Differential Geometry – Classical and Discrete

Metrization of Gauss Curvature

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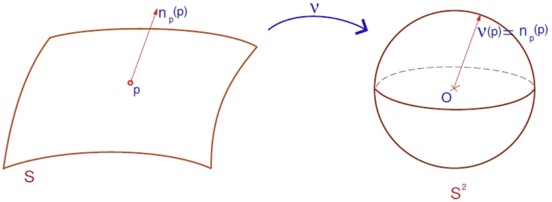
Wald Curvature

Instead of producing a metrization of *principal curvatures* (as the Haantjes and Menger curvatures), a more powerful approach stems from *Gauss' original method of comparing surface curvature to a standard, model surface* (i.e. the unit sphere in $\mathbb{R}^3$).

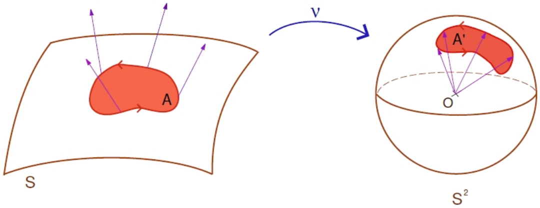
Recall that Gauss achieved this by considering the *normal mapping* $\nu : S \to \mathbf{S}^2$.



Wald Curvature - cont.

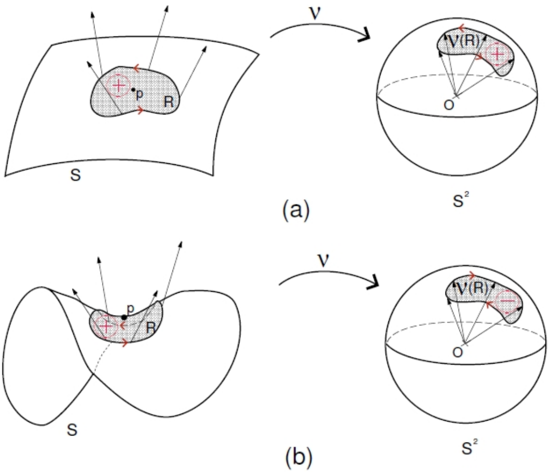
Then the *Gauss curvature* of S at p is defined as :

$$K_S(p) = \lim_{diam(R)\to 0} \frac{Area(\nu(R))}{Area(R)} .$$



Wald Curvature - cont.

A sign is attached to $K(p)$ in a natural way (for a notion defined by a integral !...) :



It was Wald's idea to use more types of *gauge surfaces* and to restrict oneself to the study of the minimal geometric figure that allows this comparison.

Definition

Let (M, d) be a metric space, and let $Q = \{p_1, ..., p_4\} \subset M$, together with the mutual distances : $d_{ij} = d_{ji} = d(p_i, p_j)$; $1 \leq i, j \leq 4$. The set Q together with the set of distances $\{d_{ij}\}_{1 \leq i,j \leq 4}$ is called a *metric quadruple*.

Definition

The *embedding curvature* $\kappa(Q)$ of the metric quadruple Q is defined to be the curvature κ of the gauge surface S_κ into which Q can be isometrically embedded.

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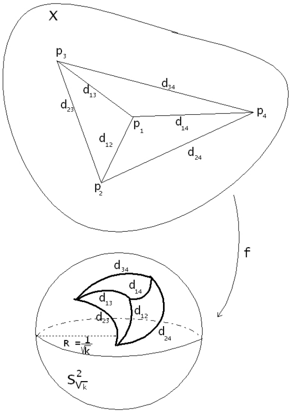

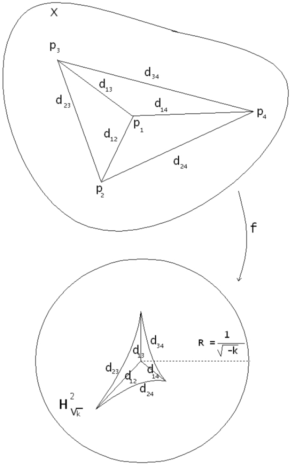

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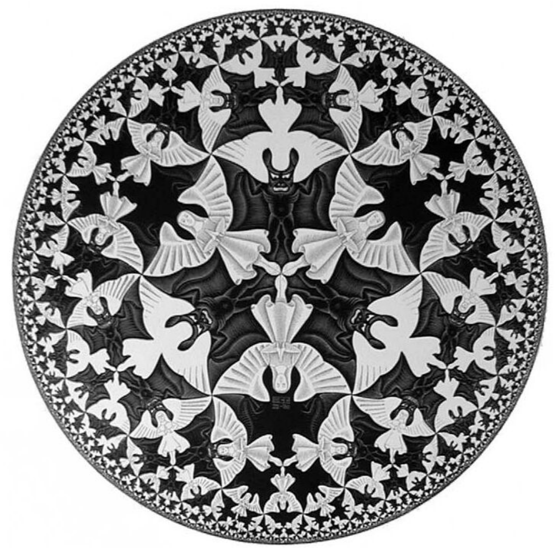

Wald Curvature - cont.

Obviously, we can define the Wald curvature **at a point** by passing to the limit :

Definition

Let (M, d) be a metric space, and let $p \in M$ be an accumulation point. Then M has (*embedding*) *Wald curvature* $\kappa_W(p)$ at the point p iff

1. Every neighbourhood of p is *non-linear*) (i.e. is not contained in a geodesic) ;
2. For any $\varepsilon > 0$, there exists $\delta > 0$ such that if $Q = \{p_1, ..., p_4\} \subset M$ and if $d(p, p_i) < \delta$, $i = 1, ..., 4$; then $|\kappa(Q) - \kappa_W(p)| < \varepsilon$.

Wald vs. Gauss

The discussion above would be nothing more than a nice intellectual exercise, where it not for the fact that the metric (Wald) and the classical (Gauss) curvatures coincide whenever the second notion makes sense, i.e for smooth (class $\geq \mathcal{C}^2$) surfaces in $\mathbb{R}^3$. More precisely the following theorem holds :

Theorem (Wald)

Let $S \subset \mathbb{R}^3$, $S \in \mathcal{C}^m$, $m \geq 2$ be a smooth surface. Then, given $p \in S$, $\kappa_W(p)$ exists and $\kappa_W(p) = \kappa_G(p)$.

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Moreover, Wald also proved the following partial reciprocal result :

Theorem

Let M be a compact and convex metric space. If $\kappa_W(p)$ exists, for all $p \in M$, then M is a smooth surface and $\kappa_W(p) = \kappa_G(p)$, for all $p \in M$.

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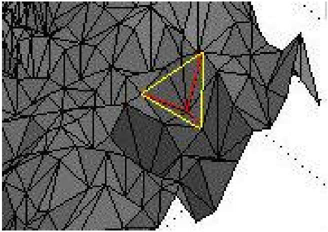
Wald Curvature Computation - cont.

Evidently, in the context of polyhedral surfaces, the natural choice for the open set U is the *open star* of a given vertex v, that is, the set $\{e_{vj}\}_j$ of edges incident to v. Therefore, for such surfaces, the set of metric quadruples containing the vertex v is finite.



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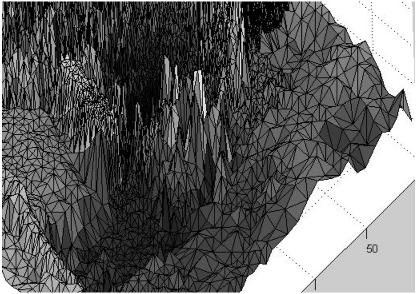

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The Cayley-Menger Determinant

To actually compute the embedding curvature, one has to make appeal to the the so called *Cayley-Menger determinant* :

Given a general metric quadruple $Q = Q(p_1, p_2, p_3, p_4)$, of distances $d_{ij} = dist(p_i, p_j), \ i = 1, ..., 4$; denote by $D(Q) = D(p_1, p_2, p_3, p_4)$ the following determinant :

$$D(p_1, p_2, p_3, p_4) = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & d_{12}^2 & d_{13}^2 & d_{14}^2 \\ 1 & d_{12}^2 & 0 & d_{23}^2 & d_{24}^2 \\ 1 & d_{13}^2 & d_{23}^2 & 0 & d_{34}^2 \\ 1 & d_{14}^2 & d_{24}^2 & d_{34}^2 & 0 \end{vmatrix}$$

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The Cayley-Menger Determinant - cont.

$D(p_1, p_2, p_3, p_4)$ is in fact the *volume* of the *Euclidean* quadruple $Q = Q(p_1, p_2, p_3, p_4)$. By developing analogous expressions of volumes as Cayley-Menger determinants, in the spherical and hyperbolical cases, one obtains the following general formula :

$$\kappa(Q) = \begin{cases} 0 & \text{if } D(Q) = 0\ ; \\ \kappa,\ \kappa < 0 & \text{if } det(\cosh\sqrt{-\kappa}\cdot d_{ij}) = 0\ ; \\ \kappa,\ \kappa > 0 & \text{if } det(\cos\sqrt{\kappa}\cdot d_{ij}) \text{ and } \sqrt{\kappa}\cdot d_{ij} \leq \pi \\ & \text{and all the principal minors of order 3} \\ & \text{are} \geq 0. \end{cases}$$

Unfortunately, the non-trivial expressions above are *transcendental*.

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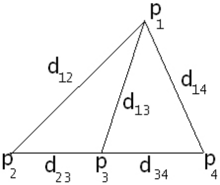
An Approximation Formula

However, for the so called *sd*-quads it is possible to develop simple (i.e. rational) formulas for the approximation of the embedding curvature :

Definition

A metric quadruple $Q = Q(p_1, p_2, p_3, p_4)$, of distances $d_{ij} = dist(p_i, p_j),\ i = 1, ..., 4$, is called *semi-dependent* (or a *sd-quad*, for brevity), iff 3 of its points are on a common geodesic, i.e. there exist 3 indices, e.g. 1,2,3, such that : $d_{12} + d_{23} = d_{13}$.



An Approximation Formula

Theorem (Robinson, 1944)

Given the metric semi-dependent quadruple $Q = Q(p_1, p_2, p_3, p_4)$, *of distances* $d_{ij} = dist(p_i, p_j),\ i, j = 1, \dots, 4$; *the embedding curvature* $\kappa(Q)$ *admits a rational approximation given by :*

$$K(Q) = \frac{6(\cos \measuredangle_0 2 + \cos \measuredangle_0 2')}{d_{24}\left(d_{12} \sin^2(\measuredangle_0 2) + d_{23} \sin^2(\measuredangle_0 2')\right)}$$

where : $\measuredangle_0 2 = \measuredangle(p_1 p_2 p_4)\,,\ \measuredangle_0 2' = \measuredangle(p_3 p_2 p_4)$ *represent the angles of the Euclidian triangles of sides* d_{12}, d_{14}, d_{24} *and* d_{23}, d_{24}, d_{34} *, respectively.*

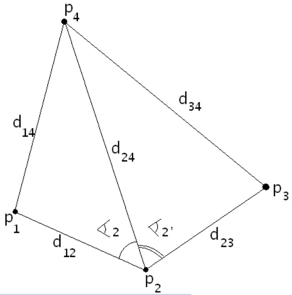
An Approximation Formula - cont.

Theorem (cont.)

Moreover absolute error R satisfies the following inequality :

$$|R| = |R(Q)| = |\kappa(Q) - K(Q)| < 4\kappa^2(Q)diam^2(Q)/\lambda(Q)\,,$$

where $\lambda(Q) = d_{24}(d_{12}\sin\measuredangle_0 2 + d_{23}\sin\measuredangle_0 2')/S^2$, *and where* $S = Max\{p, p'\};\ 2p = d_{12} + d_{14} + d_{24}\,,\ 2p' = d_{32} + d_{34} + d_{24}$.



An Important Remark

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The problem with the definition of Wald curvature resides in the fact that, then the very existence of $\kappa(Q)$ is not assured*, and, moreover,* even if a quadruple admits an embedding curvature, it still may be not unique *(even if Q is not linear).*

However, the uniqueness of the embedding curvature is assured for semi-dependent quadruples.

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Experimental Results

We present some incipient experimental results metric approximations of *Gauss* (and *mean*) *curvature* for

- Analytic surfaces

and on

- Real data
  - Standard test images (i.e. "Lena")
  - Medical images

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The chosen "toy" surfaces was the *"round" torus*, equipped with the *standard square grid in the parametric plane*, because

- Simple mesh, easy to perform computations upon

so that

- Computational error can be precisely assessed

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- Has regions of positive, zero and negative Gauss curvature.

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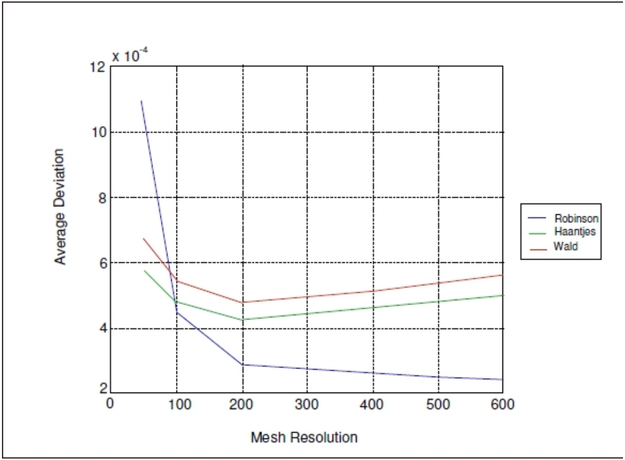

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If, in addition, all the diagonals, in one direction, in the square grid mentioned above, s.t. the edges of the triangulation will best coincide with geodesic lines of the surface (to better compute Finsler-Haantjes and Robinson approximations), then better convergence is observed :

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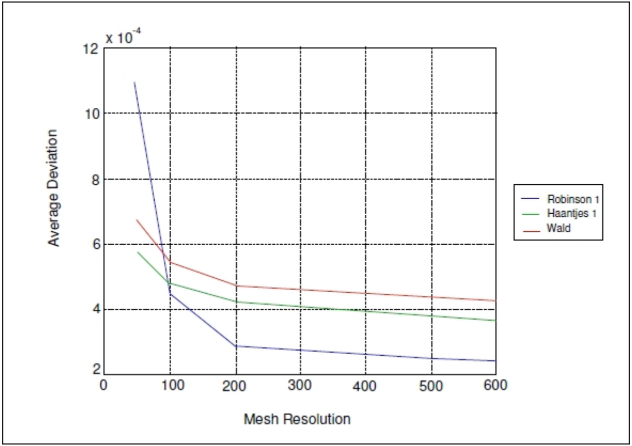

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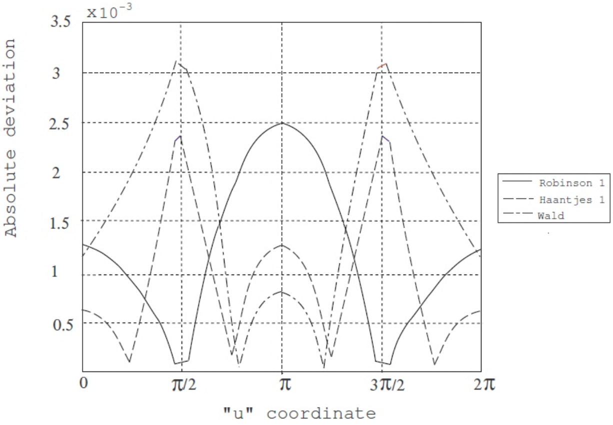
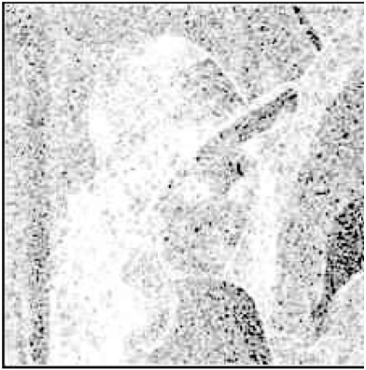


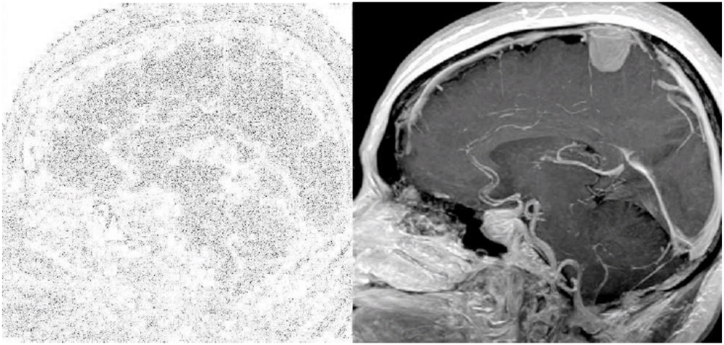
Average deviation for the torus of revolution:
$\mathbf{T}^2 = \mathbf{T}^2(u, v) = \big((R + r\cos u)\cos v, (R + r\cos u)\sin v, r\sin u\big)$.

Experimental Results - cont.



Mean curvature, computed using the "normalized" Haantjes curvature (left) and Wald curvature (right), of the grayscale "Lena".

Experimental Results - cont.



Wald curvature (right) of a cerebral radiographic image (left).

Experimental Results - cont.

So, what can we conclude regarding the *practical* side of the metric approach to sectional curvature ?

- The convergence rates of these metric methods is rather slow.
- Numerical instability is high.

So, what to do ?

- Improve aspect-ratio ("shape", "thickness") of *simplices* (triangles).
- Devise, for the Wald curvature, better ways or "sampling" the directions on the surfaces.
- Use "filtering" (natural in Imaging !) : Consider larger neighbourhoods to obtain an averaging effect (analogous to the one produced by the eye/brain.)

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