UNIVERSITY OF RUHUNA
DEPARTMENT OF MATHEMATICS

BACHELOR OF SCIENCE (GENERAL) DEGREE (LEVEL II)
MATHEMATICS
MAT 211β : LINEAR ALGEBRA

Tutorial No: 10 **Semester I, 2020**

Try All the Questions

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1. Let $V = W = \mathbb{R}$.

(a) Define $T(x) = mx$, where m is a fixed real number.
Show that T is a linear transformation.

(b) For $x \in V$, define $T(x) = mx + b$, where m and b are real numbers and $b \neq 0$.
Show that T is not a linear transformation.

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2. (a) Consider the basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ for $\mathbb{R}^3$, where $\mathbf{v}_1 = (1,2,1), \mathbf{v}_2 = (2,9,0)$, and $\mathbf{v}_3 = (3,3,4)$ and let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation for which

$$T(\mathbf{v}_1) = (1,0), \qquad T(\mathbf{v}_2) = (-1,1), \qquad T(\mathbf{v}_3) = (0,1).$$

Find a formula for $T(x_1, x_2, x_3)$, and use that formula to find $T(7,13,7)$.

(b) Let $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ be vectors in a vector space V, and let $T : V \to \mathbb{R}^3$ be a linear transformation for which

$$T(\mathbf{v}_1) = (1,-1,2), \qquad T(\mathbf{v}_2) = (0,3,2), \qquad T(\mathbf{v}_3) = (-3,1,2).$$

Find $T(2\mathbf{v}_1 - 3\mathbf{v}_2 + 4\mathbf{v}_3)$.

(c) Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$.

(i) Write an expression for T_A.
(ii) Find $T_A(1,0)$ and $T_A(0,1)$.
(iii) Find all points (x,y) such that $T_A(x,y) = (1,0)$.

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3. (a) Let $T : E^2 \to E^1$ be defined by

$$T(x_1, x_2) = x_1^2 + x_2^2.$$

Show that T is not linear even though $T(\mathbf{0}) = \mathbf{0}$.

(b) $T : P_2 \to P_2$ be defined by

$$T(a_0 + a_1 x + a_2 x^2) = a_0 + a_1(x+1) + a_2(x+1)^2.$$

Determine whether the mapping T is a linear transformation, and if so find its kernel.

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4. (a) Find $\ker(T)$, where $T : E^3 \to E^2$ is defined by,

$$T(x_1, x_2, x_3) = (x_1 + x_2, x_2 - x_3).$$

(b) Define T from E^3 to E^3 by

$$T(a, b, c) = (a - b + c, 2a + b - c, -a - 2b + 2c).$$

(i) Determine range T and dim(range T).

(ii) Find two vectors in range T and two vectors not in range T.

(iii) Find a basis for range T.

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5. Let $T : M_{2\times 2} \longmapsto \mathbb{R}^2$ be defined by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{pmatrix} a - b \\ c + d \end{pmatrix}.$$

It is given that T is a linear transformation. Find basis for Ker(T) and Im(T).

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6. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

It is given that T is a linear transformation.

(i) Is T onto? Justify your answer.

(ii) Is T one to one? Justify your answer.

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7. Let : $\mathbb{R}^2 \longmapsto \mathbb{R}^2$ be defined by

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{pmatrix} x + y \\ x - y \end{pmatrix}.$$

Show that T is an isomorphism.

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8. Let $\{ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \}$ be a basis for a vector space V.

(a) Let $T : V \to W$ be a linear transformation. Prove that if

$$T(\mathbf{v}_1) = T(\mathbf{v}_2) = \ldots = T(\mathbf{v}_n) = \mathbf{0}$$

then T is the zero transformation.

(b) Let $T : V \to V$ be a linear transformation. Prove that if

$$T(\mathbf{v}_1) = \mathbf{v}_1, \qquad T(\mathbf{v}_2) = \mathbf{v}_2 \quad \ldots \quad T(\mathbf{v}_n) = \mathbf{v}_n$$

then T is the identity transformation on V.

"The important thing to remember about mathematics is not to be frightened"

$\sim$ Richard Dawkins,

Tutorial 10

①

(I) we must show that T is additive and homogeneous. For the additivity, we let x and y be in E' and calculate.

$$T(x+y) = m(x+y) = mx+my$$
$$T(x) + T(y) = mx + my.$$

since, $T(x+y) = T(x) + T(y)$, we know That T is additive,

Also T is homogeneous since

$$T(rx) = m(rx) = r(mx) = r\,T(x)$$

Thus, T is a linear transformation.

(II) First we check additivity, notting $F(\cdot) = m(\cdot) + b$;

$$F(x+y) = m(x+y) + b$$
$$= mx + my + b \quad \text{—①}$$

However,

$$F(x) + F(y) = (mx+b) + (my+b)$$
$$= mx + my + 2b$$

since $b \neq 0$, $2b \neq b$ so $F(x+y) \neq F(x) + F(y)$ for all $x, y \in V$ and F is not linear.

(2)

(a)

$V = (x_1, x_2, x_3) \in \mathbb{R}^3$ - arbitrary vector.

it can be expressed as a linear combination of V_1, V_2, V_3:

s.t,

$V = aV_1 + bV_2 + cV_3$

$(x_1, x_2, x_3) = a(1,2,1) + b(2,9,0) + c(3,3,4)$

$(x_1, x_2, x_3) = (a + 2b + 3c,\ 2a + 9b + 3c,\ a + 4c)$

$$\left.\begin{aligned} x_1 &= a + 2b + 3c \\ x_2 &= 2a + 9b + 3c \\ x_3 &= a + 4c \end{aligned}\right\} \begin{aligned} a &= -36x_1 + 8x_2 + 21x_3 \\ b &= 5x_1 - x_2 - 3x_3 \\ c &= 9x_1 - 2x_2 - 5x_3 \end{aligned}$$

$$\begin{aligned} T(v) &= T(aV_1 + bV_2 + cV_3) \\ &= aT(V_1) + bT(V_2) + cT(V_3) \\ &= a(1,0) + b(-1,1) + c(0,1) \end{aligned}$$

$= (-36x_1 + 8x_2 + 21x_3)(1,0) + (5x_1 - x_2 - 3x_3)(-1,1) + (9x_1 - 2x_2 - 5x_3)(0,1)$

$= (-36x_1 + 8x_2 + 21x_3 - 5x_1 + x_2 + 3x_3,\ 5x_1 - x_2 - 3x_3 + 9x_1 - 2x_2 - 5x_3)$

$= (-41x_1 + 9x_2 + 24x_3,\ 14x_1 - 3x_2 - 8x_3)$

$$T(7,13,7) = (-41(7) + 9(13) + 24(7), 14(7) - 3(13) - 8(\ldots$$

$T(v_1) = (1, -1, 2)$

$T(v_2) = (0, 3, 2)$

$T(v_3) = (-3, 1, 2)$

$$T(2v_1 - 3v_2 + 4v_3) = 2\,T(v_1) - 3\,T(v_2) + 4\,T(v_3)$$

$$= 2(1, -1, 2) - 3(0, 3, 2) + 4(-3, 1, 2)$$

$$= (-10, -7, 6)$$

2

c

I $Tm(x,y) = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$= \begin{bmatrix} x+2y \\ 3x+7y \end{bmatrix}$

$= (x+2y, 3x+7y)$

II $Tm(1,0) = ((1)+2(0), 3(1)+7(0))$

$= (1,3)$

$Tm(0,1) = (0+2(1), 3(0)+7(1))$

$= (2,7)$

III we have $Tm(x,y) = (x+2y, 3x+7y) = (1,0)$, hence the simultaneous equations

$x+2y = 1$

$3x+7y = 0$

solving these equations yields $x=7$, $y=-3$ and this is the only sol^n.

③

I we have $T(\theta) = T((0,0)) = 0^2 + 0^2 = \theta$, which is the zero of E'. This allows no conclusion; the definition of linearity must be used. To check additivity we calculate.

$$T(x+y) = T((x_1, x_2) + (y_1, y_2))$$
$$= T((x_1 + y_1), (x_2 + y_2))$$
$$= (x_1 + y_1)^2 + (x_2 + y_2)^2$$
$$= x_1^2 + 2x_1y_1 + y_1^2 + x_2^2 + 2x_2y_2 + y_2^2$$

and,

$$T(x) + T(y) = T((x_1, x_2)) + T((y_1, y_2))$$
$$= (x_1^2 + x_2^2) + (y_1^2 + y_2^2)$$

Since,

$T(x+y) \neq T(x) + T(y)$, we know that T is not linear.

(b)

$T: P_2 \to P_2$ be defined by,

$$T(a_0 + a_1 x + a_2 x^2) = a_0 + a_1(x+1) + a_2(x+1)^2$$

$$v = b_0 + b_1 x + b_2 x^2 \quad , \quad w = c_0 + c_1 x + c_2 x^2$$

$$\cancel{w = c_0 + c_1(x+1) + c_2(x+1)^2}$$

$$\begin{aligned}
T(v+w) &= T(b_0 + b_1 x + b_2 x^2 + c_0 + c_1 x + c_2 x^2) \\
&= T[(b_0 + c_0) + (b_1 + c_1)x + (b_2 + c_2)x^2] \\
&= (b_0 + c_0) + (b_1 + c_1)(x+1) + (b_2 + c_2)(x+1)^2 \\
&= \underbrace{[b_0 + b_1(x+1) + b_2(x+1)^2]}_{T(v)} + \underbrace{[c_0 + c_1(x+1) + c_2(x+1)^2]}_{T(w)} \\
&= T(v) + T(w)
\end{aligned}$$

$$\begin{aligned}
T(\alpha v) &= T[\alpha(b_0 + b_1 x + b_2 x^2)] \\
&= T[(\alpha b_0 + \alpha b_1 x + \alpha b_2 x^2)] \\
&= \alpha b_0 + \alpha b_1(x+1) + \alpha b_2(x+1)^2 \\
&= \alpha[b_0 + b_1(x+1) + b_2(x+1)^2] \\
&= \alpha T(v)
\end{aligned}$$

④ ⓐ Since $\ker T = \{x \mid T(x) = 0\}$, we must solve $T((x_1, x_2, x_3)) = (0,0)$ that is,

$$(x_1 + x_2, x_2 - x_3) = (0,0)$$

The resulting equations are,

$$x_1 + x_2 = 0$$
$$x_2 - x_3 = 0$$

which have solution $(-k, k, k)$. Therefore

$$\ker T = \{v \in E^3 \mid v = k(-1,1,1)\} = \text{span}\{(-1,1,1)\}$$

(*) Here the kernel of the given linear transformation was a subspace of the domain. In fact, a basis for $\ker T$ was $\{(-1,1,1)\}$. The kernel of a linear transformation is always a vector space.

ⓑ $T(a,b,c) = (a-b+c,\ 2a+b-c,\ -a-2b+2c)$

$$T(1,0,0) = (1,2,-1)$$
$$T(0,1,0) = (-1,1,-2)$$
$$T(0,0,1) = (1,-1,2)$$

$$\begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & -2 \\ 1 & -1 & 2 \end{bmatrix} \xrightarrow{R_2 \to R_1 + R_2} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & -3 \\ 1 & -1 & 2 \end{bmatrix} \xrightarrow{R_3 \to R_1 + R_3} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & -3 \\ 0 & 3 & -3 \end{bmatrix}$$

$$\xrightarrow{R_3 \times \frac{1}{3} \text{ and } R_2 \times \frac{1}{3}} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{R_3 \to R_2 + R_3} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\dim(\mathrm{Im}(T)) = 2$

basis of range$(T) = (1, 2, -1), (0, 1, -1)$

(6) (5)

we can verify that T represents a linear transformation.

must show that all matrices A such that $T(A)=0$.

suppose,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$ then,

$$T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a-b \\ c+d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

then,

$$a-b=0 \quad (1)$$
$$c+d=0 \quad (2)$$

let,

$a=s$ then from (1) $b=s$ where s, t are scalars.

and $c=t$ then from (2) $d=-t$

$$\ker(T) = \left\{\begin{bmatrix} s & s \\ t & -t \end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}\right\}$$

it clearly shows that the set is linearly independent and therefore forms a basis for $\ker(T)$.

The basis for im(T),

$$im(T) = \left\{\begin{bmatrix} a-b \\ c+d \end{bmatrix}\right\}$$

this can be written as,

$$span\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$$

However this is not linearly independent, so by removing vectors from the set to create independent set gives a basis of im(T).

$$\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$$

(6)

let take,

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

then,

$x+y=a$ — (1)

$x+2y=b$ — (2)

augmented matrix;

$$\left[\begin{array}{cc|c} 1 & 1 & a \\ 1 & 2 & b \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left(\begin{array}{cc|c} 1 & 1 & a \\ 0 & 1 & b-a \end{array}\right] \xrightarrow{R_1 \to R_1 - R_2} \left[\begin{array}{cc|c} 1 & 0 & 2a-b \\ 0 & 0 & b-a \end{array}\right]$$

now you can see ~~that~~ from this point the system has a sol^n. Therefore, we have shown that for any a, b there is a $\begin{bmatrix} x \\ y \end{bmatrix}$ such that $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$ Thus T is onto.

one-to-one,

ms. $A\vec{x}=0$

$\therefore$ $A\vec{x} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore$ $x+y=0$ — (1)
$x+2y=0$ — (2)

Augmented matrix :

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

this tells that,

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

then,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

in other words. $AX=0$ implies that $x=0$, then

A is one to one.

II

1

To prove that T is an isomorphism we must show:

1. T is a linear transformation
2. T is one to one:
3. T is onto.

then,

1. T is linear transformation:

let k, p be scalars,

$$T\left(k\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + p\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} kx_1 \\ ky_1 \end{bmatrix} + \begin{pmatrix} px_2 \\ py_2 \end{pmatrix}\right)$$

$$= T\left(\begin{bmatrix} kx_1 + px_2 \\ ky_1 + py_2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} (kx_1 + px_2) + (ky_1 + py_2) \\ (kx_1 + px_2) - (ky_1 + py_2) \end{bmatrix}$$

$$= \begin{bmatrix} (kx_1 + ky_1) + (px_2 + py_2) \\ (kx_1 - ky_1) + (px_2 - py_2) \end{bmatrix}$$

$$= \begin{bmatrix} kx_1 + ky_1 \\ kx_1 - ky_1 \end{bmatrix} + \begin{bmatrix} px_2 + py_2 \\ px_2 - py_2 \end{bmatrix}$$

$$= k\begin{bmatrix} x_1 + y_1 \\ x_1 - y_1 \end{bmatrix} + p\begin{bmatrix} x_2 + y_2 \\ x_2 - y_2 \end{bmatrix}$$

$$= k\,T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) + p\,T\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right)$$

2. T is one to one:

we need to show that $T(\vec{x}) = \vec{0}$ for a vector $\underline{x} \in \mathbb{R}^2$, then it follows that $\underline{x} = \underline{0}$. let $\underline{x} = \begin{bmatrix} x \\ y \end{bmatrix}$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x-y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This provides a system of equations given by,

$$x+y=0$$
$$x-y=0$$

you can verify that the solution to this system if $x=y=0$. Therefore,

$$\underline{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and T is one to one.

3. T is onto.

let a, b be scalars, we want to check if there is alwas a solution to

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x-y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

This can be represented as the system of equations

$$x+y = a$$
$$x-y = b$$

setting up the augmented matrix and row reducing gives,

$$\left[\begin{array}{cc|c} 1 & 1 & a \\ 1 & -1 & b \end{array}\right] \longrightarrow \ldots$$

This has a solution for all a, b and therefore T is onto

Therefore T is an isomorphism.

(8)

I

let $v_1, v_2, v_3 \in V$ and take r be a Scalar

Then we are given that,

$$T(v_1) = T(v_2) = \cdots = 0$$

by the

since, $v_1 + v_2 + \cdots + v_n \in V$, $r v_1 \in V$

we have,

$$T(v_1 + v_2 + v_3 + \cdots + v_n) = \cancel{0 + 0 +} 0 = 0 + 0 + \cdots 0$$
$$= T(v_1) + T(v_2) + \cdots + T(v_n)$$

$$T(r v_1) = 0 = r.0 = r\,T(v_1)$$

Since above conditions holds, then T is zero Transformat.

II

if $\{v_1, \ldots, v_n\}$ is a basis for V, then any $x \in V$ can be written as $x = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$, with $\alpha_i \in \mathbb{R}$. Since T is linear, we have then that,

$$T(x) = T(\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \cdots + \alpha_n v_n)$$
$$= \alpha_1 T(v_1) + \alpha_2 T(v_2) + \alpha_3 T(v_3) + \cdots$$
$$= \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = x$$

so T maps any vector $x \in V$ to itself, and thus the identity operator.