Differential Geometry – Classical and Discrete

Distinguished Curves Revisited

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We first concentrate on the lines of curvatures and explore them using the new tools introduced above.

One can prove a number of nice results regarding lines of curvature that are not only interesting in themselves, but have important applications, some of them quite far-reaching. We begin with

Theorem (Bonnet)

Let c be a curve on the surface S. Then c is a line of curvature of S if Σ_c is planar, where Σ_c denotes the surface formed by the normals to S along c.

(Recall that S is called *flat* if $K \equiv 0$.)

Lines of Curvature Revisited - cont.

Before bringing the proof of this theorem, we need to introduce a new definition, namely

Definition

A surface S is said to be *ruled* if it is given by a parametrization of the form $f(s,t) = c(s) + t\delta(s)$, where c,δ are curves. The curve c is called the *directrix* of the surface.

Examples

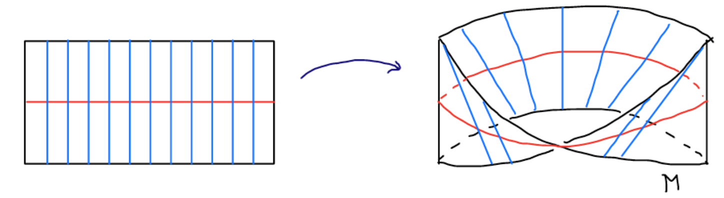
Beside the trivial example of the plane, we mention here the following well known surfaces :

- *Cylinders (not necessarily right circular ones) ;*
- *Cones (again, not only the right, circular ones) ;*

Lines of Curvature Revisited - cont.

- Helicoids ;

  and
- Catenoids ;
- The Möbius strip.



Lines of Curvature Revisited - cont.

Using the formulary developed above, one easily obtains the following formula for the Gaussian curvature of a ruled surface :

$$K = -\frac{[\dot{\mathbf{c}} \cdot (\delta \times \dot{\delta})]^2}{||(\dot{\mathbf{c}} + t\dot{\delta}) \times \delta||^2}\,.$$

Exercise

(a) Prove the formula above.

(b) Compute the Gaussian curvature of the helicoid.

We can now proceed and prove Bonnet's theorem :

Lines of Curvature Revisited - cont.

Proof of Bonnet Theorem

We begin by observing that Σ_c is a ruled surface, more precisely that $\Sigma_c = c(s) + tN(c(s))$.

Then from formula for the Gauss curvature of ruled surfaces follows that Σ_c is flat $\Leftrightarrow \dot{\mathbf{c}} \cdot (\delta \times \dot{\delta}) = 0$

$$\Leftrightarrow c(\dot{s}) \cdot \left(N(c(s)) \times \frac{dN(c(s))}{ds}\right) \Leftrightarrow \frac{dN(c(s))}{ds} \sim c(\dot{s})$$

i.e. if c is a line o curvature.

□

Bonnet's theorem has an important (even if somewhat particular) consequence :

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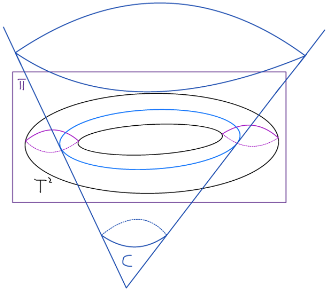
Bonnet's theorem has an important (even if somewhat particular) consequence :

Corollary

Let S be a surface of revolution. Then its meridians and parallels are lines of curvature.

This is immediate once we notice that the corresponding ruled surfaces are planes and cones.



Lines of Curvature Revisited - cont.

It turns ot that the property of being a line of curvature on a surface is strong enough to have implications for a second surface, if the given curve is the intersection of the two surfaces.

More precisely, we have the following (sadly) not very well known theorem :

Theorem (Joachimsthal)

Let S_1, S_2 be two surfaces and let $c = S_1 \cap S_2$, such that C is a line of curvature in S_1. Then c is also a line of curvature in S_2 iff S_1 and S_2 intersect at a constant angle along c.

Proof

The condition in the statement of the theorem is formally expressed as $\measuredangle\left(N_{S_1}(c(s)), N_{S_2}(c(s))\right) = \text{const.}$.
By differentiation we obtain that

$$\frac{d}{ds}\left(N_{S_1} \cdot N_{S_2}\right) = \frac{dN_{S_1}}{ds} \cdot N_{S_1} + N_{S_1} \cdot \frac{dN_{S_2}}{ds}\,.$$

Since c is a line of curvature on S_1, the last expression equals

$$-k(s)\frac{d(c(s))}{ds}\dot{N}_{S_2} + N_{S_1} \cdot \frac{dN_{S_2}}{ds}\,.$$

But $c \subset S_2$, therefore the expression above becomes

$$0 + N_{S_1} \cdot \frac{dN_{S_2}}{ds}\,.$$

Lines of Curvature Revisited - cont.

We can now proceed to the proof itself :

$(\Longrightarrow)$ c is a line of curvature in $S_2 \Rightarrow N_1 \cdot \frac{N_2}{ds} = 0$

(since c is also a line of curvature in S_1.)

From here it follows that

$\frac{d}{ds}(N_1 \cdot N_2) = 0 \Leftrightarrow \measuredangle(N_1, N_2) = \text{const.}.$

$(\Longleftarrow)$ $\measuredangle(N_1, N_2) = \text{const.} \Leftrightarrow \frac{d}{ds}(N_1 \cdot N_2) = 0 \Rightarrow dN_2 \perp N_1$.

But $N_2 \perp N_2$, therefore it follows that

1. $N_1 \nsim N_2$

   or

2. $N_1 \sim N_2$.

Lines of Curvature Revisited - cont.

- In the first case $dN_2(c(s)) = \lambda \dot{c}(s)(s)$, thence it follows that c is a line of curvature in S_2.
- In the second case $N_1(c(s)) = \pm N_2(c(s))$.

$\square$

Exercise

Using Joachimsthal's Theorem give an alternative proof of the fact that the meridians and parallels of the torus are lines of curvature.

Lines of Curvature Revisited - cont.

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Lines of Curvature Revisited - cont.

Exercise

Let $c = S_1 \cap S_2$. Prove that the curvature of c is given by

$$k_c = \frac{1}{\sin\varphi}\sqrt{k_{n1}^2 + k_{n2}^2 - 2k_{n1}k_{n2}\cos\varphi}$$

where k_{n1}, k_{n2} denote the curvatures of the normal sections of S_1, S_2, respectively and $\varphi = \measuredangle(N_1, N_2)$.

Moreover, lines of curvature can characterize intersections even of triples of surfaces. More precisely, we have the following theorem :

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Lines of Curvature Revisited - cont.

Theorem (Dupin)

The intersection curves (lines) of a triply orthogonal family are lines of curvatures on the surfaces.

Here *triply orthogonal families* are formally defined as follows :

Definition

A *triply orthogonal family* of surfaces is a diffeomorphism $F : W = \mathrm{int} W \subset \mathbb{R}^3 \to \mathbb{R}^3$, such that

$dF(u, v, w) : T_{(u,v,w)}\mathbb{R}^3 \to T_{F(u,v,w)}\mathbb{R}^3$ is a bijection ;

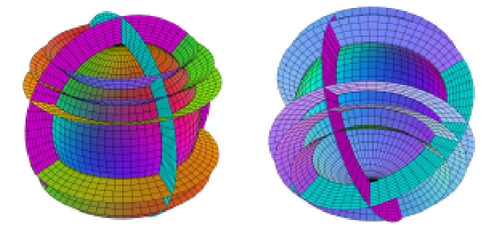
and such that

$$F_u \cdot F_v = F_u \cdot F_w = F_v \cdot F_w = 0 \,.$$

Lines of Curvature Revisited - cont.

Examples

- *The planes parallel to the coordinate planes ;*
- *Circular cylinders with Oz as axis, as well as planes through Oz and also the planes parallel to the plane xOz ;*
- *The three families of surfaces consisting of $\mathcal{F}_1$ – the spheres centered at the origin ; $\mathcal{F}_2$ – the planes through Oz ; $\mathcal{F}_3$ – the cones generated by lines through O and making a constant angle with the axis Oz.*



Lines of Curvature Revisited - cont.

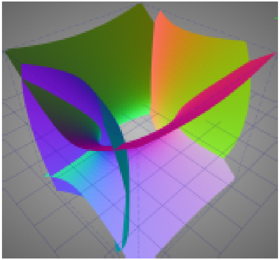
- The family $\mathcal{F}_\lambda$, where

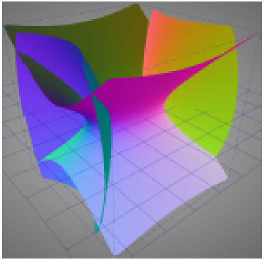
  $\mathcal{F}_\lambda = \frac{x^2}{a^2-\lambda^2} + \frac{x^2}{a^2-\lambda^2} + \frac{x^2}{a^2-\lambda^2}\,; 0 < a^2 < b^2 < c^2.$

  The family consists of

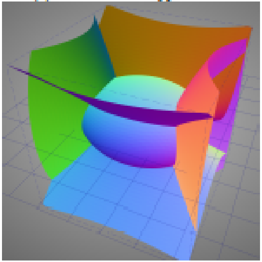
  - ellipsoids for $\lambda < a^2$;
  - of hyperboloids with one sheet, for $a^2 < \lambda < b^2$;
  - hyperboloids with with two sheets for $b^2 < \lambda < c^2$.

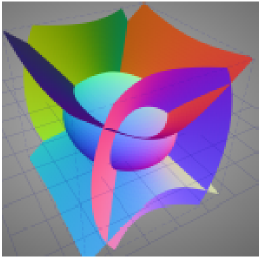
Lines of Curvature Revisited - cont.





(a). double sheet hyperboloid (b). single sheet hyperboloid





(c). ellipsoid (d). ellipsoid

Lines of Curvature Revisited - cont.

Exercise

Prove that the surfaces in the last example above indeed represent a triply orthogonal family.

It is precisely this last example, augmented by the observation that any ellipsoid can be realized as one of the surfaces in a triply orthogonal family like the one in the last example, that motivated Dupin's Theorem.

Proof of Dupin's Theorem

By differentiating $F_u \cdot F_v = F_u \cdot F_w = F_v \cdot F_w = 0$ by u, v and w we obtain

$$\begin{cases} F_{uv} \cdot F_w + F_{wu} \cdot F_v = 0 \\ F_{vw} \cdot F_u + F_{uv} \cdot F_w = 0 \\ F_{wu} \cdot F_v + F_{vw} \cdot F_u = 0 \end{cases}$$

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Lines of Curvature Revisited - cont.

By subtracting the first equation from the third we obtain $F_{vw} \cdot F_u - F_{wu} \cdot F_v = 0$.

From here, and from the last equation in the system above it follows that $F_{vw} \cdot F_u = F_{wu} \cdot F_v = 0$.

From these equalities, by applying again the condition $F_u \cdot F_v = F_u \cdot F_w = F_v \cdot F_w = 0$, we obtain that $F_u, F_v, F_{uv} \perp F_w$, hence that F_u, F_v, F_{uv} are coplanar.

Therefore, we have that $F_u \cdot (F_v \times F_{uv}) = \det(F_u, F_v, F_{uv}) = 0$.

Since the diagonal element M of the second fundamental form is given by $\frac{\det(F_u, F_v, F_{uv})}{\sqrt{EG - F^2}}$, we have shown that, for $w = \text{const.} = w_0$, we have that $M = M(u, v) = 0$.

It follows that the curves $\{u = \mathrm{const.}\}$ and $\{v = \mathrm{const.}\}$ are perpendicular, therefore $F = F_u \cdot F_v = 0$.

Lines of Curvature Revisited - cont.

Thus the curves $\{u = \text{const.}\}$ and $\{v = \text{const.}\}$ are lines of curvature on the surface $w = \text{const.}$. Analogously one proves that the similar fact holds for the surfaces $u = \text{const.}$ and $v = \text{const.}$.

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$\square$

Lines of Curvature Revisited - cont.

Exercise

In the proof above we made appeal to the following fact :

Lemma

$$\{u = \text{const.}\} \perp \{v = \text{const.}\} \iff M = F = 0.$$

Prove the lemma above.

Remark

The reciprocal of Dupin's Theorem does not hold, as the following counterexample shows :

Counterexample

Non-circular cylinders with Oz as axis, as well as planes through Oz and also the planes parallel to the plane xOz.

Lines of Curvature Revisited - cont.

However, a partial converse result does hold. More precisely, we have the following

Theorem (Darboux)

If two families of surfaces are orthogonal and, moreover, their curves of intersections are lines of curvature, then there exists a third family of surfaces orthogonal two the first two families

Remark

Dupin's Theorem might appear at best as an interesting curiosity.

However, this is not the case, as Dupin's Theorem is essential in the proof of Liouville's Theorem on the characterization of conformal mappings in space .

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Asymptotic Lines Revisited

- While, as we have seen, using the differential equation of asymptotic lines might be far less useful than one might have expected, one can still characterize asymptotic lines and, moreover, in a much more geometric manner than their differential equation might have had.

The first step in this direction is given by

Theorem (Beltrami-Enneper Theorem, the weak version)

Let $c \subset S$ be an asymptotic line, and let $p = c(0)$, such that $k(0) \neq 0$. Then

$$|\tau(0)| = \sqrt{-k(0)}\,.$$

(Note that, since S admits admits an asymptotic line at $p = c(0)$, $k(0) \leq 0$.)

Remark

The result above is called "the weak version" because it provides no insight on the sign of $\tau(0)$. (However, as its name also suggests, we shall be able to prove a strong version shorty.)

Proof of Beltrami-Enneper Theorem, the weak version

As usually, we may presume that c is parameterized by arc length.

Furthermore, we can naturally set $N = \mathbf{t} \times \mathbf{n}$.

Since, by the second of the Serret-Frènet Formulas, $-\dot{\mathbf{b}} = \mathbf{t} \times \mathbf{n}$, the matrix of the linear mapping $-dN : T_p(S) \to T_p(S)$ can be written in the basis $\{\mathbf{t}(0), \mathbf{n}(0)\}$, as :

Asymptotic Lines Revisited - cont.

$$M = \begin{pmatrix} 0 & \tau(0) \\ \tau(0) & 0 \end{pmatrix}.$$

.

Since $K(p) = \det(-M)$, it follows that $K(p) = -\tau^2(0)$.
(Recall that, since c is an asymptotic line, $K|_c \leq 0$.)

$\square$

To attain the strong version of the Beltrami-Enneper we need to introduce yet another concept, namely that *Darboux frame* :

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Definition

Let $c : I \to \mathbb{R}^3$ be a parametrized curve. Then the *Darboux frame* $\mathbf{t}, \mathbf{u}, \mathbf{v}$ is defined as follows :

$$\mathbf{t}(s) = \dot{\mathbf{c}}(s)$$

$$\mathbf{v}(s) = \mathbf{t}(s) \times \mathbf{u}(s)$$

where $\mathbf{u}(s)$ is defined in the following manner :

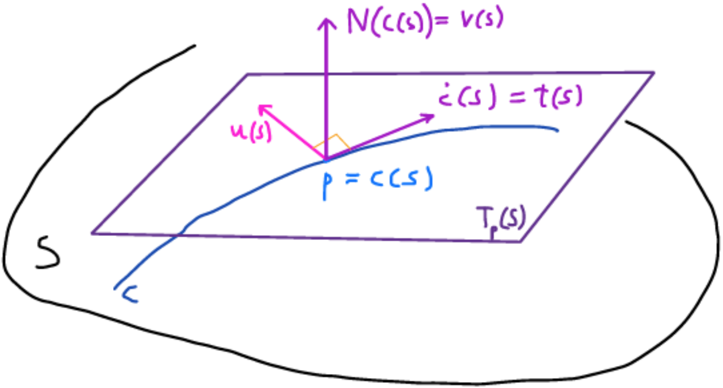
$$(i) \qquad ||\mathbf{u}|| = 1$$

$$(ii) \qquad \mathbf{u} \perp \dot{\mathbf{c}}(s)$$

$$(iii) \qquad \text{The frame } \langle \dot{\mathbf{c}}(s), \mathbf{u}(s) \rangle \text{ is positively oriented}$$

$$(iv)\ \dot{\mathbf{c}}(s) \times \mathbf{u}(s) = N(s)$$

Asymptotic Lines Revisited - cont.



The Darboux frame.

Asymptotic Lines Revisited - cont.

While, in contrast with the Frènet frame, defining the Darboux frame requires the curve c to be embedded in a surface, thus appears to be more restricted, it is also definable even when $k(s) = 0$, thus is applicable for more general curves than the more common Frènet frame.

It's efficiency, so to say, for the study of curves on surfaces becomes evident by considering the natural adaptation of the Serret-Frènet Formulas to the Darboux frame :

$$\begin{cases} \dot{\mathbf{t}} = \phantom{-k_g\mathbf{t}} \quad k_g\mathbf{u} + k_n\mathbf{v} & (*) \\ \dot{\mathbf{u}} = -k_g\mathbf{t} \qquad\quad + \tau_g\mathbf{v} & (**) \\ \dot{\mathbf{v}} = -k_n\mathbf{t} - \tau_g\mathbf{u} & (***) \end{cases}$$

Asymptotic Lines Revisited - cont.

The factor k_n appearing in formulas $(*)$ and $(**)$ is the normal curvature that we already studied, while k_g and τ_g are called the *geodesic curvature* and *geodesic torsion*, respectively.

Exercise

Prove the formulas above and show that the factor k_n appearing therein is, indeed, the normal curvature of c.

Exercise

Let $c \subset S$ be a curve. Prove that the following statements are equivalent :

1. *c is an asymptotic curve.*
2. *$k = \pm k_g$.*

Asymptotic Lines Revisited - cont.

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Asymptotic Lines Revisited - cont.

Exercise

Let $c \subset S$ be a curve. Prove that the following statements are equivalent :

1. *c is a line of curvature.*
2. $\tau_g = 0$.

It is easy to check that, given a unit vector $\mathbf{x} \in T_p(S)$ the following formulas hold :

$$\begin{cases} k_n = -dN(\mathbf{x}) \cdot \mathbf{x} & (*) \\ \tau_g = -dN(\mathbf{x}) \cdot \mathbf{x}^\perp & (**) \end{cases}$$

where $\mathbf{x}^\perp \in T_p(S), \mathbf{x}^\perp \perp \mathbf{x}, ||\mathbf{x}^\perp|| = 1$ and the frame $\langle \mathbf{x}, \mathbf{x}^\perp \rangle$ is positively oriented.

Asymptotic Lines Revisited - cont.

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Exercise

Prove the formulas above.

As might be expected from the similar form of the formulas for k_n and τ_g defined above, a result similar to Euler's formula can be proved for the geodesic torsion. More precisely, we have

Proposition

Let e_1, e_2 be the principal directions at $p = c(0) \in S$, with corresponding principal curvatures k_1 and k_2, and let $\theta = \measuredangle(e_1, \mathbf{x})$, where $\mathbf{x} \in T_p(S), ||\mathbf{x}|| = 1$. Then

$$\tau_g(\mathbf{x}) = (k_2 - k_1)\cos\theta\sin\theta$$

Asymptotic Lines Revisited - cont.

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Exercise

Prove the formula above.

Using the new formulas, one can easily prove the following results relating the torsion of an asymptotic line on a surface (an intrinsic notion) to its geodesic torsion (an extrinsic notion) :

Proposition

Let $c \in S$ be an asymptotic line with curvature k and torsion τ, respectively. If $k \neq 0$, then $\tau = \tau_g$.

Exercise

Prove the Proposition above.

Asymptotic Lines Revisited - cont.

We are now equipped with the necessary tools that allow us to formulate and prove the strong version of The Beltrami-Enneper Theorem :

Theorem

Beltrami-Enneper Theorem, the strong version Let $c \in S$ be an asymptotic line with curvature k, such that $c(0) = p$ and $k(0) \neq 0$. Then

$$|\tau(0)| = \sqrt{-k(0)}\,.$$

Furthermore, if $K(p) < 0$ and both asymptotic lines at p, c_1 and c_2 have non-zero curvatures at p, then $\tau_{c_1} = -\tau_{c_2}$.

Proof

If c is an asymptotic line, then $\tau = \tau_g$, by the proposition above.

Therefore, by the previous formula,

$$\tau_g(0) = (k_2 - k_1)\cos\theta\sin\theta$$

where τ is as above.

Moreover, $\mathbf{x} = \dot{c}(s)$ is an asymptotic vector, therefore by Euler's formula

$$k_{min}\cos^2\theta + k_{min}\sin^2\theta = 0\,.$$

Asymptotic Lines Revisited - cont.

From the two formulas above we immediately obtain that

$$k_{Max} = -\kappa_{min}\frac{\cos^2\theta}{\sin^2\theta}\,.$$

From this last formula and the first one we have

$$\tau(0) = -k_{min}\left(\frac{\cos^2\theta}{\sin^2\theta}+1\right))\cos\theta\sin\theta = -k_{min}\frac{\cos\theta}{\sin\theta}\,.$$

However, from the third formula obtained and from the fact that $K(p) = \kappa_{min}k_{Max}$ we obtain that $K(p) = -\kappa^2_{min}\frac{\cos^2\theta}{\sin^2\theta}$, i.e.

$$K(p) = -\tau^2(0)\,.$$

as desired.

Asymptotic Lines Revisited - cont.

Moreover, the first formula we derived shows that $\tau(0)$ changes sign when we replace θ by $-\theta$, i.e. when passing from one asymptotic line to the other.

$\square$