C^0(U)} + \|f\|_{C^{k-2,\alpha}(U)}\right)$$

Overall strategy of the proofs:
for interior regularity

- The result in the constant coefficient case ($a^{jk} = const$).
- Method of freezing the coefficients: elliptic regularity is local + regularity of a^{jk}, then we can approximate the general problem by constant coefficient problems.

For boundary regularity: apart form the above two steps, we

- locally straighten the boundary to reduce to the case of half balls

How to do step 1 (constant coefficient case)?

- Method A: Littlewood-Paley theory
- Method B: compactness + contradiction

Theorem 4.5 (constant coefficient Schauder estimates).

$$Pu = -\partial_j(a_0^{jk}\partial_k u) = -a_0^{jk}\partial_j\partial_k u$$

a_0^{jk} is constant on $\mathbb{R}^d$. $a_0 \geq \lambda I$ ($\Lambda \geq \lambda > 0$), $|a_0^{jk}| \leq \Lambda$. For $u \in C_c^{k,\alpha}$, $f \in C^{k-2,\alpha}(\mathbb{R}^d)$, $k = 1, 2$ s.t. $Pu = f$, then

$$\|u\|_{C^{k,\alpha}(\mathbb{R}^d)} \leq C\left(\|f\|_{C^{k-2,\alpha}(\mathbb{R}^d)}\right)$$

Proof Using Littlewood-Paley theory.

Definition 4.2 (L-P projections).

$$\chi_{\leq 0}(\xi) = \begin{cases} 1, & on\ |\xi| \leq 1 \\ 0, & on\ |\xi| \leq 2 \\ & \geq 0, \quad \forall \xi \end{cases}$$

$$\chi_{\leq k}(\xi) = \chi_{\leq 0}(\xi/2^k)$$

$$\chi_k(\xi) = \chi_{\leq k+1}(\xi) - \chi_{\leq k}(\xi)$$

We have $\operatorname{supp}\chi_k \subset \{\xi, 2^k \leq |\xi| \leq 2^{k+2}\}$*. For $v \in \mathcal{J}'(\mathbb{R}^d)$, we define*

$$P_k v = \mathcal{F}^{-1}\left(\chi_k(\xi)\hat{v}\right)$$

$$P_{\leq k} v = \mathcal{F}^{-1}\left(\chi_{\leq k}(\xi)\hat{v}\right)$$

Then

$$v = P_{\leq k_0} v + \sum_{k > k_0} P_k v, \forall v \in \mathcal{J}'(\mathbb{R}^d)$$

Note $\xi \sim 2^k$ on ξ_k.

Lemma 4.1 (L-P characterization of $C^{0,\alpha}(\mathbb{R}^d)$). *For $v \in C^{0,\alpha}(\mathbb{R}^d)$, then*

$$[v]_{C^{0,\alpha}} = \sup_{x \neq y} \frac{|v(x) - v(y)|}{(x-y|^\alpha} \sim \sup_{k \in \mathbb{Z}} 2^{k\alpha}\|P_k v\|_{L^\infty}$$

Proof. both semi-norms are scale invariant

For ($\geq$), it suffices to show that for $k = 0$, $|P_0(v)| \leq\sim [v]_{C^{0,\alpha}}$.

$$P_0 v(x) = \int \hat{\chi}_0(x-y)v(y)\mathrm{d}y = \int \hat{\chi}_0(x-y)(v(y) - v(x))\mathrm{d}y \leq \int \hat{\chi}_0(x-y)|x-y|^\alpha [v]_{C^{0,\alpha}}\mathrm{d}y$$

where we use $\int \hat{\chi}_0(y)\mathrm{d}y = 0$ since $\chi_0(0) = 0$.

For ($\leq$),

$$v(x) - v(y) = P_{\leq k_0} v(x) - P_{\leq k_0} v(y) + \sum_{k > k_0} P_k v(x) - P_k v(y)$$

Choose k_0 so that $\sum_{k \geq k_0} \|P_k v\|_{L^\infty} \leq \sum_{k \geq k_0} 2^{-k\alpha}[v]_{C^{0,\alpha}} \sim L^\alpha [v]_{C^{0,\alpha}}$ where $L = |x-y|$.

Ands $|P_{\leq k_0} v(x) - P_{\leq k_0} v(y)| \leq \|\nabla P_{\leq k_0} v\|_{L^\infty} L \leq \sum_{k \leq k_0} \|\nabla P_k v\|_{L^\infty} L \leq\sim$ ... □

Now, $P(P_k u) = P_k f$, then $a^{jl}\xi_j\xi_l(\widehat{P_k u}) = \widehat{P_k f}$, then

$$\widehat{P_k u} = \frac{2^{2k}}{a^{jl}\xi_j\xi_l}\widehat{P_k f}\widetilde{\chi}_k \frac{1}{2^{2k}} = \frac{1}{2^{2k}}\eta)k(\xi)\widehat{P_k f}$$

where we let $\eta_k(\xi) = \frac{2^{2k}}{a^{jl}}\hat{\chi}_k$. Since $\|\hat{\eta}_k\|_{L^1} \leq C$ idependent of k, then $P_k u = 2^{-2k}\widehat{\eta_k} * P_k f$, and $\|P_k u\|_{L^\infty} \leq C 2^{-2k}\|P_k f\|_{LU^\infty}$ □

fgd

5 Maximum Principle

Maximum theory is a theory for solutions to elliptic PDE in term of their pointwise values, (inherently, scalar):

$$u : U \to \mathbb{R}$$

It is more convenient to consider the operators in non-divergence form:

$$Pu = -a^{jk}\partial_j\partial_k u + b^j\partial_j u + cu$$

where $a \geq \lambda I$, $\lambda > 0$, $a, b, c \in L^\infty$. Often, we will start with $c = 0$.

Maximum theory can be understood as the theory of convex functions:

Theorem 5.1. *For 1-d convex functions $u : I \to \mathbb{R}$, we have*

$$\max_I u = \max_{\partial I} u$$

To generalize 1-d convex functions to higher dimensions, one way is to define higher-dimensional convex functions. Another way, which is more generalized and will be introduced later, is to consider the $s$ubsolutions to elliptic PDE's.

Definition 5.1 (subsolution)**.** *For $u \in C^2(U)$, we say u is a (classical) subsolution if*

$$Pu \leq 0 \text{ in } U$$

When $d = 1$, if $P = -a\partial_x^2$, then $Pu \leq 0 \Leftrightarrow u'' \geq 0$

Theorem 5.2 (weak maximum principle)**.** *For U being open, bounded and connected subset of $\mathbb{R}^d$, and $u \in C^2(U) \cap C(\bar{U})$, $Pu \leq 0$, $c = 0$, then*

$$\max_{\bar{U}} u = \max_{\partial U} u$$

Proof. Step 1: for strict subsolutions $Pu < 0$, we will show that interior maximum is impossible.

Suppose, for contradiction, that $x_0 \in U$ is a (local) maximum, then

$$Du(x_0) = 0, \quad D^2(u_0) \leq 0$$

then

$$\begin{aligned} 0 > Pu(x_0) &= -a^{jk}\ \partial_j\partial_k u\big|_{x=x_0} + b^j\partial_j u\big|_{x=x_0} + cu, \quad \partial_j u\big|_{x=x_0} = 0, c = 0 \\ &= -a^{jk}\ \partial_j\partial_k u\big|_{x=x_0} = -\operatorname{tr}(aD^2u) \geq 0 \end{aligned}$$

Step 2: upgrade to subsolutions

$$u_\varepsilon = u + \varepsilon v$$

where $Pv < 0$, $v \in C^2(U) \cap C(\bar{U})$, then $u_\varepsilon \to u$ uniformly on $\bar{U}$, and

$$Pu_\varepsilon = Pu + \varepsilon Pv \leq \varepsilon Pv < 0$$

we can apply step 1.

But how to construct v? Let $v = \exp(\mu x^1)$, we have

$$-a^{jk}\partial_j\partial_k(\exp(\mu x^1)) = -\mu^2 a^{11}\exp(\mu x^1) \leq -\lambda\mu^2\exp(\mu x^1)$$

$$|b^j\partial_j(\exp(\mu x^1))| \leq \sup|b|\mu\exp(\mu x^1)$$

We only need a large enough μ. □

Theorem 5.3 (weak minimum principle)**.** *For U being open, bounded and connected subset of $\mathbb{R}^d$, and $u \in C^2(U) \cap C(\bar{U})$, $Pu \geq 0$, $c = 0$, then*

$$\min_{\bar{U}} u = \min_{\partial U} u$$

If u is a solution, i.e. u is both subsolution and supsolution, then under the same assumption and $Pu = 0$, we have

$$\max_{\bar{U}} |u| = \min_{\partial U} |u|$$

Now let us look at $c \geq 0$:

Theorem 5.4 (Weak maximum principle)**.** *For U being open, bounded and connected subset of $\mathbb{R}^d$, and $u \in C^2(U) \cap C(\bar{U})$, $c \geq 0$, then*

- $Pu \leq 0 \Rightarrow \max_{\bar{U}} u \leq \max_{\partial U} u^+$;
- $Pu \geq 0 \Rightarrow \min_{\bar{U}} u \leq \min_{\partial U}(-u^-)$.

where

$$u^+ = \begin{cases} u, & \text{if } u > 0 \\ 0, & \text{if } u \leq 0 \end{cases}, \quad u^- = \begin{cases} 0, & \text{if } u \geq 0 \\ -u, & \text{if } u < 0 \end{cases}$$

and as a result, $Pu = 0 \Rightarrow \max_{\bar{U}} |u| = \max_{\partial U} |u|$.

Proof for max part. Let $V = \{x \in U : u(x) > 0\}$, and

$$Qu = Pu - cu \leq -cu \leq 0 \quad \text{in } V$$

Using weak maximum principle for Q on V, we have $\max_{\bar{V}} u \leq \max_{\partial V} u$. □

Another way to think about this is the following comparison principle:

Theorem 5.5 (Comparison principle)**.** *For U being open, bounded and connected subset of $\mathbb{R}^d$, and $u, v \in C^2(U) \cap C(\bar{U})$, $c \geq 0$, and*

$$Pu \leq 0, Pv \geq 0 \quad \text{in } U, \quad u \leq v \text{ on } \partial U \Rightarrow u \leq v \text{ in } U$$

The proof is the application of the previous theorem to $u - v$.
Now we turn to the so-called strong maximum principle:

Theorem 5.6 (Strong maximum principle)**.** *For U being open, bounded and connected subset of $\mathbb{R}^d$, and $u \in C^2(U) \cap C(\bar{U})$, $c = 0$, and $Pu \leq 0$. If u has an max at $x_0 \in U$, i.e. $u(x_0) = \max_{\bar{U}} u$, then $u = \text{const}$ in U.*

Theorem 5.7 (Hopf's lemma)**.** *For U being open bounded and connected set $\subset \mathbb{R}^d$, $Pu \leq 0$ in U, and assume that there is $x_0 \in \partial U$, for which we have:*

- $\exists x_1 \in U$, $r_1 > 0$, s.t. $B_{r_1}(x_1) \subset U$ and $\overline{B_{r_1}(x_1)} \cap \partial U = \{x_0\}$.
- $u(x_0) \geq u(x)$ in $\overline{B_{r_1}(x_1)}$, $u(x_0) > u(x)$ in $B_{r_1}(x_1)$.

Then, we have

$$\left.\frac{\partial u}{\partial \nu}\right|_{x=x_0} > 0, \text{ where } \nu \text{ is with respect to } B_{r_1}(x_1)$$

provided that $c = 0$ or $c \geq 0, u(x^0) \geq 0$.

Proof. W.L.O.G, $x_1 = 0$. Let $v = \exp(-\mu|x|^2) - \exp(-\mu r_1^2)$. It is easy to verify that $Pv \leq 0$ in $B_{r_1} - B_{r_1/2}$ with μ large. For u we have $Pu \leq 0$. Let $w = u(x_0) - \varepsilon v$, then $Pw \geq Pv \geq 0$.

Consider $V = B_{r_1} - B_{r_1/2}$, $\partial V = \partial B_{r_1} \cup \partial B_{r_1/2}$. On ∂B_{r_1}, $w = u(x_0) \geq u$. On $\partial B_{r_1/2}$, $w = u(x_0) - \varepsilon v \geq u$ for ε small enough, i.e. we have $w \geq u$ on ∂V. Through comparison principle, we have $w \geq u$ on V. Then

$$\left.\frac{\partial u}{\partial \nu}\right|_{x=x_0} \geq \left.\frac{\partial w}{\partial \nu}\right|_{x=x_0} > 0$$

□

Proof of strong maximum principle. Let $V = \{x \in U : u(x) < M\}$, $M = \sup_{\bar{U}} u$, $x_0 \in U$, $u(x_0) = M$, $\Rightarrow V \subsetneq U$. Assume for contradiction that $V \neq \emptyset$. Find x_1 near ∂V and biggest r s.t, $B_r(x_1) \subset V$. Let x_0' be the point that is in $B_r(x_1) \cap \partial V$. Then the Hopf's lemma is applicable, therefore we have $\frac{\partial}{\partial \nu} u\big|_{x=x_0'} \neq 0$. But this contradicts $u(x_0') = M \Rightarrow Du = 0$. □

Similarly, we can prove the following strong maximum principle with $c \geq 0$:

Theorem 5.8 (Strong maximum principle with $c \geq 0$)**.** *For U being open, bounded and connected subset of $\mathbb{R}^d$, and $u \in C^2(U) \cap C(\bar{U})$, $c \geq 0$, then we have*

- *if $Pu \leq 0$. and u has a nonnegative maximum at $x_0 \in U$, then $u = \text{const}$ in U.*
- *if $Pu \geq 0$. and u has a nonpositive minimum at $x_0 \in U$, then $u = \text{const}$ in U.*

6 General Boundary Value Problems for Elliptic PDE

We still consider

$$Pu = -\partial_j(a^{jk}\partial_k u) + b^j\partial_j u + cu$$

$$(BVP)\begin{cases} Pu = f \text{ in } U \\ Bu|_{\partial U} = g \text{ on } \partial U \end{cases}$$

So far we focused on Dirichlet B.C.:

$$\begin{cases} Pu = f \text{ in } U \\ \quad u|_{\partial U} = g \text{ on } \partial U \end{cases}$$

By introducing an extension $\widetilde{g}$ of g to U, we could set, without loss of generality that $g = 0$, and we deal with

$$\begin{cases} Pu = f \text{ in } U \\ \quad u|_{\partial U} = 0 \text{ on } \partial U \end{cases}$$

Our goal now is to generalize elliptic theory to other boundary conditions.

Unlike ODE's or wave equation or Cauchy-Kovalevskaya, for $-\Delta u = f$, we do not prescribe the full u, $\frac{\partial}{\partial\nu}u$ on ∂U.

In order to justify this rigorously, we can use uniqueness via energy method: recall the weak formalism of Dirichlet boundary conditions:

$$u \in H^1(U), \begin{cases} Pu = f \text{ in } U \\ \quad u|_{\partial U} = 0 \text{ on } \partial U \end{cases} \Leftrightarrow \forall\varphi \in H_0^1(U), \int_U a^{jk}\partial_j u\partial_k\varphi + b^j\partial_j u\varphi + cu\mathrm{d}x = \int_U f\varphi\mathrm{d}x$$

Let us look at the following Neumann B.C, with $b, c = 0$ (for now):

$$\begin{cases} Pu = f \text{ in } U \\ a^{jk}\nu_k\partial_j u|_{\partial U} = g \text{ on } \partial U \end{cases} \rightarrow \begin{cases} Pu = f \text{ in } U \\ a^{jk}\nu_k\partial_j u|_{\partial U} = 0 \text{ on } \partial U \end{cases}$$

Note that $a^{jk}\nu_k\partial_j u|_{\partial U}$ is the natural boundary condition if we consider a^{jk} as a Riemannian metric, and it will appear naturally in later calculations. Let us look at its weak formulation:

$$\int -\partial_j(a^{jk}\partial_k u)\varphi = \int_U f\varphi\mathrm{d}x$$

$$= \int_U a^{jk}\partial_j u\partial_k\varphi\mathrm{d}x - \int \nu_j a^{jk}\partial_k u\varphi\mathrm{d}A = \int_U a^{jk}\partial_j u\partial_k\varphi\mathrm{d}x$$

Therefore we define the weak formulation of Neumann Boundary value problem to be

$$u \in H^1, \begin{cases} Pu = f \text{ in } U \\ a^{jk}\nu_k\partial_j u|_{\partial U} = 0 \text{ on } \partial U \end{cases} \Leftrightarrow \forall\varphi \in H^1(U), \int_U a^{jk}\partial_j u\partial_k\varphi = \int f\varphi\mathrm{d}x$$

Now we are ready to state the theorem:

Theorem 6.1 (Neumann Boundary Value Problem)**.** *Suppose ∂U is in C^1, $a \geq \lambda I$ in U and $a \in L^\infty$, $(b, c \in L^\infty)$, we have*

1. *For any $\mu \in \mathbb{R}$, the following problem*
$$\begin{cases} Pu - \mu u = f \text{ in } U \\ a^{jk}\nu_k \partial_j u|_{\partial U} = 0 \text{ on } \partial U \end{cases}$$
is Fredholm with index μ, $H^1(U) \to H^{-1}(U)$, i.e.
either
$$\forall f \in L^2(U), \exists \text{ unique } u \in H^1 \text{ is solution to Neumann problem with index } \mu$$
or
$$\exists v \neq 0 \text{ is solution to Neumann problem with index } \mu \text{ with } f = 0$$
Further more, for $\mu >> 1$, the former alternative appears.

2. *If ∂U is C^k, and $a, b, c \in C^k$, then*
$$\|u\|_{H^{l+k}(U)} \leq C(\|f\|_{H^{k-1}(U)} + \|u\|_{H^k})$$

An example is that $P = -\Delta$. $-\Delta u = 0, u|_{\partial U} = 0$ has a nontrivial solution $v = const \neq 0$. Solvability for $f \perp \text{Ker} \Leftrightarrow \int f \mathrm{d}x = 0$.

We can have a similar definition of Robin boundary value problems.

Another example: oblique D.B.C, $b = c = 0$:
$$\begin{cases} Pu = f \text{ in } U \\ X^j \partial_j u = 0 \text{ on } \partial U \end{cases}$$
X is transversal to ∂U, outward. Actually,
$$X = X^\perp + X^T, X^\perp // a^{jk}\nu_j e_k$$
and then we have
$$\int_U a^{jk}\partial_j u \partial_k \varphi \mathrm{d}x + \int_{\partial U} X^T u \varphi \mathrm{d}A = \int_U f\varphi \mathrm{d}x$$

Other than energy method, we have another approach that is more powerful. The reference is Taylor's PDE book, Volume I, 5.11).

Consider the microlocal formulation as followed.
$$-\Delta_{z,x} u = 0 \text{ in } \mathbb{R}^d_+$$
Take FFT in x, we have
$$(-\partial_z^2 + |\xi|^2)\hat{u} = 0, \quad \hat{u}|_{z=0} = g, \quad \partial_z \hat{u}|_{z=0} = h$$
then
$$\hat{u}(z, \xi) = a_+(\xi)\mathrm{e}^{+|\xi|z} + a_-(\xi)\mathrm{e}^{-|\xi|z}$$
In order to have boundary regularity, we require $a_+(\xi)$ to be zero. This explains why we are only left with half of the full freedom to choose $\hat{g}$ and $\hat{h}$.

The idea is to use freezing coefficients, formulate the notion of regular elliptic BVP, for which we have elliptic regularity and Fredholm property, based on the constant coefficient computation.

We adopt the following assumptions: $a, b, c \in C^\infty(\bar{U})$ and ∂U is C^∞.

Definition 6.1. *For $k \geq 1$, $k \in \mathbb{Z}$,*
$$H^{k-\frac{1}{2}}(U) = \{g = v|_{\partial U} : v \in H^k(U)\}$$
and $\|g\|_{H^{k-\frac{1}{2}}} = \inf_{u|_{\partial u} = g} \|u\|_{H^k}$

$$\begin{cases} Pu = f \text{ in } U \\ Bu|_{\partial U} = g \text{ on } \partial U \end{cases}$$

$P : C^\infty(U) \to C^\infty(U)$, $B()|_{\partial U} : C^\infty(U) \to C^\infty(\partial U)$. Given $x_0 \in \partial U$, $\exists$ a boundary straightening map near x_0, in these variables ,we have:

$$P = -\partial_z^2 + P_1(y, z, Dy)\partial_z + P_0(y, z, Dy, Dy^2)$$

$$B = b\partial_z + B_0(y, z, \partial_y)$$

$$P_{x_0} = -\partial_z^2 + P_1(0, 0, Dy)\partial_z + P_0(0, 0, Dy, Dy^2)$$

$$B_{x_0} = b(0, 0)\partial_z + B_0(0, 0, \partial_y)$$

Definition 6.2. *The previous BVP is a regular elliptic BVP if* $\forall x_0 \in \partial U, \forall \zeta, \forall \xi \in \mathbb{R}^{d-1}$ $\exists$ *unique bounded solution to the ODE*

$$P_{x_0}\hat{u}(z, \xi) = 0, \quad B_{x_0}\hat{u}(z, \xi) = \zeta$$

which is an ODE in z*. This is called the Laputinsky-Shapivo condition.*

Theorem 6.2. *For a regular elliptic BVP,* $H^{k+2}(U) \to u \to (Pu, Bu) \leftarrow (H^k(U), H^{k-1/2-order(B)}(\partial U))$ *is Fredholm and we have elliptic (boundary) regularity:*

$$\|u\|_{H^{k+2}(U)} \leq C\|f\|_{H^k(U)} + \|u\|_{H^{k+1}} + \|Bu\|_{H^{k-1/2-order(B)}(\partial U)}$$

7 Unique continuation

Theorem 7.1 (Aronszajn)**.** *Let* U *be open and connected.* $Pu = -\partial_j(a^{jk}\partial_k u) + b^j\partial_j u + cu$, $a^{jk}, b^j, c \in C^\infty(U)$, $a \geq \lambda I$ *in* U*. If* $Pu = 0$ *in* U*,* $u \in H^1(U)$ *and* $u = 0$ *in a nonempty open subset* $W \subset U$*, we have* $u = 0$ *in* U*.*

To prove this, we need the Carleman estimate. The reference is Carleman inequalities by Lerner Fragnelli [2021].

References

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