Differential Geometry – Classical and Discrete

Geodesics and Curvature

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Preamble

- Recall that, if K has constant sign, then K's expression in geodesic polar coordinates is

$$K = -\frac{(\sqrt{G})_{rr}}{\sqrt{G}}\,,$$

and, moreover

$$\lim_{\rho\to 0}(\sqrt{G})_{\rho} = 1\,,$$

and

$$\lim_{\rho\to 0}(\sqrt{G})_{\rho\rho} = -K\sqrt{G}\,.$$

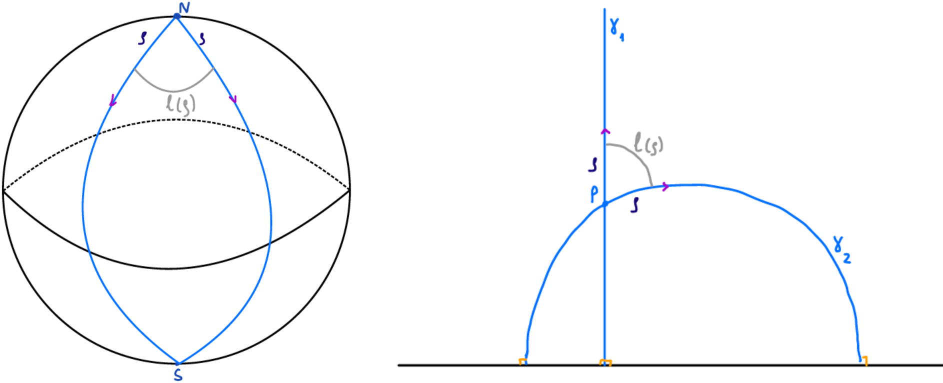
Preamble - cont.

- Moreover, if $\ell(\rho)$ denotes the length of the arc on the curve $\{\rho = \mathrm{const.}\}$ comprised, at distance ρ from a point of intersection between the geodesics γ_0, γ_1, be two geodesics determined the angles ϕ_0 and ϕ_1, respectively, then

$$\ell(\rho) = \int_{\phi_0}^{\phi_1} \sqrt{G(\rho, \phi)}d\phi\,.$$

- Therefore, if
  - $K < 0$, $\ell(\rho)$ increases with ρ, that is γ_0 and γ_1 grow farther and farther away from each other.
  - If $K > 0$, then typically the geodesics get closer to each other, however this is not always the case, depending on K.

Preamble - cont.



Left : Geodesics on the sphere starting from the North Pole first separate, maximum distance between them being attended on the equator, then they converge again to meet again at the South Pole.

Right : Geodesics in the Hyperbolic Plane keep diverging.

This first observation can be made more precise as follows :

Theorem (Bertrand-Puiseux, 1848)

$$\text{length}\, C(p,\varepsilon) = 2\pi\varepsilon - \frac{\pi}{3}K(p)\varepsilon^3 + o(\varepsilon^3)\,,$$

This theorem clearly admits a straightforward generalization to segments of circle of measure (central angle α) :

$$\text{length}\, C(p,\varepsilon,\alpha) = \alpha\varepsilon - \frac{\alpha}{3\sin^2\alpha}K(p)\varepsilon^3 + o(\varepsilon^3)$$

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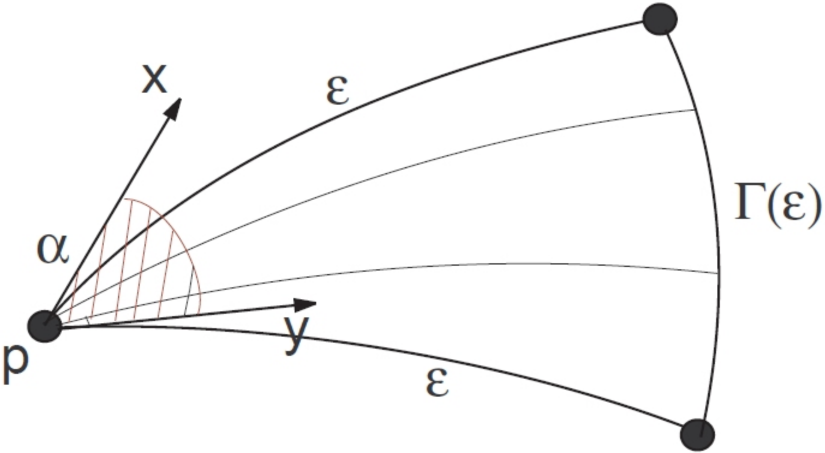
x
ε
Γ(ε)
α
p
y
ε

Furthermore, one quite naturally [1] can prove a similar result

Theorem (Diquet, 1848)

$$\text{area}\, B(p,\varepsilon) = \pi\varepsilon - \frac{\pi}{12}K(p)\varepsilon^4 + o(\varepsilon^4)\,.$$

- The role of the sign of K becomes now quite manifest :
  - In the vicinity of a point of positive curvature (hence on a surface with $K > 0$) circles are both shorter and "less massive" than their Euclidean counterparts ;
  - In the neighbourhood of a point of negative curvature (thus, a fortiori, on a surface with $K < 0$), they are longer and "heavier" than euclidean circles (and disks) having the same radius.

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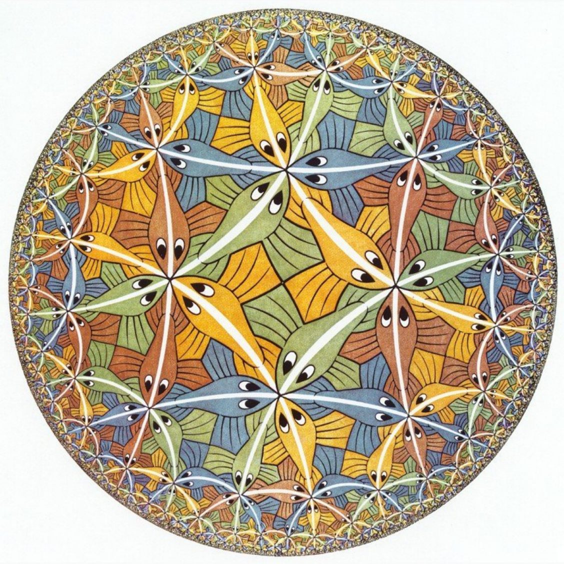

- However, a stick points in two opposite directions, and we can express, by virtue of the Theorems of Bertrand-Puiseux and Diquet, Gaussian curvature at a point both in terms of length and aria of a geodesic circle :

$$K(p) = \lim_{\varepsilon \to 0} \frac{3}{\pi} \frac{2\pi\varepsilon - \mathit{length}\, C(p,\varepsilon)}{\varepsilon^3} = \lim_{\varepsilon \to 0} \frac{12}{\pi} \frac{\pi\varepsilon^2 - \mathrm{area}\, B(p,\varepsilon)}{\varepsilon^4}$$

- Thus the formulas of Bertrand-Puiseux and Diquet can be used to *define* curvature in terms of lengths and areas of "small" circles/disks.
- This is extremely pertinent in discrete contexts, when there are no true ε-circles exist.

- This is far better understood (that not always realized) as the small, classical "puzzle" below illustrates.

How would the circle look if $\pi = 3$?

- The typical such setting is that of graphs/networks, where $\varepsilon = 1$, i.e. one use at the 1-link of a vertex.

  If the networks under scrutiny are (edge and/or vertex) weighted, one can define a discrete curvature at a vertex that takes into account the weights of the nodes/edges, viewed as measures, i.e. generalized areas.

  In fact, the extensions to higher dimensions of these theorems allow us to define notions of curvature in such spaces.

- We now return to the proofs of the Bertrand-Puiseux and Diquet Theorems.

  We begin with the following

Lemma

In geodesic polar coordinates

$$\sqrt{G} = r - \frac{r^3}{6}K(r) + R\,,$$

where $\lim_{r\to 0}\frac{R(r,\phi)}{r^3} = 0$ *uniformly in* ϕ.

Proof

Since

$$K = -\frac{(\sqrt{G})_{rr}}{\sqrt{G}}\,,$$

it follows that

$$\frac{\partial^3(\sqrt{G})}{\partial r^3} = -K(\sqrt{G})_r - K_r(\sqrt{G})\,,$$

and, since as we have seen, $\lim_{r\to 0}\sqrt{G} = 0$, also that

$$-K(p) = \lim_{r\to 0}\frac{\partial^3(\sqrt{G})}{\partial r^3}\,.$$

By developing $\sqrt{G} = \sqrt{G(r,\phi)}$ into Maclaurin series and substituting into it the obtained formulas for G and its derivatives, we obtain the required formula (R being the remainder in the Maclaurin series).

$\square$

We can now give the proofs of the theorems themselves.

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Proof of Bertrand-Puiseux Theorem

For $r = \varepsilon$, a simple computation gives

$$\text{length}\, C(p,\varepsilon) = \int_0^{2\pi} \sqrt{G(r,\phi)}d\phi = 2\pi r - \frac{\pi}{3} r^3 K(p) + o(\varepsilon^3)\,,$$

where $\lim_{\varepsilon \to 0} \frac{o(\varepsilon^3)}{\varepsilon^3} = 0$.

□

Proof of Diquet Theorem

A similar computation gives

$$\text{area}\, B(p,\varepsilon) = \int_0^{2\pi}\int_0^{\varepsilon} \sqrt{G(r,\phi)}drd\phi = \int_0^{2\pi}\int_0^{\varepsilon} \left(r - \frac{1}{6}K(p)r^3\right)drd\phi$$

$$+ \int_0^{2\pi}\int_0^{\varepsilon} o(r^3)drd\phi = 2\pi\left(\frac{1}{2} - \frac{K(p)}{24}r^4\right) + o(r^4)\,,$$

whence the desired formula immediately follows.

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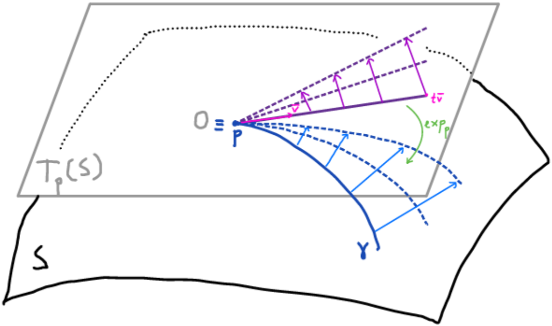
- This approach can be augmented, allowing us to obtain even stronger results by making appeal to the so called *Jacobi* (*vector*) *fields* that are obtained by the *variation* of a geodesics.
    - This is a notion that we do not make more precise here, since even the basic discussion would become too technical, but we rather rely on the suggestive figure below.



- Note that the vector field $\bar{\mathbf{y}}(t)$ is orthogonal along the geodesic γ if the initial value $\bar{\mathbf{y}}(0)$ is.
- Far less intuitive, but easier to operate with, certainly in the classical, smooth surfaces setting, is the characterization of Jacobi fields as solution of the *Jacobi equation* :

$$\ddot{\bar{\mathbf{y}}}(t) + K(\gamma(t))\bar{\mathbf{y}}(t) = 0\,.$$

Exercise

Write the the Jacobi equation for the geodesics on a sphere of radius R and determine its general solution.

- While clearly not intuitively pleasing, the characterization above has the advantage of emphasizing the role of Gauss curvature in the behaviour of families of geodesics.

- An other advantage is the fact that (15) is an ordinary differential equation [2], thus one can apply staple results on such equations to prove

Proposition

Given a geodesic γ, there exists precislely one Jacobi field $\bar{\mathbf{y}}(t)$ along γ, with initial conditions $\bar{\mathbf{y}}(0) = \bar{\mathbf{y}}_0$ and $\dot{\bar{\mathbf{y}}}\,(0) = \dot{\bar{\mathbf{y}}}_0$.

Furthermore, $\bar{\mathbf{y}}(t)$ can be realized by a one parameter family $\{\gamma_\alpha\}$ of geodesics.

2. an *Euler equation*, more precisely

On surfaces of constant curvature K the Jacobi field equation has the the special, standard form

$$\ddot{\bar{\mathbf{y}}}\,(t) = -K\bar{\mathbf{y}}(t)\,,$$

which, for the (canonical) initial conditions $\mathbf{y}(0) = \bar{0}$ and $||\bar{\mathbf{y}}(0)|| = 1$ has the well known solutions

$$\bar{\mathbf{y}}(t) = \begin{cases} \frac{\sin\sqrt{K}t}{\sqrt{K}} & K > 0\,; \\ t\mathbf{y} & K = 0\,; \\ \frac{-\sinh\sqrt{-K}t}{\sqrt{-K}} & K < 0\,. \end{cases}$$

(Note the similarity with some of the formulas used in the chapter on metric Gauss curvature.)

From here the expression of the length element in geodesic coordinates is easily to obtain :

$$ds^2 = d\rho^2 + \begin{cases} \frac{\sin^2 \sqrt{K}\rho}{K} d\sigma^2 & K > 0\,; \\ \rho^2 d\sigma^2 & K = 0\,; \\ \frac{-\sinh^2 \sqrt{-K}t}{K} d\sigma^2 & K < 0\,. \end{cases}$$

where $d\sigma$ denotes the metric of the unit circle $\mathbb{S}^1 \subset T_p(S)$.

Remark

Strangely enough, given that geodesics themselves have only rather late been treated with success in Graphics and its related fields, Jacobi fields have been discretized a long time ago (by Stone), precisely in the PL setting on which the field of Graphics rests.

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• Another classical comparison result goes back to Gauss.

• Motivated by geodesy, he wished to compare the angles and areas of geodesic triangle on a surface S with that of its Euclidean model.

• More precisely, given a geodesic triangle $T \in S$, sides a, b, c and angles α, β, γ (accordingly) consider the triangle $T' \in \mathbb{R}^2$, with sides equal to those of T, and corresponding angles α', β', γ'.

Then the angle α' is given by the following formula :

$$\alpha = \alpha' + \frac{\sigma}{12}(2K(p) + K(q) + K(r)) + o(a^4 + b^4 + c^4)\,;$$

where, to preserve Gauss' notation, we set $\sigma = \mathrm{Area}(T)$.

Clearly, one can write the similar formulas for β' and γ'.

Note that, for $K \equiv$ const., e.g. for $K \equiv 1/R^2$ (that is, for S being a sphere of radius R), the formula above reduces to the *Legendre formula*

$$\alpha = \alpha' + \frac{\sigma}{3}K + o(a^4 + b^4 + c^4)\,,$$

which was already classical even in Gauss' time.

Moreover, he obtained a similar comparison result for the areas :

$$\sigma = \sigma'\left[K(p)(s-a^2) + K(q)(s-b^2) + K(r)(s-c^2)\right] + o(a^4+b^4+c^4)\,;$$

where $\sigma' = \mathrm{Area}(T')$.

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Remark

Clearly, the formulas above are both of high potential applicative value.[a]

a. This is quite expected, given they were motivated, as noted above, by a practical problem.

The solution for te length element obtained above can be extended to include the case of varying curvature, which again hold in any dimension, as shown in the following pair of celebrated theorems due to Rauch, formulated here for the 2-dimensional case :

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Theorem (Rauch 1)

Let S be a surface such that having Gauss curvature bounded from below, that is $K \geq K_1$, for some K_1, and let $\bar{\mathbf{y}}(t)$ be a solution of

$$\ddot{\bar{\mathbf{y}}}(t) = -K\bar{\mathbf{y}}(t)\,,$$

satisfying the initial conditions $\mathbf{y}(0) = \bar{0}$ and $||\bar{\mathbf{y}}(0)|| = 1$.

Then

$$||\bar{\mathbf{y}}(t)|| \leq \begin{cases} \dfrac{\sin t\sqrt{K_1}}{\sqrt{K_1}} & K_1 > 0\,; \\ t & K_1 = 0\,; \\ \dfrac{\sinh t\sqrt{-K_1}}{\sqrt{-K_1}} & K_1 < 0\,. \end{cases}$$

which holds if $\bar{\mathbf{y}}(t) \neq \bar{0}$, for any $s \in (0, t)$.

Theorem (Rauch 2)

Let S be a surface such that having Gauss curvature bounded from above, that is $K \leq K_2$, for some K_1, and let $\bar{\mathbf{y}}(t)$ be a solution of

$$\ddot{\bar{\mathbf{y}}}\,(t) = -K\bar{\mathbf{y}}(t)\,,$$

satisfying the initial conditions $\mathbf{y}(0) = \bar{0}$ and $||\bar{\mathbf{y}}(0)|| = 1$.

Then

$$||\bar{\mathbf{y}}(t)|| \geq \begin{cases} \dfrac{\sin t\sqrt{K_2}}{\sqrt{K_2}} & K_2 > 0\,; \\ t & K_2 = 0\,; \\ \dfrac{\sinh t\sqrt{-K_2}}{\sqrt{-K_2}} & K_2 < 0 \end{cases}$$

for any

$$0 \leq r \leq \begin{cases} \dfrac{\pi}{\sqrt{K_2}} & K > 0\,; \\ \infty & K_2 \leq 0\,. \end{cases}$$

• The proofs of the Rauch Theorems are quite complicated and we do not bring them here.

Remark

- *The theorems above are classical examples of Comparison Geometry results, as are the Theorems of Bertrand-Puiseux and Diquet.*

  We have encountered already this approach in the definition of Wald curvature.
- *However, the full force of this approach is apparent when one employs them to actually define curvature in higher dimensions and to obtain discretizations and generalizations of classical notions of curvature.*

Theorem (Bonnet)

Let S be a surface such that

$$K \geq \frac{1}{k^2} > 0\,;$$

where $k = \text{const.} > 0$.

Then S is compact and

$$\text{diam}(S) \leq \pi k\,.$$

Remark

The hypothesis that $K \geq \frac{1}{k^2} > 0$ can not be weakened, as the paraboloid $S = \{(x, y, z) \,|\, z = x^2 + y^2\}$ demonstrates.

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Exercise

Verify the counterexample above.

Remark

The bound $\mathrm{diam}(S) \leq \pi k$ *is tight, as the example of the unit sphere demonstrates.*

Exercise

Validate the counterexample above.

Remark

The condition that S is complete can not be discarded. Indeed, one has the following

Counterexample

$S = \mathbb{R} \times (-\pi/2, \pi/2)$ *endowed with the metric*
$ds^2 = dx^2 + \cos^2 x dy$.

Gauss Curvature and Parallel Transport

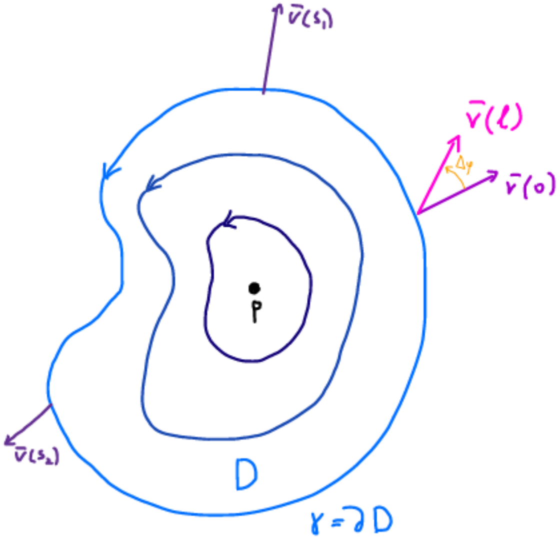
• Before concluding this chapter we should bring yet another connection between Gaussian curvature and geodesics, or rather with parallel transport.

• Consider a point p on a surface S, such that p is included in a simple region D , such that $\gamma = \partial D$ is a closed curved, contained in a coordinate patch.

• Suppose (for technical reasons) that the said coordinate patch is parametrized by an isothermal parametrization $f = f(u, v)$.

• Furthermore, suppose that γ is parametrized by arc length.

• Consider a vector $\vec{\mathbf{v}}$ tangent to S at $\vec{\mathbf{v}}(0)$ and parallel transport along γ.


$\bar{v}(s_1)$
$\bar{v}(\ell)$
$\Delta\varphi$
$\bar{v}(0)$
p
$\bar{v}(s_2)$
D
$\gamma = \partial D$

Then, on the one hand, clearly

$$\int_\gamma k(g)dl = 0\,.$$

On the other hand, as we have seen,

$$\int_\gamma k(g)dl = \int_0^{l(\gamma)} \frac{1}{2\sqrt{EG}}\left(G_u\frac{dv}{ds}ds - E_v\frac{du}{ds}\right) + \int_0^{l(\gamma)}\frac{d\varphi}{ds}\,.$$

By applying well known Green Theorem we obtain that

$$\int_\gamma k(g)dl = -\int\int_D KdA + \left(\varphi(l(\gamma)) - \varphi(0)\right).$$

(Here, as before, $\varphi(s)$ denotes a smooth determination of the angle between f_u and $\vec{\mathbf{v}}(s)$.)

Gauss Curvature and Parallel Transport - cont.

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Gauss Curvature and Parallel Transport - cont.

To summarize, we have obtained that

$$\Delta\varphi = \varphi(l(\gamma)) - \varphi(0) = \int\int_D K dA\,.$$

• Thus, the integral of Gaussian curvature over D equals precisely the discrepancy between the initial and final determination of the angle φ, as induced by the parallel transport along ∂D.

• Given that $\Delta(\varphi)$ does not depend neither on $\vec{\mathbf{v}}(0)$, nor on γ, we can pass to the limit – using a natural approach, given original Gauss' definition of curvature – to conclude that

$$K(p) = \lim_{D\to 0} \frac{\Delta\phi}{\mathrm{Area}(D)}\,.$$

Gauss Curvature and Parallel Transport - cont.

Remark

- *The difference $\Delta(\varphi)$ is called the holonomy of the loop γ.*
- *This interpretation of K in terms of parallel transport might prove itself useful in defining holonomy in such discrete settings where curvature is concentrated at points, notably in the case of nodes in networks.*

Gauss Curvature and Parallel Transport - cont.