Internship - PEU 399

3rd Summer School and Internship Programme at CTP-BUE

Candidate

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GRTensor Project Report

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Contents

Abstract

The 3rd Summer School and Internship Programme at the Centre for Theoretical Physics (CTP) was mainly divided into two parts. The first part was several lecture series and the second part was the complementing projects. Two of the lecture series were mainly important in relation to the complementing project that I took part in. The first lecture series was about Physical Cosmology and Galaxies which was lectured by Dr. Amr El-Zant. The second lecture series was specifically designed for General Relativity (GR) and was delivered by Dr. Adel Awad. And the complementing project that I took part in was the GRTensor project which was perfectly aligned with my current interests. The goal of the project was to learn to use Maple with the GRTensor package in order to be able to solve Einstein's Field Equations in different contexts.

Introduction

While GR is easy in its heart, it needs a lot of effort in dealing with the equations. The difficulty of its geometrical feature manifests in the tensorial representations. Fortunately, using the GRTensor package with Maple engine makes it a lot easier in calculating tensor components and incorporating tensors into differential equations, especially in the GR context. Therefore, during this project, Maple is used to solve Einstein's equations in different spacetime configurations.

The lectures associated with the project were advocated to get familiar with Maple by completing three exercises:

- Flat Spacetime Configuration
  In this exercise, we investigated the Minkowski metric and Minkowski space and calculated the connections. It was evident from our calculations that connections do not transform like tensors when describing the configuration in different coordinate systems, however, all the Riemann tensor components remained Zeros as expected.
- The Schwarzchild Metric
  In this exercise, we defined the spherically symmetric Schwarzchild metric and obtained its general solution.
- The FLRW metric
  In this exercise, we investigated the FLRW metric describing an isotropic and homogeneous spacetime configuration whose spatial scale factor is evolving with time. We channeled our attention to a configuration with zero curvature (Flat). We then obtained the scale factor as a function of time for three different contexts: Matter-dominated, Radiation-dominated, and Vacuum-dominated spacetime configurations.

The work of the project was then divided into three additional tasks that are to be discussed in the following sections.

Methods and Results

3.1 Task(I): The Schwarzschild Metric

In this task, building upon our exercises as discussed, we are to exactly determine the form of the Schwarzschild Metric.

3.1.1 The General Solution

The Schwarzchild metric can be described by the line element expression as follows:

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2(d\theta^2 + sin^2(\theta)d\phi^2) \tag{3.1}$$

maintaining the spherical symmetry for a spacetime configuration where the functions $A(r)$ and $B(r)$ are to be solved for.

In Maple, in the corresponding file, we solved for the two functions and obtained the following:

$$A(r) = c_1 + \frac{c_2}{r} \tag{3.2}$$

$$B(r) = \frac{c_1}{c_1 + \frac{c_2}{r}} \tag{3.3}$$

using the boundary conditions (Flatness condition):

$$\lim_{r\to\infty} A(r) = \lim_{r\to\infty} B(r) = 1 \tag{3.4}$$

we obtain the following:

$$A(r) = 1 + \frac{c_2}{r} \tag{3.5}$$

$$B(r) = \frac{1}{1 + \frac{c_2}{r}} \tag{3.6}$$

But in order to be able to fully assess the two functions we need to follow a somewhat more abstract procedure which is to be tackled through the next subsection.

3.1.2 The Newtonian Limit

It is helpful to keep in mind the geodesic equation,

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0 \tag{3.7}$$

Now, the Newtonian Limit is defined by three characteristics,

1. Particles move with non-relativistic speeds ($v \ll c$) that translates to $\frac{dx^i}{d\tau} \ll \frac{dt}{d\tau}$, for $i \in \{1,2,3\}$ taking τ (the proper time) to be our affine parameter.
2. The gravitational field is static.
3. The gravitational field influence is weak.

Using (1), the second term in the geodesic equation now becomes:

$$\begin{aligned}\Gamma^{\mu}_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} &= \Gamma^{\mu}_{00}\left(\frac{dt}{d\tau}\right)^2 + \sum_{\sigma=1}^{3}\Gamma^{\mu}_{0\sigma}\frac{dt}{d\tau}\frac{dx^\sigma}{d\tau} + \sum_{\rho=1}^{3}\Gamma^{\mu}_{\rho 0}\frac{dx^\rho}{d\tau}\frac{dt}{d\tau} + \sum_{\rho=1,\sigma=1}^{3}\Gamma^{\mu}_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} \\ &= \left(\frac{dt}{d\tau}\right)^2\left(\Gamma^{\mu}_{00} + \sum_{\sigma=1}^{3}\Gamma^{\mu}_{0\sigma}\frac{\left(\frac{dx^\sigma}{d\tau}\right)}{\left(\frac{dt}{d\tau}\right)} + \sum_{\rho=1}^{3}\Gamma^{\mu}_{\rho 0}\frac{\left(\frac{dx^\rho}{d\tau}\right)}{\left(\frac{dt}{d\tau}\right)} + \sum_{\rho=1,\sigma=1}^{3}\Gamma^{\mu}_{\rho\sigma}\frac{\left(\frac{dx^\rho}{d\tau}\right)}{\left(\frac{dt}{d\tau}\right)}\frac{\left(\frac{dx^\sigma}{d\tau}\right)}{\left(\frac{dt}{d\tau}\right)}\right) \\ &\approx \Gamma^{\mu}_{00}\left(\frac{dt}{d\tau}\right)^2\end{aligned} \tag{3.8}$$

so, the geodesic equation now becomes,

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^{\mu}_{00}\left(\frac{dt}{d\tau}\right)^2 = 0 \tag{3.9}$$

Now, we need to calculate the factor Γ^{μ}_{00}, so we get,

$$\Gamma^{\mu}_{00} = \frac{1}{2}g^{\mu\lambda}(\partial_0 g_{\lambda 0} + \partial_0 g_{0\lambda} - \partial_\lambda g_{00}) \tag{3.10}$$

using (2), since the field is static (i.e., $\partial_0 g_{\mu\nu} = 0$, the metric is not an explicit function of the time coordinate), the first two terms drop and we obtain,

$$\Gamma^{\mu}_{00} = -\frac{1}{2}g^{\mu\lambda}\partial_\lambda g_{00} \tag{3.11}$$

Finally, in considering the ***Weak-Field Limit***, we can express the metric tensor as a perturbation of the Minkowski metric decomposing it into

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, |h_{\mu\nu}| \ll 1. \tag{3.12}$$

We can also propose,

$$g^{\mu\nu} = \eta^{\mu\nu} + ah^{\mu\nu} \tag{3.13}$$

since it is plausible for the contravariant metric to be a perturbation of the same background spacetime manifold.

Keeping it only to the first order of h, we can approximate the expression

$$\begin{aligned}g^{\mu\sigma}h_{\sigma\nu} &= h^{\mu}_{\nu} \\ &= (\eta^{\mu\sigma} + ah^{\mu\sigma})\, h_{\sigma\nu} \\ &\approx \eta^{\mu\sigma}h_{\sigma\nu}\end{aligned} \tag{3.14}$$

Using the definition of the inverse (contravariant) metric, $g^{\mu\alpha}g_{\alpha\nu} = \delta^{\mu}_{\nu}$, we get

$$\begin{aligned}g^{\mu\sigma}g_{\sigma\nu} &= \delta^{\mu}_{\nu} \\ &= (\eta^{\mu\sigma} + ah^{\mu\sigma})(\eta_{\sigma\nu} + h_{\sigma\nu}) \\ &\approx \eta^{\mu\sigma}\eta_{\sigma\nu} + \eta^{\mu\sigma}h_{\sigma\nu} + ah^{\mu\sigma}\eta_{\sigma\nu} \\ &= \delta^{\mu}_{\nu} + (1+a)h^{\mu}_{\nu} \\ &\longrightarrow a = -1.\end{aligned} \tag{3.15}$$

therefore we obtain,

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \tag{3.16}$$

Now, substituting 3.12 and 3.16 into 3.11,

$$\begin{aligned}
\Gamma^{\mu}_{00} &= -\frac{1}{2} g^{\mu\lambda} \partial_\lambda g_{00} \\
&= -\frac{1}{2}(\eta^{\mu\lambda} - h^{\mu\lambda}) \partial_\lambda (\eta_{00} + h_{00}) \\
&= -\frac{1}{2}(\eta^{\mu\lambda} - h^{\mu\lambda}) \partial_\lambda (-1 + h_{00}) \\
&= -\frac{1}{2}(\eta^{\mu\lambda} - h^{\mu\lambda}) \partial_\lambda h_{00} \\
&= -\frac{1}{2} \eta^{\mu\lambda} \partial_\lambda h_{00} + \frac{1}{2} h^{\mu\lambda} \partial_\lambda h_{00}
\end{aligned} \tag{3.17}$$

keeping it only to the first order of h, we obtain,

$$\Gamma^{\mu}_{00} = -\frac{1}{2} \eta^{\mu\lambda} \partial_\lambda h_{00} \tag{3.18}$$

and the geodesic equation now becomes,

$$\frac{d^2 x^\mu}{d\tau^2} = \frac{1}{2} \eta^{\mu\lambda} \partial_\lambda h_{00} \left(\frac{dt}{d\tau} \right)^2 \tag{3.19}$$

It is important to note that, from (2),

$$\begin{aligned}
\partial_0 g_{\mu\nu} &= 0 \\
&= \partial_0 (\eta_{\mu\nu} + h_{\mu\nu}) \\
&= \partial_0 h_{\mu\nu}
\end{aligned} \tag{3.20}$$

since $\eta_{\mu\nu}$ is in its canonical form.

Solving for the first component in 3.19 ($\mu = 0$)

$$\begin{aligned}
\frac{d^2 x^0}{d\tau^2} &= \frac{d^2 t}{d\tau^2} \\
&= \frac{1}{2} \eta^{0\lambda} \partial_\lambda h_{00} \left(\frac{dt}{d\tau} \right)^2 \\
&= \frac{1}{2} \eta^{00} \partial_0 h_{00} \left(\frac{dt}{d\tau} \right)^2
\end{aligned} \tag{3.21}$$

using 3.20

$$\frac{d^2 t}{d\tau^2} = 0 \tag{3.22}$$

therefore, we conclude,

$$\frac{dt}{d\tau} = constant \tag{3.23}$$

Using $\eta^{ii} = 1$ for $i \in \{1, 2, 3\}$ and solving for the spatial components in 3.19 we get

$$\begin{aligned} \frac{d^2x^i}{d\tau^2} &= \frac{1}{2}\eta^{i\lambda}\partial_\lambda h_{00}\left(\frac{dt}{d\tau}\right)^2 \\ &= \frac{1}{2}\eta^{ii}\partial_i h_{00}\left(\frac{dt}{d\tau}\right)^2 \\ &= \frac{1}{2}\partial_i h_{00}\left(\frac{dt}{d\tau}\right)^2 \end{aligned} \tag{3.24}$$

rearranging the sides of 3.24 and using 3.23,

$$\begin{aligned} \frac{d^2x^i}{d\tau^2}\frac{d\tau}{dt}\frac{d\tau}{dt} &= \frac{d}{d\tau}\left(\frac{dx^i}{d\tau}\frac{d\tau}{dt}\right)\frac{d\tau}{dt} \\ &= \frac{d}{dt}\left(\frac{dx^i}{dt}\right) \\ &= \frac{1}{2}\partial_i h_{00} \end{aligned} \tag{3.25}$$

and again, from 3.20, we may propose that,

$$h_{00} = -2\Phi(x^i) \tag{3.26}$$

and thus,

$$g_{00} = \eta_{00} + h_{00} = -(1 + 2\Phi) \tag{3.27}$$

therefore, we obtain,

$$\frac{d^2x^i}{dt^2} = -\partial_i\Phi \tag{3.28}$$

hence, Newton's form for the acceleration in a gravitational field with a potential function Φ is retrieved, namely,

$$\vec{a} = -\vec{\nabla}\Phi \tag{3.29}$$

for unit mass, where classically,

$$\Phi = -\frac{GM}{r} \tag{3.30}$$

for a single gravitating body.

3.1.3 The Schwarzchild Solution Revisited

In the weak-field limit, from 3.27 and 3.30, we were able to show that

$$g_{00} = -\left(1 - \frac{2GM}{r}\right) \tag{3.31}$$

Further, the Schwarzchild metric should approach the same behavior in the same limit (characterized this time by $r \to \infty$), yet astoundingly, comparing 3.31 to 3.5 reveals that they are in the exact same form, we only need to identify that in 3.5 and 3.6,

$$c_2 = -2GM \tag{3.32}$$

hence, the Schwarzchild metric expressed in its line element finally becomes,

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1}dr^2 + r^2d\Omega^2 \tag{3.33}$$

3.2 Task(II): The Kerr Metric

In this task, it was required to verify that the Kerr metric satisfies Einstein's field equations. The Kerr metric is in the following form in its line-element expression,

$$ds^2 = \left(\frac{r^2 + a^2 cos^2(\theta)}{r^2 - 2mr + a^2}\right) dr^2 + (r^2 + a^2 cos^2(\theta)) d\theta^2 + sin^2(\theta) \left(r^2 + a^2 + \frac{2mra^2 sin^2(\theta)}{r^2 + a^2 cos^2(\theta)}\right) d\phi^2$$
$$- \left(\frac{4marsin^2(\theta)}{r^2 + a^2 cos^2(\theta)}\right) d\phi dt + \left(-1 + \frac{2mr}{r^2 + a^2 cos^2(\theta)}\right) dt^2$$

To fulfill the task, it suffices to show that the Ricci tensor vanishes, which is already Einstein's equation in vacuum,

$$R_{\mu\nu} = 0 \tag{3.34}$$

which was already shown using Maple in the corresponding file.

Default spacetime = Kerr

For the Kerr spacetime:

Coordinates

x(*up*)

$x^a = [t, r, \theta, \phi]$

Line element

$$ds^2 = \left(-1 + \frac{2\,m\,r}{r^2 + a^2\cos(\theta)^2}\right) dt^2 - \frac{4\,m\,a\,r\sin(\theta)^2\, dt\, d\phi}{r^2 + a^2\cos(\theta)^2} + \left(\frac{r^2}{r^2 - 2\,m\,r + a^2} + \frac{a^2\cos(\theta)^2}{r^2 - 2\,m\,r + a^2}\right) dr^2 + (r^2 + a^2\cos(\theta)^2)\, d\theta^2 + \left(\sin(\theta)^2 r^2 + \sin(\theta)^2 a^2 + \frac{2\sin(\theta)^4\, m\,r\,a^2}{r^2 + a^2\cos(\theta)^2}\right) d\phi^2$$

Figure 3.1: The Kerr Metric

For the Kerr spacetime:

Covariant Ricci

R(*dn*, *dn*)

$R_{a\,b}$ = *All components are zero*

Figure 3.2: The Ricci Tensor

3.3 Task(III): The FLRW Metric

The Friedmann–Lemaître–Robertson–Walker (FLRW) metric as given by its line-element expression,

$$ds^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + sin^2(\theta)d\phi^2)\right) \tag{3.35}$$

describes an isotropic and homogeneous spacetime configuration, where k is the sign of spatial curvature, and $a(t)$ is the scale factor as a function of time which describes how a distance between two points (particles) in the space evolves with time.

The task is to calculate $a(t)$ by utilizing Einstein's field equations for three cases of $k = 1, -1, 0$ and for three cases for the energy density in the universe $\rho \propto a^{-3}, \rho \propto a^{-4}, \rho = constant$ using an equation of state of the form

$$p(t) = w\rho(t) \tag{3.36}$$

The task was carried out using Maple and the following results were obtained.

3.3.1 Matter case

Substituting $w = 0$ in 3.36, we get:

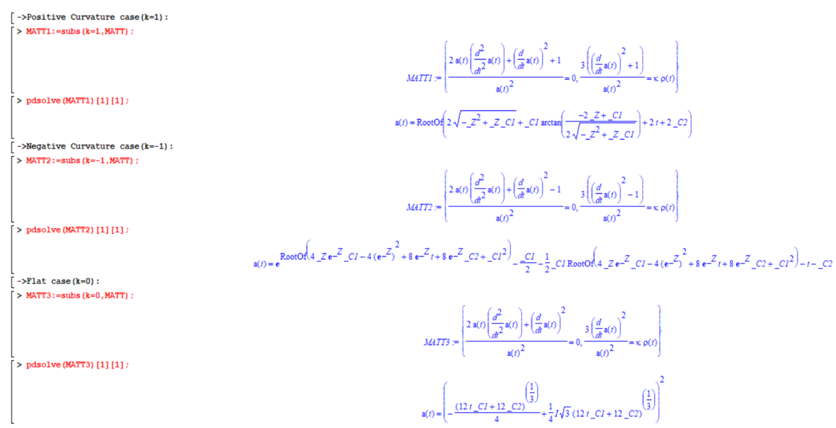
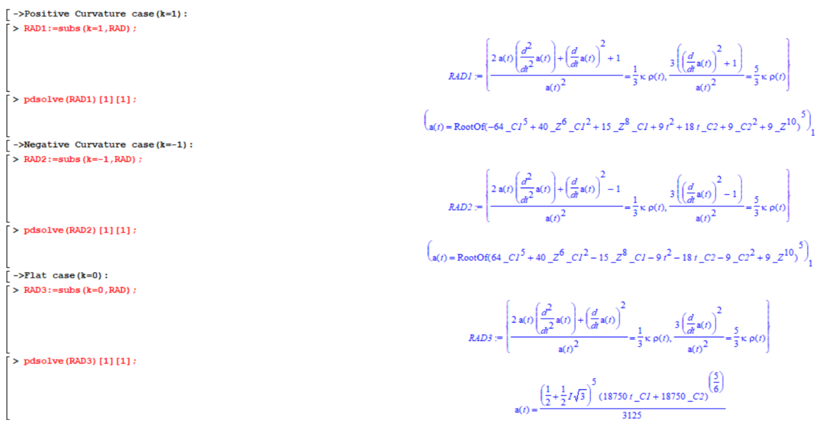


Figure 3.3: For Dust Domination

3.3.2 Radiation case

Substituting $w = \frac{1}{3}$ in 3.36, we get:



```
->Positive Curvature case(k=1):
> RAD1:=subs(k=1,RAD);
```

$$RAD1 := \left\{ \frac{2\,a(t)\left(\frac{d^2}{dt^2}a(t)\right)+\left(\frac{d}{dt}a(t)\right)^2+1}{a(t)^2} = \frac{1}{3}\kappa\,\rho(t), \frac{3\left(\left(\frac{d}{dt}a(t)\right)^2+1\right)}{a(t)^2} = \frac{5}{3}\kappa\,\rho(t) \right\}$$

```
> pdsolve(RAD1)[1][1];
```

$$\left(a(t) = \mathrm{RootOf}(-64\_C1^5+40\_Z^6\_C1^2+15\_Z^8\_C1+9t^2+18t\_C2+9\_C2^2+9\_Z^{10})^5\right)_1$$

```
->Negative Curvature case(k=-1):
> RAD2:=subs(k=-1,RAD);
```

$$RAD2 := \left\{ \frac{2\,a(t)\left(\frac{d^2}{dt^2}a(t)\right)+\left(\frac{d}{dt}a(t)\right)^2-1}{a(t)^2} = \frac{1}{3}\kappa\,\rho(t), \frac{3\left(\left(\frac{d}{dt}a(t)\right)^2-1\right)}{a(t)^2} = \frac{5}{3}\kappa\,\rho(t) \right\}$$

```
> pdsolve(RAD2)[1][1];
```

$$\left(a(t) = \mathrm{RootOf}(64\_C1^5+40\_Z^6\_C1^2-15\_Z^8\_C1-9t^2-18t\_C2-9\_C2^2+9\_Z^{10})^5\right)_1$$

```
->Flat case(k=0):
> RAD3:=subs(k=0,RAD);
```

$$RAD3 := \left\{ \frac{2\,a(t)\left(\frac{d^2}{dt^2}a(t)\right)+\left(\frac{d}{dt}a(t)\right)^2}{a(t)^2} = \frac{1}{3}\kappa\,\rho(t), \frac{3\left(\frac{d}{dt}a(t)\right)^2}{a(t)^2} = \frac{5}{3}\kappa\,\rho(t) \right\}$$

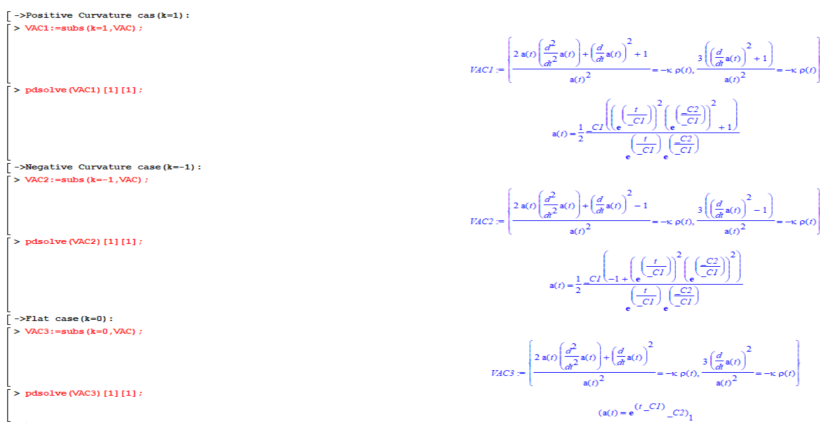
```
> pdsolve(RAD3)[1][1];
```

$$a(t) = \frac{\left(\frac{1}{2}+\frac{1}{2}I\sqrt{3}\right)^5 (18750\,t\,\_C1+18750\,\_C2)^{\left(\frac{5}{6}\right)}}{3125}$$

Figure 3.4: For Radiation Domination

3.3.3 Vacuum case

Substituting $w = -1$ in 3.36, we get:



```
->Positive Curvature cas(k=1):
> VAC1:=subs(k=1,VAC);
```

$$VAC1 := \left\{ \frac{2\,a(t)\left(\frac{d^2}{dt^2}a(t)\right)+\left(\frac{d}{dt}a(t)\right)^2+1}{a(t)^2} = -\kappa\,\rho(t), \frac{3\left(\left(\frac{d}{dt}a(t)\right)^2+1\right)}{a(t)^2} = -\kappa\,\rho(t) \right\}$$

```
> pdsolve(VAC1)[1][1];
```

$$a(t) = \frac{1}{2}\frac{\_C1\left(\left(e^{\left(\frac{t}{\_C1}\right)}\right)^2\left(e^{\left(\frac{\_C2}{\_C1}\right)}\right)^2+1\right)}{e^{\left(\frac{t}{\_C1}\right)}\,e^{\left(\frac{\_C2}{\_C1}\right)}}$$

```
->Negative Curvature case(k=-1):
> VAC2:=subs(k=-1,VAC);
```

$$VAC2 := \left\{ \frac{2\,a(t)\left(\frac{d^2}{dt^2}a(t)\right)+\left(\frac{d}{dt}a(t)\right)^2-1}{a(t)^2} = -\kappa\,\rho(t), \frac{3\left(\left(\frac{d}{dt}a(t)\right)^2-1\right)}{a(t)^2} = -\kappa\,\rho(t) \right\}$$

```
> pdsolve(VAC2)[1][1];
```

$$a(t) = \frac{1}{2}\frac{\_C1\left(-1+\left(e^{\left(\frac{t}{\_C1}\right)}\right)^2\left(e^{\left(\frac{\_C2}{\_C1}\right)}\right)^2\right)}{e^{\left(\frac{t}{\_C1}\right)}\,e^{\left(\frac{\_C2}{\_C1}\right)}}$$

```
->Flat case(k=0):
> VAC3:=subs(k=0,VAC);
```

$$VAC3 := \left\{ \frac{2\,a(t)\left(\frac{d^2}{dt^2}a(t)\right)+\left(\frac{d}{dt}a(t)\right)^2}{a(t)^2} = -\kappa\,\rho(t), \frac{3\left(\frac{d}{dt}a(t)\right)^2}{a(t)^2} = -\kappa\,\rho(t) \right\}$$

```
> pdsolve(VAC3)[1][1];
```

$$\left(a(t) = e^{(t\,\_C1)}\,\_C2\right)_1$$

Figure 3.5: For the Vacuum Case

Conclusion

We were able to easily fulfill the tasks using Maple that otherwise would have been difficult to handle (not impossible, though). We were mainly interested in Einstein's equation in Vacuum for the Schwarzchild and Kerr blackhole solutions. That is plausible since we were interested in the influence of curvature outside the gravitating body.

We also obtained the spatial scale factor as a function of time for an isotropic and homogeneous spacetime configuration in nine different contexts for curvature and energy density combined. However, I believe that case needs more investigation for the complex expressions results are strange for interpretation.

Acknowledgements

I thank Dr. Amr El-Zant for his great lectures that helped in grasping some of the information that was presented about Cosmology and Galaxies and for illustrating the GR treatments in a simplified manner. I also admire and extremely appreciate the efforts of Dr. Adel Awad in his lectures that were specifically designed for GR. It was one of the most wonderful and concise lecture series that I have ever attended. And I present my sincerest gratitude to Dr. Waleed Elhanafy for his efforts in complementing the Cosmology and GR lectures with the GRTensor project. I thank him for this wonderful opportunity, his help in learning to use Maple with the GRTensor package in such a short amount of time, and for his emanant guidance and experience from the project files that he shared with us which were of great help.

Reference

Carroll, S. M. (2004). *Spacetime and geometry: An introduction to general relativity.* Addison Wesley.