H.J. Sieburgh

Simplicial Coalgebras for Concurrent Regular Languages

Bachelor thesis

July 1st, 2025

Leiden University
Mathematical Institute

Abstract

This thesis introduces a construction of automata for concurrent languages. This is done by defining ranked hypergraphs, hypergraphs with interfaces that can be composed associatively. A simplicial set over these graphs is defined and we define F-coalgebras which give a nondeterministic transition model over the cells. The union of all paths of a tree resulting from a certain coalgebra gives a language of traces that support concurrent and sequential composition through the found operations on ranked hypergraphs.

Contents

1 Introduction

In this section we give an introduction to the problem addressed in this thesis.

1.1 The problem

1.2 Earlier research

1.3 Thesis overview

2 Background

2.1 Presheaves

Definition 2.1. Let C,D be categories, and $F, G : C \to D$ be functors. Then a *natural transformation* $\mu : F \Rightarrow G$ from F to G is a family of morphisms such that:

1. for all $x \in C$ there exists a morphism $\mu_x : F(x) \to G(x)$ called the component at x

2. for all $x, y \in C$ and all morphisms $f : x \to y$ the following diagram commutes:

$$
\begin{array}{ccc}
F(x) & \xrightarrow{F(f)} & F(y) \\
\downarrow{\scriptstyle \mu_x} & & \downarrow{\scriptstyle \mu_y} \\
G(x) & \xrightarrow{G(f)} & G(y)
\end{array}
\tag{1}
$$

Definition 2.2. Let C be a category, a *presheaf* is a functor

$$F : C^{op} \to \mathbf{Set}$$

The collection of all presheaves on a category forms a category with as morphisms the natural transformations between the presheaves.

2.2 Simplicial sets

Definition 2.3. The simplex category Δ has as objects

$$[n] = \{0 < 1 < \cdots < n\}$$

and as morphisms the order-preserving functions between them.

For all $n \in \mathbb{N}$, $i \in [n]$ we define the coface map $\delta_i^n : [n-1] \to [n]$ as the injective order preserving map which has $(\delta_i^n)^{-1}(i) = \varnothing$. So the map 'misses' i and all other elements of $[n]$ are 'hit'.

For all $n \in \mathbb{N}$, $i \in [n]$ we define the coface map $\sigma_i^n : [n-1] \to [n]$ as the surjective order preserving map which has $|(\sigma_i^n)^{-1}(i)| = 2$, $|(\sigma_i^n)^{-1}(j)| = 1 \ \forall j \neq i$. So the map 'hits' i twice, and all other j once.

All morphisms of Δ are generated from the coface and codegeneracy maps.

Notation: δ_i^n, σ_i^n and so on, are just referred to as δ_i, σ_i etc. As the exact n is not necessary for most applications.

Definition 2.4. A *simplicial set* is a presheaf on Δ, which means a simplicial set is a functor

$$X : \Delta^{op} \to \mathbf{Set}$$

Thus, a simplicial set X assigns to each $[n] \in \Delta$ a set $X_n = X([n])$ of n-simplices, and to each morphism $\theta \colon [m] \to [n]$, a function $X(\theta) \colon X_n \to X_m$.

In particular, we have $X(\delta_i) = d_i$ where $d_i : X_n \to X_{n-1}$ omits the i-th element from an n-simplex. And $X(\sigma_i) = s_i$ where $s_i : X_n \to X_{n+1}$ repeats the i-th element in the simplex.

These satisfy the *simplicial identities*:

$$d_i d_j = d_{j-1} d_i \qquad \qquad \text{if } i < j,$$
$$s_i s_j = s_{j+1} s_i \qquad \qquad \text{if } i \leq j,$$
$$d_i s_j = \begin{cases} s_{j-1} d_i & \text{if } i < j, \\ \text{id} & \text{if } i = j \text{ or } i = j+1, \\ s_j d_{i-1} & \text{if } i > j+1. \end{cases}$$

The presheaf category of simplicial sets is called sSet and its natural transformations the *simplicial morphisms*.

Given a collection of sets $S = S_0 \sqcup S_1 \sqcup \ldots$ and functions $d_i : S_n \to S_{n-1}$, $s_i : S_n \to S_{n+1}$ satisfying the simplicial identities there is a unique simplicial set which has the same face and degeneracy maps.

Lemma 2.5. Let $X, Y \in$ sSet, a collection of morphisms $f_0 : X_0 \to Y_0, f_1 : X_1 \to Y_1, \ldots$ is a simplicial morphism if and only if the following diagrams commute.

$$
\begin{array}{ccc}
X_n & \xrightarrow{d_i^X} & X_{n-1} \\
\downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} \\
Y_n & \xrightarrow{d_i^Y} & Y_{n-1}
\end{array}
\qquad
\begin{array}{ccc}
X & \xrightarrow{s_j^X} & X_{n+1} \\
\downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n+1}} \\
Y_n & \xrightarrow{s_j^Y} & Y_{n+1}
\end{array}
$$

Proof. Given that δ_i, σ_i generate the morphisms in Δ, diagram 1 automatically commutes for all f if it does so for δ and σ. $\qquad \square$

2.3 Subobject Classifiers

Definition 2.6. Let f, g be morphisms defined by

$$
\begin{array}{ccc}
 & & Y \\
 & & \downarrow{\scriptstyle g} \\
X & \xrightarrow{f} & Z
\end{array}
$$

Then a *pullback* is an object P together with morphisms p_1, p_2 such that

$$
\begin{array}{ccc}
P & \xrightarrow{p_2} & Y \\
\downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle g} \\
X & \xrightarrow{f} & Z
\end{array}
$$

commutes and is universal. The universal property means that for any other object Q, with morphisms $q_1 : Q \to X$, $q_2 : Q \to Y$ such that they commute just like the above, there needs to be a morphism $\beta : Q \to P$ such that this whole diagram commutes.

$$
\begin{array}{ccc}
Q & \xrightarrow{q_2} & \\
& \beta \searrow & \\
& P \xrightarrow{p_2} Y & \\
q_1 & \downarrow p_1 \quad\quad \downarrow g & \\
& X \xrightarrow{f} Z &
\end{array}
$$

Definition 2.7. Let C be a category with a terminal object $*$ and finite limits. A *subobject classifier* Ω is an object in C such that there exists a morphism $\top : * \to \Omega$ which has the property that for all objects A, X and monomorphisms $f : A \to X$ there exists a unique morphism $\chi_A : X \to \Omega$ that makes the diagram commute and be a pullback

$$
\begin{array}{ccc}
A & \longrightarrow & * \\
\downarrow f & & \downarrow \top \\
X & \xrightarrow{\chi_A} & \Omega
\end{array}
$$

Example 2.8. In the category Set where objects are sets and morphisms are functions, the subobject classifier is $\Omega = \{0, 1\}$ and if $f = \mathrm{id}$ then χ_A is the characteristic map of the subset $A \subseteq X$.

Lemma 2.9 ([Joh02, Lemma. 1.6.5])**.** For any small category C, the functor category $[C, \mathrm{Set}]$ has a subobject classifier.

Corollary 2.10. sSet has a subobject classifier Ω.

Proof. Given a category C the presheaf category is the same as the functor category $[C^{op}, \mathrm{Set}]$. Δ is a small category (in this thesis we won't go into detail what exactly constitutes a small category), and therefore so is its dual. From 2.9 we find that sSet $= [\Delta^{op}, \mathrm{Set}]$ has a subobject classifier Ω. $\square$

2.4 Coalgebras

In this thesis we primarily work with F-coalgebras, which we will introduce next, these are a special type of coalgebra which fits our usecase very well. A general definition therefore is not necessary as the definition of F-coalgebras is self-contained.

Definition 2.11. Given a category C and an endofunctor $F : C \to C$. An F-coalgebra (A, α) consists of an object $A \in C$ called the carrier and a morphism $\alpha : A \to FA$.

Example 2.12 (Labelled transition systems)**.** Take as carrier a state space X, and a set of actions (or characters in the context of DFA) A.

Take as endofunctor

$$FX = \mathcal{P}(A \times X)$$

Then given a state $x \in X$, $\alpha(x) = \{(a_1, x_1), (a_2, x_2), \dots\}$ denotes possible transitions to states $x_1, x_2, \dots$ with corresponding outputs of $a_1, a_2, \dots$.
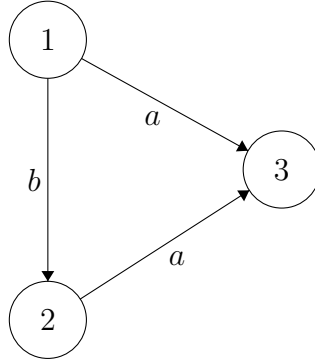

Figure 1: A labelled transition system

In figure 2.12 a labelled transition system is given where $\alpha(1) = \{(a, 3), (b, 2)\}$, $\alpha(2) = \{(a, 3)\}$, $\alpha(3) = \varnothing$.

3 Ranked Hypergraphs

Definition 3.1. A directed *hypergraph* (V, H) is a finite set of vertices V and a set of *hyperarcs* $H \subseteq V \times \mathcal{P}(V)$.

Notation: A directed hypergraph containing no cycles is a Directed Acyclic Hypergraph (DAH).

Definition 3.2. A *ranked hypergraph* $(g, r, o, \mathcal{L}, A)$ consists of:

- A DAH g,

- Finite sequences $r = (r_i)_{i \leq |r|}$, $o = (o_i)_{i \leq |o|}$ $r_i, o_i \in \mathcal{P}(V)$ denoting the root and variable interfaces. o_i contains only maximal vertices. We refer to $(|r|, |o|)$ as the *rank* of this graph.

- An action set A and a hyperarc labelling function $\mathcal{L} : H \rightarrow A$

Notation: In this thesis we refer to ranked hypergraphs as just hypergraphs as we will only be working with this kind. $HG(n, m)$ is the set of ranked term hypergraphs of rank (n, m)

A ranked hypergraph with $|r| = |o|$ is called symmetric.

Example 3.3. In figure 2 a ranked hypergraph is drawn. Left is the root interface, of rank 3, on the right is the output interface of rank 2.

The full definition of this graph is as follows

- $V = \{1, 2, 3, 4, 5, 6\}$

- $H = \{(1, \{4\}), (2, \{5, 6\})\}$

- $r = (\{1\}, \{2\}, \{3\}), o = (\{4\}, \{3, 6\})$

- $A = \{a, b\}, \mathcal{L}((\{1\}, \{4\})) = b, \mathcal{L}((\{2\}, \{5, 6\})) = a$

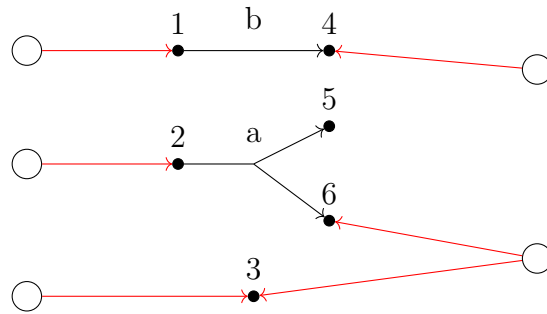Figure 2: A Directed Acyclic Hypergraph

Definition 3.4. Let G and H be ranked hypergraphs of equal rank (n, m),

An *isomorphism* between G and F is a bijection

$$f : V^G \to V^F$$

such that

$$(u, U') \in H^G \iff (f(u), f[U']) \in H^F$$
$$v \in r_i^G \iff f(v) \in r_i^F \quad \forall i \leq n$$
$$v \in o_i^G \iff f(v) \in o_i^F \quad \forall i \leq m$$

3.1 Composition of ranked hypergraphs

Definition 3.5. Let G, F be hypergraphs such that $|o^G| = |r^F|$, their composition is defined as follows:

$$G; F = (g', r', (i_F[o_i^F])_{i \leq |o^F|}, \mathcal{L}^G \sqcup \mathcal{L}^F, A^G \cup A^F) \tag{2}$$

We obtain $g' = (V, H)$ by the following procedure:

Define

$$V = (V^G + V^F) \setminus i_G[\bigcup_{i \leq |o^G|} o_i^G]$$

To get the hyperarcs we keep all elements but replace a vertex if it exists in an output, to do this neatly we define a pair of functions:

$$\psi_{r^F, o^G}(v) := \begin{cases} i_F[\bigcup_{\substack{i \leq |o^G| \\ v \in o_i^G}} r_i^F] & \text{if } \exists i \leq |o^G| \text{ such that } v \in o_i^G \\ \{i_G(v)\} & \text{otherwise} \end{cases}$$

$$\Psi_{r^F, o^G}(U') := \bigcup_{v \in U'} \psi_{r^F, o^G}(v)$$

$$H := \left\{ \big(i_G(u), \Psi_{r^F, o^G}(U')\big) : (u, U') \in H^G \right\} \cup i_F[H^F]$$

So for all i, in each arc v that ends in a vertex in o_i^G, we replace that vertex in the arc with r_i^F.

And we obtain r' by taking over the original r^G and 'connecting through' for vertices which are both minimal and maximal:

$$r_i' = \Psi_{r^F, o^G}(r_i^G)$$

This composition allows for a left identity id_n namely $id_n = ((\{1, 2, \ldots, n\}, \varnothing), (\{i\})_{i \in n}, (\{i\})_{i \in n})$.

Lemma 3.6. Let G, K, F be a DAH and $U \subseteq V^G$ a set of vertices. It holds that:

$$\Psi_{r^F, o^K} \circ \Psi_{r^K, o^G}(U) = \Psi_{r^F; r^K, o^G}(U)$$

Proof. We prove the lemma for a singleton, this then extends trivially to all subsets $U \subseteq V^G$. Let $v \in V^G$, we proceed by using the definitions:

$$\Psi_{r^F, o^K} \circ \psi_{r^K, o^G}(v) = \begin{cases} \{v\} & v \notin \cup_i o_i^G \\ \Psi_{r^F, o^K} \left(\bigcup_{\substack{i \leq |r^K| \\ v \in o_i^G}} r_i^K \right) & \text{else} \end{cases}$$

Because both unions are finite we can exchange them:

$$= \begin{cases} \{v\} & v \notin \cup_i o_i^G \\ \bigcup_{\substack{i \leq |r^K| \\ v \in o_i^G}} \Psi_{r^F, o^K}(r_i^K) & \text{else} \end{cases}$$

By definition $r_i^{K;F} := \Psi_{r^F, o^K}(r_i^K)$ and hence

$$= \begin{cases} \{v\} & v \notin \cup_i o_i^G \\ \bigcup_{\substack{i \leq |r^{K;F}| \\ v \in o_i^G}} r_i^{K;F} & \text{else} \end{cases}$$

$$= \psi_{r^F; r^K, o^G}(v)$$

Taking unions of singletons then yields:

$$\Psi_{r^F, o^K} \circ \Psi_{r^K, o^G}(U) = \Psi_{r^F; r^K, o^G}(U)$$

$\square$

Which lets us prove the following theorem:

Theorem 3.7. The sequential composition of ranked hypergraphs has the following properties:

1. ; is associative

2. $id_n; G \cong G$

Proof. Let G, K, F be ranked hypergraphs. It is clear from the definition that $(G; K); F = G; (K; F)$ if and only if their graphs and root interfaces are equal.

Let r, r' be the root interfaces for $(G; K); F$, $G; (K; F)$ respectively. We expand the definition and use lemma 3.6 to show that $r = r'$. Let $i \leq |r^G|$, we get:

$$r_i = \Psi_{r^F, o^k}(r_i^{G;K}) = \Psi_{r^F, o^k}(\Psi_{r^K, o^G}(r_i^G)) = \Psi_{r^{K;F}, o^G}(r_i) = r_i'$$

Let $g = (V, H)$, $g' = (V', H')$ be the graphs for $(G; K); F$, $G; (K; F)$ respectively. We expand the definition:

$$V = (V^{G;K} + V^F) \setminus \bigcup_{i \leq |o^K|} o_i^K$$

$$= (((V^G + V^K) \setminus \bigcup_{i \leq |o^G|} o_i^G) + V^F) \setminus \bigcup_{i \leq |o^K|} o_i^K$$

From disjointness the inclusion of V^G, V^K, and V^F into their coproduct we get

$$= (V^G + ((V^K + V^F) \setminus \bigcup_{i \leq |o^K|} o_i^K)) \setminus \bigcup_{i \leq |o^G|} o_i^G = V'$$

Lastly for the arcs:

$$H = \left\{ \left(u, \Psi_{r^F, o^K}(U')\right) : (u, U') \in H^{G;K} \right\} \cup H^F$$
$$= \left\{ \left(u, \Psi_{r^F, o^K}(\Psi_{r^K, o^G}(U'))\right) : (u, U') \in H^G \right\} \cup \left\{ \left(u, \Psi_{r^F, o^K}(U')\right) : (u, U') \in H^K \right\} \cup H^F$$

By lemma 3.6 we have

$$= \left\{ \left(u, \Psi_{r^{F;K}, o^G}(U')\right) : (u, U') \in H^G \right\} \cup \left\{ \left(u, \Psi_{r^F, o^K}(U')\right) : (u, U') \in H^K \right\} \cup H^F$$
$$= \left\{ \left(u, \Psi_{r^{K;F}, o^G}(U')\right) : (u, U') \in H^G \right\} \cup H^{K;F}$$
$$= H'$$

$\square$

3.2 Generating hypergraphs

Definition 3.8. First define the notion of predecessor of a vertex as:

$$\text{pred}(v) = \{w \in V : \exists (w, U') \text{ s.t. } v \in U'\}$$

Then we can define the depth of a vertex recursively

$$d : V \to \mathbb{N}, \quad d(v) = \begin{cases} \max\{d(w) + 1 : w \in \text{pred}(v)\} & \text{pred}(v) \neq \varnothing \\ 0 & else \end{cases}$$

which extends to a notion of depth for an entire ranked hypergraph:

$$D(G) = \max\{d(v) : v \in V^G\}$$

Theorem 3.9. The set of ranked hypergraphs is generated by the subset

$$HG_1 := \{G \in HG : D(G) \leq 1\}$$

That is, for all $G \in HG$ there exists an $n \in \mathbb{N}_{\geq 1}$ and a sequence $(G_1, G_2, \ldots, G_n) \in HG_1^n$, such that $G_1; G_2; \ldots; G_n \cong G$

Proof. To prove this by induction we first need the following result:
Claim: for any hypergraph G of depth $n \geq 2$ there exist hypergraphs $G_1 \in HG$ with $D(G_1) = n - 1$, $G_2 \in HG_1$ such that $G \cong K; F$.

First take $r^1 = r$, $o^2 = o$, $\mathcal{L}^1 = \mathcal{L}^2 = \mathcal{L}$, $A^1 = A^2 = A$.

Then let

$$V^1 = \{v \in V : d(v) \leq n - 1 \text{ or } \exists w \in \text{pred}(v) \text{ s.t. } d(w) < n - 1\}$$
$$V^2 = \{v \in V : n - 1 \leq d(v) \leq n\} \cup \bigcup_{i \leq |o^G|} o_i^G$$

$$H^1 = \{(u, U') \in H : d(u) < n - 1\}$$
$$H^2 = \{(u, U') \in H : d(u) = n - 1\}$$

Note that for all $(u, U') \in H_1$ we have $u \in V^1 \setminus V^2$ because its degree is $< n - 1$ and it isn't maximal and therefore not part of o. Furthermore, note that (H^1, H^2) partitions H.

Finally let $o^1 = r^2$ be any enumeration of $V^1 \cap V^2$.

Let $i_1 : V^1 \to V^1 + V^2$, $i_2 : V^2 \to V^1 + V^2$ be the coproduct inclusions. We prove by construction of $G_1; G_2$ that the following function is an isomorphism

$$f : V \to (V^1 + V^2) \setminus i_1[\bigcup_{i \leq |o^1|} o_i^1], \quad f(v) = \begin{cases} i_1(v) & v \in V^1 \setminus V^2 \\ i_2(v) & \text{else} \end{cases}$$

But first,

$$V^{G_1; G_2} := (V^1 + V^2) \setminus i_1[\bigcup_{i \leq |o^1|} o_i^1] = (V^1 + V^2) \setminus i_1[V^1 \cap V^2] = f[V]$$

which confirms that the function has the right domain and image.

First work out

$$\psi_{r^2, o^1}(v) := \begin{cases} i_2[\bigcup_{\substack{i \leq |o^1| \\ v \in o_i^1}} r_i^2] & \text{if } \exists i \leq |o^1| \text{ such that } v \in o_i^1 \\ \{i_1(v)\} & \text{otherwise} \end{cases}$$

by $o^1 = r^2$ we get

$$
= \begin{cases} \{i_2(v)\} & \text{if } v \in V^1 \cap V^2 \\ \{i_1(v)\} & \text{otherwise} \end{cases}
$$
$$
= \{f(v)\}
$$

Therefore

$$
H^{G_1;G_2} := \{(i_1(u), f[U']) : (u, U') \in H^1\} \cup i_2[H^2]
$$
$$
= \{(f(u), f[U']) : (u, U') \in H^1\} \cup \{(f(u), f[U'] : (u, U') \in H^2)\}
$$
$$
= \{(f(u), f[U'] : (u, U') \in H)\}
$$

and $r^{G_1;G_2} = (f[r_i^1])_{i \leq |r|} = (f[r_i])_{i \leq |r|}$, $o^{G_1;G_2} = (i_2[o_i^2])_{i \leq |o^F|} = (f[o_i])_{i \leq |o|}$

Which proves f to be an isomorphism. Hence we get $G_1; G_2 \cong G$ as claimed, furthermore from the chosen H^1 we obtain $D(G_1) = n - 1$, $D(G_2) = 1 \implies G_2 \in HG_1$.

Now, let $G \in HG$ and assume $D(G) \leq 1$. This is trivially generated by HG_1 as $G \in HG_1$.

Proceed by induction, assume all $G \in HG$ with $D(G) = n$ are generated by HG_1. Take $G \in HG$ with $D(G) = n+1$. From the just proven result there exist $G_1 \in HG$, $G_2 \in HG_1$ such that $D(G_1) = n$. Therefore there exists a sequence $(G_1 1, \ldots, G_{1n}) \in$ such that $G_1 1; \ldots; G_{1n} \cong G_1$. Take as sequence $(G_1 1, \ldots, G_{1n}, G_2)$, then $G_1 1; \ldots; G_{1n}; G_2 \cong G_1; G_2 \cong G$. Thus all ranked hypergraphs of depth $n + 1$ are generated by HG.

By induction the statement holds for all n. Therefore HG is generated by HG_1. $\square$

Definition 3.10. Let G, F be hypergraphs such that $|o^G| \leq |r^F|$ we extend the notion of the sequential composition given by 3.5 to asymmetric graph compositions. Write

$$
G;^+ F = (g', r', o^F, \mathcal{L}^G \sqcup \mathcal{L}^F, A^G \cup A^F)
$$

for the extended composition.

We obtain $g' = (V, H)$ by the following procedure:

$$
V := (V^G + V^F) \setminus \bigcup_{i \leq |o^G|} o_i^G
$$

$$
H := \{(U, \Psi_{r^F, o^G}(U')) : (U, U') \in H^G\} \cup H^F
$$

And do the same as ; for the interfaces except for the excess ones which we just copy:

$$
r_i' = \begin{cases} \Psi_{r^F, o^G}(r_i^G) & i \leq |o^G| \\ r_i^F & else \end{cases}
$$

By separation of cases we trivially extend theorem 3.7 to the following result:

Corollary 3.11. $;^+$ is associative

4 Simplicial set over ranked term graphs

Definition 4.1. Let V be the vertex set of a ranked hypergraph.
We define the monoid $\mathcal{M} = (\mathcal{P}(V)^2, (\varnothing, \varnothing), \cup \times \cup)$.

From this monoid we define a simplicial set using the nerve construction.

Definition 4.2. The *nerve* $N(\mathcal{M})$ of the monoid $\mathcal{M}$ is the simplicial set where:

$$N(\mathcal{M})_n = \mathcal{M}^n$$

$$d_i^{\mathcal{M}}(m_1, \ldots, m_n) = \begin{cases} (m_1, \ldots, m_i \cup \times \cup m_{i+1}, \ldots, m_n) & 0 < i < n \\ (m_2, \ldots, m_n) & i = 0 \\ (m_1, \ldots, m_{n-1}) & i = n \end{cases}$$

$$s_i^{\mathcal{M}}(m_1, \ldots, m_n) = (m_1, \ldots, m_i, (\varnothing, \varnothing), m_{i+1}, \ldots, m_n)$$

Definition 4.3. Define the simplicial set $\mathcal{H}$ by $\mathcal{H}_n = HG(n, n)$. And the face and degeneracy maps:

$$d_i^{\mathcal{H}}((g, (U_1, \ldots, U_n), (W_1, W_2, \ldots, W_n), \mathcal{L}, A))$$

$$= \begin{cases} (g, (U_1, \ldots, U_i \cup U_{i+1}, \ldots U_n), (W_1, \ldots, W_i \cup W_{i+1}, \ldots W_n), \mathcal{L}, A) & 0 < i < n \\ (g, (U_2, \ldots, U_n), (W_1, W_2, \ldots, W_n), \mathcal{L}, A) & i = 0 \\ (g, (U_1, \ldots, U_{n-1}), (W_1, \ldots, W_{n-1}), \mathcal{L}, A) & i = n \end{cases}$$

$$s_i^{\mathcal{H}}((g, (U_1, \ldots, U_n), (W_1, W_2, \ldots, W_n), \mathcal{L}, A))$$

$$= (g, (U_1, \ldots U_i, \varnothing, U_{i+1}, \ldots, U_n), (W_1, \ldots W_i, \varnothing, W_{i+1}, \ldots, W_n), \mathcal{L}, A)$$

Lemma 4.4. $\mathcal{H}$ is indeed a simplicial set.

Proof. Let $\pi_n : \mathcal{H}_n \to N(\mathcal{M})_n$ be the projection onto the interfaces given by $\pi_n((g, r, o, \mathcal{L}, A)) = ((r_i, o_i))_{i \in [n]}$.

Claim: the following diagrams commute.

$$
\begin{array}{ccc}
\mathcal{H}_n & \xrightarrow{d_i^{\mathcal{H}}} & \mathcal{H}_{n-1} \\
\downarrow{\scriptstyle \pi_n} & & \downarrow{\scriptstyle \pi_{n-1}} \\
N(\mathcal{M})_n & \xrightarrow{d_i^{\mathcal{M}}} & N(\mathcal{M})_{n-1}
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{H}_n & \xrightarrow{s_j^{\mathcal{H}}} & \mathcal{H}_{n+1} \\
\downarrow{\scriptstyle \pi_n} & & \downarrow{\scriptstyle \pi_{n+1}} \\
N(\mathcal{M})_n & \xrightarrow{s_j^{\mathcal{M}}} & N(\mathcal{M})_{n+1}
\end{array}
$$

Face maps
$i = 0$:

$$\pi_{n-1}(d_0^{\mathcal{H}}((g, (U_1, \ldots, U_n), (W_1, \ldots, W_n), \mathcal{L}, A)))$$
$$= \pi_{n-1}((g, (U_2, \ldots, U_n), (W_2, \ldots, W_n), \mathcal{L}, A))$$
$$= ((U_2, W_2), \ldots, (U_n, W_n))$$
$$= d_0^{\mathcal{M}}((U_1, W_1), \ldots, (U_n, W_n))$$
$$= d_0^{\mathcal{M}}(\pi_n((g, (U_1, \ldots, U_n), (W_1, \ldots, W_n), \mathcal{L}, A)))$$

$0 < i < n$:

$$\pi_{n-1}(d_i^{\mathcal{H}}((g, (U_1, \ldots, U_n), (W_1, \ldots, W_n), \mathcal{L}, A)))$$
$$= \pi_{n-1}((g, (U_1, \ldots, U_i \cup U_{i+1}, \ldots, U_n), (W_1, \ldots, W_i \cup W_{i+1}, \ldots, W_n), \mathcal{L}, A))$$
$$= ((U_1, W_1), \ldots, (U_i \cup U_{i+1}, W_i \cup W_{i+1}), \ldots, (U_n, W_n))$$
$$= d_i^{\mathcal{M}}((U_1, W_1), \ldots, (U_n, W_n))$$
$$= d_i^{\mathcal{M}}(\pi_n((g, (U_1, \ldots, U_n), (W_1, \ldots, W_n), \mathcal{L}, A)))$$

$i = n$:

$$\pi_{n-1}(d_n^{\mathcal{H}}((g, (U_1, \ldots, U_n), (W_1, \ldots, W_n), \mathcal{L}, A)))$$
$$= \pi_{n-1}((g, (U_1, \ldots, U_{n-1}), (W_1, \ldots, W_{n-1}), \mathcal{L}, A))$$
$$= ((U_1, W_1), \ldots, (U_{n-1}, W_{n-1}))$$
$$= d_n^{\mathcal{M}}((U_1, W_1), \ldots, (U_n, W_n))$$
$$= d_n^{\mathcal{M}}(\pi_n((g, (U_1, \ldots, U_n), (W_1, \ldots, W_n), \mathcal{L}, A)))$$

Degeneracy maps

$$\pi_{n+1}(s_i^{\mathcal{H}}((g, (U_1, \ldots, U_n), (W_1, \ldots, W_n), \mathcal{L}, A)))$$
$$= \pi_{n+1}((g, (U_1, \ldots, U_i, \varnothing, U_{i+1}, \ldots, U_n), (W_1, \ldots, W_i, \varnothing, W_{i+1}, \ldots, W_n), \mathcal{L}, A))$$
$$= ((U_1, W_1), \ldots, (U_i, W_i), (\varnothing, \varnothing), (U_{i+1}, W_{i+1}), \ldots, (U_n, W_n))$$
$$= s_i^{\mathcal{M}}((U_1, W_1), \ldots, (U_n, W_n))$$
$$= s_i^{\mathcal{M}}(\pi_n((g, (U_1, \ldots, U_n), (W_1, \ldots, W_n), \mathcal{L}, A)))$$

Thus π_n is a simplicial morphism. Therefore since $N(\mathcal{M})$ is a simplicial set by definition the simplicial identities also hold for $d^{\mathcal{H}}$ and $s^{\mathcal{H}}$. Therefore $\mathcal{H}$ is a simplicial set. $\square$

5 Coalgebraic behaviour

To add behaviour to the hypergraph simplicial set we define pointed F-coalgebras by the endofunctor

Definition 5.1.
$$F : sSet \to sSet, \quad FX = \Omega \times \mathcal{P}_\omega(X \sqcup \uparrow X)^{\mathcal{H}}$$

Let Δ be the simplex category, we define $(-)^{\triangleright}$ to be the functor which adds a new maximal element to a powerset. Precomposing this with the presheaf X gives $\uparrow X = X; (-)^{\triangleright}$. Thus from this construction, given an $x \in X_n$ we can only transition to elements in X_m where $m \geq n$.

We also define $F(f) = \mathcal{B} \times \mathcal{P}_\omega(f \sqcup \uparrow f)^{\mathcal{H}}$, which just applies f to all transitioned-to elements.

Definition 5.2. A pointed F-coalgebra over a functor F is a triple $(X, \alpha : X \to FX, x_0)$ where X is the *carrier set* and x_0 is the *base* or in our case an *initial state*.

Through this definition we find out what a coalgebra does on our set. If we follow the repeated iteration of this coalgebra
$$X \xrightarrow{\alpha} FX \xrightarrow{F\alpha} FFX \xrightarrow{FF\alpha} \ldots$$

We get by definition of $F(f)$ that α gets recursively applied to the transitioned-to elements. This will look like

$$\alpha(x_0) = (0, \{g_{01} \mapsto x_{01}, g_{02} \mapsto x_{02}, \ldots\})$$
$$\implies \alpha \circ \alpha(x_0) = (0, \{(g_{01} \mapsto (0, \{g_{011} \mapsto x_{011}, g_{012} \mapsto x_{012}\})),$$
$$(g_{02} \mapsto (0, \{g_{021} \mapsto x_{021}, g_{022} \mapsto x_{022}\})), \ldots\})$$

This is a tree structure with x_0 as root, $x_{01}, x_{02}, \ldots$ as children etcetera. The transitions are through the hypergraphs. Non-accepting states are red, and accepting states are indicated by a green node. In figure 3 a tree is visualized for some coalgebra. Here x_{012} is an accepting state.

Definition 5.3. Given coalgebras $(X, \alpha : X \to FX, x_0)$, $(Y, \beta : Y \to FY, y_0)$, a function $f : X \to Y$ is a homomorphism if and only if

$$F(f) \circ \alpha = \beta \circ f$$
$$f(x_0) = y_0$$

While a strict characterisation of the homomorphisms has evaded me thus far. It is known that the following property holds:

Lemma 5.4. The homomorphisms on $FX = \mathcal{H} \times \mathcal{P}_\omega(X \sqcup \uparrow X)$ preserve transitions. That is, if $\alpha(x) = (g_x, \{x_1, x_2, \ldots\})$ then $\beta(f(x)) = (g_x, \{f(x_1), f(x_2), \ldots\})$
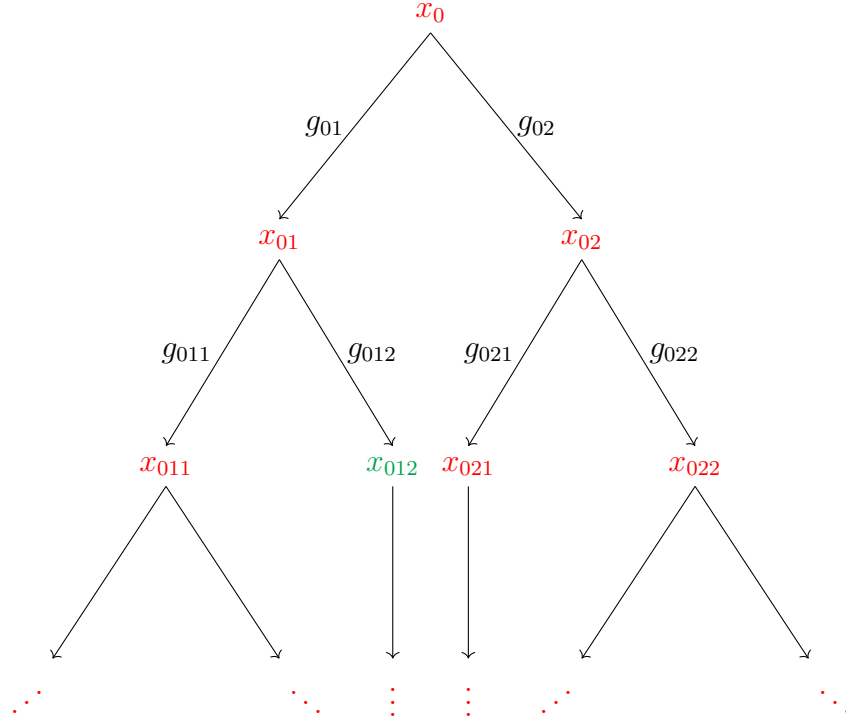
Figure 3: Tree from iteration of some F-coalgebra

6 Related Work

7 Conclusions and Further Research

References

[Joh02] Peter T. Johnstone. *Sketches of an elephant a topos theory compedium.*, volume 1. Clarendon Press, 2002.