Differential Geometry – Classical and Discrete

Curves 3: Curvature (cont.)

Emil Saucan

Ort Braude College, Karmiel

Dalian University of Technology

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Menger curvature - cont.

To define the Menger curvature at a given point on a curve, one passes to the limit (precisely like in the classical osculating circle definition) and formulate the somewhat cumbersome definition below :

Definition

Let (M, d) be a metric space and let $p \in M$ be an accumulation point. We say that M has at p *Menger curvature* $\kappa_M(p)$ iff for any $\varepsilon > 0$, there exists $\delta > 0$, such that for any triple of points p_1, p_2, p_3, satisfying $d(p, p_i) < \delta,\ i = 1, 2, 3$; the following inequality holds : $|\kappa_M(p_1, p_2, p_3) - \kappa_M(p)| < \varepsilon$.

Menger curvature - cont.

• $\kappa_M \geq 0$ and that $\kappa_m \equiv 0$ on a Euclidean line, as the reader can easily heck.

• However, Menger curvature is not necessarily defined at all the points of any curve/of a metric space, as the following simple example demonstrates :

Example

Let (X, d) the metric space consisting of three rays $\overrightarrow{PX}, \overrightarrow{PY}, \overrightarrow{PZ}$ in $\mathbb{R}^2$ having P as the only common point, endowed with the metric d, where $d(A, B)$ is the usual Euclidean distance d_2 if A, B are on the same ray, and $d(A, B) = d_2(A, P) + d_2(P, B)$ if A, B are on different rays.
Then $\kappa_M(A) = 0$ for any point $A \in X, A \neq 0$, but $\kappa_M(P)$ is not defined.

Menger curvature - cont.

- However, the very existence of Menger curvature at every point of a curve guarantees that the curve satisfies an important property of "well-behavior" :

Proposition

If a continuum C in a metric space (X, d) has finite Menger curvature at a point $p \in C$, then C is a rectifiable arc in a neighbourhood of p.
If $\kappa_M(p)$ exists and it is finite, at every point $p \in C$, then C is a rectifiable arc or a rectifiable simple closed curve.

Haantjes Curvature – Motivation

- As we have already underlined before, both Menger curvature suffers from the same impediment, namely that it imposes a Euclidean geometry on the studied space/data.
- Even if one allows for the Spherical and Hyperbolic versions, one still operates using a constant background geometry.
- Furthermore, both curvatures can take into account, by their very definitions, only triangles, a fact that represents a serious limitation in many real life applications.
- Fortunately, a much more flexible notion of metric curvature for 1-dimensional geometric objects is available, namely the so called *Haantjes curvature* or *Finsler-Haantjes curvature*.

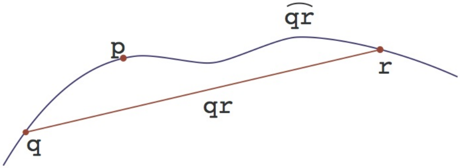
Haantjes curvature

Definition (Haantjes curvature)

If c is a curve in a metric space with metric d, and p, q, r are points on c, p between q and r, the *Haantjes curvature* of c at p is defined as

$$\kappa_H^2(p) = 24 \lim_{q,r \to p} \frac{l(\widehat{qr}) - d(q,r)}{\big(l(q,r)\big)^3} \;;$$

where $l(\widehat{qr})$ denotes the length of the arc $\widehat{qr}$.



Haantjes curvature - cont.

This is a very intuitive definition.

Haantjes curvature - cont.

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Haantjes curvature - cont.

- Haantjes curvature can be construed as a quantitative version of the following theorem :

Theorem (A. Schur)

Let γ_1, γ_2 two arcs parametrized by arch length, of ends p_1, q_1 and p_2, q_2, and curvatures κ_1, κ_2, respectively.

If $\gamma_1 \cup p_1q_1$ is a simple closed convex curve, and if $\kappa_2 \leq \kappa_1$, then

$$d(p_2, q_2) \geq d(p_1, q_1)$$

.

Haantjes curvature - cont.

• The connection with Menger curvature can be seen through the following consequence of Schur's Theorem :

Theorem (H. A. Schwarz)

Let $p, q \in \mathbb{R}^2$ and let $r \geq d(p,q)/2$, let C the circle of radius r passing through p, q, and let γ be a curve connecting p and q, such that $\kappa_\gamma \leq 1/r$.

Then $\ell(\gamma) \leq \ell(C_m)$ or $\ell(\gamma) \leq \ell(C_M)$, where C_m, C_M denote the lesser, respectively greater arcs of C determined by p and q.

Haantjes curvature - cont.

Alternatively, since for points/arcs where Haantjes curvature exists, $\frac{l(\widehat{qr})}{d(q,r)} \to 1$, as $d(q,r) \to 0$; κ_H can be defined by

$$\kappa_H^2(p) = 24 \lim_{q,r \to p} \frac{l(\widehat{qr}) - d(q,r)}{\left(d(q,r)\right)^3} \, ;$$

In applications it is this alternative form of the definition of Haantjes curvature that will prove to be more malleable, as we shall illustrate shortly.

Haantjes curvature - cont.

- Unfortunately, the expression of κ_H is far from intuitive.
  - Let us observe that it is proportional to $1/l$ (or $1/d$), which hints to the radius of curvature (and to Menger curvature).
  - Less transparent and definitely more cumbersome is the factor of "24" appearing in the definition.

    However, the proof of our next theorem show that the two are interrelated and that the "24" factor arises naturally.

Theorem

Let $\gamma \in \mathcal{C}^3$ be smooth curve in $\mathbb{R}^3$ and let $p \in \gamma$ be a regular point. Then the metric curvature $\kappa_H(p)$ exists and equals the classical curvature of γ at p.

Haantjes curvature - cont.

It turns out that if all these metric notions of metric curvature are applicable, then they coincide :

Theorem (Haantjes)

Let γ be a rectifiable arc in a metric space (M, d), and let $p \in \gamma$. If κ_A (κ_M) and κ_H exist, then they are equal.

Démonstration.

We begin by denoting by κ any of the curvatures κ_M or κ_A that exists.
Let $q, r \in \gamma$ be two points on the same side of p, such that $d(p, q) = d(q, r) = d$, and let $d(p, r) = a$. Then

$$\kappa^2(p) = \lim_{d \to 0} \frac{a^2(2d + a)(2d - a)}{a^2 d^4} = \lim_{d \to 0} \frac{4}{d^2}\left(1 - \frac{a}{2d}\right)\left(1 + \frac{a}{2d}\right).$$

Haantjes curvature - cont.

Given that $\kappa^2(p)$ is defined, it follows that $\lim_{d\to 0}\frac{a}{2d} = 1$, therefore

$$\kappa^2(p) = 4\lim_{d\to 0}\frac{2d-a}{d^3}\,.$$

Therefore from the existence of $\kappa_H(p)$ it follows that

$$\frac{1}{4!}\kappa^2(p) = \lim_{d\to 0}\frac{l(\widehat{pq})-d}{l(\widehat{pq})^3} = \lim_{d\to 0}\frac{l(\widehat{qr})-d}{l(\widehat{qr})^3} = \lim_{d\to 0}\frac{l(\widehat{pq})+l(\widehat{qr})-a}{[l(\widehat{pq})+l(\widehat{qr})]^3}\,.$$

If we simplify the notation by putting $l(\widehat{pq}) = l_1, l(\widehat{qr}) = l_2$, we obtain that

$$\frac{l_1+l_2-a}{(l_1+l_2)^3} = \frac{l_1-d}{l_1^3}\,\frac{l_1^3}{(l_1+l_2)^3} + \frac{l_2-d}{l_2^3}\,\frac{l_2^3}{(l_1+l_2)^3} + \frac{2d-a}{d^3}\,\frac{d^3}{(l_1+l_2)^3}\,.$$

Haantjes curvature - cont.

Since

$$\lim_{d\to 0}\frac{l_1}{d}=\lim_{d\to 0}\frac{l_2}{d}=1\,,$$

it follows that

$$\kappa^2(p)=4\lim_{d\to 0}\frac{2d-a}{d^3}=\kappa^2(p)\,.$$

$\square$

Haantjes curvature - cont.

Remark

Apparently, Haantjes curvature is more restricted then Menger curvature, since it requires rectificability. However, as we have seen, existence of Menger curvature at the points of a metric arc ensures its rectifiability, thus Haantjes curvature is also applicable.

Remark

- *The existence of Alt curvature does not imply the existence of Haantjes curvature.*

Counterexample

Let $C \subset \mathbb{R}^2$, $C = \{(x, y) \,|\, y = 0 \text{ if} x = 0; \text{and } y = x^4 \sin 1/x, \text{if } x \neq 0\}$. *Then* $\kappa_A(0)$ *exists, while* $\kappa_H(0)$ *is not defined.*

Haantjes curvature - cont.

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Haantjes curvature - cont.

- The existence of $\kappa_M(p)$ implies the existence of $\kappa_H(p)$.
- The existence of $\kappa_H(p)$ does not imply the existence of $\kappa_M(p)$.

Counterexample

Let $C = [0, 1]$, with the metric $d(x, y) = t - \frac{1}{3!}t^3 + \frac{1}{4!}t^4 \sin\frac{1}{t}$, where $t = |x - y|$, $x \neq y$. Then (C, d) is a rectifiable metric arc, with $\kappa_H(p)$ existing at every point $p \in C$, while $\kappa_A(p), \kappa_M(p)$ are not defined at any point $p \in C$.

Haantjes curvature – Applications : Wavelet Scale

- The notion of scale is fundamental in many mathematical and applicative discussions.
- However, while it has a clear intuitive meaning, but is hard to define mathematically.
- Therefore, the question of finding a measure for calculating the local scale in signals and images it is important in scale space analysis and wavelets transform, for :
  - Image matching and registration
  - Computer vision systems analyzing an unknown scene (no way to know a priori what scales are appropriate), in particular for
    - Blob, corner, ridge and edge detection
    - Texture segmentation

Haantjes curvature – Applications : Wavelet Scale (cont.)

However, curvature is, as we have seen, a natural – and easy to compute – notion for images, as well as more general signals. It is therefore natural to ask whether a connection exists between scale and curvature.

For (classical = 1-dimensional) wavelets, a natural (obvious) candidate is the Haantjes curvature.

Haantjes curvature – Applications : Wavelet Scale (cont.)

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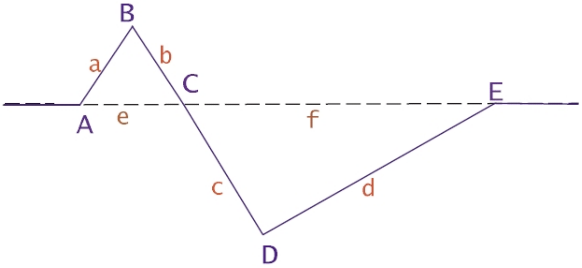
This is easy to understand for a *PL* signal :



Haantjes curvature – Applications : Wavelet Scale (cont.)

We have :

$$l(\widehat{AE}) = a + b + c + d \,, d(A,B) = e + f \,.$$

Then :

$$\kappa_H^2 = 24[(a + b + c + d) - (e + f)]/(e + f)^3 \,.$$

We can also look at the curvature at the "peaks" B and D

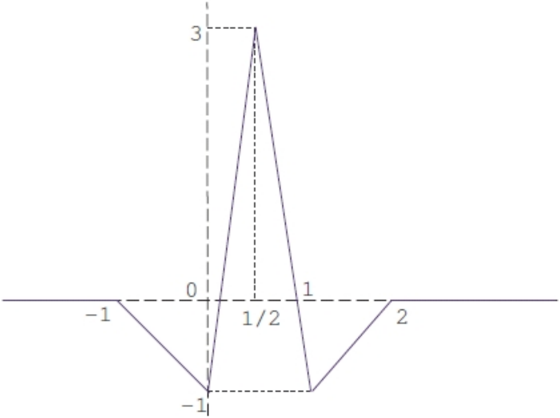
$$\kappa_B^2 = 24[a + b - e]/e^3 \,,$$

$$\kappa_D^2 = 24[(c + d - f]/f^3 \,.$$

as well as the *mean curvature* of $\wp$:

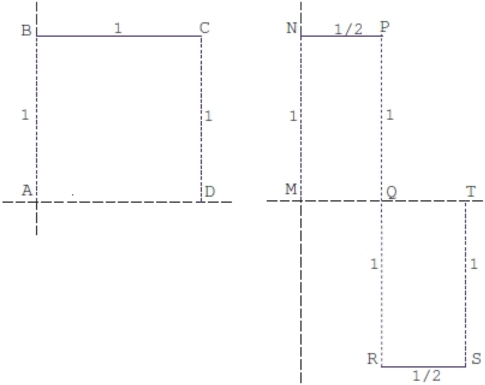
Haantjes curvature – Applications : Wavelet Scale (cont.)

An important *PL* wavelet is the Meyer wavelet :



Haantjes curvature – Applications : Wavelet Scale (cont.)

But what about more *fundamental* and widely used types of wavelets ? !...



The Haar scaling function and wavelet.

Haantjes curvature – Applications : Wavelet Scale (cont.)

For the scaling function :

$$l(\widehat{AD}) = d(A,B) + d(B,C) + d(C,D) = 3, d(A,d) = 1.$$

Therefore

$$\kappa_H^2(s_H) = 6\,.$$

For the wavelet :

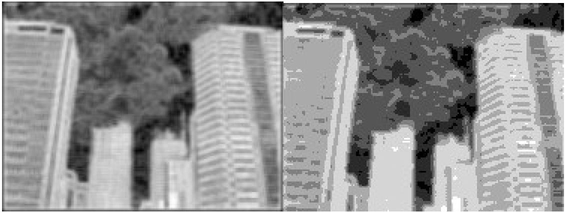
$$l(\widehat{MT}) = d(M,N) + d(N,P) + d(P,R) + d(R,S) + d(S,T) = 5, d(M,T) = 1\,,$$

and the formula for $\kappa_H^2(s_H)$ is now easily obtainable.

Haantjes curvature – Applications : Wavelet Scale (cont.)

Haantjes-Finsler curvature of an image with respect to different scales.

Haantjes curvature – Applications : Wavelet Scale (cont.)



Texture segmentation of an urban landscape image: Average Haantjes curvature (bottom, left) and the texture segmentation of image (bottom, right), using 7 scales.

Haantjes curvature - Networks

- For networks $\widehat{qr}$ is replaced by a path $\pi = v_0, v_1, \ldots, v_n$, and the subtending chord by $\bar{e} = \overline{v_0 v_n}$.
- Clearly, the limiting process has no meaning in this discrete case.
- Furthermore, the normalizing constant 24 is superfluous in this setting.

This leads to the following definition of the *Haantjes curvature of a path* π :

$$\kappa_H^2(\pi) = \frac{l(\pi) - l(v_0 v_n)}{l(v_0 v_n)^3} \; ;$$

where, if the graph is a metric graph, $l(v_0 v_n) = d(v_0 v_n)$.

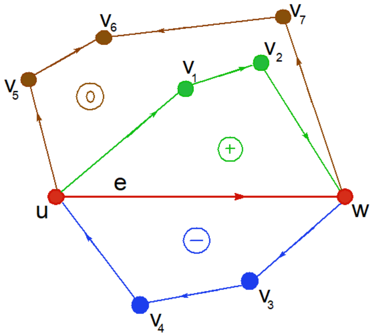
(In particular, for the combinatorial metric, $\kappa_H(\pi) = \sqrt{n-1}$.)

Haantjes curvature - Networks - cont.

Clearly, one can extend the above definition to directed paths

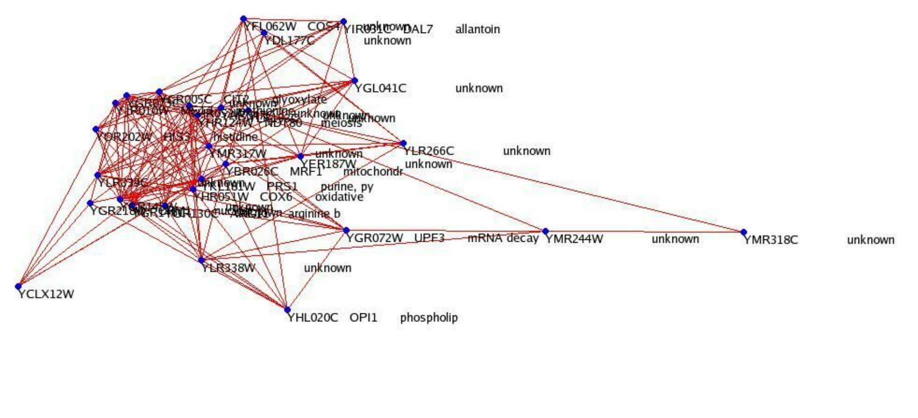
$$\kappa_{H,O}(T) = \varepsilon(\pi) \cdot \kappa_H(T)\,;$$

for every directed path π, where $\varepsilon \in \{-1, 0, +1\}$ denotes the orientation of π.

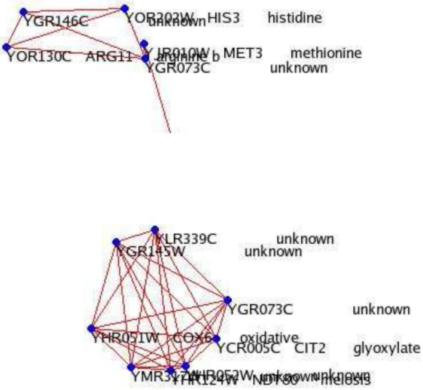


Haantjes curvature – Applications

- Clustering in gene networks



Haantjes curvature – Applications



Combinatorial (up) and Metric Curvature (below) based Clusterings: The results of the clustering as processed on a part of the yeast gene expression, for $T_{cur} = 0.6$ and correlation threshold $T_{cor} = 0.85$.

Haantjes curvature – Applications

Combinatorial (left) and Metric Curvature (right) based Clusterings: The results of the clustering as processed on a part of the yeast gene expression, for $T_{cur} = 0.7$ and correlation threshold $T_{cor} = 0.85$.

Haantjes curvature – Applications

FIGURE: The Haantjes-Ricci curvature based sampling of "Cameraman". Here 20% of the edges were retained and the main features of the image are still clearly visible. Note the blue coloring of the resulting curvature image, showing that the Haantjes-Ricci curvature is positive.

Haantjes curvature – Applications

In the setting of semantic networks [1] we introduced the

Definition (*The simplified and modified Haantjes curvature*)

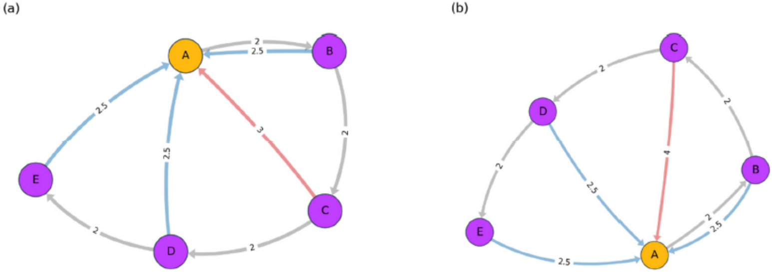
$$\lambda_H(\vec{\pi})_{0,v_0} = \frac{\ell(\pi) - \max_{i=1,\dots,n}(w(v_0, v_1))}{\max_{i=1,\dots,n}(w(v_0, v_1))}$$

where $\max_{i=1,\dots,n}(w(v_0, v_1))$ represents the maximal distance back to the starting point.

- $\max_{i=1,\dots,n}(w(v_0, v_1))$ enables us to estimate the amount of "effort" needed to v_0.
- We can now allow cases in which the path curvature is negative.

1. Cohen, Nachson, Naim, Maril, Jost and S., *A Path-Curvature Measure for Word-Based Strategy Searches in Semantic Networks*, Symmetry, **14**(8), 1737, 2022.

Haantjes curvature – Applications



Toy example for the simplified and modified Haantjes curvature. In both figures, $\pi = (A, B, C, D, E)$ and $\ell(\pi) = 8$ are represented by gray edges, and $\max_{i=1,\dots,n}(w(v_0, v_1))$ is attained on the red edge. The value λ_H of the path in (a) is $\frac{8-3}{3} = 1.25$ and for (b) is $\frac{8-4}{4} = 1$. Hence, the path in (a) is more curved and circulates in a smaller "radius" compared to the path length in (b).