UNIVERSITY OF RUHUNA
DEPARTMENT OF MATHEMATICS

BACHELOR OF SCIENCE (GENERAL) DEGREE (LEVEL I)
MATHEMATICS
MAT 121β : ALGEBRA

Tutorial No:4 **Semester II, 2020**

Submit answer sheets on or before : 03/02/2020

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1. (a) Let A and B be two sets. Prove the followings.

(i) $A \subseteq B$ iff $B^c \subseteq A^c$

(ii) $(A \cup B)^c = A^c \cap B^c$

(b) Determine the power set of each of the following sets.

(i) A={a,b,c,d}

(ii) A={1,2,3}

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2. (a) Let $U = \{1, 2, 3, ..., 9\}$ be the universal set.
Let $A = \{2, 4, 5, 7\}$, $B = \{1, 2, 8, 9\}$, $C = \{3, 6, 7\}$.
Find,

(i) $A \cap B$ and $A \cap C$

(ii) $A \cup B$ and $B \cup C$

(iii) A^c and B^c

(iv) $A \backslash B$ and $A \backslash C$

(v) $(A \cup C) \backslash B$

(b) Considering the sets
$A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6, 7\}$, and $C = \{6, 7, 8, 9\}$.
Find,

(i) $A \backslash B$

(ii) $B \backslash A$

(iii) $B \backslash C$

(iv) $A \triangle B$

(v) $B \triangle C$

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3. (a) Suppose $n(U) = 70, n(A) = 40, n(B) = 35$ and $n(A \cap B) = 10$.
Find,

(i) $n(A \cup B)$

(ii) $n(A^c)$ and $n(B^c)$

(iii) $n(A^c \cap B^c)$

(iv) $n(A \cap B)^c$

(v) $n(B^c)^c$

(b) In a shop 380 people buy socks, 150 people buy shoes and 200 people buy belts. If there are total 580 people who bought either socks or shoes or belts. Only 30 people bought all the 3 things. How many people bought exactly 2 things.

4. Let $\{A_n\}$ be an infinite sequence of sunsets of a non-empty set Ω.

Prove that $\bigcup\limits_{n=1}^{\infty}\left(\bigcup\limits_{k=n}^{\infty} A_k\right) \subset \bigcap\limits_{n=1}^{\infty}\left(\bigcap\limits_{k=n}^{\infty} A_k\right)$.

Let $\Omega = R$ and define

$$A_n = \begin{cases} \left[0, 1+\dfrac{1}{n}\right] ; & n \textit{ is an even positive integer}; \\ \\ \left[-1-\dfrac{1}{n}, 0\right] ; & n \textit{ is an odd positive integer}; \end{cases}$$

Determine $\bigcup\limits_{n=1}^{\infty}\left(\bigcup\limits_{k=n}^{\infty} A_k\right)$ and $\bigcap\limits_{n=1}^{\infty}\left(\bigcap\limits_{k=n}^{\infty} A_k\right)$.

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5. (i) Let A and B be two subsets of a non-empty set Ω. Show that $A \cup (B \backslash A) = A \cup B$

(ii) Let $\{A_n\}_{n=1}^{\infty}$ be an infinity sequence of subsets of a non-empty set Ω such that $A_n \subseteq A_{n+1}$ for all $n \in Z^+$. Let $\{B_n\}_{n=1}^{\infty}$ be an infinite sequence of subsets defined by :

$B_1 = A_1$
$B_n = A_n \backslash A_{n-1}$ for all $n \in Z^+ \backslash \{1\}$

Using the Principal of Mathematical Induction or otherwise, show that

$A_n = \bigcup\limits_{i=1}^{n} B_i$ for all $n \in Z^+$.

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6. Let $\{A_n\}_{n=1}^{\infty}$ be an infinity sequence of subsets of a non-empty set Ω such that $A_{n+1} \subseteq A_n$ for all $n \in Z^+$. Let $\{B_n\}_{n=1}^{\infty}$ be an infinite sequence of subsets defined by :

$B_n = A_1 \backslash A_n$ for all $n \in Z^+$.

Show that

(i) $B_n \subseteq B_{n+1}$ for all $n \in Z^+$,

(ii) $\bigcup\limits_{n=1}^{\infty} B_n = A_1 \backslash \left(\bigcap\limits_{n=1}^{\infty} A_n\right)$.

Now, let $A_n = \left(0, \dfrac{1}{n}\right]$ for all $n \in Z^+$. Show that $\bigcap\limits_{n=1}^{\infty} A_n = \phi$ and verify that

$\bigcup\limits_{n=1}^{\infty} B_n = A_1 \backslash \left(\bigcap\limits_{n=1}^{\infty} A_n\right)$.

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Tutorial No : 04

1 (a) (i) $A \subseteq B \iff B^c \subseteq A^c$

Assume that $A \subseteq B$ which means

if $x \in A \Rightarrow x \in B \longrightarrow$ ①

By contrapositive of ①

$\sim (x \in B) \Rightarrow \sim (x \in A)$

$x \notin B \Rightarrow x \notin A$

i.e if $x \in B^c \Rightarrow x \in A^c$

Hence $B^c \subseteq A^c$

Similarily assume that

$B^c \subseteq A^c$

Let $x \in B^c \Rightarrow x \in A^c \longrightarrow$ ②

By contrapositive of ②

$\sim (x \in A^c) \Rightarrow \sim (x \in B^c)$

$x \notin A^c \Rightarrow x \notin B^c$

i.e if $x \in A \Rightarrow x \in B$

Hence $A \subseteq B$

(ii) $(A \cup B)^c = A^c \cap B^c$

Let $x \in (A \cup B)^c \iff x \notin (A \cup B)$

$\iff x \notin A$ and $x \notin B$

$\iff x \in A^c$ and $x \in B^c$

$\iff x \in A^c \cap B^c$

$\therefore (A \cup B)^c \subseteq A^c \cap B^c \longrightarrow$ 1

By considering the above step backwards

$A^c \cap B^c \subseteq (A \cup B)^c \longrightarrow$ 2

From 1 and 2

$(A \cup B)^c = A^c \cap B^c$

(b) (i) $A = \{a, b, c, d\}$

$P(A) = \{\{a, b, c, d\}, \{a, b, c\}, \{b, c, d\}, \{a, c, d\}, \{a, b, d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a\}, \{b\}, \{c\}, \{d\}, \{\}\}$

no: of elements in power set $= 2^n$

(ii) $A = \{1, 2, 3\}$

$P(A) = \{\{1, 2, 3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1\}, \{2\}, \{3\}, \{\}\}$

2] (a) $U = \{1, 2, 3, \dots, 9\}$

$A = \{2, 4, 5, 7\}$ $\quad B = \{1, 2, 8, 9\}$

$C = \{3, 6, 7\}$

(i) $A \cap B = \{2\}$

$A \cap C = \{7\}$

(ii) $A \cup B = \{1, 2, 4, 5, 7, 8, 9\}$

$B \cup C = \{1, 2, 3, 6, 7, 8, 9\}$

(iii) $A^c = \{1, 3, 6, 8, 9\}$

$B^c = \{3, 4, 5, 6, 7\}$

(iv) $A \setminus B = A - B = \{4, 5, 7\}$

$A \setminus C = A - C = \{2, 4, 5\}$

(v) $A \cup C = \{2, 3, 4, 5, 6, 7\}$

$(A \cup C) \setminus B = \{3, 4, 5, 6, 7\}$

(b) $A = \{1, 2, 3, 4\}$

$B = \{3, 4, 5, 6, 7\}$

$C = \{6, 7, 8, 9\}$

(i) $A \setminus B = \{1, 2\}$

(ii) $B \setminus A = \{5, 6, 7\}$

(iii) $B \setminus C = \{3, 4, 5\}$

(iv) $A \triangle B = (A \cup B) - (A \cap B)$

$A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$

$A \cap B = \{3, 4\}$

$A \Delta B = \{1, 2, 5, 6, 7\}$

(v) $B \Delta C = (B \cup C) - (B \cap C)$

$B \cup C = \{3, 4, 5, 6, 7, 8, 9\}$

$B \cap C = \{6, 7\}$

$B \Delta C = \{3, 4, 5, 8, 9\}$

3] (a) $n(U) = 70$, $n(A) = 40$, $n(B) = 35$, $n(A \cap B) = 10$

(i) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$= 40 + 35 - 10$

$= 65$

(ii) $n(A^c) = n(U) - n(A)$

$= 70 - 40$

$= 30$

$n(B^c) = n(U) - n(B)$

$= 70 - 35$

$= 35$

(iii) $n(A^c \cap B^c) = n(A \cup B)^c$

$= n(U) - n(A \cup B)$

$= 70 - 65$

$= 5$

(iv) $n(A \cap B)^c = n(U) - n(A \cap B)$

$= 70 - 10$

$= 60$

(v) $n(B^c)^c = n(B)$

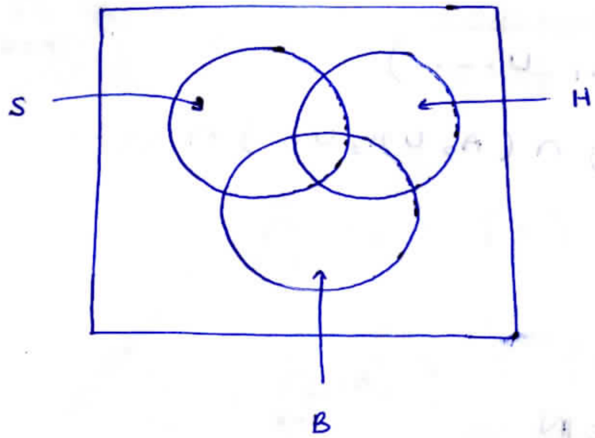
$= 35$

(b) Let

S - no: of people who bought socks

H - no: of people " " shoes

B - " " " " belts



$n(S) = 380$

$n(H) = 150$

$n(B) = 200$

$n(S \cup H \cup B) = 580$

$n(S \cap H \cap B) = 30$

$\therefore n(S \cup H \cup B) = n(S) + n(H) + n(B) - n(S \cap H) - n(H \cap B) - n(B \cap S) + n(S \cap H \cap B)$

$580 = 380 + 150 + 200 - n(S \cap H) - n(H \cap B) - n(B \cap S) + 30$

$n(S \cap H) + n(H \cap B) + n(B \cap S) = 180$

But this includes the no: of people who bought all the 3 item. So we have to deduct these no: of people from it.

Let $n(S \cap H \cap B) = a$

no: of people who bought exactly 2 things $\Big\} = n(S \cap H) - a + n(H \cap B) - a + n(B \cap S) - a$

$= n(S \cap H) + n(H \cap B) + n(B \cap S) - 3a$

$= 180 - 90$

$= 90$

4]
$$\bigcup_{n=1}^{\infty}\left(\bigcap_{k=n}^{\infty} A_k\right) = \bigcup_{n=1}^{\infty}\left(A_n \cap A_{n+1} \cap \cdots\right)$$

$$= (A_1 \cap A_2 \cap A_3 \cap \cdots) \cup (A_2 \cap A_3 \cap \cdots) \cup \cdots$$

$$\bigcap_{n=1}^{\infty}\left(\bigcup_{k=n}^{\infty} A_k\right) = \bigcap_{n=1}^{\infty}\left(A_n \cup A_{n+1} \cup \cdots\right)$$

$$= (A_1 \cup A_2 \cup A_3 \cup \cdots) \cap (A_2 \cup A_3 \cup \cdots) \cap \cdots$$

Take any $x \in \bigcup_{n=1}^{\infty}\left(\bigcap_{k=n}^{\infty} A_k\right)$

$\Rightarrow$ $x \in \bigcap_{k=n_0}^{\infty} A_k$ for some $n_0 \in \mathbb{N}$

$\Rightarrow$ $x \in A_k$ for all $n \geq n_0$ and some $n_0 \in \mathbb{N}$

$x \in \bigcup_{k=n}^{\infty} A_k \quad \forall n \in \mathbb{N}$

Since this is true for all $n \in \mathbb{N}$

$$x \in \left(\bigcap_{n=1}^{\infty}\left(\bigcup_{k=n}^{\infty} A_k\right)\right)$$

So
$$\bigcup_{n=1}^{\infty}\left(\bigcap_{k=n}^{\infty} A_k\right) \subset \bigcap_{n=1}^{\infty}\left(\bigcup_{k=n}^{\infty} A_k\right) \quad //$$

$$A_n = \begin{cases} \left[0, 1+\frac{1}{n}\right] & ; \ n \text{ even +ve integer} \\ \left[-1-\frac{1}{n}, 0\right] & ; \ n \text{ odd +ve integer} \end{cases}$$

$$\{A_n\}_{n=1}^{\infty} = \left\{ [-2, 0],\ \left[0, 1+\tfrac{1}{2}\right],\ \left[-1-\tfrac{1}{3}, 0\right],\ \left[0, 1+\tfrac{1}{4}\right], \ldots \right\}$$

$$\bigcap_{k=n}^{\infty} A_n =$$

$$\Rightarrow \therefore \bigcap_{k=n}^{\infty} A_n = \{0\} \quad \forall n \in \mathbb{N}$$

$$\therefore \bigcup_{n=1}^{\infty} \left(\bigcap_{k=n}^{\infty} A_n \right) = \{0\}$$

$$\bigcup_{k=n}^{\infty} A_n = \left[-1-\frac{1}{n}, 0\right] \cup \left[0, 1+\frac{1}{n}\right]$$

$$\bigcap_{n=1}^{\infty} \left(\bigcup_{k=n}^{\infty} A_n \right) = [-1, 0] \cup [0, 1]$$

$$= [-1, 1]$$

$\boxed{5}$ (i) $A \cup (B \setminus A) = A \cup B$

Let $A, B \in \Omega$

$$\Rightarrow A \cup (B \setminus A) = A \cup (B \cap A^c)$$

$$= (A \cup B) \cap (A \cup A^c)$$

$$= (A \cup B) \cap \Omega$$

$$= A \cup B$$

(ii) $\{A_n\}_{n=1}^{\infty} \subseteq \Omega$ s.t. $A_n \subseteq A_{n+1}$ $\forall n \in \mathbb{Z}^+$

$$B_1 = A_1$$

$$B_n = A_n \setminus A_{n-1} \quad \forall n \in \mathbb{Z}^+ \setminus \{1\}$$

ms $\quad A_n = \bigcup_{i=1}^{n} B_i \quad \forall n \in \mathbb{Z}^+$

Let use mathematical induction on n

When $n = 1 \Rightarrow$

$$A_1 = \bigcup_{i=1}^{1} B_i = B_1 \quad (\text{given})$$

The result is true for $n = 1$.

Assume that the assertion is true for $n = p \in \mathbb{Z}^+$

i.e. $\quad A_p = \bigcup_{i=1}^{p} B_i$

Consider

$$\bigcup_{i=1}^{p+1} B_i = \bigcup_{i=1}^{p} B_i \cup B_{p+1}$$

$$= A_p \cup B_{p+1}$$

$$= A_p \cup (A_{p+1} \setminus A_p) \quad (\because B_n = A_n \setminus A_{n-1} \text{ given})$$

$$= \cancel{A_p \cup (A_{p+1} \cap A_p^c)}$$

$$= A_p \cup A_{p+1} \quad (\because \text{part (i)} \ A \cup (B \setminus A) = A \cup B)$$

$$= A_{p+1} \quad (\because A_p \subseteq A_{p+1})$$

The assertion is true for $n = p \in \mathbb{Z}^+$. If the assertion is true for $n = p$, the assertion is true for $n = p+1$.

Hence by M.I, we have the proof.