The analytic solution to four-wave mixing in nonlinear fibers in Weierstrass notation

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Abstract

The general analytic solution to continuous wave four-wave mixing in nonlinear optical fibers is presented in terms of Weierstrass elliptic $\wp, \sigma, \zeta$ functions. Solutions are provided for the full complex envelopes for all four frequency modes, under all initial conditions, and without any undepleted pump approximation.

1 Introduction

Four-wave mixing (FWM) in nonlinear optical fibers is one of the fundamental parametric processes enabled by the Kerr nonlinearity. It underpins a wide range of applications, including wavelength conversion, parametric amplification, frequency-comb generation, and quantum light sources (see [1] and references therein). Despite this broad relevance, obtaining closed-form analytic descriptions of FWM remains challenging because the underlying coupled-wave equations are nonlinear, phase-sensitive, and generally require numerical integration. As a result, most textbook treatments rely on simplifying assumptions such as undepleted pumps, identical effective mode areas, negligible phase mismatch, or weak signal and idler powers.

Two notable works have previously presented relevant analytic solutions for four-wave mixing. In [2] solutions for the power in each of the frequency modes is derived by inverting elliptic integrals to obtain elliptic functions. In [3] elliptic integrals are given for the phases for each mode. Both works use Jacobi elliptic functions. In contrast, this work derives complete solutions for the complex mode envelopes themselves, avoiding decomposition into amplitude and phase or real and imaginary parts, and does so using Weierstrass elliptic functions.

The structure of the resulting analytic solutions naturally motivates a transformation to a new set of canonical coordinates. Remarkably, although this transformation depends explicitly on the propagation variable z, it leaves the coupled differential equations structurally invariant, preserving the characteristic interaction terms of four-wave mixing dynamics. At the same time, the transformation leads to a substantial simplification of both the analysis and the solutions which become single-valued meromorphic functions, specifically Kronecker theta functions. The emergence of these functions establishes a direct connection between four-wave mixing and other nonlinear optical systems [4]. Within this framework, the four-wave mixing system is cast as a canonical Hamiltonian system,

and the associated conservation laws are shown to arise from the Frobenius–Stickelberger determinant formula.

2 The continuous wave four-wave mixing system

The following coupled system of ordinary differential equations is taken from [1] which credits [5] with its creation. The system describes four-wave mixing in the continuous wave limit, i.e., in the absence of time derivatives:

$$
\begin{aligned}
\frac{dA_1}{dz} &= \frac{in_2\omega_1}{c}\left[\left(f_{11}|A_1|^2 + 2\sum_{k\neq 1}f_{1k}|A_k|^2\right)A_1 + 2f_{1234}A_2^*A_3A_4e^{i\Delta kz}\right], \\
\frac{dA_2}{dz} &= \frac{in_2\omega_2}{c}\left[\left(f_{22}|A_2|^2 + 2\sum_{k\neq 2}f_{2k}|A_k|^2\right)A_2 + 2f_{2134}A_1^*A_3A_4e^{i\Delta kz}\right], \\
\frac{dA_3}{dz} &= \frac{in_2\omega_3}{c}\left[\left(f_{33}|A_3|^2 + 2\sum_{k\neq 3}f_{3k}|A_k|^2\right)A_3 + 2f_{3412}A_1A_2A_4^*e^{-i\Delta kz}\right], \\
\frac{dA_4}{dz} &= \frac{in_2\omega_4}{c}\left[\left(f_{44}|A_4|^2 + 2\sum_{k\neq 4}f_{4k}|A_k|^2\right)A_4 + 2f_{4312}A_1A_2A_3^*e^{-i\Delta kz}\right].
\end{aligned}
\tag{2.1}
$$

where:

- A_j are the slowly-varying complex field amplitudes for waves $j = 1, 2, 3, 4$

- z is the propagation distance

- ω_j are the angular frequencies of the respective waves

- c is the speed of light in vacuum

- n_2 is the nonlinear refractive index

- f_{jj} are the self-phase modulation (SPM) coefficients

- f_{jk} (for $j \neq k$) are the cross-phase modulation (XPM) coefficients

- f_{jklm} are the four-wave mixing (FWM) coefficients

- A^* denotes complex conjugation

- $\Delta k = \beta_1 + \beta_2 - \beta_3 - \beta_4$ is the phase mismatch

- $\beta_j = n(\omega_j)\omega_j/c$ are the propagation constants

The modal overlap integrals are defined in terms of the transverse distribution of the fiber mode $F_j(x, y)$ as:

$$
f_{jklm} = \frac{\langle F_j^* F_k^* F_l F_m \rangle}{\sqrt{\langle |F_j|^2\rangle\langle |F_k|^2\rangle\langle |F_l|^2\rangle\langle |F_m|^2\rangle}},
\tag{2.2}
$$

$$
f_{jk} = f_{kj} = f_{jjkk}
\tag{2.3}
$$

where $\langle \cdots \rangle = \iint_{-\infty}^{\infty} (\cdots) \, dx \, dy$ denotes the transverse spatial integral.

Agrawal says of (2.1) that the equations "are quite general in the sense that they include the effects of SPM, XPM, and pump depletion on the FWM process; a numerical approach is necessary to solve them exactly." That said, herein, they are solved analytically in full, as written, without any further approximation.

3   Simplifying parameter dependence

From (2.2), we observe the following symmetries among the wave mixing coefficients:

$$
\begin{aligned}
f_{1234} &= |f_{1234}| e^{i\theta}, \\
f_{2134} &= f_{1234} = |f_{1234}| e^{i\theta}, \\
f_{3412} &= f_{1234}^* = |f_{1234}| e^{-i\theta}, \\
f_{4312} &= f_{1234}^* = |f_{1234}| e^{-i\theta},
\end{aligned}
\tag{3.1}
$$

where θ is the phase of f_{1234}. Consequently, we can conveniently renormalise the functions, and also absorb the phase θ as a global phase rotation of the modes, in such a way that the wave mixing coefficients all become equal to one. To do so, we introduce the following redefinition of the mode functions (take care to note that, while not crucial, the complex conjugates are shared among u, v in this choice of labels so as to later conveniently give one product over u and one over v):

$$
T = \sqrt{\frac{2 \, n_2 \, |f_{1234}|}{c}} \sqrt{\prod_{k=1}^{4} \omega_k},
\tag{3.2}
$$

$$
u_1(z) = \frac{T \, e^{-i\pi/4} \, e^{-i\theta/4}}{\sqrt{\omega_1}} \, A_1(z) \, e^{iz\beta_1},
$$

$$
u_2(z) = \frac{T \, e^{-i\pi/4} \, e^{-i\theta/4}}{\sqrt{\omega_2}} \, A_2(z) \, e^{iz\beta_2},
$$

$$
u_3(z) = \frac{T \, e^{i\pi/4} \, e^{-i\theta/4}}{\sqrt{\omega_3}} \, A_3^*(z) \, e^{-iz\beta_3},
$$

$$
u_4(z) = \frac{T \, e^{i\pi/4} \, e^{-i\theta/4}}{\sqrt{\omega_4}} \, A_4^*(z) \, e^{-iz\beta_4},
$$

$$
v_1(z) = \frac{T \, e^{-i\pi/4} \, e^{i\theta/4}}{\sqrt{\omega_1}} \, A_1^*(z) \, e^{-iz\beta_1},
$$

$$
v_2(z) = \frac{T \, e^{-i\pi/4} \, e^{i\theta/4}}{\sqrt{\omega_2}} \, A_2^*(z) \, e^{-iz\beta_2},
$$

$$
v_3(z) = \frac{T \, e^{i\pi/4} \, e^{i\theta/4}}{\sqrt{\omega_3}} \, A_3(z) \, e^{iz\beta_3},
$$

$$
v_4(z) = \frac{T \, e^{i\pi/4} \, e^{i\theta/4}}{\sqrt{\omega_4}} \, A_4(z) \, e^{iz\beta_4}.
\tag{3.3}
$$

We subsequently redefine the remaining phase modulation parameters as:

$$a_j = -i\,s(j)\beta_j, \tag{3.4}$$

$$a_{j,k} = a_{k,j} = -\frac{(\delta_{jk} - 2)\,s(j)\,s(k)\,f_{j,k}\,\omega_j\,\omega_k}{2\,|f_{1,2,3,4}|\,\sqrt{\prod_{l=1}^{4}\omega_l}} \tag{3.5}$$

with Kronecker δ, and $s(j)$ defined such that $s(1) = s(2) = 1, s(3) = s(4) = -1$.

The four-wave mixing system in (2.1) is thus recast as:

$$\frac{d}{dz}u_j(z) = -\left(a_j + \sum_{k=1}^{4} a_{j,k}\,u_k v_k\right) u_j + \prod_{k=1, k\neq j}^{4} v_k,$$

$$\frac{d}{dz}v_j(z) = \left(a_j + \sum_{k=1}^{4} a_{j,k}\,u_k v_k\right) v_j - \prod_{k=1, k\neq j}^{4} u_k. \tag{3.6}$$

4   Conserved quantities

Based on prior experience of similar systems we are prompted to ask if the system in (3.6) is a canonical Hamiltonian system, and indeed we confirm that it is with:

$$H(u_1, \ldots, u_4, v_1, \ldots, v_4) = -\sum_{j=1}^{4} a_j\,u_j v_j - \frac{1}{2}\sum_{j,k=1}^{4} a_{j,k}\,u_j v_j u_k v_k + \prod_{j=1}^{4} u_j + \prod_{j=1}^{4} v_j, \tag{4.1}$$

$$\frac{d}{dz}u_j(z) = \frac{\partial H}{\partial v_j}, \tag{4.2}$$

$$\frac{d}{dz}v_j(z) = -\frac{\partial H}{\partial u_j}, \tag{4.3}$$

$$\frac{d}{dz}H = 0 \tag{4.4}$$

The conservation of H is to be expected as a consequence of (4.1) lacking an explicit z dependence. The pair u_j, v_j are Hamiltonian conjugates, and each pair represents one of four degrees of freedom in the system. We refer to their product $u_j v_j$ as the modal power and it evolves according to:

$$\frac{d}{dz}u_j v_j = \prod_{k=1}^{4} v_k - \prod_{k=1}^{4} u_k \tag{4.5}$$

As the right hand side of (4.5) is the same for all j, we may define constants γ_j and function $\rho(z)$ such that:

$$u_j(z)v_j(z) = \gamma_j - \rho(z) \tag{4.6}$$

from which it follows that there are 3 (read as $j > k$ to avoid over counting) intermodal power conservation laws of the form:

$$u_j(z)v_j(z) - u_k(z)v_k(z) = \gamma_j - \gamma_k. \tag{4.7}$$

The system has 4 degrees of freedom and four conserved quantities which is a requirement for integrability in the Liouville-Arnold sense. The γ_j constants can be determined

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from (4.7) and initial conditions, however, this provides three equations for four unknowns and thus a choice is available for normalisation. We choose to impose the constraint that:

$$\sum_{j=1}^{4} \gamma_j = 0. \tag{4.8}$$

Remark: Appendix-C shows that (4.1) is the Frobenius-Stickelberger determinant.

5   Solutions for modal powers in terms of Weierstrass $\wp$ elliptic functions

The fundamental Weierstrass elliptic function theory used in coming sections can be found in [6]. One key thing to note in order to obtain elliptic functions solutions in wave mixing dynamics is that the derivative of the modal power in (4.5) is proportional to the difference of the two wave mixing product terms, and that the Hamiltonian in (4.1) contains their sum. We then note the simple but important identity:

$$\left(\prod_{k=1}^{4} v_k - \prod_{k=1}^{4} u_k\right)^2 - \left(\prod_{k=1}^{4} v_k + \prod_{k=1}^{4} u_k\right)^2 = -4 \prod_{j=1}^{4} u_j v_j \tag{5.1}$$

which enables us to square (4.5) and replace wave mixing terms with phase modulation terms.

To proceed in this manner let us introduce the function Q which represents the phase modulation part of the Hamiltonian:

$$Q(u_1(z)v_1(z), \ldots, u_4(z)v_4(z)) = a_0 + \sum_{j=1}^{4} a_j \, u_j v_j + \frac{1}{2} \sum_{j,k=1}^{4} a_{j,k} \, u_j v_j u_k v_k,$$

$$= \sum_{l=0}^{2} b_l \, \rho\left(z\right)^l \tag{5.2}$$

with $a_0 = H$, and observe that by squaring (4.5) and substituting (4.1), (4.6), (5.1), and (5.2), we obtain:

$$\left(\frac{d}{dz}\rho(z)\right)^2 = Q^2(\gamma_1 - \rho(z), \ldots, \gamma_4 - \rho(z)) - 4 \prod_{j=1}^{4} (\gamma_j - \rho(z)), \tag{5.3}$$

$$\left(\frac{d}{dz}\rho(z)\right)^2 = \left(\sum_{l=0}^{2} b_l \, \rho\left(z\right)^l\right)^2 - 4 \prod_{j=1}^{4} (\gamma_j - \rho(z)), \tag{5.4}$$

$$\left(\frac{d}{dz}\rho(z)\right)^2 = \sum_{l=0}^{4} d_l \, \rho\left(z\right)^l, \tag{5.5}$$

$$\left(\frac{d}{dz}\rho(z)\right)^2 = d_4 \prod_{l=1}^{4} (\rho\left(z\right) - \lambda_l), \tag{5.6}$$

where b_l, and d_l are given in terms of other parameters and initial conditions in Appendices A and B, and where λ_l are the roots of the quartic polynomial in $\rho(z)$, i.e.,

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$\sum_{l=0}^{4} d_l \lambda^l = 0$. We now transform (5.6) from quartic to the standard cubic form of the $\wp$ function using the classical trick which can be conceptualised in three steps:

$$\rho(z) = q(z) + \lambda_1 \qquad\qquad \text{shift so rhs is 0 for } q(z) = 0,$$
$$s(z) = 1/q(z) \qquad\qquad \text{invert to make the quartic a cubic,}$$
$$w(z) = C_1 s(z) + C_0 \quad \text{shift and scale with } C_j \text{ to match Weierstrass coefficients.} \quad (5.7)$$

After choosing a root, λ_1, the procedure sketched in (5.7) is implemented in the following single transformation:

$$\rho(z) = \lambda_1 + \frac{d_4}{-4w(z)\prod_{l=1}^{3}\Omega_l + \frac{d_4}{3}\sum_{l=1}^{3}\Omega_l}, \tag{5.8}$$

$$\Omega_l = \frac{1}{\lambda_{l+1} - \lambda_1},$$

$$\left(\frac{d}{dz}w(z)\right)^2 = 4\,w(z)^3 - g_2\,w(z) - g_3, \tag{5.9}$$

$$g_2 = d_0 d_4 - \frac{d_1 d_3}{4} + \frac{d_2^2}{12}, \tag{5.10}$$

$$g_3 = \frac{d_0 d_2 d_4}{6} - \frac{d_0 d_3^2}{16} - \frac{d_1^2 d_4}{16} + \frac{d_1 d_2 d_3}{48} - \frac{d_2^3}{216} \tag{5.11}$$

where the constants g_2 and g_3 are known as Weierstrass elliptic invariants.

Equation (5.9) defines the Weierstrass elliptic $\wp$ function and the solution is:

$$w(z) = \wp(z - z_0, g_2, g_3) \tag{5.12}$$

where z_0 is a constant chosen to match initial conditions. This constant z_0, and others that we will now introduce, can be obtained by inverting $\wp$ using an elliptic integral, (e.g. Carlson's symmetric R_F integral). As $\wp$ is an even function, it is neccessary to also specify a corresponding condition for the derivative when inverting, i.e., to find z from known x, y we give conditions such as $\wp(z) = x, \wp'(z) = y$, where $\wp'$ is the derivative of $\wp$ known as Weierstrass p-prime. The points obtained during such an inversion are determined modulo the period lattice of the doubly periodic $\wp$. Let us proceed to define the point z_0, and introduce the point z_1 as the pole of $\rho(z)$, and points μ_j as zeroes of $u_j(z)v_j(z)$:

$$\wp(z_0) = \frac{d_2}{12} + \frac{d_3 \lambda_1}{4} + \frac{d_4 \lambda_1^2}{2} + \frac{-d_1 - 2d_2 \lambda_1 - 3d_3 \lambda_1^2 - 4d_4 \lambda_1^3}{4(-\rho(0) + \lambda_1)},$$

$$\wp'(z_0) = \frac{(d_1 + 2d_2 \lambda_1 + 3d_3 \lambda_1^2 + 4d_4 \lambda_1^3)}{4(\rho(0) - \lambda_1)^2} \left.\frac{d}{dz}\rho(z)\right|_{z=0},$$

$$\wp(z_1) = \frac{d_2}{12} + \frac{d_3 \lambda_1}{4} + \frac{d_4 \lambda_1^2}{2},$$

$$\wp'(z_1) = \frac{(-d_1 - 2d_2 \lambda_1 - 3d_3 \lambda_1^2 - 4d_4 \lambda_1^3)\sqrt{d_4}}{4},$$

$$\wp(\mu_j - z_0) = \frac{d_2}{12} + \frac{d_3 \lambda_1}{4} + \frac{d_4 \lambda_1^2}{2} - \frac{-d_1 - 2d_2 \lambda_1 - 3d_3 \lambda_1^2 - 4d_4 \lambda_1^3}{4(\gamma_j - \lambda_1)},$$

$$\wp'(\mu_j - z_0) = -\frac{(b_0 + b_1 \gamma_j + b_2 \gamma_j^2)(d_1 + 2d_2 \lambda_1 + 3d_3 \lambda_1^2 + 4d_4 \lambda_1^3)}{4(\gamma_j - \lambda_1)^2}. \tag{5.13}$$

The solutions for modal powers are then:

$$u_j(z)v_j(z) = \rho(\mu_j) - \rho(z),$$

$$= \frac{\wp'(z_1)}{\sqrt{d_4}\left(\wp(z_1) - \wp(\mu_j - z_0)\right)} - \frac{\wp'(z_1)}{\sqrt{d_4}\left(\wp(z_1) - \wp(z - z_0)\right)},$$

$$= \frac{\wp'(z_1)}{\sqrt{d_4}\left(\wp(z_1) - \wp(\mu_j - z_0)\right)} \frac{\left(\wp(z - z_0) - \wp(\mu_j - z_0)\right)}{\left(\wp(z - z_0) - \wp(z_1)\right)} \tag{5.14}$$

6 Solutions for modes in terms of Weierstrass σ, ζ functions

Through substitution of (4.1), (4.5), (4.6), and (4.7) into (3.6), it can be shown that (3.6) can be written as logarithmic derivatives such that the right hand side is a function of ρ and its derivative ρ' in the form:

$$\frac{\frac{\partial}{\partial z}u_j(z)}{u_j(z)} = \frac{1}{2}\frac{\rho'(z) - \rho'(\mu_j)}{\rho(z) - \rho(\mu_j)} + \rho(z)\Lambda_{1,j} + \Lambda_{0,j},$$

$$\frac{\frac{\partial}{\partial z}v_j(z)}{v_j(z)} = \frac{1}{2}\frac{\rho'(z) + \rho'(\mu_j)}{\rho(z) - \rho(\mu_j)} - \rho(z)\Lambda_{1,j} - \Lambda_{0,j}, \tag{6.1}$$

$$\Lambda_{0,j} = -a_j - \frac{\gamma_j}{4}\sum_{k,l=1}^{4}a_{k,l} - \sum_{k=1}^{4}a_{j,k}\gamma_k + \frac{1}{2}\sum_{k=1}^{4}\gamma_k\sum_{l=1}^{4}a_{k,l} + \frac{1}{2}\sum_{k=1}^{4}a_k, \tag{6.2}$$

$$\Lambda_{1,j} = \sum_{k=1}^{4}a_{j,k} - \frac{1}{4}\sum_{k,l=1}^{4}a_{k,l}. \tag{6.3}$$

We substitute (5.14) into (6.1) and use the elliptic function identity:

$$\frac{\wp'(x, g_2, g_3)}{\wp(x, g_2, g_3) - \wp(y, g_2, g_3)} = \zeta(x + y, g_2, g_3) + \zeta(x - y, g_2, g_3) - 2\zeta(x, g_2, g_3) \tag{6.4}$$

to write the right hand side in terms of ζ and constants:

$$\frac{\frac{\partial}{\partial z}u(z, \mu_j)}{u(z, \mu_j)} = \frac{\left(\zeta(z - z_0 + z_1) - 2\zeta(z_1) - \zeta(z - z_0 - z_1)\right)\Lambda_{1,j}}{\sqrt{d_4}}$$

$$+ \zeta(z - 2z_0 + \mu_j) - \frac{\zeta(z - z_0 - z_1)}{2} - \frac{\zeta(z - z_0 + z_1)}{2}$$

$$- \frac{\zeta(\mu_j - z_0 - z_1)}{2} - \frac{\zeta(\mu_j - z_0 + z_1)}{2} + \Lambda_{0,j} + \Lambda_{1,j}\lambda_1$$

$$\frac{\frac{\partial}{\partial z}v(z, \mu_j)}{v(z, \mu_j)} = -\frac{\left(\zeta(z - z_0 + z_1) - 2\zeta(z_1) - \zeta(z - z_0 - z_1)\right)\Lambda_{1,j}}{\sqrt{d_4}}$$

$$+ \zeta(z - \mu_j) - \frac{\zeta(z - z_0 - z_1)}{2} - \frac{\zeta(z - z_0 + z_1)}{2}$$

$$+ \frac{\zeta(\mu_j - z_0 - z_1)}{2} + \frac{\zeta(\mu_j - z_0 + z_1)}{2} - \Lambda_{0,j} - \Lambda_{1,j}\lambda_1 \tag{6.5}$$

Equations (6.5) can be integrated by noting that the Weierstrass ζ function is the logarithmic derivative of the σ function:

$$\zeta(z, g_2, g_3) = \frac{\partial}{\partial z} \sigma(z, g_2, g_3).$$ (6.6)

Performing the integration and taking exponentials gives the solutions for modes u_j, v_j:

$$u_j(z) = \frac{\alpha_j \sqrt{W_j}\, \sigma(z - 2z_0 + \mu_j) \exp\left(zr_{0,j} + \log\left(\frac{\sigma(z - z_0 + z_1)}{\sigma(z - z_0 - z_1)}\right) r_{1,j}\right)}{\sqrt{\wp(z_1) - \wp(z - z_0)}\, \sigma(\mu_j - z_0)\sigma(z - z_0)},$$

$$v_j(z) = \frac{\sqrt{W_j}\, \sigma(z - \mu_j) \exp\left(-zr_{0,j} - \log\left(\frac{\sigma(z - z_0 + z_1)}{\sigma(z - z_0 - z_1)}\right) r_{1,j}\right)}{\alpha_j \sqrt{\wp(z_1) - \wp(z - z_0)}\, \sigma(\mu_j - z_0)\sigma(z - z_0)}$$ (6.7)

where α_j is the integration constant that can be fixed by initial conditions to capture any phase offset between a mode and its conjugate, and where the other constants are:

$$W_j = \frac{\wp'(z_1)}{(\wp(z_1) - \wp(\mu_j - z_0))\sqrt{d_4}},$$ (6.8)

$$r_{0,j} = \Lambda_{0,j} + \Lambda_{1,j}\lambda_1 - \frac{2\zeta(z_1)\Lambda_{1,j}}{\sqrt{d_4}} - \frac{\zeta(\mu_j - z_0 - z_1)}{2} - \frac{\zeta(\mu_j - z_0 + z_1)}{2},$$ (6.9)

$$r_{1,j} = \frac{\Lambda_{1,j}}{\sqrt{d_4}}.$$ (6.10)

7 When quartic terms cancel to leave a cubic

There will be no $\rho(z)^4$ term in the differential equation when $d_4 = 0$. For $d_4 = 0$, we require $b_2 \pm 2$, which is a parameter requirement, i.e., it cannot be achieved in general through tailored initial conditions of the modes. This case is simpler to solve in some ways but it is not easily obtained by taking the limit $d_4 \to 0$ because d_4 appears in the denominator in several places. When $d_4 = 0$, we get $\wp'(z_1) = 0$, i.e., z_1 is congruent to a half-period of the elliptic function, commonly denoted ω_i, for $i = 1, 2, 3$. It can then be shown either by solving the differential equation starting from the cubic, or by applying elliptic function identities to intermediate results in the quartic derivation, that the solutions for modes simplify to:

$$u_j(z) = \frac{\alpha_j\, 2\sigma(z - 2z_0 + 2z_1 + \mu_j) \exp\left(z\left(\Lambda_{0,j} + \frac{d_2}{3d_3}\Lambda_{1,j} - \zeta(\mu_j - z_0 + z_1)\right)\right)}{\sqrt{d_3}\, \sigma(\mu_j - z_0 + z_1)\sigma(z - z_0 + z_1)},$$

$$v_j(z) = \frac{2\sigma(z - \mu_j) \exp\left(-z\left(\Lambda_{0,j} + \frac{d_2}{3d_3}\Lambda_{1,j} - \zeta(\mu_j - z_0 + z_1)\right)\right)}{\alpha_j \sqrt{d_3}\, \sigma(\mu_j - z_0 + z_1)\sigma(z - z_0 + z_1)}$$ (7.1)

where the integration constant α_j is to be determined by initial conditions using (7.1) and is therefore not the same α_j as in (6.7), but where all other parameters retain their prior definitions. The modal power simplies to:

$$u_j(z)v_j(z) = \frac{4\wp(\mu_j - z_0 + z_1)}{d_3} - \frac{4\wp(z - z_0 + z_1)}{d_3}$$ (7.2)

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8 Transforming to canonical coordinates

In this section we will demonstrate an application of the analytic solution by showing how its form guides us towards a canonical coordinate system for the analysis of four-wave mixing. In the canonical coordinates the system is simplified in the following ways:

- there is no cross-phase modulation XPM

- differential equations for modal powers involve cubic not quartic polynomials

- system parameters are greatly reduced and if desired completely removable

- solutions are single-valued meromorphic Kronecker theta functions

The transformations preserve the following:

- conservation of the Hamiltonian and intermodal power laws

- intermodal power ratios

- the cross-ratio between any four distinct modal powers,

- the differential equations structure in regards to recognisable four-wave mixing terms

8.1 Removing cross-phase modulation with a gauge transform

To begin, let us consider a transform of the system in (3.6) of the following form:

$$u_j(z) = \hat{u}_j(z)e^{-\phi_j(z)}$$
$$v_j(z) = \hat{v}_j(z)e^{\phi_j(z)} \tag{8.1}$$

$$\sum_{j=1}^{4} \phi_j(z) = 0 \tag{8.2}$$

where transforming the conjugate mode with the opposing phase leaves modal powers unchanged, and where the sum over ϕ_j being zero ensures we do not encounter any new terms appearing in the exponents of wave mixing terms. We make the following choice for ϕ_j composed of a linear part (L) and a nonlinear part (NL) (where L and NL are function labels not exponents):

$$\phi_j(z) = \phi_j^L(z) + \phi_j^{NL}(z) \tag{8.3}$$

$$\phi_j^L(z) = z\left(a_j - \frac{1}{4}\sum_{m=1}^{4} a_m - \frac{\gamma_j}{4}\sum_{l,k=1}^{4} a_{l,k}\right) \tag{8.4}$$

$$\phi_j^{NL}(z) = \sum_{k=1}^{4}\left(a_{j,k} - \frac{1}{4}\sum_{l=1}^{4} a_{l,k}\right)\int \hat{u}_k(z)\hat{v}_k(z)\,dz \tag{8.5}$$

$$\sum_{j=1}^{4} \phi_j(z) = \sum_{j=1}^{4} \phi_j^L(z) = \sum_{j=1}^{4} \phi_j^{NL}(z) = 0. \tag{8.6}$$

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This transforms (3.6) into the following system:

$$\frac{\partial}{\partial z}\hat{u}_j = \left(\frac{b_1}{4} - \frac{b_2}{2}\hat{u}_j\hat{v}_j\right)\hat{u}_j + \prod_{k=1,k\neq j}^{4} \hat{v}_k$$

$$\frac{\partial}{\partial z}\hat{v}_j = \left(-\frac{b_1}{4} + \frac{b_2}{2}\hat{u}_j\hat{v}_j\right)\hat{v}_j - \prod_{k=1,k\neq j}^{4} \hat{u}_k \qquad (8.7)$$

for which the conserved canonical Hamiltonian is:

$$\hat{H} = \prod_{l=1}^{4}\hat{u}_l + \prod_{l=1}^{4}\hat{v}_l - \frac{1}{4}\sum_{l=1}^{4}\left(b_2\hat{u}_l^2\hat{v}_l^2 - b_1\hat{u}_l\hat{v}_l\right) \qquad (8.8)$$

and the intermodal power conservation laws are unchanged such that $\hat{u}_j\hat{v}_j - \hat{u}_k\hat{v}_k = \gamma_j - \gamma_k$. In comparison to (3.6), we see that in (8.7) we now have a single phase velocity coefficient for all modes $b_1/4$, a single self-phase modulation coefficient $b_2/2$, and no cross-phase modulation terms. In terms of the solutions to the system, the effect of the ϕ_j^{NL} transform is to remove the z dependent log terms weighted by $r_{1,j}$ in (6.7).

8.2 Reducing the quartic to a cubic via local normalisation

Next, we show how we can perform a coordinate transformation that involves a local, i.e. z dependent, normalisation of the modes themselves and, at the same time, induces a Mobius transform on modal powers which implements the classic trick that reduces the quartic to the cubic discussed in Section 5. However, here that same effect is delivered at the level of modes u_j, v_j not only at the mode power level $u_j v_j$ so that we can observe a corresponding effect on the coupled system. When we do this, we will see that it further reduces the number of parameters in the system such that the phase velocity coefficient becomes the reciprocal of the SPM coefficient.

Let us define the local normalisation function $h(z)$ and its associated coordinate transform $\hat{u}_j, \hat{v}_j \rightarrow \bar{u}_j, \bar{v}_j$ (note the hat to bar change in notation):

$$\hat{u}_j(z) = 2\sqrt{\lambda_1(\gamma_j - \lambda_1)}\frac{\bar{u}_j(z)e^{-\theta_j z}}{\sqrt{h(z)}}$$

$$\hat{v}_j(z) = 2\sqrt{\lambda_1(\gamma_j - \lambda_1)}\frac{\bar{v}_j(z)e^{\theta_j z}}{\sqrt{h(z)}}$$

$$h(z) = d_5 - \sum_{l=1}^{4}(\gamma_l - \lambda_1)\bar{u}_l\bar{v}_l = 4\lambda_1\bar{u}_j\bar{v}_j - \frac{\lambda_1 d_5}{\gamma_j - \lambda_1} \qquad (8.9)$$

where θ_j and d_5 are given in Appendix A. The θ_j parameter is not involved in the reduction of the quartic to the cubic but its inclusion keeps the phase velocity coefficient the same for all modes after the transform. The key parameter in the transformation is the root λ_1 previously defined for the purpose of quartic to cubic reduction.

The transformation in (8.9) sends (8.7) to:

$$\frac{\partial}{\partial z}\bar{u}_j = -\left(\frac{1}{\chi} - 4\chi\,\bar{u}_j\bar{v}_j\right)\bar{u}_j - 4\chi\prod_{k=1,k\neq j}^{4}\bar{v}_k,$$

$$\frac{\partial}{\partial z}\bar{v}_j = \left(\frac{1}{\chi} - 4\chi\,\bar{u}_j\bar{v}_j\right)\bar{v}_j + 4\chi\prod_{k=1,k\neq j}^{4}\bar{u}_k \qquad (8.10)$$

for which the conserved canonical Hamiltonian is:

$$\bar{H} = \sum_{l=1}^{4} \left(\frac{\chi}{8} \bar{u}_l^2 \bar{v}_l^2 - \frac{1}{\chi} \bar{u}_l \bar{v}_l \right) - \frac{\chi}{4} \prod_{l=1}^{4} \bar{u}_l - \frac{\chi}{4} \prod_{l=1}^{4} \bar{v}_l. \tag{8.11}$$

and the intermodal power conservation laws are:

$$\bar{u}_j(z)\bar{v}_j(z) - \bar{u}_k(z)\bar{v}_k(z) = \frac{d_5}{4\left(\gamma_j - \lambda_1\right)} - \frac{d_5}{4\left(\gamma_k - \lambda_1\right)} \tag{8.12}$$

The modal power in these coordinates evolves according to:

$$\frac{d}{dz}\bar{u}_j\bar{v}_j = 4\chi \left(\prod_{k=1}^{4} \bar{v}_k - \prod_{k=1}^{4} \bar{u}_k \right) \tag{8.13}$$

Remarkably, the terms in (8.9) are the same as those in (8.7) only with different coefficients. We thus say that the system is invariant under the transformation, which is quite suprising given the z dependence of the normalisation. Similarly, by comparing (8.13) to (4.5) we see that (4.5) is invariant up to a constant factor.

To solve this system, we may define the function $\bar{w}(z)$ in terms of h by:

$$h(z) = \frac{\left(d_2 + 3d_3\lambda_1 + 6d_4\lambda_1^2\right)\lambda_1}{3} - 4\bar{w}(z)\lambda_1 \tag{8.14}$$

and we find then that:

$$\left(\frac{d}{dz}\bar{w}(z) \right)^2 = 4\,\bar{w}(z)^3 - g_2\,\bar{w}(z) - g_3,$$

$$\bar{w}(z) = \wp(z - z_0) \tag{8.15}$$

$$h(z) = 4\lambda_1\left(\wp(z_1) - \wp(z - z_0)\right) \tag{8.16}$$

$$\bar{u}_j(z)\bar{v}_j(z) = \wp(\mu_j - z_0) - \wp(z - z_0) \tag{8.17}$$

where z_0, z_1, μ_j, g_2, and g_3 retain their previous definitions from the original coordinates. We see that as a result of the transformation in (8.9), the right hand side of (8.15) is cubic in $\bar{w}$ and thus any differential equation in h or $\bar{u}_j\bar{v}_j$ is also cubic in those functions as they are linearly related. The elliptic function form of $h(z)$ is not a surprise as it was purposely constructed in the transformation so as to remove the corresponding term from the denominator in (6.7).

The solutions for modes $\bar{u}_j, \bar{v}_j$ are:

$$\bar{u}_j(z) = \frac{\bar{\alpha}_j\sigma(z - 2z_0 + \mu_j)\exp\left(z\bar{\kappa}_j\right)}{\sigma(z - z_0)\sigma(\mu_j - z_0)}$$

$$\bar{v}_j(z) = \frac{\sigma(z - \mu_j)\exp\left(-z\bar{\kappa}_j\right)}{\bar{\alpha}_j\sigma(z - z_0)\sigma(\mu_j - z_0)} \tag{8.18}$$

where the Weierstrass σ functions use the same g_2, g_3 from the original coordinates, $\bar{\kappa}_j$ is given in Appendix A and $\bar{\alpha}_j$ is an integration constant fixed by initial conditions and used to capture any relative phase offset between a mode $\bar{u}_j$ and its conjugate $\bar{v}_j$. The solutions in (8.18) are Kronecker theta functions (up to constant and z dependent exponential factors), which simply put are a ratio of two Weierstrass σ functions shifted relative to

each other in z. This form is similar to that obtained in other systems in nonlinear optics such as two and three wave mixing in quadratic nonlinear media, polarisation dynamics in nonlinear fibers, and parity-time symmetric nonlinear couplers [4]. The exponentials and σ functions are entire functions and this ratio is a single valued meromorphic function, unlike the solutions in (6.7). The solutions are quasi-periodic and modulo the lattice they have zeros where the argument of σ vanishes in the numerator, and poles where the argument of σ vanishes in the denominator.

8.3 A parameterless system via rescaling

It is now simple to remove the remaining parameter via a scaling of the modes and the length variable as follows (note the bar to tilde notation change):

$$\bar{u}_j(z) = -\frac{2i}{\chi}\tilde{u}_j\left(\frac{z}{\chi}\right) \tag{8.19}$$

$$\bar{v}_j(z) = \frac{2i}{\chi}\tilde{v}_j\left(\frac{z}{\chi}\right) \tag{8.20}$$

$$z = \chi\xi \tag{8.21}$$

This substitution transforms (8.10) to:

$$\frac{\partial}{\partial\xi}\tilde{u}_j = -\left(1 - \tilde{u}_j\tilde{v}_j\right)\tilde{u}_j - \prod_{k=1,k\neq j}^{4}\tilde{v}_k$$

$$\frac{\partial}{\partial\xi}\tilde{v}_j = \left(1 - \tilde{u}_j\tilde{v}_j\right)\tilde{v}_j + \prod_{k=1,k\neq j}^{4}\tilde{u}_k \tag{8.22}$$

for which the conserved canonical Hamiltonian is:

$$\tilde{H} = \sum_{l=1}^{4}\left(\frac{1}{2}\tilde{u}_l^2\tilde{v}_l^2 - \tilde{u}_l\tilde{v}_l\right) - \prod_{l=1}^{4}\tilde{u}_l - \prod_{l=1}^{4}\tilde{v}_l \tag{8.23}$$

9 Plotting analytic vs numeric solutions

To check the validity of the solutions, we evaluate the analytic solutions using Sympy in Python and we plot them against numeric solutions of the corresponding system of differential equations found using the DOP853 Runge-Kutta algorithm from SciPy in Python.

In Figures 1 and 2 we plot the real and imaginary parts, respectively, of the product of u_jv_j in an abstract non-physical u, v scenario. The analytic dashed lines are shown to be in agreement with the numeric solutions shown as symbols.
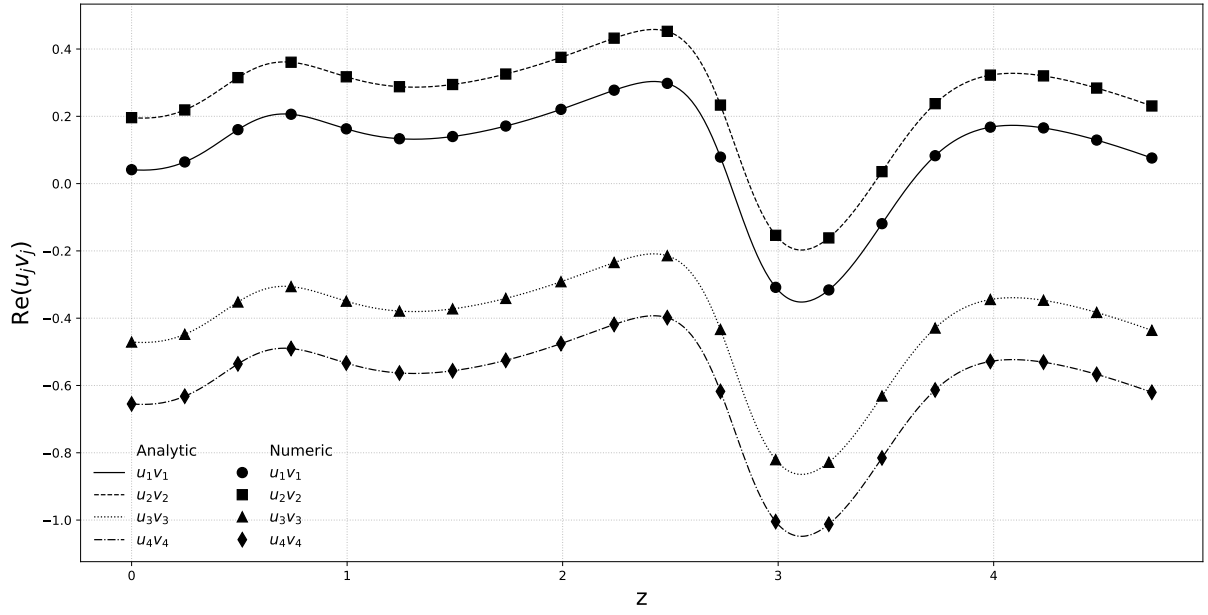
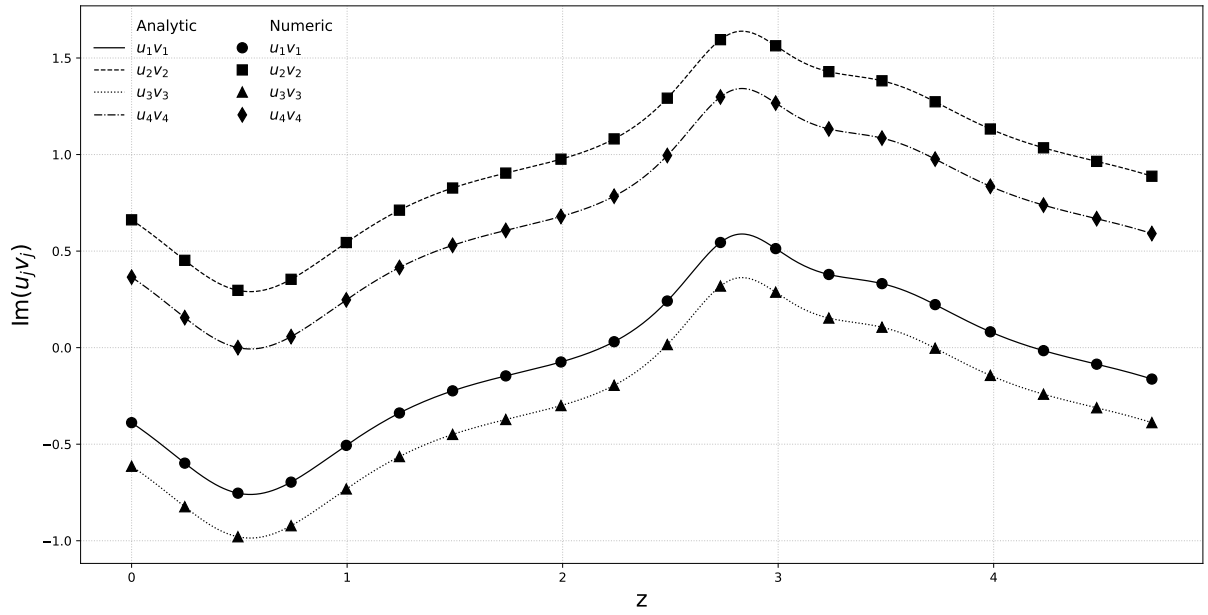Figure 1: Real part of analytic Equation (5.14) against numeric solutions.

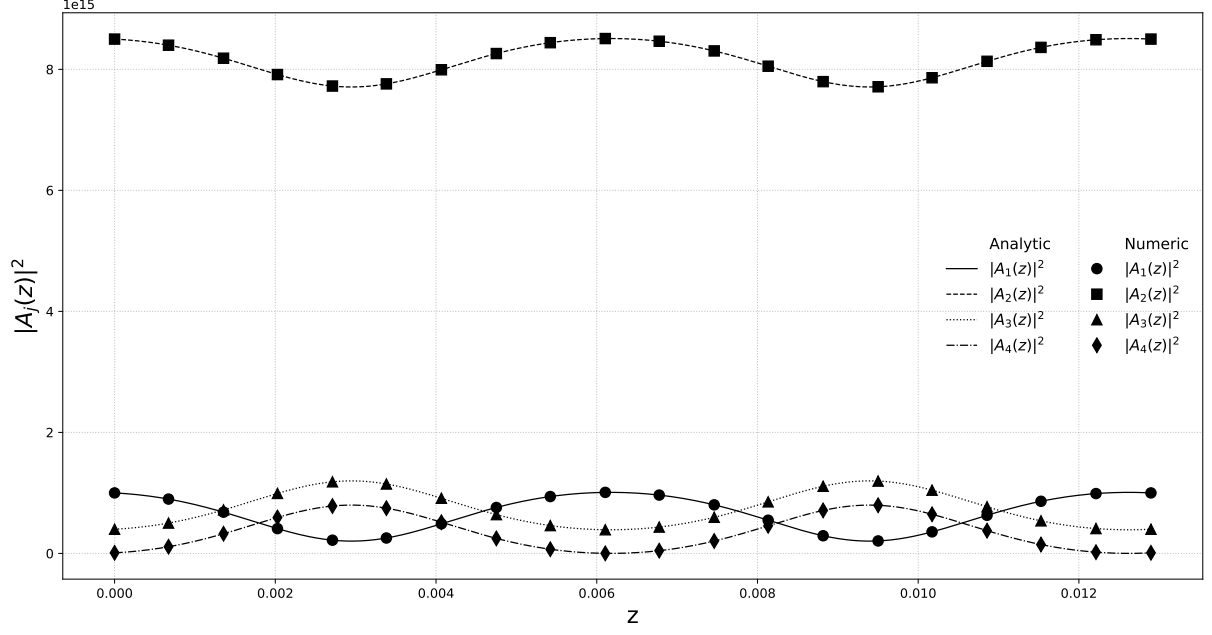Figure 2: Imaginary part of analytic Equation (5.14) against numeric solutions.

Figure 3: $|A_j|^2$ using analytic Equation (6.7) in (3.3) against numeric solutions.
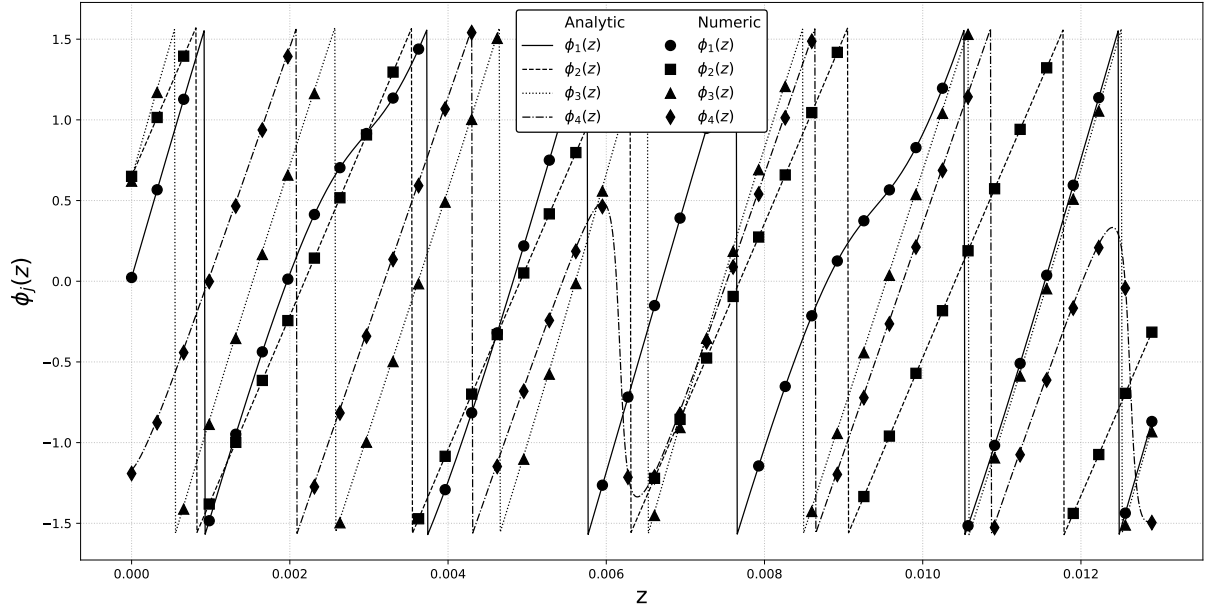

Figure 4: Phase ϕ_j using analytic Equation (6.7) in (3.3) against numeric solutions.

In Figures 3 and 4 we plot the absolute value squared and phase, respectively, of A_j in a physical four-wave mixing case. The value of A_j was found using (6.7) to first obtain u_j, v_j, before converting to A_j, A_j^* using (3.3). The analytic dashed lines are shown to be in agreement with the numeric solutions shown as symbols. As expected in nonlinear fiber optic four-wave mixing, periodic intermodal power exchange is seen between modes in Figure 3, while the phase plot in Figure 4 is dominated by linear phase velocity effects with some nonlinear effects visible as areas of high curvature in the plotted lines.

10   Conclusion

We have derived a complete analytic solution to the continuous-wave four-wave mixing system. The solutions are expressed in terms of Weierstrass elliptic functions, and we have demonstrated how they may be evaluated numerically using open-source Python code. Motivated by the structure of these analytic solutions, we investigated a class of coordinate transformations that yield a canonical coordinate system in which the governing differential equations simplify substantially and the solutions become single-valued, taking the form of Kronecker theta functions.

Through this exploration of coordinate transformations, we showed that the full four-wave mixing system can be reduced to a significantly simpler, parameter-free system. Remarkably, the structural form of the differential equations is preserved under these transformations, despite the fact that the transformations themselves depend explicitly on the independent variable z.

Four-wave mixing provides an important and challenging test case for mathematical analysis, and is arguably among the more intricate examples of dynamical systems admitting elliptic (genus-one) function solutions due to the presence of quartic polynomials. In this work, we have demonstrated in detail how these difficulties can be systematically addressed. The methods developed here may be transferable to the analysis of related nonlinear systems, such as the interaction of modes in multimode nonlinear fiber optics (see e.g. Chapter 4 in [7]).

In forthcoming work, we intend to present a unified framework for integrable nonlinear coupled ordinary differential equations with elliptic and hyperelliptic function solutions, for which the four-wave mixing system studied here will serve as a central example.

Acknowledgements

A Parameter definitions

$$b_0 = a_0 + \sum_{j=1}^{4} a_j \gamma_j + \frac{1}{2} \sum_{j,k=1}^{4} a_{j,k} \gamma_j \gamma_k,$$

$$b_1 = -\sum_{j=1}^{4} a_j - \frac{1}{2} \sum_{j,k=1}^{4} (\gamma_j + \gamma_k) a_{j,k},$$

$$b_2 = \frac{1}{2} \sum_{j,k=1}^{4} a_{j,k},$$

$$c_0 = \prod_{j=1}^{4} \gamma_j,$$

$$c_1 = -\sum_{k=1}^{4} \prod_{\substack{j=1 \\ j \neq k}}^{4} \gamma_j,$$

$$c_2 = \frac{3}{2} \sum_{j=1}^{4} \gamma_j^2 - \frac{1}{4} \sum_{j,k=1}^{4} (\gamma_j - \gamma_k)^2 = -\frac{1}{2} \sum_{j=1}^{4} \gamma_j^2,$$

$$c_3 = 0,$$

$$c_4 = 1,$$

$$d_0 = b_0^2 - 4c_0,$$

$$d_1 = 2b_0 b_1 - 4c_1,$$

$$d_2 = 2b_0 b_2 + b_1^2 - 4c_2,$$

$$d_3 = 2b_1 b_2 - 4c_3,$$

$$d_3 = 2b_1 b_2 - 4c_3,$$

$$d_4 = b_2^2 - 4c_4,$$

$$d_5 = d_1 + 2d_2 \lambda_1 + 3d_3 \lambda_1^2 + 4d_4 \lambda_1^3,$$

$$\chi = \frac{8 \left(b_0 + b_1 \lambda_1 + b_2 \lambda_1^2 \right)}{d_5} = \frac{16}{d_5} \sqrt{\prod_{j=1}^{4} \gamma_j - \lambda_1},$$

$$\theta_j = \frac{b_0 + b_1 \lambda_1 + b_2 \lambda_1^2}{2 \left(\gamma_j - \lambda_1 \right)} + \frac{b_1}{4} + \frac{b_2 \gamma_j}{2} + \frac{b_2 \lambda_1}{2} - \frac{1}{\chi}, \quad \sum_{j=1}^{4} \theta_j = 0,$$

$$\bar{\kappa}_j = -\zeta(-z_0 + \mu_j, g_2, g_3) - b_2 \gamma_j + 2\theta_j + \frac{1}{\chi} \tag{A.1}$$

In equating the two equivalent definitions of χ we have chosen the positive branch of the square root when evaluating (5.4) at $\rho(z) = \lambda_1$.

B Initial condition relations

$$\rho(0) = -\frac{1}{4} \sum_{j=1}^{4} u_j(0) v_j(0),$$

$$\gamma_j = u_j(0) v_j(0) - \frac{1}{4} \sum_{j=1}^{4} u_j(0) v_j(0),$$

$$\left. \frac{d}{dz} \rho(z) \right|_{z=0} = \prod_{j=1}^{4} u_j(0) - \prod_{j=1}^{4} v_j(0) \tag{B.1}$$

C The Frobenius-Stickelberger determinant

The Frobenius-Stickelberger (FS) determinant formula is an elliptic function identity that relates products of σ functions to a multivariate determinant of higher order derivatives of $\wp$ [8, 6]. In this section we show that the conservation of the Hamiltonian can be viewed as a manifestation of FS. We are interested in the 4×4 case of FS in the form:

$$\frac{C \, \sigma(z+\nu_1)\sigma(z+\nu_2)\sigma(z+\nu_3)\sigma(z-\nu_1-\nu_2-\nu_3)}{\sigma^4(z)} = \det M \tag{C.1}$$

where:

$$C = \frac{12\sigma(\nu_1 - \nu_2)\sigma(\nu_1 - \nu_3)\sigma(\nu_2 - \nu_3)}{\sigma(\nu_1)\sigma^4(\nu_2)\sigma^4(\nu_3)} \tag{C.2}$$

$$\det M = \begin{vmatrix} 1 & \wp(z) & \frac{\partial \wp(z)}{\partial z} & \frac{\partial^2 \wp(z)}{\partial z^2} \\ 1 & \wp(\nu_1) & -\frac{\partial \wp(\nu_1)}{\partial \nu_1} & \frac{\partial^2 \wp(\nu_1)}{\partial \nu_1^2} \\ 1 & \wp(\nu_2) & -\frac{\partial \wp(\nu_2)}{\partial \nu_2} & \frac{\partial^2 \wp(\nu_2)}{\partial \nu_2^2} \\ 1 & \wp(\nu_3) & -\frac{\partial \wp(\nu_3)}{\partial \nu_3} & \frac{\partial^2 \wp(\nu_3)}{\partial \nu_3^2} \end{vmatrix}$$

$$= -6 \left(\wp(\nu_1) - \wp(z) \right) \left(\wp(\nu_2) - \wp(z) \right) \left(\wp(\nu_1) - \wp(\nu_2) \right) \wp'(\nu_3)$$
$$+ 6 \left(\wp(\nu_1) - \wp(z) \right) \left(\wp(\nu_3) - \wp(z) \right) \left(\wp(\nu_1) - \wp(\nu_3) \right) \wp'(\nu_2)$$
$$- 6 \left(\wp(\nu_2) - \wp(z) \right) \left(\wp(\nu_3) - \wp(z) \right) \left(\wp(\nu_2) - \wp(\nu_3) \right) \wp'(\nu_1)$$
$$- 6 \left(\wp(\nu_1) - \wp(\nu_2) \right) \left(\wp(\nu_1) - \wp(\nu_3) \right) \left(\wp(\nu_2) - \wp(\nu_3) \right) \wp'(z) \tag{C.3}$$

The starting point for us is the following relation established via (4.1), (4.5), and (5.2):

$$2 \prod_{j=1}^{4} u_j = -\frac{\partial}{\partial z} u_j \, v_j + a_0 + \sum_{j=1}^{4} u_j v_j a_j + \frac{1}{2} \sum_{j,k=1}^{4} u_j v_j u_k v_k \, a_{j,k},$$

$$= \rho' + \sum_{l=0}^{2} b_l \, \rho\,(z)^l. \tag{C.4}$$

We will show how (C.4) can be transformed into (C.1)

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C.1 The left hand side

From (5.14), the right hand side of (C.4) is a doubly periodic elliptic function, so too then is the left. Thus for integers m, n and half-periods ω_1, ω_3 we must have:

$$\prod_{j=1}^{4} \frac{u_j(z)}{u_j(2m\omega_3 + 2n\omega_1 + z)} = 1. \tag{C.5}$$

We substitute our solution (6.7) into (C.5) and use the quasi-periodicity of σ:

$$\sigma(2m\omega_3 + 2n\omega_1 + z) = (-1)^{mn+m+n} \sigma(z) e^{(2m\omega_3 + 2n\omega_1 + 2z)\zeta(m\omega_3 + n\omega_1)} \tag{C.6}$$

to reduce the product in (C.5) to an exponential which, as it is equal to one, must have argument $2\pi i N(n, m)$, with $N(n, m)$ an integer that varies with n, m. This leads to:

$$2i\pi N(n, m) = -(2m\omega_3 + 2n\omega_1) \sum_{j=1}^{4} r_{0,j} - 2\zeta(m\omega_3 + n\omega_1) \sum_{j=1}^{4} \nu_j, \tag{C.7}$$

$$\sum_{j=1}^{4} \nu_j = 2N(1, 0)\omega_3 - 2N(0, 1)\omega_1 = 0 \quad (\text{mod lattice}), \tag{C.8}$$

$$\sum_{j=1}^{4} r_{0,j} = 2\zeta(N(0, 1)\omega_1 - N(1, 0)\omega_3) \tag{C.9}$$

where $\nu_j = \mu_j - z_0$, and in moving from (C.7) to (C.8) and (C.9) we made use of the identity $\zeta(\omega_3)\omega_1 - \zeta(\omega_1)\omega_3 = i\pi/2$. Ultimately, equations (C.6), (C.8) and (C.9) allow us to make the following substitution in the left hand side of (C.4):

$$\frac{\sigma(z + \nu_4)}{\sigma(\nu_4)} \exp\left(z \sum_{j=1}^{4} r_{0,j}\right) = -\frac{\sigma(z - \nu_1 - \nu_2 - \nu_3)}{\sigma(\nu_1 + \nu_2 + \nu_3)} \tag{C.10}$$

C.2 The right hand side

The following algebraic identity follows from Waring-Lagrange interpolation and holds for any values of the variables and functions:

$$\sum_{l=0}^{2} b_l \rho(z)^l = \frac{(\gamma_1 - \rho(z))(\gamma_2 - \rho(z))}{(\gamma_1 - \gamma_3)(\gamma_2 - \gamma_3)} \sum_{l=0}^{2} b_l \gamma_1^l +$$

$$\frac{(\gamma_1 - \rho(z))(\gamma_3 - \rho(z))}{(\gamma_1 - \gamma_2)(\gamma_3 - \gamma_2)} \sum_{l=0}^{2} b_l \gamma_2^l +$$

$$\frac{(\gamma_2 - \rho(z))(\gamma_3 - \rho(z))}{(\gamma_1 - \gamma_2)(\gamma_1 - \gamma_3)} \sum_{l=0}^{2} b_l \gamma_3^l \tag{C.11}$$

where in our case $\gamma_j = \rho(\mu_j)$ and the sum over $b_l \gamma_j$ can be written using (5.4) as:

$$\sum_{l=0}^{2} b_l \gamma_j^l = \rho'(\mu_j). \tag{C.12}$$

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We can then use (5.14) to express ρ in terms of $\wp$ so that we can substitute all variables in (C.11) for values in terms of $\wp, \wp'$. We then divide both sides of (C.4) by:

$$\frac{\wp'(z_1)}{\sqrt{d_4}\left(\wp(z_1) - \wp(z - z_0)\right)^2} \tag{C.13}$$

and observe that it takes the equivalent form to (C.1) up to constant factors that may look different but can be shown to be identical by multiplying by $\sigma(z)^4$ and taking $z \to 0$.

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