Ruhr-University Bochum — Modern Techniques in Hadron Physics

MTHS24 – Exercise sheet 1

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Lecture material

Discussed topics:

- Lattice discretization
- Simulation algorithms
- Correlation functions

References:

- Quantum Chromodynamics on the Lattice (C. Gattringer, C. Lang), available here
- Lattice Quantum Chromodynamics: Practical Essentials (F. Knechtli, M. Günther, M. Peardon), available here

Exercices

1.1 Fermion Doubling

Consider the Fourier transform of the naive Dirac operator for a free field with the gauge links all set to unity.

$$\tilde{D}(p|q) = \frac{1}{|\Lambda|} \sum_{x,y} \mathrm{e}^{-ip\cdot x}\, D(x|y)\, \mathrm{e}^{iq\cdot y} \tag{1}$$

where $|\Lambda|$ denotes the number of infinite lattice sites.

(a) Verify that $\tilde{D}(p|q) = \delta^4(p-q)\tilde{D}(p)$, with

$$\tilde{D}(p) = m + \frac{i}{a} \sum_{\mu} \gamma_\mu \sin(ap_\mu). \tag{2}$$

(b) Use the identity (with $c,\, b_\mu \in \mathbb{R}$)

$$\left(c + i \sum_{\mu} \gamma_\mu b_\mu \right)^{-1} = \frac{c - i \sum_\mu \gamma_\mu b_\mu}{c^2 + \sum_\mu b_\mu^2} \tag{3}$$

to show that

$$\tilde{D}(p)^{-1} = \frac{m - i a^{-1} \sum_\mu \gamma_\mu \sin(ap_\mu)}{m^2 + a^{-2} \sum_\mu \sin^2(ap_\mu)}. \tag{4}$$

Verify that this propagator has 16 poles at all corners of the Brilloin zone.

(c) Repeat the above exercies for the Wilson Dirac operator $D_{\mathrm{W}}(x|y)$, which is the naive Dirac operator plus the Wilson term. To this end, show that the Fourier transform of the free Wilson Dirac operator is

$$\tilde{D}_{\mathrm{W}}(p) = m + \frac{i}{a} \sum_{\mu} \gamma_\mu \sin(ap_\mu) + \frac{1}{a} \sum_{\mu} (1 - \cos(ap_\mu)). \tag{5}$$

Use this to compute $\tilde{D}_{\mathrm{W}}^{-1}(p)$ and verify that the doublers are given a mass $m + \frac{2\ell}{a}$ where ℓ is the number of non-zero components.

1.2 Symanzik improvement

Consider the five operators which can contribute at dimension five to the Symanzik EFT for Wilson fermions $\mathcal{L}_1^{(1)}(x), \ldots, \mathcal{L}_5^{(1)}(x)$.

(a) Using the field equation $(\gamma_\mu D_\mu + m)\psi(x) = 0$ (and the corresponding one for $\bar{\psi}(x)$), show that

$$\mathcal{L}_1^{(1)} - \mathcal{L}_2^{(1)} + 2\mathcal{L}_5^{(1)} = 0, \quad \mathcal{L}_4^{(1)} + 2\mathcal{L}_5^{(1)} = 0 \tag{6}$$

so that two of these operators (say $\mathcal{L}_2^{(1)}$ and $\mathcal{L}_4^{(1)}$) can be ignored. Note that the Euclidean gamma matrices satisfy $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$

(b) Show that $\mathcal{L}_3^{(1)}$ and $\mathcal{L}_5^{(1)}$ merely shift the coefficients of terms already present in $\mathcal{L}^{(0)}$.

(c) Repeat the above exercise for the improvement of the axial current, to show

$$A^a_{\mathrm{I},\mu}(x) = A^a_\mu(x) + c_\mathrm{A} a \hat{\partial}_\mu P^a(x) \tag{7}$$

where $\hat{\partial}_\mu f(x) = [f(x + a\hat{\mu}) - f(x - a\hat{\mu})]/2a$ is the symmetric derivative.

1.3 Wick contraction

Consider the (zero-momentum) pion interpolator

$$O_{\pi^+}(t) = \sum_{\mathbf{x}} \bar{d}(x)\, \gamma_5\, u(x). \tag{8}$$

This exercise concerns the computation of correlation functions using Wick's theorem, and the signal to noise problem.

(a) Compute the correlation function $C(t) = \langle \bar{O}_{\pi^+}(t)\, O_{\pi^+}(0)\rangle$ in terms of the light quark propagator $D_l^{-1}(x-y)$. Use that result to conclude that (for asymptotically large separations)

$$|D_l^{-1}(x-y)| \sim \mathrm{e}^{-m_\pi |x-y|}. \tag{9}$$

Use this to argue that $C(t)$ has no signal-to-noise problem, that is $C(t)/\sigma(t)$ (where $\sigma^2(t)$ is variance of $C(t)$) is asymptotically independent of t.

(b) Next compute the correlation function for the ρ-meson interpolator, which is the same as the operator above, but with $\gamma_5 \to \gamma_i$. How does the signal-to-noise ratio behave as a function of t now?

(c) Finally, consider the η-meson interpolator

$$O_\eta(t) = \frac{1}{\sqrt{2}} \sum_{\mathbf{x}} \left(\bar{u}(x)\, \gamma_5\, u(x) + \bar{d}(x)\, \gamma_5\, d(x)\right) \tag{10}$$

and compute the correlation function in terms of quark propagators. Note that this correlator requires 'disconnected' diagrams, where the quark propagators start and end at the same time. How does the signal-to-noise ratio behave here? Does $\sigma(t)$ fall off with t?

1.4 Hybrid Monte Carlo Simulations

This exercise requires the openQCD software suite, available here.

(a) Make sure the `Open MPI` and `OpenMP` libraries are installed.

(b) In the openQCD source code, edit the file `main/Makefile`. Near the top, add a line `GCC=<>` to specify the C-compiler on your system. Also specify `MPI_HOME` to provide the location of the MPI libraries on your system, and `MPI_INCLUDE` for the location of the mpi header files.

(c) In `include/global.h`, set `L0` to `L3` to simulate a 16^4 lattice. The number of threads can be controlled by editing `L0_TRD` to `L3_TRD`. Return to the `main` directory, and execute the `make` command.

(d) In a directory to be used for running, create the `log`, `dat`, and `cnfg` directories, and download modified input file `ym1.in`.

(e) Run a simulation using the `ym1` executable using the Wilson gauge action at $\beta = 5.789$ approximately corresponding to a lattice spacing $a = 0.14\,\mathrm{fm}$. Follow the plaquette to trace the thermalization process. Also, monitor the smoothed global topological charge. How many trajectories does it typically take for the charge to change sign?

(f) Now change to $\beta = 6.0$. To maintain an approximately constant physical volume, a 24^4 lattice is required. Can you observe a 'slowdown' in the global topological charge?