

Assignment for the course  
*The statistical analysis of time series*

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# 1 Theoretical aspects

[ 1. ]

[ 1.1 ] Compute the autocovariance function  $\gamma_X(k) = \text{cov}(X_t, X_{t+k})$  for  $k = 0, 1, \dots$  Is  $X$  stationary? Do we need a restriction on  $\theta$ ?

$$\begin{aligned}
 \gamma_X(k) &= \text{cov}(X_t, X_{t+k}) \\
 &= E((W_t + \frac{1}{\theta}W_{t-1})(W_{t+k} + \frac{1}{\theta}W_{t+k-1})) - E(W_t + \frac{1}{\theta}W_{t-1})E(W_{t+k} + \frac{1}{\theta}W_{t+k-1}) \\
 &= E((W_t + \frac{1}{\theta}W_{t-1})(W_{t+k} + \frac{1}{\theta}W_{t+k-1})) \\
 &= E(W_t W_{t+k} + \frac{1}{\theta}W_t W_{t+k-1} + \frac{1}{\theta}W_{t-1} W_{t+k} + \frac{1}{\theta^2}W_{t-1} W_{t+k-1}) \\
 &= \begin{cases} \sigma_W^2(1 + \frac{1}{\theta^2}) & h = 0 \\ \frac{\sigma_W^2}{\theta} & |h| = 1 \\ 0 & \text{else} \end{cases}
 \end{aligned}$$

$\Rightarrow X_t$  is weakly stationary.

Apart from  $\theta \neq 0$ , to avoid  $1 + |\frac{1}{\theta}| \rightarrow \infty$ , no conditions for  $\frac{1}{\theta}$  necessary to get stationarity on MA(q) processes.

[ 1.2 ] Use the definition of Fourier transform of  $\gamma_X$  and derive the expression of the spectral density,  $f_X(\lambda)$ ,  $\lambda \in [-1/2, 1/2]$ .

$$\begin{aligned}
 f_X(\lambda) &= \sum_{k=-\infty}^{\infty} \gamma_X(k) \exp(-2\pi i \lambda k) \\
 &= \sum_{k=-2}^{-\infty} \gamma_X(k) \exp(-2\pi i \lambda k) + \gamma_X(-1) \exp(-2\pi i \lambda \cdot (-1)) \\
 &\quad + \gamma_X(0) \exp(-2\pi i \lambda \cdot 0) + \gamma_X(1) \exp(-2\pi i \lambda \cdot 1) + \sum_{k=2}^{\infty} \gamma_X(k) \exp(-2\pi i \lambda k) \\
 &= \sigma_W^2 \left( \frac{\theta(\exp(2\pi i \lambda) + \exp(-2\pi i \lambda)) + \theta^2 + 1}{\theta^2} \right) \\
 &= \sigma_W^2 \left( \frac{\theta(\cos(2\pi \lambda) + i \sin(2\pi \lambda) + \cos(2\pi \lambda) - i \sin(2\pi \lambda)) + \theta^2 + 1}{\theta^2} \right) \\
 &= \frac{\sigma_W^2}{\theta^2} (2\theta \cos(2\pi \lambda) + \theta^2 + 1)
 \end{aligned}$$

[ 2. ]

[ 2.1 ] Compute the autocovariance function  $\lambda_Y(k) = \text{cov}(Y_t, Y_{t+k})$  for

$$k = 0, 1, \dots$$

$$\begin{aligned}\gamma_Y(k) &= \text{cov}(Y_t, Y_{t+k}) \\ &= E\left(\left(\frac{1}{3}W_{t-1} + \frac{1}{3}W_t + \frac{1}{3}W_{t+1}\right)\left(\frac{1}{3}W_{t+k-1} + \frac{1}{3}W_{t+k} + \frac{1}{3}W_{t+k+1}\right)\right) \\ &= \begin{cases} \frac{\sigma_W^2}{3} & h = 0 \\ \frac{2\sigma_W^2}{9} & |h| = 1 \\ \frac{\sigma_W^2}{9} & |h| = 2 \\ 0 & \text{else} \end{cases}\end{aligned}$$

[ 2.2 ] Use the definition of Fourier transform  $\lambda_Y$  and derive the expression of the spectral density  $f_Y(\lambda)$ ,  $\lambda \in [-1/2, 1/2]$

$$\begin{aligned}f_Y(\lambda) &= \sum_{k=-\infty}^{\infty} \gamma_Y(k) \exp(-2\pi i \lambda k) \\ &= \gamma_Y(-2) \exp(4\pi i \lambda) + \gamma_Y(-1) \exp(2\pi i \lambda) + \gamma_Y(0) \\ &= +\gamma_Y(1) \exp(\pi i \lambda) + \gamma_Y(2) \exp(-4\pi i \lambda) \\ &= \sigma_W^2 \left( \frac{\exp(4\pi i \lambda) + \exp(-4\pi i \lambda) + 2\exp(2\pi i \lambda) + 2\exp(-2\pi i \lambda) + 3}{9} \right) \\ &= \sigma_W^2 \left( \frac{2\cos(4\pi \lambda) + 4\cos(2\pi \lambda) + 3}{9} \right)\end{aligned}$$

[ 3. ]

[ 3.1 ] Derive the expression for joint autocovariance function  $\lambda_{WY}(k) = \text{cov}(W_{t+k}, Y_t)$ .

$$\begin{aligned}\gamma_{WY}(k) &= \text{cov}(W_{t+k}, Y_t) \\ &= E\left(W_{t+k} \left(\frac{1}{3}(W_{t-1} + W_t + W_{t+1})\right)\right) \\ &= \frac{1}{3} E(W_{t+k} W_{t-1} + W_{t+k} W_t + W_{t+k} W_{t+1}) \\ &= \begin{cases} \frac{\sigma_W^2}{3} & h = 0 \text{ or } |h| = 1 \\ 0 & \text{else} \end{cases}\end{aligned}$$

[ 3.2 ] Derive the expression of the cross-spectrum

$$\begin{aligned}f_{WY}(\lambda) &= \sum_{k=-\infty}^{\infty} \gamma_{WY}(k) \exp(-2\pi i \lambda k) \\ &= \gamma_{WY}(-1) \exp(2\pi i \lambda) + \gamma_{WY}(0) \\ &= +\gamma_{WY}(1) \exp(\pi i \lambda) \\ &= \sigma_W^2 \left( \frac{\exp(2\pi i \lambda) + \exp(-2\pi i \lambda) + 1}{3} \right) \\ &= \sigma_W^2 \left( \frac{\cos(2\pi \lambda) + i\sin(2\pi \lambda) + \cos(-2\pi \lambda) + i\sin(-2\pi \lambda) + 1}{3} \right) \\ &= \sigma_W^2 \left( \frac{2\cos(2\pi \lambda) + 1}{3} \right)\end{aligned}$$

[ 3.3 ] Let  $f_W$  be the spectral density of the white noise process. Making use of the result in point 3.2, rearrange the terms and express  $f_{WY}(\lambda)$  and provide the form of the term  $|A_{WY}(\lambda)|^2$

$$\begin{aligned}
 f_W(\lambda) &= \sum_{k=-\infty}^{\infty} \gamma_W(k) \exp(-2\pi i \lambda k) = \gamma_W(0) = \sigma_W^2 \\
 f_{WY}(\lambda) &= \sigma_W^2 \left( \sum_{k=-1}^1 \left( \frac{1}{\sqrt{3}} \exp(2\pi i \lambda) \right)^2 \right) \\
 &= \sigma_W^2 |A_{WY}(\lambda)|^2 \\
 |A_{WY}(\lambda)|^2 &= \left( \sum_{k=-1}^1 \left( \frac{1}{\sqrt{3}} \exp(2\pi i \lambda) \right)^2 \right)
 \end{aligned}$$

[ 4. ] Prove that  $f_S(\lambda) = |A_{SQ}(\lambda)|^2 f_Q(\lambda)$  with  $A_{SQ}(\lambda) = \sum_{j=-\infty}^{\infty} a_j \exp(-2\pi i \lambda j)$

[ 4.1 ] Show that

$$\begin{aligned}
 \gamma_S(k) &= \text{cov}(S_{t+k}, S_t) \\
 &= E \left( \sum_{j=-\infty}^{\infty} a_j Q_{t+k-j} \cdot \sum_{j=-\infty}^{\infty} a_j Q_{t-j} \right) \\
 &= E \left( (...a_{-1} Q_{t+k+1} + a_0 Q_{t+k} + a_1 Q_{t+k-1} ...) \right. \\
 &\quad \cdot (...a_{-1} Q_{t+1} + a_0 Q_t + a_1 Q_{t-1} ...) \left. \right) \\
 &= ...E(a_{-1} Q_{t+k+1} a_{-1} Q_{t+1}) + E(a_{-1} Q_{t+k+1} a_0 Q_t) \\
 &\quad + E(a_{-1} Q_{t+k+1} a_1 Q_{t-1}) ... \\
 &= E \left( \sum_r \sum_s a_s Q_{t+k-r} a_r Q_{t-s} \right) = E \left( \sum_r \sum_s a_r a_s Q_{t+k-r} Q_{t-s} \right) \\
 &= \sum_r \sum_s a_r a_s E(Q_{t+k-r} Q_{t-s}) = \sum_r \sum_s a_r a_s \gamma_Q(k-r+s) \\
 &= \sum_r \sum_s a_r a_s \left( \int_{-1/2}^{1/2} \exp(2\pi i \lambda (k-r+s)) f_Q(\lambda) d\lambda \right)
 \end{aligned}$$

[ 4.2 ] Prove that

$$\begin{aligned}
|A_S Q(\lambda)|^2 &= \left| \sum_{j=-\infty}^{\infty} a_j \exp(-2\pi i \lambda j) \right|^2 \\
&= \sum_r \sum_s a_r a_s \exp(-2\pi i \lambda r) \exp(2\pi i \lambda s) \\
&= \sum_r a_r \exp(-2\pi i \lambda r) \sum_s a_s \exp(2\pi i \lambda s) \\
\gamma_S(k) &= \sum_r \sum_s a_r a_s \left( \int_{-1/2}^{1/2} \exp(2\pi i \lambda (k - r + s)) f_Q(\lambda) d\lambda \right) \\
&= \sum_r \sum_s a_r a_s \left( \int_{-1/2}^{1/2} \exp(2\pi i \lambda k) \exp(-2\pi i \lambda r) \exp(2\pi i \lambda s) f_Q(\lambda) d\lambda \right) \\
&= \int_{-1/2}^{1/2} \sum_r a_r \exp(-2\pi i \lambda r) \sum_s a_s \exp(2\pi i \lambda s) \exp(2\pi i \lambda k) f_Q(\lambda) d\lambda \\
&= \int_{-1/2}^{1/2} |A_S Q(\lambda)|^2 \exp(2\pi i \lambda k) f_Q(\lambda) d\lambda
\end{aligned}$$

$$\begin{aligned}
&\text{By definition } \gamma_S(k) = \int_{-1/2}^{1/2} \exp(2\pi i \lambda k) f_S(\lambda) d\lambda \\
\text{Then } \int_{-1/2}^{1/2} \exp(2\pi i \lambda k) f_S(\lambda) d\lambda &= \int_{-1/2}^{1/2} |A_S Q(\lambda)|^2 \exp(2\pi i \lambda k) f_Q(\lambda) d\lambda \\
f_S(\lambda) &= |A_S Q(\lambda)|^2 \frac{\exp(2\pi i \lambda k)}{\exp(2\pi i \lambda k)} f_Q(\lambda) \\
&= |A_S Q(\lambda)|^2 f_Q(\lambda)
\end{aligned}$$

## 2 Numerical exercises

Please see *Assignment\_TS.rmd* for the analysis (code and comments).