Assignment for the course
The statistical analysis of time series

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1 Theoretical aspects

[1.]

[1.1] Compute the autocovariance function $\gamma_X(k) = \text{cov}(X_t, X_{t+k})$ for $k = 0, 1, \ldots$ Is χ stationary? Do we need a restriction on θ?

$$
\begin{aligned}
\gamma_X(k) &= \text{cov}(X_t, X_{t+k}) \\
&= \text{E}((W_t + \frac{1}{\theta}W_{t-1})(W_{t+k} + \frac{1}{\theta}W_{t+k-1})) - \text{E}(W_t + \frac{1}{\theta}W_{t-1})\text{E}(W_{t+k} + \frac{1}{\theta}W_{t+k-1}) \\
&= \text{E}((W_t + \frac{1}{\theta}W_{t-1})(W_{t+k} + \frac{1}{\theta}W_{t+k-1})) \\
&= \text{E}(W_t W_{t+k} + \frac{1}{\theta}W_t W_{t+k-1} + \frac{1}{\theta}W_{t-1}W_{t+k} + \frac{1}{\theta^2}W_{t-1}W_{t+k-1}) \\
&= \begin{cases} \sigma_W^2(1 + \frac{1}{\theta^2}) & h = 0 \\ \frac{\sigma_W^2}{\theta} & |h| = 1 \\ 0 & \text{else} \end{cases}
\end{aligned}
$$

$\Rightarrow X_t$ is weakly stationary.
Apart from $\theta \neq 0$, to avoid $1 + |\frac{1}{\theta}| \to \infty$, no conditions for $\frac{1}{\theta}$ necessary to get stationarity on MA(q) processes.

[1.2] Use the definition of Fourier transform of γ_X and derive the expression of the spectral density, $f_X(\lambda)$, $\lambda \in [-1/2, 1/2]$.

$$
\begin{aligned}
f_X(\lambda) &= \sum_{k=-\infty}^{\infty} \gamma_X(k)\exp(-2\pi i\lambda k) \\
&= \sum_{k=-2}^{-\infty} \gamma_X(k)\exp(-2\pi i\lambda k) + \gamma_X(-1)\exp(-2\pi i\lambda \cdot (-1)) \\
&+ \gamma_X(0)\exp(-2\pi i\lambda \cdot 0) + \gamma_X(1)\exp(-2\pi i\lambda \cdot 1) + \sum_{k=2}^{\infty} \gamma_X(k)\exp(-2\pi i\lambda k) \\
&= \sigma_W^2 \left(\frac{\theta(\exp(2\pi i\lambda) + \exp(-2\pi i\lambda)) + \theta^2 + 1}{\theta^2} \right) \\
&= \sigma_W^2 \left(\frac{\theta(\cos(2\pi\lambda) + i\sin(2\pi\lambda) + \cos(2\pi\lambda) - i\sin(2\pi\lambda)) + \theta^2 + 1}{\theta^2} \right) \\
&= \frac{\sigma_W^2}{\theta^2}(2\theta\cos(2\pi\lambda) + \theta^2 + 1)
\end{aligned}
$$

[2.]

[2.1] Compute the autocovariance function $\lambda_Y(k) = \text{cov}(Y_t, Y_{t+k})$ for

$k = 0, 1, \ldots$

$$
\begin{aligned}
\gamma_Y(k) &= \text{cov}(Y_t, Y_{t+k}) \\
&= \text{E}((\frac{1}{3}W_{t-1} + \frac{1}{3}W_t + \frac{1}{3}W_{t+1})((\frac{1}{3}W_{t+k-1} + \frac{1}{3}W_{t+k} + \frac{1}{3}W_{t+k+1}))) \\
&= \begin{cases} \frac{\sigma_W^2}{3} & h = 0 \\ \frac{2\sigma_W^2}{9} & |h| = 1 \\ \frac{\sigma_W^2}{9} & |h| = 2 \\ 0 & \text{else} \end{cases}
\end{aligned}
$$

[2.2] Use the definition of Fourier transform λ_Y and derive the expression of the spectral density $f_Y(\lambda)$, $\lambda \in [-1/2, 1/2]$

$$
\begin{aligned}
f_Y(\lambda) &= \sum_{k=-\infty}^{\infty} \gamma_Y(k)\exp(-2\pi i\lambda k) \\
&= \gamma_Y(-2)\exp(4\pi i\lambda) + \gamma_Y(-1)\exp(2\pi i\lambda) + \gamma_Y(0) \\
&= +\gamma_Y(1)\exp(\pi i\lambda) + \gamma_Y(2)\exp(-4\pi i\lambda) \\
&= \sigma_W^2 \left(\frac{\exp(4\pi i\lambda) + \exp(-4\pi i\lambda) + 2\exp(2\pi i\lambda) + 2\exp(-2\pi i\lambda) + 3}{9} \right) \\
&= \sigma_W^2 \left(\frac{2\cos(4\pi\lambda) + 4\cos(2\pi\lambda) + 3}{9} \right)
\end{aligned}
$$

[3.]

[3.1] Derive the expression for joint autocovariance function $\lambda_{WY}(k) = \text{cov}(W_{t+k}, Y_t)$.

$$
\begin{aligned}
\gamma_{WY}(k) &= \text{cov}(W_{t+k}, Y_t) \\
&= \text{E}(W_{t+k}(\frac{1}{3}(W_{t-1} + W_t + W_{t+1}))) \\
&= \frac{1}{3}\text{E}(W_{t+k}W_{t-1} + W_{t+k}W_t + W_{t+k}W_{t+1}) \\
&= \begin{cases} \frac{\sigma_W^2}{3} & h = 0 \text{ or } |h| = 1 \\ 0 & \text{else} \end{cases}
\end{aligned}
$$

[3.2] Derive the expression of the cross-spectrum

$$
\begin{aligned}
f_{WY}(\lambda) &= \sum_{k=-\infty}^{\infty} \gamma_{WY}(k)\exp(-2\pi i\lambda k) \\
&= \gamma_{WY}(-1)\exp(2\pi i\lambda) + \gamma_{WY}(0) + \gamma_{WY}(1)\exp(\pi i\lambda) \\
&= \sigma_W^2 \left(\frac{\exp(2\pi i\lambda) + \exp(-2\pi i\lambda) + 1}{3} \right) \\
&= \sigma_W^2 \left(\frac{\cos(2\pi\lambda) + i\sin(2\pi\lambda) + \cos(-2\pi\lambda) + i\sin(-2\pi\lambda) + 1}{3} \right) \\
&= \sigma_W^2 \left(\frac{2\cos(2\pi\lambda) + 1}{3} \right)
\end{aligned}
$$

[3.3] Let f_W be the spectral density of the white noise process. Making use of the result in point 3.2, rearrange the terms and express $f_{WY}(\lambda)$ and provide the form of the term $|A_{WY}(\lambda)|^2$

$$
\begin{aligned}
f_W(\lambda) &= \sum_{k=-\infty}^{\infty} \gamma_W(k)\exp(-2\pi i\lambda k) = \gamma_W(0) = \sigma_W^2 \\
f_{WY}(\lambda) &= \sigma_W^2 \left(\sum_{k=-1}^{1} (\frac{1}{\sqrt{3}}\exp(2\pi i\lambda))^2 \right) \\
&= \sigma_W^2 |A_{WY}(\lambda)|^2 \\
|A_{WY}(\lambda)|^2 &= \left(\sum_{k=-1}^{1} (\frac{1}{\sqrt{3}}\exp(2\pi i\lambda))^2 \right)
\end{aligned}
$$

[4.] Prove that $f_S(\lambda) = |A_{SQ}(\lambda)|^2 f_Q(\lambda)$ with $A_{SQ}(\lambda) = \sum_{j=-\infty}^{\infty} a_j \exp(-2\pi i\lambda j)$

[4.1] Show that

$$
\begin{aligned}
\gamma_S(k) &= \mathrm{cov}(S_{t+k}, S_t) \\
&= \mathrm{E}(\sum_{j=-\infty}^{\infty} a_j Q_{t+k-j} \cdot \sum_{j=-\infty}^{\infty} a_j Q_{t-j}) \\
&= \mathrm{E}\left((...a_{-1}Q_{t+k+1} + a_0 Q_{t+k} + a_1 Q_{t+k-1}...) \right. \\
&\quad \left. \cdot\, (...a_{-1}Q_{t+1} + a_0 Q_t + a_1 Q_{t-1}...) \right) \\
&= ...\mathrm{E}(a_{-1}Q_{t+k+1}a_{-1}Q_{t+1}) + \mathrm{E}(a_{-1}Q_{t+k+1}a_0 Q_t) \\
&\quad + \mathrm{E}(a_{-1}Q_{t+k+1}a_1 Q_{t-1})... \\
&= \mathrm{E}(\sum_r \sum_s a_s Q_{t+k-r} a_r Q_{t-s}) = \mathrm{E}(\sum_r \sum_s a_r a_s Q_{t+k-r} Q_{t-s}) \\
&= \sum_r \sum_s a_r a_s \mathrm{E}(Q_{t+k-r} Q_{t-s}) = \sum_r \sum_s a_r a_s \gamma_Q(k - r + s) \\
&= \sum_r \sum_s a_r a_s \left(\int_{-1/2}^{1/2} \exp(2\pi i\lambda(k - r + s)) f_Q(\lambda) \mathrm{d}\lambda \right)
\end{aligned}
$$

[4.2] Prove that

$$
\begin{aligned}
|A_S Q(\lambda)|^2 &= |\sum_{j=-\infty}^{\infty} a_j \exp(-2\pi i \lambda j)|^2 \\
&= \sum_r \sum_s a_r a_s \exp(-2\pi i \lambda r) \exp(2\pi i \lambda s) \\
&= \sum_r a_r \exp(-2\pi i \lambda r) \sum_s a_s \exp(2\pi i \lambda s) \\
\gamma_S(k) &= \sum_r \sum_s a_r a_s \left(\int_{-1/2}^{1/2} \exp(2\pi i \lambda (k - r + s)) f_Q(\lambda) \mathrm{d}\lambda \right) \\
&= \sum_r \sum_s a_r a_s \left(\int_{-1/2}^{1/2} \exp(2\pi i \lambda k) \exp(-2\pi i \lambda r) \exp(2\pi i \lambda s) f_Q(\lambda) \mathrm{d}\lambda \right) \\
&= \int_{-1/2}^{1/2} \sum_r a_r \exp(-2\pi i \lambda r) \sum_s a_s \exp(2\pi i \lambda s) \exp(2\pi i \lambda k) f_Q(\lambda) \\
&= \int_{-1/2}^{1/2} |A_S Q(\lambda)|^2 \exp(2\pi i \lambda k) f_Q(\lambda)
\end{aligned}
$$

$$
\begin{aligned}
\text{By definition } \gamma_S(k) &= \int_{-1/2}^{1/2} \exp(2\pi i \lambda k) f_S(\lambda) d\lambda \\
\text{Then } \int_{-1/2}^{1/2} \exp(2\pi i \lambda k) f_S(\lambda) d\lambda &= \int_{-1/2}^{1/2} |A_S Q(\lambda)|^2 \exp(2\pi i \lambda k) f_Q(\lambda) \mathrm{d}\lambda \\
f_S(\lambda) &= |A_S Q(\lambda)|^2 \frac{\exp(2\pi \lambda k)}{\exp(2\pi \lambda k} f_Q(\lambda) \\
&|A_S Q(\lambda)|^2 f_Q(\lambda)
\end{aligned}
$$

2 Numerical exercises

Please see *Assignment_TS.rmd* for the analysis (code and comments).