UNIVERSITY OF RUHUNA
DEPARTMENT OF MATHEMATICS

BACHELOR OF SCIENCE (GENERAL) DEGREE (LEVEL II)
MATHEMATICS
MAT 121β : ALGEBRA

Tutorial No:07 **Semester II, 2020**

Submit answer sheets on: 16/03/2020

---

1. (a) Identify the degree, leading coefficient and the leading term of each of the following polynomials.

(i) $3x^5 + x^4 - 4/3x^2 + 4x - 12$
(ii) $5y - 9 - 2y^4 - 6y^3$

(b) Let $p(x) = 3x^4 - 2x^2 + x - 1$ and $q(x) = 7x^5 + 2x^2$
Find

(i) $p(x) + q(x)$
(ii) $p(x) - q(x)$
(iii) $2p(x) - 3q(x)$

(c) The polynomials $P(x), Q(x)$ and $R(x)$ are given by
$P(x) = x^3 - 2x^2 + x - 1$, $Q(x) = 3x^3 - 2x^2$, $R(x) = -x^4 + 2x^3 - 3x^2$
Find

(i) $P(x)Q(x)$
(ii) $Q(x)R(x)$

(d) Divide the given polynomial

(i) $9x^3 + 15x^2 - 9x + 1 \div 3x - 1$
(ii) $3x^3 + 4x + 11 \div x^2 - 3x + 2$
(iii) $2x^3 - 4x + 7x^2 + 7 \div x^2 + 2x - 1$

---

2. State and prove the remainder theorem.

(a) Use the remainder theorem to find the remainder of each $p(x)$ when it is divided by $q(x)$.

(i) $p(x) = 5x^4 - 8x^3 + 3x^2 - x - 1$; $q(x) = x - 2$
(ii) $p(x) = x^5 + 3x^4 - x^3 - 2x^2 + x - 6$; $q(x) = x + 3$
(iii) $p(x) = x^4 - 3x^2 - 10x + 2$; $q(x) = 2x - 1$

(b) The polynomial $p(x) = x^5 - 7x^3 + ax + 1$ has remainder 13 after division by $x - 1$. Find the value of coefficient a.

(c) If $x^3 + 8x^2 + mx - 5$ is divided by $x + 1$ the remainder is n, express m in terms of n.

---

3. Write down the factor theorem.

(a) Which following are factors of $p(x) = x^3 - 6x^2 + 11x - 6$?

(i)$(x-2)$

(ii)$(x+1)$

(iii)$(x-1)$

(reminder)
$r = 0$

(b) The polynomial $p(x) = 3x^6 - 5x^3 + ax^2 + bx + 10$ is divisible by $x+1$ and $x-2$. Find the value of coefficient a and b.

(c) Identify the factor of polynomial $f(y) = y^4 - 9y^3 + 28y^2 - 36y + 18$.

---

4. (a) Find the rational zeros of

(i)$2x^4 + x^3 - 19x^2 - 9x + 9$

(ii) $x^4 - x^3 + x^2 - 3x - 6$

(b) Write $f(x) = x^5 + x^3 + 2x^2 - 12x + 8$ as the product of linear factors and list all of its zeros.

(c) Find the rational roots of $2x^3 + 3x^2 - 8x + 3$.

---

5. (a) If the sum of two roots of the equation $x^4 + px^3 + qx^2 + rx + s = 0$ equals the sum of the other two. Prove that $p^3 + 8r = 4pq$.

(b) If the product of two roots of the equation $x^4 + px^3 + qx^2 + rx + s = 0$ equals the product of the other two. Prove that $r^2 = p^2 s$.

---

6. (a) If $\alpha,\ \beta,\ \gamma$ are the roots of equation $x^3 + px + q = 0$, express $\alpha^4 + \beta^4 + \gamma^4$ in terms of p and q.

(b) Find the sum of forth powers of the roots of the equation. $f(x) = x^3 - 5x^2 + 6x - 1$.

****

ALGEBRA

MAT 121 β

Tutorial No: 07 - Answers

(01)(a) (i) $3x^5 + x^4 - \frac{4}{3}x^2 + 4x - 12$

degree = 5

leading coefficient = 3

leading term = x^5

(ii) $5y - 9 - 2y^4 - 6y^3$

degree = 4

leading coeficient = -2

leading term = y^4

(c) $P(x) = x^3 - 2x^2 + x - 1$ $\quad Q(x) = 3x^3 - 2x^2$

$R(x) = -x^4 + 2x^3 - 3x^2$

(i) $P(x)Q(x) = (x^3 - 2x^2 + x - 1)(3x^3 - 2x^2)$

$= x^3(3x^3 - 2x^2) - 2x^2(3x^3 - 2x^2)$

$+ x(3x^3 - 2x^2) - (3x^3 - 2x^2)$

$= 3x^6 - 2x^5 - 6x^5 + 4x^4 + 3x^4 - 2x^3$

$- 3x^3 + 2x^2$

$= 3x^6 - 8x^5 + 7x^4 - 5x^3 + 2x^2$

(ii) $Q(x)R(x) = (3x^3-2x^2)(-x^4+2x^3-3x^2)$

$= 3x^3(-x^4+2x^3-3x^2) - 2x^2(-x^4+2x^3-3x^2)$

$= -3x^7+6x^6-9x^5+2x^6-4x^5+6x^4$

$= -3x^7+8x^6-13x^5+6x^4$

(b) $P(x) = 3x^4-2x^2+x-1$ $\quad q(x) = 7x^5+2x^2$

(i) $P(x)+q(x) = 7x^5+3x^4+x-1$

(ii) $P(x)-q(x) = -7x^5+3x^4-4x^2+x-1$

(iii) $2P(x)-3q(x) = 2(3x^4-2x^2+x-1)-3(7x^5+2x^2)$

$= 6x^4-4x^2+2x-2-21x^5-6x^2$

$= -21x^5+6x^4-10x^2+2x-2$

(d) (i)

$$
\begin{array}{r|l}
 & 3x^2+6x-1 \\
\hline
3x-1 & 9x^3+15x^2-9x+1 \\
 & 9x^3-3x^2 \\
\hline
 & 18x^2-9x+1 \\
 & 18x^2-6x \\
\hline
 & -3x+1 \\
 & -3x+1 \\
\hline
 & 0
\end{array}
$$

answer $= 3x^2+6x-1$

(ii)

$$
\begin{array}{r|l}
 & 3x+9 \\
\hline
x^2-3x+2 & 3x^3+4x+11 \\
 & 3x^3-9x^2+6x \\
\hline
 & 9x^2-2x+11 \\
 & 9x^2-27x+18 \\
\hline
 & 25x-7
\end{array}
$$

answer $= (3x+9) + \dfrac{25x-7}{x^2-3x+2}$

(iii)

$$\begin{array}{r|l} & 2x+3 \\ \hline x^2+2x-1 & 2x^3-4x+7x^2+7 \\ & 2x^3+4x^2-2x \\ \hline & 3x^2-2x+7 \\ & 3x^2+6x-3 \\ \hline & -8x+10 \end{array}$$

answer $= (2x+3) + \dfrac{-8x+10}{x^2+2x-1}$

(02) (a) (i) $P(2) = 5(2)^4 - 3(2^3) + 3(2^2) - 2 - 1$

$= 25$

(ii) $P(-3) = (-3)^5 + 3(-3)^4 - (-3)^3 - 2(-3)^2 - 3 - 6$

$= 0$

(iii) $P(\frac{1}{2}) = (\frac{1}{2})^4 - 3(\frac{1}{2})^2 - 10(\frac{1}{2}) + 2$

$= \frac{1}{16} - \frac{3}{4} - 5 + 2$

$= \frac{-59}{16}$

(b) $P(1) = (1)^5 - 7(1)^3 + a(1) + 1 = 13$

$1 - 7 + a + 1 = 13$

$a - 5 = 13$

$a = 18$

(c) $f(x) = x^3 + 3x^2 + mx - 5$

(1) $f(-1) = (-1)^3 + 3(-1)^2 + m(-1) - 5 = n$

$-1 + 3 - m - 5 = n$

$2 - m = n$

$n + m = 2$

$m = 2 - n$

(03) (a) (i) $P(2) = (2)^3 - 6(2)^2 + 11(2) - 6$
$= 8 - 24 + 22 - 6$
$= 0$

$\therefore (x-2)$ is a factor of $p(x)$

(ii) $P(-1) = (-1)^3 - 6(-1)^2 + 11(-1) - 6$
$= 1 - 6 - 11 - 6$
$= -22 \neq 0$

$\therefore (x+1)$ is not a factor of $p(x)$

(iii) $P(1) = (1)^3 - 6(1)^2 + 11(1) - 6$
$= 1 - 6 + 11 - 6$
$= 0$

$\therefore (x-1)$ is a factor of $p(x)$

(b) If $p(x)$ divisible by $(x+1)$ and $(x-2)$, $P(-1) = 0$ and $P(2) = 0$.

(b) $P(-1) = 0$

$P(-1) = 3(-1)^6 - 5(-1)^3 + a(-1)^2 + b(-1) + 10 = 0$

$3 + 5 + a - b + 10 = 0$

$a - b = -18$ —— (1)

$P(2) = 0$

$P(2) = 3(2)^6 - 5(2)^3 + a(2)^2 + b(2) + 10 = 0$

$= 192 - 40 + 4a + 2b + 10 = 0$

$2a + b = -81$ —— (2)

① + ② $\quad 3a = -99$

$a = -33$

by ① $\rightarrow$ $b = -33 + 18$

$= -15$

(C). $f(y) = y^4 - 9y^3 + 28y^2 - 36y + 18$

$P/q = \pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \pm 18$

$f(1) = (1)^4 - 9(1)^3 + 28(1) - 36(1) + 18 = 2 \neq 0$

$f(2) = (2)^4 - 9(2^3) + 28(2)^2 - 36(2) + 18 = 2 \neq 0$

$f(3) = (3)^4 - 9(3)^3 + 28(3)^2 - 36(3) + 18 = 0$

$f(6) = 162 \neq 0$

$f(9) \neq 0$

$f(18) \neq 0$

$\vdots$

$\therefore (y-3)$ is the factor of polynomial $f(y)$

(04) (a) (i) $P(x) = 2x^4 + x^3 - 19x^2 - 9x + 9$

factors of constant term : $p = \pm 1, \pm 3, \pm 9$

factors of leading coefficient : $q = \pm 1, \pm 2$

Possible values of $\frac{p}{q} = \pm 1, \pm \frac{1}{2}, \pm 3, \pm \frac{3}{2}, \pm 9, \pm \frac{9}{2}$

$P(1) = 2(1)^4 + (1)^3 - 19(1)^2 - 9(1) + 9 = -16$

$P(-1) = 2(-1)^4 + (-1)^3 - 19(-1)^2 - 9(-1) + 9 = 0$

$\therefore (x+1)$ is a factor of $p(x)$

$P(1/2) = 2(1/2)^4 + (1/2)^3 - 19(1/2)^2 - 9(1/2) + 9 = 0$

$\therefore (x - 1/2)$ is a factor of $p(x)$

$P(-1/2) = 2(-1/2)^4 + (-1/2)^3 - 19(-1/2)^2 - 9(-1/2) + 9 = \frac{35}{4}$

~~$\therefore (x + 1/2)$ is a factor of $p(x)$~~

$P(3) = 2(3)^4 + (3)^3 - 19(3)^2 - 9(3) + 9 = 0$

$\therefore (x-3)$ is a factor of $p(x)$.

$P(-3) = 2(-3)^4 + (-3)^3 - 19(-3)^2 - 9(-3) + 9 = 0$

$\therefore (x+3)$ is a factor of $p(x)$

$\therefore$ rational zeros are $-1, 1/2, 3, -3$

(ii) $P(x) = x^4 - x^3 + x^2 - 3x - 6$

factors of constant term: $p: \pm 1, \pm 2, \pm 3, \pm 6$

factors of leading coeficient: $q: \pm 1$

$$\frac{p}{q} = \pm 1, \pm 2, \pm 3, \pm 6$$

$P(1) = 1 - 1 + 1 - 3 - 6 = -8 \neq 0$

$P(-1) = 1 + 1 + 1 + 3 - 6 = 0$

$P(2) = 16 - 8 + 4 - 6 - 6 = 0$

$P(-2) = 16 + 8 + 4 + 6 - 6 = 28 \neq 0$

$P(3) = 81 - 27 + 9 - 9 - 6 = 48 \neq 0$

$P(-3) = 81 + 27 + 9 + 9 - 6 = 120 \neq 0$

$P(6) = 1296 - 216 + 36 - 18 - 6 \neq 0$

$P(-6) = 1296 + 216 + 36 + 18 - 6 \neq 0$

~~$x^4 - x^3 + x^2 - 3x - 6 = (x+1)(x-2) f(x)$~~

$(x+1)$ and $(x-2)$ are factors of the $p(x)$

$\therefore$ rational zeros are -1 and 2.

(b) $f(x) = x^5 + x^3 + 2x^2 - 12x + 8$

If $x = P/q$ is a root,

$p = \pm 1, \pm 2, \pm 4, \pm 8$

$q = \pm 1$

$P/q = \pm 1, \pm 2, \pm 4, \pm 8$

$f(1) = 0$

$f(-1) = 20 \neq 0$

$f(2) = 32 \neq 0$

$f(-2) = 0$

$f(4) = 1080 \neq 0$

$f(-4) = -1000 \neq 0$

$f(8) \neq 0$

$f(-8) \neq 0$

$x^5 + x^3 + 2x^2 - 12x + 8 = (x-1)(x+2)(x^3 - x^2 + 4x - 4)$

$= (x-1)(x+2)(x^2(x-1) + 4(x-1))$

$= (x-1)^2 (x+2)(x^2+4)$

zeros of $f(x)$ are $1, -2, 2i, -2i$

(c) $f(x) = 2x^3 + 3x^2 - 3x + 3$

If $x = p/q$ is a root

$p = \pm 1, \pm 3$

$q = \pm 1, \pm 2$

$p/q = \pm 1, \pm 3, \pm 1/2, \pm 3/2$

$f(1) = 0$ $\qquad$ $f(-1) = 12 \neq 0$

$f(3) = 60 \neq 0$ $\qquad$ $f(-3) = 0$

$f(1/2) = 0$ $\qquad$ $f(-1/2) \neq 0$

$f(3/2) \neq 0$ $\qquad$ $f(-3/2) \neq 0$

$f(x) = (x-1)(2x-1)(x+3)$

$\therefore$ rational roots of $f(x)$ are $1, 1/2, -3$

(05) (a) Let $\alpha, \beta, \gamma, \delta$ be the roots of the equation $x^4 + px^3 + qx^2 + rx + s = 0$,

Then

$$\alpha + \beta = \gamma + \delta \quad \text{--- (1)}$$

By vieta's formula

$$\alpha + \beta + \gamma + \delta = -p \quad \text{--- (2)}$$

$$\alpha\beta + \beta\gamma + \gamma\delta + \alpha\delta + \beta\delta + \alpha\gamma = q \quad \text{--- (3)}$$

$$\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \alpha\beta\delta = -r \quad \text{--- (4)}$$

$$\alpha\beta\gamma\delta = s \quad \text{--- (5)}$$

from (1) and (2)

$$2(\gamma + \delta) = -p$$

$$\gamma + \delta = -p/2$$

(4) $\rightarrow$

$$\alpha\beta(\gamma + \delta) + \gamma\delta(\alpha + \beta) = -r$$

$$\alpha\beta(-p/2) + \gamma\delta(-p/2) = -r$$

$$\alpha\beta + \gamma\delta = \frac{2r}{p}$$

(3) $\rightarrow$

$$\alpha\beta + \gamma\delta + \alpha(\gamma + \delta) + \beta(\gamma + \delta) = q$$

$$\cancel{\alpha\beta + \gamma\delta}$$

$$\frac{2r}{p} + (\alpha + \beta)(\gamma + \delta) = q$$

$$\frac{2r}{p} + \frac{p^2}{4} = q$$

$$p^3 + 8r + 4pq$$

(b) Let $\alpha, \beta, \gamma, \delta$ be the roots of the equation.

$x^4 + px^3 + qx + rx + s = 0$

$\alpha\beta = \gamma\delta$ —— (1)

$\alpha + \beta + \gamma + \delta = -p$ —— (2)

$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = q$ —— (3)

$\alpha\beta\gamma + \alpha\beta\delta + \beta\gamma\delta + \alpha\gamma\delta = -r$ —— (4)

$\alpha\beta\gamma\delta = s$ —— (5)

from (1) and (5)

$(\alpha\beta)^2 = s$

from (4)

$\alpha\beta(\gamma + \delta) + \underbrace{\gamma\delta}_{\alpha\beta}(\alpha + \beta) = -r$

$\alpha\beta(\alpha + \beta + \gamma + \delta) = -r$

$\alpha\beta(-p) = -r$

$\alpha\beta = \frac{r}{p}$

$(\alpha\beta)^2 = \frac{r^2}{p^2}$

$s = \frac{r^2}{p^2}$

$r^2 = p^2 s$

(06)

(a) $x^3 + px + q = 0$

$a_0 = q \quad a_1 = p \quad a_2 = 0 \quad a_3 = 1$

$P_1 = \alpha + \beta + \gamma \qquad P_3 = \alpha^3 + \beta^3 + \gamma^3$

$P_2 = \alpha^2 + \beta^2 + \gamma^2 \qquad P_4 = \alpha^4 + \beta^4 + \gamma^4$

$n = 1 \Rightarrow a_3 P_1 + 1 \cdot a_2 = 0$
$P_1 + 0 = 0$ $\Big\}$ $P_1 = 0$

$n = 2 \Rightarrow a_3 P_2 + a_2 P_1 + 2 a_1 = 0$
$P_2 + 0 + 2p = 0$ $\Big\}$ $P_2 = -2p$

$n = 3 \Rightarrow a_3 P_3 + a_2 P_2 + a_1 P_1 + 3 a_0 = 0$
$P_3 + 0 + 0 + 3q = 0$ $\Big\}$ $P_3 = -3$

$n = 4 \Rightarrow a_3 P_4 + a_2 P_3 + a_1 P_2 + a_0 P_1 = 0$
$P_4 + 0 + p \times (-2p) + 0 = 0$ $\Big\}$ $P_4 = 2$

$\therefore \alpha^4 + \beta^4 + \gamma^4 = 2p^2$

(b) $x^3 - 5x^2 + 6x - 1$

$a_0 = -1 \ , \quad a_1 = 6 \quad a_2 = -5 \quad a_3 = 1$

$S_1 = \alpha + \beta + \gamma$
$S_2 = \alpha^2 + \beta^2 + \gamma^2$
$S_3 = \alpha^3 + \beta^3 + \gamma^3$
$S_4 = \alpha^4 + \beta^4 + \gamma^4$

$r = 1 \rightarrow a_3 S_1 + 1 a_2 = 0$
$S_1 - 5 = 0$ $\Big\}$ $S_1 = 5$

$r = 2 \longrightarrow \quad a_3 S_2 + a_2 S_1 + 2a_1 = 0$

$$\left.\begin{array}{l} a_3 S_2 + a_2 S_1 + 2a_1 = 0 \\ S_2 + (-5 \times 5) + 2 \times 6 = 0 \end{array}\right\} S_2 = 13$$

$r = 3 \longrightarrow$

$$\left.\begin{array}{l} a_3 S_3 + a_2 S_2 + a_1 S_1 + 3a_0 = 0 \\ S_3 + (-5 \times 13) + (6 \times 5) + (-3) = 0 \end{array}\right\} S_3 = 3$$

$r = 4 \Rightarrow$

$$\left.\begin{array}{l} a_3 S_4 + a_2 S_3 + a_1 S_2 + a_0 S_1 = 0 \\ S_4 + (-5 \times 38) + (6 \times 13) + (-5) = 0 \end{array}\right\} S_4 = 11$$

$\therefore \alpha^4 + \beta^4 + \gamma^4 \cancel{+\delta^4} = 117$