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# Parallel sorting algorithms

## Theory and implementation

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# Overview


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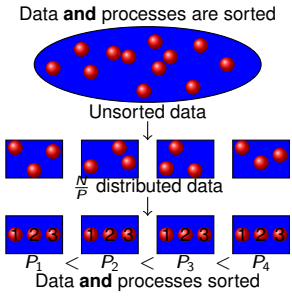
# Complexity of sorting algorithms

## Basic operations

- **Compare algorithm** : Comparaison algorithm complexity is supposed $\mathcal{O}(1)$. But in distributed parallel context, one must consider the distribution of the initial data to insert the cost of data exchange between processes !
- **Exchange algorithm** : Exchange algorithm complexity is supposed $\mathcal{O}(1)$. But same consideration to do as **compare algorithm** ;
- Sequential “compare–and–exchange” algorithm :

```
if (a>b) { // Comparaison
  // Exchange
  tmp = a;
  a   = b;
  b   = tmp;   }
```

# Potential speed-up




- Best sequential sorting algorithms ( for arbitrary sequences of numbers ) have average time complexity $O(n \log n)$
- hence, the best speedup one can except from using $n$ processors is $\frac{O(n \log n)}{n} = O(\log n)$
- there are such parallel algorithms, but the hidden constant is very large ( F. Thomson Leighton : Introduction to parallel algorithms and architectures (1991) )
- Generally, a pratical useful $O(\log n)$ algorithm may be difficult to find.

Beware, it may be a bad idea to take $n$ processes to sort $n$ data (granularity).

# Parallelization of a naive algorithm

## Naive algorithm

- Count the number of numbers that are smaller than a number *a* in the list
- this gives the position of *a* in the sorted list
- this procedure has to be repeated for all elements of the list ; hence the time complexity is $n(n-1) = O(n^2)$ ( not so good sequential algorithm )

## Implementation

```
for ( i = 0; i < n; i++ ) {// For each value
  x = 0;
  for ( j = 0; j < n; j++ )// Computing the new pos.
    if (a[i] > a[j]) x++;
  b[x] = a[i];
}
```

Work well if there are no repetitions of the numbers in the list ( in the case of repetitions one has to change slightly the code ).

# Rank sort : Parallel code

## Embarrasingly “ideal” algorithm

Parallel code, using $n$ processes ( for $n$ values to sort )

```
x = 0;
for ( j = 0; j < n; j++ )
  if ( a[rank] > a[j] ) x++;
b[x] = a[rank];
```

## Complexity

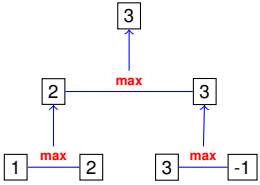
- $n$ processors work in parallel to find the ranks of all numbers of the list ;
- Parallel time complexity is $O(n)$, better than any sequential sorting algorithm !
- Usable for GPGPU units.

# More parallelization...

**Parallel code using $n^2$ processes ( for $n$ values to sort )**

## Parallel algorithm

- In the case $n^2$ processes may be used, the comparison of each `a[0],...,a[n-1]` with `a[i]` may be done in parallel as well
- Incrementing the counter is still sequential, hence the overall computation requires $1 + n$ steps ;
- If a tree structure is used to increment the counter, then the overall computation time is $O(\log_2 n)$




( but, as one expects, processor efficiency is very low )

There are just theorical results : it is not efficient to use $n$ or $n^2$ processors to sort $n$ numbers.

# Data partitionning

## Context

- Usually the number $n$ of values is much larger than the number $p$ of processes ;
- In such cases, each process will handle a part of the data (a sublist of the data)

## Distributed sorted container

- local container is sorted ;
- if $p_i < p_j$ then $\forall a_i \in p_i, \forall a_j \in p_j, a_i \leq a_j$

## Global scheme of parallel sort algorithm

For a process :

- Sort his local data ;
- Run a merge sort algorithm to concatenate its list with that received from another process ;
- Keep the bottom half (or the top half) of the sorted list.

# Parallel compare and exchange operations

## Asymmetric algorithm

- Process $p_i$ sends local value $A$ to process $p_j$ ;
- Process $p_j$ compares value $A$ with some local values $B_j$ ;
- Send the $B_j$ which are larger (or lesser) than $A$. If none $B_j$ is larger (or lesser) than $A$, sends back $A$ ;

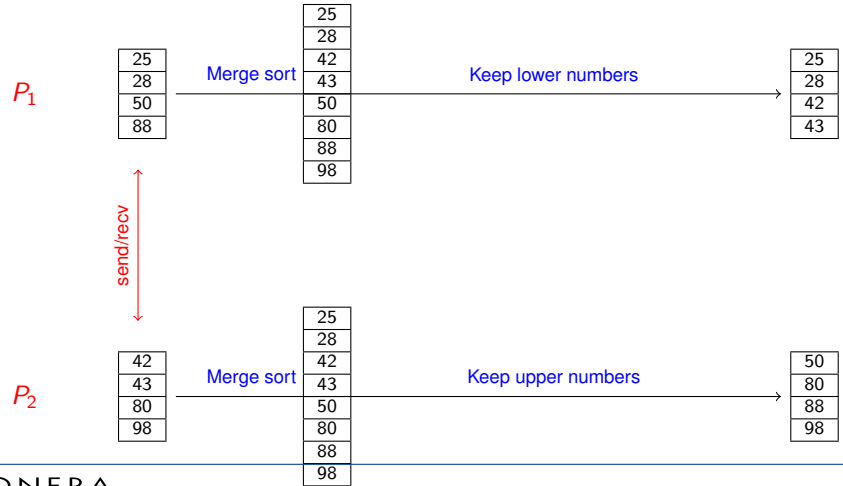
## Symmetric algorithm

- Processes $p_i$ and $p_j$ sends some value to the other ;
- Each process compares his value with the received value ;
- Each process keeps his value or the received value relative to the comparaison result ;

## Remarks

- Data exchanges between processes is very expensive, so find some algorithms which minimize data exchanges ;
- Generally, the receive operation doesn't know the number of values to receive $\Rightarrow$ one must probe the received message to get the number of data to receive, allocate the relative buffer and receive the data !

# Scheme of a general algorithm for parallel sort algorithm



# Overview

2 Parallel sort algorithms

1 Theory of parallel sorting algorithms

# Sequential bubble sort algorithm

## Bubble sort algorithm

- Simplest, but not so efficient sequential sorting algorithm ;
- Compare/exchange complexity :

$$\sum_{i=1}^{n-1} i = \frac{n(n-1)}{2} = O(n^2)$$

Figure – Analogic bubble sort

## Sequential code

```
for (int i=n-1; i>0; --i)
    for (int j=0; j<i; ++j) {
      k = j+1;
      if (a[j]>a[k]) std::swap(a[j],a[k]);
    }
```

# Odd-Even sort algorithm

- Parallelized bubble sort
- Based on idea that the bodies of the main loop may be overlapped

## “scalar” Algorithm : Iteration between even and odd phase

- Even phase


```
if (rank%2==0) {
  recv(&temp, (rank+1)%nbp);
  send(&value,(rank+1)%nbp);
  if (temp < A) A = temp;   }
```

```
if (rank%2==1) {
  send(&value,rank-1);
  recv(&temp, rank-1);
  if (temp > A) A = temp;   }
```

- Odd phase


```
if (rank%2==0) {
  recv(&temp, (rank+nbp-1)%nbp);
  send(&value,(rank+nbp-1)%nbp);
  if (temp > A) A = temp;   }
```

```
if (rank%2==1) {
  send(&value,(rank+1)%nbp);
  recv(&temp, (rank+1)%nbp);
  if (temp < A) A = temp;   }
```

# Example of even–odd parallel bubble sort

Example : Sorting 8 numbers on 8 processes

| Step | $P_0$ | | $P_1$ | | $P_2$ | | $P_3$ | | $P_4$ | | $P_5$ | | $P_6$ | | $P_7$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 4 | ↔ | 2 | | 7 | ↔ | 8 | | 5 | ↔ | 1 | | 3 | ↔ | 6 |
| 1 | 2 | | 4 | ↔ | 7 | | 8 | ↔ | 1 | | 5 | ↔ | 3 | | 6 |
| 2 | 2 | ↔ | 4 | | 7 | ↔ | 1 | | 8 | ↔ | 3 | | 5 | ↔ | 6 |
| 3 | 2 | | 4 | ↔ | 1 | | 7 | ↔ | 3 | | 8 | ↔ | 5 | | 6 |
| 4 | 2 | ↔ | 1 | | 4 | ↔ | 3 | | 7 | ↔ | 5 | | 8 | ↔ | 6 |
| 5 | 1 | | 2 | ↔ | 3 | | 4 | ↔ | 5 | | 7 | ↔ | 6 | | 8 |
| 6 | 1 | ↔ | 2 | | 3 | ↔ | 4 | | 5 | ↔ | 6 | | 7 | ↔ | 8 |
| 7 | 1 | | 2 | ↔ | 3 | | 4 | ↔ | 5 | | 6 | ↔ | 7 | | 8 |

# Odd-even parallel algorithm per block

## Per block algorithm

- Replace a value per process with a sorted set of values per process
- Use sort-fusion algorithm to exchange values
- Data comparaison complexity :
  $\frac{N}{nbp} \log_2 \left( \frac{N}{nbp} \right) + (nbp - 1).\frac{2N}{nbp}$
- Data communication complexity :
  $(nbp - 1).\frac{2N}{nbp}$

## Implementation

```
sort(values);// Quick sort of local values
for (it=0; it<nbp-1; ++it) {// Odd-even algorithm
  if (it is odd) {
    if (rank is even and rank > 0) {
      receive(buffer, rank-1); send(values, rank-1);
      values = fusionSort(buffer, values, keepMax);
    } else if (rank is odd and rank < nbp-1) {
      send(values, rank+1); recv(buffer, rank+1);
      values = fusionSort(buffer, values, keepMin);
    }
  } else if (it is even) {
    if (rank is even and rank < nbp-1) {
      recv(buffer, rank+1); send(values, rank+1);
      values = fusionSort(buffer, values, keepMin);
    } else if (rank is odd) {
      send(values, rank-1); recv(buffer, rank-1);
      values = fusionSort(buffer, values, keepMax);
    }
  }
}
```

# Shear sort algorithm

Two dimensional sorting

## Basic idea

- Look at the array as a two-dimensional array (one row per process)
- The goal is to sort this 2D array in snakelike style : even rows increasing, odd rows decreasing ;
- Two phases : In **even phase**, sort per row, in odd phase,, sort per column increasing from top to bottom ;
- After $\log_2(N) + 1$ phases, the array is snakelike-style sorted.

## Example

| | | | |
|---|---|---|---|
| 4 | 14 | 8 | 2 |
| 10 | 3 | 13 | 16 |
| 7 | 15 | 1 | 5 |
| 12 | 6 | 11 | 9 |

Original number

## Remarques

- Embarrassingly parallel algorithm for shared memory !
- But not well adapted for distributed parallel architecture as is ;
- How change this algorithm for distributed parallel architecture ?

# Shear sort algorithm

Two dimensional sorting

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- Look at the array as a two-dimensional array (one row per process)
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- After $\log_2(N) + 1$ phases, the array is snakelike-style sorted.

## Example

| | | | |
|---|---|---|---|
| 2 | 4 | 8 | 14 |
| 16 | 13 | 10 | 3 |
| 1 | 5 | 7 | 15 |
| 12 | 11 | 9 | 6 |

Phase 1 – row sort

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# Shear sort algorithm

Two dimensional sorting

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- Two phases : In **even phase**, sort per row, in odd phase,, sort per column increasing from top to bottom ;
- After $\log_2(N) + 1$ phases, the array is snakelike-style sorted.

## Example

| | | | |
|---|---|---|---|
| 1 | 4 | 7 | 3 |
| 2 | 5 | 8 | 6 |
| 12 | 11 | 9 | 14 |
| 16 | 13 | 10 | 15 |

Phase 2 – col sort

## Remarques

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- How change this algorithm for distributed parallel architecture ?

# Shear sort algorithm

Two dimensional sorting

## Basic idea

- Look at the array as a two-dimensional array (one row per process)
- The goal is to sort this 2D array in snakelike style : even rows increasing, odd rows decreasing ;
- Two phases : In **even phase**, sort per row, in odd phase,, sort per column increasing from top to bottom ;
- After $\log_2(N) + 1$ phases, the array is snakelike-style sorted.

## Example

| | | | |
|---|---|---|---|
| 1 | 3 | 4 | 7 |
| 8 | 6 | 5 | 2 |
| 9 | 11 | 12 | 14 |
| 16 | 15 | 13 | 10 |

Phase 3 : row sort

## Remarques

- Embarrassingly parallel algorithm for shared memory !
- But not well adapted for distributed parallel architecture as is ;
- How change this algorithm for distributed parallel architecture ?

# Shear sort algorithm

Two dimensional sorting

## Basic idea

- Look at the array as a two-dimensional array (one row per process)
- The goal is to sort this 2D array in snakelike style : even rows increasing, odd rows decreasing ;
- Two phases : In **even phase**, sort per row, in odd phase,, sort per column increasing from top to bottom ;
- After $\log_2(N) + 1$ phases, the array is snakelike-style sorted.

## Example

| | | | |
|---|---|---|---|
| 1 | 3 | 4 | 2 |
| 8 | 6 | 5 | 7 |
| 9 | 11 | 12 | 10 |
| 16 | 15 | 13 | 14 |

Phase 4 : col sort

## Remarques

- Embarrassingly parallel algorithm for shared memory !
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- How change this algorithm for distributed parallel architecture ?

# Shear sort algorithm

Two dimensional sorting

## Basic idea

- Look at the array as a two-dimensional array (one row per process)
- The goal is to sort this 2D array in snakelike style : even rows increasing, odd rows decreasing ;
- Two phases : In **even phase**, sort per row, in odd phase,, sort per column increasing from top to bottom ;
- After $\log_2(N) + 1$ phases, the array is snakelike-style sorted.

## Example

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 8 | 7 | 6 | 5 |
| 9 | 10 | 11 | 12 |
| 16 | 15 | 14 | 13 |

Final 5 : row sort

## Remarques

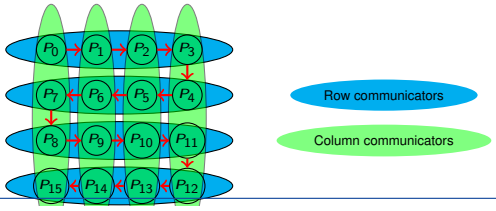
- Embarrassingly parallel algorithm for shared memory !
- But not well adapted for distributed parallel architecture as is ;
- How change this algorithm for distributed parallel architecture ?

# Shear sort algorithm for parallel distributed memory architecture

## Implementation ideas

- Same principal as odd-even algorithm : replace a value with some sets of values $S_i$ (one set per process) ;
- Define a relation order : $S_i < S_j$ iff $\max(S_i) < \min(S_j)$ (In set, values ordered as increasing order)
- Use odd-even algorithm to parallelize the phase of sorting per row or column ;
- Grouping processes in new communicators per rows and per columns ;
- Play with rank numbering to alternate between increasing order and decreasing order for rows ;

## Processes repartition




- Use `MPI_Comm_split(comm,color,key,&newcomm)` to define row and columns communicators ;
- Processes calling this function with same color are inside the same new communicator ;
- key is a value to numbering the processes inside the new communicator.