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# Simplicial Coalgebras for Concurrent Regular Languages

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**Abstract**

This thesis introduces a construction of automata for concurrent languages. This is done by defining ranked hypergraphs, hypergraphs with interfaces that can be composed associatively. A simplicial set over these graphs is defined and we define F-coalgebras which give a nondeterministic transition model over the cells. Using a coalgebra we can parse a ranked hypergraph, and find out if it is part of the coalgebra's language.

# Contents

# 1 Introduction

Finite automata and regular languages form the foundation of classical formal language theory. Deterministic finite automata (DFA) provide a simple and elegant model for accepting regular languages, with Kleene's theorem stating their equivalence to regular expressions. In this setting, words are built sequentially from single letters in an alphabet, and state transitions are deterministic.

However, many systems in practice exhibit inherently concurrent behaviour, where actions could occur in parallel with complex dependencies. In a previous attempt to model this behaviour, higher-dimensional automata (HDA) have been proposed. In HDA, concurrent actions are modelled using paths through hypercubes. But this combination of structure and behaviour is constricting and leads to just a small set of modelled languages.

In this thesis, we develop an approach to model concurrent regular languages which separates structure from behaviour: simplicial coalgebras over ranked hypergraphs.

We introduce ranked hypergraphs with interfaces that allow associative composition. The interfaces give a way to connect dependencies. This composition allows for a generator which consists of graphs of depth 1.

The generator allows for piece-by-piece parsing of a graph. Mirroring how finite automata parse a word character by character. With the alphabet being the generator.

We will first introduce the necessary background on simplicial sets, subobject identifiers and coalgebras. We then formalize ranked hypergraphs and their compositions, describe the simplicial structure, and finally define the coalgebraic behaviour function which gives the parsing semantics.

# 2 Background

## 2.1 Presheaves

**Definition 2.1.** Let $C$,$D$ be categories, and $F, G : C \to D$ be functors. Then a *natural transformation* $\mu : F \Rightarrow G$ from $F$ to $G$ is a family of morphisms such that:

1. for all $x \in C$ there exists a morphism $\mu_x : F(x) \to G(x)$ called the component at $x$
2. for all $x, y \in C$ and all morphisms $f : x \to y$ the following diagram commutes:

$$\begin{array}{ccc} F(x) & \xrightarrow{F(f)} & F(y) \\ \downarrow{\scriptstyle \mu_x} & & \downarrow{\scriptstyle \mu_y} \\ G(x) & \xrightarrow{G(f)} & G(y) \end{array} \tag{1}$$

**Definition 2.2.** Let $C$ be a category, a *presheaf* is a functor

$$F : C^{op} \to \mathbf{Set}$$

The collection of all presheaves on a category forms a category with as morphisms the natural transformations between the presheaves.

## 2.2 Simplicial sets

**Definition 2.3.** The simplex category $\Delta$ has as objects

$$[n] = \{0 < 1 < \cdots < n\}$$

and as morphisms the order-preserving functions between them.

For all $n \in \mathbb{N}$, $i \in [n]$ we define the coface map $\delta_i^n : [n-1] \to [n]$ as the injective order preserving map which has $(\delta_i^n)^{-1}(i) = \varnothing$. So the map 'misses' $i$ and all other elements of $[n]$ are 'hit'.

For all $n \in \mathbb{N}$, $i \in [n]$ we define the coface map $\sigma_i^n : [n-1] \to [n]$ as the surjective order preserving map which has $|(\sigma_i^n)^{-1}(i)| = 2$, $|(\sigma_i^n)^{-1}(j)| = 1\ \forall j \neq i$. So the map 'hits' $i$ twice, and all other $j$ once.

All morphisms of $\Delta$ are generated from the coface and codegeneracy maps.

*Notation:* $\delta_i^n$, $\sigma_i^n$ and so on, are just referred to as $\delta_i$, $\sigma_i$ etc. As the exact $n$ is not necessary for most applications.

**Definition 2.4.** A *simplicial set* is a presheaf on $\Delta$, which means a simplicial set is a functor

$$X : \Delta^{\mathrm{op}} \to \mathbf{Set}$$

Thus, a simplicial set $X$ assigns to each $[n] \in \Delta$ a set $X_n = X([n])$ of *$n$-simplices*, and to each morphism $\theta\colon [m] \to [n]$, a function $X(\theta)\colon X_n \to X_m$.

In particular, we have $X(\delta_i) = d_i$ where $d_i : X_n \to X_{n-1}$ omits the $i$-th element from an $n$-simplex. And $X(\sigma_i) = s_i$ where $s_i : X_n \to X_{n+1}$ repeats the $i$-th element in the simplex.

These satisfy the *simplicial identities*:

$$\begin{aligned}
d_i d_j &= d_{j-1} d_i && \text{if } i < j,\\
s_i s_j &= s_{j+1} s_i && \text{if } i \leq j,\\
d_i s_j &= \begin{cases} s_{j-1} d_i & \text{if } i < j,\\ \mathrm{id} & \text{if } i = j \text{ or } i = j+1,\\ s_j d_{i-1} & \text{if } i > j+1. \end{cases}
\end{aligned}$$

The presheaf category of simplicial sets is called sSet and its natural transformations the *simplicial morphisms*.

Given a collection of sets $S = S_0 \sqcup S_1 \sqcup \dots$ and functions $d_i : S_n \to S_{n-1}$, $s_i : S_n \to S_{n+1}$ satisfying the simplicial identities there is a unique simplicial set which has the same face and degeneracy maps.

**Lemma 2.5.** Let $X, Y \in$ sSet, a collection of morphisms $f_0 : X_0 \to Y_0, f_1 : X_1 \to Y_1, \dots$ is a simplicial morphism if and only if the following diagrams commute.

$$\begin{tikzcd}
X_n \arrow[r, "d_i^X"] \arrow[d, "f_n"] & X_{n-1} \arrow[d, "f_{n-1}"] & X \arrow[r, "s_j^X"] \arrow[d, "f_n"] & X_{n+1} \arrow[d, "f_{n+1}"] \\
Y_n \arrow[r, "d_i^Y"] & Y_{n-1} & Y_n \arrow[r, "s_j^Y"] & Y_{n+1}
\end{tikzcd}$$

*Proof.* Given that $\delta_i$, $\sigma_i$ generate the morphisms in $\Delta$, diagram 1 automatically commutes for all $f$ if it does so for $\delta$ and $\sigma$. ☐

## 2.3 Semilattices

**Definition 2.6.** Let $P$ be a partially ordered set (poset), the join (or least upper bound) is a binary operator $\vee$ such that for all $x, y, z \in P$:

- $x \leq x \vee y$
- if $x \leq a$ and $y \leq a$ then $x \vee y \leq a$

the meet (or greatest lower bound) is a binary operator $\wedge$ such that for all $x, y, z \in P$:

- $x \wedge y \leq x$
- if $z \leq x$ and $z \leq y$ then $z \leq x \wedge y$

**Definition 2.7.** A *join-semilattice* is a triple $(A, \vee, \perp)$ consisting of

- A partially ordered set $A$
- A binary join $\vee : A \times A \to A$ which is commutative, associative, and idempotent: $a \vee a = a$
- A unit $\perp$ such that $a \vee \perp = a$

The collection of all join-semilattices forms a category JSL.

A *meet-semilattice* is the same as a join-semilattice, except we have as operator a meet instead of a join.

An object that is both a meet-, and a join-semilattice is called a *lattice*.

**Definition 2.8.** Let $X$ be a set. Let $\mathcal{P}_\omega(X)$ be the set of finite subsets of $X$, then $(\mathcal{P}_\omega(X), \cup, \varnothing)$ is a JSL.

**Definition 2.9.** The free join-semilattice on a set $X$ is the unique JSL $F(X)$ together with a function $\eta : X \to F(X)$ such that for all $A \in$ JSL, $f : X \to A$ there exists a unique JSL morphism $\hat{f} : F(X) \to A$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\eta"] & A \\ F(X) \arrow[ur, "\hat{f}"'] & \end{tikzcd}$$

This definition is a special case of the more general notion of a free object [nLa25].

**Lemma 2.10.** Let $X \in \mathbf{Set}$, $\mathcal{P}_\omega(X)$ is the free JSL on $X$.

*Proof.* Take $\eta(x) = \{x\}$. Let $A$ be a join-semilattice, $f : X \to A$ a function and let $\hat{f}(\{x_1, \dots, x_n\}) = f(x_1) \vee \dots \vee f(x_n)$.

Then $(\hat{f} \circ \eta)(x) = \hat{f}(\{x\}) = f(x)$. And thus the finite powerset is the unique free JSL. $\square$

**Definition 2.11.** A Heyting algebra H is a lattice together with the operation of implication $\Rightarrow: H \times H \to H$ which for all $x, y, z \in H$ satisfies:

$$x \wedge y \leq z \iff x \leq y \Rightarrow z$$

Moreover, a Heyting algebra where the *principle of excluded middle* holds, which states every proposition is either true or false, is a Boolean algebra. When writing $\vdash$ for $\leq$ you get precisely the classical notion of propositional logic.

## 2.4 Subobject Classifiers

**Definition 2.12.** Let $f, g$ be morphisms defined by

$$\begin{tikzcd} & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

Then a *pullback* is an object $P$ together with morphisms $p_1, p_2$ such that

$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

commutes and is universal. The universal property means that for any other object $Q$, with morphisms $q_1 : Q \to X$, $q_2 : Q \to Y$ such that they commute just like the above, there needs to be a morphism $\beta : Q \to P$ such that this whole diagram commutes.

$$\begin{tikzcd} Q \arrow[dr, dashed, "\beta"] \arrow[drr, bend left, "q_2"] \arrow[ddr, bend right, "q_1"'] & & \\ & P \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"'] & Z \end{tikzcd}$$

**Definition 2.13.** Let $C$ be a category with a terminal object $*$ and finite limits. A *subobject classifier* $\Omega$ is an object in $C$ such that there exists a morphism $\top : * \to \Omega$ which has the property that for all objects $A, X$ and monomorphisms $f : A \to X$ there exists a unique morphism $\chi_A : X \to \Omega$ that makes the diagram commute and be a pullback

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & * \arrow[d, "\top"] \\ X \arrow[r, "\chi_A"] & \Omega \end{tikzcd}$$

**Example 2.14.** In the category Set where objects are sets and morphisms are functions, the subobject classifier is $\Omega = \mathbb{B} := \{0, 1\}$ and if $f = \mathrm{id}$ then $\chi_A$ is the characteristic map of the subset $A \subseteq X$.

**Lemma 2.15** ([Joh02, Lemma. 1.6.6])**.** For any small category $C$, the functor category $[C, \mathrm{Set}]$ has a subobject classifier.

**Corollary 2.16.** sSet has a subobject classifier $\Omega$.

*Proof.* Given a category $C$, the presheaf category is the same as the functor category $[C^{op}, \mathrm{Set}]$. $\Delta$ is a small category (in this thesis we won't go into detail what exactly constitutes a small category), and therefore so is its dual. From 2.15 we find that $\mathrm{sSet} = [\Delta^{op}, \mathrm{Set}]$ has a subobject classifier $\Omega$. □

**Lemma 2.17** ([Joh02, Lemma. 1.6.3.ii])**.** If C is coherent, then $\Omega$ is an internal Heyting algebra.

**Corollary 2.18.** The subobject classifier $\Omega$ of sSet is an internal Heyting algebra.

## 2.5 Coalgebras

In this thesis we primarily work with $F$-coalgebras, which we will introduce next, these are a special type of coalgebra which fits our use-case very well. A general definition therefore is not necessary as the definition of $F$-coalgebras is self-contained.

**Definition 2.19.** Given a category $C$ and an endofunctor $F : C \to C$. An $F$-coalgebra $(A, \alpha)$ consists of an object $A \in C$ called the carrier and a morphism $\alpha : A \to FA$.

**Example 2.20** (Labelled transition systems)**.** Take as carrier a state space $X$, and a set of actions (or characters in the context of DFA) $A$.

Take as endofunctor

$$FX = \mathcal{P}(A \times X)$$

Then given a state $x \in X$, $\alpha(x) = \{(a_1, x_1), (a_2, x_2), \dots\}$ denotes possible transitions to states $x_1, x_2, \dots$ with corresponding outputs of $a_1, a_2, \dots$.

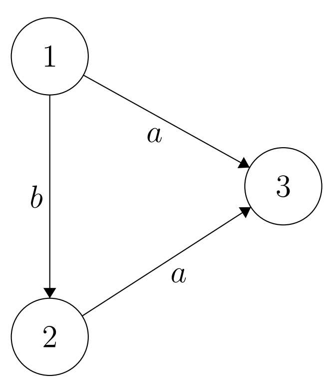


Figure 1: A labelled transition system

In figure 2.20 a labelled transition system is given where $\alpha(1) = \{(a, 3), (b, 2)\}$, $\alpha(2) = \{(a, 3)\}$, $\alpha(3) = \varnothing$.

**Definition 2.21.** A pointed $F$-coalgebra over a functor $F$ is a triple $(X, \alpha : X \to FX, x_0)$ where $X$ is the carrier as usual and $x_0$ is the *base* or in our case an *initial state.*

# 3 Ranked Hypergraphs

**Definition 3.1.** A directed *hypergraph* $(V, H)$ is a finite set of vertices $V$ and a set of *hyperarcs* $H \subseteq V \times \mathcal{P}(V)$.

*Notation:* A directed hypergraph containing no cycles is a Directed Acyclic Hypergraph (DAH).

**Definition 3.2.** A *ranked hypergraph* $(g, r, o, \mathcal{L}, A)$ consists of:

- A DAH $g$,
- Finite sequences $r = (r_i)_{i \leq |r|}$, $o = (o_i)_{i \leq |o|}$ $r_i, o_i \in \mathcal{P}(V)$ denoting the root and variable interfaces. $o_i$ contains only maximal vertices. We refer to $(|r|, |o|)$ as the *rank* of this graph.
- An action set $A$ and a hyperarc labelling function $\mathcal{L} : H \to A$

*Notation:* In this thesis we refer to ranked hypergraphs as just hypergraphs as we will only be working with this kind. $HG(n, m)$ is the set of ranked term hypergraphs of rank $(n, m)$

**Example 3.3.** In figure 2 a ranked hypergraph is drawn. Left is the root interface, of rank 3, on the right is the output interface of rank 2.

The full definition of this graph is as follows

- $V = \{1, 2, 3, 4, 5, 6\}$
- $H = \{(1, \{4\}), (2, \{5, 6\})\}$
- $r = (\{1\}, \{2\}, \{3\})$, $o = (\{4\}, \{3, 6\})$
- $A = \{a, b\}$, $\mathcal{L}((1, \{4\})) = b$, $\mathcal{L}((2, \{5, 6\})) = a$

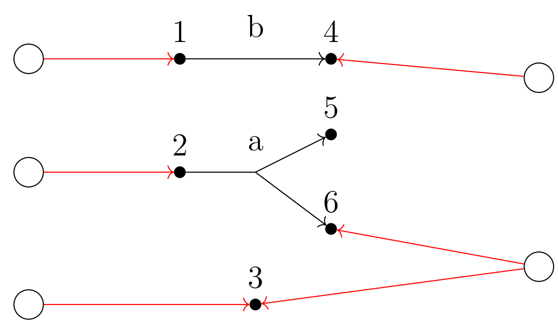


Figure 2: A Directed Acyclic Hypergraph

**Definition 3.4.** Let $G$ and $H$ be ranked hypergraphs of equal rank $(n, m)$, An *isomorphism* between $G$ and $F$ is a bijection

$$f : V^G \to V^F$$

such that

$$\begin{aligned}(u, U') \in H^G &\iff (f(u), f[U']) \in H^F \\ v \in r_i^G &\iff f(v) \in r_i^F \quad \forall i \leq n \\ v \in o_i^G &\iff f(v) \in o_i^F \quad \forall i \leq m\end{aligned}$$

**Remark 3.5.** When proving a property which involves multiple hypergraphs, we can assume disjointness of vertex sets. But then the property will only hold up to isomorphism. This is because renaming of vertices is an isomorphism in the sense of definition 3.4.

## 3.1 Composition of ranked hypergraphs

**Definition 3.6.** Let $G, F$ be hypergraphs such that $|o^G| = |r^F|$, their composition is defined as follows:

$$G; F = (g', r', i_F[o^F] := (i_F[o_i^F])_{i \leq |o^F|}, \mathcal{L}^G \sqcup \mathcal{L}^F, A^G \cup A^F) \tag{2}$$

We obtain $g' = (V, H)$ by the following procedure:

Define

$$V = (V^G + V^F) \setminus i_G[\bigcup_{i \leq |o^G|} o_i^G]$$

To get the hyperarcs we keep all elements but replace a vertex if it exists in an output, to do this neatly we define a pair of functions:

$$\psi_{G,F}(v) := \begin{cases} i_F[\bigcup_{\substack{i \leq |o^G| \\ v \in o_i^G}} r_i^F] & \text{if } \exists i \leq |o^G| \text{ such that } v \in o_i^G \\ \{i_G(v)\} & \text{else} \end{cases}$$

$$\Psi_{G,F}(U') := \bigcup_{v \in U'} \psi_{G,F}(v)$$

$$H := \left\{(i_G(u), \Psi_{G,F}(U')) : (u, U') \in H^G\right\} \cup i_F[H^F]$$

So for all $i$, in each arc $v$ that ends in a vertex in $o_i^G$, we replace that vertex in the arc with $r_i^F$.

And we obtain $r'$ by taking over the original $r^G$ and 'connecting through' for vertices which are both minimal and maximal:

$$r_i' = \Psi_{G,F}(r_i^G)$$

**Remark 3.7.** When vertex sets are disjoint, we can simply take set union as coproduct. Which makes the inclusions identity maps.

This composition allows for a left identity $id_n$ namely $id_n = ((\{1,2,\dots,n\},\varnothing),(\{i\})_{i\in n},(\{i\})_{i\in n})$.

**Lemma 3.8.** Let $G, K, F$ be DAH and $U \subseteq V^G$ a set of vertices, it holds that:

$$\Psi_{G;K,F} \circ \Psi_{G,K}(U) \cong \Psi_{G,F;K}(U)$$

*Proof.* Let $G, K, F$ be DAH and $U \subseteq V^G$ a set of vertices. Assume disjoint vertex sets.

We prove the lemma for a singleton, this then extends trivially to all subsets $U \subseteq V^G$. Let $v \in V^G$, we proceed by using the definitions:

$$\Psi_{G;K,F} \circ \psi_{G,K}(v) = \begin{cases} \{v\} & v \notin \cup_i o_i^G \\ \Psi_{G;K,F}\left(\bigcup_{\substack{j\le |r^K| \\ v\in o_j^G}} r_j^K\right) & \text{else} \end{cases}$$

Because both unions are finite we can exchange them:

$$= \begin{cases} \{v\} & v \notin \cup_i o_i^G \\ \bigcup_{\substack{j\le |r^K| \\ v\in o_j^G}} \Psi_{G;K,F}(r_j^K) & \text{else} \end{cases}$$

By definition $r_i^{K;F} := \Psi_{K,F}(r_i^K)$ and hence

$$= \begin{cases} \{v\} & v \notin \cup_i o_i^G \\ \bigcup_{\substack{j\le |r^{K;F}| \\ v\in o_j^G}} r_j^{K;F} & \text{else} \end{cases}$$
$$= \psi_{G,K;F}(v)$$

Taking unions of singletons then yields:

$$\Psi_{K,F} \circ \Psi_{G,K}(U) = \Psi_{G,K;F}(U)$$

By remark 3.5 the equality holds up to isomorphism for all $G, K, F$, regardless of vertex sets. ∎

**Theorem 3.9.** Let G,K,F be ranked hypergraphs. The following properties hold:

1. $G;(K;F) \cong (G;K);F$
2. $id_{|r^G|}; G \cong G$

*Proof.* $G;(K;F) \cong (G;K);F$
Let $G, K, F$ be ranked hypergraphs with disjoint vertex sets. Use remark 3.7. It is clear from the definition that $(G;K);F = G;(K;F)$ if and only if their graphs and root interfaces are equal.

Let $r$, $r'$ be the root interfaces for $(G;K);F$, $G;(K;F)$ respectively. We expand the definition and use lemma 3.8 to show that $r = r'$. Let $i \leq |r^G|$, we get:

$$r_i = \Psi_{r^F,o^k}(r_i^{G;K}) = \Psi_{r^F,o^k}(\Psi_{r^K,o^G}(r_i^G)) = \Psi_{r^{K;F},o^G}(r_i) = r_i'$$

Let $g = (V, H)$, $g' = (V', H')$ be the graphs for $(G;K);F$, $G;(K;F)$ respectively.

We expand the definition:

$$\begin{aligned} V &= (V^{G;K} + V^F) \setminus \bigcup_{i \leq |o^K|} o_i^K \\ &= (((V^G + V^K) \setminus \bigcup_{i \leq |o^G|} o_i^G) + V^F) \setminus \bigcup_{i \leq |o^K|} o_i^K \end{aligned}$$

From disjointness the inclusion of $V^G$, $V^K$, and $V^F$ into their coproduct we get

$$= (V^G + ((V^K + V^F) \setminus \bigcup_{i \leq |o^K|} o_i^K)) \setminus \bigcup_{i \leq |o^G|} o_i^G = V'$$

Lastly for the arcs:

$$\begin{aligned} H &= \left\{\left(u, \Psi_{r^F,o^K}(U')\right) : (u, U') \in H^{G;K}\right\} \cup H^F \\ &= \left\{\left(u, \Psi_{r^F,o^K}(\Psi_{r^K,o^G}(U'))\right) : (u, U') \in H^G\right\} \cup \left\{\left(u, \Psi_{r^F,o^K}(U')\right) : (u, U') \in H^K\right\} \cup H^F \end{aligned}$$

By lemma 3.8 we have

$$\begin{aligned} &= \left\{\left(u, \Psi_{r^{F;K},o^G}(U')\right) : (u, U') \in H^G\right\} \cup \left\{\left(u, \Psi_{r^F,o^K}(U')\right) : (u, U') \in H^K\right\} \cup H^F \\ &= \left\{\left(u, \Psi_{r^{K;F},o^G}(U')\right) : (u, U') \in H^G\right\} \cup H^{K;F} \\ &= H' \end{aligned}$$

Which proves equality for disjoint graphs. Which in turn gives $G;(K;F) \cong (G;K);F$ for any $G, K, F$ by using remark 3.5. □

*Proof.* $id_n; G \cong G$
Let $G$ be a ranked hypergraph with $|r^G| = n$ and

$$id_n = ((\{1, 2, \ldots, n\}, \varnothing), (\{i\})_{i \in n}, (\{i\})_{i \in n})$$

Assume disjoint vertex sets. Using remark 3.7 we get:

$$V = (\{1, 2, \ldots, n\} \cup V^G) \setminus \bigcup_{i \in n} \{i\} = V^G$$

$$H = \left\{(u, \Psi_{id_n;G}(U')) : (u, U') \in H^{id_n}\right\} \cup H^G = H^G$$

$$\psi_{id_n;G}(i) = r_i^G \quad \forall i \in \{1, 2, \ldots, n\}$$

Therefore

$$r'_i = \Psi_{id_n;G}(r_i^{id_n}) = \psi_{id_n;G}(i) = r_i^G$$

From 3.5 we conclude $id_n; G \cong G$ for any ranked hypergraph $G$. □

## 3.2 Generating hypergraphs

**Definition 3.10.** First define the notion of predecessors of a vertex as:

$$\text{pred}(v) = \{w \in V : \exists (w, U') \text{ s.t. } v \in U'\}$$

Then we can define the depth of a vertex recursively

$$d : V \to \mathbb{N}, \quad d(v) = \begin{cases} \max\{d(w) + 1 : w \in \text{pred}(v)\} & \text{pred}(v) \neq \varnothing \\ 0 & \textit{else} \end{cases}$$

which extends to a notion of depth for an entire ranked hypergraph:

$$D(G) = \max\{d(v) : v \in V^G\}$$

**Theorem 3.11.** The set of ranked hypergraphs is generated by the subset

$$HG_1 := \{G \in HG : D(G) \leq 1\}$$

That is, for all $G \in HG$ there exists an $n \in \mathbb{N}_{\geq 1}$ and a sequence $(G_1, G_2, \ldots, G_n) \in HG_1^n$, such that $G_1; G_2; \ldots; G_n \cong G$

*Proof.* To prove this by induction we first need the following result:
*Claim:* for any hypergraph $G$ of depth $n \geq 2$ there exist hypergraphs $G_1 \in HG$ with $D(G_1) = n - 1$, $G_2 \in HG_1$ such that $G \cong K; F$.

First take $r^1 = r$, $o^2 = o$, $\mathcal{L}^1 = \mathcal{L}^2 = \mathcal{L}$, $A^1 = A^2 = A$.

Then let

$$V^1 = \{v \in V : d(v) \leq n - 1 \text{ or } \exists w \in \text{pred}(v) \text{ s.t. } d(w) < n - 1\}$$
$$V^2 = \{v \in V : n - 1 \leq d(v) \leq n\} \cup \bigcup_{i \leq |o^G|} o_i^G$$

$$H^1 = \{(u, U') \in H : d(u) < n-1\}$$
$$H^2 = \{(u, U') \in H : d(u) = n-1\}$$

Note that for all $(u, U') \in H_1$ we have $u \in V^1 \setminus V^2$ because its degree is $< n-1$ and it isn't maximal and therefore not part of $o$. Furthermore, $(H^1, H^2)$ partitions $H$.

Finally let $o^1 = r^2$ be any enumeration of $V^1 \cap V^2$.

Let $i_1 : V^1 \to V^1 + V^2$, $i_2 : V^2 \to V^1 + V^2$ be the coproduct inclusions. We prove by construction of $G_1; G_2$ that the following function is an isomorphism

$$f : V \to (V^1 + V^2) \setminus i_1[\bigcup_{i \leq |o^1|} o^1_i], \quad f(v) = \begin{cases} i_1(v) & v \in V^1 \setminus V^2 \\ i_2(v) & \text{else} \end{cases}$$

First work out

$$\psi_{r^2, o^1}(v) := \begin{cases} i_2[\bigcup_{\substack{i \leq |o^1| \\ v \in o^1_i}} r^2_i] & \text{if } \exists i \leq |o^1| \text{ such that } v \in o^1_i \\ \{i_1(v)\} & \text{otherwise} \end{cases}$$

by $o^1 = r^2$ we get

$$\begin{aligned} &= \begin{cases} \{i_2(v)\} & \text{if } v \in V^1 \cap V^2 \\ \{i_1(v)\} & \text{otherwise} \end{cases} \\ &= \{f(v)\} \quad \forall v \in V^1 \end{aligned}$$

Therefore

$$\begin{aligned} H^{G_1;G_2} &:= \left\{(i_1(u), f[U']) : (u, U') \in H^1\right\} \cup i_2[H^2] \\ &= \left\{(f(u), f[U']) : (u, U') \in H^1\right\} \cup \{(f(u), f[U'] : (u, U') \in H^2)\} \\ &= \{(f(u), f[U'] : (u, U') \in H)\} \end{aligned}$$

and $r^{G_1;G_2} = (f[r^1_i])_{i \leq |r|} = (f[r_i])_{i \leq |r|}$, $o^{G_1;G_2} = (i_2[o^2_i])_{i \leq |o^F|} = (f[o_i])_{i \leq |o|}$

Which proves $f$ to be an isomorphism. Hence we get $G_1; G_2 \cong G$ as claimed, furthermore from the chosen $H^1$ we obtain $D(G_1) = n-1$, $D(G_2) = 1 \implies G_2 \in HG_1$.

Now, let $G \in HG$ and assume $D(G) \leq 1$. This is trivially generated by $HG_1$ as $G \in HG_1$.

Proceed by induction, assume all $G \in HG$ with $D(G) = n$ are generated by $HG_1$. Take $G \in HG$ with $D(G) = n+1$. From the just proven result there exist $G_1 \in HG$, $G_2 \in HG_1$ such that $D(G_1) = n$. Therefore there exists a sequence $(G_{11}, \ldots, G_{1n}) \in$ such that $G_{11}; \ldots; G_{1n} \cong G_1$.

Take as sequence $(G_{11}, \dots, G_{1n}, G_2)$, then $G_11; \dots; G_{1n}; G_2 \cong G_1; G_2 \cong G$. Thus all ranked hypergraphs of depth $n+1$ are generated by $HG$.

By induction the statement holds for all $n$. Therefore $HG$ is generated by $HG_1$. $\square$

**Definition 3.12.** Let $G, F$ be hypergraphs such that $|o^G| \leq |r^F|$ we extend the notion of the sequential composition given by 3.6 to asymmetric graph compositions. Write

$$G;^+ F = (g', r', o^F, \mathcal{L}^G \sqcup \mathcal{L}^F, A^G \cup A^F)$$

for the extended composition.

We obtain $g' = (V, H)$ by the following procedure:

$$V := (V^G + V^F) \setminus \bigcup_{i \leq |o^G|} o_i^G$$

$$H := \left\{ \left(U, \Psi_{r^F, o^G}(U')\right) : (U, U') \in H^G \right\} \cup H^F$$

And do the same as ; for the interfaces except for the excess ones which we just copy:

$$r'_i = \begin{cases} \Psi_{r^F, o^G}(r_i^G) & i \leq |o^G| \\ r_i^F & else \end{cases}$$

By separation of cases we trivially extend theorem 3.9 to the following result:

**Corollary 3.13.** $;^+$ is associative

# 4 Simplicial set over ranked term graphs

**Definition 4.1.** Let $V$ be the vertex set of a ranked hypergraph.
We define the monoid $\mathcal{M} = (\mathcal{P}(V)^2, (\varnothing, \varnothing), \cup \times \cup)$.

From this monoid we define a simplicial set using the nerve construction.

**Definition 4.2.** The *nerve* $N(\mathcal{M})$ of the monoid $\mathcal{M}$ is the simplicial set where:

$$N(\mathcal{M})_n = \mathcal{M}^n$$

$$d_i^{\mathcal{M}}(m_1, \ldots, m_n) = \begin{cases} (m_1, \ldots, m_i \cup \times \cup m_{i+1}, \ldots, m_n) & 0 < i < n \\ (m_2, \ldots, m_n) & i = 0 \\ (m_1, \ldots, m_{n-1}) & i = n \end{cases}$$

$$s_i^{\mathcal{M}}(m_1, \ldots, m_n) = (m_1, \ldots, m_i, (\varnothing, \varnothing), m_{i+1}, \ldots, m_n)$$

**Definition 4.3.** Define the simplicial set $\mathcal{H}$ by $\mathcal{H}_n = HG(n, n)$. With face and degeneracy maps:

$$\begin{aligned} & d_i^{\mathcal{H}}((g, (U_1, \ldots, U_n), (W_1, W_2, \ldots, W_n), \mathcal{L}, A)) \\ & \quad = \begin{cases} (g, (U_1, \ldots, U_i \cup U_{i+1}, \ldots U_n), (W_1, \ldots, W_i \cup W_{i+1}, \ldots W_n), \mathcal{L}, A) & 0 < i < n \\ (g, (U_2, \ldots, U_n), (W_1, W_2, \ldots, W_n), \mathcal{L}, A) & i = 0 \\ (g, (U_1, \ldots, U_{n-1}), (W_1, \ldots, W_{n-1}), \mathcal{L}, A) & i = n \end{cases} \\ & s_i^{\mathcal{H}}((g, (U_1, \ldots, U_n), (W_1, W_2, \ldots, W_n), \mathcal{L}, A)) \\ & \quad = \quad (g, (U_1, \ldots U_i, \varnothing, U_{i+1}, \ldots, U_n), (W_1, \ldots W_i, \varnothing, W_{i+1}, \ldots, W_n), \mathcal{L}, A) \end{aligned}$$

**Lemma 4.4.** $\mathcal{H}$ is indeed a simplicial set.

*Proof.* Let $\pi_n : \mathcal{H}_n \to N(\mathcal{M})_n$ be the projection onto the interfaces given by $\pi_n((g, r, o, \mathcal{L}, A)) = ((r_i, o_i))_{i \in [n]}$.

*Claim:* the following diagrams commute.

$$\begin{tikzcd} \mathcal{H}_n \arrow[r, "d_i^{\mathcal{H}}"] \arrow[d, "\pi_n"] & \mathcal{H}_{n-1} \arrow[d, "\pi_{n-1}"] \\ N(\mathcal{M})_n \arrow[r, "d_i^{\mathcal{M}}"] & N(\mathcal{M})_{n-1} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{H}_n \arrow[r, "s_j^{\mathcal{H}}"] \arrow[d, "\pi_n"] & \mathcal{H}_{n+1} \arrow[d, "\pi_{n+1}"] \\ N(\mathcal{M})_n \arrow[r, "s_j^{\mathcal{M}}"] & N(\mathcal{M})_{n+1} \end{tikzcd}$$

**Face maps**
$i = 0$:

$$\begin{aligned}
&\pi_{n-1}(d_0^{\mathcal{H}}((g,(U_1,\dots,U_n),(W_1,\dots,W_n),\mathcal{L},A))) \\
&= \pi_{n-1}((g,(U_2,\dots,U_n),(W_2,\dots,W_n),\mathcal{L},A)) \\
&= ((U_2,W_2),\dots,(U_n,W_n)) \\
&= d_0^{\mathcal{M}}((U_1,W_1),\dots,(U_n,W_n)) \\
&= d_0^{\mathcal{M}}(\pi_n((g,(U_1,\dots,U_n),(W_1,\dots,W_n),\mathcal{L},A)))
\end{aligned}$$

$0 < i < n$:

$$\begin{aligned}
&\pi_{n-1}(d_i^{\mathcal{H}}((g,(U_1,\dots,U_n),(W_1,\dots,W_n),\mathcal{L},A))) \\
&= \pi_{n-1}((g,(U_1,\dots,U_i\cup U_{i+1},\dots,U_n),(W_1,\dots,W_i\cup W_{i+1},\dots,W_n),\mathcal{L},A)) \\
&= ((U_1,W_1),\dots,(U_i\cup U_{i+1},W_i\cup W_{i+1}),\dots,(U_n,W_n)) \\
&= d_i^{\mathcal{M}}((U_1,W_1),\dots,(U_n,W_n)) \\
&= d_i^{\mathcal{M}}(\pi_n((g,(U_1,\dots,U_n),(W_1,\dots,W_n),\mathcal{L},A)))
\end{aligned}$$

$i = n$:

$$\begin{aligned}
&\pi_{n-1}(d_n^{\mathcal{H}}((g,(U_1,\dots,U_n),(W_1,\dots,W_n),\mathcal{L},A))) \\
&= \pi_{n-1}((g,(U_1,\dots,U_{n-1}),(W_1,\dots,W_{n-1}),\mathcal{L},A)) \\
&= ((U_1,W_1),\dots,(U_{n-1},W_{n-1})) \\
&= d_n^{\mathcal{M}}((U_1,W_1),\dots,(U_n,W_n)) \\
&= d_n^{\mathcal{M}}(\pi_n((g,(U_1,\dots,U_n),(W_1,\dots,W_n),\mathcal{L},A)))
\end{aligned}$$

**Degeneracy maps**

$$\begin{aligned}
&\pi_{n+1}(s_i^{\mathcal{H}}((g,(U_1,\dots,U_n),(W_1,\dots,W_n),\mathcal{L},A))) \\
&= \pi_{n+1}((g,(U_1,\dots,U_i,\varnothing,U_{i+1},\dots,U_n),(W_1,\dots,W_i,\varnothing,W_{i+1},\dots,W_n),\mathcal{L},A)) \\
&= ((U_1,W_1),\dots,(U_i,W_i),(\varnothing,\varnothing),(U_{i+1},W_{i+1}),\dots,(U_n,W_n)) \\
&= s_i^{\mathcal{M}}((U_1,W_1),\dots,(U_n,W_n)) \\
&= s_i^{\mathcal{M}}(\pi_n((g,(U_1,\dots,U_n),(W_1,\dots,W_n),\mathcal{L},A)))
\end{aligned}$$

Thus $\pi_n$ is a simplicial morphism. Therefore since $N(\mathcal{M})$ is a simplicial set by definition the simplicial identities also hold for $d^{\mathcal{H}}$ and $s^{\mathcal{H}}$. Therefore $\mathcal{H}$ is a simplicial set. □

# 5 Coalgebraic behaviour

To add behaviour to the hypergraph simplicial set we define pointed F-coalgebras by the endofunctor

**Definition 5.1.**

$$F : sSet \to sSet, \quad FX = \Omega \times \mathcal{P}_\omega(X \sqcup \uparrow X)^{HG_1}$$

Through this definition we find out what a coalgebra does on our set. If we follow the repeated iteration of this coalgebra

$$X \xrightarrow{\alpha} FX \xrightarrow{F\alpha} FFX \xrightarrow{FF\alpha} \dots$$

We get by definition of $F(f)$ that $\alpha$ gets recursively applied to the transitioned-to elements. This will look like

$$\begin{aligned} \alpha(x_0) &= (0, \{g_{01} \mapsto x_{01}, g_{02} \mapsto x_{02}, \dots\}) \\ \implies \alpha \circ \alpha(x_0) &= (0, \{(g_{01} \mapsto (0, \{g_{011} \mapsto x_{011}, g_{012} \mapsto x_{012}\})), \\ &\qquad\qquad (g_{02} \mapsto (0, \{g_{021} \mapsto x_{021}, g_{022} \mapsto x_{022}\})), \dots\}) \end{aligned}$$

Which is a tree structure with $x_0$ as root, $x_{01}, x_{02}, \dots$ as children etcetera. The transitions are through the hypergraphs. In figure 3 a tree is visualized for some coalgebra. Non-accepting states are red, and accepting states are indicated by a green node. Here $x_{012}$ is an accepting state.

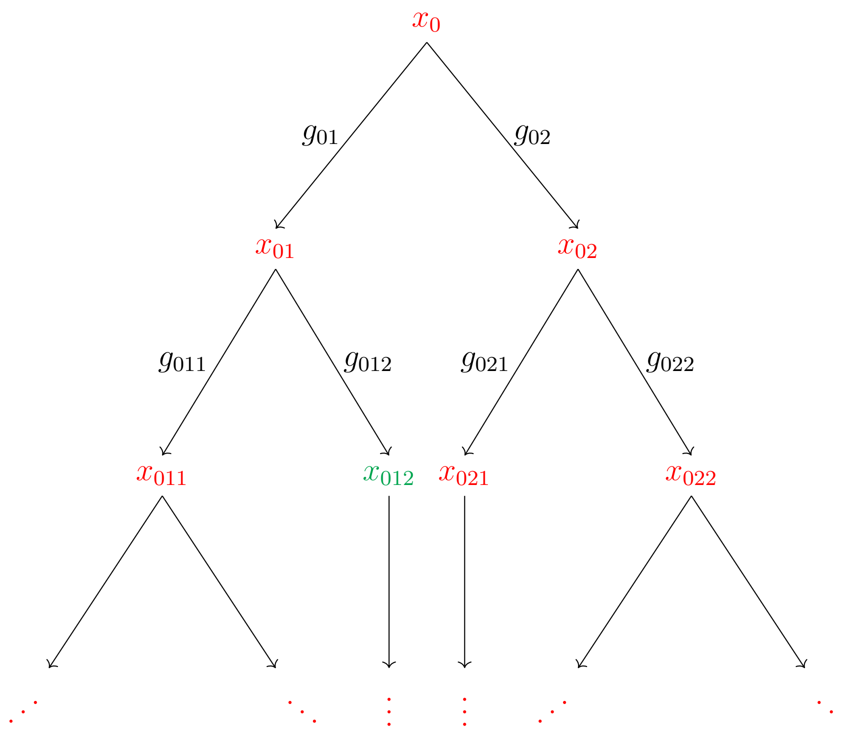


Figure 3: Tree from iteration of some $F$-coalgebra

**Definition 5.2.** Let $(A, \vee, \perp)$ be a join-semilattice. We define the JSL morphism $\bigvee : \mathcal{P}_\omega(A) \to A$ as the map making the following commute:

$$\begin{tikzcd} A \arrow[r, "\mathrm{id}_A"] \arrow[d, "\eta"] & A \\ \mathcal{P}_\omega(A) \arrow[ur, "\bigvee"'] & \end{tikzcd}$$

by lemma 2.10 $\mathcal{P}_\omega(A)$ is the free JSL, hence $\bigvee$ exists and is unique.

**Definition 5.3.** Let $ev : A \times [A, B] \to B$, $(a, f) \mapsto f(a)$ be the evaluation morphism. We extend this to evaluate finite powersets by taking

$$\mathcal{P}_\omega ev : \mathcal{P}_\omega(A) \times [A, B] \to \mathcal{P}_\omega(B)$$
$$\mathcal{P}_\omega ev(\{a_1, \ldots, a_n\}, f) = \{f(a_1), \ldots, f(a_n)\}$$

To find out if a ranked hypergraph is part of a coalgebras language we split it up into parts of depth 1 using theorem 3.11. Which are then used left to right to transition between states. If any of the final states is accepting, the graph is part of the language.

To formalise this we define a behaviour function.

**Definition 5.4.** Let $(X, c, x_0)$ be a pointed F-coalgebra of 5.1. Define the so called *behaviour* *function* as

$$\begin{aligned} \overline{c} &: \mathcal{H} \to [X, \Omega] \\ \overline{c} &= \overline{(\overline{c}_0, \overline{c}_1)} \end{aligned}$$

Where $\overline{c}$ is the inductive closure of $\overline{c}_0, \overline{c}_1$:

$$\begin{array}{ll} \overline{c}_0 : \mathbb{1} \to [X, \Omega] & \overline{c}_0 = \lambda(\pi_1 \circ c) \\ \overline{c}_1 : HG_1 \times [X, \Omega] \to [X, \Omega] & \overline{c}_1 = \lambda_2 K \end{array}$$

Where $K$ is a morphism defined by the following commuting diagram

$$\begin{tikzcd} HG_1 \times X \times [X, \Omega] \arrow[r, "K"] \arrow[d, "\widetilde{\pi_2 \circ c} \times \mathrm{id}"'] & \Omega \\ \mathcal{P}_\omega(X \sqcup \uparrow X) \times [X, \Omega] \arrow[r, "\mathcal{P}_\omega ev"] & \mathcal{P}_\omega(\Omega) \arrow[u, "\bigvee"'] \end{tikzcd}$$

**Definition 5.5.** Let $(X, c, x_0)$ be a pointed F-coalgebra of 5.1. Define the so called *behaviour function* as

$$\begin{aligned} \overline{c} &: \mathcal{H} \to \Omega^X \\ \overline{c} &= \overline{(\overline{c}_0, \overline{c}_1)} \end{aligned}$$

Where $\overline{c}$ is the inductive closure of $\overline{c}_0, \overline{c}_1$:

$$\begin{array}{ll} \overline{c}_0 : \mathbb{1} \to [X, \Omega] & \overline{c}_0 = \lambda(\pi_1 \circ c) \\ \overline{c}_1 : HG_1 \times [X, \Omega] \to [X, \Omega] & \overline{c}_1 = \lambda_2 K \end{array}$$

Where $K$ is a morphism defined by the following commuting diagram

$$\begin{tikzcd} HG_1 \times X \times [X, \Omega] \arrow[r, "K"] \arrow[d, "\widetilde{\pi_2 \circ c} \times \mathrm{id}"'] & \Omega \\ \mathcal{P}_\omega(X \sqcup \uparrow X) \times [X, \Omega] \arrow[r, "\mathcal{P}_\omega ev"] & \mathcal{P}_\omega(\Omega) \arrow[u, "\bigvee"'] \end{tikzcd}$$

# 6 Related Work

# 7 Conclusions and Further Research

## References

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[nLa25] nLab authors. free object. https://ncatlab.org/nlab/show/free+object, July 2025. Revision 12.