COMPSCI/SFWRENG 2FA3

Discrete Mathematics with Applications II

Winter 2020

Assignment 8

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Assignment 8 consists of two problems. You must write your solutions to the problems using LaTeX.

Please submit Assignment 8 as two files, `Assignment_8_`*`YourMacID`*`.tex` and `Assignment_8_`*`YourMacID`*`.pdf`, to the Assignment 8 folder on Avenue under Assessments/Assignments. *`YourMacID`* must be your personal MacID (written without capitalization). The `Assignment_8_`*`YourMacID`*`.tex` file is a copy of the LaTeX source file for this assignment (`Assignment_8.tex` found on Avenue under Contents/Assignments) with your solution entered after each problem. The `Assignment_8_`*`YourMacID`*`.pdf` is the PDF output produced by executing

```
pdflatex Assignment_8_YourMacID
```

This assignment is due **Sunday, March 22, 2020 before midnight.** You are allow to submit the assignment multiple times, but only the last submission will be marked. **Late submissions and files that are not named exactly as specified above will not be accepted!** It is suggested that you submit your preliminary `Assignment_8_`*`YourMacID`*`.tex` and `Assignment_8_`*`YourMacID`*`.pdf` files well before the deadline so that your mark is not zero if, e.g., your computer fails at 11:50 PM on March 22.

Although you are allowed to receive help from the instructional staff and other students, your submission must be your own work. Copying will be treated as academic dishonesty! If any of the ideas used in your submission were obtained from other students or sources outside of the lectures and tutorials, you must acknowledge where or from whom these ideas were obtained.

Problems

1. **[10 points]** Let $G = (N, \Sigma, P, S)$ be the CFG where $N = \{S\}$, $\Sigma = \{a, b\}$, and P contains the following productions:

$$S \to aSb \mid \epsilon.$$

For $x \in \Sigma^*$, let $Q(x)$ be the property that $S \xrightarrow[G]{*} x$ iff $x = a^n b^n$ for some $n \geq 0$. Prove

$$\forall\, x \in \Sigma^* \,.\, Q(x)$$

by weak induction on the length of the derivation $S \xrightarrow[G]{*} x$ for the $(\Rightarrow)$ direction and by strong induction on the length of x for the $(\Leftarrow)$ direction.

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Solution for Problem 1:

Proof. Let $x \in \Sigma^*$. Define $Q_{\Rightarrow}(x)$ to be $(S \xrightarrow[G]{*} x \Rightarrow x = a^n b^n)$ and $Q_{\Leftarrow}(x)$ to be $(x = a^n b^n \Rightarrow S \xrightarrow[G]{*} x)$ for some $n \geq 0$. We will prove $\forall\, x \in \Sigma^* \,.\, Q(x)$ by proving $\forall\, x \in \Sigma^* \,.\, Q_{\Rightarrow}(x)$ and $\forall\, x \in \Sigma^* \,.\, Q_{\Leftarrow}(x)$.

We will prove $\forall\, x \in \Sigma^* \,.\, Q_{\Rightarrow}(x)$ by weak induction on the length of the derivation of $S \xrightarrow[G]{*} x$. Let $P_{\Rightarrow}(d) \equiv \forall\, x \in \Sigma^* \,.\, S \xrightarrow[G]{d} x \implies x = a^n b^n$ for some $n \geq 0$. $\forall\, x \in \Sigma^* \,.\, Q_{\Rightarrow}(x)$ follows immediately from $\forall\, d \in \mathbb{N} \,.\, P_{\Rightarrow}(d)$. So we will prove $\forall\, d \in \mathbb{N} \,.\, P_{\Rightarrow}(d)$ by weak induction.

Base Case 1: $S \xrightarrow[G]{1} x$. Then x must be ϵ and obviously $n = 0$. Hence $P_{\Rightarrow}(1)$ holds.

Inductive Step: $S \xrightarrow[G]{d+1} x$ where $d \geq 1$. Assume $P_{\Rightarrow}(d)$. Then $S \xrightarrow[G]{d} y \xrightarrow[G]{1} x$. This implies $S \xrightarrow[G]{d} y$, and so $y = a^n b^n$ by the induction hypothesis.
Then we have:

$$\begin{aligned} x &= ayb && y \xrightarrow[G]{1} x \\ &= aa^n b^n b && y = a^n b^n \\ &= a^{n+1} b^{n+1} && \text{Arithmetic} \end{aligned} \tag{1}$$

Then, x must be $a^{n+1}b^{n+1}$ and $n = n + 1$. Hence $P_{\Rightarrow}(d + 1)$ holds.
Therefore, $\forall\, d \in \mathbb{N} \,.\, P_{\Rightarrow}(d)$ holds by weak induction.

We will prove $\forall\, x \in \Sigma^* \,.\, Q_{\Leftarrow}(x)$ by strong induction on $|x|$. Let $P_{\Leftarrow}(len) \equiv \forall\, x \in \Sigma^* \,.\, x = a^n b^n \implies S \xrightarrow[G]{*} x$ for some $n \geq 0$ with $len = |x|$.

Case 1. $|x|$ is even

$\forall\, x \in \Sigma^*\, .\, Q_{\Leftarrow}(x)$ follows immediately from $\forall\, len\, .\, P_{\Leftarrow}(len)$, where $len = 2k$ and k $\in \mathbb{N}$. So we will prove $\forall\, len\, .\, P_{\Leftarrow}(len)$, where $len = 2k$ and k $\in \mathbb{N}$ by strong induction.

Base Case 1: $len = 0$. Assume $x = a^n b^n$ with $|x| = 0$. Then $x = \epsilon$ and obviously $S \xrightarrow[G]{*} x$. Hence $P_{\Leftarrow}(0)$ holds.

Base Case 2: $len = 2$. Assume $x = a^n b^n$ with $|x| = 2$. Then $x = ab$ and obviously $S \xrightarrow[G]{*} x$. Hence $P_{\Leftarrow}(2)$ holds.

Inductive Step: $len > 2$. Assume $P_{\Leftarrow}(m)$ for all m, where $m = 2j$ and $j \in \mathbb{N}$ with $m < len$. Assume $x = a^n b^n$ with $|x| = len$. So $x = ayb$ with $|y| + 2 = |x| = len$ and $y = a^{n-1} b^{n-1}$. Then we have:

$$
\begin{aligned}
|y| &= len - 2 && |y| + 2 = len;\ \text{arithmetic} \\
&= 2k - 2 && len = 2k \\
&= 2(k-1) && \text{Arithmetic}
\end{aligned} \tag{2}
$$

Therefore, $m = |y| = 2j$ where $j = k - 1$ and $|y| < len$. This implies $S \xrightarrow[G]{*} y$ by the induction hypothesis. Therefore, $S \xrightarrow[G]{*} ayb \xrightarrow[G]{1} x$, and so $S \xrightarrow[G]{*} x$ when $|x|$ is even. Hence $P_{\Leftarrow}(len)$ holds when $|x|$ is even.

Case 2. $|x|$ is odd

When $|x|$ is odd, neither $x = a^n b^n$ nor $S \xrightarrow[G]{*} x$ hold for some $n \geq 0$. Hence $P_{\Leftarrow}(len)$ holds when $|x|$ is odd.

Therefore, $\forall\, len \in \mathbb{N}\, .\, P_{\Leftarrow}(len)$ holds by strong induction. $\square$

2. **[10 points]** Let $\Sigma = (\mathcal{B}, \mathcal{C}, \mathcal{F}, \mathcal{P}, \tau)$ be a signature of MSFOL where:

   $\mathcal{B} = \{\alpha, \beta\}$.
   $\mathcal{C} = \{a, b\}$ with $\tau(a) = \alpha$ and $\tau(b) = \beta$.
   $\mathcal{F} = \{f, g\}$ with $\tau(f) = \alpha \times \beta \to \alpha$ and $\tau(g) = \beta \to \beta$.
   $\mathcal{P} = \{p, q\}$ with $\tau(p) = \alpha \to \mathbb{B}$ and $\beta \times \beta \to \mathbb{B}$.

   Write a context-free grammar in BNF form that generates the set of Σ-formulas. Assume $\mathcal{V} = \{u, v, w, x, y, z\}$.

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   BNF:

   $\langle formula \rangle ::= \langle equality \rangle \mid \langle predicate \rangle \mid \langle negation \rangle \mid \langle implication \rangle \mid \langle forall \rangle$
   $\langle equality \rangle ::= \langle formula \rangle = \langle formula \rangle$
   $\langle predicate \rangle ::= p(\langle \alpha - term \rangle) \mid q(\langle \beta - term \rangle, \langle \beta - term \rangle)$
   $\langle negation \rangle ::= \neg \langle formula \rangle$
   $\langle implication \rangle ::= \langle formula \rangle \Rightarrow \langle formula \rangle$
   $\langle forall \rangle ::= \forall\ \langle var \rangle\ .\ \langle formula \rangle$
   $\langle term \rangle ::= \langle \alpha - term \rangle \mid \langle \beta - term \rangle$
   $\langle \alpha - term \rangle ::= f(\langle \alpha - term \rangle, \langle \beta - term \rangle) \mid \langle a \rangle \mid \langle var \rangle$
   $\langle \beta - term \rangle ::= g(\langle \beta - term \rangle) \mid \langle b \rangle \mid \langle var \rangle$
   $\langle a \rangle ::= a$
   $\langle b \rangle ::= b$
   $\langle var \rangle ::= u \mid v \mid w \mid x \mid y \mid z$