Differential Geometry – Classical and Discrete

Curves 1

Parametrized curves, regular curves, arc length

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Motivation

Why begin the study of Differential Geometry with that of curves ?

Because...

- They are the simplest objects in the field
- They provide *tools* and *intuition* for the study of surfaces, etc.
- They are *needed* in the study of such geometric objects
- They are also needed in applications (Graphics, Imaging, etc.)
- They provide *ideas* for developing deeper, more general notions in Geometry
- They give us the first glimpse into *discretizations* rather than *approximations*
- These discretizations are useful and important in applications : Complex Networks, Imaging, Wavelets, etc.

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Basic facts

- We begin by noting that curves are geometric objects, i.e. subsets, say, of the plane.
- However, since we wish to study *differentiable* properties of curves, that is to say we want to be able to apply ideas and techniques of Calculus.
- To this end, we clearly need to be able to make appeal to functions.

  We are therefore conducted to the notion of parameterized curves :

Parameterized Curves

Definition (Parameterized Curves)

Let $I \subseteq \mathbb{R}$ (or $\mathbb{R}^2$) be an open interval, and let $c : I \to \mathbb{R}^3$ be a differentiable function. Then c is called a (*differentiable*) *parameterized curve* c, while $c(I)$ is called the *image* (or *trace*) of the curve. Furthermore, the variable $t \in I$, i.e. such that $c(t) = (x(t), y(t), z(t))$ is called the (eponymous) *parameter*.

Remark

- *Note that since we wish to ensure differentiability at every point, here we consider open intervals.*
- *Also, if not specified otherwise, we shall I will denote the unit interval $I = (0, 1)$ (or $I = [0, 1]$ if closed intervals needed).*
- *By "differentiable" we mean $\mathcal{C}^\infty$, which means, in fact, that the function is as smooth as needed, while in practice $\mathcal{C}^3$ usually suffices.*

Parameterized Curves - cont.

- Most importantly, note the distinction made between the curve c and its image $c(I)$.
  While in practice we often refer to the second simply as a ''curve", the distinction is essential, since the former is a function, while the later is a set (in plane or in space).
- It is this later geometric object that we wish to study, and to this end we make appeal to the first notion.
  Clearly, this might generate confusion, which one tries as best as possible to avoid by using the proper terminology of "parametrized curve".
- This points to the limitations of classical Differential Geometry with its dissociation between the geometric object one wishes to investigate, and a differentiable structure imposed on it for convenience reasons.

  However, passing to the much more general case of curves in metric spaces has the advantage of allowing us to concentrate on the geometry of the object itself, without the need parameterizations

Parameterized Curves - cont.

Remark

The artificiality of the parametrization is made manifest by the fact that different parameterizations may give rise to the same image.

Example

$c : \mathbb{R} \to \mathbb{R}^2$, $c(t) = (\cos t, \sin t)$ *and* $\gamma : \mathbb{R} \to \mathbb{R}^2$, $\gamma(t) = (\cos 2t, \sin 2t)$ *have the same image : The unit circle* $\mathbb{S}^1$*.*

This problem is compound by the fact that not all curves in the plane, not even some simple ones, are differentiable.

Counterexample

The graph of the absolute value function, i.e. the image of curve $c : \mathbb{R} \to \mathbb{R}^2$, $c(t) = (y, |t|)$ *is not a parameterized differentiable curve, because the function* $t \mapsto |t|$ *is not differentiable at* $t = 0$*.*

Parameterized Curves - cont.

We do not bring more examples of curves here, as some of the most interesting ones will illustrate important definitions and interesting case later on.

In the case of metric spaces one replaces open intervals with the generalizations of closed ones, i.e *continua*, where a continuum is defined as follows :

Definition

Let (X, d) be a metric space and let $A \subseteq X$, $|A| \geq 2$. Then A is called a (*metric*) *continuum* if it is compact and connected.

The definition above renders the most general extension of curve that we shall study here. However, in many case we shall consider only *metric arcs* :

Parameterized Curves - cont.

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Parameterized Curves - cont.

Definition

Let (X, d) be a metric space and let $\varphi : I \xrightarrow{\sim} X$ be a homeomorphism of the closed (unit) interval. Then $\varphi(I)$ is called a (*metric*) *arc*.

Furthermore, intervals themselves admit a generalization to the setting of general metric spaces. More precisely, we have the following

Definition

Let (X, d) be a metric space and let $A \subseteq X$. Then A is called a *metric segment* if it is isometric to a segment $[a, b] \subset \mathbb{R}$.

Parameterized Curves - cont.

After this digression into metric curves, we return to parameterized differentiable curves :

Definition

Given the differentiable parameterized curve $c : I \to \mathbb{R}^3$, $\dot{c}(t) = (\dot{x}(t), \dot{y}(t), \dot{z}(t))$ is called the *tangent* (or *velocity*) vector at c in the point $c(t)$.

Clearly, if $\dot{c}(t) \neq 0$, then the line $l(t) = \dot{c}(t)t + c(t)$ is the *tangent line* to c at t. Since in Differential Geometry curves need to have tangents at every point, we are conduced to formulate

Definition

Let $c : I \to \mathbb{R}^3$, be a parameterized differentiable curve, and let $t \in I$ such that $\dot{c}(t) = 0$. Then t is called a *singular point*. Otherwise, t is called a *regular point*. The curve c itself is called *regular* if all its points are regular, i.e. if $\dot{c}(t) \neq 0$, for all $t \in I$.

Parameterized Curves - cont.

As we know from Calculus, knowledge of $\dot{c}(t)$ allows us to compute the length of the curve, in a manner that we formalize as

Definition

Let $c : I \to \mathbb{R}^3$, be a parameterized differentiable curve. Given $t_0 \in I$, the *arc length* of c from the point t_0 is defined as

$$s(t) = \int_{t_0}^{t} ||\dot{c}(t)||dt\,;$$

where $||\dot{c}(t)|| = \sqrt{\dot{x}^2(t), \dot{y}^2(t), \dot{z}^2(t)}$.

The notation $s(t)$ is not our own, but it rather is the standard one. The use of a standardized notation points out to the fact that this simple definition is quite important in the sequel.

Parameterized Curves - cont.

Indeed, this is the case, and it is justified by the following simple observation : If $t \equiv s$, then $ds/dt = ||\dot{c}(t)|| = 1$ and, reciprocal, if $||\dot{c}(t)|| = 1$ then

$$s = \int_{t_0}^{t} ||\dot{c}(t)||dt = t - t_0\,,$$

that is t is precisely the arc length of c measured from t_0. Clearly, having curves parameterized such at every point the *speed* $||\dot{c}(t)||$ equals 1 is advantageous, a fact that will be made most evident in the sequel. Therefore, it would be most advantageous if we could ensure that any curve can be thus parameterized. This turns out to be the case, at least for regular curves,[1] as guaranteed by

1. This being one more reason why classical Differential Geometry concentrates on their study.

Parameterized Curves - cont.

Proposition

Any regular differentiable curve $c : I \to \mathbb{R}^3$ can be parameterized by arc length.

From the proposition above allows us to presume that any regular curve is parametrized by arc length (which we shall do, unless otherwise specified).

Exercise

Prove Proposition 1.

Metric Spaces

Our approach to ensure a *minimal* - but *relevant* – Geometry

Consider metric spaces :

Definition

Let (X, d) be a set, and let $d : X \times X \to \mathbb{R}_+$ such that :

1. $d(x, y) = 0$ iff $x = y$;
2. $d(x, y) = d(y, x)$, for all $x, y \in X$;
3. $d(x, z) \leq d(x, y) + d(y, z)$, for all for all $x, y, z \in X$ (*triangle inequality*).

Then d is called a *metric* (on X) and (X, d) is called a *metric space*.

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Metric Spaces - Examples

Some elementary (but important) examples :

- $X = \mathbb{R}$, $d(x, y) = |x - y|$;

  Analysis, Everywhere

- $X = \mathbb{R}^2$, $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$, where $x = (x_1, x_2), y = (y_1, y_2)$;

  or, slightly more generally : $X = \mathbb{R}^3$,

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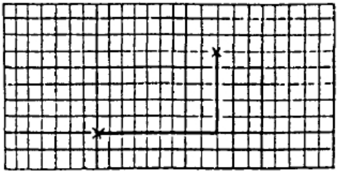
Metric Spaces - Examples - cont.

- *The discrete metric* X any set, $d(x,y) = 0$ if $x = y$, and $d(x,y) = 1$, if $x \neq y$.

  Analysis, Computer Science

- *Manhattan distance* $X = \mathbb{R}^2$, $d(x,y) = |x_1 - y_1| + |x_2 - y_2|$;



Analysis, Architecture

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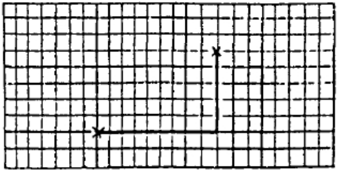

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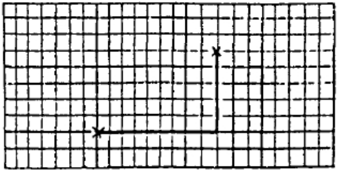

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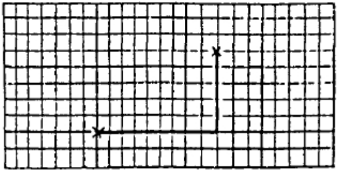
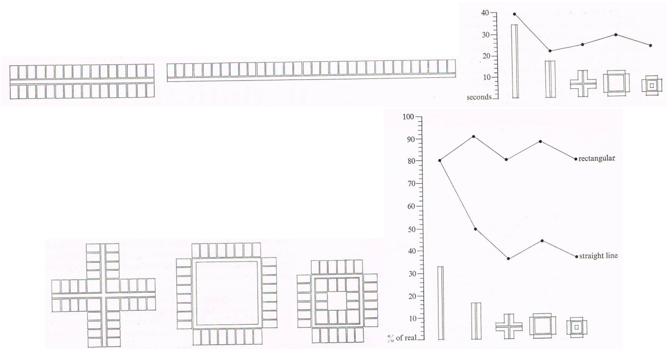


Analysis, Architecture

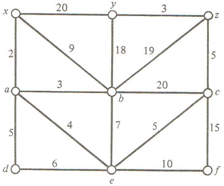
Metric Spaces - Examples - cont.

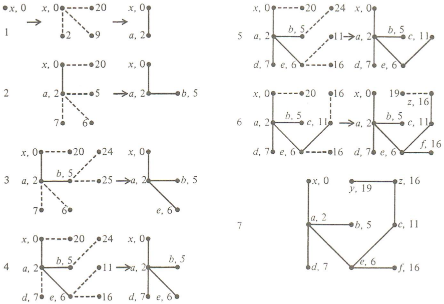
Architecture : For instance, in evaluating floor plans.



Metric Spaces - Examples - cont.

- X is an edge weighted graph, i.e. $X = G = (V, E; W)$, with positive edge weights $w_i = w(e_i)$, $d(u, v) = \min \sum_1^n w_i$, where $e_1 = (u, v_1), \ldots, e_n = (v_{n-1}, v)$.





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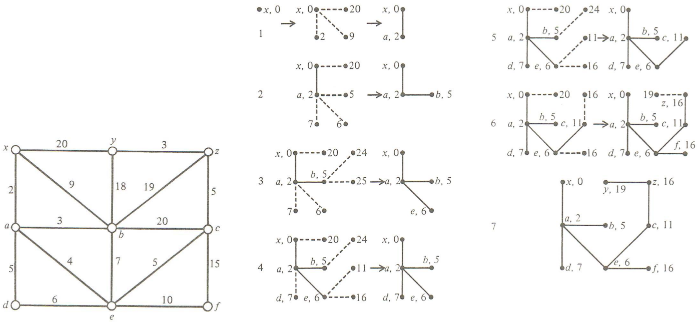
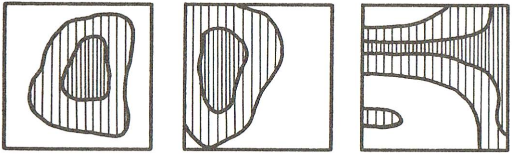


Graph Theory, Networks, Architecture

Metric Spaces - Examples - cont.

Architecture : For instance, in determining optimal routes.



1	1	2	3	2
1	2	6	6	3
2	4	7	5	2
2	4	4	4	1
1	1	2	3	1

Population (*P*)

3	3	4	2	1
4	8	6	2	1
5	8	4	1	1
4	7	3	1	1
3	4	1	1	1

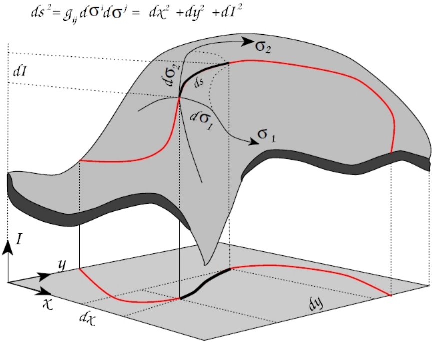
Natural restrictions (*N*)

1	1	1	3	6
5	5	5	4	6
1	1	2	6	6
2	1	1	2	5
2	1	1	2	5

Other land use (*L*)

Metric Spaces - Examples - cont.

- *Riemannian metrics*
  $ds^2 = g_{11}dx^2 + 2g_{12}dxdy + g_{22}dy^2$, etc.



Metric Spaces - Examples - cont.

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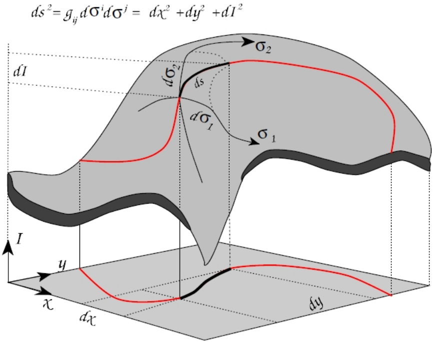


Differential Geometry, Imaging, Graphics, Manifold Learning

The Two Poincaré Models

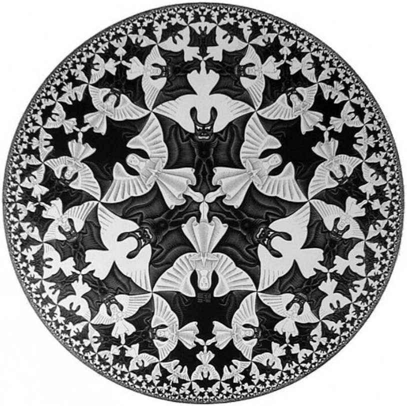
- **The Upper Half-Plane Model**

$X = \mathbb{H}_+ = \{(x, y) \in \mathbb{R}^2 \,|\, y > 0\}$, $ds^2 = \frac{dx^2+dy^2}{y^2}$.

Hyperbolic Plane – The Two Poincaré Models - cont.

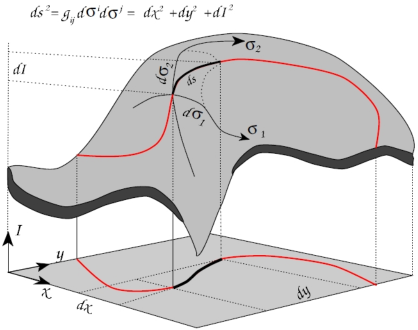
- **The Disk Model**

$X = \mathbb{B}^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$, $ds^2 = 4\frac{dx^2+dy^2}{(1-x^2-y^2)^2}$.



Hyperbolic Plane – The Two Poincaré Models - cont.

- These are *conformal*, *isometric* models.
- Are particular cases of *Riemannian metrics* $ds^2 = g_{11}dx^2 + 2g_{12}dxdy + g_{22}dy^2$, etc.

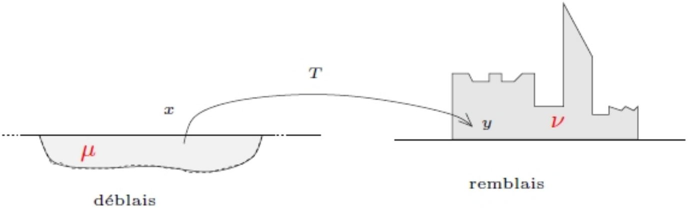


Metric Spaces - Examples - cont.

- Use the cost function, usually

$$c(x, y) = d(x, y)^2$$

(viewed as the price of moving a unit of mass – say, a standard pebble/brick/etc. – from point x to point y)



Metric Spaces - Examples - cont.

to define [2] the *Wasserstein distance* (of order 2) on $P(X_0)$ is defined as

$$W_2(\mu, \nu) = \left(\inf \int_X d(x,y)^2 d\pi(x,y) \right)^{\frac{1}{2}},$$

where the infimum is taken over all the *transference* (*transport*) *plans* $\pi \in P(X \times Y)$ between μ and ν, with *marginals* μ and ν.

Here $X = P(X)$ – the space of probability measures on X and μ, ν should be viewed, intuitively, as the above "piles of dirt".

Optimization, Image Processing

2. (sadly, quite technically...)

Metric Spaces - Examples - cont.

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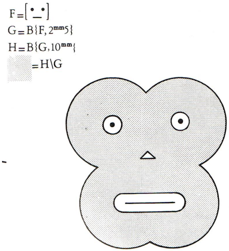
The Hausdorff distance

Definition

Let (X, d) be a metric space and let $A, B \subseteq X$. We define the *Hausdorff distance* between A and B as :

$$d_H(A, B) = \inf\{r > 0 \,|\, A \subset U_r(B),\, B \subset U_r(A)\}\,,$$

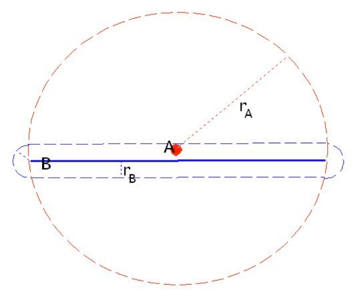
where $U_r(A)$ is the r-neighborhood of A, $U_r(A) \triangleq \bigcup_{a\in A} B_r(a)$.



The Hausdorff distance - cont.

Remark

The apparent asymmetry induced by the double inclusion in the definition above is necessary :



The Hausdorff distance - cont.

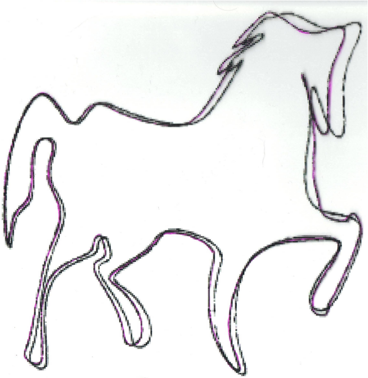
Remark

Another equivalent way of defining the Hausdorff distance is as follows :

$$d_H(A,B) = \max\{\sup_{a\in A} d(a,B),\ \sup_{b\in B} d(b,A)\}\,.$$

The Hausdorff distance - cont.

Advantage : "Ready-made" for image comparison :



Clearly close to each other (in the Hausdorff metric)

The Hausdorff distance - cont.

Advantage : “Ready-made” for image comparison :

Clearly far from each other (in the Hausdorff metric)

The Hausdorff distance - cont.

However...

...not so clear...

(In fact these horses is are identical, so a proper distance between them should be 0...)

The Hausdorff distance - cont.

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The Hausdorff distance - cont.

There are some theoretical advantages as well :

Even though in general $(P(X), d_H))$ is **not** a metric spaces, when restricted to *compact* subsets, it is. We can state this formally as a

Theorem

Let $K(X)$ denote the sets of all compact subsets of a metric space (X, d). Then $(K(X), d_H)$ is a metric space. Moreover, if X is complete (resp. compact), than is is complete $K(X)$ (resp. compact).

This is very encouraging. However, the following corollary is far more important for our (practical !) goals :

The Hausdorff distance - cont.

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Even though in general $(P(X), d_H))$ is **not** a metric spaces, when restricted to *compact* subsets, it is. We can state this formally as a

Theorem

Let $K(X)$ denote the sets of all compact subsets of a metric space (X, d). Then $(K(X), d_H)$ is a metric space. Moreover, if X is complete (resp. compact), than is is complete $K(X)$ (resp. compact).

This is very encouraging. However, the following corollary is far more important for our (practical !) goals :

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For Computer and Imaging Sciences – where in practice only *finite* sets are encountered – this fact is highly relevant. In particular, it implies that one need not compute the distance between say, images (*smooth, theoretically infinitely many points needed*) – suffices to use finitely many points (*samples, pixels*).

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