Differential Geometry – Classical and Discrete Surfaces – Introduction

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Introduction

We now concentrate on the differential geometry of surfaces, which not only represents the central part of any introductory course in Differential Geometry, it also provides most of the essential theoretical tools essential in Graphics, CAGD and Imaging.

- Our basic modus operandi, at least in the basic, incipient stages is largely that encountered in Calculus 2.
- That means that, at least in the beginning, we concentrate only on the *local theory* of surfaces.
- We need to deal with surfaces in a manner that allows for derivatives computation, i.e. that are smooth enough or, in technical term specific to the field *regular*.
- Furthermore, for practical computations we will need our surfaces to be precisely like those in Calculus 2, namely *parametrized surfaces* in Euclidean Space $\mathbb{R}^3$.

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Basics

Definition

A set $S \subset \mathbb{R}^3$ is called a (*regular*) *surface* if, for any $p \in S$, there exists a neighbourhood V of p in $\mathbb{R}^3$, and an open set $U \in \mathbb{R}^2$ together with a surjection $f : U \to V \cap S$, such that

1. $f \in \mathcal{C}^3$;
2. f is a homeomorphism ;
3. f is regular at any point $q \in U$, i.e. $df_q : \mathbb{R}^2 \to \mathbb{R}^3$ is injective.

Then f is called a *parametrization* or f^{-1} is called a (*local*) *coordinate system*[a] (which is also continuous), and $V \cap S$ is called a *coordinate neighborhood.*

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a. Sometimes the later name is attached, especially in Differential Topology, to f^{-1} .

Basics - cont.

Remark

The main subject of Differential Geometry is, as we already know, curvature, which can be seen as generalization or geometrization of the second derivative, thus it is not surprising that class $\mathcal{C}^2$ suffices for most results in the sequel. However, it would be false to presume this is always the case, and we shall point out in due time where the higher differentiability class is absolutely need.

Remark

Condition (3) is equivalent[a] to the condition that the Jacobian matrix $J_f(q)$ has rank 2 for any $q \in U$. This can be expressed in a more geometric manner as $\partial f/\partial u \times \partial f/\partial u \neq \bar{0}$. In other words, S admits a tangent plane at any point, thus it does nowhere reduce to a point or a curve (so it can't be expressed solely in terms of u or v).

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Basics - cont.

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Basics - cont.

Remark

Condition (2) ensures that S has no self intersections, while Condition (3) prevents q to be the vertex of cone (see also our previous remark).

Examples

The simplest examples are

1. A plane ;
2. An open disk.
   The first nontrivial example is given by
3. The sphere $\mathbb{S}^2$.

Basics - cont.

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Basics - cont.

The sphere - cont.

1. The simplest way is to consider the *stereographic projection* from the North Pole $N = N(0,0,1)$ and from the South Pole $S = S(0,0,-1)$. Recall that (standard) stereographic projection from the North Pole, $\Pi_N : \mathbb{S}^2 \to \mathbb{R}^2$, is given by

$$(x,y,z) \in \mathbb{S}^2 \longmapsto (u,v) \in \mathbb{R}^2\,;\; u = \frac{x}{1-z}\,, v = \frac{y}{1-z}\,.$$

Exercise

There is another version of the stereographic projection in which N is taken to be $(0,0,2)$ and $S = (0,0,0)$. Determine the formulas of Π_N and Π_S for this variant of the stereographic projection.

Basics - cont.

The sphere - cont.

1. Another method of finding a parametrization of the sphere, again with two coordinate neighborhoods, is the one encountered in Calculus 2, of geographical (and astronomical) inspiration, namely taking as one parametrization the mapping $\phi : (0,\pi) \times (0, 2\pi) \rightarrow \mathbb{S}^2$, given by

$$\phi(u,v) = (\sin v \cos v, \sin u \sin v, \cos u)\,.$$

2. The easiest to use in practice way to parametrize the sphere, in the sense that it is natural from its expression as $x^2 + y^2 + z^2$ uses **six** coordinate neighborhoods (thus emphasizing the fact that their number is not canonical). For example, $U_1 = x^2 + y^2 < 1$ and $f_1(x,y) = (x, y, \sqrt{1 - x^2 - y^2})$.

Basics - cont.

Clearly, the definition of a surface we brought above is not always an easy one to operate with in general. Therefore, one might hope to be able to view surfaces as graphs of functions (again, as one does in Calculus). This is, indeed, the case, and its proof is left as an exercise.

Exercise

Let $f : U \to \mathbb{R}$, $U = \mathrm{int}U \subset \mathbb{R}^2$ be a $\mathcal{C}^3$ function. Then its graph is a surface.

While this nomenclature is less commonly used, such surfaces are called *simple surfaces* and f is called a *Monge parametrization*. Note that any surface in $\mathbb{R}^3$ can be obtained by "gluing" (or "stitching") simple surfaces.

Basics - cont.

Another canonical and useful manner to obtain surfaces (again taught in Calculus courses) is to express them in terms of *regular values* of functions, which are defined as follows :

Definition

Let $F : U = \mathrm{int}\, U \subset \mathbb{R}^3 \to \mathbb{R}^3$ be a *diffeomorphism*. Then $p \in U$ is called a *critical point* (of or for F) if $d_pF : \mathbb{R}^3 \to \mathbb{R}^3$ is not surjective. In this case, $F(p)$ is called a *critical value*. Otherwise, p is called a *regular value*.

Remark

Clearly the definition can be extended for mappings from $\mathbb{R}^m$ to $\mathbb{R}^n$, for any m, n. To understand the inspiration for this theorem, the reader is invited to solve the following simple

Basics - cont.

Exercise

What are the critical points and values and the regular points in the case $m = n = 1$?

Remark

It turns there are "very few" critical values, thus "most" values are regular values. (This is a intuitive formulation of the classical Sard – or Morse-Sard Theorem.)

This last remark is extremely important in view of the following

Theorem

Let $f : U = \mathrm{int}\, U \subset \mathbb{R}^3 \to \mathbb{R}$ be a $\mathcal{C}^3$ function, and let $a \in f(U)$ be a regular value. Then $f^{-1}(a)$ is a regular surface (in $\mathbb{R}^3$).

Basics - cont.

The proposition above is extremely useful in practice, since it allows to easily produce regular surfaces, as the following examples demonstrate :

Examples

- *The ellipsoid* $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ *can be viewed as* $f^{-1}(0)$*, for* $f(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$.
- *The hyperboloid of two sheets* $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ *can be obtained as* $f^{-1}(0)$*, for* $f(x,y,z) = \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} - 1$.

Remark

This last example demonstrates that a surface is not necessarily connected. However, connected surfaces can be characterized by the property in the following elementary exercise :

Basics - cont.

Examples (Cont.)

Exercise

Prove that $f : S \subset \mathbb{R}^3 \to \mathbb{R}$ is a continuous function such that $f(p)f(q) > 0$, for any $p, q \in S$, where S is a connected surface, then $\mathrm{sign}\, f$ is constant.

- Let $\mathbb{R}^3 \to \mathbb{R}$, $f(x, y, z) = z^2 + (\sqrt{x^2 + y^2} - a)^2$. Then r^2 is a regular value for any $r < a$, thence the torus of revolution $z^2 = z^2 - (\sqrt{x^2 + y^2} - a)^2$ is a regular surface.

Exercise

Check the assertion above.

Basics - cont.

There also exists a local converse of Theorem 3 :

Proposition

Let $S \subset \mathbb{R}^3$ be a regular surface and let $p \in S$. Then there exists a neighbourhood $U \subset S$ of p, such that U is the graph of a function of one of the following types :

$$z = f(x, y)\,,\; y = g(x, z)\,,\; x = h(y, z)\,.$$

(As the reader might have noticed, this result is one encountered usually in Calculus 2.)

Exercise

Prove that the (one-sheeted) cone $z = \sqrt{x^2 + y^2}\,, (x, y) \in \mathbb{R}^2$, is not a surface.

Basics - cont.

The proposition above can be used to reduce the number the conditions in our definition of surfaces, as can be seen by solving the following

Exercise

1. *Show that Condition (2) in the definition of regular surfaces is redundant.*
2. *Using (1) above prove that the torus of revolution* $f(u,v) = ((R + r\cos u)\cos v, (R + r\cos u)\sin v, r\sin u)$ *is a surface.*

To conclude this primer on the notion of surface, as employed in Differential Geometry, we bring yet another definition, which expresses surfaces in terms of mappings into, rather then subset of $\mathbb{R}^3$:

Basics - cont.

Definition

A *parametrized surface* $f : U = \mathrm{int}\, U \subset \mathbb{R}^2 \to \mathbb{R}^3$ is a $\mathcal{C}^3$ mapping. f is called *regular* if its differential $d_p f : \mathbb{R}^2 \to \mathbb{R}^3$ is injective for all $p \in U$. A point $q \in U$, such that $d_q f$ is not regular is called a *singular point* (of/for f).

It is precisely this definition, which generalizes our definition of curves and, in turn, will be generalized itself later to yield the notion of manifold.

Remark

A regular parametrized surface can self intersect.

Basics - cont.

We wind up this succinct overview of the basic facts regarding surfaces with the following result which shows that we can actually consider, at least locally, regular parametrized surfaces instead of surfaces as defined before :

Proposition

Let $f : U = \mathrm{int}\, U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a regular parametrized surface, and let $p \in U$. Then there exists a neighbourhood $V \in U$ of p, such that $f(V) \subset \mathbb{R}^3$ is a surface.

Remark

We restricted here only to the bare minimum. Other ideas and definitions, such as the tangent plane [a] will be introduced when necessary.

a. Recall that this specific notion was also introduced in Calculus 2.