Differential Geometry – Classical and Discrete

Curves 2: Curvature

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Motivation

- Why study curvature ?
- Because, to cite the regretted Robert Brooks :

"The fundamental notion of differential geometry is the concept of curvature."

- We can take this even further and, only slightly exaggerating, maintaining Differential Geometry is nothing but the study of curvature.

Motivation - cont.

- Thus the "hero" of this course is curvature, and the "story" it presents is its development, from curves to surfaces, thence to higher dimensional manifolds, and from smooth structures, to metric spaces, weighted manifolds and complexes, to images, meshes and networks.

First Definitions

We begin with the following

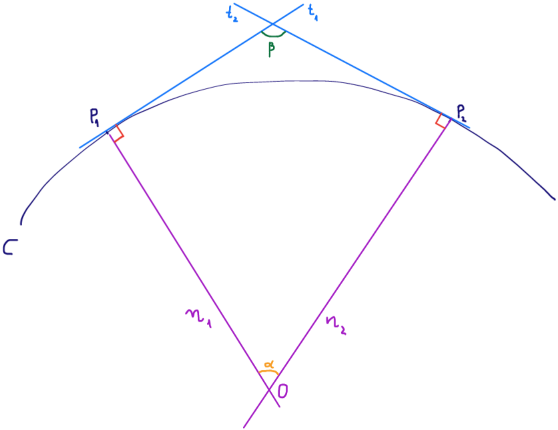
Remark

It is a well known fact of Calculus 101 that the derivative of a function measures rate of change, thence the second derivative measures the rate of change of the (first) derivative.

This somewhat physical observation has a an important geometric meaning : Since the derivative represents the slope of the tangent (at a given point), the change of the derivative between the points P_1 and P_2 is represented by the angle $\alpha = \measuredangle(t_1, t_2)$ between the tangents t_1 and t_2 (at the points P_1, P_2, respectively).

But $\alpha = \pi - \beta = \measuredangle(n_1, n_2)$ – the angle between the normals at the points P_1, P_2, therefore $\sin\measuredangle(t_1, t_2) = \sin\measuredangle(n_1, n_2)$.

First Definitions - cont.



First Definitions - cont.

It follows that, if $P_2 \to P_1$, we have

$$\lim_{P_2 \to P_1} \frac{\measuredangle(n_1, n_2)}{P_1P_2} = \lim_{P_2 \to P_1} \frac{\sin \measuredangle(n_1, n_2)}{P_1P_2} = \lim_{P_2 \to P_1} \frac{P_1P_2}{OP_1 \cdot P_1P_2}$$

$$= \lim_{P_2 \to P_1} \frac{1}{OP_1} = \frac{1}{R} = \kappa\,.$$

First Definitions - cont.

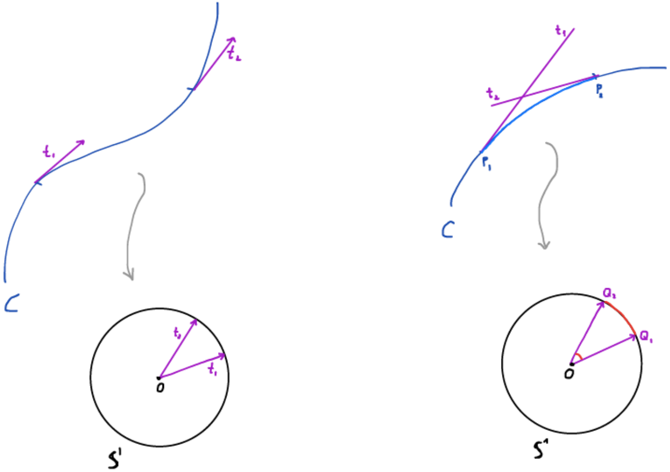
Remark

The geometric importance of the observation above is further revealed by the following observation :

Since the curve may be presumed to be parametrized by arc-length, i.e. the tangent vectors at each point may be supposed to have length one, they may be viewed as points on the unit circle $\mathbb{S}^1$.

Thus the arc P_1P_2 *is mapped via this* tangential mapping *to the the corresponding arc* Q_1Q_2 *on the unit circle.*

First Definitions - cont.



Curvature measures the ratio between the length of an infinitesimal arc length and its tangential image.

First Definitions - cont.

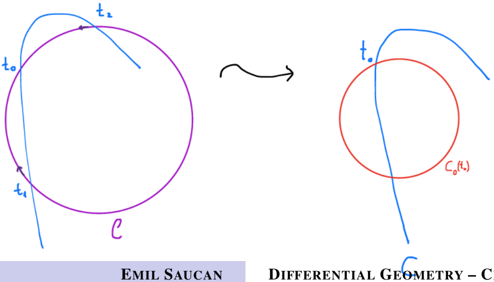
Since $\measuredangle(t_1, t_2) = \measuredangle(Q_1 O Q_2)$ and, moreover, $\measuredangle(Q_1 O Q_2)$ equals the length of the arc $Q_1 Q_2$, we conclude (and write quite informally), that

$$\kappa = \lim \frac{\text{small curve arc}}{\text{its tangent image}}\,.$$

The Osculating Circle

We can make the first approach quite concrete and computational as follows :

Let $c : I \to \mathbb{R}^2\,; c(t) = (x(t), y(t))$, and let $t_0 \in I$, such that $\kappa(t_0) \neq 0$ (so we can talk about curvature, at least in the analytic approach. Let us also consider the circle passing through three points P_0, P_1, P_2, where $P_0 = c(t_0), P_1 = c(t_1)$, $P_2 = c(t_2)$, and let the points P_1, P_2 tend to $P_0 = P_0(x_0, y_0)$, that is the osculating circle at P_0.

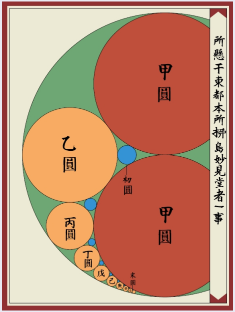


The Osculating Circle

Remark

"Osculatory" comes from the Latin "oscluare" = "to kiss".

"Kissing circles" are also quite famous, and a formula by Soddy[a] exists – in terms of curvature !...



a. A Nobel-winning chemist !...

The Osculating Circle - cont.

Since the equation of a circle cam be written in the following form :

$$(\mathcal{C}) : x^2 + y^2 - 2ax - 2by + c = 0\,,$$

the intersection between a circle and the given curve is given by

$$(c) \bigcap (\mathcal{C}) : x^2(t) + y^2(t) - 2ax(t) - 2by(t) + c = 0\,.$$

We make the substitution $t \mapsto t_0 + h$, we then obtain the following expansion into series :

$$\begin{cases} x(t_0 + h) = x(t_0) + \frac{h}{1!}\dot{x}(t_0) + \frac{h^2}{2!}\ddot{x}(t_0) + \ldots \\ y(t_0 + h) = y(t_0) + \frac{h}{1!}\dot{y}(t_0) + \frac{h^2}{2!}\ddot{y}(t_0) + \ldots \end{cases}$$

(Suffices to restrict to second derivatives, since the circle is a quadric, thus there is no need (or value) in passing beyond quadratic terms.)

The Osculating Circle - cont.

From here and from the fact that the curve and the circle intersect we get

$$x_0^2 + y_0^2 - 2ax_0 - 2by_0 + c + 2\frac{h}{1!}(x_0\dot{x}_0 + y_0\dot{y}_0 - a\dot{x}_0 - b\dot{y}_0)$$

$$+h^2(\dot{x}_0^2 + \dot{y}_0^2 + x_0\ddot{x}_0^2 + y_0\ddot{y}_0^2 - a\ddot{x}_0 - b\ddot{y}_0) + \ldots .$$

(Here we put, for simplicity, $x_0 = x_0(t), \dot{x}_0(t)$, etc.)

Then the circle $\mathcal{C}$ intersects the curve c in three coinciding points iff $h = 0$ is a triple root of the second equation above, i.e. if and only if

$$\begin{cases} x_0^2 + y_0^2 - 2ax_0 - 2by_0 + c = 0 \\ x_0\dot{x}_0 + y_0\dot{y}_0 - a\dot{x}_0 - b\dot{y}_0 = 0 \\ \dot{x}_0^2 + \dot{y}_0^2 + x_0\ddot{x}_0^2 + y_0\ddot{y}_0^2 - a\ddot{x}_0 - b\ddot{y}_0 = 0 \end{cases}$$

The Osculating Circle - cont.

But $\kappa \neq 0 \iff \dot{x}_0\ddot{y}_0 - \dot{y}_0\ddot{x}_0 \neq 0$, that is $\dot{c}, \ddot{c}$ are linearly independent. Therefore we obtain that

$$a = x_0 + \dot{y}\frac{x_0^2 + y_0^2}{\dot{x}_0\ddot{y}_0 - \dot{y}_{00}\ddot{x}_0}\,; \quad b = y_0 - \dot{x}_0\frac{x_0^2 + y_0^2}{\dot{x}_0\ddot{y}_0 - \dot{y}_0\ddot{x}_0}\,;$$

and

$$c = x_0^2 + y_0^2 + \frac{2(x_0\dot{y}_0 - \dot{x}_0 y_0)(\dot{x}_0^2 + \dot{y}_0^2)}{\dot{x}_0\ddot{y}_0 - \dot{y}_0\ddot{x}_0}\,.$$

From the formulas above we immediately obtain that the radius of the limiting circle.

$$R_O = R_O(t_0) = a^2 + b^2 - c^2 = \frac{1}{\kappa(t_0)}\,.$$

(Also, (a, b) represent the coordinates of the osculating circle, also called the *center of curvature*.)

The Osculating Circle - cont.

Exercise

Complete the computations and explicate the formula for $\kappa(t_0)$.

Remark

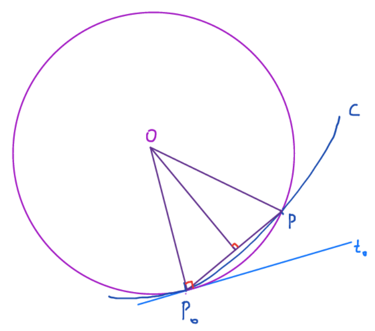
Since the considered limit does not depend on the specific way the points P_1, P_2 are selected, we can chose P_1 such that it will be much closer (orders of magnitude closer) to P_0 than P_2, thus P_0 and P_1 will coincide before P_2 does.

But saying that P_0, P_1 coincide is to say (as we well know from our basic course in calculus), that the segment P_0P_1 tends to the tangent t_0 at the point P_1.

Thus the center of the osculating circle, that is the center of curvature *is the limit of the centers of circles tangent to the curve.*

The Osculating Circle - cont.

Therefore, by taking in the limiting process defining the osculating circle, as we did, P_1 to be much closer to P_0 than P_2, we have shown that the osculation circle represents the limit of tangent circles at P_0.



The Osculating Circle - cont.

Remark

The simples quadratic curve one encounters in his elementary mathematical studies, e.g. in Algebra or Calculus, is the parabola, whose equation is far simpler than that of the circle. This familiarity and simplicity conduce one, quite naturally ask her/himself whether one can define curvature via the osculating parabola, instead of the osculating circle. This is indeed possible and offer this as a problem :

Problem

Define the osculating parabola (of a curve c at a point p) and study its properties.

The Osculating Circle - cont.

There also standard formulas for computing the curvature of a planar regular curve, which allow us to compute the curvature for a number of classical, essential types of curves. [1]

Proposition

Let $c : I \to \mathbb{R}^2, c(t) = (x(t), y(t))$ *be a regular curve. Then*

$$\kappa(t) = \frac{\det(\dot{c}(t), \ddot{c}(t))}{||\dot{c}(t)||^3} = \frac{||\dot{c}(t) \times \ddot{c}(t)||}{||\dot{c}(t)||^3}\,;$$

which in coordinates can be written as

$$\kappa(t) = \frac{\dot{x}(t)\ddot{y}(t) - \ddot{x}(t)\dot{y}(t)}{(\dot{x}^2(t) + \dot{y}^2(t))^{3/2}}\,.$$

1. We shall shortly return to the more geometric definitions of curvature and see how one actually compute curvature in the frame of that paradigm.

The Osculating Circle - cont.

This can be more compactly written as

$$\kappa = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x} + \dot{y})^{3/2}}\,.$$

(This makes sense, of course, only when $(\dot{x}, \ddot{y}) \neq (0,0)$.)

For curves parameterized by *arc length* this becomes :

$$k = \dot{x}\ddot{y} - \dot{y}\ddot{x} = \det(\dot{c}, \ddot{c})\,.$$

The Osculating Circle - cont.

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The Osculating Circle - cont.

Remark

In fact, we can write the equation of the osculatory circle in modern, vectorial notation, as follows :

$$\vec{\mathbf{r}} = \vec{\mathbf{r}}_O + R\vec{\mathbf{n}}\,.$$

The Osculating Circle - cont.

Exercise

Prove the proposition above.

Example

The curvature of the ellipse $\gamma(t) = (c\cos t, b\sin t)\ , a > b > 0$ *is given by* $\kappa(t) = \frac{ab}{\left(a^2\sin^2 t + b^2\cos^2 t\right)^{3/2}}$. *In particular, the curvature at its vertices are* $\kappa(0) = \frac{a}{b^2}$ *and* $\kappa(\frac{\pi}{2}) = \frac{b}{a^2}$. *Moreover, if* $a = b$, *then* $\kappa(t) \equiv \frac{1}{a}$, *that is we recover the expected curvature of the circle (who is, after all, it's own osculating circle !).*

Exercise

Verify the computations above.

The Osculating Circle - cont.

Remark

The formulas above show that the notion of curvature is not defined at points where $||\dot{c}|| = 0$. This fits our geometric intuition that at point of inflection the very notion of circle of curvature is not definable.

Remark

If $||\dot{c}|| = 1$, then $\ddot{c} \perp \dot{c}$ and, moreover, $|\kappa| = ||\ddot{c}||$, thus $|\kappa|$ represents the area of the rectangle of sides $||\dot{c}||$ and $||\ddot{c}||$.

We conclude the presentation of classical curvature of curves with

Exercise

Show that, if the curve c is given in parametric form, i.e. $c(t) = \rho e^{it}$, then $\kappa(t)\frac{|\rho^2+2\dot{\rho}^2-\rho\ddot{\rho}|}{|\rho^2+(\dot{rho})^2|^{3/2}}$. Particular case : The curvature of the curve given by $\rho(t) = 1 + 2\cos t$.

The Osculating Circle - cont.

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Menger Curvature

- We can actually easily compute the radius R of the circle circumscribed to the triangle $\triangle(a, b, c)$.
- This will prove to be quite useful in applications (to images, networks, etc.)
- It allows us to extend the notion of curvature, beyond $\mathbb{R}^2$, to general metric spaces.

Menger curvature

Definition (Menger curvature)

Given a triangle T in a metric space (M, d), with sides of lengths a, b, c, the *Menger curvature* of T is defined as $1/R(T)$, where $R(T)$ is the radius of the circle circumscribed to the triangle (the circumradius). More precisely we have

$$\kappa_{M,E}(T) = \frac{1}{R(T)} = \frac{abc}{4\sqrt{p(p-a)(p-b)(p-c)}}\,,$$

where $p = (a + b + c)/2$.

Menger Curvature - cont.

This approach is based upon two most familiar high school formulas for the aria of the triangle of sides a, b, c :

- Heron's Formula

$$S = \sqrt{p(p-a)(p-b)(p-c)};$$

and

- $$S = \frac{abc}{4R}\,.$$

Menger Curvature - cont.

- There exists an alternative formulas for the Menger curvature of a triangle $T = T(a,b,c)$ of sides a,b,c and angles α,β,γ :

$$\kappa_M(T) = \frac{2\sin\alpha}{bc} = \frac{2\sin\beta}{ac} = \frac{2\sin\gamma}{ab}\,.$$

- Furthermore,

$$\mathrm{Area}\,T(a,b,c) = \sqrt{-16D(a,b,c)}\,;$$

where

$$D(a,b,c) = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & c^2 & b^2 \\ 1 & c^2 & 0 & a^2 \\ 1 & b^2 & a^2 & 0 \end{vmatrix}\,.$$

Therefore,

$$\kappa_M(T) = \frac{\sqrt{-D(a,b,c)}}{abc}\,.$$

Menger Curvature - cont.

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Menger Curvature – Applications

- Theory : Menger curvature has been employed with considerable success to the study of such problems as finding estimates (obtained via the *Cauchy integral*) for the regularity of fractals and the flatness of sets in the plane.[2].

- Practical Applications :
  - Menger curvature has been used – in conjunction with the *traveling salesman* algorithm – for curve reconstruction[3].
  - Approximation of principal curvatures of surface, in Graphics and Imaging tasks.

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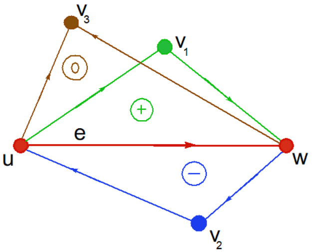
2. Pajot, Shvartsman
3. Giesing

Menger Curvature Applications – Networks

- Adapts easily to networks : No passing to the limit.
- Problem : As defined, the Menger curvature is positive.
- Impediment when dealing with *directed* networks.
- However, for such networks a sign $\varepsilon(T) \in \{-1, 0, +1\}$ is naturally attached to a directed triangle T.



- The *Menger curvature* of the directed triangle is defined as

$$\kappa_{M,O}(T) = \varepsilon(T) \cdot \kappa_M(T) .$$

Menger Curvature Applications – Imaging

- Understanding and classification of textures, especially of the so called *stochastic* textures.

Note the pure blue coloring of the resulting curvature maps, due to the fact that, the Menger curvature itself is always positive, as are areas (squared diameters) contributing to the measure.

Being a measure, this type of curvature, is less sensitive to local (pixel level) noise, thus is a better distinguisher of texture types and, in consequence, also a better edge detector than the Graph Forman-Ricci curvature.

Menger Curvature Applications – Imaging

- In conjunction with *non-local gradients* to the segmentation of textures.

A standard test image (left), its Menger curvature measure (middle) and the segmentation it renders combined with non-local diffusion (right). Note that it is a good distinguisher of texture types and, in consequence, an excellent edge detector.

Menger curvature measure

- In these applications – and in the theoretical ones on fractals – one uses, in fact, the *Menger curvature measure* :

Definition

Let T be a (metric) triangle. Its *Menger curvature measure* $\mu(T)$ is defined as

$$\mu(T) = \kappa_M(T) \cdot (\mathrm{diam}(T))^2 \,.$$

Moreover, if $\mathcal{T}$ is a triangulation (e.g. of a domain in plane or on a surface) its Menger curvature measure is naturally defined as follows :

$$\mu(\mathcal{T}) = \sum_{T \in \mathcal{T}} \kappa_M(T) \cdot (\mathrm{diam}(T))^2 \,.$$

Menger curvature measure - cont.

Remark

The definition above is the basic one. One can generalize it in three manners :

1. *Replace* $\kappa_M(T)$ *by* $\kappa_M^p(T)$*, for some* $p > 1$ *;*
2. *Instead of* $(\mathrm{diam}(T))^2$ *use any other fitting measure* $\nu(T)$ *associated to the triangle* T *;*
3. *Extend the very definition of Menger curvature to include tetrahedra.*

1. *Used in the theoretical works and in the analysis of fractal-like textures, which can thus be studied at many scales.*
2. *Ideal in the case on has to deal with some probability measure, but also when general measures that can be attached to a texture.*
3. *Devised to be used for volumetric data.*

Menger curvature measure - cont.

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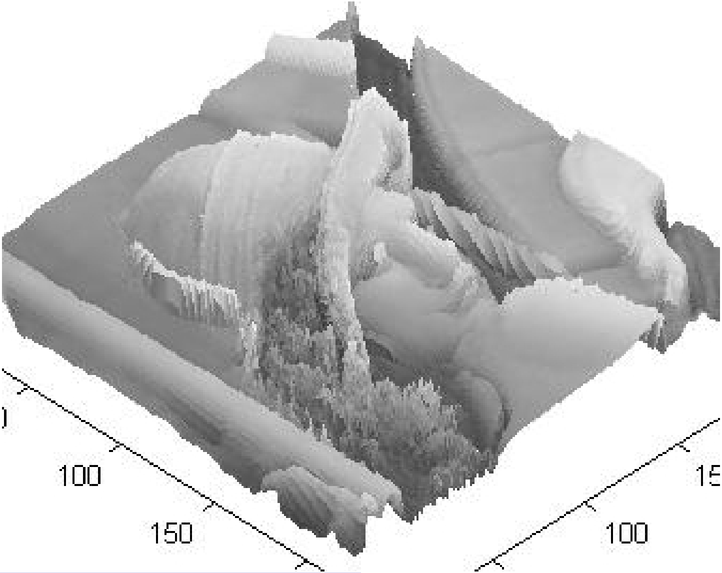
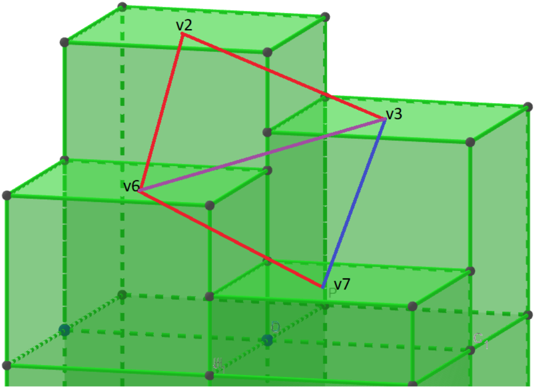
Menger curvature measure - cont.

- Here we use the classical *stick model* of an image.

Menger curvature measure - cont.

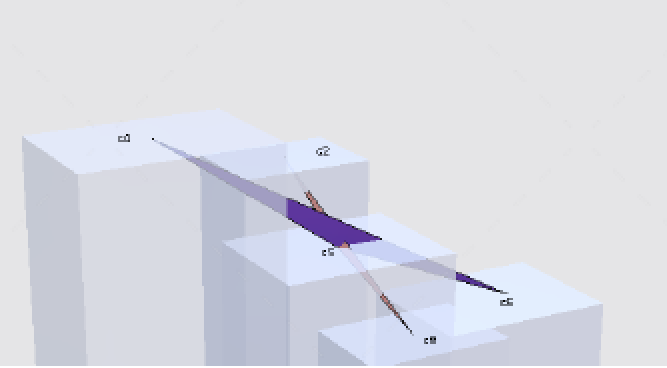


Menger curvature measure - cont.



One possible triangles choice for the computation of the Menger curvature.

Menger curvature measure - cont.



Better triangles choice for the computation of the Menger curvature measure.

Menger Curvature - cont.

- **Problem** : Menger curvature imposes an *Euclidean* geometry on the metric space.

However, it can be extended to

Spherical triangles

$$\kappa_{M,S}(T) = \frac{1}{\tan R(T)} = \frac{\sqrt{\sin p \sin(p-a)\sin(p-b)\sin(p-c)}}{2\sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2}}.$$

Menger Curvature - cont.

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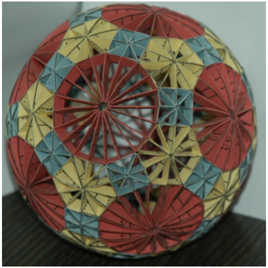
Menger Curvature - cont.

Example : A Spherical Archimedean polyehdron



• The 1-skeleton of the triangulation of the sphere consisting of the fundamental triangles of a spherical *Archimedean polyehdron*.

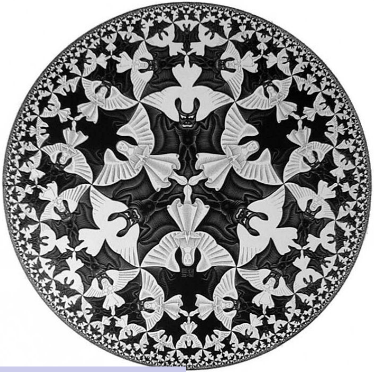
• The side of the spherical faces is taken to be 2 units. The fundamental triangle of an n-gonal face has angles $(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{n})$, $n = 4, 6, 10$. Hence, the remaining sides of the triangles and κ_M, can be computed using *spherical trigonometry*.

Menger Curvature - cont.

... and also to

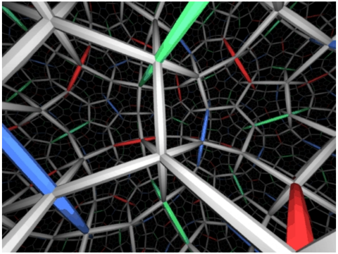
Hyperbolic triangles

$$\kappa_{M,H}(T) = \frac{1}{\tanh R(T)} = \frac{\sqrt{\sinh p \sinh(p-a)\sinh(p-b)\sinh(p-c)}}{2\sinh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2}};$$



Menger Curvature - cont.

Example : Tessellation of $\mathbb{H}^3$ with regular dodecahedra



- All face angles equal to $\frac{\pi}{2}$.
- The fundamental triangle of the faces has angles $(\frac{\pi}{5}, \frac{\pi}{4}, \frac{\pi}{2})$.
- Normalizing the sides of the dodecahedra, such that half of its length be equal to 1, using the *hyperbolic law of sinuses*, $\kappa_{M,H}(T)$ of the triangle can be readily computed.