Differential Geometry – Classical and Discrete

Curves 4: Torsion

Emil Saucan

Ort Braude College, Karmiel

Dalian University of Technology

March 19 & 26, 2023

Skew curves

• We have seen that curvature measures the departure from a planar curve of being a straight line.

• In analogy with it, it is desirable to have at our disposal a quantity that measures the degree (or speed) of departure of a general curve (or *skew* curve) from being planar. In order to define it we need first to better understand curves in space.

Let $c : I \to \mathbb{R}^3$ let be a curve parametrized by arc length, i.e. $||\dot{c}|| \equiv 1$, and let us denote (as usual) $\mathbf{t} = \dot{c}$. Denote (as before) by

$$\mathbf{n} = \frac{\dot{\mathbf{t}}}{|\mathbf{t}|}$$

the *normal vector*. Clearly

$$(*) \qquad \dot{\mathbf{t}}(s) = \kappa(s)\mathbf{n}(s)$$

Skew curves - cont.

We also denote

$$\mathbf{b} = \mathbf{t} \times \mathbf{n}$$

the *binormal vector* (to c at $c(s)$).

The triple $\mathbf{t}, \mathbf{n}, \mathbf{b}$ constitutes the so called *Frènet frame*.

By their very definitions these vectors satisfy the following relations :

$$\begin{cases} \mathbf{t}^2 = \mathbf{n}^2 = \mathbf{b}^2 = 1 \\ \mathbf{t} \cdot \mathbf{n} = \mathbf{n} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{t} = 0 \\ \mathbf{t} = \mathbf{n} \times \mathbf{b} \quad \mathbf{n} = \mathbf{b} \times \mathbf{t} \quad \mathbf{b} = \mathbf{t} \times \mathbf{n} \end{cases}$$

Skew curves - cont.

We also denote

$$\mathbf{b} = \mathbf{t} \times \mathbf{n}$$

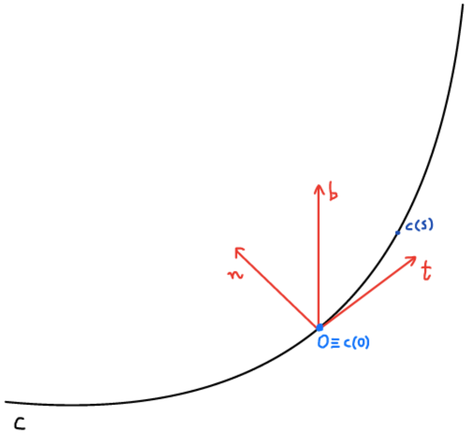
the *binormal vector* (to c at $c(s)$).

The triple $\mathbf{t}, \mathbf{n}, \mathbf{b}$ constitutes the so called *Frènet frame*.

By their very definitions these vectors satisfy the following relations :

$$\begin{cases} \mathbf{t}^2 = \mathbf{n}^2 = \mathbf{b}^2 = 1 \\ \mathbf{t} \cdot \mathbf{n} = \mathbf{n} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{t} = 0 \\ \mathbf{t} = \mathbf{n} \times \mathbf{b} \quad \mathbf{n} = \mathbf{b} \times \mathbf{t} \quad \mathbf{b} = \mathbf{t} \times \mathbf{n} \end{cases}$$

Skew curves - cont



Skew curves - cont

By differentiating we obtain that

$$\dot{\mathbf{b}} = \dot{\mathbf{t}} \times \mathbf{n} + \mathbf{t} \times \dot{\mathbf{n}}\,.$$

Since $\dot{\mathbf{t}} \times \mathbf{n} = 0$, it follows that

$$\dot{\mathbf{b}} = \mathbf{t} \times \dot{\mathbf{n}} = \mathbf{t} \times \dot{\mathbf{n}}\,.$$

From here and from the fact that $\dot{\mathbf{b}}(s) \perp \mathbf{b}$, it follows that $\dot{\mathbf{b}}(s) \parallel \mathbf{n}$, i.e. $\dot{\mathbf{b}}(s)$ is proportional to $\mathbf{n}$, a fact which is written as

$$(***) \qquad \dot{\mathbf{b}}(s) = -\tau(s)\mathbf{n}(s)\,.$$

Torsion

We can now formally introduce our next

Definition

Let $c : I \rightarrow \mathbb{R}^3$ be a regular curve, such that $||\dot{c}|| \equiv 1$ and such that $\ddot{c}(s) \neq 0$, for any $s \in I$. Then $\tau(s)$ in Formula (***) above is called the *torsion* of the curve c at the point s.

Remark

Note that here and above we have used the notation "s" for the parameter of c, instead of the traditional "t". Given that $||\dot{c}|| \equiv 1$, i.e. the curve is parametrized by arc length, this is fully justified.

Torsion - cont.

Moreover, since $\mathbf{b} = \mathbf{t} \times \mathbf{n}$ can be written also as $\mathbf{n} = \mathbf{b} \times \mathbf{t}$, we obtain by derivation that

$$\dot{\mathbf{n}} = \dot{\mathbf{b}} \times \mathbf{t} + \mathbf{b} \times \dot{\mathbf{t}}\,,$$

that is

$$(**) \qquad \dot{\mathbf{n}}(s) = -\kappa(s)\mathbf{t} + \tau(s)\mathbf{b}\,.$$

The formulas $(*) - (***)$ we obtained above are grouped as

Torsion - cont.

Moreover, since $\mathbf{b} = \mathbf{t} \times \mathbf{n}$ can be written also as $\mathbf{n} = \mathbf{b} \times \mathbf{t}$, we obtain by derivation that

$$\dot{\mathbf{n}} = \dot{\mathbf{b}} \times \mathbf{t} + \mathbf{b} \times \dot{\mathbf{t}}\,,$$

that is

$$(**) \qquad \dot{\mathbf{n}}(s) = -\kappa(s)\mathbf{t} + \tau(s)\mathbf{b}\,.$$

The formulas $(*) - (***)$ we obtained above are grouped as

The Serret-Frènet Formulas

$$\begin{cases} \dot{\mathbf{t}} = \kappa \mathbf{n} & (*) \\ \dot{\mathbf{n}} = -\kappa \mathbf{t} + \tau \mathbf{b} & (**) \\ \dot{\mathbf{b}} = -\tau \mathbf{n} & (***) \end{cases}$$

The formulas above are called the *Serret-Frènet Formulas*, and their importance in the understanding of spatial curves can hardly be underestimated, as it will become evident in the sequel.

Remark

The vector $\mathbf{d} = \kappa \mathbf{b} + \tau \mathbf{t}$ is called the Darboux vector. It's usefulness resides in the fact that $\dot{\vec{v}} = \mathbf{d} \times \vec{v}$, thus differentiation along the curve may be written formally as $\frac{d}{ds} = \mathbf{d} \times$.

The Serret-Frènet Formulas

$$\begin{cases} \dot{\mathbf{t}} = \kappa \mathbf{n} & (*) \\ \dot{\mathbf{n}} = -\kappa \mathbf{t} + \tau \mathbf{b} & (**) \\ \dot{\mathbf{b}} = -\tau \mathbf{n} & (***) \end{cases}$$

The formulas above are called the *Serret-Frènet Formulas*, and their importance in the understanding of spatial curves can hardly be underestimated, as it will become evident in the sequel.

Remark

The vector $\mathbf{d} = \kappa \mathbf{b} + \tau \mathbf{t}$ is called the Darboux vector. It's usefulness resides in the fact that $\dot{\vec{v}} = \mathbf{d} \times \vec{v}$, thus differentiation along the curve may be written formally as $\frac{d}{ds} = \mathbf{d}\times$.

Torsion - cont.

We shall return shortly to the S.F. formulas and their applications, but first be bring a few observations regarding torsion :

- The condition that $\ddot{c}(s) \neq 0$, i.e. that $\kappa(s) \neq 0$ is essential. In its absence, it is not clear which plane the curve is departing from – see the counterexample below.

Counterexample

Consider the curve $c : \mathbb{R} \to \mathbb{R}^3$,

$$c(t) = \begin{cases} (t, 0, e^{-1/t^2}) \,; & t > 0 \\ (0, 0, 0) \,; & t = 0 \\ (t, e^{-1/t^2}, 0) & t < 0 \end{cases}$$

.

Torsion - cont.

We shall return shortly to the S.F. formulas and their applications, but first be bring a few observations regarding torsion :

- The condition that $\ddot{c}(s) \neq 0$, i.e. that $\kappa(s) \neq 0$ is essential. In its absence, it is not clear which plane the curve is departing from – see the counterexample below.

Counterexample

Consider the curve $c : \mathbb{R} \to \mathbb{R}^3$,

$$c(t) = \begin{cases} (t, 0, e^{-1/t^2}) \; ; & t > 0 \\ (0, 0, 0) \; ; & t = 0 \\ (t, e^{-1/t^2}, 0) & t < 0 \end{cases}$$

.

Torsion - cont.

- Note that, contrary to curvature, torsion can take negative values. We shall soon see the geometric interpretation of the sign of τ.
- It is easy to see that (the sign of) τ is invariant under change of orientation. (Check !)

We can now return to the applications of the S.F. formulas, and we begin with

Torsion - cont.

- Note that, contrary to curvature, torsion can take negative values. We shall soon see the geometric interpretation of the sign of τ.
- It is easy to see that (the sign of) τ is invariant under change of orientation. (Check !)

We can now return to the applications of the S.F. formulas, and we begin with

The Serret-Frènet Formulas - cont.

Corollary

Let $c : I \to \mathbb{R}^3$, $||\dot{c}|| = 1$. Then $\dot{c}, \ddot{c}, \dddot{c}$ are linearly independent iff c is a plane curve.

Proof

Recall that, given vectors u, v, w, their *mixed* (or *triple*) *product* $u \cdot (v \times w)$ represents the signed volume of the parallelepiped constructed on the three vectors. Therefore, u, v, w are linearly independent $\Leftrightarrow u \cdot (v \times w) = 0$.

Since (a) $\dot{c} = \mathbf{t}$, it follows from $(*)$ that (b) $\ddot{c} = \kappa\mathbf{n}$, thence $\dddot{c} = (\kappa\dot{\mathbf{n}}) = \dot{\kappa}\mathbf{n} + \kappa\dot{\mathbf{n}}$. From the last expression and $(**)$ it follows, (after a couple of manipulations that the reader can provide for he/himself) that (c) $\dddot{c} = \dot{\kappa}\mathbf{n} + \kappa(\tau\mathbf{b} - \kappa\mathbf{t})$.

The Serret-Frènet Formulas - cont.

Then, by using formulas (a), (b) and (c) that we just derived, we obtain that

$$\dot{c} \cdot (\ddot{c} \times \dddot{c}) = \kappa \mathbf{b} \cdot \dot{\kappa} \mathbf{n} + \kappa^2 \tau \mathbf{t} \cdot (\mathbf{n} \times \mathbf{b}) - \kappa^2 \mathbf{t} \cdot (\mathbf{n} \times \mathbf{t})$$

From the definition of $\mathbf{t}, \mathbf{n}, \mathbf{t}$ it follows that the last expression equals $0 + \kappa^2 \tau \mathbf{t} \cdot \mathbf{t} - 0 = \kappa^2 \tau$.

Therefore $\dot{c} \cdot (\ddot{c} \times \dddot{c}) = 0 \Leftrightarrow \kappa^2 \tau = 0$, that is either $\kappa = 0$, i.e. c is straight line, hence planar, or $\tau = 0$, that is c is contained in a plane.

□

The Serret-Frènet Formulas - cont.

A result similar in spirit to the corollary above, that is, again, both simple to prove but extremely important, is the following

Corollary

Let $c : I \to \mathbb{R}^3$, such that $||\dot{c}|| \equiv 1$. Then

1. $\kappa \equiv 0$ *iff c is (part of) a straight line.*
2. $\tau \equiv 0$ *iff c is planar (i.e. it is included in a plane).*

The Serret-Frènet Formulas - cont.

Proof

1. If $\kappa = 0$, then, by $(*)$ it follows that $\dot{\mathbf{t}} = 0 \Leftrightarrow \ddot{c} = 0$, thence it follows that $\dot{c} = \alpha = \text{const.}$, where α is a constant vector. From here it immediately follows that $c(s) = \alpha s + \beta$, where β is a constant vector.
2. If $\tau = 0$, then, by $(***)$ it follows that $\dot{\mathbf{b}} = 0$, thence $\mathbf{b} = \lambda = (\lambda_1, \lambda_2, \lambda_3) = \text{const.}$. By scalar multiplying this last equality by $\dot{c}$ we obtain that $\mathbf{b} \cdot \dot{c} = \lambda \cdot \dot{c} \Leftrightarrow \mathbf{b} \cdot \mathbf{t} = \lambda \cdot \dot{c}$, therefore that $\lambda \cdot \dot{c} = 0$. By integrating this last expression we obtain that $\lambda \cdot c = a$, where a is a constant scalar. This represents, as it is well known from a Linear Algebra course, the equation of a plane, which can be written in coordinates as $\lambda_1 x(s) + \lambda_2 y(s) + \lambda_3 z(s) = a$. Thus, c is, indeed, a planar curve.

$\square$

The Serret-Frènet Formulas - cont.

The following simple, yet geometrically interesting results (since they characterize some classical, important types of curves), can be viewed as exercises :

Proposition

Let c be a curve such that all the tangents to c pass trough a fixed point. Then c is (included in) a straight line.

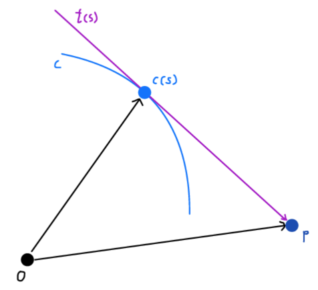
Proof

Let p be the fixed point. Then $c(s) = p - \lambda\dot{c}(s)$, $(\dot{c} \neq 0)$.
By differentiating the equation above we obtain, using the first of the Serret-Frènet formulas, that

$$(1 + \dot{\lambda})\dot{c} = -\lambda\kappa\mathbf{n} \Leftrightarrow ((1 + \dot{\lambda})\dot{c}) \cdot \mathbf{n} = -\lambda\kappa\mathbf{n} \cdot \mathbf{n}\,.$$

The Serret-Frènet Formulas - cont.

Since $\dot{c} \cdot \mathbf{n} = 0$ and $\mathbf{n} \cdot \mathbf{n} = 1$, we obtain that $\kappa \equiv 0$, i.e. c is (part of) a line.



$\square$

The Serret-Frènet Formulas - cont.

Proposition

Let c be a curve such that all the (principal) normals to c pass trough a fixed point. Then c is (included in) a circle.

Again, let p denote the fixed point. Then the condition of statement can be written as

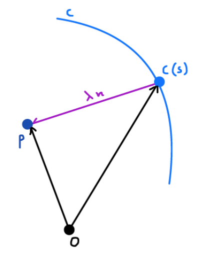
$$c(s) + \lambda(s)\mathbf{n}(s) = p$$

$$\Leftrightarrow \dot{c} = \dot{\lambda}\mathbf{n} + \lambda\dot{\mathbf{n}} = 0 \Leftrightarrow \mathbf{t} + \dot{\lambda}\mathbf{n} + \lambda(-\kappa\mathbf{t} + \tau\mathbf{b}) = 0$$
$$\Leftrightarrow \mathbf{t}(1 - \lambda\kappa) + \dot{\lambda}\mathbf{n} - \tau\lambda\mathbf{b} = 0$$

By the orthogonality of $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ it follows that $1 - \kappa\lambda = 0$, hence (i) $\lambda = 1/\kappa$.

The Serret-Frènet Formulas - cont.

Furthermore, by the same argument, we have that $\dot{\lambda} = 0$, thence (ii) $\lambda = \mathrm{const.}$. From (i) and (ii) above it follows that (iii) $\kappa = \mathrm{const.}$. Moreover, again by the orthogonality of **t**, **n**, **b**, we have that $\tau\lambda = 0$, hence, since $\lambda \neq 0$, $\tau = 0$, therefore (iv) c is included in a plane. From (iii) and (iv) above we immediately reach the desired conclusion.



$\square$

The Serret-Frènet Formulas - cont.

The next result represents a characterization of spherical curves :

Proposition

Let $c : I \to \mathbb{R}^3$ be a parametrized curve, such that $\tau(s) \neq 0, \kappa'(s) \neq 0, \forall s \in I$.
Then c is a spherical curve $\Longleftrightarrow R^2 + (R')^2 T^2 = \mathrm{const.}$,
where we used the following notation : $R = 1/\kappa$, $T = 1/\tau$.

The proof is technical and rather lengthy. Therefore we omit it and concentrate on the more intuitive results as well, as more novel and applicative matters.

However, let us note that the "$\Rightarrow$" of the proof above admits a simple, geometrical argumentation.
Hint : If c lies on a sphere S, then the radius of S represents the osculating radius of c.

Torsion - cont.

Before bringing our next result, let us bring our first – and belated ! – example on the computation of torsion :

Example (The circular helix)

Consider the circular helix, that is the cylindrical curve $c(t) = (a\cos t, a\sin t, bt)$, where $a, b \neq 0$. Then its torsion is $\frac{b}{a^2+b^2}$ (and its curvature is $\frac{|a|}{a^2+b^2}$).

Remark

Note that the pitch of the helix– that is the distance between two consecutive intersections of the helix with a generator of the cylinder (i.e. "the height of one complete helix turn, measured parallel to the axis of the helix") – equals $2\pi b$.

The Serret-Frènet Formulas - cont.

We can now bring the following characterization of helixes :

Proposition

Let $c : I \to \mathbb{R}^3$ be a (parametrized) curve in general position, such that$\kappa(t)\tau(t) \neq 0, \forall t \in I$. Then the following conditions are equivalent :

1. *The tangents to the curve form a constant angle with fixed direction.*
2. *The principal normals to the curve are perpendicular to a fixed direction.*
3. *The binormals to the curve are perpendicular to a fixed direction.*
4. $\kappa(t) = \lambda\tau(t), \forall t \in I$, *where* $\lambda = \mathrm{const.}$.

Haantjes Curvature Revisited

We can now return to the Haantjes and Menger curvatures and (finally !) understand them better in rapport with classical theory.

Theorem

Let $c \subset \mathbb{R}^3; c(s) = (x_1(s), x_2(s), x_3(s))$ be a curve parametrized by arc length, such that $x_i \in \mathcal{C}^3, i = 1, 2, 3$. Then κ_H exists and $\kappa_H \equiv \kappa$ – the classical curvature of c.

Proof

We begin by selecting a mobile coordinate system who's origin O is at p – an arbitrary point on c, and who's axes x_1, x_2, x_3 are along the tangent, normal and binormal at p, respectively. Also, let $q \neq r \neq p \in c$, and denote $l(\widehat{pq}) = u, l(\widehat{qr}) = v$. (If one wishes to taken orientation into account, then choose the points such that $v > 0$.)

Haantjes Curvature Revisited - cont.

Then, by standard expansion into Maclaurin series, we can write

$$x_i(u+v) - x_i(u) = v x_i'(u) + \frac{1}{2}v^2 x_i''(u) + \frac{1}{6}v^3 x_i'''(u+\xi_i v)\,;$$

where $0 < \xi_i, i = 1,2,3.$. Then

$$d(q,r) = \sqrt{\sum_{i=1}^{3}(x_i(u+v) - x_i(u))^2}$$

$$= v\sqrt{\sum_{i=1}^{3}\left(x_i'(u)\frac{1}{2}v^2 x_i''(u) + \frac{1}{6}v^3 x_i'''(u+\xi_i v)\right)^2}\,,$$

where d denotes the usual Euclidean metric in the plane.

Haantjes Curvature Revisited - cont.

Therefore

$$\frac{l(\widehat{qr}) - d(q,r)}{l^3(\widehat{qr})} = \frac{v - d(q,r)}{v^3}$$

$$= \frac{1 - \sqrt{\sum_{i=1}^{3}\left(x_i'(u)\frac{1}{2}v^2x_i''(u) + \frac{1}{6}v^3x_i'''(u+\xi_i v)\right)^2}}{v^2}\,.$$

After some technical manipulations of the right hand side, and noting that $q, r \to p \Leftrightarrow v \to 0$, we obtain that

$$\lim_{q,r\to p}\frac{l(\widehat{qr}) - d(q,r)}{l^3(\widehat{qr})} = -\frac{1}{6}\sum_{i=1}^{3}x_i'(0)x_i'''(0) - \frac{1}{8}\sum_{i=1}^{3}\left(x_i''(0)\right)^2\,.$$

Haantjes Curvature Revisited - cont.

Given our choice of axis at $O \equiv p$, we have that $x_1'(0) = 1, x_2'(0) = x_3'(0) = 0$. By using the Serret-Frènet formulas we obtain that $x_1'''(0) = -\kappa^2$ and $\sum_{i=1}^{3}\left(x_i''(0)\right)^2 = \kappa^2$. Therefore we obtain that

$$\lim_{q,r\to p} \frac{l(\widehat{qr}) - d(q,r)}{l^3(\widehat{qr})} = \frac{1}{6}\kappa^2 - \frac{1}{8}\kappa^2 = \frac{1}{24}\kappa^2,$$

that is $\kappa_H(p) = \kappa$.

□

Remark

Given that, by its very definition, $\kappa_M(p) = \kappa$, the theorem above could have been formulated in terms of Menger curvature, instead of Haantjes curvature.

Haantjes Curvature Revisited - cont.

Given our choice of axis at $O \equiv p$, we have that $x_1'(0) = 1, x_2'(0) = x_3'(0) = 0$. By using the Serret-Frènet formulas we obtain that $x_1'''(0) = -\kappa^2$ and $\sum_{i=1}^{3}\left(x_i''(0)\right)^2 = \kappa^2$. Therefore we obtain that

$$\lim_{q,r\to p} \frac{l(\widehat{qr}) - d(q,r)}{l^3(\widehat{qr})} = \frac{1}{6}\kappa^2 - \frac{1}{8}\kappa^2 = \frac{1}{24}\kappa^2,$$

that is $\kappa_H(p) = \kappa$.

$\square$

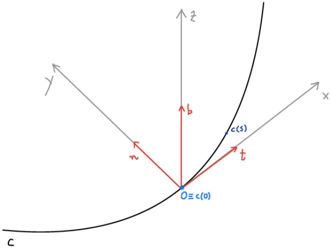
Remark

Given that, by its very definition, $\kappa_M(p) = \kappa$, the theorem above could have been formulated in terms of Menger curvature, instead of Haantjes curvature.

The Local Canonical Form

It is natural to try and express the curve not in the absolute coordinates of the ambient space, but rather into local ones given by the *moving frame* $\mathbf{t}, \mathbf{n}, \mathbf{b}$. [1]

To this end, let us suppose, that $c : I \to \mathbb{R}^3$ is a curve parametrized by arc length, such that $\dot{c}, \ddot{c} \neq 0$; and let us suppose that the base point of reference $p \in c$ is $p = c(0)$.



1. Intuitively, this would like the feeling of a person in a roller-coaster ride who is trying to find his bearing.

The Local Canonical Form - cont.

Then, by developing into Maclaurin series, one can write

$$c(s) = c(0) + \dot{c}(0)s + \frac{\ddot{c}(0)}{2!}s^2 + \frac{\dddot{c}(0)}{3!}s^3 + \omega(s^4)\,;$$

where $\omega(s^4) \xrightarrow[s \to 0]{} 0$.

Since $\dot{c}(0) = \mathbf{t}, \ddot{c}(0) = \mathbf{n}$ and $\dddot{c} = \dot{\kappa}\mathbf{n} + \kappa(\tau\mathbf{b} - \kappa\mathbf{t})$, we get that

$$c(s) - c(0) = \left(s - \frac{\kappa^2 s^3}{3!}\right)\mathbf{t} + \left(\frac{s^2\kappa}{2} + \frac{s^3\dot{\kappa}}{3!}\right)\mathbf{n} - \frac{s^3}{3!}\kappa\tau\mathbf{b} + \omega(s^4)\,.$$

(Here $\kappa = \kappa(0)$, etc.)

Writing, in the coordinates system considered above, $c(s) = (x(s), y(s), z(s))$, one obtains the following system :

The Local Canonical Form - cont.

$$\begin{cases} x(s) = s - \frac{\kappa^2 s^3}{3!} + \omega(x) \\ y(s) = \frac{s^2\kappa}{2} + \frac{s^3\dot{\kappa}}{3!} + \omega(y) \\ z(s) = \frac{s^3}{3!}\kappa\tau + \omega(z) \end{cases}$$

where $\omega(s^4) = \omega(x)\mathbf{t} + \omega(y)\mathbf{n} + \omega(z)\mathbf{b}$.

Thus

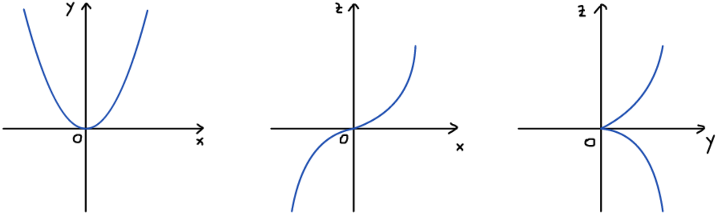
$$\begin{cases} x(s) = s + \alpha(s^3) \\ y(s) = \frac{s^2\kappa}{2} + \beta(s^4) \\ z(s) = \frac{s^3}{3!}\kappa\tau + \gamma(s^4) \end{cases}$$

By simple algebraic manipulations we obtain the following connections between the coordinate functions (in the moving coordinate frame) :

The Local Canonical Form - cont.

$$\begin{cases} y = \frac{\kappa}{2}x^2 \text{ (up to order 2 terms in the osculating plane)} \\ z = \frac{\kappa\tau}{6}x^3 \text{ (up to order 3 terms in the rectifying plane)} \\ z^2 = \frac{2}{9}\frac{\tau^2}{\kappa}y^3 \text{ (up to order 3 terms in the normal plane)} \end{cases}$$

In other words, the projections of the curve on the osculating, rectifying and normal planes have (up to higher order terms) the forms depicted below



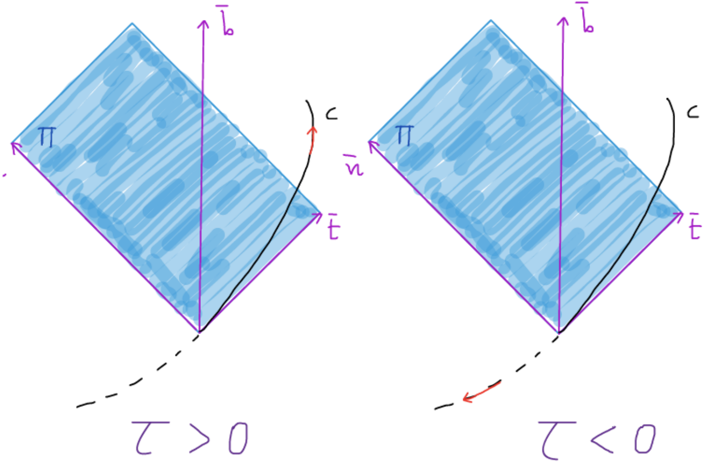
The Local Canonical Form - cont.

One can discern, by looking at the shapes, hence properties, of the projections above, that, as already noted above, the projection on the osculating plane approximates the curve in the best manner, followed, in descending order, by projection on the rectifying and normal planes. (Indeed, the projection onto the normal plane is not even differentiable at 0, as it has a cusp.)

Beyond this simple observation, the local canonical form offers some immediate consequences :

- The first such conclusion is nothing less than the geometric interpretation of torsion :
  It follows from the third equation above that (for small enough s) if $\tau > 0$, then z grows, as s grows ; and if $\tau < 0$, then z decreases, as s grows.

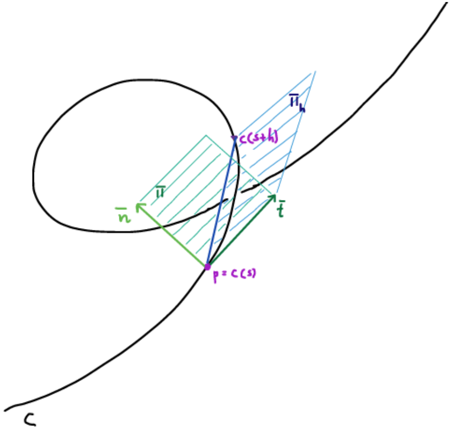
The Local Canonical Form - cont.



The geometric interpretation of the sign of τ.

The Local Canonical Form - cont.

- Considers the planes $\pi_h = \langle \mathbf{t}, c(s+h) \rangle$. Then the osculating plane at p, is the limiting position of the planes π_h, that is $\pi(p) = \lim_{h \to 0} \pi_h$.



Existence and Uniqueness Theorem

The Serret-Frènet formulas not only allow us to prove such results on spatial curves as we have seen above, but also encapsulate a deeper significance, as by using them one can show that the curvature and torsion function uniquely determine the curve (up to isometry) :

Theorem (*The Existence and Uniqueness Theorem of Curves a.k.a. The Fundamental Theorem of the Local Theory of Curves*)

Given differentiable functions $\kappa(s) > 0$ and $\tau(s), s \in I$, there exists a regular parametrized curve $c : I \to \mathbb{R}^3$, such that s represents its arc length, and $\kappa(s), \tau(s)$ are the curvature, resp. torsion of c. Moreover, c is unique up to Euclidean isometry.

(Recall that an *Euclidean isometry* (or *rigid motion*) can be written as an orthogonal linear map of positive determinant O and a translation T of vector $\vec{\mathbf{v}}$.)

Existence and Uniqueness Theorem - cont.

Remark

The full proof of the Existence and Uniqueness Theorem requires making appeal – not surprisingly, given its name – to the Existence and Uniqueness Theorem for Ordinary Differential Equations.

Therefore, we do not bring the proof here, moreover so since our focus is placed on discretizations, rather than on the classical theory.

Metric Torsion

- Given the success (theoretical and practical) of curvature metrization, it is natural to look for a metrization of torsion, too.
- Unfortunately, no simple, intuitive approach to the metrization of the notation of torsion is available.
- However, this does not mean that there exists no such geometrically motivated definition.

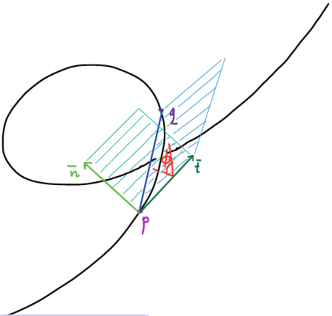
  It is based on the following

Metric Torsion - cont.

Theorem

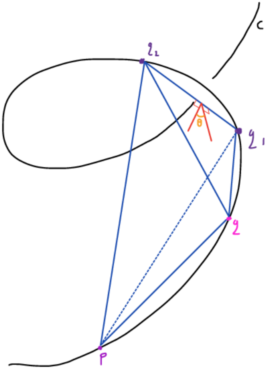
Let ϕ be the angle between the osculating plane to the curve c at the point p, and the plane $\pi(\mathbf{t}, pq)$, where $q \in c, q \neq p$. Then

$$\frac{1}{|\tau(p)|} = 3 \lim_{q \to p} \frac{\sin \phi}{pq}$$



Metric Torsion - cont.

Consider points $q_1, q_2 \in c$, as well as the planes $\pi(p, q_1, q_2)$ and $\pi(q, q_1, q_2)$, and denote by θ the smallest of the two dihedral angles determined by $\pi(p, q_1, q_2)$ and $\pi(q, q_1, q_2)$.



(This tetrahedron is called the osculating tetrahedron.)

Metric Torsion - cont.

Then

$$\frac{\sin\phi}{pq} = \lim_{q_1,q_2\to p}\frac{\sin\theta}{pq}\,.$$

Therefore

$$\frac{1}{|\tau(p)|} = 3\lim_{q\to p}\lim_{q_1,q_2\to q}\frac{\sin\measuredangle(\pi(p,q_1,q_2),\pi(q,q_1,q_2))}{pq}\,.$$

The metrization of torsion is obtained by expressing the right-side of the formula above, in purely metric terms, using its geometric meaning as a dihedral angle in the tetrahedron $T = T(p,q,q_1,q_2)$.

Metric Torsion - cont.

Again, as already seen for Menger's curvature, this is done via the determinant

$$D(p,q,q_1,q_2) = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & pq^2 & pq_1^2 & pq_2^2 \\ 1 & pq^2 & 0 & qq_1^2 & qq_2^2 \\ 1 & pq_1^2 & qq_1^2 & 0 & q_1q_2^2 \\ 1 & pq_2^2 & qq_2^2 & q_1q_2^2 & 0 \end{vmatrix} .$$

By elementary row (or column) operations and by making appeal to the law of cosines, one obtains that

$$D(p,q,q_1,q_2) = 8pq^2 \cdot pq_1^2 \cdot pq_2^2 \cdot \Delta$$

where

$$\Delta = \begin{vmatrix} 1 & \cos\measuredangle pq_1q & \cos\measuredangle pq_1q_2 \\ \cos\measuredangle pq_1q & 1 & \cos\measuredangle qq_1q_2 \\ \cos\measuredangle pq_1q_2 & \cos\measuredangle qq_1q_2 & 1 \end{vmatrix} .$$

Metric Torsion - cont.

Again, as already seen for Menger's curvature, this is done via the determinant

$$D(p,q,q_1,q_2) = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & pq^2 & pq_1^2 & pq_2^2 \\ 1 & pq^2 & 0 & qq_1^2 & qq_2^2 \\ 1 & pq_1^2 & qq_1^2 & 0 & q_1q_2^2 \\ 1 & pq_2^2 & qq_2^2 & q_1q_2^2 & 0 \end{vmatrix} .$$

By elementary row (or column) operations and by making appeal to the law of cosines, one obtains that

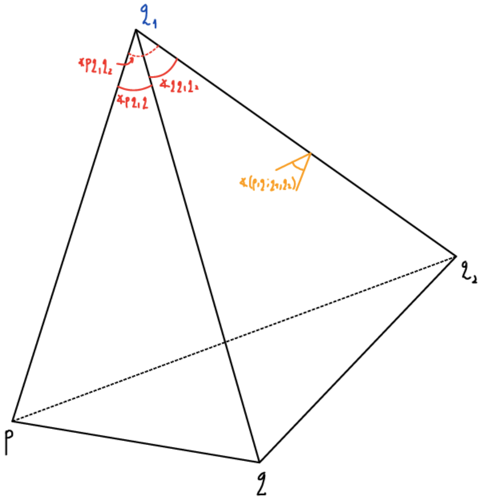
$$D(p,q,q_1,q_2) = 8pq^2 \cdot pq_1^2 \cdot pq_2^2 \cdot \Delta$$

where

$$\Delta = \begin{vmatrix} 1 & \cos\measuredangle pq_1q & \cos\measuredangle pq_1q_2 \\ \cos\measuredangle pq_1q & 1 & \cos\measuredangle qq_1q_2 \\ \cos\measuredangle pq_1q_2 & \cos\measuredangle qq_1q_2 & 1 \end{vmatrix} .$$

Metric Torsion - cont.

The geometric meaning of the entries of the last determinant can be seen below :



Metric Torsion - cont.

Further manipulations of the determinant above, and application of the spherical law of cosines, renders

$$\Delta = \sin^2 \measuredangle(p, q_1, q_2) \cdot \sin^2 \measuredangle(q, q_1, q_2) \cdot \sin^2 \measuredangle(q_1, q_2; p, q)\,;$$

where $\sphericalangle(q_1, q_2; p, q)$ represents the dihedral angle between the faces $T(p, q_1, q_2)$ and $T(q, q_1, q_2)$ of the tetrahedron $T(p, q, q_1, q_2)$.

Since, by a high-school formula,

$$\mathrm{Area}(T(p, q_1, q_2)) = \frac{1}{2} pq_1 \sin \measuredangle(p, q_1, q_2), \mathrm{Area}(T(q, q_1, q_2)) = \frac{1}{2} qq_2 \sin \measuredangle(q, q_1, q_2);$$

Metric Torsion - cont.

Further manipulations of the determinant above, and application of the spherical law of cosines, renders

$$\Delta = \sin^2 \measuredangle(p, q_1, q_2) \cdot \sin^2 \measuredangle(q, q_1, q_2) \cdot \sin^2 \measuredangle(q_1, q_2; p, q)\,;$$

where $\sphericalangle(q_1, q_2; p, q)$ represents the dihedral angle between the faces $T(p, q_1, q_2)$ and $T(q, q_1, q_2)$ of the tetrahedron $T(p, q, q_1, q_2)$.

Since, by a high-school formula,

$$\begin{aligned}\mathrm{Area}(T(p, q_1, q_2)) &= \frac{1}{2} p q_1 \sin \measuredangle(p, q_1, q_2), \mathrm{Area}(T(q, q_1, q_2)) \\ &= \frac{1}{2} q q_2 \sin \measuredangle(q, q_1, q_2);\end{aligned}$$

Metric Torsion - cont.

and, moreover

$$-16\mathrm{Area}^2(T(p,q_1,q_2)) = D(p,q_1,q_2)\,,$$

$$-16\mathrm{Area}^2(T(q,q_1,q_2)) = D(q,q_1,q_2);$$

we obtain that

$$\sin\sphericalangle(q_1,q_2;p,q) = q_1q_2\sqrt{\frac{18D(p,q,q_1,q_2)}{D(p,q_1,q_2)D(q,q_1,q_2)}}\,.$$

Metric Torsion - cont.

In consequence, we attained the desired metric form of torsion of curves in $\mathbb{R}^3$, namely

$$\frac{1}{|\tau(p)|} = 3 \lim_{q \to p} \lim_{p_1, p_2 \to q} \frac{p_1 p_2}{pq} \sqrt{\frac{18 D(p, q, p_1, p_2)}{D(p, p_1, p_2) D(q, p_1, p_2)}} \, .$$

Note that, since we assumed that $\tau(p)$ is defined, the triples considered above are not collinear, it follows that $D(p, p_1, p_2), D(q, p_1, p_2) < 0$ and $D(p, q, p_1, p_2) \geq 0$, therefore the right-hand side of the formula above is well defined.

Blumenthal Metric Torsion

We can now proceed and present Blumenthal's metrization of torsion, devised, in 1939 :

Let $p_1, p, p_3, p_4 \subset \mathbb{R}^3$ points in general position. We define *Blumenthal metric torsion* of the quadruple p_1, p, p_3, p_4 by

Blumenthal metric torsion 1

$$\tau_B^2(p_1, p, p_3, p_4) = \frac{18 D(p_1, p_2, p_3, p_4)}{D(p_1, p_2, p_3) D(p_1, p_3, p_4) D(p_1, p_2, p_4) D(p_2, p_3, p_4)}\,.$$

Blumenthal Metric Torsion - cont.

To define the Blumenthal metric curvature at an accumulation point p of a metric space, we pass, as usual for is by now, to the limit :

Blumenthal metric torsion 2

$$\tau_B(p) = \lim_{p_1, p_2, p_3 \to p} \frac{\sqrt{18|D(p, p_1, p_2, p_3)|}}{\sqrt[4]{D(p_1, p_2, p_3) D(p_1, p_3, p_4) D(p_1, p_2, p_4) D(p_2, p_3, p_4)}} .$$

Metric Torsion - cont.

It turns out that this is a "correct" metrization of torsion and, more precisely we have the following

Theorem

Let $c \subset \mathbb{R}^3$ be a regular smooth curve and let $p \in c$. Then $\tau_B(p)$ exists and $\tau_B(p) = |1/\tau(p)|$.

Hint of Proof :

Use the geometric interpretation of the determinants $D(p_1, p_2, p_3)$, etc. and their connection with Menger curvature to first prove that

$$\tau_B(p_1, p_2; p_3, p_4) = 3 \lim_{p_2, p_3, p_4 \to p_1} \frac{\sin \measuredangle(p_1, p_2; p_3, p_4)}{p_3 p_4}\,.$$

Metric Torsion - cont.

One is also conduced to the following *strong form of Blumenthal's metric torsion* :

Blumenthal metric torsion 3

$$\tau_B^*(p) = \lim_{p_i \to p} \frac{\sqrt{18|D(p_1,p_2,p_3,p_4)|}}{\sqrt[4]{D(p_1,p_2,p_3)D(p_1,p_3,p_4)D(p_1,p_2,p_4)D(p_2,p_3,p_4)}}\,;$$

$$(i = 1,2,3,4)\,.$$

Metric Torsion - cont.

The relationship between the two types of Blumenthal metric torsion are summarized in the exercise below :

Proposition

Prove that

1. *The existence of $\tau_B(p)$ does not imply the existence of $\tau_B^*(p)$.*
2. *$\tau_B^*(p)$ may be defined, even though $\tau_B(p)$ is not.*

One can show (following J. W. Sawyer) that $\tau_B^*(p)$ exists at any point of a regular smooth curve in $\mathbb{R}^3$, and therefore it equals $1/|\tau(p)|$.

Metric Torsion - cont.

Moreover, it can be shown that if $p \in c$, where c is a continuum in $\mathbb{R}^3$, such that $\kappa_M(p) \neq 0$ and $\tau_B^*(p)$ exists, then

$$\tau_B^*(p) = 3 \lim_{p_i \to p} \frac{\sin \measuredangle(p_1, p_2; p_3, p_4)}{p_3 p_4}\, , i = 1, 2, 3, 4\, ;$$

where the points p_1, p_2, p_3, p_4 are in general position.

Metric Torsion - cont.

Using the formula above one can prove the following result

Theorem

If p is a point on metric continuum such that $\tau_B^*(p)$ is defined, and such that $\kappa_M(p) \neq 0$, then there exists a plane $\pi(p)$ such that $\pi(p_1, p_2, p_3) \rightarrow \pi(p)$, that is c has an osculating plane at p, in the generalized sense (which coincides with the classical one for smooth curves in Euclidean space).

Metric Torsion - cont.

Furthermore, metric torsion and curvature also satisfy the following property that shows that they do represent, indeed, proper generalizations of the classical notions, as hoped for :

Theorem

Let $c \subset \mathbb{R}^3$ a continuum such that, for any $p \in C$, the following conditions hold : (a) $\kappa_M(p) > 0$; and (b) $\tau_B^*(p) > 0$. Then c is contained in a plane.

In fact, if a metric curve for which κ_M, τ_B^* are defined and positive at each point p satisfies all the essential properties of a regular curve, such as rectifiability, existence of tangents, etc.

Metric Existence and Uniqueness Theorem of Curves

- Given that the metric curvatures and torsions were developed to mimic the classical notion and, moreover, since they coincide with the classical notions in the case of smooth curves in Euclidean plane and space, it is only natural to ask oneself whether an existence and uniqueness theorem might be formulated in terms of metric curvature and torsion.
- This is is not just a question raised by simple curiosity ; it is, in fact, much deeper, since it probes how well the metric notions in question generalize the classical ones. In other words, whether they are separate discretizations or if they do, indeed, integrate into a metric theory of spatial curves.
- The answer is again, positive. More precisely, we have the following metric generalization of the Fundamental Theorem (of the local theory of curves) :

Metric Existence and Uniqueness Theorem of Curves - cont.

Theorem (Gadum, 1953)

Let $c, \gamma \subset \mathbb{R}^3$ be two smooth metric arcs such that (a) κ_M and τ_B^* are defined and positive at all the points of the two metric arcs. Then the c, γ are isometric iff there exists a bijective application $\Phi : c \to \gamma$ that preserves

1. *Arc length ;*
2. *The metric curvature κ_M ;*
3. *The metric torsion τ_B^* .*

The proof is, unfortunately, quite technical and we therefore omit it.

Metric Existence and Uniqueness Theorem of Curves - cont.

We conclude the presentation of (metric) torsion with a further result illustrating again the parallels between classical and metric curve theory.

Proposition

Let $c, \gamma \subset \mathbb{R}^3$ be two smooth metric arcs such that (a) κ_M and τ_B^* are defined and positive at all the points of the two metric arcs, and let $p \in c$.

Then $\kappa_M(p)$ is equal to the curvature of p of the normal projection of c on the osculating plane of c at p.[a]

a. Recall that the existence of the osculating plane is guaranteed by the Theorem on Slide 50.

Higher Dimensional Curves

- The differential geometry of curves we presented here extends, rather automatically, to curves in $\mathbb{R}^n$, for $n > 3$. However
  - It is not only less intuitive, it also is of far less use in most practical applications.
  - Furthermore, the theory of metric curvatures that we have introduced above transcends, in generality, this simple dimensionality extension.

As such, we do not dwell on it in any detail, but only bring here the most basic facts, for the sake of completeness.

Higher Dimensional Curves - cont.

By mimicking the construction of a Frènet frame for curves in $\mathbb{R}^3$, one constructs a (distinguished) Frènet frame in the n dimensional case as follows : First define

$$e_1(t) = \frac{\dot{c}(t)}{||\dot{c}(t)||}\,;$$

and

$$e_j(t) = \frac{\tilde{e}_j(t)}{||\tilde{e}_j(t)||}\, 2 \leq j \leq n\,;$$

where $\tilde{e}_j(t)$ is defined inductively as follows :

$$\tilde{e}_j(t) = -\sum_{k=1}^{j-1} \left(c^{(j)}(t) \cdot e_k(t)\right) e_k(t) + c^{(j)}(t)\,.$$

Higher Dimensional Curves - cont.

A moving frame $\{e_i(t)\}_1^n$ associated to a curve $c(t)$ in $\mathbb{R}^n$ satisfies the following equations :

$$\dot{c}(t) = \sum_i \alpha_i(t) e_i(t)$$

$$\dot{e}_i(t) = \sum_j \omega_{ij}(t) e_j(t)$$

where

$$(\star) \quad \omega_{ij}(t) = \dot{e}_i(t) \cdot e_j(t) = -\omega_{ij}(t)\,.$$

Higher Dimensional Curves - cont.

If, furthermore, $\{e_i(t)\}_1^n$ is the distinguished Frènet frame we just defined, it also satisfies the fitting Serret-Frènet equations, namely

$$(\star\star_1) \quad \alpha_1 = ||\dot{c}(t)||\,, \;\; \alpha_1 = 0\,, i > 1\,;$$

and

$$(\star\star_2) \quad \omega_{ij}(t) = 0\,, \;\; j > i + 1\,.$$

Higher Dimensional Curves - cont.

It is important to notice how the differential equations above behave under changes of variables. To this effeect we have

Proposition

The coefficients ω_{ij} are invariant under isometries. Furthermore, if the isometry is orientation preserving, so are $\frac{\omega_{ij}(t)}{||\dot{c}(t)||}$.

The importance of the last fact reveals itself in light of our next definition :

Definition

Let $c :\to \mathbb{R}^n$ be a curve admiting a distinguished Frènet frame. Then

$$\kappa_i(t) = \frac{\omega_{ij}(t)}{||\dot{c}(t)||}\,; i = 1, 2, \ldots, n-1\,.$$

is called the *i-th curvature of c*.

Higher Dimensional Curves - cont.

We have the following

Proposition

$\kappa_i(t) > 0$ *for* $1 \leq i \leq n-2$ *.*

In particular, if

- $n = 2$, then there is only one curvature function, $\kappa_1 \equiv \kappa$, which is always positive ;
- $n = 3$, there are only two curvature functions : $\kappa_1 \equiv \kappa$ – the (classical) curvature, which, as we know, is always positive ; and $\kappa_2 \equiv \tau$ – the (classical) torsion, which, of course, can be both positive and negative.
- Out of the $n-1$ curvature functions, the first $n-2$ are positive, thus can be viewed as proper curvatures, while the $(n-1)$-curvature function, which can be positive or negative, corresponds to the classical torsion of curves in $\mathbb{R}^3$.

Higher Dimensional Curves - cont.

We add a homework exercise, for further edification :

Exercise

Consider the curve $c : \mathbb{R} \to \mathbb{R}^4$, $c(t) = (\cos t, \sin t, t, t)$ *.*

1. *Write the distinguished Frènet frame of* c*.*
2. *Compute the curvatures of* c*.*

Remark

Interesting enough, even though the definitions extend, as we have seen, immediately, not all classical types of curves have their immediate higher dimensional analogues.

Higher Dimensional Curves - cont.

We add a homework exercise, for further edification :

Exercise

Consider the curve $c : \mathbb{R} \to \mathbb{R}^4$, $c(t) = (\cos t, \sin t, t, t)$.

1. *Write the distinguished Frènet frame of* c.
2. *Compute the curvatures of* c.

Remark

Interesting enough, even though the definitions extend, as we have seen, immediately, not all classical types of curves have their immediate higher dimensional analogues.