ECE580 FunWork#1

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1 Question 1

$\epsilon = \cos\frac{2\pi}{3} + j\sin\frac{2\pi}{3} = e^{j\frac{2\pi}{3}}$
$\epsilon^2 = e^{j\frac{4\pi}{3}} = -e^{j\frac{\pi}{3}}$
$\epsilon^3 = e^{j\frac{6\pi}{3}} = 1$
$\epsilon^4 = e^{j\frac{8\pi}{3}} = e^{j\frac{2\pi}{3}}$

$$\begin{vmatrix} 1 & 1 & \epsilon \\ 1 & 1 & \epsilon^2 \\ \epsilon^2 & \epsilon & 1 \end{vmatrix} = 1 + \epsilon^4 + \epsilon^2 - \epsilon^3 - \epsilon^3 - 1 = \epsilon^4 - 2\epsilon^3 + \epsilon^2 = e^{j\frac{2\pi}{3}} - 2 - e^{j\frac{\pi}{3}}$$

2 Question 2

$$\begin{bmatrix} 4bc & ac & -2ab & 0 \\ 5bc & 3ac & -4ab & -abc \\ 3bc & 2ac & -ab & +4abc \end{bmatrix} = \begin{bmatrix} bc & -ac & -ab & -4abc \\ 0 & ac & -4ab & -10abc \\ 0 & 0 & 1 & 3c \end{bmatrix} \implies \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a \\ 2b \\ 3c \end{bmatrix}$$

3 Question 3

The two following matrices A and B *are not* similar.

$$A = \begin{bmatrix} 3 & 2 & -5 \\ 2 & 6 & -10 \\ 1 & 2 & -3 \end{bmatrix},\ B = \begin{bmatrix} 6 & 20 & -34 \\ 6 & 32 & -51 \\ 4 & 20 & -32 \end{bmatrix}$$

We need to find a matrix P such that $P^{-1}AP = B$. The strategy is to find two invertible matrices V_A and V_B such that $V_A^{-1}AV_A = V_B^{-1}BV_B = D$ with D being a diagonal matrix. In this case, D will include eigenvalues of the two matrices. And then considering $P = V_A V_B^{-1}$. For this, both A and B need to be diagonalizeable. According to the following theorem, we need to find independent eigenvetors of the two matrices.

Theorem: Let A be an $n \times n$ matrix. Then A is diagonalizable if and only if A has n linearly independent eigenvectors.

Polynomial functions of matrices A and B for finding eigenvalues are $p_A(\lambda) = p_B(\lambda) = \lambda^3 - 6\lambda^2 + 12\lambda - 8$. Hence the two matrices have the same characteristic polynomial, thus the same eigenvalues: $\lambda = 2$

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

A has only two eigenvectors $\boldsymbol{v}_1^T = [5\ 0\ 1]$ and $\boldsymbol{v}_2^T = [-2\ 1\ 0]$, which means that A is not diagonalizeable. Also B has only two eigenvectors $\boldsymbol{w}_1^T = [-5\ 1\ 0]$ and $\boldsymbol{w}_2^T = [17\ 0\ 2]$, which means that B is not diagonalizeable either. Therefore, V_A and V_B are not invertible. And this follows that $P = V_A V_B^{-1}$ does not exist.

4 Question 4

$$\begin{bmatrix} 1 & 1 & -2 & 3 \\ 1 & 2 & -1 & 2 \\ 1 & -1 & -4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix}$$

Rank of the the coefficient 3×4 matrix is 2. Therefore, we can arbitrarily set 4 - 2 = 2 variables. Setting $x_3 = x_4 = 0$ we obtain a general solution $[1\ 2\ 0\ 0]^T$

5 Question 5

Reducing the following matrix:
$\begin{bmatrix} 3 & 2 & 1 \\ 5 & 4 & 3 \\ 4 & 3 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$
With the equations $2x_1 + x2 = 0$ and $x_2 + 2x_3 = 0$, we have $2x_1 = -x_2 = 2x_3$ which yield a general solution $[2\ \ -1\ 2]^T$.

6 Question 6

In this problem, X must be a 2×2 matrix. The leaniar equation format of this matrix can be written as follows: $2x_1 + x3 = 1$
$2x_2 + x4 = 1$
Therefore, general solution is $[1\ 1\ \ -1\ \ -1]$.

7 Question 7

(a)
$Df(\mathbf{x}) = x^T(A + A^T) - [-2 \quad 3] = [2x_1 + 8y + 2 \qquad 8x + 4y - 3]$

(b)
$F(\mathbf{x}) = \begin{bmatrix} 2 & 8 \\ 8 & 4 \end{bmatrix}$

8 Question 8

$f = f(x_1, x_2) = 5e^{x_1^3 x_2} + \frac{1}{x_1 x_2^2}$

(a)
$\nabla f(x_1, x_2) = \begin{bmatrix} 15x_1^2 x_2 e^{x_1^3 x_2} - \frac{1}{x_1^2 x_2^2} \\ 5x_1^3 e^{x_1^3 x_2} - \frac{2}{x_1 x_2^3} \end{bmatrix}$

$\nabla f(1, 1) = \begin{bmatrix} 15e - 1 \\ 5e - 2 \end{bmatrix}$

(b)
$\mathbf{d} = [-3 \quad 4]^T$
$\mathbf{d} \cdot \nabla f(1, 1) = -25e - 5$

(c)
$||\mathbf{d}|| = 5$

$\frac{\mathbf{d}}{||\mathbf{d}||} \cdot \nabla f(1, 1) = -5e - 1$

9 Question 9

$f = f(x_1, x_2) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + 2x_1x_2 + \frac{1}{3}x_2^3 + x_2 + 5$

(a)
$\nabla f(x_1, x_2) = \begin{bmatrix} x_1 + 2x_2 \\ x_2 + 2x_1 + x_2^2 + 1 \end{bmatrix} = 0 \implies x_1 = -3 \pm \sqrt{5}, x_2 = \frac{3 \mp \sqrt{5}}{2}$

(b)
x^* is strict local minimizer if $F(x^*) > 0$, or $F(x^*)$ is p.d. For this we need to have all principal minors greater than 0:
$F(x_1, x_2) = \begin{bmatrix} 1 & 2 \\ 2 & 1+2x_2 \end{bmatrix}$
$\Delta_1 = 1 > 0$,
$\Delta_2 = \begin{vmatrix} 1 & 2 \\ 2 & 1+2x_2 \end{vmatrix} > 0 \implies x_2 > \frac{3}{2}$
Hence the pair of (x_1, x_2) supporting the above inequality is $x_1 = -3 - \sqrt{5}, x_2 = \frac{3+\sqrt{5}}{2}$.

10 Question 10

(a)
$f(x_1, x_2, x_3, x_4) = x_1^2 + x_3^2 + 2x_1x_3 + 4x_1x_4$
$Q' = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 4 & 0 & 2 & 0 \\ 8 & 0 & 0 & 0 \end{bmatrix} \implies Q = Q^T = \begin{bmatrix} 2 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ 4 & 0 & 0 & 0 \end{bmatrix}$

(b)
$f(x_1, x_2, x_3) = x_2^2 + x_1x_2 - x_1x_3$
$Q' = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 2 & 0 \\ -2 & 0 & 0 \end{bmatrix} \implies Q = Q^T = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

(c)
$f(x_1, x_2, x_3) = x_1^2 + x_2^2 + 6x_1x_2$
$Q' = \begin{bmatrix} 2 & 0 & 0 \\ 12 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \implies Q = Q^T = \begin{bmatrix} 2 & 6 & 0 \\ 6 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$