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Simplicial Coalgebras for Concurrent Regular Languages

Bachelor thesis

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Abstract

This thesis introduces a construction of automata for concurrent languages. This is done by defining ranked hypergraphs, hypergraphs with interfaces that can be composed associatively. A simplicial set over these graphs is defined and we define F-coalgebras which give a nondeterministic transition model over the cells. The union of all paths of a tree resulting from a certain coalgebra gives a language of traces that support concurrent and sequential composition through the found operations on ranked hypergraphs.

Contents

1 Introduction

In this section we give an introduction to the problem addressed in this thesis.

1.1 The problem

1.2 Earlier research

1.3 Thesis overview

2 Background

2.1 Presheaves

Definition 2.1. *Let C,D be categories, and $F, G : C \to D$ be functors. Then a* natural transformation *$\mu : F \Rightarrow G$ from F to G is a family of morphisms such that:*

1. *for all $x \in C$ there exists a morphism $\mu_x : F(x) \to G(x)$ called the component at x*
2. *for all $x, y \in C$ and all morphisms $f : x \to y$ the following diagram commutes:*

$$\begin{array}{ccc} F(x) & \xrightarrow{F(f)} & F(y) \\ \downarrow{\scriptstyle \mu_x} & & \downarrow{\scriptstyle \mu_y} \\ G(x) & \xrightarrow{G(f)} & G(y) \end{array}$$

Definition 2.2. *Let C be a category, a* presheaf *is a functor*

$$F : C^{op} \to \mathbf{Set}$$

The collection of all presheaves on a category forms a category with as morphisms the natural transformations between the presheaves.

2.2 Simplicial sets

Definition 2.3. *The simplex category Δ has as objects*

$$[n] = \{0 < 1 < \cdots < n\}$$

and as morphisms the order-preserving functions between them.

For all $n \in \mathbb{N}$, $i \in [n]$ we define the coface map $\delta_i^n : [n-1] \to [n]$ as the injective order preserving map which has $(\delta_i^n)^{-1}(i) = \varnothing$. So the map 'misses' i and all other elements of $[n]$ are 'hit'.

For all $n \in \mathbb{N}$, $i \in [n]$ we define the coface map $\sigma_i^n : [n-1] \to [n]$ as the surjective order preserving map which has $|(\sigma_i^n)^{-1}(i)| = 2$, $|(\sigma_i^n)^{-1}(j)| = 1 \; \forall j \neq i$. So the map 'hits' i twice, and all other j once.

All morphisms of Δ are generated from the coface and codegeneracy maps.

Notation: δ_i^n, σ_i^n and so on, are just referred to as δ_i, σ_i etc. As the exact n is not necessary for most applications.

Definition 2.4. *A* simplicial set *is a presheaf on Δ, which means a simplicial set is a functor*

$$X \colon \Delta^{\mathrm{op}} \to \mathbf{Set}$$

Thus, a simplicial set X assigns to each $[n] \in \Delta$ a set $X_n = X([n])$ of n-simplices, *and to each morphism $\theta \colon [m] \to [n]$, a function $X(\theta) \colon X_n \to X_m$.*

In particular, we have $X(\delta_i) = d_i$ *where* $d_i : X_n \to X_{n-1}$ *omits the* i*-th element from an* n*-simplex. And* $X(\sigma_i) = s_i$ *where* $s_i : X_n \to X_{n+1}$ *repeats the* i*-th element in the simplex.*

These satisfy the simplicial identities*:*

$$
\begin{aligned}
d_i d_j &= d_{j-1} d_i && \text{if } i < j, \\
s_i s_j &= s_{j+1} s_i && \text{if } i \leq j, \\
d_i s_j &= \begin{cases} s_{j-1} d_i & \text{if } i < j, \\ \mathrm{id} & \text{if } i = j \text{ or } i = j+1, \\ s_j d_{i-1} & \text{if } i > j+1. \end{cases}
\end{aligned}
$$

Given a collection of sets $S = S_0 \sqcup S_1 \sqcup \ldots$ *and functions* $d_i : S_n \to S_{n-1}$*,* $s_i : S_n \to S_{n+1}$ *satisfying the simplicial identities there is a unique simplicial set which has the same face and identity maps. This fact gives a second way to define simplicial sets, these two definitions are used interchangeably in this thesis.*

3 Definitions

3.1 Ranked Hypergraphs

Definition 3.1. *A directed* hypergraph (V, H) *is a finite set of vertices* V *and a set of* hyperarcs $H \subseteq \mathcal{P}(V)^2$.

Notation: A directed hypergraph containing no cycles is a Directed Acyclic Hypergraph (DAH).

Definition 3.2. *A* ranked term hypergraph $(g, r, o, \mathcal{L}, A)$ *consists of:*

- *A DAH* g*,*
- *Finite sequences* $r = (r_i)_i$*,* $o = (o_i)_i$ $r_i, o_i \in \mathcal{P}(V)$ *denoting the root and variable interfaces.* o_i *contains only maximal vertices. We refer to* $(|r|, |o|)$ *as the* rank *of this graph.*
- *An action set* A *and a hyperarc labelling function* $\mathcal{L} : H \to A$

Notation: In this thesis we refer to ranked term hypergraphs as just hypergraphs as we will only be working with this kind. $HG(n, m)$ is the set of ranked term hypergraphs of rank (n, m)

A ranked hypergraph with $|r| = |o|$ is called symmetric.

Example 3.1. *In figure 1 a ranked hypergraph is drawn. Left is the root interface, of rank 3, on the right is the output interface of rank 2.*

The full definition of this graph is as follows

- $V = \{1, 2, 3, 4, 5, 6\}$
- $H = \{(\{1\}, \{4\}), (\{2\}, \{5, 6\})\}$
- $r = (\{1\}, \{2\}, \{3\})$*,* $o = (\{4\}, \{3, 6\})$
- $A = \{a, b\}$*,* $\mathcal{L}((\{1\}, \{4\})) = b$*,* $\mathcal{L}((\{2\}, \{5, 6\})) = a$

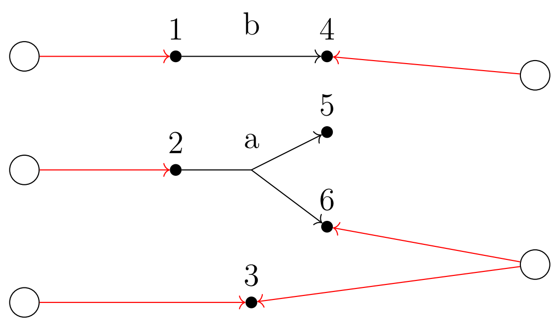

Figure 1: A Directed Acyclic Hypergraph

3.2 Composition of ranked hypergraphs

Definition 3.3. *Let G, F be hypergraphs such that $|o^G| = |r^F|$, their composition is defined as follows:*

$$G \otimes F = (g', r', o^F, \mathcal{L}^G \sqcup \mathcal{L}^F, A^G \cup A^F) \tag{1}$$

We obtain $g' = (V, H)$ by the following procedure:

Define

$$V = (V^G + V^F) \setminus \bigcup_{i \leq |o^G|} o_i^G$$

To get the hyperarcs we keep all elements but replace a vertex if it exists in an output, to do this neatly we define a pair of functions:

$$\psi_{r^F, o^G}(v) := \begin{cases} \bigcup\limits_{\substack{i \leq n \\ v \in o_i^G}} r_i^F & \textit{if } \exists i \in [n] \textit{ such that } v \in o_i^G \\ \{v\} & \textit{otherwise} \end{cases}$$

$$\Psi_{r^F, o^G}(U') := \bigcup_{v \in U'} \psi_{r^F, o^G}(v)$$

$$H := \left\{ \left(U, \Psi_{r^F, o^G}(U')\right) : (U, U') \in H^G \right\} \cup H^F$$

So for all i, in each arc v that ends in a vertex in o_i^G, we replace that vertex in the arc with r_i^F.

And we obtain r' by taking over the original r^G and 'connecting through' for vertices which are both minimal and maximal:

$$r_i' = \Psi_{r^F, o^G}(r_i^G)$$

This composition allows for a left identity id_n namely $id_n = ((\{1, 2, \ldots, n\}, \varnothing), (\{i\})_{i \in n}, (\{i\})_{i \in n})$.

Lemma 3.1. *Let G, K, F be a DAH and $U \subseteq V^G$ a set of vertices. It holds that:*

$$\Psi_{r^F, o^K} \circ \Psi_{r^K, o^G}(U) = \Psi_{r^F \otimes r^K, o^G}(U)$$

Proof. We prove the lemma for a singleton, this then extends trivially to all subsets $U \subseteq V^G$. Let $v \in V^G$, we proceed by using the definitions:

$$\Psi_{r^F, o^K} \circ \psi_{r^K, o^G}(v) = \begin{cases} \{v\} & v \notin \cup_i o_i^G \\ \Psi_{r^F, o^K} \left(\bigcup_{\substack{i \leq |r^K| \\ v \in o_i^G}} r_i^K \right) & \text{else} \end{cases}$$

Because both unions are finite we can exchange them:

$$= \begin{cases} \{v\} & v \notin \cup_i o_i^G \\ \bigcup_{\substack{i \le |r^K| \\ v \in o_i^G}} \Psi_{r^F, o^K}(r_i^K) & \text{else} \end{cases}$$

By definition $r_i^{K \otimes F} := \Psi_{r^F, o^K}(r_i^K)$ and hence

$$\begin{aligned} &= \begin{cases} \{v\} & v \notin \cup_i o_i^G \\ \bigcup_{\substack{i \le |r^{K \otimes F}| \\ v \in o_i^G}} r_i^{K \otimes F} & \text{else} \end{cases} \\ &= \psi_{r^F \otimes r^K, o^G}(v) \end{aligned}$$

Taking unions of singletons then yields:

$$\Psi_{r^F, o^K} \circ \Psi_{r^K, o^G}(U) = \Psi_{r^F \otimes r^K, o^G}(U)$$

□

Which lets us prove the following theorem:

Theorem 3.1. *The sequential composition of ranked hypergraphs has the following properties:*

1. $\otimes$ *is associative*
2. $id_n \otimes G = G$

Proof. Let G, K, F be ranked hypergraphs. It is clear from the definition that $(G \otimes K) \otimes F = G \otimes (K \otimes F)$ if and only if their graphs and root interfaces are equal.

Let r, r' be the root interfaces for $(G \otimes K) \otimes F$, $G \otimes (K \otimes F)$ respectively. We expand the definition and use lemma 3.1 to show that $r = r'$. Let $i \in [|r^G|]$, we get:

$$r_i = \Psi_{r^F, o^k}(r_i^{G \otimes K}) = \Psi_{r^F, o^k}(\Psi_{r^K, o^G}(r_i^G)) = \Psi_{r^{K \otimes F}, o^G}(r_i) = r_i'$$

Let $g = (V, H)$, $g' = (V', H')$ be the graphs for $(G \otimes K) \otimes F$, $G \otimes (K \otimes F)$ respectively.

We expand the definition:

$$\begin{aligned} V &= (V^{G \otimes K} + V^F) \setminus \bigcup_{i \in [|o^K|]} o_i^K \\ &= (((V^G + V^K) \setminus \bigcup_{i \in [|o^G|]} o_i^G) + V^F) \setminus \bigcup_{i \in [|o^K|]} o_i^K \end{aligned}$$

From disjointness the inclusion of V^G, V^K, and V^F into their coproduct we get

$$= (V^G + ((V^K + V^F) \setminus \bigcup_{i\in[|o^K|]} o_i^K)) \setminus \bigcup_{i\in[|o^G|]} o_i^G = V'$$

Lastly for the arcs:

$$\begin{aligned} H &= \left\{\left(U, \Psi_{r^F,o^K}(U')\right) : (U,U') \in H^{G\otimes K}\right\} \cup H^F \\ &= \left\{\left(U, \Psi_{r^F,o^K}(\Psi_{r^K,o^G}(U'))\right) : (U,U') \in H^G\right\} \cup \left\{\left(U, \Psi_{r^F,o^K}(U')\right) : (U,U') \in H^K\right\} \cup H^F \end{aligned}$$

By lemma 3.1 we have

$$\begin{aligned} &= \left\{\left(U, \Psi_{r^{F\otimes K},o^G}(U')\right) : (U,U') \in H^G\right\} \cup \left\{\left(U, \Psi_{r^F,o^K}(U')\right) : (U,U') \in H^K\right\} \cup H^F \\ &= \left\{\left(U, \Psi_{r^{K\otimes F},o^G}(U')\right) : (U,U') \in H^G\right\} \cup H^{K\otimes F} \\ &= H' \end{aligned}$$

□

3.3 Simplicial set over ranked term graphs

Definition 3.4. *Let V be the vertex set of a ranked hypergraph.*
We define the monoid $\mathcal{M} = (\mathcal{P}(V)^2, (\varnothing,\varnothing), \cup\times\cup)$.

From this monoid we define a simplicial set using the nerve construction.
Definition 3.5. *The* nerve *$N(\mathcal{M})$ of the monoid $\mathcal{M}$ is the simplicial set where:*

$$N(\mathcal{M})_n = \mathcal{M}^n$$

$$d_i^{\mathcal{M}}(m_1,\dots,m_n) = \begin{cases} (m_1,\dots,m_i \cup\times\cup m_{i+1},\dots,m_n) & 0<i<n \\ (m_2,\dots,m_n) & i=0 \\ (m_1,\dots,m_{n-1}) & i=n \end{cases}$$

$$s_i^{\mathcal{M}}(m_1,\dots,m_n) = (m_1,\dots,m_i,(\varnothing,\varnothing),m_{i+1},\dots,m_n)$$

Definition 3.6. *Define the simplicial set $\mathcal{H}$ by $\mathcal{H}_n = HG(n,n)$. And the face and degeneracy maps:*

$$\begin{aligned} &d_i^{\mathcal{H}}((g,(U_1,\dots,U_n),(W_1,W_2,\dots,W_n),\mathcal{L},A)) \\ &\quad= \begin{cases} (g,(U_1,\dots,U_i\cup U_{i+1},\dots U_n),(W_1,\dots,W_i\cup W_{i+1},\dots W_n),\mathcal{L},A) & 0<i<n \\ (g,(U_2,\dots,U_n),(W_1,W_2,\dots,W_n),\mathcal{L},A) & i=0 \\ (g,(U_1,\dots,U_{n-1}),(W_1,\dots,W_{n-1}),\mathcal{L},A) & i=n \end{cases} \\ &s_i^{\mathcal{H}}((g,(U_1,\dots,U_n),(W_1,W_2,\dots,W_n),\mathcal{L},A)) \\ &\quad= (g,(U_1,\dots U_i,\varnothing,U_{i+1},\dots,U_n),(W_1,\dots W_i,\varnothing,W_{i+1},\dots,W_n),\mathcal{L},A) \end{aligned}$$

Lemma 3.2. *$\mathcal{H}$ is indeed a simplicial set.*

Proof. Let $\pi_n : \mathcal{H}_n \to N(\mathcal{M})_n$ be the projection onto the interfaces given by $\pi_n((g, r, o, \mathcal{L}, A)) = ((r_i, o_i))_{i\in[n]}$.

Claim: the following diagrams commute.

$$\begin{tikzcd} \mathcal{H}_n \arrow[r, "d_i^{\mathcal{H}}"] \arrow[d, "\pi_n"] & \mathcal{H}_{n-1} \arrow[d, "\pi_{n-1}"] \\ N(\mathcal{M})_n \arrow[r, "d_i^{\mathcal{M}}"] & N(\mathcal{M})_{n-1} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{H}_n \arrow[r, "s_j^{\mathcal{H}}"] \arrow[d, "\pi_n"] & \mathcal{H}_{n+1} \arrow[d, "\pi_{n+1}"] \\ N(\mathcal{M})_n \arrow[r, "s_j^{\mathcal{M}}"] & N(\mathcal{M})_{n+1} \end{tikzcd}$$

Face maps

$i = 0$:

$$\begin{aligned}
&\pi_{n-1}(d_0^{\mathcal{H}}((g, (U_1, \ldots, U_n), (W_1, \ldots, W_n), \mathcal{L}, A))) \\
&= \pi_{n-1}((g, (U_2, \ldots, U_n), (W_2, \ldots, W_n), \mathcal{L}, A)) \\
&= ((U_2, W_2), \ldots, (U_n, W_n)) \\
&= d_0^{\mathcal{M}}((U_1, W_1), \ldots, (U_n, W_n)) \\
&= d_0^{\mathcal{M}}(\pi_n((g, (U_1, \ldots, U_n), (W_1, \ldots, W_n), \mathcal{L}, A)))
\end{aligned}$$

$0 < i < n$:

$$\begin{aligned}
&\pi_{n-1}(d_i^{\mathcal{H}}((g, (U_1, \ldots, U_n), (W_1, \ldots, W_n), \mathcal{L}, A))) \\
&= \pi_{n-1}((g, (U_1, \ldots, U_i \cup U_{i+1}, \ldots, U_n), (W_1, \ldots, W_i \cup W_{i+1}, \ldots, W_n), \mathcal{L}, A)) \\
&= ((U_1, W_1), \ldots, (U_i \cup U_{i+1}, W_i \cup W_{i+1}), \ldots, (U_n, W_n)) \\
&= d_i^{\mathcal{M}}((U_1, W_1), \ldots, (U_n, W_n)) \\
&= d_i^{\mathcal{M}}(\pi_n((g, (U_1, \ldots, U_n), (W_1, \ldots, W_n), \mathcal{L}, A)))
\end{aligned}$$

$i = n$:

$$\begin{aligned}
&\pi_{n-1}(d_n^{\mathcal{H}}((g, (U_1, \ldots, U_n), (W_1, \ldots, W_n), \mathcal{L}, A))) \\
&= \pi_{n-1}((g, (U_1, \ldots, U_{n-1}), (W_1, \ldots, W_{n-1}), \mathcal{L}, A)) \\
&= ((U_1, W_1), \ldots, (U_{n-1}, W_{n-1})) \\
&= d_n^{\mathcal{M}}((U_1, W_1), \ldots, (U_n, W_n)) \\
&= d_n^{\mathcal{M}}(\pi_n((g, (U_1, \ldots, U_n), (W_1, \ldots, W_n), \mathcal{L}, A)))
\end{aligned}$$

Degeneracy maps

$$\begin{aligned}
&\pi_{n+1}(s_i^{\mathcal{H}}((g, (U_1, \ldots, U_n), (W_1, \ldots, W_n), \mathcal{L}, A))) \\
&= \pi_{n+1}((g, (U_1, \ldots, U_i, \varnothing, U_{i+1}, \ldots, U_n), (W_1, \ldots, W_i, \varnothing, W_{i+1}, \ldots, W_n), \mathcal{L}, A)) \\
&= ((U_1, W_1), \ldots, (U_i, W_i), (\varnothing, \varnothing), (U_{i+1}, W_{i+1}), \ldots, (U_n, W_n)) \\
&= s_i^{\mathcal{M}}((U_1, W_1), \ldots, (U_n, W_n)) \\
&= s_i^{\mathcal{M}}(\pi_n((g, (U_1, \ldots, U_n), (W_1, \ldots, W_n), \mathcal{L}, A)))
\end{aligned}$$

Thus π_n is a simplicial morphism. Therefore since $N(\mathcal{M})$ is a simplicial set by definition the simplicial identities also hold for $d^{\mathcal{H}}$ and $s^{\mathcal{H}}$. Therefore $\mathcal{H}$ is a simplicial set. □

3.4 Coalgebraic behaviour

To add behaviour to the hypergraph simplicial set we define pointed F-coalgebras by the endofunctor

Definition 3.7.

$$F : sSet \to sSet, \quad FX = \mathcal{B} \times \mathcal{P}_\omega(X \sqcup \uparrow X)^{\mathcal{H}}$$

Here each state therefore outputs a boolean in $\mathcal{B}$ indicating if it's an accepting state, and transition maps $\mathcal{H} \to \mathcal{P}_\omega(X \sqcup \uparrow X)$. Here the $\mathcal{P}_\omega$ is the finite powerset on simplicial sets $\mathcal{P}_\omega : sSet \to sSet$ applied to $X \sqcup \uparrow X$.

Let Δ be the simplex category, we define $(-)^{\triangleright}$ to be the functor which adds a new maximal element to a powerset. Precomposing this with the presheaf X gives $\uparrow X = X \otimes (-)^{\triangleright}$. Thus from this construction, given an $x \in X_n$ we can only transition to elements in X_m where $m \geq n$.

We also define $F(f) = \mathcal{B} \times \mathcal{P}_\omega(f \sqcup \uparrow f)^{\mathcal{H}}$, which just applies f to all transitioned-to elements.

Definition 3.8. *A pointed F-coalgebra over a functor F is a triple $(X, \alpha : X \to FX, x_0)$ where X is the* carrier set *and x_0 is the* base *or in our case an* initial state.

Through this definition we find out what a coalgebra does on our set. If we follow the repeated iteration of this coalgebra

$$X \xrightarrow{\alpha} FX \xrightarrow{F\alpha} FFX \xrightarrow{FF\alpha} \dots$$

We get by definition of $F(f)$ that α gets recursively applied to the transitioned-to elements. This will look like

$$\begin{aligned} \alpha(x_0) &= (0, \{g_{01} \mapsto x_{01}, g_{02} \mapsto x_{02}, \dots\}) \\ \implies \alpha \circ \alpha(x_0) &= (0, \{(g_{01} \mapsto (0, \{g_{011} \mapsto x_{011}, g_{012} \mapsto x_{012}\})), \\ &\qquad (g_{02} \mapsto (0, \{g_{021} \mapsto x_{021}, g_{022} \mapsto x_{022}\})), \dots\}) \end{aligned}$$

This is a tree structure with x_0 as root, $x_{01}, x_{02}, \dots$ as children etcetera. The transitions are through the hypergraphs. Non-accepting states are red, and accepting states are indicated by a green node. In figure 2 a tree is visualized for some coalgebra. Here x_{012} is an accepting state.

Definition 3.9. *Given coalgebras $(X, \alpha : X \to FX, x_0)$, $(Y, \beta : Y \to FY, y_0)$, a function $f : X \to Y$ is a homomorphism if and only if*

$$\begin{aligned} F(f) \otimes \alpha &= \beta \otimes f \\ f(x_0) &= y_0 \end{aligned}$$

While a strict characterisation of the homomorphisms has evaded me thus far. It is known that the following property holds:

Lemma 3.3. *The homomorphisms on $FX = \mathcal{H} \times \mathcal{P}_\omega(X \sqcup \uparrow X)$ preserve transitions. That is, if $\alpha(x) = (g_x, \{x_1, x_2, \dots\})$ then $\beta(f(x)) = (g_x, \{f(x_1), f(x_2), \dots\})$*

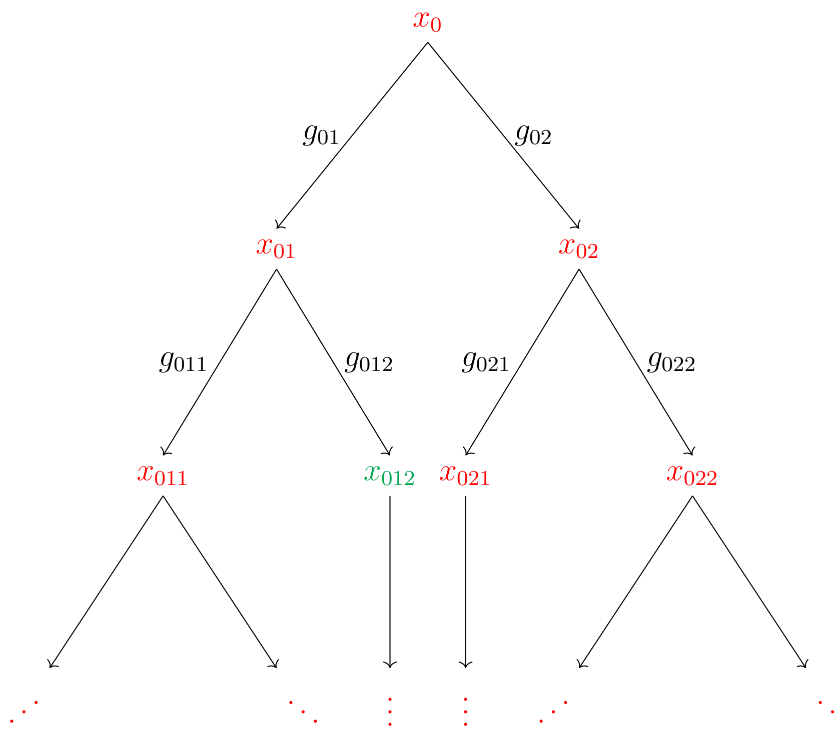

Figure 2: Tree from iteration of some F-coalgebra

4 Related Work

5 Conclusions and Further Research

References