Assignment for the course
The statistical analysis of time series

Jan, 2020

- You are not allowed to contact other groups
- You are not allowed to contact either the Professor or the Teaching Assistant. If you think there is a typo in the assignment, then propose your correction, go ahead with the derivation and propose your solution
- You are allowed to check only the slides of the course and the help of the R packages
- Equally share the workload among the members of each group
- Write your answer on a separate document, which can be e.g. in LaTeX, or in Microsoft Words, or in Apple Pages, in a pdf document, or even hand-written
- Send me the code via email along with the answers to the other questions in this assignment
- Good luck!!

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1 The basic ingredients

Let $\mathcal{W} = \{W_t\}$ be a white noise with mean zero and variance σ_W^2. Recall that for a general time series $\mathcal{Z} := \{Z_t\}$ having autocovariance function γ_Z and spectral density f_Z, we have that:

$$\gamma_Z(k) = \int_{-1/2}^{1/2} \exp\{2\pi i\lambda k\} f_Z(\lambda) d\lambda \tag{1}$$

$$f_Z(\lambda) = \sum_{k=-\infty}^{\infty} \gamma_Z(k) \exp\{-2\pi i\lambda k\} \tag{2}$$

where $i^2 = -1$.

2 Theoretical aspects

[1.] Define the new process $\mathcal{X} = \{X_t\}$, whose dynamics is

$$X_t = W_t + (1/\theta)W_{t-1}, \tag{3}$$

with $\theta \in \Theta \subset \mathbb{R}$. For the process $\mathcal{X} = \{X_t\}$:

1.1 Compute the autocovariance function $\gamma_X(k) = \text{cov}(X_t, X_{t+k})$ for $k = 0, 1, ...$ Is $\mathcal{X}$ stationary? Do we need a restriction on θ?

1.2 Use the definition of Fourier transform of γ_X (see Eq. (1)-(2)) and derive the expression of the spectral density, $f_X(\lambda)$, $\lambda \in [-1/2, 1/2]$.

[2.] Now consider a new process $\mathcal{Y} = \{Y_t\}$, whose dynamics at time t is

$$Y_t = \frac{1}{3}(W_{t-1} + W_t + W_{t+1}).$$

2.1 Compute the autocovariance function $\gamma_Y(k) = \text{cov}(Y_t, Y_{t+k})$ for $k = 0, 1, ..$

2.2 Use the definition of Fourier transform γ_Y (see Eq. (1)-(2)) and derive the expression of the spectral density, $f_Y(\lambda)$, $\lambda \in [-1/2, 1/2]$.

[3.] Now consider jointly the time series $\mathcal{W}$ and $\mathcal{Y}$, with $\mathcal{Y}$ as in [2.].

3.1 Derive the expression for joint autocovariance function $\gamma_{WY}(k) = \text{cov}(W_{t+k}, Y_t)$.

3.2 Derive the expression of the cross-spectrum defined as

$$f_{WY}(\lambda) = \sum_{k=-\infty}^{\infty} \gamma_{WY}(k) \exp\{-2\pi i\lambda k\},$$

for $\lambda \in [-1/2, 1/2]$.

3.3 Let f_W be the spectral density of the white noise process. Making use of the result in point 3.2, rearrange the terms, express $f_{WY}(\lambda)$ as

$$f_{WY}(\lambda) = |A_{WY}(\lambda)|^2 f_W(\lambda), \tag{4}$$

and provide the form of the term $|A_{WY}(\lambda)|^2$.

[4.] Now let us generalize the relation in (4) to linear processes. Consider a linear filter which uses a set of specified coefficients a_j , for $j = 0, \pm 1, \pm 2, ...$, to transform an input series $\{Q_t\}$ producing an output series, $\{S_t\}$, of the form, for every t,

$$S_t = \sum_{j=-\infty}^{\infty} a_j Q_{t-j}, \qquad \sum_{j=-\infty}^{\infty} |a_j| < \infty. \tag{5}$$

Set the complex number

$$A_{SQ}(\lambda) = \sum_{j=-\infty}^{\infty} a_j \exp\{-2\pi i\lambda j\}.$$

Next, we are going to prove that

$$f_S(\lambda) = |A_{SQ}(\lambda)|^2 f_Q(\lambda), \tag{6}$$

with f_Q representing the spectral density of $\{Q_t\}$.

4.1 To prove (6) we proceed as in [1.],[2.] and [3.]. Thus, first for S_t as in (5) show that

$$\gamma_S(k) = \mathrm{cov}(S_{t+k}, S_t) = \sum_r \sum_s a_r a_s \left(\int_{-1/2}^{1/2} \exp\{2\pi i\lambda(k - r + s)\} f_Q(\lambda) \right)$$

[Hint: make use of the relation between the autocovariance function and the spectral density as in (1).]

4.2 Now prove that

$$|A_{SQ}(\lambda)|^2 = \left(\sum_r a_r \exp\{-2\pi i\lambda r\} \right) \left(\sum_s a_s \exp\{2\pi i\lambda s\} \right)$$

and making use of this result conclude that

$$\sum_r \sum_s a_r a_s \left(\int_{-1/2}^{1/2} \exp\{2\pi i\lambda(k - r + s)\} f_Q(\lambda) \right) = \int_{-1/2}^{1/2} |A_{SQ}(\lambda)|^2 \exp\{2\pi i\lambda k\} f_Q(\lambda) d\lambda. \tag{7}$$

Explain why (7) yields the desired result in (6).

3 Numerical exercices

[5.] Set $\theta = 2$ in (3) and simulate the process in (3), starting from $W_0 = 0$ and with sample size $n = 2000$. Plot the resulting trajectory.

[6.] Now let us apply the R routine `arima` (https://stat.ethz.ch/R-manual/R-patched/library/stats/html/arima.html), available in the R-package `stats`. Specifically, consider your simulated data in [6.] and estimate via `arima`, with the `''css''` option, the model parameter of an **MA(1)** model. Label the estimated parameter $\hat{\theta}$.

[7.] For the MA process in (3), write your own R code to compute γ_X, as derived in [1.1]. Then, compare the plot of the function γ_X computed using the theoretical $\theta = 1.5$ to the plot of the same function computed using the estimated $\hat{\theta}$. To compare the two plots, draw them in the same figure. Comment on the graphical output.