FYSB21 - Hand-in 1

Hervé Schmit-Veiler

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i)

When entering a highway, cars will typically sustain multiple seconds of acceleration in order to reach the required speed (110-130 km/h). A Volkswagen Polo can go from 0-100 km/hr in just under 15 s. Assuming constant acceleration a, this means :

$$a \approx 1.85\,\mathrm{m\cdot s^{-2}}$$

If a car enters the highway at an initial speed of 30 km/h, the time T it would take to accelerate to a speed of 120 km/h is:

$$T = \frac{90}{3.6 \cdot 1.85} \approx 14\,\mathrm{s}$$

ii)

Consider a pendulum attached to the car, this pendulum is composed of a ball of mass m attached at the end of a perfectly rigid rod of negligeable mass and constant length l. In polar coordinates, we express the acceleration of the ball within the car frame of reference $\boldsymbol{a_{R'}}$:

$$\boldsymbol{a_{R'}} = l\ddot{\phi}\hat{\boldsymbol{\phi}} - l\dot{\phi}^2\hat{\boldsymbol{r}} \tag{1}$$

Where $\hat{\boldsymbol{\phi}}$ and $\hat{\boldsymbol{r}}$ are the tangential and radial unit vectors respectively. This ball is subject to it's own weight $\boldsymbol{F_g}$, a drag force $\boldsymbol{f}$ as well as tension from the rod $\boldsymbol{T}$. As the accelerating car consists of a non-inertial frame, we have to consider an addition translational ficticious force $\boldsymbol{F_p}$ well writing Newton's second law:

$$m\boldsymbol{a_{R'}} = \boldsymbol{F_g} + \boldsymbol{f} + \boldsymbol{T} + \boldsymbol{F_p} \tag{2}$$

Where:

$$\boldsymbol{F_g} = -mg\hat{\boldsymbol{y}} = mg\cos(\phi)\hat{\boldsymbol{r}} - mg\sin(\phi)\hat{\boldsymbol{\phi}} \tag{3}$$

$$\boldsymbol{f} = -ml\gamma\dot{\phi}\hat{\boldsymbol{\phi}} \tag{4}$$

Assuming the car undergoes uniform acceleration a along the direction $\hat{\boldsymbol{x}}$:

$$\boldsymbol{F_p} = -ma\hat{\boldsymbol{x}} = -ma\sin(\phi)\hat{\boldsymbol{r}} - ma\cos(\phi)\hat{\boldsymbol{\phi}} \tag{5}$$

Considering the rod is perfectly rigid, there should be no net acceleration in the radial direction. Hence the tension $\boldsymbol{T}$ provided by the rod should counteract the radial components of the other two forces. In other words, there is no radial degree of freedom, so projecting (2) along $\hat{\boldsymbol{\phi}}$ we get:

$$ml\ddot{\phi} = -mg\sin(\phi) - ml\gamma\dot{\phi} - ma\cos(\phi) \tag{6}$$

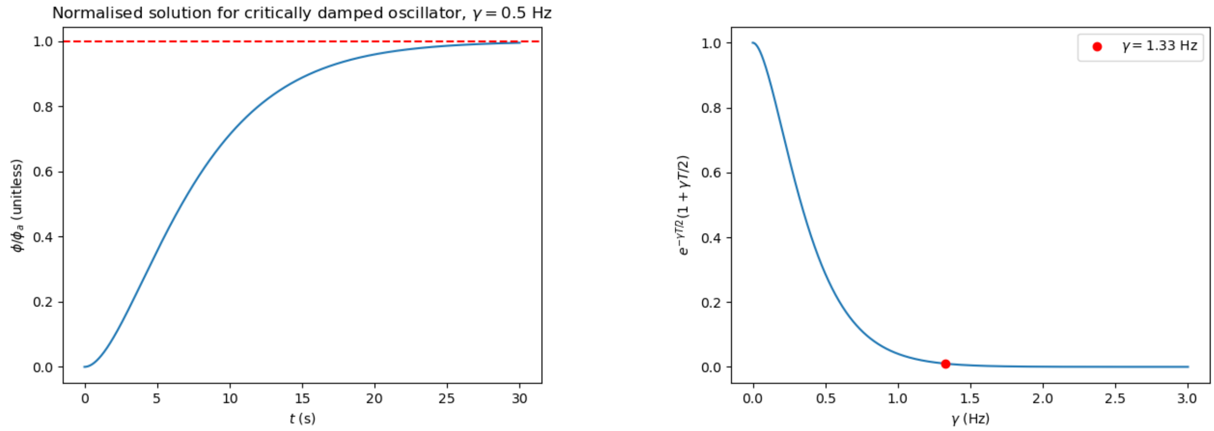
We find the equilibrium angle ϕ_a such that $\dot{\phi}_a = \ddot{\phi}_a = 0$:

$$\phi_a = -\arctan\left(\frac{a}{g}\right) \sim -\frac{a}{g} \tag{7}$$

As $\tan(\phi) \sim \phi$ for small ϕ. This then means that imposes a limit on a, beyond which the small angle approximation will no longer hold.

If we take the threshold for the small angle approximation to be $\phi_{max} = 10\,^\circ \approx 0.175\,\mathrm{rad}$, then this gives us an upper bound for the allowed acceleration of the car while staying within the small angle approximation:

$$a_{max} = \frac{\pi\phi_{max}g}{180} \approx 1.71\,\mathrm{m\cdot s^{-2}} \tag{8}$$



(a) **(b)**

Figure 1: (a) Normalised solution to critically damped oscillator where $\gamma = 0.5$ and initial conditions $\phi(0) = \dot{\phi}(0) = 0$. **(b)** Numerical determination for the minimum damping coefficient γ where $\phi(T) = 1.01\phi_a$ for $T = 10$ s.

Which coincidently, is just under the estimated acceleration of our Volkswagen Polo $a \approx 1.85\,\mathrm{m \cdot s^{-2}}$, we are therefore operating just slightly over the boundarily of applicability of the small angle approximation. For simplicity (and out of laziness), we will continue using the values of T and a found in section i), but it is useful to remember that the small angle approximation would be more accurate for a vehicle with lower acceleration (e.g a tractor).

iii)

After applying the small angle approximation, we get the following linear differential equation:

$$\ddot{\phi} + \gamma\dot{\phi} + \omega_0^2\phi = \frac{a}{l} \qquad \text{Where } \omega_0^2 = \frac{g}{l} \tag{9}$$

This is the damped harmonic oscillator with a constant forcing term. The particular solution to this equation is actually our equilibrium angle ϕ_a. We know that the homogenous equation admits three classes of solution depending on the value of γ w.r.t $2\omega_0$. Out of these three classes of solutions, the critically damped solution where $\gamma = 2\omega_0$ decays the fastest. Since our objective is to measure ϕ_a in order to extract the acceleration a of the car, this is the case we will consider.

Note that this will require controlling the value of γ, we should therefore immerse the pendulum in a medium more viscous than air, such as oil.

The general solution of the critically dampled harmonic oscillator is as follows:

$$\phi(t) = Ae^{-\frac{\gamma}{2}t} + Bte^{-\frac{\gamma}{2}t} + \phi_a \tag{10}$$

If we impose the following initial conditions:

$$\phi(0) = \phi_0 \qquad \dot{\phi}(0) = 0 \tag{11}$$

Then:

$$\phi(t) = (\phi_0 - \phi_a)(1 + \frac{\gamma}{2}t)e^{-\frac{\gamma}{2}t} + \phi_a \tag{12}$$

Note that because the growth of the exponential function e^{-at} dominates over t for large t, $\phi(t)$ does tend towards ϕ_a as t tends towards infinity. A plot of this is shown in fig. 1a.

We seek a value for γ (and therefore also l) such that $\phi(T)$ is within 1% of ϕ_a. This means:

$$|(\phi_0 - \phi_a)(1 + \gamma T/2)e^{-\frac{\gamma}{2}T}| \leq 0.01\phi_a \tag{13}$$

The initial conditions don't actually interest us very much, we set $\phi_0 = 0$ to simplify the inequality for ourselves. Then, we search for γ which satisfies:

$$e^{-\frac{\gamma}{2}T}(1 + \gamma T/2) = 0.01 \tag{14}$$

This is actually not such an easy equation to resolve analytically. We therefore did so analytically, the results are plotted in fig. 1b and it was found that the minimum value was approximately $\gamma = 1.33\,\mathrm{Hz}$.

Assuming that the pendulum is immersed oil of a known viscosity, we wish to convert this constraint on γ into a constraint on the radius of the steel ball attached to the end of the pendulum. First we equate the viscous drag coefficient for a sphere of radius r $6\pi\eta r$ to $ml\gamma$, which nets us:

$$\gamma = \frac{6\pi\eta r}{ml} \tag{15}$$

Where η is the viscosity of the oil. We know the expression mass of a homogenous steel ball of radius r and density ρ:

$$m = \rho\frac{4}{3}\pi r^3 \tag{16}$$

Substituting into 15 and making use of the critical damping condition $\gamma = 2\sqrt{g/l}$, we get:

$$r = \frac{3}{2\sqrt{2}}\sqrt{\frac{\eta\gamma}{\rho g}} \tag{17}$$

The viscosity of sesame oil at 26 °C is roughly $0.05\,\mathrm{Pa\cdot s}$ (the car is going to smell quite interesting) and the density of iron is roughly $7850\,\mathrm{kg\cdot m^{-3}}$. Considering the minimum value of $\gamma = 1.33\,\mathrm{Hz}$ found earlier, we can find the minimum radius:

$$r_{min} \approx 0.99\,\mathrm{mm}$$

iv)

Setting $\gamma = 1.33\,\mathrm{Hz}$ actually imposes a pendulum length of $l = 22.2\,\mathrm{m}$, which is obviously not going to fit inside or outside of a human-sized car. Thankfully, our work has not been wasted as this is the maximum pendulum length given the critical damping condition:

$$l = \frac{4g}{\gamma^2} \tag{18}$$

If we impose a much more reasonable length of 10 cm, then we find that the required damping coefficient is $\gamma = 19.8\,\mathrm{Hz}$. This imposes a ball radius of $r \approx 3.8\,\mathrm{mm}$, which is very reasonable! Additionally, with this larger damping factor the pendulum converges to its equilibrium position much faster too (the decay towars equilibrium is exponential).

2

i)

$s(t) = s_0\sin(\omega_0 t)$ when $t \in [0, T]$ and $s(t) = 0$ otherwise. We can compute the Fourier transform $s(\omega)$

$$\begin{aligned} s(\omega) &= \int_{-\infty}^{\infty} s(t)e^{i\omega t}dt \\ &= \frac{s_0}{2i}\int_0^T \left(e^{i(\omega+\omega_0)t} - e^{i(\omega-\omega_0)t}\right)dt \\ &= \frac{s_0}{\omega^2-\omega_0^2}\left[-\omega_0 + e^{i\omega T}\left(\omega_0\cos(\omega_0 T) - i\omega\sin(\omega_0 T)\right)\right] \end{aligned} \tag{19}$$

Where $\omega = \frac{2\pi}{T}$.

ii)

Plotting $|s(\omega)|^2$ against ω shows that as expected, this function posseses a peak around in the neighbourhood of $\omega = \omega_0$ (fig. 2).

We recall the approximate inequality relating the temporal dispersion T of a signal to its spectral dispersion Ω:

$$\Omega \cdot T \geq \frac{\pi}{2}\,\mathrm{rad} \tag{20}$$

We associate T to the period during which the note is played, and Ω to the FWHM of $|s(\omega)|^2$. For $T = 1\,\mathrm{s}$ we see that $\Omega = 5.56\,\mathrm{rad}$ indeed verifies inequality (20).

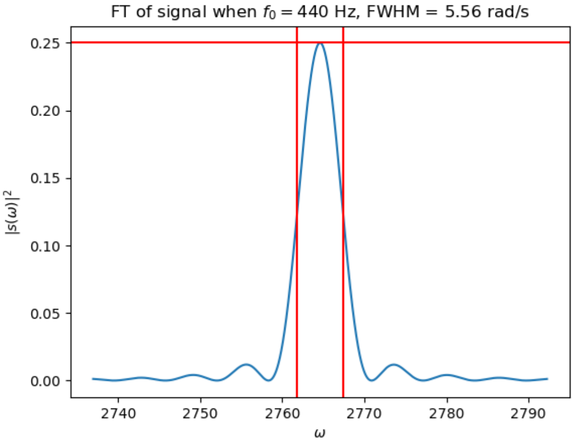
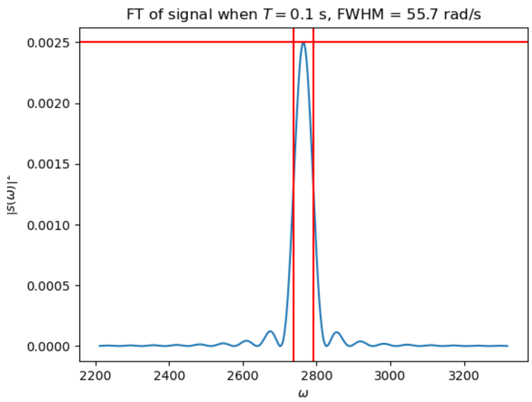
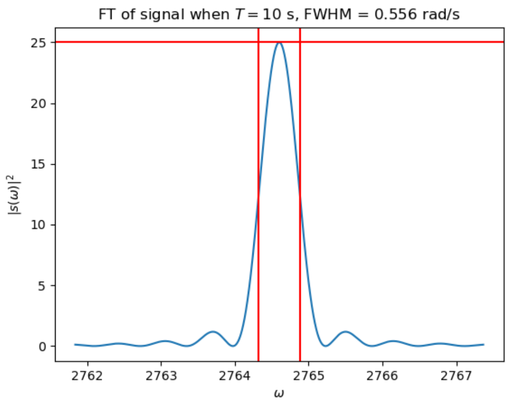


Figure 2: Plot of $|s(\omega)|^2$ for $f_0 = 440\,\text{Hz}$ and $T = 1\,\text{s}$



(a)



(b)

Figure 3: Plots of $|s(\omega)|^2$ for $T = 0.1\,\text{s}$ **(a)** and for $T = 10\,\text{s}$**(b)**

iii)

We expect Ω to change in the most interesting manner when T changes, as per inequality (20). Same as before we plot $|s(\omega_0)|^2$ this time for different T and constant frequency $f_0 = \omega_0/2\pi = 440\,\text{Hz}$. Cases $T = 0.1$ s and $T = 10$ s are included in fig. 3.

After plotting for various T in the $0.1 - 10\,\text{s}$ range, we plot the different Ω values obtained against the corresponding T (4). From (20), we expect the relationship between Ω and T to be a hyperbola, and the fit that is obtained with the data confirms this.

In order to satisfy the uncertainty relation, Ω must decrease as T increases and vice-versa.

iv)

I am not using the exact uncertainty relation as T and Ω aren't exactly standard deviations. Since the approximate dispersion relation 20 gives a larger lower limit, it is a safer assumption I believe.

In order to resolve a note of frequency f_0, then its spectral dispersion Ω needs to be smaller than the closest semi-tone. Since neighbouring semi-tones are multiplied by a $\sqrt[12]{2}$ factor, we can find the lower limit for T, under the condition that $\Omega \leq (\sqrt[12]{2} - 1)\omega_0$.

$$T_{min} = \frac{\pi}{2\omega_0(\sqrt[12]{2} - 1)} \tag{21}$$

Which is once again the equation of a hyperbola. Since the bass plays at at a lower frequency than a violin, the minimum period that must be played by a bass is longer than that of the violin - which explains why the violin

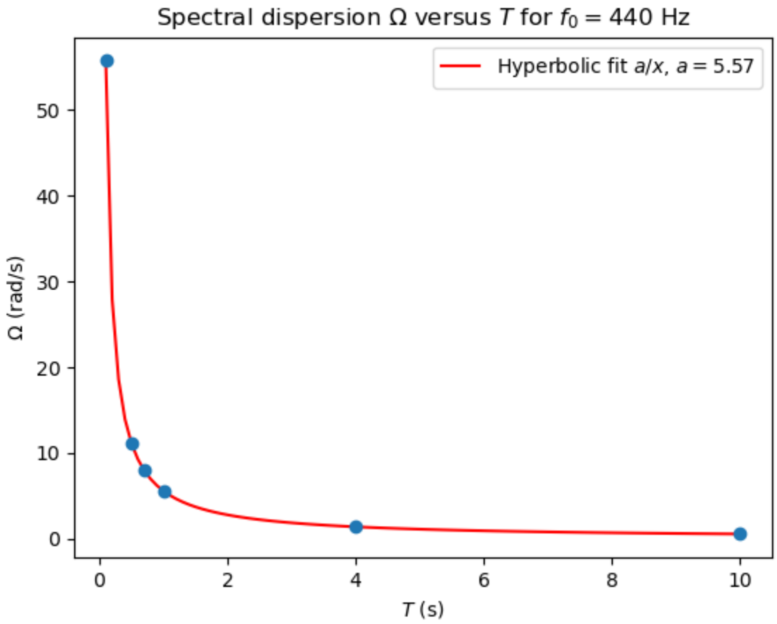


Figure 4: Scatter plot of the different FWHM values against period T of the signal. Best fit curve is a hyperbola.

typically plays at a much faster tempo. In fact, if we compare the notes A1 (played on bass) to A4 (played on violin), the bass's minimum period is over 7 times higher (plot in fig.).

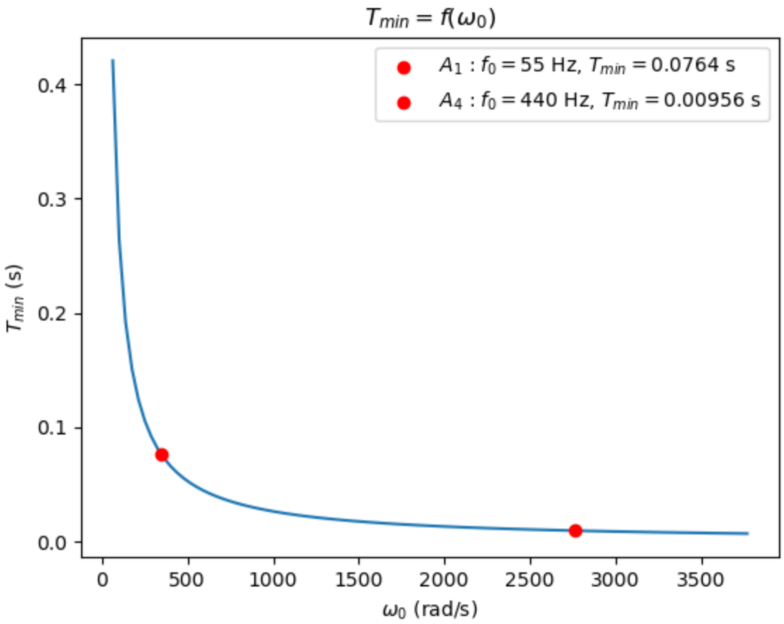


Figure 5: Plot of the minimum period T_{min} required in order to still resolve the nearest semi-tone, as a function of the angular frequency of the sound ω_0.

3

i)

We define the Fourier expansion of $p(t)$:

$$p(t) = \sum_{n=-\infty}^{\infty} a_n e^{-in\omega t} \qquad \text{where } \omega = \frac{2\pi}{T} \tag{22}$$

Since $p(t)$ is basically a cut-off sine function of angular frequency $\omega_0 = \frac{1}{\tau}$, we should expect that the most important coefficient would be a_n for n such that $\frac{2\pi}{T}n \approx \frac{1}{\tau}$.

ii)

$$\begin{aligned} a_n &= \frac{1}{T}\int_0^T p(t)e^{n\omega t}dt \\ &= \frac{1}{T}\int_0^{2\pi\tau} \sin(t/\tau)e^{n\omega t}dt \\ &= \frac{-1}{2T}\left[\frac{(n\omega - \omega_0)e^{in2\pi\frac{\omega}{\omega_0}} - (n\omega + \omega_0)e^{-in2\pi\frac{\omega}{\omega_0}} + 2\omega_0}{(n\omega)^2 - \omega^2}\right] \end{aligned} \tag{23}$$

Where $\omega_0 = \frac{1}{\tau}$.