Laplace Transforms for Systems

An Example

Laplace transforms are also useful in analyzing systems of differential equations. As we will see, the use of Laplace transforms reduces the problem of solving a system to a problem in algebra and, of course, the use of tables, paper or electronic. We illustrate with a simple example of an initial value problem for a 2×2 system.

$$\dot{x} = 2x + y, \quad x(0) = 1$$
$$\dot{y} = 3x + 4y, \quad y(0) = 0.$$

We begin by applying the Laplace tranform to both sides of *each* equation:

$$\mathcal{L}(\dot{x}) = 2\mathcal{L}(x) + \mathcal{L}(y)$$
$$\mathcal{L}(\dot{y}) = 3\mathcal{L}(x) + 4\mathcal{L}(y)$$

Setting $X(s) = \mathcal{L}(x(t))$ and $Y(s) = \mathcal{L}(y(t))$ and using the result that $\mathcal{L}(f')(s) = s\mathcal{L}(f)(s) - f(0)$, we find that the differential system is transformed into a pair of simultaneous linear algebraic equations for the functions X and Y:

$$sX(s) - 1 = 2X(s) + Y(s)$$
$$sY(s) = 3X(s) + 4Y(s).$$

which can be rearranged as

$$(s - 2)X(s) - Y(s) = 1$$
$$-3X(s) + (s - 4)Y(s) = 0$$

which has solutions

$$X(s) = \frac{(s - 4)}{(s - 1)(s - 5)}, \quad Y(s) = \frac{3}{(s - 1)(s - 5)}.$$

We now wish to find the inverse transform of the functions X and Y which will then be solutions of the system. To do this, we use a table of Laplace transforms after decomposing the expressions into sums of simple fractions. Using the method of partial fraction expansions leads to

$$X(s) = \frac{\frac{3}{4}}{(s-1)} + \frac{\frac{1}{4}}{(s-5)}$$

$$Y(s) = \frac{\frac{-3}{4}}{(s-1)} + \frac{\frac{3}{4}}{(s-5)}$$

The second entry in the table that we handed out can then be used to find the solution of the system:

$$x(t) = (\frac{3}{4})e^t + (\frac{1}{4})e^{5t}$$

$$y(t) = (\frac{-3}{4})e^t + (\frac{3}{4})e^{5t}$$