Differential Geometry – Classical and Discrete

Gauss Curvature – First Definition

Emil Saucan

Braude College, Karmiel

Dalian University of Technology

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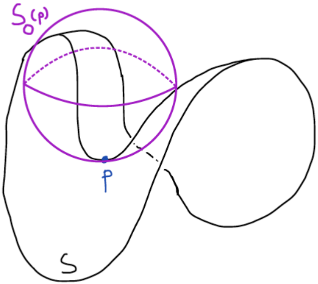
The Naive Approach

- When passing from planar curves to surfaces in Euclidean space, it is very tempting to try and follow the same route, namely to define curvature via the *osculating sphere*.
- However, while this approach would certainly still be valid for convex surfaces, one can not avoid but noticing that it is not applicable when the considered surfaces has *saddle points*



The Naive Approach - cont.

Exercise

Properly define the osculating sphere at a point on a surface, determine its center and radius and write its equation.

Remark

Precisely like in the curve's case, one might be naturally conduced to the idea that it might be possible to define surface curvature by approximating them with paraboloids, rather then with spheres. This turns, indeed, to be a successful strategy, for surfaces as well, not just for planar curves.

Problem

Define the osculating paraboloid and explore its properties.

The Naive Approach - cont.

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Properly define the osculating sphere at a point on a surface, determine its center and radius and write its equation.

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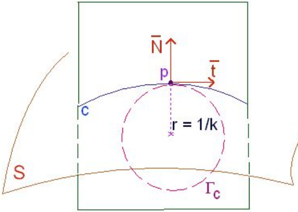
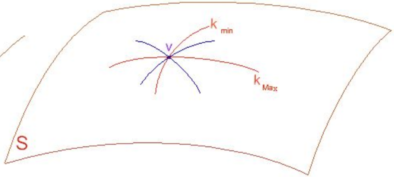
Problem

Define the osculating paraboloid *and explore its properties.*

The "Natural" Approach

- This represents, in Spivak's formulation, "what they knew about surfaces before Gauss".
- To do this, one "cuts" the surface with *normal* planes Π, each such "cut" producing a curve $C = S^2 \cap \Pi$, having curvature $k_C = k_C(p)$ at the point p.





(Note that $\Pi =< N_p, t_c >$, where N_p denotes the normal vector to S at p and t_C represent the tangent vector to C at p; hence the importance of "cutting" the surface only with normal planes.)

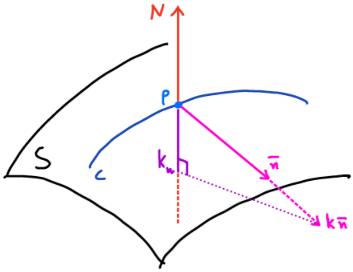
The "Natural" Approach - cont.

- The first such result (at least from a logical) viewpoint shows how to compute the curvature of a curve obtained by sectioning the surface by a non-normal plane, thus showing that the choice of *normal sections* is not truly restrictive. But first, let us introduce a new definition :

Definition

Let C be a regular curve passing through a point p on a surface S, and let $k = k_C(p)$ denote its curvature at p. Denote by θ the angle between the normal N to the surface at P, i.e. the angle given by $\cos(\theta) = n \cdot N$, where n represents the normal vector to C at p. The length of projection of the vector kn on N, i.e. $k\cos(\theta)$ is denoted by k_n and is called the *normal curvature* (of C at the point p).

The "Natural" Approach - cont.

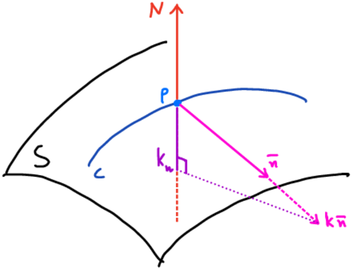


As the notation clearly suggests, the definition of normal curvature is independent of the choice of the curve C :

Theorem (Meusnier, 1976)

All curves on a surface S passing through a given point $p \in S$, and having the same tangent vector at p, have the same normal curvature $k_n = k_n(p)$.

The "Natural" Approach - cont.



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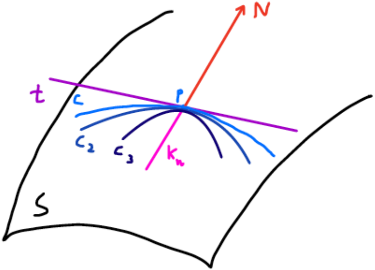
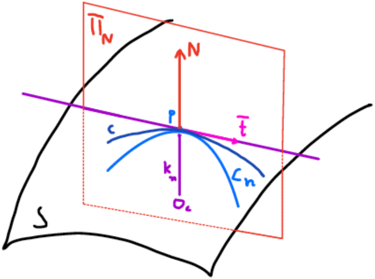
The "Natural" Approach - cont.

In other words, if Π_φ is a plane intersecting the surface S at the point p, such that $\measuredangle(\Pi_\varphi, \Pi) = \varphi$, then the curvature (at p) k_φ of the curve $\gamma = S \cap \Pi_\varphi$ is related to the normal curvature via the following simple relation :

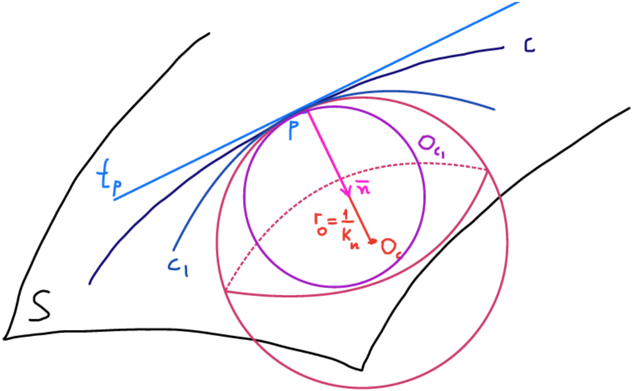
$$k_n = k_\varphi \cdot \cos\varphi\,.$$





The "Natural" Approach - cont.

The geometric interpretation of Meusnier's Theorem is that the osculating circles of the plane sections though the same tangent to the surface are contained in a sphere .



The "Natural" Approach - cont.

Another geometric insight into Meusnier's Theorem is given in the following

Proposition

Let c be a curve on a surface S. Then its center of curvature at a point p is the normal projection onto the osculating plane at the surface at S of the center of curvature of c_n – the normal section at p.

We leave the proof as an exercise.

- It follows that one can study only the curvature of normal sections, a fact that will prove important in the sequel.
- Normal curvature at a point p of a curve c on a surface S, does not depend on the orientation of C (since the curvature of planar curves does not), but it does depend on the orientation of S (which it is defined via the normal at p).

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The "Natural" Approach - cont.

Of special import are the extremal values of k_n. It turns out that not only are the minimal and maximal values $k_{\min}(p), k_{\max}(p)$ of k_n attained, they occur, in fact, in orthogonal directions – let they be given by the orthonormal basis $\{e_1, e_2\}$ of $T_p(S)$. This warrants a formal definition :

Definition

The minimal and maximal values $k_{\min}(p), k_{\max}(p)$ of normal curvature are called the *principal curvatures* (at p). The corresponding directions given by e_1, e_2 are called the *principal directions* (at p).

Remark

Traditionally the principal directions are denoted as $k_1 = k_{\min}(p)$, $k_2 = k_{\max}(p)$, thus the associated principal directions are numbered fittingly.

The "Natural" Approach - cont.

Examples

A number of important, if immediate, examples must be listed :

1. *Obviously, for any point in the plane* $k_{\text{min}}(p) = k_{\text{max}}(p) = 0$.
2. *The principal curvatures are clearly equal also at any point of the sphere and equal to* $1/R$, *where* R *is the radius of the sphere (since they are the reciprocal of the osculating radius of a great circle).*
3. *On a cylinder, the principal curvatures are* $1/R$, *where* R *is the radius of the base circle and* 0 *– the curvature of the generatrix. Thus all other sections (which are ellipses) have curvatures between* 0 *and* $1/R$. *(The reader is encouraged to provide for him/herself the accompanying drawing.)*
4. *The principal curvatures of the saddle surface (hyperbolic paraboloid)* $z = x^2 - y^2$ *at* $O = (0,0)$ *are opposite to each other.*

The "Natural" Approach - cont.

- Having obtained the possible information regarding the curvature of one curve, passing through a given point on a surface, one can study the curvatures of all the members of the family of curves on S passing through p.
- It turns out that one can even compute the normal curvature of any curve on a surface, given that the principal curvatures (at the given point) are known. More precisely, we have the following theorem :

Theorem (Euler, 1760)

Let $p \in S$ be such that $k_{\min}(p) \neq k_{\max}(p)$, and let $\mathbf{v} \in \vec{\mathbf{T}}_{\mathbf{p}}(\mathbf{S})$ such that $\measuredangle(\vec{\mathbf{v}}, \mathbf{e_1}) = \theta$. Then

$$k_n = k_{\min} \cos^2 \theta + k_{\max} \sin^2 \theta \,.$$

The "Natural" Approach - cont.

Remark

1. *One can consider the angle φ with e_2, rather than with e_1, given that $\varphi = \pi/2 - \theta$, hence $\sin\varphi = \cos\theta, \sin\theta = \cos\varphi$.*
2. *One can choose the vector $-\vec{\mathbf{v}}$, which will accordingly affect the signs of the principal curvatures. (Check !)*

Remark

A proof in the spirit of Euler's original one is quite lengthy.

To give a more modern one once we need the contemporary mathematical tool of the two fundamental forms.

Exercise

What can be said if $k_{\min}(p) = k_{\max}(p)$?

The "Natural" Approach - Gauss and mean Curvature

The principal curvatures, in conjunction, define the curvature at a point of the surface. More precisely, we have

Definition (Gauss curvature)

The product $K = k_{\min}k_{\max}$ is called the *Gauss* (or *Gaussian curvature* of the surface (at a point p).

While this definition is natural (and it seemed so in Gauss' own time), it is not less (or even perhaps even more so) logical to consider the arithmetic, rather than (the square of) the geometric mean.

Definition (Mean curvature)

$H = (k_{\min} + k_{\max})/2$ is called the *mean curvature* of the surface (at a point p).

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Gauss and mean Curvature

- Clearly the second definition is simpler to handle.
- However, we shall see that Gaussian curvature is more "powerful", so to say.
- This is not to say that mean curvature is not an extremely important notion, with a plethora of practical applications.
- For now let us content with the fact that Gaussian curvature is the one that allows us to classify the points on a surface.

Definition

A p be a point on a (smooth) surface $S \subset \mathbb{R}^3$ is called

1. *Elliptic*, if $K(p) > 0 \Leftrightarrow k_{\text{min}} \cdot k_{\text{Maz}} > 0$;
2. *Hyperbolic*, if $K(p) < 0 \Leftrightarrow k_{\text{min}} \cdot k_{\text{Maz}} < 0$;
3. *Parabolic*, if $K(p) = 0$, but not both principal curvatures are 0 ;
4. *Planar*, if both principal curvatures are 0 .

Gauss and mean Curvature - cont.

It should be not too surprising, after going through a Calculus 2 course, that the four types of points encode the basic possible shapes of surfaces in a vicinity of a point. More precisely, in the neigbourhood of a point p a surface is

1. An elliptic paraboloid, if p is an elliptic point ;
2. A hyperbolic paraboloid if p is a hyperbolic point ;
3. A parabolic cylinder, if p is a parabolic point ;
4. In this last case, the situation is more complicated : Clearly a plane satisfies it, but so does, for instance, the surface generated by the rotation of the graph of $z = y^4$ around the Oz axis, as well as the "monkey saddle" $z = x^3 - 3xy^2$.

Gauss and mean Curvature - cont.

The classical quadratic surfaces above do not appear by pure chance. Instead, they are the result of a way of circumventing, at least partially our failure to understand curvature via the osculating sphere. The idea is to use, instead, the *osculating paraboloid* which approximates the surface up to order 2, at the point p. It is defined, using our freshly gained knowledge of principal curvatures, as

$$\psi(u, v) = \frac{1}{2}(k_{\min} u^2 + k_{\max} v^2)\,.$$

(Here we identify p with $0 = (0, 0) \in \mathbb{R}^2$.)

Gauss and mean Curvature - cont.

By restricting this expression to the tangent plane at p and considering the *asymptotic directions* (i.e. having director vectors equal to the *principal vectors*), we obtain the so called Dupin indicatrix :

$$k_{\min} u^2 + k_{\max} v^2 = \pm 1$$

It is, as expected, an ellipse, if p is an elliptic point and a hyperbola if p is a hyperbolic point, while for a planar point it degenerates into a pair of parallel lines. We shall encounter the osculating paraboloid again shortly, after introducing the *second fundamental form*.

Lines of curvature.

- There are two types of “interesting” types of curves on surfaces which we can introduce quite early in the course. [1]
- The first such kind of curves is quite natural to define :

Definition

Let $c \subset S$ a regular, connected curve, such that $\mathbf{t}(p)$ – the unit tangent to c at p – is a principal direction for any $p \in c$ is called a *line of curvature* of S.

1. The definition of a third one, much more important, will have to be postponed for a while.

Lines of curvature - cont.

Lines of curvature admit the following characterization :

Theorem (O. Rodrigues, 1813)

A regular, connected curve c on a surface S is a line of curvature if and only iff

$$N'(t) = \alpha(t)\tilde{c}\,'(t)\,;$$

where $\tilde{c}(t)$ is any parametrization of c, $N'(t) = N \circ \tilde{c}$, and and where $\alpha(t)$ is a differentiable function (of t). Moreover, in this case, $-\alpha(t)$ represents the principal curvature in the direction of $\tilde{c}\,'(t)$.

The proof is quite simple and short, therefore we leave it as an exercise.

Umbilics

- Let us note that in the plane and on the sphere all directions at any point are principal ones, whereas this special situation certainly does not happen at any point of the cylinder ot parabolic hyperboloid, for instance.
- Thus points where all directions are principal directions are clearly special, thus deserving a definition of their own :

Definition

Let $p \in S$, such that $k_{\min}(p) = k_{\max}(p)$. Then p is called an *umbilical point* (or an *umbilic*[a]).

a. from the Latin word for "navel".

Umbilics - cont.

Examples

1. *As we have seen, all the points of the sphere and the plane are umbilics.*
2. *The point* $O = (0, 0, 0)$ *is an umbilical point of the paraboloid* $z = x^2 + y^2$.
3. *The point* $O = (0, 0, 0)$ *is an umbilical point of the monkey saddle* $z = x^3 - 3xy^2$. *(In this case* $k_{\min}(O) = k_{\max}(O) = 0$.*)*
4. *On the other hand, the torus of revolution has no umbilics.*

Umbilics - cont.

- The natural question arises whether there exist other surfaces, apart from the plane and the sphere, on which all point are umbilics ?
- The answer is "No" – this property is characteristic for the plane and the sphere :

Theorem

Let S be a connected surface such that all its points are umbilics. Then S is a subset of a plane or of a sphere.

Umbilics - cont.

1. One can still hope to easily find many umbilical points on a variety of surfaces.
2. Indeed, as we have seen, the ellipsoid has four umbilics, while the monkey's saddle has one umbilical point, for which $k_{\min}(p) = k_{\max}(p) = 0$. (Thus one says that p is a *planar umbilical point*.)
3. On the other hand, the torus of revolution has no umbilical points.
4. Still, one might hope that (closed) surfaces other than the torus resemble more the ellipsoid and do have umbilics. Indeed, we have the following

Proposition

Let S be a smooth surface in $\mathbb{R}^3$ which is mot a torus of revolution. Then S has at least one umblical point.

Umbilics - cont.

- One is entitled to believe that, due to symmetry reasons, and motivated by the case of the ellipsoid, if not all, then, at least surfaces homeomorphic to the sphere have at least two umbilical points.
- Most surprisingly, this is not a settled matter, and at this point in time it still remains a conjecture :

Conjecture (Caratheodory)

If $S \subset \mathbb{R}^3$ is a compact surface, homeomorphic to the sphere, then S has at least two umbilics.

While the conjecture has been proved for real analytic surfaces, the proof is very involved and, moreover, the conjecture has not yet been settled in its generality.

Asymptotic lines

- The second type of remarkable lines on surfaces that we can present already are the so called *asymptotic lines* (or *asymptotic curves*) :

Definition

Let $c : I \to S$ be a curve. If $\vec{\mathbf{n}}(s) \perp \vec{N}(s)$, for all $s \in I$, then c is called an asymptotic line (or *curve*).

In other words, asymptotic lines are characterized by the property of being perpendicular to their spherical image.

Equivalently, **n** is always contained in the tangent plane which coincides with the osculating plane[2].

2. when it is defined

Asymptotic lines - cont.

Exercise

Show that c is an asymptotic line $\Longleftrightarrow k_n \equiv 0$.

As expected, straight lines are asymptotic lines. More precisely, we have

Proposition

Let $l \subset S$ be a straight line. Then

1. *l is an asymptotic line.*
2. $K|_l \equiv 0 \Leftrightarrow N|_l \equiv 0$.

We leave the proof as a straightforward exercise.

Asymptotic lines - cont.

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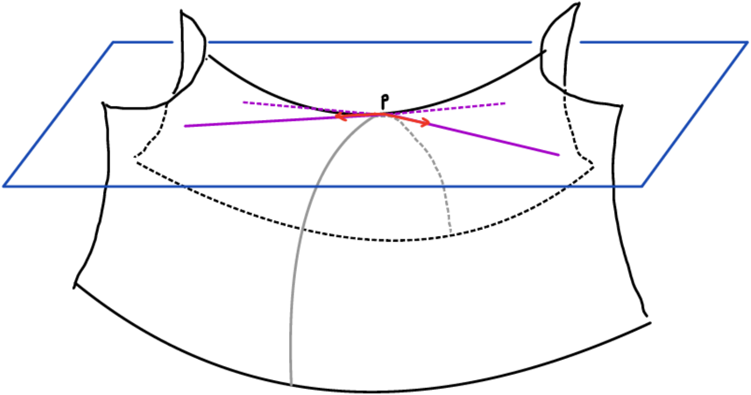
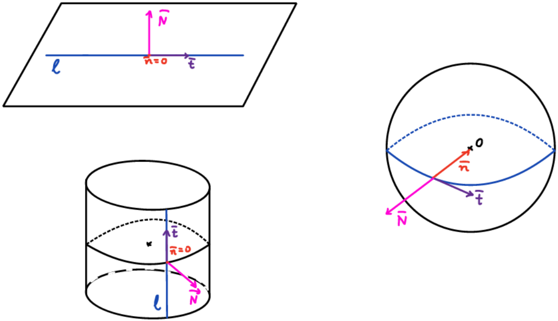
Asymptotic lines - cont.

As an immediate consequence of the result above we remark that the asymptotic lines of the hyperbolic paraboloid $S : x^2 - y^2 = 2z$ are its rectilinear generators, i.e. the curves $\{x = \mathrm{const.}\}$ and $\{y = \mathrm{const.}\}$.



Asymptotic lines - cont.

The example above conforms with our intuition, as well as with the previous proposition, that straight lines – and asymptotic lines in general – tend “fast” to infinity.



Left : Straight lines as asymptotic lines ; they tend fast to infinity.

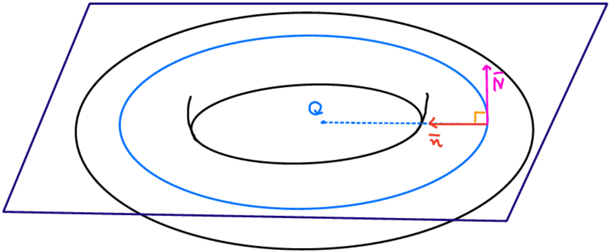
Right : Big circles on the sphere do not tend at all to infinity.

Asymptotic lines - cont.

It is less intuitive, perhaps that the upper and lower meridians of torus of revolution are also asymptotic lines, since they are, of course, circles, thus far from intuitively being lines.

Yet they – and their normals – are contained in the tangent plane to the torus.

In fact, one can view them as being lines "at infinity" since they are contained, entirely, in tangent plane.



Asymptotic lines - cont.

We conclude – for now – the discussion on special lines on surfaces with

Remark

Asymptotic directions, *that is directions which are tangent vectors to asymptotic lines exist only at points p such that $K(p) \leq 0$.*

Exercise

Prove the (simple) assertion above.

Classical Formulas

- The classical case of smooth surfaces, i.e. of class $\mathcal{C}^k, k \geq 2$, deserves, of course, special attention :

Let $U = \text{int}(U)$ be an open set and let $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ be a smooth function (i.e. $f \in \mathcal{C}^k, k \geq 2$). Then the expression of K in local coordinates is :

$$K = \frac{eg - f^2}{EG - F^2}\,;$$

where

$$E = f_u \cdot f_u\,,\ F = f_u \cdot f_v\,,\ G = f_v \cdot f_v\,;$$

and

$$e = \frac{det(f_u, f_v, F_{uu})}{\sqrt{EG - F^2}}\,,\ f = \frac{det(f_u, f_v, F_{uv})}{\sqrt{EG - F^2}}\,,\ g = \frac{det(f_u, f_v, F_{vv})}{\sqrt{EG - F^2}}\,;$$

Classical Formulas - cont.

Definition

$$I_f = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

and

$$II_f = \begin{pmatrix} e & f \\ f & g \end{pmatrix}$$

are called *the first*, respectively *the second fundamental form* of S.

We next bring more modern (and formal) approach to the essential geometric ideas above, which allows us to obtain a series of important results (albeit at the cost of lack of adaptability to discrete settings).

The First Fundamental Form

The *first fundamental form* of a surface encodes the manner in which a surface $S \subset \mathbb{R}^3$ inherits the (natural) inner product of the ambient space $\mathbb{R}^3$, thus the manner in which infinitesimal lengths are defined on the surface, and thus computing lengths of curves and angles between them, as well as areas of domains on the surface, in an *intrinsic* manner, i.e. without making appeal each time to the metric in the surrounding $\mathbb{R}^3$.

More precisely, given a parameterized surface $S = f(D), f : D \subseteq \mathbb{R}^2 \to \mathbb{R}^3$, and a point $p \in S$, consider $w_1, w_2 \in T_p(S)$. Then $w_1, w_2 \in \mathbb{R}^3$, therefore the inner product $w_1 \cdot w_2$ is well defined, and we use this simple observation to define the *first fundamental form* by lifting the inner product on $\mathbb{R}^2$ to $T_p(S)$

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The First Fundamental Form - cont.

Definition

Let S and p be as above. The *first fundamental form* of S (at p) $I_p : T_p(S) \to \mathbb{R}$ is defined as $I_p(w) = w \cdot w = ||w||^2 \geq 0$.

In matricial form the first fundamental form is written as $I_p = I = (g_{ij})$, this being the form which we shall use mainly later on or, as already introduced above as

$$I = \begin{pmatrix} E & F \\ F & G \end{pmatrix} .$$

Remark

While I is a bilinear form, $I_p : T_P(S)^2 \times T_P(S^2 \to \mathbb{R}$, it is symmetric (given defined via the internal product of $\mathbb{R}^3$), thus not a differential form (which has to be, by definition, antisymmetric.)

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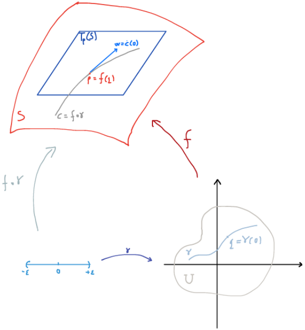
The First Fundamental Form - cont.

We wish to express I_p in terms of the (canonical) base $< \frac{\partial f}{\partial u}, \frac{\partial f}{\partial v} >$:

Given $w \in T_p(S)$, we can write (by its very definition) w as the tangent vector at p to a curve $c \in S$ passing through p, that is $w = \dot{c}(0)$



Then :

The First Fundamental Form - cont.

$$I_p(\dot{c}(0)) = (\dot{c}(0)\cdot\dot{c}(0))_p = \left(\frac{\partial f}{\partial u}\dot{u} + \frac{\partial f}{\partial v}\dot{v}\right)\cdot\left(\frac{\partial f}{\partial u}\dot{u} + \frac{\partial f}{\partial v}\dot{v}\right)$$

$$= \left(\frac{\partial f}{\partial u}\cdot\frac{\partial f}{\partial u}\right)(\dot{u})^2 + 2\left(\frac{\partial f}{\partial u}\cdot\frac{\partial f}{\partial v}\right)\dot{u}\dot{v} + \left(\frac{\partial f}{\partial v}\cdot\frac{\partial f}{\partial v}\right)(\dot{v})^2\,.$$

$$= E(\dot{u})^2 + 2F\dot{u}\dot{v} + G(\dot{v})^2\,.$$

We can now list the following formulary :

The First Fundamental Form - cont.

$$I_p(\dot{c}(0)) = (\dot{c}(0)\cdot\dot{c}(0))_p = \left(\frac{\partial f}{\partial u}\dot{u} + \frac{\partial f}{\partial v}\dot{v}\right)\cdot\left(\frac{\partial f}{\partial u}\dot{u} + \frac{\partial f}{\partial v}\dot{v}\right)$$

$$= \left(\frac{\partial f}{\partial u}\cdot\frac{\partial f}{\partial u}\right)(\dot{u})^2 + 2\left(\frac{\partial f}{\partial u}\cdot\frac{\partial f}{\partial v}\right)\dot{u}\dot{v} + \left(\frac{\partial f}{\partial v}\cdot\frac{\partial f}{\partial v}\right)(\dot{v})^2\,.$$

$$= E(\dot{u})^2 + 2F\dot{u}\dot{v} + G(\dot{v})^2\,.$$

We can now list the following formulary :

The First Fundamental Form - cont.

A. *Curve length* Given a curve $c : I \to S$, it is a classical Calculus fact that

$$\ell(c(t)) = \int_0^t ||\dot{c}(t)||dt = \int_0^t \sqrt{I(\dot{c(t)})}dt\,.$$

By denoting $c(t) = f(u(t), v(t))$, we obtain

$$\ell(c(t)) = \int_0^t \sqrt{E(\dot{u})^2 + 2F\dot{u}\dot{v} + E(\dot{u})^2(\dot{v})^2}dt$$

The length element of a curve on a surface $d\ell(t)$ is usually denoted simply by ds and we have

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2\,.$$

The First Fundamental Form - cont.

B. *Angles* Since the angle between the curves c and γ at an intersection point $p_0 = c(t_0) = \gamma(t_0)$ is given by

$$\cos(\theta) = \frac{\dot{c}(t_0) \cdot \dot{\gamma}(t_0)}{||\dot{c}(t_0)|| \cdot ||\dot{\gamma}(t_0)||}\,,$$

we obtain, with the same notation as above, that

$$\cos(\theta) = \frac{\frac{\partial f}{\partial u} \cdot \frac{\partial f}{\partial v}}{||\frac{\partial f}{\partial u}|| \cdot ||\frac{\partial f}{\partial v}||} = \frac{F}{\sqrt{EG}}$$

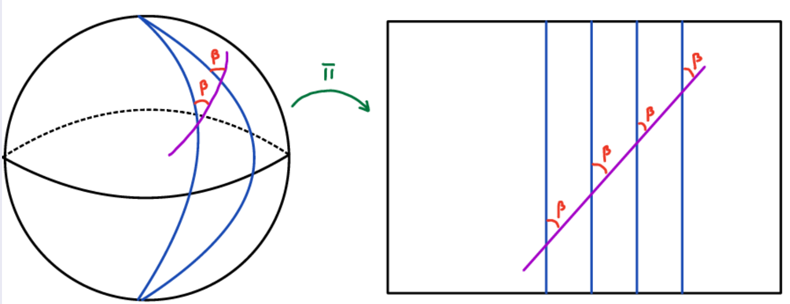
Remark

Note that from the last formula above it follows immediately that the coordinate curves are orthogonal iff $F = F(u, v) \equiv 0$.

The First Fundamental Form - cont.

Exercise

Determine the loxodromes of the sphere.



The meridians and loxodromes of the sphere are mapped by the Mercator projection onto straight lines intersecting at a constant angle, since the Mercator projection is conformal.

The First Fundamental Form - cont.

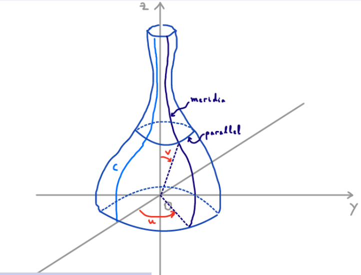
C. *Areas* Precisely like we have seen with regard to lengths, the area formula is also, essentially, the one from Calculus 2, namely

$$\mathrm{Area}(R) = \iint_Q \left\| \frac{\partial f}{\partial u} \times \frac{\partial f}{\partial v} \right\| dudv = \iint_Q \sqrt{EG - F^2} dudv$$

The First Fundamental Form - cont.

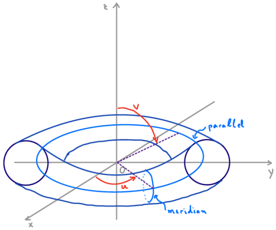
Examples

- Surfaces of Revolution *These are surfaces generated by the rotation, around the Oz axis of a curve* $c : I \to \mathbb{R}, c(t) = (\varphi(t), 0, \psi(t))$, *where* $\dot{\varphi}^2(t) + \dot{\psi}^2(t) > 0$ *and* $\varphi(t) \neq 0$ *for any* $t \in I$. *In other words, these are surfaces parametrized by a function* $f(u, v) = (\varphi(v)\cos(u), \varphi(v)\sin(u), \psi(v)), u \in (0, 2\pi), v \in (a, b), \varphi(v) \neq 0$.



The First Fundamental Form - cont.

- $\varphi(v) = \sin v, \psi(t) = \cos(v)$, i.e. for the *unit sphere*, $E = 1, F = 0, G = \sin^2 v$.
- $\varphi(v) = R + r\cos(v), \psi(t) = r\sin(v), 0 < r < R$, i.e. for the *torus (of revolution)*, $E = r^2, F = 0, G = (R + r\cos(v))^2$.



- $\varphi(v) = r\sin v, \psi(v) = r\ln|\tan(v/2)| + \cos(v)$, that is if the generating curve is the *tractrix*, the resulting surface is the *pseudosphere*. Then $E = r^2\cot^2(x), F = 0, G = r^2\sin^2(x)$.

The First Fundamental Form - cont.

Exercise

Let $f : \mathbb{R}^2 \to \mathbb{R}^3, f(u,v) = (u\cos v, u\sin v, 2u^2)$*, and the planar curves* $c_i : \mathbb{R} \to \mathbb{R}^2, i = 1,2,3$ *;*
$c_1(t) = (t^2, t), c_2(t) = (-t^2, t), c_3(t) = (t, 1)$.

1. *Compute the first fundamental form of* $S = f(\mathbb{R})$.
2. *Compute the lengths of the sides of the triangle* T *determined on* S *by the curves* $\gamma_i = f \circ c_i, i = 1,2,3$ *; and*
3. *The angles of* T.

The First Fundamental Form - cont.

Examples (cont.)

- *We conclude with an important example that is not a surface of revolution, namely the helicoid, which is the surface generated by lines perpendicular to the Oz axis and passing through the points of the (circular) helix.*

  Its parametrization is easily seen to be $f(u, v) = (v \cos u, v \sin u, bu)$, *where* $b \neq 0$. *Then* $E = b^2 + v^2, F = 0, G = 1$. *(Check !)*

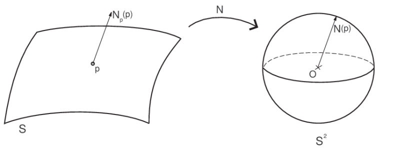
The Second Fundamental Form

- The other essential ingredient in defining the curvature of a surface is the *second fundamental form*. It rests upon the *Gauss map*, that attaches to each point on S its unit normal :

Definition (Gauss map)

Let $S \subset \mathbb{R}^3$ be an oriented surface, and let N denote its outward normal (i.e. such that $(f_u \times f_v) \cdot N > 0$). The *Gauss* (or *normal*) *map* of S is defined as

$$N(p) = \frac{f_u \times f_v}{||f_u \times f_v||}(p)$$



The Second Fundamental Form - cont.

It is is to verify that the Gauss map is differentiable. Moreover, given that $T_p(S) \equiv T_N(p)(\mathbb{S}^2)$ as vector spaces, one can view the differential dN_p as a linear map defined on $T_p(S)$ with values on itself.

If $c : I \to S$ is a parametrized curve, then

$$\dot{N}(0) = dN_p(\dot{c}(0))$$

Thus dN_p measures th rate of change of N along the curve c (at $t_0 = 0$), that is it measures the rate of change (in a neighborhood of p.) of N along c.

This is analogy the with the role of curvature for curves, however in this case N is not a scalar.

The Second Fundamental Form - cont.

Exercise

Find the image $N(c)$ of the curve c on a surface S and $N(S)$, when

1. *A plane.*
2. $S = \mathbb{S}^2$ *and* c *is (a) A great circle ; (b) The* 60% *parallel.*
3. S *is a circular cylinder and* c *is (a) A meridian ; (b) A parallel.*
4. S *is a torus of revolution and* c *is (a) A meridian ; (b) A parallel.*
5. S *is the cone* $z = \sqrt{x^2 + y^2}$ *and* c *is (a) A generator ; (b) The circle of height* $z_0 = 1$.
6. S *is the hyperbolic paraboloid* $z = y^2 - x^2$ *and* c *is a small circle around the point* $p = (0, 0, 0)$.
7. S *is the paraboloid* $z = x^2 + y^2$ *and* $c = S \cap \{y = 0\}$.

The Second Fundamental Form - cont.

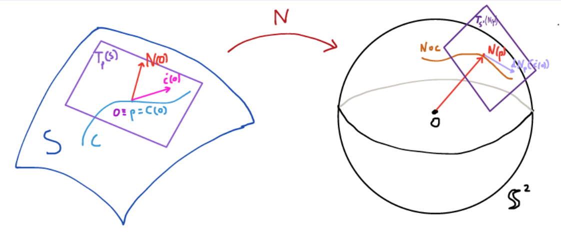
We are now ready to define the *second fundamental form* :

Definition

The bilinear form $II_p : T_p(S) \times T_P(S) \to \mathbb{R}$, given by

$$II_p(\mathbf{v}, \mathbf{v}) = -dN(\mathbf{v}) \cdot \mathbf{v}$$

is called the *second fundamental form* of the surface S.



The Second Fundamental Form - cont.

We can reinterpret Meusnier's Theorem in terms of the second fundamental form as :

Proposition

$$II_c(s)(\dot{c}(s), \dot{c}(s)) = k_c(s)N(s) \cdot n(s)$$

i.e.

$$II(\dot{c}, \dot{c}) = k_n(c)\,.$$

In local coordinates, the second fundamental form is written as

$$II_c(s)(\dot{c}(s), \dot{c}(s)) = e(\dot{u})^2 + 2f\dot{u}\dot{v} + g(\dot{v})^2\,.$$

We leave the proof as an exercise.

The Second Fundamental Form - cont.

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We leave the proof as an exercise.

The Second Fundamental Form - cont.

Examples

1. *Let us return to the important case of surfaces of revolution.*

   In the generic case, it is not hard to verify that

   $$e = -\varphi\dot{\psi}, f = 0, g = \dot{\psi}\ddot{\varphi} - \dot{\varphi}\ddot{\psi}\,.$$

   Therefore, it follows that

   $$K = -\frac{\ddot{\varphi}}{\varphi}\,.$$

Exercise

Verify the formulas above.

The Second Fundamental Form - cont.

In particular, for

1. For the unit sphere,
$e = \sin^2(v), f = 0, g = \sin(v)(\sin(v) + \cos(v)), K == +1$,
as expected.
2. For the torus of revolution, $K \sim \frac{\cos(v)}{R + r\cos(v)}$.
An immediate consequence of this formula is the fact that the torus has no umbilics.
Another is that $K = 0$, if $v = \pi/2$ or $3\pi/2$; $K > 0$, for $v \in (0, \pi/2) \cap (3\pi/2, 2\pi)$; and $K < 0$ for $v \in (\pi/2, 3\pi/2)$. Thus, on the exterior of the torus curvature is positive, on it's interior ("close" to the hole) it is negative, and it is zero on the upper and lower parallels.
3. For the pseudosphere, $e = -r\cot x, f = 0, g = r\sin x\cos x$, therefore, using the earlier computations for the First Fundamental form earlier we see that the pseudosphere's Gauss curvature is $K \equiv -1$.

The Second Fundamental Form - cont.

Remark

This result has prompted some to offer the pseudosphere as a model of Hyperbolic Geometry.

It is tempting to view the pseudosphere as a "materialization" into Euclidean space of a sphere of radius i (since $K \equiv -1 = i^2$), this is holds only locally.

In truth, the pseudosphere fails to fulfill two of the fundamentals geometric conditions of a plane :

- *It is not a* complete *surface, since it has a singular curve at* $r = 0$ *;*
- *It is not* simply connected, *given that it has the topology of cylinder.*

In fact, such an embedding is not possible, due to a famous theorem of Hilbert, from 1901 that states that no such smooth embedding exists in $\mathbb{R}^3$. *(The minimal embedding dimension is* $n = 5$.)

The Second Fundamental Form - cont.

The power of the discrete approach to Differential Geometry is shown here :

While, as we have seen, no model of a true Hyperbolic plane is possible in Euclidean space, a *PL* paper, made of gluing 7 equilateral triangles at each vertex is easy to produce. Since the discrete (defect) curvature at each vertex is $2\pi - 7\pi/3 = -\pi/3 \approx -1$, this is, indeed, a simple model of the Hyperbolic plane.

The Second Fundamental Form - cont.

As a further example of handling the second fundamental form we bring here the (promised) proof of Euler's Formula :

Proof of Euler's Theorem

Let $\mathbf{v} \in T_p(S), ||\mathbf{v} = 1$. and let e_1, e_2 be an orthonormal basis of $T_p(S)$. Then $\mathbf{v} = e_1 \cos\theta + e_2 \sin\theta$. Therefore

$$k_n = II_p(\mathbf{v},\mathbf{v}) = -dN_p(\mathbf{v}) \cdot \mathbf{v}$$

$$= -dN_p(e_1 \cos\theta + e_2 \sin\theta) \cdot (e_1 \cos\theta + e_2 \sin\theta)\,.$$

Recalling that the eigenvalues of $-dN_p$ are $k_{\min}$ and k_{Max}, it follows that

$$k_n = (k_{\min} e_1 \cos\theta + k_{\mathrm{Max}} e_2 \sin\theta) \cdot (e_1 \cos\theta + e_2 \sin\theta)$$

$$= k_{\min} \cos^2\theta + k_{\mathrm{Max}} \sin^2\theta\,.$$

□

The Second Fundamental Form - cont.

Problem

Determine all the surfaces of revolution for whom $K \equiv \text{const.}$ *(Hint : For instance, to imagine surfaces of constant positive curvature apart from the sphere, think of spindles and of wedding rings.* [a]

a. While we certainly hope the student will at least attempt to solve this not overly difficult problem by his/herself, we note that a full solution can be found in Spivak's book.

Problem

Determine all the surfaces S^2 *for whom* $|\nabla K| \equiv 1$.

The Second Fundamental Form - cont.

We have the alternative classification of points in terms of dN_p, which is less intuitive than the one formulated in terms of Gaussian curvature, but easier to explore in the planar case :

Definition

A p be a point on a (smooth) surface $S \subset \mathbb{R}^3$ is called

1. *Elliptic*, if $\det(dN_p) > 0$;
2. *Hyperbolic*, if $\det(dN_p) < 0$;
3. *Parabolic*, if $\det(dN_p) = 0$, and $dN_p \neq 0$;
4. *Planar*, if $dN_p = 0$.

The Second Fundamental Form - cont.

- As we have seen, $dN_p : T_p(S) \to T_p(S)$ (or $-dN_p$, as it appears in the definition of the second fundamental form), is it itself a measure of curvature, called the *Weingarten map* or, more suggestively, the *shape operator*.

  Both the "+" and "-" signs can (and, alternatively, are) be used to define th shape operator. While the "+" would seem more intuitive, we have seen that using the "-" one is natural in explaining the geometric signification of the second fundamental form via Meusnier's Theorem. Therefore this will be convention that we shall use.

  The intuition behind this definition is quite straightforward : The amount of change of N_p in the direction $\mathbf{v}$ measures the variation of the tangent planes to the surface in that direction, thus gauging the (directional) departure of S from being a plane, i.e. of its degree of curving.

The Second Fundamental Form - cont.

Exercise

Find the shape operator of the following surfaces :

1. *A plane ;*
2. *The unit sphere ;*
3. *A circular cylinder ;*
4. *The saddle surface $z = xy$.*

The following algebraic fact regarding dN_p (or $-dN_p$) is quite significant from a geometric viewpoint as well :

Proposition

$dN_p : T_p(S) \to T_p(S)$ is a self-adjoint linear map.

(Recall that a linear mapping $L : V \to W$ is *self-adjoint* if $\langle L\mathbf{v}, \mathbf{w}\rangle = \langle \mathbf{v}, L\mathbf{w}\rangle$. (Which, in $\mathbb{R}^3$ is simply $L(\mathbf{v}) \cdot \mathbf{w} = \mathbf{v} \cdot L(\mathbf{w})$.)

The Second Fundamental Form - cont.

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The Second Fundamental Form - cont.

Exercise

Prove the proposition above.

The fact that dN_p (or $-dN_p$) is self-adjoint is important due to

Theorem

Let $L : V \to W$ be a self-adjoint linear map. Then there exists an orthonormal basis $\{e_1, e_2\}$ of V such that e_1, e_2 are eigenvectors of V.

The importance of the result above is that, for $L = -dN_p$, the eigenvalues corresponding to e_1, e_2, respectively, are k_{min}, k_{Max}.

This result formally prove that the maximal and minimal sectional curvatures occur in orthogonal directions.

The Second Fundamental Form - cont.

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Prove the proposition above.

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The Second Fundamental Form - cont.

Furthermore, since the eigenvalues of a linear operator satisfy its *characteristic polynomial*, it follows that the following equation holds :

$$k^2 + 2(k_{min} + k_{Max})k + k_{min} \cdot k_{Max} = 0$$

that is

$$k^2 + 2Hk + K = 0\,;$$

Thus we have algebraically recuperated (or even could have defined) the two curvatures of a surface.

The Second Fundamental Form - cont.

Given that any linear mapping can be written as a matrix (which, in this case is diagonal), it follows that K and H can be written as

$$K(p) = \det A_{dN_p}\,;\, H(p) = \frac{1}{2}\mathrm{trace} A_{dN_p}\,;$$

where A_{dN_p} is the (2×2) matrix of $-dN_p$.

The Second Fundamental Form - cont.

Furthermore, to any self-adjoint mapping $L : V \to V$ corresponds a quadratic form Q in V, namely $Q\mathbf{v} = L\mathbf{v} \cdot \mathbf{v}$.

Then the eigenvalues λ_1, λ_2 corresponding to e_1, e_2 above, are the minimum, respective maximum, of Q on the unit circle in V.

In our case $L = -dN_p$ and Q is nothing else than the second fundamental form II_p, thus indeed the $\lambda_1 = k_{min}$, $\lambda_2 = k_{Max}$ are obtained on orthogonal directions in the unit circle of $T_p(S)$.

The Second Fundamental Form - cont.

Exercise

Let $S = f(U)$, $f : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ *be a parametrized surface.*

1. *Prove that*

$$\begin{cases} N_u = a_{11}f_u + a_{21}f_v \\ N_v = a_{12}f_u + a_{22}f_v \end{cases}$$

   (These are called the equations of Weingarten.)

2. *Show that*

$$\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = -\begin{pmatrix} e & f \\ f & g \end{pmatrix}\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1},$$

   and compute a_{ij} ; $1 \leq i, j \leq 2$.

The Second Fundamental Form - cont.

The coefficients e, f, g of the second fundamental form play an analogous role to the curvature and torsion of a curve, at least in the sense that they allow us to understand how well surfaces are (locally) approximated up to order two.

We begin by recalling that any surface can be expressed as the graph of a function $f(x, y, z) = (x, y, \varphi(x, y))$. Then we have the following simple formula for K :

$$K = \frac{\varphi_{xx} - \varphi_{xy}^2}{\sqrt{1 + \varphi_x^2 + \varphi_y^2}} ;$$

(The expression for H is more complicated, so we omit it here.)

The Second Fundamental Form - cont.

No less important (and perhaps more, at least at this stage) is the fact that in this case

$$II_p(x,y) = \varphi_{xx}(0,0)x^2 + 2\varphi_{xy}(0,0)xy + \varphi_{yy}(0,0)y^2 \,.$$

In other words, the second fundamental form is nothing but the Hessian of φ. This strengthens and concertizes our rather vague observation that curvature *is* second curvature (in some avatar or another).

The Second Fundamental Form - cont.

Remark

While for any practical purposes, at least in a first course in Differential Geometry, the essential first two fundamental forms suffice, it is interesting to note that a third fundamental form also exists, $III_p : T_p(S) \times T_p(S) \to \mathbb{R}$, which is defined as

$$III_p(p)(\mathbf{v},\mathbf{w}) = -dN_p(\mathbf{v}) \cdot (-dN_p(\mathbf{w})) = dN_p(\mathbf{v}) \cdot dN_p(\mathbf{w})\,;\,\mathbf{v},\mathbf{w} \in T_p(S).$$

Exercise

Prove that

$$III_p(p)(\mathbf{v},\mathbf{w}) = (dN_p(\mathbf{v}))^2 \cdot \mathbf{w}\,.$$

The Second Fundamental Form - cont.

While the third fundamental form has its uses, it is, in fact, expressible in terms of the first and second fundamental forms (and the Gaussian and mean curvatures), and we have

$$III - 2H \cdot II + K \cdot I = 0\,.$$

Exercise

Prove the formula above. (Hint : Recall that $-dN_p$ satisfies its characteristic polynomial.)

The importance of fundamental forms of higher order is quite modest. However, it might be instructive to show that, for instance, a *forth fundamental form*, can be defined as follows :

$$IV_p(p)(\mathbf{v},\mathbf{w}) = (dN_p(\mathbf{v}))^3 \cdot \mathbf{w}\,.$$

Distinguished Curves Revisited

The Differential Equation of the Lines of Curvature

Recall that c is a line of curvature iff

$$dN(\dot{c}(t)) = \lambda(t)\dot{c}(t)\,.$$

By expressing dN in terms of the coefficients of the fundamental forms we obtain that the equation above is equivalent to

$$\begin{cases} \frac{fF-eG}{\sqrt{\Delta}}\dot{u} + \frac{gF-fG}{\sqrt{\Delta}}\dot{v} = \lambda\dot{u} \\ \\ \frac{eF-fE}{\sqrt{\Delta}}\dot{u} + \frac{fF-gE}{\sqrt{\Delta}}\dot{v} = \lambda\dot{v} \end{cases}$$

(Here, as before, $\Delta = EG - F^2$.)

Distinguished Curves Revisited - cont.

By eliminating λ from both equations we obtain

Differential Equation of the Lines of Curvature I

$$(fF - eG)(\dot{u})^2 + (gF - fG)\dot{u}\dot{v} + (gF - fG)(\dot{v})^2 = 0\,,$$

which can be written in determinant (and easily to remember) form as

Differential Equation of the Lines of Curvature II

$$\begin{vmatrix} (\dot{v})^2 & -\dot{u}\dot{v} & (\dot{u})^2 \\ E & F & G \\ e & f & g \end{vmatrix} = 0\,.$$

Distinguished Curves Revisited - cont.

From here the following lemma follows easily :

Lemma

The parametric curves are lines of curvature iff

$$F = f = 0\,.$$

Exercise

Prove the lemma above.

Corollary

The parametric curves of a surface of revolution are lines of curvature.

In particular, the parallels and meridians of the torus of revolution are lines of curvature, as we already noted.

Distinguished Curves Revisited - cont.

The Differential Equation of the Asymptotic Lines

Since c is a line of curvature iff $II(\dot{c}, \dot{c}) \equiv 0$, the differential equations of the lines of curvature is

The Differential Equation of the Asymptotic Lines

$$e(\dot{u})^2 + 2f\dot{u}\dot{v} + g(\dot{u})^2 = 0\,.$$

From here we immediately obtain

Corollary

If $eg - f^2 < 0$ (i.e. at hyperbolic points) the coordinate curves are asymptotic lines iff

$$e = g = 0\,.$$

Distinguished Curves Revisited - cont.

Exercise

Determine the asymptotic lines of the surface given by $f : \mathbb{R}^2 \to \mathbb{R}^3$, $f(x, y) = (x + y, x - y, 2xy)$. *(See also the example of the parabolic hyperboloid we gave before.)*

Remark

The apparent simplicity of differential equations of the curvature lines and of the asymptotic lines represents a habitual trap for the CS or EE students who try to determine the special lines on a generic surface, by solving the equations in question, a task that is practically impossible (except in such simple cases as above) by using only the customary tricks such students are learning in their prevalent ODE courses.