Differential Geometry – Classical and Discrete

The Tube Formula and Approximation of Surface Curvatures

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Introduction

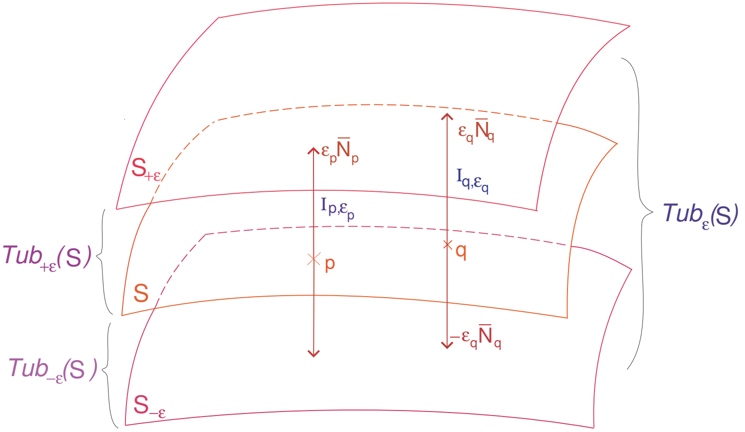
- Recall that, while there are many successful ways of computing (or, rather, approximating) Gauss curvature, the same can not be said about mean curvature.
- This is hardly surprising, given that mean curvature is *extrinsic*, in contrast with Gauss curvature (therefore purely metric, for instance are difficult, if not impossible, to apply).
- Partly to illustrate this phenomenon, but also for the sake of the method itself and the ideas and theories behind it we bring below what we believe it is the simplest, computationally method (yet hardly trivial at a theoretical level).
- This method of approximating the mean curvature is based upon the so called *Tube Formula*.

Tubes

Let us start with some necessary definitions and notation :

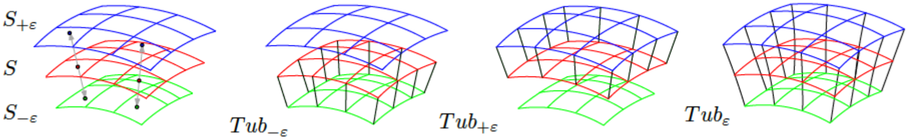
- Let $S \subset \mathbb{R}^3$ be an orientable surface and let N_p denote, as usual, the unit normal of S at p.
- For each $p \in S$ consider, in the direction of N_p, the open symmetric interval of length $2\varepsilon_p,\ I_{p,\varepsilon_p}$, where ε_p is chosen to be small enough such that $I_{p,\varepsilon_p} \cap I_{q,\varepsilon_q} = \emptyset$, for any $p, q \in S$ such that $||p - q|| > \xi \in \mathbb{R}_+$.
- Then, $\mathrm{Tub}_\varepsilon(S) = \bigcup_{p \in S} I_{p,\varepsilon_p}$ is an open set that contains S and such that for any point $x \in \mathrm{Tub}_\varepsilon(S)$, there exists a unique normal line to S through x. $\mathrm{Tub}_\varepsilon(S)$ is called a *tubular neighborhood* of S or just a *tube*.
- The two surfaces $S_{\pm\varepsilon} = S \pm \varepsilon \bar{N}$ are called the *offset surfaces* of S with offset distance ε. We shall consider sets of constant *offset* ε.

Tubes - cont.



$S_{\pm\varepsilon} \overset{not}{=} S \pm \varepsilon_p \bar{N}_p$ (Offset Surfaces)

Tubes - cont.



$S_{+\varepsilon}$
S
$S_{-\varepsilon}$
$Tub_{-\varepsilon}$
$Tub_{+\varepsilon}$
Tub_{ε}

Tubes - cont.

- We can derive the following formula :

$$Vol\big(Tub_\varepsilon(S)\big) = 2\,\varepsilon Area(S) + \frac{2\varepsilon^3}{3}\int_S KdA$$

- If S is compact, we get – by applying the Gauss-Bonnet Theorem :

$$Vol\big(Tub_\varepsilon(S)\big) = 2\,\varepsilon Area(S) + \frac{4\pi\varepsilon^3}{3}\chi(S)$$

where $\chi(S)$ represents the Euler characteristic of S.

Tubes - cont.

- Note the absence of H in these formulae. Thus, the Tube Formula cannot be employed to compute the mean curvature.
- Moreover, in the case of triangulated surfaces computing K by means of the Tube Formula reduces to approximating $K(p)$ by the angle defect at the point p.
- Nevertheless, not everything is lost, since for *Half-Tubes* the following formula holds :

$$Vol(Tub_{\pm\varepsilon}(S)) = \varepsilon Area(S) \mp \varepsilon^2 \int_S HdA + \frac{\varepsilon^3}{3} \int_S KdA$$

Tubes - cont.

- What actually allows one to employ the Half Tube Formula in the computation of H are its roots :
  - The Tube Formula is, in fact, a generalization of the classical

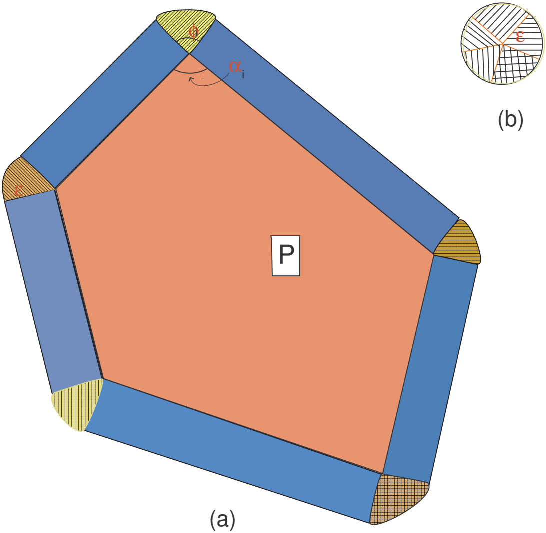
Theorem (Steiner-Minkowski Theorem)

Let $P \subset \mathbf{R}^n$, $n = 2, 3$ be a compact, convex polyhedron and let $N_\varepsilon(P) = \{x \in \mathbf{R}^n \,|\, dist(x, P) \leq \varepsilon\}$, $n = 2, 3$.

If $n = 2$, then

$$Area(N_\varepsilon(P)) = Area(P) + \varepsilon Length(\partial P) + \pi\varepsilon^2$$

where ∂P denotes the perimeter of P.



(a)

Tubes - cont.

If $n = 3$, then

$$Vol(N_\varepsilon(P)) = Vol(P) + \varepsilon Area(\partial_2 P) + C\varepsilon^2 Length(\partial_1 P) + \frac{4\pi\varepsilon^3}{3}$$

where :

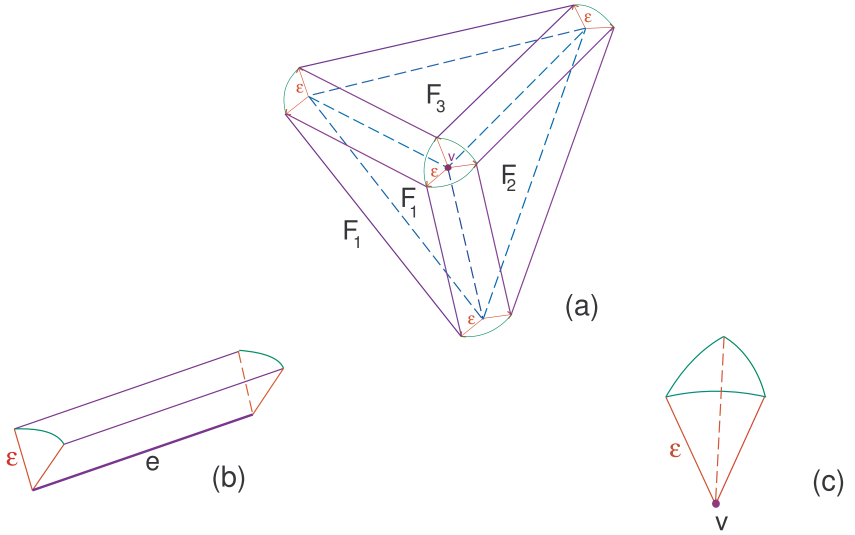
• $\partial_2 P$ denotes the faces of P

• $\partial_1 P$ denotes the edges of P

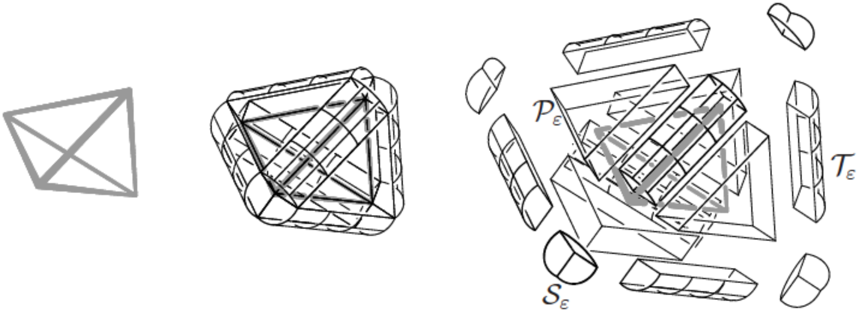
• the last term contains the 0-dimensional volume contribution of the vertices of P, where • $Vol(\partial_0 P) = \mid V_P \mid$

and where $C = C(P)$ is a scalar value that encapsulates $\int\int_S H$ and that essentially depends on the dihedral angles of P.

Tubes - cont.

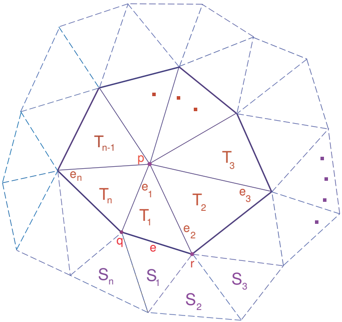


Tubes - cont.



(a) (b) (c)

- This last formula extends also to any piecewise linear surface (not only to convex ones), *with or without boundary*, thus enabling us to refine the results obtained so far by extending our method to the second [1] ring of triangles :



1. or even more rings

Tubes - cont.

- Consider – for instance – the edge e_1 :
  - It is incident to both p and q, therefore its contribution to $H(p)$ will be half of the dihedral angle associated with it.
  - Moreover, since the boundary edge e is common to T_1 and the second-ring triangle S_1, it's contribution to each of the triangles is half of the associated dihedral angle.
- Thus we receive the following formula for the computation of $H(p)$:

$$(\bigstar) \quad H(p) = \frac{1}{2}\sum_{i=1}^{n}\varphi(T_i, T_{i+1})e_i + \frac{1}{4}\sum_{i=1}^{n}\varphi(T_i, S_i^*)e_i$$

where S_i^* is the second ring triangle having in common with T_i the edge e_i and where, for instance, $\varphi(T_i, T_{i+1})$, $\varphi(T_i, S_i^*)$ denote the dihedral angles between the triangles T_i and T_{i+1} and between T_i and S_i^*, respectively.

Tubes – Algorithm

Algorithm 1 Estimates the mean curvature at vertex p

$RingArea \Leftarrow 0;$ ▷ The area of the first ring around vertex p
$ContribSum \Leftarrow 0;$ ▷ The sum of the contributions of the triangles from the first ring around vertex p
$n \Leftarrow |Ring_1(p)|;$ ▷ Number of triangles in the first ring around vertex p
for $i \leftarrow 0$ **to** $n-1$ **do**
  $RingArea \mathrel{+}= Area(T_i);$
  $ContribSum \mathrel{+}= \varphi(T_i, T_{(i+1) \bmod n})|e_i| + \frac{1}{2}\varphi(T_i, S_i)|\overline{e_i}|;$
end for
Return $\frac{ContribSum}{2RingArea};$

Tubes – Algorithm (cont.)

- If the user requires the mean curvature for many vertices of the model, this algorithm can be implemented more efficiently as follows :
  - First calculate the contribution of each edge : For each edge, e_i, that is adjacent to triangles T_i and T_{i+1}, we assign an attribute called *EdgeContrib* that is equal to $\frac{1}{4}\varphi(T_i, T_{i+1})|e|$.
  - Then, for each triangle we assign an attribute called *TriangleContrib* that is the sum of all *EdgeContrib* of its' edges.
  - Finally, we can compute the mean curvature at vertex p using the following improved algorithm :

Tubes – Algorithm (cont.)

Algorithm 2 Estimates the mean curvature at vertex p. Efficient version.

$RingArea \Leftarrow 0$; ▷ The area of the first ring around vertex p

$ContribSum \Leftarrow 0$; ▷ The sum of the contributions of the triangles from the first ring around vertex p

$n \Leftarrow |Ring_1(p)|$; ▷ Number of triangles in the first ring around vertex p

for $i \leftarrow 0$ to $n-1$ **do**

  $RingArea \mathrel{+}= Area(T_i)$;

  $ContribSum \mathrel{+}= T_i.TriangleContrib$;

end for

Return $\frac{ContribSum}{2RingArea}$;

Experimental Results

We tested the Half Tube Formula based Algorithm on triangular meshes that represent tesselations of the following 7 synthetic models of NURB surfaces :

- A Cylinder
- A Cone
- A Sphere
- An Ellipsoid
- A Hyperbolic Surface of Revolution
- The Body

  and

- The Spout

  of

  The Infamous Utah Teapot

Experimental Results - cont.

- The tesselations of each model were produced for several different resolutions :

  From $\approx$ 100 triangles to $\approx$ 5000 triangles,
- This allows us to gain some insight into the the convergence rate of the algorithms.
- We compared our Algorithm's performance with those of the following previously tested ones :
  - Gauss-Bonnet /Angle Deficiency
  - Taubin
  - Watanabe

  and the classical
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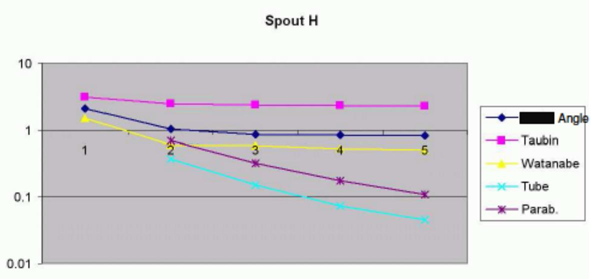
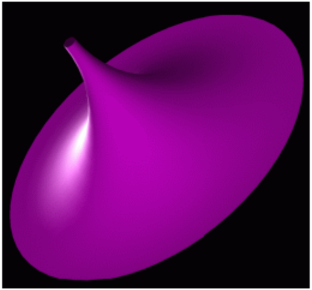
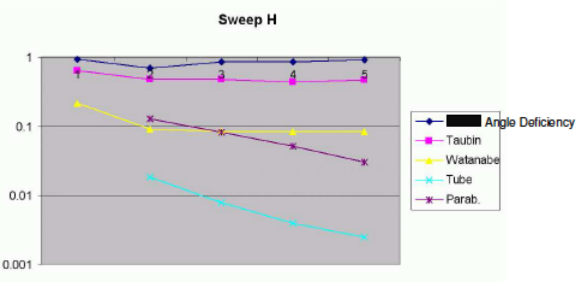
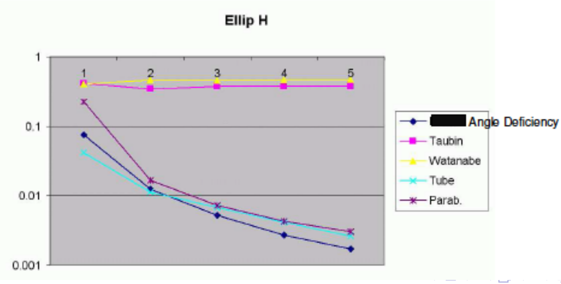


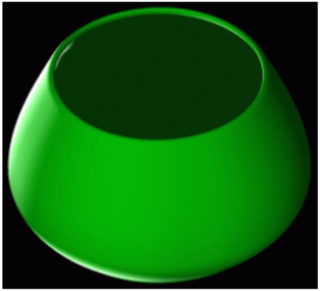


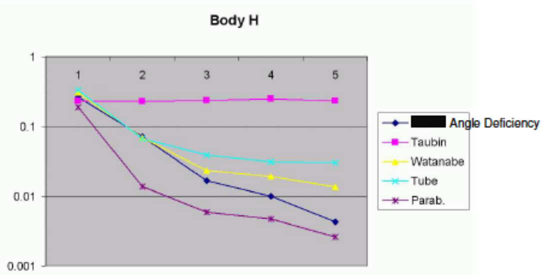

Sweep H
1
0.1
0.01
0.001
1
2
3
4
5
[redacted] Angle Deficiency
Taubin
Watanabe
Tube
Parab.

Experimental Results - cont.



Experimental Results - cont.

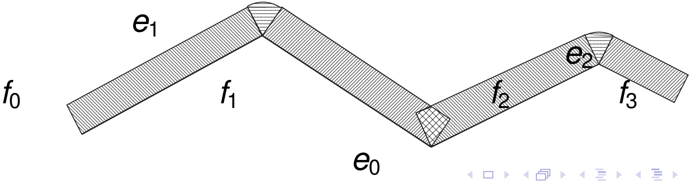




Experimental Results - cont.

- Conclusion
  - While *good results* in general for H
  - *Best results* for Hyperbolic Surfaces or mixed sign Gauss curvature.

    This is due, apparently, to the averaging effect of the tubes (more generally, *normal cycles*, which represent a generalization of Steiner's approach on convex polyhedra to a much larger class of geometrical objects, in particular to smooth and piecewise-linear manifolds in any dimension) above edges adjacent to vertices of negative curvature



Applications

- An immediate practical application of our method of approximating mean curvature is for the computation for of *Willmore (elastic) energy* of triangulated surfaces, where the the Willmore $W(S)$ energy of be a smooth, compact surface S in $\mathbb{R}^3$ is defined as :

Wilmore Energy

$$W(S) = \int_S H^2 dA - \int_S K dA\,.$$

Remark

From the very definitions of H and K it follows immediately that

$$W(S) = \frac{1}{4}\int_S (k_1 - k_2) dA\,.$$

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Applications - cont.

- Since, as we shall prove later on, $\int_S KdA$ is a constant that depends only on the *genus* of the closed surface S, it follows that, for all practical purposes, suffices to compute

$$W_0(S) = \int_S H^2 dA\,.$$

- It turns out that the Willmore energy is a *conformal invariant*, i.e. it is invariant under *conformal* (i.e. *angle preserving*) mappings/
  - Therefore, comparing the Willmore energy under before and after deformation (of a given triangular mesh) would represent a measure of the departure of this deformation from being conformal.

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Applications - cont.

- A first important application of this fact (and of or algorithm) is in Medical Imaging.
- Furthermore, its computation, as well as of the related *Willmore flow*

$$\frac{dS}{dt} = -\nabla W(S)\,;$$

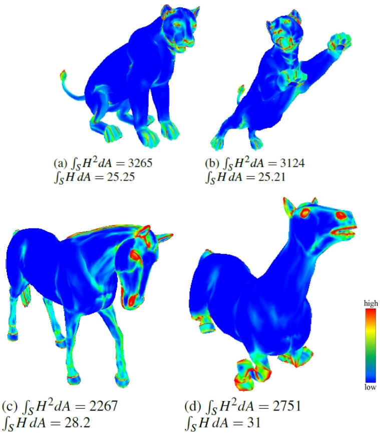
where dS denotes the differential of f, $S = f(u, v)$, is also relevant in a variety of fields and their applications, such as in

  - Conformal Geometry
  - Variational Surface Modeling
  - Thin Structures

Applications - cont.

- While the conformal invariance of the Willmore energy – thus of $\int_S H^2 dA$ – is a classical and well known fact, is a much more recent and surprising result of Almgren ('99) that the integral $\int_S H$ is, in fact, invariant under bending.
- Therefore, the computation of this simple integral is useful useful in applications involving *PL isometric embeddings* and as measure of the departure of deformations from being isometric.

Applications - cont.



(a) $\int_S H^2 dA = 3265$
$\int_S H\, dA = 25.25$

(b) $\int_S H^2 dA = 3124$
$\int_S H\, dA = 25.21$

(c) $\int_S H^2 dA = 2267$
$\int_S H\, dA = 28.2$

(d) $\int_S H^2 dA = 2751$
$\int_S H\, dA = 31$

Above : $\Delta W = 4.3\%$, $\Delta \int H = 0.1\%$. Below : $\Delta W = 21\%$, $\Delta \int H \simeq 10\%$.
It follows that the deformation the “Panther” is almost isometric (and conformal), while the one of the “Horse” fails on both accounts.

Applications - cont.

Applications - cont.

Applications - cont.