Subject: Date: Sa Su Mo Tu We Th Fr

1.

a) $P(\omega_1) = P(\omega_2)$, $x = (a_1 + a_2)/2$

$$\Rightarrow P(\omega_1|x) = P(x|\omega_1)P(\omega_1) = \frac{1}{\pi b}\frac{1}{1+(\frac{x-a_1}{b})^2}P(\omega_1)$$

$$\Rightarrow \frac{1}{\pi b}\frac{1}{1+(\frac{a_2-a_1}{2b})^2}P(\omega_2) = \frac{1}{\pi b}\frac{1}{1+(\frac{x-a_2}{b})^2}P(\omega_2) = \boxed{P(\omega_2|x)}$$ ✓

b) $$P(error) = 1 - \sum_{C=1}^{2}\int_{R_c} f(x|\omega_c)P(\omega_c)dx = 1 - \int_{-\infty}^{\frac{a_1+a_2}{2}} f(x|\omega_1)P(\omega_1)dx - \int_{\frac{a_1+a_2}{2}}^{\infty} f(x|\omega_2)P(\omega_2)dx$$

$$= 1 - \frac{1}{2\pi}\left(tg^{-1}\left(\frac{a_2-a_1}{2b}\right) + \frac{\pi}{2} - tg^{-1}\left(\frac{a_1-a_2}{2b}\right) + \frac{\pi}{2}\right) = \boxed{\frac{1}{2} - \frac{1}{\pi}tg^{-1}\left(\frac{a_2-a_1}{2b}\right)}$$ ✓

c) $$P_{max}(error) = \frac{\partial f(x)}{\partial x} = 0 \Rightarrow \left(\frac{x-a_i}{b}\right)\left(\frac{1}{1+(\frac{x-a_i}{2})^2}\right) = 0 \Rightarrow \boxed{x = a_i}$$

$$\Rightarrow max\, P(error)\Big|_{x=a_1} = 1 - \frac{1}{2\pi}\left(tg^{-1}\left(\frac{x-a_1}{b}\right)\right)\Big|_{-\infty}^{a_1} + tg^{-1}\left(\frac{x-a_2}{b}\right)\Big|_{a_1}^{\infty} = \boxed{\frac{1}{2} + \frac{1}{2}tg^{-1}\left(\frac{a_1-a_2}{b}\right)}$$

$$max\, P(error)\Big|_{x=a_2} = \frac{1}{2} - \frac{1}{2\pi}tg^{-1}\left(\frac{a_2-a_1}{b}\right)$$

marginal $\Rightarrow$ $$max\, P(error)\Big|_{x=\infty} = 1 - \frac{1}{2\pi}\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = \boxed{\frac{1}{2}}$$

در دو حالت اول کمتر از $\frac{1}{2}$ می‌شود پس حداکثر احتمال خطا در $x = \infty$ است و برابر $\frac{1}{2}$ است.

d. $$f(x|\omega_1)P(\omega_1) \underset{R_1}{\overset{R_2}{\lessgtr}} f(x|\omega_2)P(\omega_2) \quad , \quad x = \frac{a_1+a_2}{2} \quad , \quad P(e) = 1 - \sum_{C=1}^{2}\int_{R_c} f(x|\omega_c)P(\omega_c)dx$$

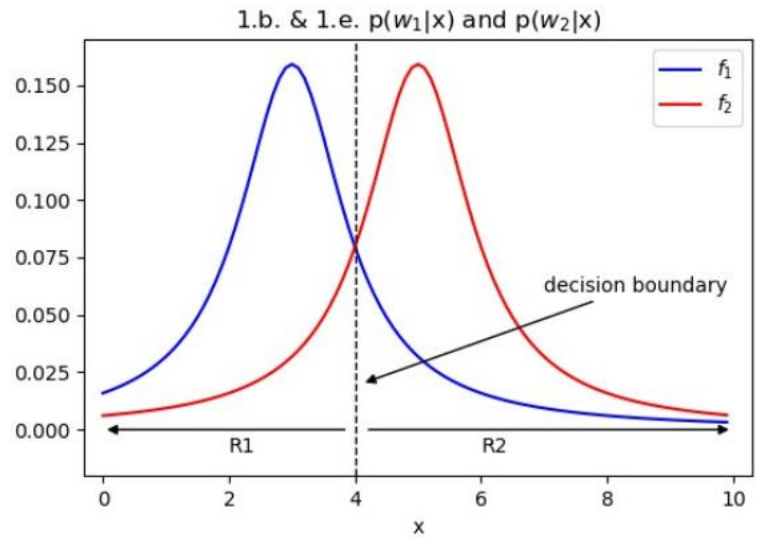
e. $$i^* = \arg\min \sum_{C=1}^{2} \lambda_{ic} P(\omega_c) f(x|\omega_c)$$

$$\Rightarrow \lambda_{12}P(\omega_2)f(x|\omega_2) = \frac{1}{2\pi b}\frac{1}{1+(\frac{x-a_2}{b})^2}$$

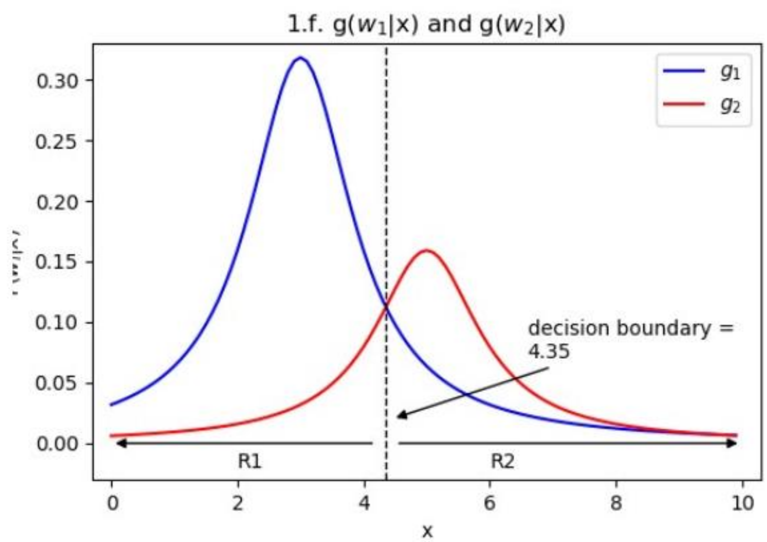
$$\lambda_{21}P(\omega_1)f(x|\omega_1) = \frac{1}{\pi b}\frac{1}{1+(\frac{x-a_1}{b})^2} \Rightarrow a_1 > a_2$$

$$\boxed{x = 4.35}$$

$$\Rightarrow \frac{1}{2\pi b}\frac{1}{1+(\frac{x-a_2}{b})^2} = \frac{1}{\pi b}\frac{1}{1+(\frac{x-a_1}{b})^2} \text{ , assume } a_1 = 3, a_2 = 5, b = 1 \Rightarrow x^2 + 10 - 6x = 2x^2 + 52 - 20x$$



1.b. نمایش مرز تصمیم و توزیع های دو دسته



1.f. مرز تصمیم و discriminant function دو توزیع

2.

$$\min(P(\text{error})) = \max\left(\sum_{c=1}^{2} \int_{R_c} p(x|\omega_c)\, p(\omega_c)\, dx\right)$$

$$p(\omega_2)\, p(x|\omega_2) = P(\omega_1)\, P(x|\omega_1) \xrightarrow{P(\omega_1)=P(\omega_2)} P(x|\omega_2) = P(x|\omega_1)$$

$$\frac{x}{\sigma_2^2} \exp\left(\frac{-x^2}{2\sigma_2^2}\right) = \frac{x}{\sigma_1^2} \exp\left(\frac{-x^2}{2\sigma_1^2}\right) \Rightarrow \ln(\sigma_1^2) + \frac{-x^2}{2\sigma_2^2} = \ln(\sigma_2^2) + \frac{-x^2}{2\sigma_1^2}$$

$$\Rightarrow x = \sqrt{(\ln(\sigma_2^2) - \ln(\sigma_1^2))\, \frac{1}{\frac{-1}{2\sigma_2^2} + \frac{1}{2\sigma_1^2}}}$$

3.

a) در این بخش با یک خط معقول روی شکل داده‌ها را جداسازی کنید

b) $\mu_{red} = [1.33\, ,\, 1.61]$, $\mu_{black} = [-0.15\, ,\, -0.15]$

$$\Sigma_{red} = \begin{bmatrix} 0.62 & 0.2 \\ 0.2 & 1.11 \end{bmatrix}, \quad \Sigma_{black} = \begin{bmatrix} 1.72 & 0.002 \\ 0.002 & 0.55 \end{bmatrix}$$

c) $P(\omega_1)\, P(x|\omega_1) = P(\omega_2)\, P(x|\omega_2)$

$$\frac{1}{\sqrt{2\pi\, \Sigma_1}} \exp\left(-\tfrac{1}{2}(x-\mu_1)^T \Sigma_1^{-1} (x-\mu_1)\right) = \frac{1}{\sqrt{2\pi\, \Sigma_2}} \exp\left(-\tfrac{1}{2}(x-\mu_2)^T \Sigma_2^{-1} (x-\mu_2)\right)$$

$$\Rightarrow \ln\left(\frac{1}{\sqrt{\Sigma_1}}\right) - \tfrac{1}{2}(x-\mu_1)^T \Sigma_1^{-1}(x-\mu_1) = \ln\left(\frac{1}{\sqrt{\Sigma_2}}\right) - \tfrac{1}{2}(x-\mu_2)^T \Sigma_2^{-1}(x-\mu_2)$$

$$\Rightarrow \ln\left(\sqrt{\frac{\Sigma_2}{\Sigma_1}}\right) = \tfrac{1}{2}\left((x-\mu_1)^T \Sigma_1^{-1}(x-\mu_1) - (x-\mu_2)^T \Sigma_2^{-1}(x-\mu_2)\right)$$

d. $$\text{risk} = \sum_{c=1}^{2} \int_{R_c} \lambda_{c0}\, P(x|\omega_c)\, P(\omega_c)\, dx = \int_{R_1} \lambda_{12}\, p(x|\omega_1) P(\omega_1)\, dx + \int_{R_2} \lambda_{21}\, P(x|\omega_2) P(\omega_2)\, dx$$

$$\Rightarrow \lambda_{12} P(\omega_1|x) - \lambda_{21} P(\omega_2|x) = 0 \Rightarrow 2a P(\omega_1|x) - a P(\omega_2|x) = 0 \Rightarrow 2P(\omega_1|x) = P(\omega_2|x)$$

$$\Rightarrow 2P(x|\omega_1)\, P(\omega_1) = P(x|\omega_2)\, P(\omega_2) \Rightarrow \ln\left(\sqrt{\frac{\Sigma_2}{\Sigma_1}}\right) = (x-\mu_1)^T \Sigma_1^{-1}(x-\mu_2) - \tfrac{1}{2}(x-\mu_2)^T \Sigma_2^{-1}(x-\mu_2)$$

e. $$\ln\left(\sqrt{\frac{\Sigma_2}{\Sigma_1}}\right) = \tfrac{1}{3}(x-\mu_1)^T \Sigma_1^{-1}(x-\mu_1) - \tfrac{2}{3}(x-\mu_2)^T \Sigma_2^{-1}(x-\mu_2)$$

4.

a) $\log P(D|\lambda) = \log \prod_{i=1}^{n} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!} = \sum_{i=1}^{n} (x_i \log\lambda - \lambda - \log x_i!) = \log\lambda \sum_{i=1}^{n} x_i - n\lambda - \sum_{i=1}^{n} \log x_i!$

$$\frac{\partial \log P(D|\lambda)}{\partial\lambda} = \frac{1}{\lambda}\sum_{i=1}^{n} x_i - n = 0 \Rightarrow \hat{\lambda}_{ML} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

b) $P(\lambda|D) = \frac{P(D|\lambda)\,P(\lambda)}{P(D)} \propto P(D|\lambda)P(\lambda)$

$$= \prod_{i=1}^{n} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!}\, c\,\lambda^{\alpha-1} e^{-\beta\lambda} = c'\,\lambda^{\sum x_i+\alpha-1}\, e^{-(n+\beta)\lambda} = c'\,\lambda^{\alpha'-1} e^{\beta'\lambda} = \mathrm{Gamma}(\lambda|\alpha',\beta')$$

$$\alpha' = \sum_{i=1}^{n} x_i + \alpha\,,\quad \beta' = n+\beta$$

c) بله زیرا توزیع پسین از جنس توزیع پیشین است.

d) $\lambda_{MAP} = \arg\max_{\lambda} P(\lambda|D) = \frac{\alpha'-1}{\beta'}$

e) $\lim_{n\to\infty} \frac{\sum_{i=1}^{n} x_i + \alpha - 1}{n+\beta} = \frac{\sum x_i}{n} = \hat{\lambda}_{ML}$ بله زیرا:

f) به طور خلاصه زمانی که داده زیاد داریم و اصل برای تخمین داده‌ها دیده شده اس MLE بهتره ولی وقتی یه دانش قبلی از توزیع داده‌ها داریم که می‌خوایم از اون استفاده کنیم MAP بهتره.