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Simplicial Coalgebras for Concurrent Regular Languages

Bachelor thesis

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Abstract

This thesis introduces a construction of automata for concurrent languages. This is done by defining ranked hypergraphs, hypergraphs with interfaces that can be composed associatively. A simplicial set over these graphs is defined and we define F-coalgebras which give a nondeterministic transition model over the cells. The union of all paths of a tree resulting from a certain coalgebra gives a language of traces that support concurrent and sequential composition through the found operations on ranked hypergraphs.

Contents

1　Introduction

In this section we give an introduction to the problem addressed in this thesis.

1.1　The problem

1.2　Earlier research

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2 Background

2.1 Simplicial sets

Definition 2.1. *A* simplicial set *is a presheaf on the simplex category Δ, which means a simplicial set is a functor*

$$X \colon \Delta^{\mathrm{op}} \to \mathbf{Set}.$$

The category Δ has as objects

$$[n] = \{0 < 1 < \cdots < n\}$$

and as morphisms the order-preserving functions between them.

Thus, a simplicial set X assigns to each $[n] \in \Delta$ a set $X_n = X([n])$ of n-simplices, *and to each morphism $\theta \colon [m] \to [n]$, a function $X(\theta) \colon X_n \to X_m$.*

In particular, arise from:

- *The* face maps $d_i \colon X_n \to X_{n-1}$ *are given by $X(\delta^i)$, where $\delta^i \colon [n-1] \to [n]$ skips i.*

- *The* degeneracy maps $s_i \colon X_n \to X_{n+1}$ *are given by $X(\sigma^i)$, where $\sigma^i \colon [n+1] \to [n]$ repeats i.*

These satisfy the simplicial identities*:*

$$
\begin{aligned}
d_i d_j &= d_{j-1} d_i && \text{if } i < j, \\
s_i s_j &= s_{j+1} s_i && \text{if } i \leq j,
\end{aligned}
$$

$$
d_i s_j = \begin{cases}
s_{j-1} d_i & \text{if } i < j, \\
\mathrm{id} & \text{if } i = j \text{ or } i = j + 1, \\
s_j d_{i-1} & \text{if } i > j + 1.
\end{cases}
$$

Given a set $S = S_0 \sqcup S_1 \sqcup \ldots$ and functions $d_i : S_n \to S_{n-1}$, $s_i : S_n \to S_{n+1}$ satisfying the simplicial identities there is a unique simplicial set which has the same face and identity maps. This fact gives a second way to define simplicial sets, these two definitions are used interchangeably in this thesis.

Notation: The notation $[n] = \{0 < 1 < \cdots < n\}$ is used frequently in this thesis, most importantly for long winded combinatorical proofs to prevent the constant use of temporary variables and unreadable subscripts.

3 Definitions

3.1 Ranked Hypergraphs

Definition 3.1. *A* directed *hypergraph* (V, H) *is a finite set of vertices* V *and a set of hyperarcs* $H \subseteq \mathcal{P}(V)^2$.

Notation: A directed hypergraph containing no cycles is a Directed Acyclic Hypergraph (DAH).

Definition 3.2. *A* ranked term hypergraph $(g, r, o, \mathcal{L}, A)$ *consists of:*

- *A DAH* g,

- *Sequences* $r = (r_i)_{i \in [|r|]}$, $o = (o_i)_{i \in [|o|]}$ $r_i, o_i \in \mathcal{P}(V)$ *denoting the root and variable interfaces.* o_i *contains only maximal vertices. We refer to* $(|r|, |o|)$ *as the* rank *of this graph.*

- *An action set* A *and a hyperarc labelling function* $\mathcal{L} : H \to A$

Notation: In this thesis we refer to ranked term hypergraphs as just hypergraphs as we will only be working with this kind. $HG(n, m)$ is the set of ranked term hypergraphs of rank (n, m)

A ranked hypergraph with $|r| = |o|$ is called symmetric.

Example 3.1. *In figure 1 a ranked hypergraph is drawn. Left is the root interface, of rank 3, on the right is the output interface of rank 2.*

The full definition of this graph is as follows

- $V = \{1, 2, 3, 4, 5, 6\}$

- $H = \{(\{1\}, \{4\}), (\{2\}, \{5, 6\})\}$

- $r = (\{1\}, \{2\}, \{3\})$, $o = (\{4\}, \{3, 6\})$

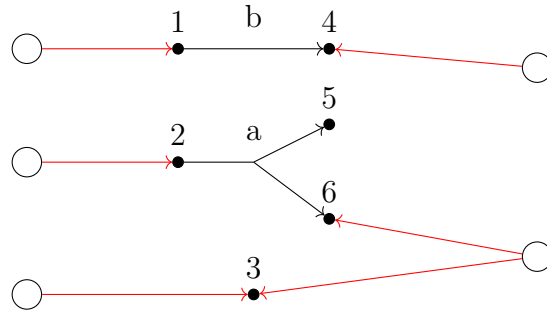- $A = \{a, b\}$, $\mathcal{L}((\{1\}, \{4\})) = b$, $\mathcal{L}((\{2\}, \{5, 6\})) = a$

Figure 1: A Directed Acyclic Hypergraph

3.2   Composition of ranked hypergraphs

Definition 3.3. *Let G, F be hypergraphs such that $|o^G| = |r^F|$, their composition is defined as follows:*

$$G \otimes F = (g', r', o^F, \mathcal{L}^G \sqcup \mathcal{L}^F, A^G \cup A^F) \tag{1}$$

We obtain $g' = (V, H)$ by the following procedure:

Define

$$V = (V^G + V^F) \setminus \bigcup_{i \in [|o^G|]} o_i^G$$

To get the hyperarcs we keep all elements but replace a vertex if it exists in an output, to do this neatly we define a pair of functions:

$$\psi_{r^F, o^G}(v) := \begin{cases} \displaystyle\bigcup_{\substack{i \in [n] \\ v \in o_i^G}} r_i^F & \text{if } \exists i \in [n] \text{ such that } v \in o_i^G \\[2em] \{v\} & \text{otherwise} \end{cases}$$

$$\Psi_{r^F, o^G}(U') := \bigcup_{v \in U'} \psi_{r^F, o^G}(v)$$

$$H := \left\{ \left(U, \Psi_{r^F, o^G}(U')\right) : (U, U') \in H^G \right\} \cup H^F$$

So for all i, in each arc v that ends in a vertex in o_i^G, we replace that vertex in the arc with r_i^F.

And we obtain r' by taking over the original r^G and 'connecting through' for vertices which are both minimal and maximal:

$$r_i' = \Psi_{r^F, o^G}(r_i^G)$$

This composition allows for an identity id_n namely $id_n = (([n], \varnothing), (\{i\})_{i \in n}, (\{i\})_{i \in n})$.

Lemma 3.1. *Let G, K, F be a DAH and $U \subseteq V^G$ a set of vertices. This identity holds:*

$$\Psi_{r^F, o^K} \circ \Psi_{r^K, o^G}(U) = \Psi_{r^F \otimes r^K, o^G}(U)$$

Proof. We prove the lemma for a singleton, this then extends to all subsets $U \subseteq V^G$. Let $v \in V^G$, we proceed by using the definitions:

$$\Psi_{r^F, o^K} \circ \psi_{r^K, o^G}(v) = \begin{cases} \{v\} & v \notin \cup_i o_i^G \\[1.5em] \Psi_{r^F, o^K}\left(\displaystyle\bigcup_{\substack{i \in [|r^K|] \\ v \in o_i^G}} r_i^K\right) & \text{else} \end{cases}$$

Because both unions are finite we can exchange them:

$$
= \begin{cases} \{v\} & v \notin \cup_i o_i^G \\ \bigcup_{\substack{i \in [|r^K|] \\ v \in o_i^G}} \Psi_{r^F, o^K}(r_i^K) & \text{else} \end{cases}
$$

By definition $r_i^{K \otimes F} := \Psi_{r^F, o^K}(r_i^K)$ and hence

$$
= \begin{cases} \{v\} & v \notin \cup_i o_i^G \\ \bigcup_{\substack{i \in [|r^{K \otimes F}|] \\ v \in o_i^G}} r_i^{K \otimes F} & \text{else} \end{cases}
$$

$$
= \psi_{r^F \otimes r^K, o^G}(v)
$$

Taking unions of singletons then yields:

$$
\Psi_{r^F, o^K} \circ \Psi_{r^K, o^G}(U) = \Psi_{r^F \otimes r^K, o^G}(U)
$$

$\square$

Which lets us prove the following theorem:

Theorem 3.1. *The sequential composition of ranked hypergraphs has the following properties:*

1. *$\otimes$ is associative*

2. *$id_n \otimes G = G = G \otimes id_m$ up to renaming of vertices*

Proof. Let G, K, F be ranked hypergraphs. It is clear from the definition that $(G \otimes K) \otimes F = G \otimes (K \otimes F)$ if and only if their graphs and root interfaces are equal.

Let r, r' be the root interfaces for $(G \otimes K) \otimes F$, $G \otimes (K \otimes F)$ respectively. We expand the definition and use lemma 3.1 to show that $r = r'$. Let $i \in [|r^G|]$, we get:

$$
r_i = \Psi_{r^F, o^k}(r_i^{G \otimes K}) = \Psi_{r^F, o^k}(\Psi_{r^K, o^G}(r_i^G)) = \Psi_{r^{K \otimes F}, o^G}(r_i) = r_i'
$$

Let $g = (V, H)$, $g' = (V', H')$ be the graphs for $(G \otimes K) \otimes F$, $G \otimes (K \otimes F)$ respectively.

We expand the definition:

$$
V = (V^{G \otimes K} + V^F) \setminus \bigcup_{i \in [|o^K|]} o_i^K
$$

$$
= (((V^G + V^K) \setminus \bigcup_{i \in [|o^G|]} o_i^G) + V^F) \setminus \bigcup_{i \in [|o^K|]} o_i^K
$$

From disjointness the inclusion of V^G, V^K, and V^F into their coproduct we get

$$= (V^G + ((V^K + V^F) \setminus \bigcup_{i \in [|o^K|]} o_i^K)) \setminus \bigcup_{i \in [|o^G|]} o_i^G = V'$$

Lastly for the arcs:

$$H = \left\{ \left(U, \Psi_{r^F, o^K}(U') \right) : (U, U') \in H^{G \otimes K} \right\} \cup H^F$$
$$= \left\{ \left(U, \Psi_{r^F, o^K}(\Psi_{r^K, o^G}(U')) \right) : (U, U') \in H^G \right\} \cup \left\{ \left(U, \Psi_{r^F, o^K}(U') \right) : (U, U') \in H^K \right\} \cup H^F$$

By lemma 3.1 we have

$$= \left\{ \left(U, \Psi_{r^{F \otimes K}, o^G}(U') \right) : (U, U') \in H^G \right\} \cup \left\{ \left(U, \Psi_{r^F, o^K}(U') \right) : (U, U') \in H^K \right\} \cup H^F$$
$$= \left\{ \left(U, \Psi_{r^{K \otimes F}, o^G}(U') \right) : (U, U') \in H^G \right\} \cup H^{K \otimes F}$$
$$= H'$$

$\square$

3.3 Simplicial set over ranked term graphs

Definition 3.4. *Let V be the vertex set of a ranked hypergraph.*
We define the monoid $\mathcal{M} = (\mathcal{P}(V)^2, (\varnothing, \varnothing), \cup \times \cup)$.

From this monoid we define a simplicial set using the nerve construction.
Definition 3.5. *The* nerve *$N(\mathcal{M})$ of the monoid $\mathcal{M}$ is the simplicial set where:*

$$N(\mathcal{M})_n = \mathcal{M}^n$$

$$d_i(m_1, \ldots, m_n) = \begin{cases} (m_1, \ldots, m_i \cup \times \cup m_{i+1}, \ldots, m_n) & 0 < i < n \\ (m_2, \ldots, m_n) & i = 0 \\ (m_1, \ldots, m_{n-1}) & i = n \end{cases}$$

$$s_i(m_1, \ldots, m_n) = (m_1, \ldots, m_i, (\varnothing, \varnothing), m_{i+1}, \ldots, m_n)$$

Definition 3.6. *Define the simplicial set $\mathcal{H}$ by $\mathcal{H}_n = HG(n, n)$. The face and degeneracy maps of $\mathcal{H}$ are defined to be the unique maps $d^{\mathcal{H}}$, $s^{\mathcal{H}}$ making the following diagrams commute:*

$$
\begin{array}{ccc}
\mathcal{H}_n & \xrightarrow{d_i^{\mathcal{H}}} & \mathcal{H}_{n-1} \\
\downarrow{\scriptstyle \pi_n} & & \downarrow{\scriptstyle \pi_{n-1}} \\
N(\mathcal{M})_n & \xrightarrow{d_i^{\mathcal{M}}} & N(\mathcal{M})_{n-1}
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{H}_n & \xrightarrow{s_j^{\mathcal{H}}} & \mathcal{H}_{n+1} \\
\downarrow{\scriptstyle \pi_n} & & \downarrow{\scriptstyle \pi_{n+1}} \\
N(\mathcal{M})_n & \xrightarrow{s_j^{\mathcal{M}}} & N(\mathcal{M})_{n+1}
\end{array}
$$

Where π_n is the projection onto the interfaces given by: $\pi_n((g, r, o, \mathcal{L}, A)) = ((r_i, o_i))_{i \in [n]}$. That is, the face and degeneracy maps of $\mathcal{H}$ are defined by the underlying monoidal nerve on the interfaces.

$\mathcal{H}$ is a simplicial set precisely because we inherit the face and degeneracy maps from $N(\mathcal{M})$:

Lemma 3.2. *$\mathcal{H}$ is indeed a simplicial set.*

Proof. π_n is a simplicial morphism by commutation of the given diagrams. Since $N(\mathcal{M})$ is a simplicial set by definition and the diagrams commute the simplicial identities also hold for $d^{\mathcal{H}}$ and $s^{\mathcal{H}}$. Therefore $\mathcal{H}$ is a simplicial set. $\qquad\square$

3.4   Coalgebraic behaviour

To add behaviour to the hypergraph simplicial set we define pointed F-coalgebras by the endofunctor

Definition 3.7.
$$F : sSet \to sSet, \quad FX = \mathcal{B} \times \mathcal{P}_\omega(X \sqcup \uparrow X)^{\mathcal{H}}$$

Here each state therefore outputs a boolean in $\mathcal{B}$ indicating if it's an accepting state, and transition maps $\mathcal{H} \to \mathcal{P}_\omega(X \sqcup \uparrow X)$. Here the $\mathcal{P}_\omega$ is the finite powerset on simplicial sets $\mathcal{P}_\omega : sSet \to sSet$ applied to $X \sqcup \uparrow X$.

Let Δ be the simplex category, we define $(-)^{\triangleright}$ to be the functor which adds a new maximal element to a powerset. Precomposing this with the presheaf X gives $\uparrow X = X \otimes (-)^{\triangleright}$. Thus from this construction, given an $x \in X_n$ we can only transition to elements in X_m where $m \geq n$.

We also define $F(f) = \mathcal{B} \times \mathcal{P}_\omega(f \sqcup \uparrow f)^{\mathcal{H}}$, which just applies f to all transitioned-to elements.

Definition 3.8. *A pointed F-coalgebra over a functor F is a triple $(X, \alpha : X \to FX, x_0)$ where X is the* carrier set *and x_0 is the* base *or in our case an* initial state.

Through this definition we find out what a coalgebra does on our set. If we follow the repeated iteration of this coalgebra
$$X \xrightarrow{\alpha} FX \xrightarrow{F\alpha} FFX \xrightarrow{FF\alpha} \dots$$

We get by definition of $F(f)$ that α gets recursively applied to the transitioned-to elements. This will look like

$$\alpha(x_0) = (0, \{g_{01} \mapsto x_{01}, g_{02} \mapsto x_{02}, \dots \})$$
$$\implies \alpha \circ \alpha(x_0) = (0, \{(g_{01} \mapsto (0, \{g_{011} \mapsto x_{011}, g_{012} \mapsto x_{012}\})),$$
$$(g_{02} \mapsto (0, \{g_{021} \mapsto x_{021}, g_{022} \mapsto x_{022}\})), \dots \})$$

This is a tree structure with x_0 as root, $x_{01}, x_{02}, \dots$ as children etcetera. The transitions are through the hypergraphs. Non-accepting states are red, and accepting states are indicated by a green node. In figure 2 a tree is visualized for some coalgebra. Here x_{012} is an accepting state.
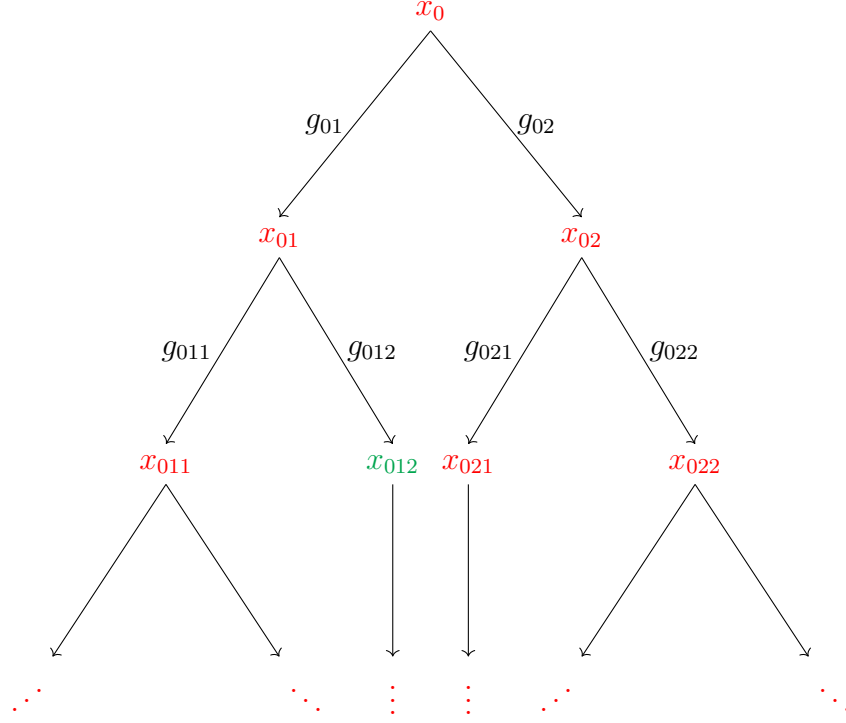
Figure 2: Tree from iteration of some F-coalgebra

Definition 3.9. *Given coalgebras* $(X, \alpha : X \to FX, x_0)$, $(Y, \beta : Y \to FY, y_0)$, *a function* $f : X \to Y$ *is a homomorphism if and only if*

$$F(f) \otimes \alpha = \beta \otimes f$$
$$f(x_0) = y_0$$

While a strict characterisation of the homomorphisms has evaded me thus far. It is known that the following property holds:

Lemma 3.3. *The homomorphisms on* $FX = \mathcal{H} \times \mathcal{P}_\omega(X \sqcup \uparrow X)$ *preserve transitions. That is, if* $\alpha(x) = (g_x, \{x_1, x_2, \dots\})$ *then* $\beta(f(x)) = (g_x, \{f(x_1), f(x_2), \dots\})$

4   Related Work

5   Conclusions and Further Research

References