



# MTHS24 – Exercise sheet 1

Morning: John Bulava

Afternoon: Daniel Winney, Wyatt Smith



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## Lecture material

### Discussed topics:

- Partial waves
- Analyticity
- Unitarity

### References:

- Book 1, [inspire](#)
- Book 2, [amazon](#)
- A good review, [inspire](#)
- A good aper, [inspire](#)

## Exercises

### 1.1 Two-body phase space

- (a) Particle  $A$  with mass  $M$  decays into two daughters with masses  $m_1$  and  $m_2$ . Derive the formula for the *break up momentum*,  $p^* = |\mathbf{p}_1| = |\mathbf{p}_2|$ , in the rest frame of  $A$ . Use your result to evaluate  $p^*$  for the decay  $\Delta(1232) \rightarrow p\pi$ .

**Solution:** First calculate  $E_1$ :

$$\begin{aligned}
 P^2 &= M^2 = (p_1 + p_2)^2 = p_1^2 + p_2^2 + 2p_1 \cdot p_2 \\
 &= m_1^2 + m_2^2 + 2(E_1 E_2 - \mathbf{p}_1 \cdot \mathbf{p}_2) \\
 &= m_1^2 + m_2^2 + 2(E_1 E_2 + p_1^2) \quad \rightarrow \quad \text{with } \mathbf{p}_1 = -\mathbf{p}_2 \\
 &= (2m_1^2 + 2p_1^2) - m_1^2 + m_2^2 + 2E_1(M - E_1) \quad \rightarrow \quad \text{with } M = E_1 + E_2 \\
 &\rightarrow E_1 = \frac{M^2 + m_1^2 - m_2^2}{2M}
 \end{aligned}$$

Now calculate  $p_1$ :

$$\begin{aligned}
 p_1^2 &= E_1^2 - m_1^2 \\
 &= (E_1 + m_1)(E_1 - m_1) \\
 &= \left( \frac{M^2 + m_1^2 - m_2^2}{2M} \right) \left( \frac{M^2 + m_1^2 - m_2^2 - 2Mm_1}{2M} \right) \\
 &\rightarrow p_1 = \frac{1}{2M} \left[ ((M + m_1)^2 - m_2^2) ((M - m_1)^2 - m_2^2) \right]^{\frac{1}{2}}
 \end{aligned}$$

I believe this is equivalent to the PDG formula:

$$p_1 = \frac{1}{2M} \left[ (M^2 - (m_1 + m_2)^2) (M^2 - (m_1 - m_2)^2) \right]^{\frac{1}{2}}$$

For the decay  $\Delta(1232) \rightarrow p\pi$ ,  $p^* \approx 230 \text{ MeV}/c$

(b) Starting from the formula for the 2-body decay rate,

$$\Gamma_{fi} = \frac{1}{2M} \int |\mathcal{M}_{fi}|^2 \frac{d^3 p_1}{(2\pi)^3 2E_1} \frac{d^3 p_2}{(2\pi)^3 2E_2} (2\pi)^4 \delta^4(P - p_1 - p_2),$$

perform integrations using  $\delta$  functions to obtain  $d\Gamma/d\Omega$  in the centre of mass frame,  $P = \begin{pmatrix} M \\ \vec{0} \end{pmatrix}$ , to obtain the 2-body phase-space factor.

*Hint: You may want to use the following property of the delta function:*

$$\int_{-\infty}^{+\infty} g(x) \delta(f(x)) dx = \sum_{x_0} \frac{g(x_0)}{|df/dx|_{x_0}}, \text{ where } x_0 \in \{x : f(x) = 0\}.$$

**Solution:** First integrate out  $\mathbf{p}_2$ , imposing  $\mathbf{p}_2 = -\mathbf{p}_1$  via the delta function:

$$\Gamma_{fi} = \frac{1}{8\pi^2 M} \int |\mathcal{M}_{fi}|^2 \delta(M - E_1 - E_2) \frac{d^3 p_1}{4E_1 E_2}$$

Substitute  $d^3 p_1 = p_1^2 dp_1 d\Omega$ :

$$\Gamma_{fi} = \frac{1}{8\pi^2 M} \int |\mathcal{M}_{fi}|^2 \delta(M - E_1 - E_2) \frac{p_1^2 dp_1 d\Omega}{4E_1 E_2}$$

Set:

$$f(\mathbf{p}_1) = M - E_1 - E_2 = M - (m_1^2 + \mathbf{p}_1^2)^{\frac{1}{2}} - (m_2^2 + \mathbf{p}_1^2)^{\frac{1}{2}}$$

$$\frac{\partial f}{\partial \mathbf{p}_1} = -\mathbf{p}_1 \left[ (m_1^2 + \mathbf{p}_1^2)^{-\frac{1}{2}} + (m_2^2 + \mathbf{p}_1^2)^{-\frac{1}{2}} \right]$$

$$g(\mathbf{p}_1) = \frac{p_1^2}{4E_1 E_2} = \frac{p_1^2}{4(m_1^2 + \mathbf{p}_1^2)^{\frac{1}{2}} (m_2^2 + \mathbf{p}_1^2)^{-\frac{1}{2}}}$$

$$\begin{aligned} \Gamma_{fi} &= \frac{1}{8\pi^2 M} \int |\mathcal{M}_{fi}|^2 \delta(f(\mathbf{p}_1)) g(\mathbf{p}_1) dp_1 d\Omega \\ &= \frac{1}{8\pi^2 M} \left| \frac{df}{d\mathbf{p}_1} \right|_{\mathbf{p}^*}^{-1} \int |\mathcal{M}_{fi}|^2 g(\mathbf{p}_1) \delta(\mathbf{p}_1 - \mathbf{p}^*) dp_1 d\Omega \\ &= \frac{1}{8\pi^2 M} \frac{E_1 E_2}{\mathbf{p}^*(E_1 + E_2)} \frac{(\mathbf{p}^*)^2}{4E_1 E_2} \int |\mathcal{M}_{fi}|^2 d\Omega \\ &= \frac{\mathbf{p}^*}{32\pi^2 M^2} \int |\mathcal{M}_{fi}|^2 d\Omega \end{aligned}$$