1. Výška devítiletých chlapců v centimetrech je náhodná veličina s normálním rozdělením $N(136, 5^2)$.

(a) Co říká o jejich výšce pravidlo 2σ?

(b) Jaká je výška chlapce na desátém percentilu? (Tj. takového, že 10 % chlapců je nižší než on.)

(c) Jaká je výška chlapce na devadesátém percentilu? (Tj. takového, že 90 % chlapců je nižší než on.)

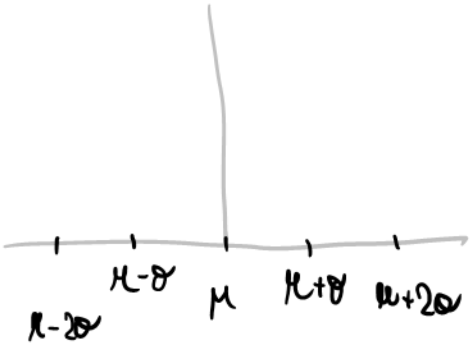
(d) S jakou pravděpodobností je devítiletý chlapec vysoký alespoň 150 cm?

$Z \sim N(0,1)$

$N \sim (\mu, \sigma^2)$ $\quad N = \mu + \sigma \cdot Z$

$z = \frac{N-\mu}{\sigma}$

a)



$$P(\mu-\sigma \le X \le \mu+\sigma) = P\left(\frac{-\sigma}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{\sigma}{\sigma}\right) = P(-1 \le Z \le 1)$$

$$= \Phi(1) - \Phi(-1) = 0{,}841 - 0{,}159 = 0{,}682$$

x	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
$\Phi(x)$	0.023	0.067	0.159	0.309	0.5	0.692	0.841	0.933	0.977

t	0.1	0.2	0.3	0.4	0.5
$\Phi^{-1}(t)$	-1.28	-0.84	-0.52	-0.25	0

$$P(\mu-2\sigma \le X \le \mu+2\sigma) = P\left(\frac{-2\sigma}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{2\sigma}{\sigma}\right)$$

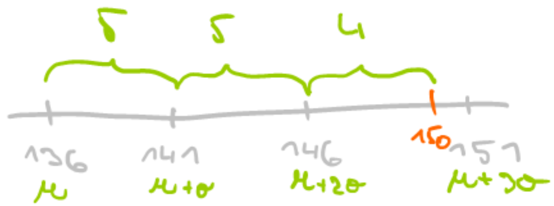
$$= \Phi(2) - \Phi(-2) = 0{,}954$$

68 % bude vyšší než 131 a nižší než 141

95 % bude vyšší než 126 a nižší než 146

d)

$$P(X \ge 150) = 1 - P(X < 150)$$

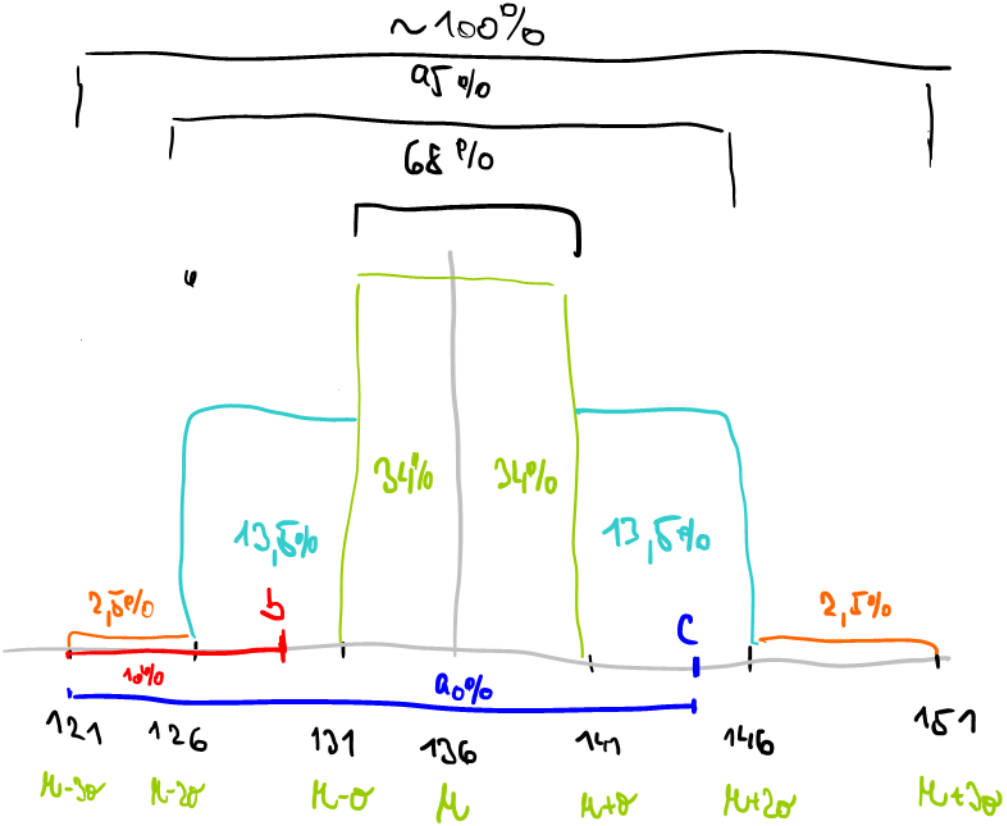


$\mu = 136$

$\sigma = 5$

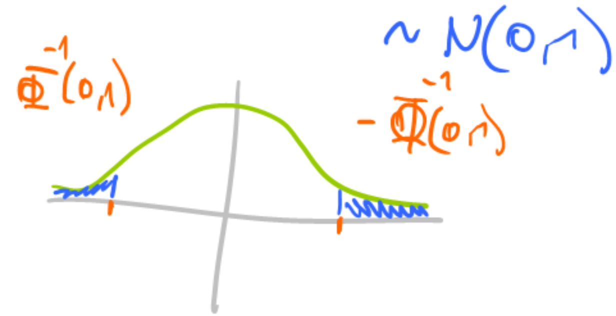
$$\Rightarrow 150 = \mu + 2\sigma + \frac{4}{5}\sigma = \mu + \frac{14}{5}\sigma$$

$$= 1 - P\left(X \le \mu + \frac{14}{5}\sigma\right) = 1 - P\left(Z \le \frac{14}{5}\right) = 1 - \Phi\left(\frac{14}{5}\right)$$



$b \in (126, 131) \ \dots \ (\mu-2\sigma, \mu-\sigma)$

$c \in (141, 146) \ \dots \ (\mu+\sigma, \mu+2\sigma)$



b)

$$P(Z \le z) = 0{,}1$$

$$z = \Phi^{-1}(0{,}1) = -1{,}28$$

$$x = 136 + (5(-1{,}28)) = \underline{\underline{129{,}6 \text{ cm}}}$$

$$Z = \frac{X-\mu}{\sigma}$$

$$X = \mu + \sigma \cdot Z$$

c)

$$P(Z \le z) = 0{,}9$$

$$z = \Phi^{-1}(0{,}9) = \text{neznám}$$

vyjádřeme s tím co mám:

$$x = \underline{\underline{136 + (5 \cdot \Phi^{-1}(0{,}9))}} \text{ cm}$$

2)

X = "počet chyb na 100 řádků"

10 chyb na 1000 řádků ... 1 chyba na 100
a protože X je vzhledem ke 100 řádkům, je $\lambda = 1$

$X \sim Poiss(\lambda)$

k = # výskytů $\quad \lambda = 1$

a)

$$P(X=1) = \frac{\lambda^k}{k!} \cdot e^{-\lambda} = \frac{\lambda}{1} \cdot e^{-\lambda} = e^{-1} \sim \underline{\underline{36\%}}$$

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b)

$$P(X \geq 2 \;\&\; X < 5) = \sum_{x=2}^{4} P(X=x) = \frac{e^{-1}}{2!} + \frac{e^{-1}}{3!} + \frac{e^{-1}}{4!} \sim \underline{\underline{26\%}}$$

3)

$R \sim Exp(1)$

a) $\mathbb{E}(R) = \frac{1}{\lambda} = \underline{1}$

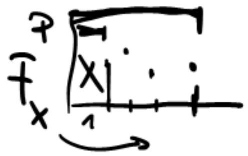
b) $P(R>1 \,\&\, R<4) = F_R(4) - F_R(1) = 1-e^{-4} - 1 + e^{-1} = e^{-1} - e^{-4} = 0{,}349 \sim \underline{35\,\%}$

c)

$$f_X(x) = \begin{cases} 0 & \text{pro } x < 0 \\ \lambda \cdot e^{-\lambda x} & \text{pro } x \geq 0 \end{cases}$$

$f_X(r) = e^{-r}$ protože $\lambda = 1$ a poloměr je vždy ≥ 0

$\text{"}= P(X = r)\text{"}$



$F_X(x) = P(X \leq x)$

$\text{"}f_X(x) = P(X=x)\text{"} = 0$

$S = g(r) = 4\pi r^2$

$f: r \rightarrow p \in [0,1]$

$g: r \rightarrow S$

$h: S \to p \in [0,1]$

$g^{-1}: s \to r$

$h(s) = f(g^{-1}(s))$

$r = g^{-1}(s) = \sqrt{\frac{s}{4\pi}} = \frac{1}{2}\sqrt{\frac{s}{\pi}}$

$f_s(s) = e^{-\frac{1}{2}\sqrt{\frac{s}{\pi}}}$

když mám $S \dots g(X)$ potom

F_S i f_S vyjádřím jako $f_S(s) = f_X(g(x))$

$F_S(s) = \int_{-\infty}^{+\infty} f_S(s)$

$\mathbb{E}(S) \int_{-\infty}^{+\infty} s \cdot f_S(s)$

$K \dots$ „povrch koule"

protože povrch je vždy ≥ 0 hustotu můžu počítat jen na tomto intervalu

$K = g(X) = 4\pi x^2$

$$f_K(g(x)) = e^{-4\pi x^2}$$

$$F_K(g(x)) = \int_{-\infty}^{x} e^{-4\pi y^2}\, dy$$ ⊛ dál neumím upravit

d) spojitý LOTUS

$$\mathbb{E}(K) = \mathbb{E}(g(x)) = \int_{-\infty}^{+\infty} g(x) \cdot f_X(x)\, dx = \int_{-\infty}^{+\infty} 4\pi x^2 \cdot e^{-x}\, dx = 4\pi \int_{-\infty}^{+\infty} x^2 \cdot e^{-x}\, dx$$

PER PARTES

$$\int u.v' = u.v - \int u'.v$$

předtím na neurčitý integrál

$$\left| \begin{matrix} u = x^2 & v' = e^{-x} \\ u' = 2x & v = -e^{-x} \end{matrix} \right| = 4\pi \left(x^2 \cdot (-e^{-x}) - \int 2x \cdot (-e^{-x}) dx \right) =$$

$$\left| \begin{matrix} u = 2x & v' = (-e^{-x}) \\ u' = 2 & v = e^{-x} \end{matrix} \right| = 4\pi \left(x^2 (-e^{-x}) - \left((2x \cdot e^{-x}) - \int 2 \cdot (e^{-x})\, dx \right) \right)$$

$$= 4\pi \left(x^2 \cdot (-e^{-x}) - (2x(e^{-x})) - 2 \cdot (-e^{-x}) \right)$$

$$= 4\pi \left(x^2 \cdot (-e^{-x}) - 2xe^{-x} + 2(-e^{-x}) \right)$$

$$= -4\pi e^{-x} (x^2 + 2x + 2)$$

návrat ma

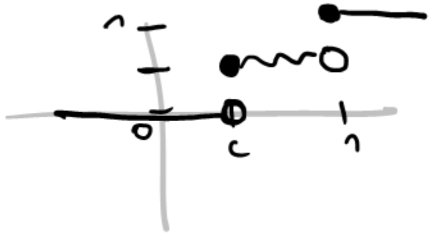
4) $X \sim Exp(\lambda)$

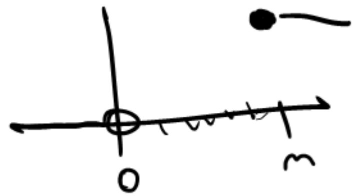
$c > 0$

$$F_X = \begin{cases} 0 & \dots \quad x < 0 \\ 1 - e^{-\lambda x} & \dots \quad x \geq 0 \end{cases}$$

$$f_X = \begin{cases} 0 & x < 0 \\ F_X' = \lambda \cdot e^{-\lambda x} & \dots \quad x \geq 0 \end{cases}$$

a)





takže řešim $P(X \in (c, x))$

pro $x > c$

$$F_{X|X>c}(x) = P(X \leq x \mid X > c) = \frac{P(X \leq x \ \& \ X > c)}{P(X > c)} = \frac{F_X(x) - F_X(c)}{1 - F_X(c)}$$

$$= \frac{1 - e^{-\lambda x} - (1 - e^{-\lambda c})}{1 - (1 - e^{-\lambda c})} = \frac{e^{-\lambda c} - e^{-\lambda x}}{e^{-\lambda c}} = \frac{\frac{1}{e^{\lambda c}} - \frac{1}{e^{\lambda x}}}{\frac{1}{e^{\lambda c}}} = \frac{e^{\lambda c}}{e^{\lambda c}} - \frac{e^{\lambda c}}{e^{\lambda x}} = 1 - \frac{e^{\lambda c}}{e^{\lambda x}}$$

$$= 1 - e^{\lambda c - \lambda x} = \underline{\underline{1 - e^{\lambda(c-x)}}}$$

pro $x \leq c$

$$F_{X|X>c}(x) = \frac{P(X \leq c \ \& \ X > c)}{P(X > c)} = \underline{\underline{0}}$$

pro $x > c$

$h(y) = e^y$

$g(x) = -\lambda x \to h(g(x))' = h'(g(x)) \cdot g(x)'$

$$f_{X|X>c}(x) = F_{X|X>c}(x)' = 1 - e^{\lambda c} \cdot e^{-\lambda x} = -e^{\lambda c} \cdot (e^{-\lambda x})' = -e^{\lambda c} \cdot (e^{-\lambda x}) \cdot (-\lambda)$$

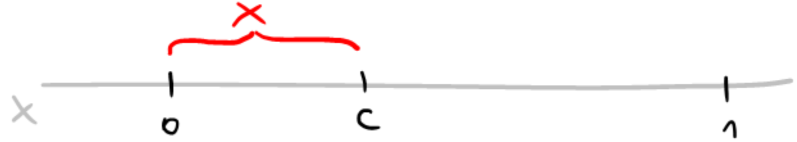
$$= \lambda \cdot e^{\lambda c - \lambda x} = \underline{\underline{\lambda \cdot e^{\lambda(c-x)}}}$$

pro $x \leq c$

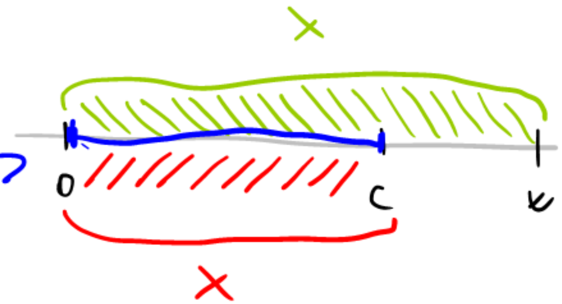
$$f_{X|X>c}(x) = F_{X|X>c}(x)' = (0)' = \underline{\underline{0}}$$

(*) platí pro $X \geq 0$, pro $X < 0$ je $F_{X|X>c} = f_{X|X>c} = 0$

b)

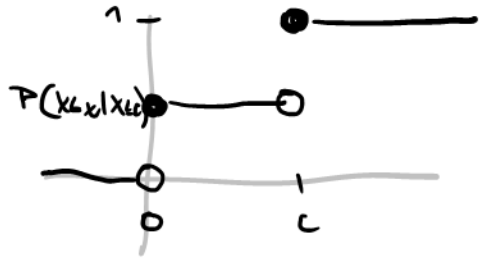


$$F_{X|X<c}(x) = P(X\le x \mid X\le c) = \frac{P(X\le x \;\&\; X\le c)}{P(X\le c)}$$



pro $x>c$ je $P(X\le x \,\&\, X\le c) = P(X\le c)$

$$= \begin{cases} \text{pro } x>c & \dfrac{P(X\le x \,\&\, X\le c)}{P(X\le c)} = \dfrac{P(X\le c)}{P(X\le c)} = \underline{\underline{1}} \\ \text{pro } x\le c & \dfrac{P(X\le x)}{P(X\le c)} = \underline{\underline{\dfrac{1-e^{-\lambda x}}{1-e^{-\lambda c}}}} \end{cases}$$



$$f_{X|X<c}(x) = F_{X|X<c}(x)' = \begin{cases} \text{pro } x>c & (1)' = \underline{\underline{0}} \\ \text{pro } x\le c & \left(\dfrac{1-e^{-\lambda x}}{1-e^{-\lambda c}}\right)' \end{cases}$$

$h(g(x))' = h'(g(x)) \cdot g(x)'$ $\quad$ $h(y) = -e^{y}$ $\quad$ $g(x) = -\lambda x$

$$\left(\frac{1-e^{-\lambda x}}{1-e^{-\lambda c}}\right)' = \frac{1}{1-e^{-\lambda c}} \cdot \left(1-e^{-\lambda x}\right)' = \frac{1}{1-e^{-\lambda c}} \left(-e^{-\lambda x} \cdot (-\lambda)\right) = \underline{\underline{\frac{\lambda \cdot e^{-x}}{1-e^{-\lambda c}}}}$$