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UNIVERSITY OF RUHUNA
DEPARTMENT OF MATHEMATICS
BACHELOR OF SCIENCE (GENERAL) DEGREE (LEVEL I)
MATHEMATICS
MAT 121β : ALGEBRA

Tutorial No:03 **Semester II, 2020**

Submit answer sheets on 20/01/2020

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1. (a) Let $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \lambda$, where $\lambda \in \Re$.

Using the properties of determinants, evaluate each of the following determinants in terms of λ:

(i) $\begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix}$, (ii) $\begin{vmatrix} a & b & c \\ d-3a & e-3b & f-3c \\ 2g & 2h & 2i \end{vmatrix}$,

(iii) $\begin{vmatrix} b & a & c \\ e & d & f \\ h & g & i \end{vmatrix}$, (iv) $\begin{vmatrix} a+\alpha a+\beta b+\gamma c & b & c \\ d+\alpha d+\beta e+\gamma f & e & f \\ g+\alpha g+\beta h+\gamma i & h & i \end{vmatrix}$.

(b) Let $A = \begin{pmatrix} 1 & 2 & -3 \\ 3 & -1 & 5 \\ 4 & 1 & a^2-14 \end{pmatrix}$, where $a \in \Re$.

Find the determinant of A in terms of a.

Consider the following system of equations:

$$\begin{aligned} x + 2y - 3z &= 4 \\ 3x - y + 5z &= 2 \\ 4x + y + (a^2-14)z &= a+2. \end{aligned}$$

For which value(s) of a does the above system have

(i) exactly one solution?
(ii) infinitely many solutions?
(iii) no solutions?

Justify your answers.

2. (a) Let $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ and let det $(A) = \alpha$, where $\alpha \in \Re \setminus \{0\}$.

Using *the properties of determinants*, evaluate each of the following determinants in terms of α:

(i) $\begin{vmatrix} -a_1 & -a_2 & -a_3 \\ -b_1 & -b_2 & -b_3 \\ -c_1 & -c_2 & -c_3 \end{vmatrix}$, (ii) $\begin{vmatrix} a_1+b_1 & a_1-b_1 & c_1 \\ a_2+b_2 & a_2-b_2 & c_2 \\ a_3+b_3 & a_3-b_3 & c_3 \end{vmatrix}$,

(iii) det $\left((2A)^{-1}\right)$. (iv) det $\left(AA'\right)$.

(b) Let $A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{pmatrix}$.

Find A^{-1} and **hence** solve the following system of equations:

$$\begin{aligned} x + y + 2z &= 9 \\ 2x + 4y - 3z &= 1 \\ 3x + 6y - 5z &= 0. \end{aligned}$$

(c) Let $A = \begin{pmatrix} 1 & 0 & 0 & 3 \\ 2 & 2\alpha & 0 & 6 \\ 0 & 6 & 3(\alpha-1) & 0 \\ 1 & 3 & 1 & 7 \end{pmatrix}$, where $\alpha \in \Re$.

Using *the properties of determinants*, find the determinant of A in terms of α.

For which value(s) of α does the following homogeneous system has infinite number of solutions:

$$\begin{aligned} p \qquad\qquad\qquad\qquad + 3s &= 0 \\ 2p + 2\alpha q \qquad\qquad\quad + 6s &= 0 \\ 6q + 3(\alpha-1)r \qquad &= 0 \\ p + 3q \qquad + r \quad + 7s &= 0 \end{aligned}$$

Justify your answer.

(01) (a)

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \lambda$$

(i)

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^T$$

$$\Rightarrow \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \lambda$$

(ii)

$$\begin{vmatrix} a & b & c \\ d-3a & e-3b & f-3c \\ 2g & 2h & 2e \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ 2g & 2h & 2e \end{vmatrix} - 3\underbrace{\begin{vmatrix} a & b & c \\ a & b & c \\ 2g & 2h & 2e \end{vmatrix}}_{0}$$

$$= 2\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & e \end{vmatrix}$$

(iii)

$$\begin{pmatrix} b & a & c \\ e & d & f \\ h & g & i \end{pmatrix} \xleftrightarrow{C_1 \leftrightarrow C_2} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$\Rightarrow \begin{vmatrix} b & a & c \\ e & d & f \\ h & g & i \end{vmatrix} = -\lambda$$

(iv) $\begin{vmatrix} a+\alpha a+\beta b+\gamma c & b & c \\ d+\alpha d+\beta e+\gamma f & e & f \\ g+\alpha g+\beta h+\gamma i & h & i \end{vmatrix}$

$$= \begin{vmatrix} a+\alpha a & b & c \\ d+\alpha d & e & f \\ g+\alpha g & h & i \end{vmatrix} + \begin{vmatrix} \beta b & b & c \\ \beta e & e & f \\ \beta h & h & i \end{vmatrix} + \begin{vmatrix} \gamma c & b & c \\ \gamma f & e & f \\ \gamma i & h & i \end{vmatrix}$$

$$= \begin{vmatrix} (1+\alpha) a & b & c \\ (1+\alpha) d & e & f \\ (1+\alpha) g & h & i \end{vmatrix} + \beta \underbrace{\begin{vmatrix} b & b & c \\ e & e & f \\ h & h & i \end{vmatrix}}_{0} + \gamma \underbrace{\begin{vmatrix} c & b & c \\ f & e & f \\ i & h & i \end{vmatrix}}_{0}$$

$$= (1+\alpha) \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$= (1+\alpha)\lambda$$

(b) $A = \begin{pmatrix} 1 & 2 & -3 \\ 3 & -1 & 5 \\ 4 & 1 & a^2-14 \end{pmatrix}$

$$\begin{vmatrix} 1 & 2 & -3 \\ 3 & -1 & 5 \\ 4 & 1 & a^2-14 \end{vmatrix} \xrightarrow[R_3 \to R_3 - 4R_1]{R_2 \to R_2 - 3R_1} \begin{vmatrix} 1 & 2 & -3 \\ 0 & -7 & 14 \\ 0 & -7 & a^2-2 \end{vmatrix}$$

$$\downarrow R_3 \to R_3 - R_2$$

$$\begin{vmatrix} 1 & 2 & -3 \\ 0 & -7 & 14 \\ 0 & 0 & a^2-16 \end{vmatrix}$$

$$= 1 \times \begin{vmatrix} -7 & 14 \\ 0 & a^2-16 \end{vmatrix}$$

$$= 7(16-a^2)$$

$x + 2y - 3z = 4$

$3x - y + 5z = 2$

$4x + y + (a^2-14)z = a+2$

$$\begin{pmatrix} 1 & 2 & -3 \\ 3 & -1 & 5 \\ 4 & 1 & (a^2-14) \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ a+2 \end{pmatrix}$$

The above system has exactly one solution if

$\det A \neq 0$

$7(16-a^2) \neq 0$

$a^2 \neq 16$

$a \neq \pm 4$

(ii) $\det A = 0$

$7(16-a^2) = 0$

$a = \pm 4$

<u>If $a = 4$</u>

Crammer determinants.

$$|A_x| = \Delta x = \begin{vmatrix} 4 & 2 & -3 \\ 2 & -1 & 5 \\ 6 & 1 & 2 \end{vmatrix} = 4(-7) - 2(-26) - 3(8) = 0$$

$$|A_y| = \Delta y = \begin{vmatrix} 1 & 4 & -3 \\ 3 & 2 & 5 \\ 4 & 6 & 2 \end{vmatrix} = 1(-26) - 4(-14) - 3(10) = 0$$

$$|A_z| = \Delta z = \begin{vmatrix} 1 & 2 & 4 \\ 3 & -1 & 2 \\ 4 & 1 & 6 \end{vmatrix} = 1(-8) - 2(10) + 4(7) = 0$$

Since the crammer determinants are all zero, the system has infinitely many solution.

(iii) <u>If $a = -4$</u>

Crammer determinats.

$$\Delta x = \begin{vmatrix} 4 & 2 & -3 \\ 2 & -1 & 5 \\ -2 & 1 & 2 \end{vmatrix} = 4(-7) - 2(14) - 3(0) \neq 0$$

So the system has no solution.

(Q2) (a)

(i)
$$\begin{vmatrix} -a_1 & -a_2 & -a_3 \\ -b_1 & -b_2 & -b_3 \\ -c_1 & -c_2 & -c_3 \end{vmatrix} = (-1)^3 \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= -\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= -\alpha$$

(ii)
$$\begin{vmatrix} a_1+b_1 & a_1-b_1 & c_1 \\ a_2+b_2 & a_2-b_2 & c_2 \\ a_3+b_3 & a_3-b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_1-b_1 & c_1 \\ a_2 & a_2-b_2 & c_2 \\ a_3 & a_3-b_3 & c_3 \end{vmatrix} + \begin{vmatrix} b_1 & a_1-b_1 & c_1 \\ b_2 & a_2-b_2 & c_2 \\ b_3 & a_3-b_3 & c_3 \end{vmatrix}$$

$$= \underbrace{\begin{vmatrix} a_1 & a_1 & c_1 \\ a_2 & a_2 & c_2 \\ a_3 & a_3 & c_3 \end{vmatrix}}_{0} - \underbrace{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}_{\alpha} + \underbrace{\begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix}}_{-\alpha}$$

$$- \underbrace{\begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix}}_{0}$$

$$= -2\alpha$$

(iii) $\det((2A)^{-1}) = \dfrac{1}{\det(2A)}$

$= \dfrac{1}{2^3 \det(A)}$

$= \dfrac{1}{8\alpha}$ //

(iv) $\det(AA') = \det(A)\det(A')$

$= \det(A)\det(A)$

$= (\det(A))^2$

$= \alpha^2$

(b) $A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{pmatrix}$

$C_{11} = (-1)^2 \begin{vmatrix} 4 & -3 \\ 6 & -5 \end{vmatrix}$

$= -20 + 18$

$= -2$

$C_{21} = (-1)^3 \begin{vmatrix} 1 & 2 \\ 6 & -5 \end{vmatrix}$

$= -(-5 - 12)$

$= 17$

$C_{12} = (-1)^3 \begin{vmatrix} 2 & -3 \\ 3 & -5 \end{vmatrix}$

$= -(-10 + 9)$

$= 1$

$C_{22} = (-1)^4 \begin{vmatrix} 1 & 2 \\ 3 & -5 \end{vmatrix}$

$= -5 - 6$

$= -11$

$C_{13} = (-1)^4 \begin{vmatrix} 2 & 4 \\ 3 & 6 \end{vmatrix}$

$= 12 - 12$

$= 0$

$C_{23} = (-1)^5 \begin{vmatrix} 1 & 1 \\ 3 & 6 \end{vmatrix}$

$= -(6 - 3)$

$= -3$

$$C_{31} = (-1)^4 \begin{vmatrix} 1 & 2 \\ 4 & -3 \end{vmatrix} = -3 - 8$$
$$= -11$$

$$C_{32} = (-1)^5 \begin{vmatrix} 1 & 2 \\ 2 & -3 \end{vmatrix} = -(-3 - 4)$$
$$= 7$$

$$C_{33} = (-1)^6 \begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} = 4 - 2$$
$$= 2$$

Cofacter matrix

$$C = \begin{pmatrix} -2 & 1 & 0 \\ 17 & -11 & -3 \\ -11 & 7 & 2 \end{pmatrix}$$

$$\text{adj } A = C^T = \begin{pmatrix} -2 & 17 & -11 \\ 1 & -11 & 7 \\ 0 & -3 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{pmatrix} \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & -7 \\ 0 & 3 & -11 \end{pmatrix}$$

$$\det A = 1 \times \begin{vmatrix} 2 & -7 \\ 3 & -11 \end{vmatrix}$$
$$= -22 + 21$$
$$= -1$$

$$A^{-1} = \frac{1}{\det A} \text{adj } A$$
$$= \frac{1}{-1} \begin{pmatrix} -2 & 17 & -11 \\ 1 & -11 & 7 \\ 0 & -3 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -17 & 11 \\ -1 & 11 & -7 \\ 0 & 3 & -2 \end{pmatrix}$$

$$x + y + 2z = 9$$
$$2x + 4y - 3z = 1$$
$$3x + 6y - 5z = 0$$

$$\begin{pmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ 1 \\ 0 \end{pmatrix}$$

$$Ax = b$$
$$\underbrace{A^{-1}A}_{I}\, x = A^{-1}b$$
$$x = A^{-1}b$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 & -17 & 11 \\ -1 & 11 & -7 \\ 0 & 3 & -2 \end{pmatrix}\begin{pmatrix} 9 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 18 - 17 \\ -9 + 11 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$x = 1$, $y = 2$, $z = 3$

(C) $D = \begin{pmatrix} 1 & 0 & 0 & 3 \\ 2 & 2\alpha & 0 & 6 \\ 0 & 6 & 3(\alpha-1) & 0 \\ 1 & 3 & 1 & 7 \end{pmatrix}$

$$\begin{vmatrix} 1 & 0 & 0 & 3 \\ 2 & 2\alpha & 0 & 6 \\ 0 & 6 & 3(\alpha-1) & 0 \\ 1 & 3 & 1 & 7 \end{vmatrix}$$

$$= 2 \times 3 \begin{vmatrix} 1 & 0 & 0 & 3 \\ 1 & \alpha & 0 & 3 \\ 0 & 2 & (\alpha-1) & 0 \\ 1 & 3 & 1 & 7 \end{vmatrix}$$

$R_2 \longrightarrow R_2 - R_1$

$R_4 \longrightarrow R_4 - R_1$

$$= 6 \begin{vmatrix} 1 & 0 & 0 & 3 \\ 0 & \alpha & 0 & 0 \\ 0 & 2 & \alpha-1 & 0 \\ 0 & 3 & 1 & 4 \end{vmatrix}$$

$$= 6 \begin{vmatrix} \alpha & 0 & 0 \\ 2 & \alpha-1 & 0 \\ 3 & 1 & 4 \end{vmatrix}$$

$$= 6 \times \alpha \begin{vmatrix} \alpha-1 & 0 \\ 1 & 4 \end{vmatrix}$$

$= 6\alpha \times 4(\alpha-1)$

$= 24\alpha(\alpha-1)$ // $\longrightarrow$ (1)

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & 3 \\ 2 & 2\alpha & 0 & 6 \\ 0 & 6 & 3(\alpha-1) & 0 \\ 1 & 3 & 1 & 7 \end{pmatrix}}_{A} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

$\det A = 24\alpha(\alpha-1)$ $(\because$ from $(1))$.

Infinite number of solution

$\det A = 0$.

$24\alpha(\alpha-1) = 0$

$\alpha = 0$ or $\alpha = 1$

<u>If $\alpha = 0$</u>

Crammer determinants

$$|A_x| = \Delta_x = \begin{vmatrix} 0 & 0 & 0 & 3 \\ 0 & \cancel{2\alpha}\ 0 & 0 & 6 \\ 0 & 6 & -3(\cancel{\alpha-1}) & 0 \\ 0 & 3 & 1 & 7 \end{vmatrix}$$

$= 0$

$$|A_y| = \Delta_y = \begin{vmatrix} 1 & 0 & 0 & 3 \\ 2 & 0 & 0 & 6 \\ 0 & 0 & 3(\alpha-1) & 0 \\ 1 & 0 & 1 & 7 \end{vmatrix} = 0$$

Similarily

$|A_z| = \Delta_z = 0$

If $\alpha = 1$

Crammer determinants

$$|A_x| = \Delta_x = \begin{vmatrix} 0 & 0 & 0 & 3 \\ 0 & 2 & 0 & 6 \\ 0 & 6 & 0 & 0 \\ 0 & 3 & 1 & 7 \end{vmatrix} = 0$$

Similary

$|A_y| = \Delta_y = 0$

and

$|A_z| = \Delta_z = 0.$

Hence, the system has infinitely many sol.^{es}