MTHS24 – Exercise sheet 5

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Lecture material

Discussed topics:
- Partial waves
- Analyticity
- Unitarity

References:
- Book 1, inspire
- Book 2, amazon
- A good review, inspire
- A good aper, inspire

Exercices

5.1 Two-body phase space

(a) Particle A with mass M decays into two daughters with masses m_1 and m_2. Derive the formula for the *break up momentum*, $\mathrm{p}^* = |\mathbf{p}_1| = |\mathbf{p}_2|$, in the rest frame of A. Use your result to evaluate p^* for the decay $\Delta(1232) \to p\pi$.

Solution: First calculate E_1:

$$
\begin{aligned}
P^2 = M^2 = (p_1+p_2)^2 &= p_1^2 + p_2^2 + 2p_1 \cdot p_2 \\
&= m_1^2 + m_2^2 + 2(E_1E_2 - \mathbf{p}_1 \cdot \mathbf{p}_2) \\
&= m_1^2 + m_2^2 + 2(E_1E_2 + \mathrm{p}_1^2) \qquad \to \qquad \text{with } \mathbf{p}_1 = -\mathbf{p}_2 \\
&= (2m_1^2 + 2\mathrm{p}_1^2) - m_1^2 + m_2^2 + 2E_1(M - E_1) \qquad \to \qquad \text{with } M = E_1 + E_2 \\
&\to E_1 = \frac{M^2 + m_1^2 - m_2^2}{2M}
\end{aligned}
$$

Now calculate p_1:

$$
\begin{aligned}
\mathrm{p}_1^2 &= E_1^2 - m_1^2 \\
&= (E_1 + m_1)(E_1 - m_1) \\
&= \left(\frac{M^2 + m_1^2 - m_2^2 + 2Mm_1}{2M}\right)\left(\frac{M^2 + m_1^2 - m_2^2 - 2Mm_1}{2M}\right) \\
\to \mathrm{p}_1 &= \frac{1}{2M}\left[\left((M+m_1)^2 - m_2^2\right)\left((M-m_1)^2 - m_2^2\right)\right]^{\frac{1}{2}}
\end{aligned}
$$

I believe this is equivalent to the PDG formula:

$$
\mathrm{p}_1 = \frac{1}{2M}\left[\left(M^2 - (m_1+m_2)^2\right)\left(M^2 - (m_1-m_2)^2\right)\right]^{\frac{1}{2}}
$$

For the decay $\Delta(1232) \to p\pi$, $p^* \approx 230$ MeV/c

(b) Starting from the formula for the 2-body decay rate,

$$\Gamma_{fi} = \frac{1}{2M} \int |\mathcal{M}_{fi}|^2 \frac{\mathrm{d}^3 p_1}{(2\pi)^3 2E_1} \frac{\mathrm{d}^3 p_2}{(2\pi)^3 2E_2} (2\pi)^4 \delta^4(P - p_1 - p_2)\,,$$

perform integrations using δ functions to obtain $\mathrm{d}\Gamma/\mathrm{d}\Omega$ in the centre of mass frame, $P = \begin{pmatrix} M \\ \vec{0} \end{pmatrix}$, to obtain the 2-body phase-space factor.

Hint: You may want to use the following property of the delta function:

$$\int_{-\infty}^{+\infty} g(x)\delta(f(x))\mathrm{d}x = \sum_{x_0} \frac{g(x_0)}{|\mathrm{d}f/\mathrm{d}x|_{x_0}}, \text{ where } x_0 \in \{x : f(x) = 0\}.$$

Solution: First integrate out $\mathbf{p}_2$, imposing $\mathbf{p}_2 = -\mathbf{p}_1$ via the delta function:

$$\Gamma_{fi} = \frac{1}{8\pi^2 M} \int |\mathcal{M}_{fi}|^2 \delta(M - E_1 - E_2) \frac{\mathrm{d}^3 p_1}{4E_1 E_2}$$

Substitute $\mathrm{d}^3 p_1 = \mathsf{p}_1^2 \mathrm{d}\mathsf{p}_1 \mathrm{d}\Omega$:

$$\Gamma_{fi} = \frac{1}{8\pi^2 M} \int |\mathcal{M}_{fi}|^2 \delta(M - E_1 - E_2) \frac{\mathsf{p}_1^2 \mathrm{d}\mathsf{p}_1 \mathrm{d}\Omega}{4E_1 E_2}$$

Set:

$$\begin{aligned}
f(\mathsf{p}_1) &= M - E_1 - E_2 = M - \left(m_1^2 + \mathsf{p}_1^2\right)^{\frac{1}{2}} - \left(m_2^2 + \mathsf{p}_1^2\right)^{\frac{1}{2}} \\
\frac{\partial f}{\partial \mathsf{p}_1} &= -\mathsf{p}_1 \left[\left(m_1^2 + \mathsf{p}_1^2\right)^{-\frac{1}{2}} + \left(m_2^2 + \mathsf{p}_1^2\right)^{-\frac{1}{2}} \right] \\
g(\mathsf{p}_1) &= \frac{\mathsf{p}_1^2}{4E_1 E_2} = \frac{\mathsf{p}_1^2}{4\left(m_1^2 + \mathsf{p}_1^2\right)^{\frac{1}{2}} \left(m_2^2 + \mathsf{p}_1^2\right)^{-\frac{1}{2}}} \\
\Gamma_{fi} &= \frac{1}{8\pi^2 M} \int |\mathcal{M}_{fi}|^2 \delta\left(f(\mathsf{p}_1)\right) g(\mathsf{p}_1) \mathrm{d}\mathsf{p}_1 \mathrm{d}\Omega \\
&= \frac{1}{8\pi^2 M} \left| \frac{\mathrm{d}f}{\mathrm{d}\mathsf{p}_1} \right|_{\mathsf{p}^*}^{-1} \int |\mathcal{M}_{fi}|^2 g(\mathsf{p}_1) \delta(\mathsf{p}_1 - \mathsf{p}^*) \mathrm{d}\mathsf{p}_1 \mathrm{d}\Omega \\
&= \frac{1}{8\pi^2 M} \frac{E_1 E_2}{\mathsf{p}^*(E_1 + E_2)} \frac{(\mathsf{p}^*)^2}{4E_1 E_2} \int |\mathcal{M}_{fi}|^2 \mathrm{d}\Omega \\
&= \frac{\mathsf{p}^*}{32\pi^2 M^2} \int |\mathcal{M}_{fi}|^2 \mathrm{d}\Omega
\end{aligned}$$