Differential Geometry – Classical and Discrete

Gauss Curvature and Theorema Egregium

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- Looking at the formula
$$K = \frac{\det(II)}{det(I)} = \frac{eg - f^2}{EG - F^2}$$
one would be conducted to presume that Gauss curvature depends on the way the surface is *embedded* in $\mathbb{R}^3$, given that it depends on the second fundamental form, hence on the normal mapping, thence on its position in space.
- In other words, *prima facie*, Gauss curvature would depend on its position in space, thus being what it is called a *extrinsic* property.
- It is therefore quite a surprising fact that, in truth, Gauss curvature is an *intrinsic* property, i.e. it depends solely on the (geometric properties of the) surface itself, and not on its specific positioning (*embedding*) in space.

- Precisely how surprising a fact this is demonstrated by the fact that Gauss himself was quite surprised when making this discovery hence he named it “Theorema Egregium”, i.e. in Latin, “The Remarkable”' (or “Excellent”) “Theorem”.

Theorem (Gauss' Theorema Egregium)

Gauss curvature is an intrinsic property.

The way one would prove this in the classical (smooth) setting – as Gauss has indeed done – is to demonstrate an atrocious formula for K that expresses it solely in terms of the first fundamental form, thus showing it to be, indeed, intrinsic (since shows that K depends only on the metric properties of the surface and not on its position in space).

Theorema Egregium

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Theorema Egregium - cont.

The most common such formula is the one below. It was first proven by Brioschi in 1852 and, independently by R. Baltzer. Gauss own original proof of 1827 is much earlier, but his version of the formula, albeit equivalent, is more complicated and much harder to arrive at.

$$K(EG-F^2)^2 = \begin{vmatrix} -\frac{1}{2}G_{uu} + F_{uv} - \frac{1}{2}E_{vv} & \frac{1}{2}E_u & F_u - \frac{1}{2}E_v \\ F_v - \frac{1}{2}G_u & E & F \\ \frac{1}{2}G_v & F & G \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & F \\ \frac{1}{2}G_u & F & G \end{vmatrix}$$

Theorema Egregium - cont.

As already mentioned before, there exist (many) other formulas.

A particularly short and symmetric one one – but, unfortunately, using Christoffel symbols for K that show that is intrinsic – is due to Liouville (1851).

However, arguably the most elegant one is the following one due to Frobenius :

$$K = -\frac{1}{4(EG-F^2)^2}\begin{vmatrix} E & E_u & E_v \\ F & F_u & F_v \\ G & G_u & G_v \end{vmatrix} - \frac{1}{\sqrt{EG-F^2}}\left(\frac{\partial}{\partial v}\frac{E_v - F_u}{\sqrt{EG-F^2}} - \frac{\partial}{\partial u}\frac{F_v - G_u}{\sqrt{EG-F^2}}\right)$$

Theorema Egregium - cont.

- While unfortunately irrational, it is much simpler and more symmetric (a fact always attractive to the mathematical eye, so to say.
- However, its charm does not resides solely on these facts and it presents an advantage also from the computational point of view :
  - Given that it only contains only order 1 derivatives of E, F, G (thus only order 3 derivatives of the given parametrization) it is numerically more stable than the Brioschi formula, that contains order 2 derivatives of E, F, G (thus order 4 derivatives of the parametrization), therefore amply compensating for the presence of the square-root.

Exercise

Prove Frobenius' formula.

Theorema Egregium - cont.

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Exercise

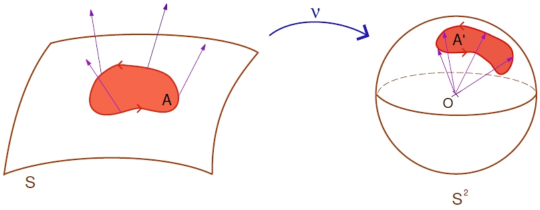
Prove Frobenius' formula.

Theorema Egregium – The Discrete Approach

- As we have stressed above, the classical proof is tedious and not very intuitive.
- However, the discrete approach has a clear advantage in this case, providing not just a simple, elementary proof, but also revealing the geometric insight behind Gauss' result.
- Fittingly, the very basic idea is much more geometric : One has to go back to the normal (Gauss) mapping.

  In fact, this shouldn't be too surprising to the reader, given that we have already seen in the previous chapter the relation between curvature and the variation of the normals in the curves setting.
- It is probable, that Gauss was well aware of the curves case and had in mind when developing the *total* (Gauss) curvature of surfaces

Given a point p on a surface S, consider a small curve γ on S containing p in its interior. Its image by the normal mapping determines a corresponding small curve on the unit sphere $\mathbb{S}^2$. Let A, A' be the interior (domains) determined by γ and $N(\gamma)$, respectively.



Denote $K(\gamma) = \frac{\mathrm{Area}(A)}{\mathrm{Area}(A')}$, and let γ collapse to p, such that $\mathrm{diam}\gamma \to 0$.
Then (obviously), both Area(A) and Area(A') tend to 0.

Question

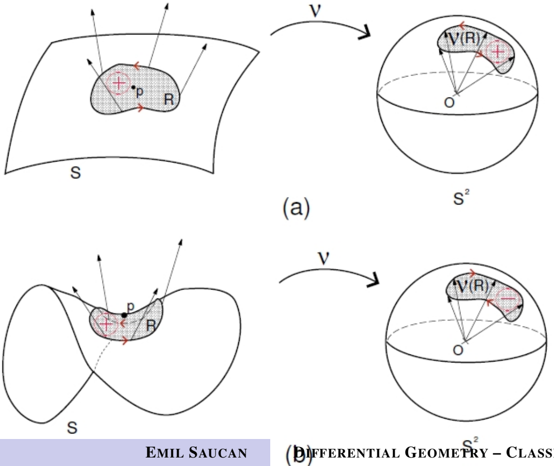
Why is it important that $\mathrm{diam}\gamma \to 0$ *and it is not sufficient to request only that* $\mathrm{Area}(A) \to 0$ *?*
(Hint : Recall some Calculus 2 arguments.)

Now define

$$K(p) = \lim_{\mathrm{diam}\gamma \to 0} K_\gamma \,.$$

Theorema Egregium – The Discrete Approach (cont.)

We still have to provide in the definition for the possible negative K, by noticing that the Gauss map preserves the sign of (small enough) curves around points with positive curvature, but reverses it for those around points of negative curvature



Theorema Egregium – The Discrete Approach (cont.)

- This is, in fact, Gauss' original definition of curvature.
- He next **proved** that this definition of curvature can be also expressed as the product of the principal curvatures !
- The fact that text-books (even us !...) define Gauss curvature as a product of the maximal and minimal sectional curvatures has two reasons :
  - The first one is historical : This is (as we have seen) the first approach to the Differential Geometry of surfaces before Gauss (and it still appears to be intuitive).
  - The second resides in the fact that, given the modern language and tools of Calculus and Linear Algebra, it is more "economical" to give first this definition.
  - The problem with this tradition is not only that it lacks geometric intuition and depth, it also not natural in such fields as Graphics and Imaging, where the discrete, geometric methods are essential.

- Before proceeding further we should verify that this definition truly makes sense, by verifying that we obtain the right (expected) answer when computing K, in this manner, for some standard, elementary surfaces.
- The simple – but special – cases of $S \equiv \mathbb{R}^2$ and $S \equiv \mathbb{S}^2$ are the ones to check first and it is easy that the results correspond to the previous computations.

Exercise

Verify, using the same type argument, that
(i) $K(\mathrm{cylinder}) \equiv 0$ *;*
(ii) $K(\mathrm{cone}) \equiv 0$*.*
Hint : *Consider the images of these images under the normal map, as discussed above.*

Theorema Egregium – The Discrete Approach (cont.)

It still remains to show that this new definition of Gauss curvature coincides with the previous ("technical") one.

Proof

$K = [N\ N_u\ N_v]/\sqrt{g}$. Since $N_u \times N_v \parallel N$, it follows that

$$|N_u \times N_v| = |K|\sqrt{g}\,.$$

On the other hand

$$\mathrm{Area}(A_1) = \int\int |\frac{\partial f}{\partial u}du \times \frac{\partial f}{\partial v}dv| = \int\int \sqrt{g}dudv\,.$$

Moreover, $\mathrm{Area}(A_2) = \int\int |N_u du \times N_v dv|$, hence it follows that $\mathrm{Area}(A_2) = \int\int |K|\sqrt{g}dudv$. Therefore

$$\lim_{\mathrm{diam}\gamma \to 0} \frac{\mathrm{Area}(A_1)}{\mathrm{Area}(A_2)} = K\,.$$

□

Theorema Egregium – The Discrete Approach (cont.)

The route to the elementary proof we shall give bellow is to consider *bendings* (or, in modern language, *local isometries*, i.e transformations (of the surface) that leave lengths of curves (and angles of intersections of curves) invariant.

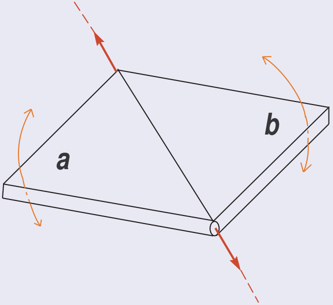
In this suggestive language, Theorema Egregium is formulated as :

> *Gauss curvature is invariant under bendings.*

The proof we bring here follows closely that of Hilbert and Cohn-Vossen (in a popularization work of real depth who, incidentally, pioneered, as far as we know, this discrete approach, at least in this context).

Hilbert and Cohn-VossenProof of Theoema Egregium

Imagine the surface to be composed of a (very large) number of extremely thin – but rigid – plates, connected at the edges by hinges around which they can swivel.



If the number of plates around each vertex is at least 4, than the "umbrella" around each vertex can change shape by "vibrating" along the hinges.

Question

Why is the condition that the number of plates around each vertex be > 3 *necessary ? Is this a "reasonable"/natural condition ?*

However, all this change in the relative position of the elements of each "umbrella" leave lengths of curves and angles between curves invariant, that is to say they are bendings.

Also, here we presume, without loss of generality, that all the curves intersect only in the interiors of the plates, and on the edges.

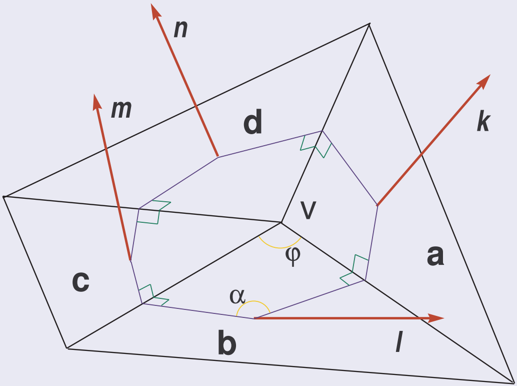
Theorema Egregium – The Discrete Approach (cont.)

Question

Why is this presumption a legitimate one ?

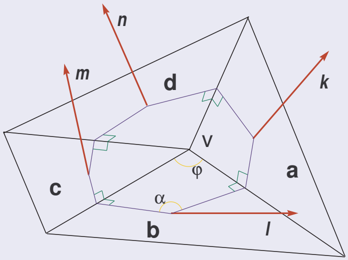
Next we consider unit normals to all the faces (triangles) adjacent (abutting) a vertex v one for each such triangle.

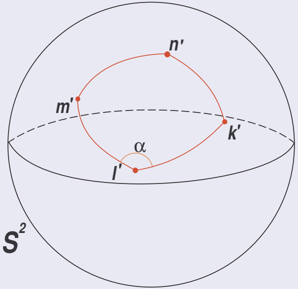
Of course, one has an infinity of possibilities for the choice of the base point for the normals, but, since they are all parallel, we can chose, for each as a base point for each triangle in such a manner that the perpendiculars from these base points to each edge fall in the same point.



We next remember the fundamental idea of passing to the normal image :

Let us denote by P the (non-planar polygon formed by these triangles, and by $\mathcal{P}$ its spherical image, that is the spherical polygon of vertices k', l', m', n' (and edges great circle arcs connecting the relevant pairs of vertices).





Theorema Egregium – The Discrete Approach (cont.)

We next prove that $\mathrm{Area}(\mathcal{P})$ is invariant under bending. To this end we need the following lemma (which we shall prove slightly later) :

Lemma

Area$(\mathcal{P}) =$ Area(interior angles of $\mathcal{P}$)*, or, simply put, that the area of* $\mathcal{P}$ *depends solely on the sum of its (interior) angles.*

Therefore, suffices to prove that the sum of the (interior) angles of $\mathcal{P}$ is invariant under bending. Since $\alpha = \pi - \varphi$ for each such angle α, the assertion follows immediately.

By considering finer and finer approximations of the given surface by such triangular plates and passing to the limit as the number of triangles tends to infinity, the desired results follows.

$\square$

Theorema Egregium – The Discrete Approach (cont.)

Before taking a second look at the deep meaning of the proof above, we give the proof of lemma above (which, by the way, is of no mean importance itself) :

Proof of the Lemma

We first determine the area of a *lunule*[a], ie. half of the area bounded by two great circles.

By examining it, it is easy to see that

$$\frac{\mathrm{Area}(L)}{\mathrm{Area}(\mathbb{S}_{\mathbb{R}}{}^2)} = \frac{\alpha}{2\pi}\mathrm{Area}(\mathbb{S}_{\mathbb{R}}{}^2)\,,$$

where $\mathbb{S}_{\mathbb{R}}{}^2$) is the sphere of radius R, therefore

$$\mathrm{Area}(L) = 2\alpha R^2\,.$$

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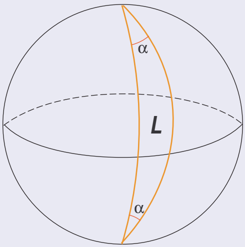
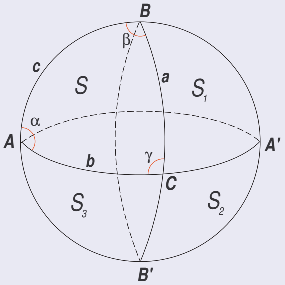
a. i.e., in Latin, a “little moon”

Theorema Egregium – The Discrete Approach (cont.)



We next consider the spherical triangle *S*, of angles α, β, γ (and sides *a*, *b*, *c*, respectively).



Then

$$S + S_1 + S_2 + S_3 = 2\pi R^2$$

(since the left hand represents the area of a hemisphere.)

Theorema Egregium – The Discrete Approach (cont.)

On the other hand it is easy to check that :

$$S + S_1 = 2\alpha R^2$$

$$S + S_2 = 2\beta R^2$$

$$S + S_3 = 2\gamma R^2$$

By summing the last three equalities we obtain

$$2(\alpha + \beta + \gamma)R^2 = 3S + (S_1 + S_2 + S_3)$$

$$= 2S + (S + S_1 + S_2 + S_3)$$

$$= 2S + 2\pi R^2$$

(By the first equality in the series.)

Theorema Egregium – The Discrete Approach (cont.)

It follows that

$$S = (\alpha + \beta + \gamma - \pi)R^2 \,.$$

(of course, the formula is even simpler for $R = 1$, i.e. for the unit sphere.)

Before continuing with our proof, let us make the following

Remark

Since area is a positive function, from the formula above we obtain the immediate – but, nevertheless, important – corollary that the (interior) angle sum of a spherical triangle is strictly greater than π.

To finish the proof of the lemma, we have only to recall that, if $\mathcal{P}$ has n sides and angles $\alpha_1, \ldots, \alpha_n$, then it is decomposable in n triangles, $S_1, \ldots, S_n$.

We obtain that

$$\mathrm{Area}(\mathcal{P}) = [(\alpha_1 + \cdots + \alpha_n) - (n-2)\pi]\, R^2 \,.$$

$\square$

Theorema Egregium – The Discrete Approach (cont.)

Remark

While the possibility of approximating arbitrarily well a surface by flat (Euclidean) triangles, and, indeed, dividing a surface in "curvilinear" triangles is a very intuitive statement, it necessitates, however, a quite non-trivial proof, at least in the general case.

How non-trivial this proof might be, is underlined by the fact that Hilbert and Cohn-Vossen only claim that their proof of Theorema Egregium holds just for convex surfaces.

We shall encounter these triangulations *further on, in the context of the* Gauss-Bonnet Theorem *and, instead of merely referring the reader to the Topology of Surfaces literature, we shall bring a quite simple and concise Differential Geometric demonstration for general surfaces, befitted for the case of smooth (at least $\mathcal{C}^2$) surfaces.*

Theorema Egregium – The Discrete Approach (cont.)

- We can now return at the proof of the Theorema Egregium itself and try to understand its deeper significance.
- The main inside that the proof provides is that the Gauss curvature of a triangulated [1] (or polyhedral) surface at a vertex is a function of the sum angles of the faces incident with that vertex.
- In the *combinatorial* case, i.e. when all the edges are given length one and all triangles (faces) aria also equal to 1, the contribution of $\mathrm{Area}(P)$ in the proof above can, therefore, be take to be always 1, thus we obtain the following simple and appealing definition :

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1. technically formulated a *PL piecewise linear* or, more precisely, a *piecewise flat* surface

Theorema Egregium – The Discrete Approach (cont.)

Definition (Combinatorial curvature)

Let $\mathcal{S}$ be a *PL* (polyhedral) surface and let v a vertex of $\mathcal{S}$. Then the *combinatorial* (*Gauss*) *curvature* (of $\mathcal{S}$ at v) is

$$K(v) = 2\pi - \sum_i \alpha_i \,,$$

where the sum is taken over all the adjacent angles at v.

The difference $2\pi - \sum_i \alpha_i$ is called the (*angular*) *defect* (of S at v) and it (obviously) measures the "distance" of a small neighbourhood of v from being a planar disk.

Theorema Egregium – The Discrete Approach (cont.)

Remark

Recall that, since $2\pi - 7\pi/3 = -\pi/3 \approx -1.046$, this definition of curvature allows us to construct – using an idea of Bill Thurston – simple and suggestive models of the Hyperbolic Plane : Start with a regular triangular grid in the plane, then cut-and-paste triangles such that, around each vertex, there will be precisely 7 triangles. From the computation above it follows that we obtain a piecewise flat surface, where all the curvature is concentrated at the vertices (and identically equal to -1, up to some small error), thus rendering a PL model of the Hyperbolic Plane, as promised.

Theorema Egregium – The Discrete Approach (cont.)

- The combinatorial definition of curvature is not only simple and intuitive, it is also quite powerful, as we shall see, for instance in defining the *Discrete Ricci Flow*.
  - However, its importance transcends the fields of Discrete Differential Geometry and Graphics.
    - Most importantly, its very combinatorial nature render it an extremely important tool in such domains where basic combinatorial objects, such as graphs, arise naturally, notably in Group Theory (via, for instance, the so called *Cayley Graphs*).
  - In Graphics, CAGD, Finite Element implementations, etc. combinatorial curvature is employed with the following weighted variation :

$$K(v) = \frac{2\pi - \sum_i \alpha_i}{\frac{1}{3}\mathrm{Area}(P)}\,.$$

(Here the same convention for the summation was used as above.)

Exercise

Justify the method described above. In particular, explain the choice of $1/3$.

- We shall return to the problem of approximating curvature of surfaces, but first let us take a second, in depth, look at the Hilbert and Cohn-Vossen proof of Theorema Egregium that we presented above.
- To this end we turn to work the work of Banchoff, who's approach, however, is quite different as it makes appeal to topological ideas, more precisely to the so called *Morse Theory*.
  - We shall present here only the polyhedral setting, but we shall also point out, by means of a number of exercises, the original, smooth definitions and results.

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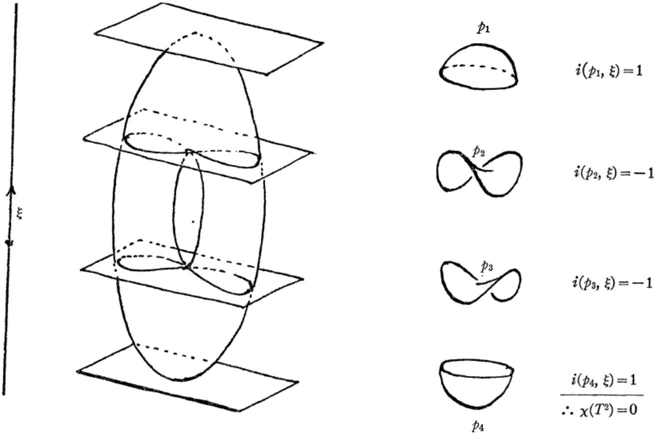
  - We shall present here only the polyhedral setting, but we shall also point out, by means of a number of exercises, the original, smooth definitions and results.

- The basic notion we need is that of *critical point of a height function*.
- To obtain the intuition necessary to develop the polyhedral ideas, let us first review the classical notions which spurned them :
  - Let S^2 be a smooth, closed surface in $\mathbb{R}^3$ and let $\vec{\mathbf{v}}$ be a and arbitrary direction in $\mathbb{R}^3$ (i.e. a unit vector, or a point on the unit sphere $\mathbb{S}^2$). We define the *height function* h as being the projection (function) of $\mathbb{R}^3$ on the line l determined by $\vec{\mathbf{v}}$.
  - A point $p \in S^2$ is called a *critical point* for h if the tangent plane to S^2 at p is perpendicular to l, otherwise it is called an *ordinary point*.

Theorema Egregium – The Discrete Approach (cont.)



Height function for the torus : All points are ordinary, except the maximum, the minimum and the two saddle points, that is a total of only four critical points.

Theorema Egregium – The Discrete Approach (cont.)

- To each critical point a numerical value is attached, namely we set $i(p, l) = +1$ if m is a local minimum or maximum, and $i(p, l) = -1$ if m is a saddle point.
- To this formal (and seemingly arbitrary) assignment we wish to find a more geometrical characterization.
  - If p is an ordinary point, then the tangent plane (to S) at p is not "horizontal" (parallel to l), therefore it meets a "small" (infinitesimal) circle (on S) around p in precisely two points.
  - In contrast, the intersection of the tangent plane with such a circle at maximum or minimum point is void, whereas at a saddle point it will intersect an infinitesimal circle in four distinct points.

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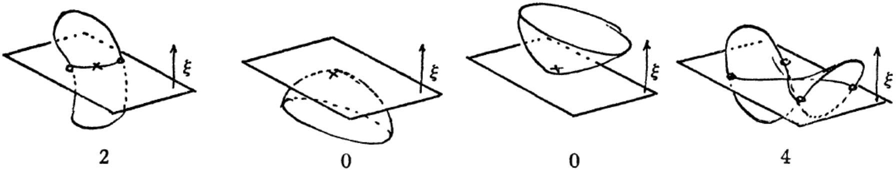
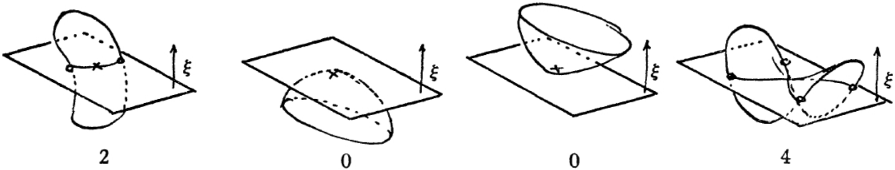


Based on the observation above one can then formally define the index in the following combinatorial manner :

$$i(p, l) = 1 - \frac{1}{2}|\{T_N(p) \cap C_\varepsilon(p)\}|\,;$$

where $T_N(p)$ is the plane through p normal to l and $C_\varepsilon(p)$ denotes an infinitesimal circle centered at p ; in other words

Theorema Egregium – The Discrete Approach (cont.)



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Theorema Egregium – The Discrete Approach (cont.)

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$$i(p, l) = 1 - \frac{1}{2}\big(\#\text{points in which the plane through } p$$

$$\text{perpendicular to } T \text{ meets a ``small circle'' about } p \text{ on } M^2\big).$$

- While very simple, this definition is not quite what one would expect from a notion on smooth surfaces, both because of the vagueness of the notion of "small circle", and because, in practice, it would be quite difficult to determine the required number of intersections on a general ("not very smooth" surface, and for a general direction l ($\vec{\mathbf{v}}$)).
- On the other hand, its form is almost what one would request from a definition befitting polyhedral surfaces.

Theorema Egregium – The Discrete Approach (cont.)

In other words

$$i(p,l) = 1 - \frac{1}{2}\big(\#\text{points in which the plane through } p \text{ perpendicular to } T \text{ meets a ``small circle'' about } p \text{ on } M^2\big).$$

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- On the other hand, its form is almost what one would request from a definition befitting polyhedral surfaces.

Theorema Egregium – The Discrete Approach (cont.)

- To justify this assertion, let us first note that, for polyhedral surfaces, the *star* $St(p)$ of a vertex p, i.e. the set of all simplices incident to p (that is the edges and faces (including their edges and vertices containing p) plays the role of a "small" disk neighbourhood centered at p, while the *link* $Lk(p)$, i.e. the polygon representing the boundary of $St(p)$ represents the polyhedral analogue of a "small circle" around p.
- Observe also that a point is *ordinary* for the height function h if the plane perpendicular to l that passes through p divides $St(p)$ into two pieces.
  - Any interior point of face of an edge is, therefore, an ordinary one for any direction *general* for the given polyhedral surface, i.e. such that $h(p) \neq h(q)$, for any two distinct vertices of S^2.

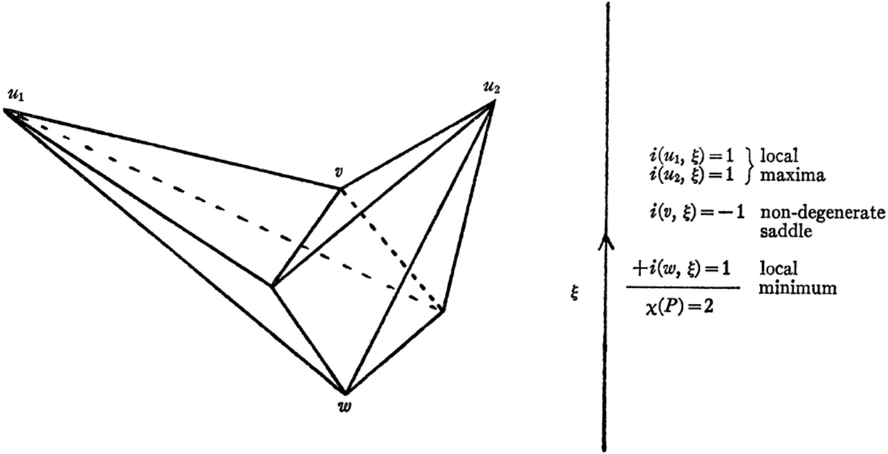
Theorema Egregium – The Discrete Approach (cont.)

• Moreover, given that M^2 has only a finite number of edges, it follows that the number of non-general directions is finite, thus our analysis is not limited by using general directions, given the fact that they are the rule, rather then one of the finite number of exceptional cases.

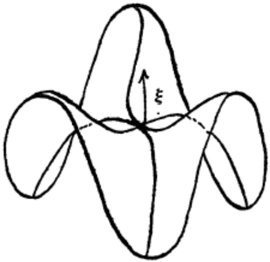
• Furthermore, this represents the precise polyhedral equivalent of a classical result, namely that almost any direction $\vec{\mathbf{v}} \in \mathbb{S}^2$, the associated height function has only a finite number of critical points, thus almost all directions (up to a set of zero measure) is general.

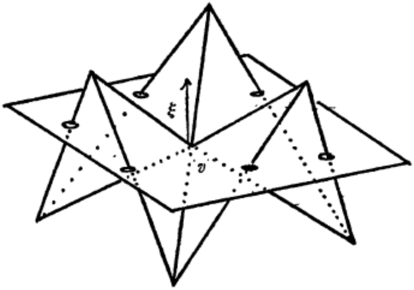
• In contrast, vertices represent critical points of all of the types arising for smooth surfaces.

Theorema Egregium – The Discrete Approach (cont.)



• In fact, while for smooth surfaces the only possible critical points are maxima, minima and non-degenerate saddle points on polyhedral surfaces *degenerate* critical points can also arise.



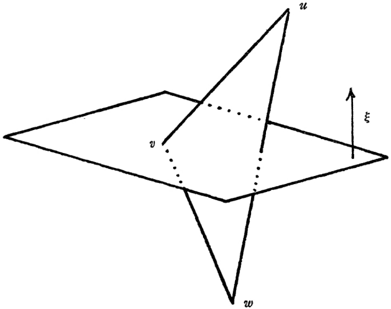


$i(v, \xi) = -2$

- Note that we said “almost” ; however we can improve this definition of the index to become purely combinatorial, by making the following observation : We still can count intersections with “small disks” – only better, because now “small disk” has a precise meaning – the star of a vertex.

- From this observation easily follows that the number of times the plane through p perpendicular to a triangle T (with vertex p) meets $\mathrm{Lk}(p)$ is then equal to $\#T$ in $\mathrm{St}(p)$, such that one of the vertices of T lies above the plane and the other lies below. In such a case v is called *the middle vertex* of T for l.

Theorema Egregium – The Discrete Approach (cont.)

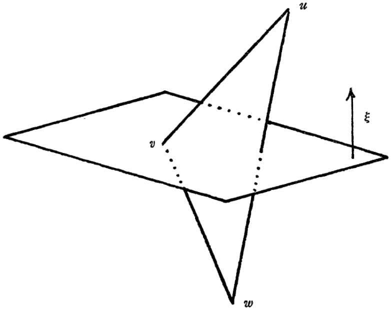


v is the middle vertex of $\triangle$ for ξ.

We are thus conducted to formulate the desired combinatorial definition of the index :

$$i(p, l) = i(p, l) = 1 - \frac{1}{2}\big(\#T \text{ s.t.} p \text{ is a middle for } l\big).$$

Theorema Egregium – The Discrete Approach (cont.)



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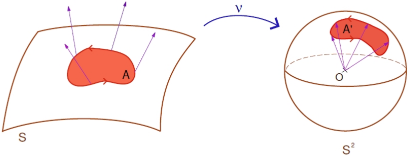
Theorema Egregium – The Discrete Approach (cont.)

- Using the above formula of the index, we can proceed to the proof of the Theorema Egregium.
- However, in order to do this, we have still to correlate between combinatorial form of the Gauss curvature at a vertex on a polyhedral surface and the vertex's index (w.r.t. a general direction).
- To this end, let us first return to the smooth case and observe that the area of A' is given by its characteristic function[2] $\chi_{A'}$, namely $\mathrm{Area}(A') = \int_{A'} \chi_{A'} dA$.



2. Recall that the *characteristic function* of a set A is defined as $\chi_A(x) = 1$ if $x \in A$, and 0 otherwise.

Theorema Egregium – The Discrete Approach (cont.)

- Furthermore, the height function w.r.t. a given direction l has a critical point at $p \in A'$ if and only if the direction vector of l, $\vec{\mathbf{v}} \in A'$. (Note that to the same direction correspond to points on the unit sphere, p and its antipodal point p^*.)
- Then, since we can chose A such that A' does not contain antipodal points, we obtain

$$K(A') = \frac{1}{2}\int_{\mathbb{S}^2} \sum_{p \in A'} i(p, l)\,.$$

- Note that we presumed that the normal mapping ν is injective on A.
- However, the formula above can be extended to the general case, but this requires a more delicate analysis, that is not required in the polyhedral case

Theorema Egregium – The Discrete Approach (cont.)

- Indeed, in this case the role of A is played by $\mathrm{St}(p)$ and, moreover, the vertex p is the only critical point in $\mathrm{St}(p)$.

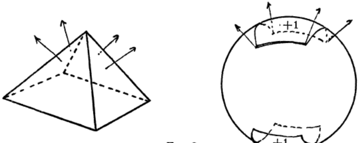


FIG. 8

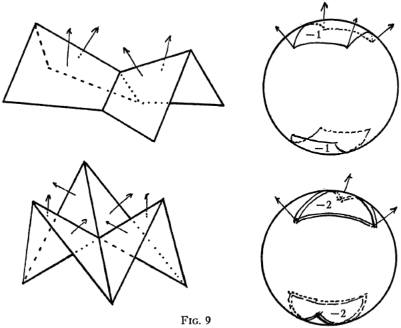


FIG. 9

Theorema Egregium – The Discrete Approach (cont.)

- Therefore we can use, in the polyhedral case as well, the index based formula for K and, moreover,

$$K(p) = \frac{1}{2}\int_{\mathbb{S}^2} i(p,l)dA\,.$$

We can now bring the

Alternative Proof of Theorema Egregium

We start by introducing yet another notation :
Let $m = m(p,T,l)$ be the function that takes the value 1, if p is a middle vertex for T, relative to l, and 0 otherwise.
Then, $i(p,l) = m(p,T,l)$, by its definition. Therefore

$$K(p) = \frac{1}{2}\int_{\mathbb{S}^2}\left(1 - \frac{1}{2}\sum_{T\in S^2} m(p,T,l)\right)dA = \frac{1}{2}\int_{\mathbb{S}^2} dA - \frac{1}{4}\int_{\mathbb{S}^2} m(p,T,l)dA\,.$$

Theorema Egregium – The Discrete Approach (cont.)

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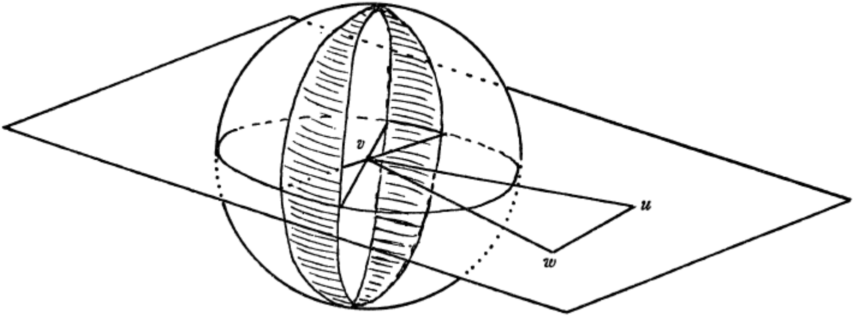
Theorema Egregium – The Discrete Approach (cont.)

- The first term in the right hand equation represents half of the area of the unit sphere, therefore it is equal to 2π, thus we only have to estimate the second term.
- Since any vector in $\vec{\mathbf{v}}$ in $\mathbb{R}^3$ can be uniquely expressed as a sum $\vec{\mathbf{v}} = \vec{\mathbf{u}} + \vec{\mathbf{v}}$, where $\vec{\mathbf{u}} \in \Pi(T)$ – the plane determined by the triangle T, and $\vec{\mathbf{v}} \perp \Pi(T)$, it follows that $m(p, T, l) = 1$ if and only if

$$\vec{\mathbf{v}} \cdot \vec{pq} > 0 > \vec{\mathbf{v}} \cdot \vec{pr} \text{ or } \vec{\mathbf{v}} \cdot \vec{pq} < 0 < \vec{\mathbf{v}} \cdot \vec{pr}\,;$$

  that is if and only if $\vec{\mathbf{v}}$ belongs to the double *lunule* of angle $\alpha = \sphericalangle(\vec{pq}, \vec{pr})$ on the unit sphere of center p, perpendicular to the equatorial plane Π_T

Theorema Egregium – The Discrete Approach (cont.)



Triangle middle vertex and double lunule on the unit sphere.

Theorema Egregium – The Discrete Approach (cont.)

- Since the area of such a double lunule is 4α it follows that the second term is equal to $\sum_T(\alpha)$, where the sum is taken over all the triangles incident with p.
- Thus, we discover yet again, by following this alternative route of proof, that $K(p) = 2\pi - \sum_T(\alpha)$, thus K is intrinsic. This concludes our alternative proof.

$\square$

The following exercises introduce, via a graduated sequence of questions, the smooth (classical) counterpart of the tools developed in Banchoff's proof :

Theorema Egregium – The Discrete Approach (cont.)

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The following exercises introduce, via a graduated sequence of questions, the smooth (classical) counterpart of the tools developed in Banchoff's proof :

Exercise

- *Let S^2 be a smooth surface, $f : S^2 \to \mathbb{R}$ a differentiable function, and let $p \in S^2$ be a critical point of p. Denote by $H_p f$ the* Hessian *of f of at p.*
    - *Show that, if f is the height function of S^2 relative to $T_p(S^2$ – the tangent plane to S^2 at p, i.e. $f(q) = \overrightarrow{(q-p)} \cdot N(p)$, $q \in S^2$., then p is a critical point of f and that $H_p f$ is well defined.*
    - *Prove that if $\vec{\mathbf{v}}$, $|\vec{\mathbf{v}}| = 1$ then $H_p f(\vec{\mathbf{v}}) = k_{n,\vec{\mathbf{v}}}$ – the normal curvature of S^2, at p, in the direction $\vec{\mathbf{v}}$.*
    - *Prove that $H_p f$ (where f is as above) is precisely the second fundamental form of S^2 at p.*

- Let S^2, f and p be as above. We say that p is a *non-degenerate* critical point for f if the self-adjoint linear mapping associated to the quadratic form H_pf is nonsingular, and *degenerate* otherwise. Furthermore, f is called a *Morse function* if has only non-degenerate critical points.
  - Let h_{p_0} be the distance function from S^2 to a fixed point $p_0 \notin S^2$, i.e. $h_{p_0}(p) = \sqrt{\overrightarrow{(p-p_0)} \cdot \overrightarrow{(p-p_0)}}$, $p \in S^2$. Prove that $p \in S^2$ is a critical point for h iff the line $l = \overleftrightarrow{pp_0} \perp S^2$.
  - Let p a critical point of h_{p_0}, and let $C \subset S^2$ a curve parameterized by arc length, trough p_0 (i.e. $C(0) = p$), having as (unit) tangent vector at p, $\dot{C}(0) = \vec{\mathbf{v}} \in T_p(S^2)$.

Show that

$$H_p h_{p_0}(\vec{\mathbf{v}}) = \frac{1}{h_{p_0}} - k_n\,;$$

where k_n is, as above, the normal curvature of S^2 in the direction $\vec{\mathbf{v}}$.

Furthermore, show that p is a degenerate critical point of h_{p_0} iff $h_{p_0} = 1/k_1$ or $h_{p_0} = 1/k_2$, where k_1, k_2 are the principal curvatures at p.

- Prove that the set $X = \{r \in \mathbb{R} \mid h_{p_0} \text{ is a Morse function}\}$ is *dense*in $\mathbb{R}^3$. (In other words, Morse functions on smooth, regular surfaces are "generic".)

Theorema Egregium – The Discrete Approach (cont.)

Remark

As Banchoff points out, his approach represents an (extensive) extension of the one in a paper of Pólya's concerning polyhedral disks in $\mathbb{R}^3$.

Remark

The method above applies, in fact, to any dimension $n \geq 2$ and allows for the extension dimensions higher than 3 of the Theorema Egregium.