Differential Geometry – Classical and Discrete

Gauss Curvature – Some Implementation Aspects

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Motivation

- We should mention, however briefly, the role the notions introduced so far play in
  - CAD,
  - Graphics
  - Imaging & Vision
  - Geometric Modeling and Manufacturing

  and the way they are computed (or rather, approximated) in practice.

- The approaches listed below (based on approximations) are both older and different in spirit from the ones we promulgate in this course (which rest on discretization), they are never the less important, and not just historically.

Implementations

- We begin with the observation that for such implementations, polygonal meshes constitute the basic representations of geometry that are employed.
- Analysis of such data sets is extremely important in many applications such as reconstruction, segmentation and recognition or even non-photorealistic rendering.
- In this context, curvature analysis plays a major role.
  - As an example, curvature analysis of 3D scanned data sets were shown quite a long time ago to be one of the best approaches to segmenting the data [Yokoya & Levine '89].

Implementations - cont.

- Let us note first that in [Hamann 93] and [Stokely 92], the principal curvatures and principal directions of a triangulated surface are estimated at each vertex by a least square fitting of an osculating paraboloid to the vertex and its neighbors.
    - In these papers references use quadratic approximation methods where the approximated surface is obtained by solving an over-determined system of linear equations.
- In [Martin 98], circular cross sections, near the examined vertex, are fitted to the surface. Then, the principal curvatures are computed using Meusnier's and Euler's theorems.

Implementations - cont.

- In [Taubin 95], the principal curvatures are estimated with the aid of an eigenvalues/vectors analysis of 3×3 symmetric matrices that are defined via an integral formulation. These can be used to approximate Gauss and mean curvature.
- Another method for the computation of Gaussian curvature based on the asymptotic analysis of the paraboloid fitting scheme is given in [Meek 00], .
- Also, in [Watanabe 01] the fact that

$$\frac{1}{2\pi}\int_0^{2\pi} k_n(\varphi)d\varphi = H \text{ and } \frac{1}{2\pi}\int_0^{2\pi} k_n(\varphi)^2 d\varphi = \frac{3}{2}H^2 - \frac{1}{2}K\,,$$

where $k_n(\varphi)$ is the normal curvature in direction φ, is used to estimate K and H, via discrete approximations of these integrals around a vertex

Implementations - cont.

- While we restrict here only to this modicum of methods, shall return to the problem of approximating surface curvature in the next chapters and we shall examine it in detail, from a theoretical, as well as practical viewpoint.
  - However, let us note that more details and comparisons regarding the above mentioned approaches can be found in [Surazhsky '03], [Belyaev '02], [Lev, Saucan and Elber].
- Moreover, not just computation of Gaussian and mean curvature are of interest, but also the extraction of other geometric features :
  - The determination of umbilics and of lines of curvature [Maekawa '96].
  - The extremalities of curvature are also of interest [Yoshizawa '21].
  - More details an all the directions presented above can be found in the monograph [Patrikalakis & Maekawa '02].

- We wouldn't like to conclude this part with this enumerative note, however important it might be.
- Instead, we would like to emphasize the importance of curvature in Imaging, Vision, CAD, and related fields, since abrupt changes in its magnitude indicate the presence of *edges* and *contours*.
  - We do not expatiate on this subject since it is intuitive and, more importantly, we already addressed with specific applications in our chapter on curves and we shall again return to it in the sequel.

Koenderink's Work

- Instead we rather consider here the sometimes forgotten subject of projections and their curvature.
- The importance of this topic, in Imaging, Vision, Graphics and GAGD should be evident :
  - 2-dimensional images (photographs in particular) are nothing but the projection of 3-dimensional object on a 2-dimensional screen, so they might be viewed as the shades of the object on the said “wall” (plane).
- Given that axonometric drawings is a more than classical tool, indeed of paramount importance, in Technical (particularly in machine) Drawing and Architecture, it is rather strange that the main result below is rather new. [1]

1. A reason for this might be not just a certain degree of conservatorism, but also the fact that people tended to concentrate on straight lines designs.

Koenderink's - cont.

To be able to enunciate it, we need the following

Definition

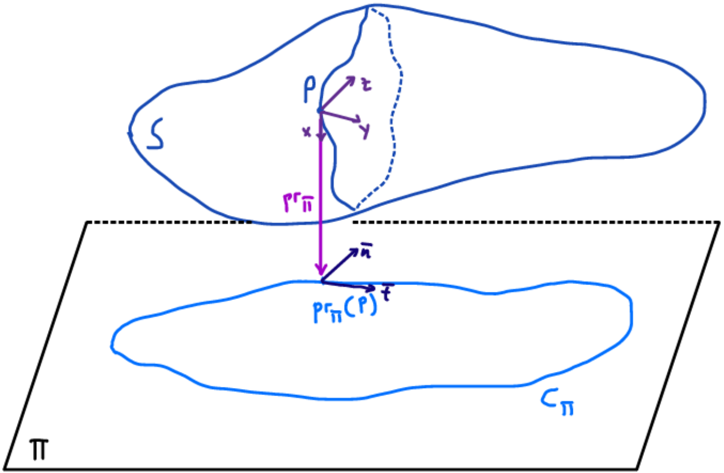
Let S be a surface and let consider the orthogonal projection pr_Π of S onto a plane Π. The contour C_Π of the projection $S_\Pi = \mathrm{pr}_\Pi(S)$ is called the *apparent contour* (of S on Π). The points of S projected onto the apparent contour constitute the *rim* of S as viewed on (or by) Π.

Theorem (Koenderink ; "Shape from contour")

Let S be a surface and consider the orthogonal projection pr_Π of S onto a plane Π. Then the Gaussian curvature K of S satisfies the following equality :

$$K = k_c k_n\,;$$

Koenderink's - cont.



Koenderink's - cont.

Exercise

Prove Koenderink's Theorem.

Hint : Approximate locally the surface by an order 2 quadric.

Remark

Koenderink also formulated and proved a version of his theorem where the projection "screen" is the unit sphere.

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