let, $u=(x_1, y_1)$, $v=(x_2, y_2) \in V$

given that,
$$\langle u, v \rangle = 6x_1x_2 + 2y_1y_2$$

(i) Symmetry Axiom:

$$\langle u, v \rangle = 6x_1x_2 + 2y_1y_2$$
$$= 6x_2x_1 + 2y_2y_1$$
$$= \langle v, u \rangle$$

(II) Additivity Axiom: let, $w=(x_3, y_3)$

$$\langle u+v, w \rangle = 6(x_1+x_2)x_3 + 2(y_1+y_2)y_3$$
$$= 6x_1x_3 + 6x_2x_3 + 2y_1y_3 + 2y_2y_3$$
$$= (6x_1x_3 + 2y_1y_3) + (6x_2x_3 + 2y_2y_3)$$
$$= \langle u, w \rangle + \langle v, w \rangle$$

Homogeneity Axiom:

$$\langle \alpha u, v \rangle = 6(\alpha x_1)x_2 + 2(\alpha y_1)y_2$$
$$= 6\alpha x_1 x_2 + 2\alpha y_1 y_2$$
$$= \alpha(6x_1x_2 + 2y_1y_2)$$
$$= \alpha(u, v)$$

Positive Definite Axiom:

$$\langle v, v \rangle = 6x_1x_1 + 2y_1y_1$$
$$= 6x_1^2 + 2y_1^2$$
$$\geq 0$$

furthermore,

$$\langle v, v \rangle = 0 \iff 6x_1^2 + 2y_1^2 = 0$$
$$\iff v_1 = 0 \text{ and } v_2 = 0$$
$$\iff v = 0$$

consequently, we have shown that $\langle u, v \rangle = 3u_1v_1 + 2u_2v_2$ is an inner product on $\mathbb{R}^2$

given that,

$$\langle u_1, u_2 \rangle = 5x_1x_2 - x_1y_2 - x_2y_1 + 5y_1y_2$$

when,

$$u_1 = (x_1, y_1) \text{ and } u_2 = (x_2, y_2)$$

axiom 1 : Symmetry:

$$\langle u_1, u_2 \rangle = 5x_1x_2 - x_1y_2 - x_2y_1 + 5y_1y_2$$

$$\langle u_2, u_1 \rangle = 5x_2x_1 - x_2y_1 - x_1y_2 + 5y_2y_1$$

$$= \langle u_1, u_2 \rangle$$

axiom 2: Additivity:

if $u_3 = (x_3, y_3)$ then,

$$\langle u_1 + u_2, u_3 \rangle = 5(x_1+x_2)x_3 - (x_1+x_2)y_3 - x_3(y_1+y_2) + 5(y_1+y_2)y_3$$

$$= 5x_1x_3 + 5x_2x_3 - x_1y_3 - x_2y_3 - x_3y_1 - x_3y_2 + 5y_1y_3 + 5y_2y_3$$

$$= (5x_1x_3 - x_3y_1 - x_1y_3 + 5y_1y_3) + (5x_2x_3 - x_3y_2 - x_2y_3 + 5y_2y_3)$$

$$= \langle u_1, u_3 \rangle + \langle u_2, u_3 \rangle$$

axiom 3 : Homogeneity:

$$\langle \lambda u_1, u_2 \rangle = 5(\lambda x_1)x_2 - x_2(\lambda y_1) - (\lambda x_1)y_2 + 5(\lambda y_1)y_2$$

$$= \lambda\left[5x_1x_2 - x_2y_1 - x_1y_2 + 5y_1y_2\right]$$

$$= \lambda\langle u_1, u_2 \rangle \quad \forall \lambda \in \mathbb{R}$$

axiom 4: Positive Definite Property:

if $v = (x, y)$ then $\langle v, v \rangle = 5x^2 - 2xy + 5y^2$

$$= 5\left(x^2 - \frac{2xy}{5} + y^2\right)$$

$$= 5\left(x - \frac{y}{5}\right)^2 + \frac{24y^2}{25}$$

$$\geq 0 \text{ for all } x, y$$

$\langle v, v \rangle = 0$ if and only if $x = y/5$ and $y = 0$, that is, if and only if $v = 0$.

let,

$$A = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \text{ and } B = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$$

we are given that,

$$\langle A, B \rangle = a_1a_2 + b_1b_2 + c_1c_2 + d_1d_2$$

axiom 1: symmetry;

$$\langle A, B \rangle = a_1a_2 + b_1b_2 + c_1c_2 + d_1d_2$$
$$= a_2a_1 + b_1b_2 + c_2c_1 + d_2d_1$$
$$= \langle B, A \rangle$$

axiom 2: additivity;

let, $C = \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix}$

Then,

$$\langle A+B, C \rangle = (a_1+a_2)a_3 + (b_1+b_2)b_3 + (c_1+c_2)c_3 + (d_1+d_2)d_3$$
$$= a_1a_3 + a_2a_3 + b_1b_3 + b_2b_3 + c_1c_3 + c_2c_3 + d_1d_3 + d_2d_3$$
$$= (a_1a_3 + b_1b_3 + c_1c_3 + d_1d_3) + (a_2a_3 + b_2b_3 + c_2c_3 + d_2d_3)$$
$$= \langle A, C \rangle + \langle B, C \rangle$$

axiom 3; Homogeneity;

$$\langle kA, B\rangle = (ka_1)a_2 + (kb_1)b_2 + (kc_1)c_2 + (kd_1)d_2$$
$$= k(a_1a_2) + k(b_1b_2) + k(c_1c_2) + k(d_1d_2)$$
$$= k(a_1a_2 + b_1b_2 + c_1c_2 + d_1d_2)$$
$$= k\langle A, B\rangle$$

axiom 4: Positivity;

$$\langle A, A\rangle = a_1^2 + b_1^2 + c_1^2 + d_1^2 \geq 0$$

and $\langle A, A\rangle = 0$ implies $A = 0$, since this is only possible when all entries A are zero.

(I) given that,

$$\langle f, g \rangle = \int_a^b f(x)\, g(x)\, dx$$

$$\langle \sin x, \cos x \rangle = \int_{-\pi}^{\pi} \sin x \cos x \, dx$$

$$= \int_0^0 u \cdot du$$

let,

$u = \sin x$

$du = \cos x \, du$

$$= 0$$

(II)

$$\text{dist}(x, x^2) = \|x - x^2\| = \sqrt{\langle x - x^2, x - x^2 \rangle}$$

$$= \sqrt{\int_0^1 (x-x^2)(x-x^2)\, dx}$$

$$= \sqrt{\int_0^1 (x^2 - 2x^3 + x^4)\, dx}$$

$$= \sqrt{\left(\frac{x^3}{3} - \frac{x^4}{2} + \frac{x^5}{5}\right|_0^1}$$

$$= \frac{1}{\sqrt{30}}$$

$$\mathrm{dist}(x^3, x^2) = \| x^3 - x^2 \| = \sqrt{\langle x^3 - x^2, x^3 - x^2 \rangle}$$

$$= \sqrt{\int_0^1 (x^3 - x^2)(x^3 - x^2)\, dx}$$

$$= \sqrt{\int_0^1 (x^6 - 2x^5 + x^4)\, dx}$$

$$= \sqrt{\left(\frac{x^7}{7} - \frac{x^6}{3} + \frac{x^5}{5}\right)\Big|_0^1}$$

$$= \frac{1}{\sqrt{105}}$$

Therefore, x^3 is closer to x^2.

$w = \text{span}\{u, v\}$

$$W = \text{span}\left\{\begin{bmatrix}-1\\0\\1\\2\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\end{bmatrix}\right\}$$

$$w = w_1 + w_2$$

$$= \text{Proj}_W w + (w - \text{Proj}_W w) \quad \text{—} ①$$

$$\text{Proj}_W w = \alpha u + \beta v$$

$$= \frac{\langle w, u \rangle}{\|u\|^2} u + \frac{\langle w, v \rangle}{\|v\|^2} v$$

$$= \frac{\langle (-1,2,6,0) . (-1,0,1,2) \rangle}{\|(-1,0,1,2).(-1,0,1,2)\|^2} (-1,0,1,2) \cancel{+ \frac{\langle (-1,2,6,0).(0,1,0,1) \rangle}{}}$$

$$+ \frac{\langle (-1,2,6,0).(0,1,0,1) \rangle}{\|(0,1,0,1).(0,1,0,1)\|^2} (0,1,0,1)$$

$$= \left(\frac{7}{6}\right)(-1,0,1,2) + (0,1,0,1)$$

$$= \left(-\frac{7}{6}, 1, \frac{7}{6}, \frac{10}{3}\right)$$

$$\text{proj}_{W^\perp} w = w - \text{proj}_W w$$

$$= (-1, 2, 6, 0) - \left(-\tfrac{7}{6}, 1, \tfrac{7}{6}, \tfrac{10}{3}\right)$$

$$= \left(\frac{1}{6}, 1, \frac{29}{6}, -\frac{10}{3}\right)$$

$$W = \text{Span}\left\{\begin{bmatrix}3\\0\\1\\1\end{bmatrix}, \begin{bmatrix}0\\2\\5\\1\end{bmatrix}\right\}$$

According to the propositions

$$3x_1 + 0x_2 + x_3 + x_4 = 0$$
$$0x_1 + 2x_2 + 5x_3 + x_4 = 0$$

$$\begin{bmatrix}3 & 0 & 1 & 1\\0 & 2 & 5 & 1\end{bmatrix} \longrightarrow \begin{bmatrix}1 & 0 & \frac{1}{3} & \frac{1}{3}\\0 & 1 & \frac{5}{2} & \frac{1}{2}\end{bmatrix}$$

$$x_1 + \frac{x_3}{3} + \frac{x_4}{3} = 0$$
$$x_2 + \frac{5x_3}{2} + \frac{x_4}{2} = 0$$
$$x_1 = -\frac{x_3}{3} - \frac{x_4}{3} \quad \text{—(1)}$$
$$x_2 = -\frac{5x_3}{2} - \frac{x_4}{2} \quad \text{—(2)}$$

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}-\frac{x_3}{3} - \frac{x_4}{3}\\ -\frac{5x_3}{2} - \frac{x_4}{2}\\ x_3\\ x_4\end{bmatrix} = x_3\begin{bmatrix}-\frac{1}{3}\\-\frac{5}{2}\\1\\0\end{bmatrix} + x_4\begin{bmatrix}-\frac{1}{3}\\-\frac{1}{2}\\0\\1\end{bmatrix}$$

$$W^{\perp} = \text{Span}\left\{\begin{bmatrix}-\frac{1}{3}\\-\frac{5}{2}\\1\\0\end{bmatrix}, \begin{bmatrix}-\frac{1}{3}\\-\frac{1}{2}\\0\\1\end{bmatrix}\right\}$$

then,

$$\begin{bmatrix} 3 & 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} -\frac{1}{3} \\ -\frac{5}{2} \\ 1 \\ 0 \end{bmatrix} = 0$$

$$\begin{bmatrix} 0 & 2 & 5 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} = 0$$

Let,

$$x_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 8 \\ 1 \\ -6 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Produces an orthogonal basis.:

$$V_1 = x_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$V_2 = x_2 - \frac{(x_2 . V_1)}{(V_1 . V_1)} V_1 = \begin{bmatrix} 8 \\ 1 \\ -6 \end{bmatrix} - \frac{\left(\begin{bmatrix} 8 \\ 1 \\ -6 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}\right)}{\left(\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}\right)} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 8 \\ 1 \\ -6 \end{bmatrix} - \frac{10}{5} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ -3 \\ -6 \end{bmatrix}$$

$$V_3 = x_3 - \frac{(x_3 . V_1)}{(V_1 . V_1)} V_1 - \frac{(x_3 . V_2)}{(V_2 . V_2)} V_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \frac{\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}\right)}{\left(\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}\right)} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - \frac{\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ -3 \\ -6 \end{bmatrix}\right)}{\left(\begin{bmatrix} 6 \\ -3 \\ -6 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ -3 \\ -6 \end{bmatrix}\right)} \begin{bmatrix} 6 \\ -3 \\ -6 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \frac{0}{5} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \cancel{+} - \frac{-6}{81} \begin{bmatrix} 6 \\ -3 \\ -6 \end{bmatrix}$$

$$= \frac{1}{9} \begin{bmatrix} 4 \\ -2 \\ 5 \end{bmatrix}$$

then,

$$\frac{V_1}{\|V_1\|} = \frac{\begin{bmatrix}1\\2\\0\end{bmatrix}}{\left\|\begin{bmatrix}1\\2\\0\end{bmatrix}\right\|} = \frac{1}{\sqrt{5}}\begin{bmatrix}1\\2\\0\end{bmatrix}$$

$$\frac{V_2}{\|V_2\|} = \frac{\begin{bmatrix}6\\-3\\-6\end{bmatrix}}{\left\|\begin{bmatrix}6\\-3\\-6\end{bmatrix}\right\|} = \frac{1}{9}\begin{bmatrix}6\\-3\\-6\end{bmatrix} = \frac{1}{3}\begin{bmatrix}2\\-1\\-2\end{bmatrix}$$

$$\frac{V_3}{\|V_3\|} = \frac{\frac{1}{9}\begin{bmatrix}4\\-2\\5\end{bmatrix}}{\left\|\frac{1}{9}\begin{bmatrix}4\\-2\\5\end{bmatrix}\right\|} = \frac{1}{3\sqrt{5}}\begin{bmatrix}4\\-2\\5\end{bmatrix}$$

So the final solⁿ. is,

$$V_1 = \frac{1}{\sqrt{5}}\begin{bmatrix}1\\2\\0\end{bmatrix}, \quad V_2 = \frac{1}{3}\begin{bmatrix}2\\-1\\-2\end{bmatrix}, \quad V_3 = \frac{1}{3\sqrt{5}}\begin{bmatrix}4\\-2\\5\end{bmatrix}$$

et,

$v_1 = (1, 3)$ and $v_2 = (-1, 2)$

$$e_1 = \frac{v_1}{\|v_1\|} = \frac{(1, 3)}{\sqrt{1^2 + 3^2}} = \left(\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right)$$

$$e_2 = \frac{v_2 - \langle v_2, e_1 \rangle e_1}{\| v_2 - \langle v_2, e_1 \rangle e_1 \|} \quad \text{—— } 1$$

$$\langle v_2, e_1 \rangle = (-1, 2) \cdot \left(\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right)$$
$$= \frac{-1}{\sqrt{10}} + \frac{6}{\sqrt{10}}$$
$$= \frac{5}{\sqrt{10}}$$

$$v_2 - \langle v_2, e_1 \rangle e_1 = (-1, 2) - \frac{5}{\sqrt{10}}\left(\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right)$$
$$= \left(-\frac{3}{2}, \frac{1}{2}\right)$$

$$\therefore e_2 = \frac{\left(-\frac{3}{2}, \frac{1}{2}\right)}{\| -\frac{3}{2}, \frac{1}{2} \|} = \frac{\left(-\frac{3}{2}, \frac{1}{2}\right)}{\sqrt{\frac{9}{4} + \frac{1}{4}}} = \frac{2}{\sqrt{10}}\left(-\frac{3}{2}, \frac{1}{2}\right)$$
$$= \left(\frac{-3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right)$$

Therefore our orthonormal set of vectors is $\{e_1, e_2\}$

$$= \left\{\left(\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right), \left(\frac{-3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right)\right\}$$

Note that $\dim(\mathbb{R}^2) = 2$ and every set of orthonormal is linearly independent ~~to~~ so indeed this set of orthonormal vectors is an orthonormal basis of $\mathbb{R}^2$.

(a)

$$A = \begin{bmatrix} -1 & 1 & 3 & 0 \\ 1 & 1 & 1 & 4 \\ 0 & 2 & 4 & 4 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 3 & 0 \\ 1 & 1 & 1 & 4 \\ 0 & 2 & 4 & 4 \end{bmatrix} \xrightarrow{\cancel{R_1 \to (-R_1)}} \cancel{\begin{bmatrix} 1 & -1 & -3 & 0 \\ 1 & 1 & 1 & 4 \\ 0 & 2 & 4 & 4 \end{bmatrix}}$$

$$\downarrow$$

$$\begin{bmatrix} -1 & 1 & 3 & 0 \\ 0 & 2 & 4 & 4 \\ 0 & 2 & 4 & 4 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} -1 & 1 & 3 & 0 \\ 0 & 2 & 4 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} -1 & 1 & 3 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Longrightarrow B_{C(A)} = \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right\}$$

$$= \{x_1, x_2\}$$

$$\Rightarrow v_1 = x_1$$

Now use the Gram- Schmidt process.

Given a set of linearly independent vectors $\{x_1, x_2, \dots\}$, a orthogonal basis $V = \{V_1, V_2\}$

$$V_1 = x_1 \quad , \quad V_2 = x_2 - \frac{x_2^T V_1}{V_1^T V_1} V_1$$

Execute the formula:

$$V_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - \frac{\begin{pmatrix} 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}{\begin{pmatrix} -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - 0$$

$$= \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

we see that the set is already 'orthogonal' i.e $\langle x_1, x_2 \rangle = 0$

Recall that a unit vector, ~~is the vector~~ &

$$\hat{u} = \frac{V}{\|V\|}$$

$$\therefore u = \left\{ \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right\}$$

$$\therefore \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix} = 0$$

The orthogonal complement is the set of all orthogonal vectors, which is just the left null space.

$$\begin{bmatrix} -1 & 1 & 3 & 0 \\ 1 & 1 & 1 & 4 \\ 0 & 2 & 4 & 4 \end{bmatrix}^T = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 1 & 2 \\ 3 & 1 & 4 \\ 0 & 4 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow N(A^T) = \left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

verifying that this vector is orthogonal to both vectors from part a.

$$u = \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right\} \perp \frac{1}{\sqrt{3}} \left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

(c) Find vectors v in $C(A)$ and w in $C(A)^\perp$ so that $[1 \ 2 \ 3]^T = v + w$

$$B_{C(A)} = \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right\}$$

from the two vectors given, we can see the sol^n is either

$$v + w = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

(a)

$$w = \alpha_1 v_1 + \alpha_2 v_2 + \cdots\cdots + \alpha_n v_n$$

$$w = \langle w_1, v_1\rangle v_1 + \langle w_2, v_2\rangle v_2 + \langle w_3, v_3\rangle v_3 + \cdots\cdots$$
$$+ \langle w_n, v_n\rangle v_n$$

then, $\|w\|^2 = \left(\sqrt{\langle w, w\rangle}\right)^2$

$$= \Big[\big((\langle w, v_1\rangle v_1 + \langle w, v_2\rangle v_2 + \langle w, v_3\rangle v_3 + \cdots + \langle w, v_n\rangle v_n) \cdot (\langle w, v_1\rangle v_1 + \langle w, v_2\rangle v_2 + \langle w, v_3\rangle v_3 + \cdots + \langle w, v_n\rangle v_n\big)^2\Big]$$

$$\cancel{= \langle w, v_1\rangle^2 + \langle w, v_2\rangle^2 + \langle w, v_3\rangle^2 + \cdots\cdots + \langle w, v_n\rangle^2}$$

$$= \Big[\langle w, v_1\rangle v_1 \cdot \langle w, v_1\rangle v_1 + \langle w, v_1\rangle v_1 \cdot \langle w, v_2\rangle v_2 + \langle w, v_1\rangle v_1 \cdot \langle w, v_3\rangle v_3 + \cdots\cdots + \langle w, v_1\rangle v_1 \cdot \langle w, v_n\rangle v_n + \cdots\cdots\Big)$$

$$= \Big[\langle w, v_1\rangle^2 v_1^2 + \langle w, v_2\rangle^2 v_2^2 + \langle w, v_3\rangle v_3^2 + \cdots\cdots + \langle w, v_n\rangle v_n^2\Big]$$

$$= \langle w, v_1\rangle^2 + \langle w, v_2\rangle^2 + \cdots\cdots + \langle w, v_n\rangle^2.$$