



X. DEFINING YIELD STRESS AND FAILURE STRESS (STRENGTH)

One of the most obvious but still most difficult problems in dealing with matters of failure is that of defining the yield stress and the failure stress, commonly known as strength. These properties are needed to calibrate failure criteria. Such properties as modulus, yield stress, and strength are widely codified and quoted. Modulus E is straightforward but the terms yield stress and strength have a somewhat ambiguous history of use. In fact, there has not even been agreement on the proper designations for these properties. The terms yield point, proportional limit, yield strength have been used for the first property. For the failure stress such terms as ultimate strength, strength, rupture, and limiting stress and many more have been used. All of these have had rather different interpretations. None have been entirely satisfactory and none universally adopted.

Modern continuum mechanics does not offer any special insight on these matters nor does traditional mechanics of materials, which with considerable irony is also called strength of materials even though it has almost nothing to say about strength, Timoshenko [1]. In materials science, Cottrell [2] gives helpful discussions but does not give operational definitions. Hull and Bacon [3] state that “the yield stress is not unique” in recognition that the plastic deformation in metals due to dislocation flow is not a singular event but a diffuse process. Similarly for strength, the usual approach is to assign the stress at which the specimen ceases to function as being the strength. The term “ceases to function” usually remains undefined.

Perhaps as the result of these conceptual difficulties, modern usage has evolved into that of an arbitrary rule, the 0.2% strain offset rule for determining the yield stress of metals. For other materials there are not even arbitrary rules, there are only individual preferences and proclivities.

For perfectly brittle materials there is no problem. The yield stress is irrelevant and the strength is obvious. But for partially ductile or very ductile materials there is uncertainty and confusion about how to determine the yield stress and the strength.

It is quite apparent that to make the best use of failure criteria, it is necessary that they be implemented and supported by consistent and meaningful definitions of their calibrating properties and correspondingly of the interpretation of their results. To this end, rational definitions of yield stress and strength will now be sought.

Yield Stress Definition

Consider a typical, ductile material stress strain curve as shown in Fig. 1

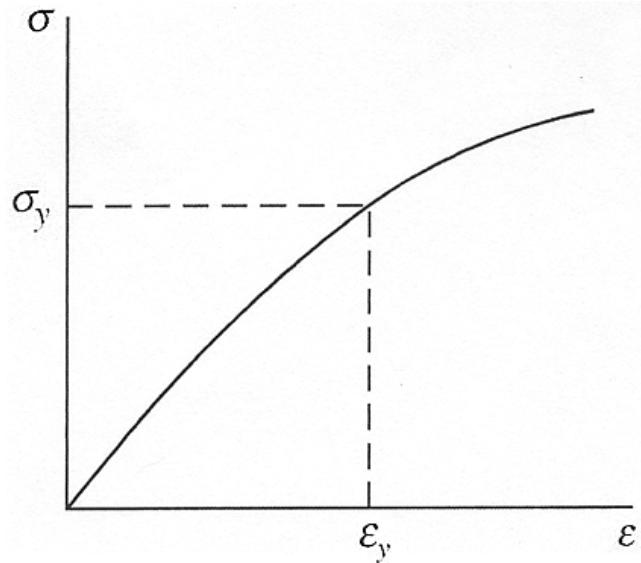


Fig. 1 Stress strain curve

The related constitutive form will be taken to be that of the strain hardening type and applicable to any standard test such as those for uniaxial tension, uniaxial compression, shear, or any proportional loading state. Now take the first and second derivatives of the stress strain curve in Fig. 1. These are shown schematically in Fig. 2

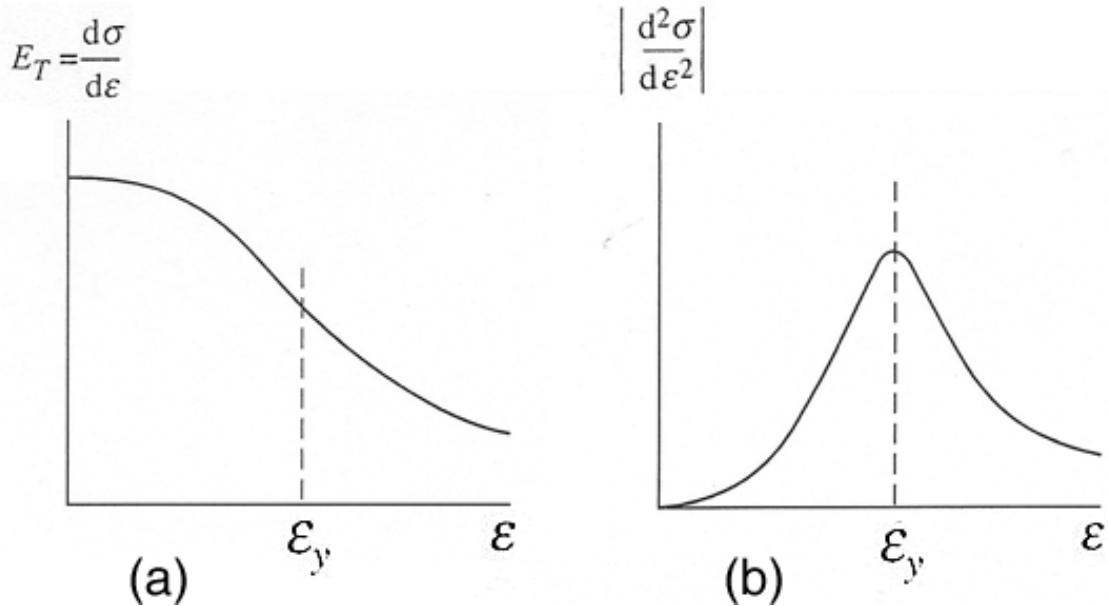


Fig. 2 Derivatives of stress strain curve

At the inflection point shown in Fig. 2a the second derivative in Fig. 2b reaches a maximum.

It is now seen that the strain $\epsilon = \epsilon_y$ in Fig. 2b at which the magnitude of the second derivative reaches a maximum is a transition point and the associated stress σ_y in Fig. 1 is the yield stress. Thus, this definition of the yield stress, σ_y , is the single point on the stress strain curve at which the tangent modulus, E_T , is changing at the greatest rate with respect to increasing strain. The criterion for the yield stress is then

$$\sigma = \sigma_y \quad \text{at} \quad \left| \frac{d^2\sigma}{d\epsilon^2} \right| = \text{maximum} \quad (1a)$$

or

$$\sigma = \sigma_y \quad \text{at} \quad \frac{d^3\sigma}{d\epsilon^3} = 0 \quad (1b)$$

This criterion for yield stress was given and examined rather recently by Christensen [4]. Surprisingly it apparently was never considered historically.

For ductile metals the location of the maximum of the second derivative is that condition at which the dislocation flow is sufficiently intense and varied as to cause this result. Such a process point involves both dislocation nucleation and the actual dynamics of the flow. For other materials such as glassy polymers, this point of the maximal second derivative correspondingly is caused by molecular rearrangement and damage at both the molecular and macroscopic scales.

The best way of viewing this yield behavior is that the first derivative $\frac{d\sigma}{d\epsilon}$ versus ϵ is constantly decreasing for increasing ϵ but it has an inflection point that identifies the transition point σ_y . This single, unique point then designates the transition from the previous nearly ideally elastic behavior to the following behavior approaching perfectly plastic flow. That the material may never truly attain the perfectly plastic state is usually caused by the intercession of the effects of flow anisotropy or localization etc.

To illustrate the process, an analytical form for typical stress strain curves will be taken so that the derivatives in (1) can be evaluated. Decomposing strain into elastic and plastic parts, take

$$\epsilon = \frac{\sigma}{E} - a \ln \left[1 - \left(\frac{\sigma}{\sigma_0} \right)^m \right] \quad (2)$$

where a , m , and σ_0 are parameters to be specified. It can be shown that as exponent m becomes very large (2) approaches that of the elastic-perfectly plastic case. Thus, except in this limiting case, the form (2) represents a continuous function with continuous derivatives in accordance with most physical observations.

Two examples will be given to show the evaluation of the yield stress using (1). The two examples with different values for exponent m are

(i)	(ii)
$E=70\text{GPa}$	$E=70\text{GPa}$
$\sigma_0=500\text{MPa}$	$\sigma_0=500\text{MPa}$
$a=0.01$	$a=0.01$
$m=5$	$m=10$

The corresponding second derivatives from (2) are shown in Fig. 3.

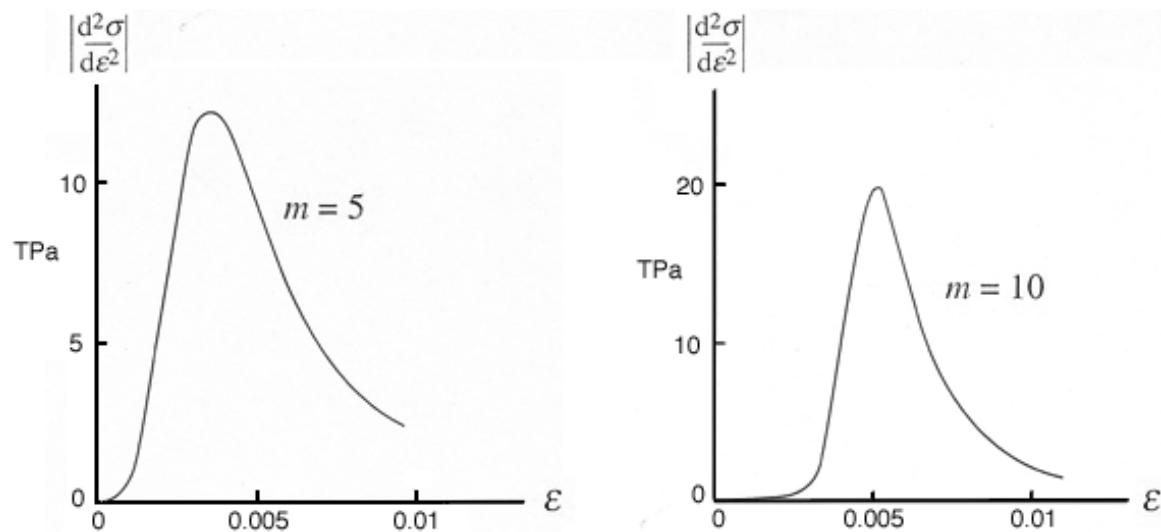


Fig. 3 Second derivatives for the examples

It is seen that these second derivatives give sharply defined maxima. The actual yield stresses were determined from (1b), the third derivative equal to zero

The full stress strain curves with the yield stresses determined by (1) are given in Fig. 4.

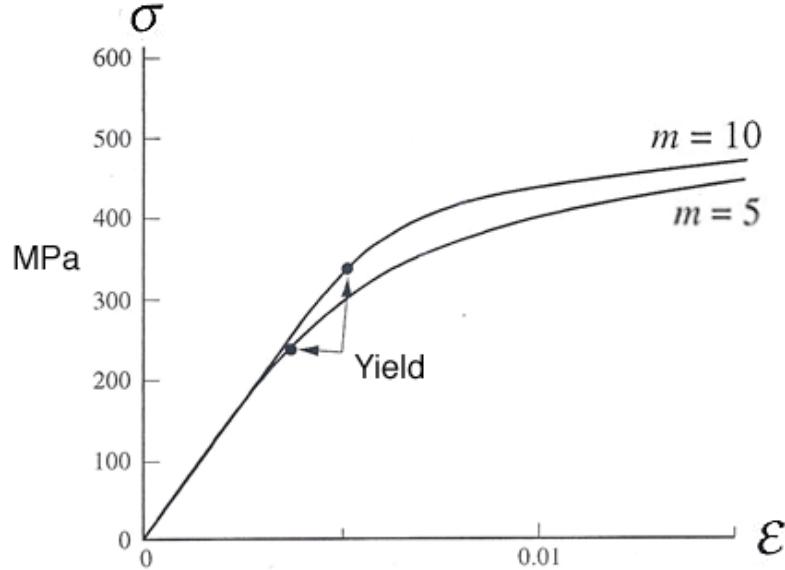


Fig. 4 Yield stresses for the examples

The yield stresses from (1) in the two examples are given by

$$m = 5 \quad \sigma_y = 236 \text{ MPa}$$

$$m = 10 \quad \sigma_y = 339 \text{ MPa}$$

It is apparent from Fig. 4 that the yield stress values are larger than the values at which nonlinearity can first be observed, but it can also be shown that they are far less than the values from the 0.2% strain offset rule. If the exponent $m=100$ were taken in these examples, the result would be almost indistinguishable from the elastic-perfectly plastic case.

This new method for determining the yield stress complies with the original objective to find a rational criterion for the operation. However, it has a major disadvantage that renders it as almost unusable. Typical stress strain curves are actually just sequences of data points. Reliably determining second and/or third derivatives from such typical data sets is prohibitively difficult. So the situation appears to have reduced to the

common observation that “the operation was a success but the patient may not survive”.

The practical means and method for determining the yield stress is still in an unsatisfactory and suspended state. Here is how to proceed further. The condition (1) is taken as the rigorous definition of the yield stress, even though it is difficult to implement. An approximation to the criterion (1) will be sought in a form that is easy and direct to use, but still retains the essential calibration of (1) in the stress strain problems of interest.

After examining several different forms the following was selected as a simple but reliable operational means of determining the yield stress. The yield stress is taken (designated) as that stress at which the actual strain is 5% greater than that of the linear elastic projection. This definition is

$$\sigma = \sigma_y \text{ at } \frac{\varepsilon_y - \varepsilon_L}{\varepsilon_y} = 0.05 \quad (3)$$

with ε_L being the linear elastic range strain shown in Fig. 5.

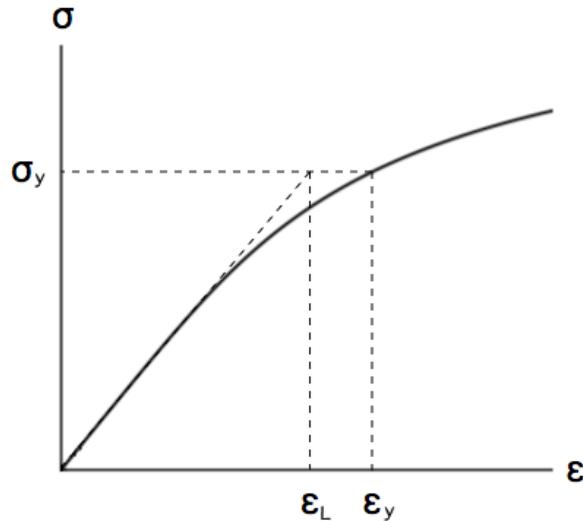


Fig. 5 Strain deviation condition

In the two examples considered earlier, the rigorous yield stress given by (1) then gives the strain form in (3) as

$$\text{For } m = 5 \quad \frac{\varepsilon_y - \varepsilon_L}{\varepsilon_y} = 0.066$$

and

$$\text{For } m = 10 \quad \frac{\varepsilon_y - \varepsilon_L}{\varepsilon_y} = 0.042$$

These values are close to the 0.05 value of the 5% rule. The 5% strain deviation rule given by (3) provides a reasonable and easy to use criterion for determining the yield stress. It must be remembered however that it is only a convenient approximation to the rigorous definition of yield stress in (1). In the examples the approximate yield stresses given directly by the 5% strain deviation rule (3) are

$$\text{For } m = 5 \quad \sigma_y \cong 220 \text{ MPa}$$

$$\text{For } m = 10 \quad \sigma_y \cong 347 \text{ MPa}$$

Definition of Strength

How should one define strength? Is it simply the stress at which a specimen ruptures or fragments into multiple pieces? If the material were very brittle then the resulting failure stress is obvious. When the material is not perfectly brittle the problem becomes much more complex

Sometimes at very large strains in uniaxial tension materials undergo anisotropic reorientation of the very small scale microstructures and very large stresses can be attained before final rupture. This one dimensional behavior probably has very little to do with usable strengths in three dimensional stress states. For example, polymers show strain induced extreme molecular orientation in 1-D but it is meaningless for 3-D stress states. This complicating but real condition must be dealt with.

First, pose the general situation as shown in Fig. 6.

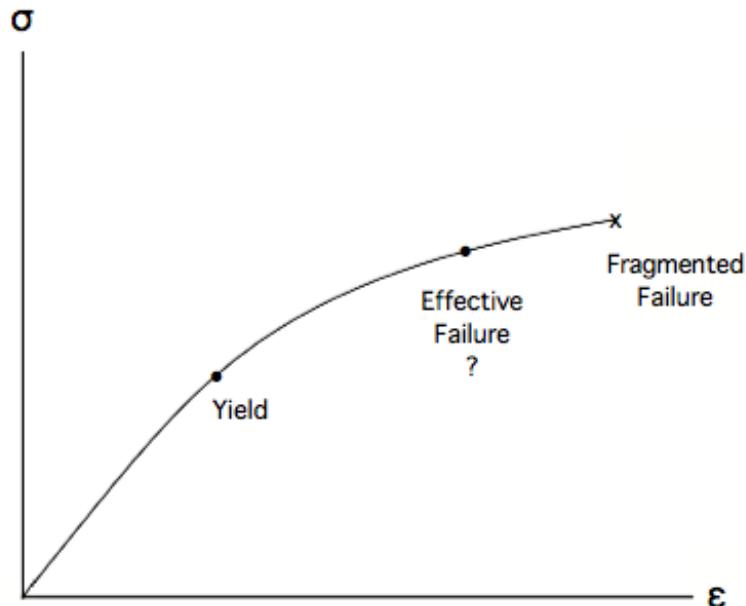


Fig. 6 Failure stress possibilities

From Fig. 6 it is surmised that yield stress as defined earlier is almost useless in specifying strength. They are completely independent properties.

To get a grasp on the failure problem start with the three idealized cases in Fig. 7.

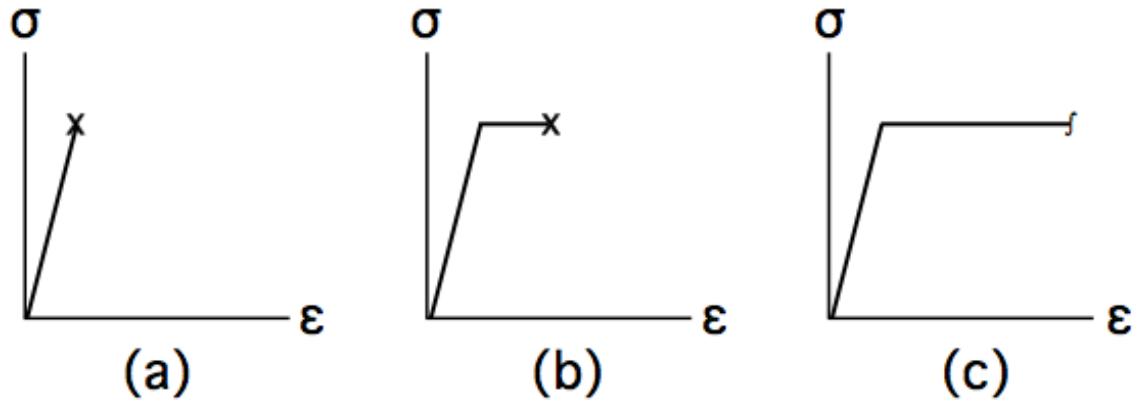


Fig. 7 Idealized stress strain cases

Case (a) is the perfectly brittle case and Case (c) is the perfectly ductile, perfectly-plastic case. What is Case (b)? Is it brittle or ductile or something in between?

Approach the problem by asking what is the failure strain in this idealized case that represents the inception of full ductility? Ultimately the answer to this question will help to define strength in the general situation.

Continuing with the idealized cases of Fig. 7, take the following designations shown in Fig. 8.

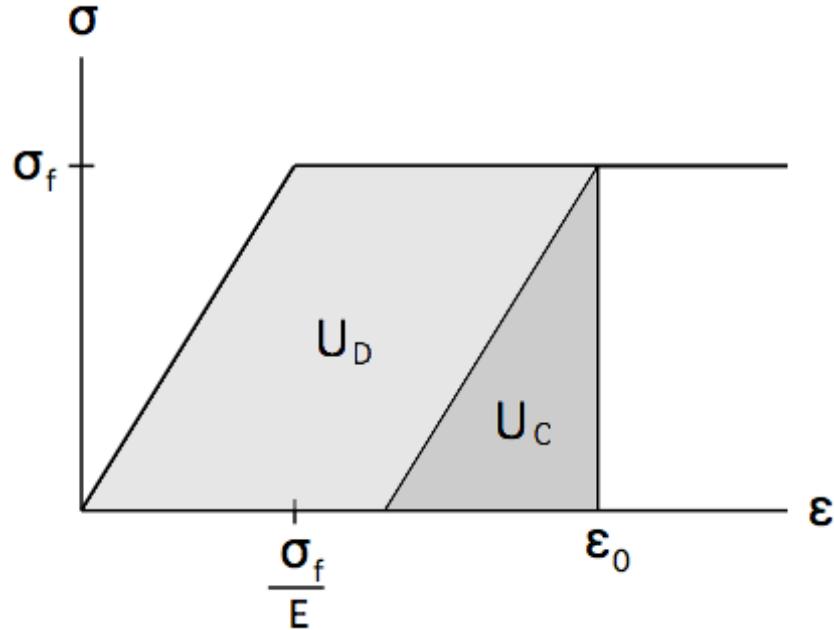


Fig. 8 Idealized cases nomenclature

U_C – *Conserved or recoverable energy*

U_D – *Dissipated energy*

The linear partition between the conserved and dissipated energies is that of the idealized unloading line controlled by the slope E.

Take,

$$(U_C - U_D)^2$$

as an indicator of the energy dominance or divergence for the two types of energy. Then

$$(U_C - U_D)^2 = U_C^2 \quad \text{at} \quad \varepsilon_0 = \frac{\sigma_f}{E}$$

$$(U_C - U_D)^2 = 0 \quad \text{at} \quad U_C = U_D$$

$$(U_C - U_D)^2 = U_D^2 \quad \text{at} \quad \varepsilon_0 \gg \frac{\sigma_f}{E}$$

where now ε_0 is taken to be the failure strain. Normalize the above expression by U_C^2 and define the energy dominance by

$$\Lambda = \left(1 - \frac{U_D}{U_C}\right)^2 \quad (4)$$

From Fig. 8 the two energy terms in (4) are given by

$$U_C = \frac{1}{2} \frac{\sigma_f^2}{E} \quad (5)$$

$$U_D = \sigma_f \left(\varepsilon_0 - \frac{\sigma_f}{E} \right)$$

Substituting (5) into (4) gives

$$\Lambda = \left[3 - 2 \frac{\varepsilon_0}{\left(\frac{\sigma_f}{E} \right)} \right]^2 \quad (6)$$

A graph of (6) showing the energy dominance Λ as a function of the failure strain ε_0 is in Fig. 9.

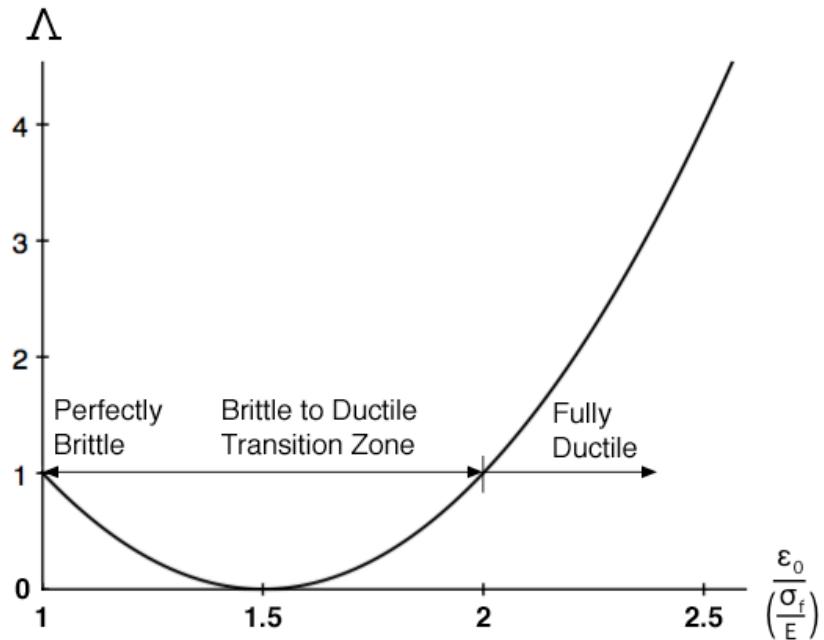


Fig. 9 Energy dominance versus failure strain

The brittle and ductile designations in Fig. 9 are with failure at $\varepsilon = \varepsilon_0$ in Fig. 8. The case of $\varepsilon_0 = \frac{\sigma_f}{E}$ in Fig. 9 is that of the perfectly brittle behavior. As ε_0 is increased beyond that, greater degrees of ductility are imparted to the material. The symmetry characteristic in Fig. 9 between

$$\varepsilon_0 = \frac{\sigma_f}{E} \quad \text{and} \quad \varepsilon_0 = 2 \frac{\sigma_f}{E}$$

is used to designate the later value of ε_0 as that at which full ductility is first attained.

$$\varepsilon_0 = 2 \frac{\sigma_f}{E} \quad \text{Inception of full ductility} \quad (7)$$

This is in accordance with the ductile/brittle failure behavior and characteristics found in Section VII. Beyond $\varepsilon_0 = 2 \frac{\sigma_f}{E}$ fully ductile states of failure are assured.

At the inception of fully ductile behavior, as defined above, then from (5)

$$U_c = \frac{1}{2} \frac{\sigma_f^2}{E} \quad (8)$$

$$U_D = \frac{\sigma_f^2}{E}$$

So at this start of fully ductile behavior there is

$$U_D = 2U_c \quad (9)$$

The original objective has been accomplished. For the idealized stress strain curve in Fig. 8 the failure strain has been found at which a state of full ductility can be designated. Furthermore, the dissipated and the conserved energies are found to satisfy (9) at this value of ε_0 giving the inception of full ductility.

Now the final step is taken to move beyond the idealized forms of Figs. 7 and 8. In actual behavior the effective failure stress is taken at the strain at which the state of full ductility first commences, as given by (9) from the idealized case. This general case is as shown in Fig. 10.

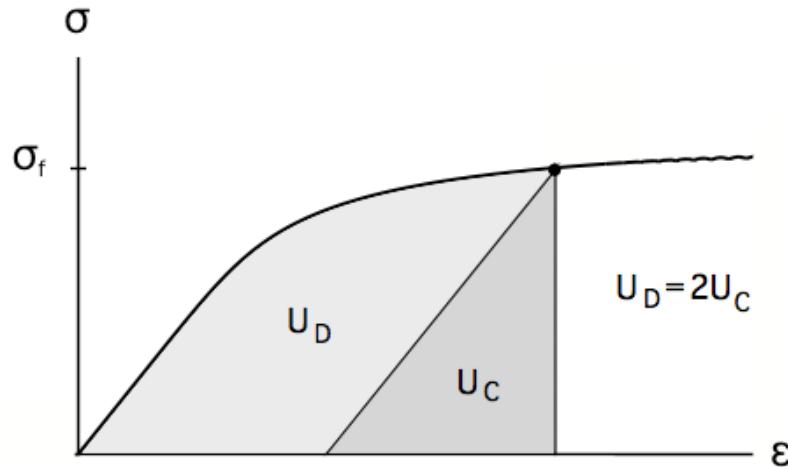


Fig. 10 Strength definition

The effective failure stress in the general case of Fig. 10 is defined by

$$\sigma = \sigma_f \quad \text{at} \quad U_D = 2U_c \quad (10)$$

where U_c and U_D are the conserved and dissipated energies shown in Fig. 10. If U_D never attains the value $2U_c$ before actual rupture then the failure stress is of course that at actual rupture. For ductile materials beyond the inception of full ductility, the material begins to develop “texture” which tends to give

false indications of usably increasing strength. In this case of ductile materials the failure stress shown in Fig. 10 is that which could be called the effective failure stress or strength since it is not the actual breaking of the specimen but rather a conceptual loss of material function. Although the terms conserved and dissipated energies are used here in the general case, they are motivated by the more precise definitions of them from the previous idealization in Fig. 8.

This definition of effective failure stress may occur before final rupture, or it may actually be that of final failure in the case of brittle behavior. Henceforth the term effective failure stress will be dropped and simply taken as the strength prescribed by the failure criterion (10). In using (10) the dissipated energy is that of all non-conservative energy expenditures through damage, dissipation and any other mechanisms.

The failure criterion (10) applies to most materials that possess a linear elastic range of behavior followed by yield and then by failure. This then excludes all elastomers, all polymeric materials above their glass transition temperatures and probably most or all biological materials. It does include all the usual engineering materials whether the ranges beyond the yield stress are strain hardening, or brittle, or almost brittle. Strain softening materials generally require special consideration although this definition of strength may usefully apply to some cases.

The criterion (10) applies to any stress state: tension, compression, shear or anything else, although it is of the most interest here in applications to uniaxial tension and compression for use in the failure theory. In this application to uniaxial tension and compression, and within the framework of isotropic failure theory there is always $T \leq C$. Equality occurs in the perfectly ductile case. Any apparent data that seems to suggest $T > C$ would not be admissible.

Significance and Conclusions

The yield stress is rigorously defined by (1), involving the maximum of the second derivative of the stress strain curve. Taking derivatives from data sets is not an easy to use or reliable procedure. Accordingly a related operational method for approximately determining the yield stress has been

found and verified. This method, embodied in (3) is that of the 5% strain deviation from linearity. It is very easy to use and far more realistic than the arbitrary 0.2% offset rule usually used with metals.

In contrast to the yield stress, the strength is derived from a drastically different energy type criterion that results in the definition

$$\sigma = \sigma_f \quad \text{at} \quad U_D = 2U_c$$

where U_c and U_D are the conserved and dissipated energies shown in Fig. 10. The consequence of this definition is that the effective failure stress for ductile materials exhibiting extensive strain hardening is not that of the final rupture stress, but rather is a somewhat lesser, intermediate value of stress. Following the reasoning used to derive this failure stress, it is really an effective failure stress, one that is physically transferable to use in other stress states through the failure criterion. This puts a wholly new and different interpretation on the definition of the strength. It is also interesting and revealing to note that this strength definition involves the elastic modulus E, or the modulus for whatever stress state being considered.

The yield stress and the strength definitions are appropriate to engineering materials, all of which have a linear range of elastic behavior. This is followed by irreversible damage or dissipation, ultimately leading to failure. The associated failure criteria are in effect the constitutive relations specifying the termination of the conservative and reversible range of the elastic behavior. Both the yield stress and strength definitions are physically direct, each is quite apparent on the usual computer graphics displays. The 5% strain deviation rule for yield stress is just a little above the apparent loss of linearity. The failure criterion $U_D = 2U_c$ for strength also is easily calibrated by appearance on the stress strain curve. Quantification of these follow directly.

The terms global and local have some meaning when considering these definitions of yield stress and strength. With regard to the full stress strain curve, the yield stress is a local property dependent upon the null value of the third derivative, as shown in the examples. The strength is a global property involving an integration operation over major portions of the stress

strain curve. When one sees both of these properties in this perspective it is immediately apparent that in most cases the strength property is of far more significance than is the yield stress property. Fortunately the strength definition only involves an integration operation that is very easy to use in quantifying its value from data.

It is not expected that these new definitions of yield stress and strength will ever come into general use. There is too much momentum embedded in the old intuitive and empirical preferences that are in general use to ever overcome that. It is strongly recommended however that in using the failure criteria derived here, they (the failure criteria) should utilize these rational definitions of failure stress or yield stress or other similarly carefully defined ones. Above all, there must be consistent usage of the definitions of yield stress and failure/strength when used as parts of a predictive failure theory

Finally, it is worth further emphasizing the difference between yield stress and strength. The failure criteria that are derived and used here are mainly intended for use with the specific conditions of failure, rather than with yielding. It may still be possible to use the failure criteria with considerations of yielding so long as both the calibrating properties and the predicted failure (yield) envelopes are consistently defined and interpreted.

References

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