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Opleiding Informatica

Simplicial Coalgebras

for Concurrent Regular Languages

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BACHELOR THESIS

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Abstract

This is where you write an abstract that concisely summarizes your thesis. Keep it short. No references here — exceptions do occur.

Contents

1 Introduction

In this section we give an introduction to the problem addressed in this thesis.

1.1 The situation

Sections may include subsections.

To make sure that this document renders correctly, execute these commands:

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Here, the `pdflatex` command may need to be executed three times in order to generate the table of contents and so on. Note that a good thesis has figures and tables; examples can be found in Figure 1 and Table 1. And every thesis has references, like [DT19].

Figure 1: Every thesis should have figures. Source: www.marxbrothers.org.

Column A	Column B
Point 1	Good
Point 2	Bad

Table 1: Every thesis should have tables.

Final reminder: this template is just an example, if you want you can make adjustments; also discuss with your supervisor which layout he or she likes. But the front page should be as it is now.

TODO: quite a lot!

1.2 Thesis overview

It is recommended to end the introduction with an overview of the thesis. This chapter contains the introduction; Section **??** includes the definitions; Section 4 discusses related work; Section 5 describes the experiments and their outcome; Section 6 concludes. By the way, different section titles are certainly possible.

Also, produce a nice sentence with "bachelor thesis", LIACS and the names of the supervisors.

2 Ranked Term Graphs

2.1 Definition and Structure

Definition 2.1. *A directed* hypergraph (V, H) *is a set of vertices* V *and a set of* hyperarcs $H \subseteq \mathcal{P}(V)^2$.

Notation: A directed hypergraph containing no cycles is a Directed Acyclic Hypergraph (DAH)

Definition 2.2. *A* ranked term hypergraph $(g, r, o, \mathcal{L}, A)$ *consists of:*

- *A DAH* g,
- *Sequences* $r = (r_i)_{i\in[|r|]}$, $o = (o_i)_{i\in[|o|]}$ $r_i, o_i \in \mathcal{P}(V)$ *denoting the root and variable interfaces.* o_i *contains only maximal vertices. We refer to* $(|r|, |o|)$ *as the* rank *of this graph.*
- *An action set* A *and a hyperarc labelling function* $\mathcal{L} : H \to A$

Notation: In this thesis we refer to ranked term hypergraphs as just hypergraphs as we will only be working with this kind. $HG(n, m)$ is the set of ranked term hypergraphs of rank (n, m)

A ranked hypergraph with $|r| = |o|$ is called symmetric.

2.2 Composition of ranked hypergraphs

Definition 2.3. *Let* G, F *be hypergraphs such that* $|o^G| = |r^F|$, *their composition is defined as follows:*

$$G \circ F = (g', r', o^F, \mathcal{L}^G \sqcup \mathcal{L}^F, A^G \cup A^F) \tag{1}$$

We obtain $g' = (V, H)$ *by the following procedure:*

$$V = (V^G \sqcup V^F) \setminus \bigcup_{i\in[|o^G|]} o_i^G$$

$$\begin{aligned} H = ((H^G \sqcup H^F) \setminus \bigcup_{i\in[|o^G|]} \{(U, U') \in H^G : o_i^G \cap U' \neq \varnothing\}) \\ \cup \bigcup_{i\in[|o^G|]} \{(U, (U' \setminus o_i^G) \cup r_i^F) : (U, U') \in H^G, o_i^G \cap U' \neq \varnothing\} \end{aligned}$$

So for all i, *in each arc* v *that ends in a vertex in* o_i^G, *we replace that vertex in the arc with* r_i^F.

And we obtain r' *by taking over the original* r^G *and 'connecting through' for vertices which are both minimal and maximal:*

$$r_i' = \{v \in r_i^G : v \textit{ is not maximal}\} \cup \{v \in r_j^F : r_i^G \cap o_j^G \neq \varnothing, j \in [|r^F|]\}$$

This composition allows for an identity id_n namely $id_n = (([n], \varnothing), (\{i\})_{i\in n}, (\{i\})_{i\in n})$.

Lemma 2.1. *The sequential composition of ranked hypergraphs has the following properties:*

1. *$\circ$ is associative*
2. *$id_n \circ G = G = G \circ id_m$ up to renaming of vertices*

Proof. Let G, K, F be ranked hypergraphs. It is clear from the definition that $(G\circ K)\circ F = G\circ(K\circ F)$ if and only if their graphs and root interfaces are equal.

Let r, r' be the root interfaces for $(G \circ K) \circ F$, $G \circ (K \circ F)$ respectively. We expand the definition to find equality.

$$\begin{gathered} r_i = \{v \in r_i^{G\circ K} : v \text{ is not maximal in } G \circ K\} \\ \cup\{v \in r_j^F : r_i^{G\circ K} \cap o_j^K \neq \varnothing, j \in [|r^F|]\} \end{gathered}$$

Expand the first term

$$\begin{gathered} \{v \in r_i^{G\circ K} : v \text{ is not maximal in } G \circ K\} = \{v \in r_i^G : v \text{ is not maximal in } G\} \\ \cup\{v \in r_j^K : r_i^G \cap o_j^G \neq \varnothing, v \text{ is not maximal in } G \circ K\} \end{gathered}$$

In the second set: a vertex in r_j^K is not maximal in $G \circ K$ iff it is not maximal in K

$$\begin{gathered} = \{v \in r_i^G : v \text{ is not maximal in } G\} \\ \cup\{v \in r_j^K : r_i^G \cap o_j^G \neq \varnothing, v \text{ is not maximal in } K\} \end{gathered}$$

Expanding the second term of r_i we get, knowing $r_i^{G\circ K} \cap o_j^K \subseteq g^K$, that we had to connect through from r_i^G to r_j^K for some j:

$$\begin{gathered} \{v \in r_j^F : r_i^{G\circ K} \cap o_j^K \neq \varnothing, j \in [|r^F|]\} \\ = \{v \in r_k^F : r_i^G \cap o_j^G \neq \varnothing, r_j^K \cap o_k^K \neq \varnothing, j \in [|r^K|], k \in [|r^F|]\} \end{gathered}$$

and finally therefore we find

$$\begin{gathered} r_i = \{v \in r_i^G : v \text{ is not maximal in } G\} \\ \cup\{v \in r_j^K : r_i^G \cap o_j^G \neq \varnothing, v \text{ is not maximal in } K, j \in [|r^F|]\} \\ \cup\{v \in r_k^F : r_i^G \cap o_j^G \neq \varnothing, r_j^K \cap o_k^K \neq \varnothing, j \in [|r^K|], k \in [|r^F|]\} \end{gathered}$$

We do this same rewrite starting from the other graph:

$$\begin{gathered} r'_i = \{v \in r_i^G : v \text{ is not maximal in } G\} \\ \cup\{v \in r_j^{K\circ F} : r_i^G \cap o_j^G \neq \varnothing, j \in [|r^{K\circ F}|]\} \\ = \{v \in r_i^G : v \text{ is not maximal in } G\} \\ \cup\{v \in r_j^K : r_i^G \cap o_j^G \neq \varnothing, v \text{ is not maximal in } K, j \in [|r^K|]\} \\ \cup\{v \in r_k^F : r_i^G \cap o_j^G \neq \varnothing, r_j^K \cap o_k^K \neq \varnothing, j \in [|r^K|], k \in [|r^F|]\} \end{gathered}$$

Which finally shows

$$r_i = r'_i \quad \forall i \in [|r^G|]$$

Let $g = (V, H)$, $g' = (V', H')$ be the graphs for $(G \circ H) \circ F$, $G \circ (H \circ F)$ respectively.

We expand the definition:

$$V = (V^{G \circ H} \sqcup) \setminus \bigcup_{i \in [|o^H|]} o_i^H$$
$$= (((V^G \sqcup V^H) \setminus \bigcup_{[|o^G|]} o_i^G) \sqcup V^F) \setminus \bigcup_{i \in [|o^H|]} o_i^H$$

From disjointness of V^G, V^H, and V^F we get

$$= (V^G \sqcup ((V^H \sqcup V^F) \setminus \bigcup_{[|o^H|]} o_i^H)) \setminus \bigcup_{i \in [|o^G|]} o_i^G = V'$$

Lastly for the arcs:

$$H = ((H^{G \circ H} \sqcup H^F) \setminus \bigcup_{i \in [|o^H|]} \{(U, U') \in H^{G \circ H} o_i^H \cap U' \neq \varnothing\})$$
$$\cup \bigcup_{i \in [|o^H|]} \{(U, (U' \setminus o_i^H) \cup r_i^F) : (U, U') \in H^{G \circ H}, o_i^H \cap U' \neq \varnothing\}$$
$$= ((((H^G \sqcup H^F) \setminus \bigcup_{i \in [|o^G|]} \{(U, U') \in H^G : o_i^G \cap U' \neq \varnothing\})$$
$$\cup \bigcup_{i \in [|o^G|]} \{(U, (U' \setminus o_i^G) \cup r_i^F) : (U, U') \in H^G, o_i^G \cap U' \neq \varnothing\}) \sqcup H^F) \setminus \bigcup_{i \in [|o^H|]} \{(U, U') \in H^{G \circ H} o_i^H \cap U' \neq \varnothing\})$$
$$\cup \bigcup_{i \in [|o^H|]} \{(U, (U' \setminus o_i^H) \cup r_i^F) : (U, U') \in H^{G \circ H}, o_i^H \cap U' \neq \varnothing\}$$

□

3 Simplicial set over ranked term graphs

3.1 Monoidal structure on interfaces

Definition 3.1. *Let V be the vertex set of a ranked hypergraph.*
We define the monoid $\mathcal{M} = (\mathcal{P}(V)^2, (\varnothing, \varnothing), \cup \times \cup)$.

From this monoid we define a simplicial set using the nerve construction.

Definition 3.2. *The* nerve $N(\mathcal{M})$ *of the monoid* $\mathcal{M}$ *is the simplicial set where:*

$$N(\mathcal{M})_n = \mathcal{M}^n$$

$$d_i(m_1, \ldots, m_n) = \begin{cases} (m_1, \ldots, m_i \cup \times \cup m_{i+1}, \ldots, m_n) 0 < i < n \\ (m_2, \ldots, m_n) i = 0 \\ (m_1, \ldots, m_{n-1}) i = n \end{cases}$$

$$s_i(m_1, \ldots, m_n) = (m_1, \ldots, m_i, (\varnothing, \varnothing), m_{i+1}, \ldots, m_n)$$

Definition 3.3. *Define the simplicial set* $\mathcal{H}$ *by* $\mathcal{H}_n = HG(n, n)$*. The face and degeneracy maps of* $\mathcal{H}$ *are defined to be the unique maps* $d^{\mathcal{H}}$, $s^{\mathcal{H}}$ *making the following diagrams commute:*

$$\begin{tikzcd} \mathcal{H}_n \arrow[r, "d_i^{\mathcal{H}}"] \arrow[d, "\pi_n"] & \mathcal{H}_{n-1} \arrow[d, "\pi_{n-1}"] \\ N(\mathcal{M})_n \arrow[r, "d_i^{\mathcal{M}}"] & N(\mathcal{M})_{n-1} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{H}_n \arrow[r, "s_j^{\mathcal{H}}"] \arrow[d, "\pi_n"] & \mathcal{H}_{n+1} \arrow[d, "\pi_{n+1}"] \\ N(\mathcal{M})_n \arrow[r, "s_j^{\mathcal{M}}"] & N(\mathcal{M})_{n+1} \end{tikzcd}$$

Where π_n *is the projection onto the interfaces given by:* $\pi_n((g, r, o, \mathcal{L}, A)) = ((r_i, o_i))_{i \in [n]}$*. That is, the face and degeneracy maps of* $\mathcal{H}$ *are defined by the underlying monoidal nerve on the interfaces.*

$\mathcal{H}$ is a simplicial set precisely because we inherit the face and degeneracy maps from $N(\mathcal{M})$:

Lemma 3.1. $\mathcal{H}$ *is indeed a simplicial set.*

Proof. π_n is a simplicial morphism by commutation of the given diagrams. Since $N(\mathcal{M})$ is a simplicial set by definition and the diagrams commute the simplicial identities also hold for $d^{\mathcal{H}}$ and $s^{\mathcal{H}}$. Therefore $\mathcal{H}$ is a simplicial set. □

4 Related Work

5 Experiments

6 Conclusions and Further Research

References

[DT19] B. Dylan and D. Trump. How to write a good thesis in three months. *International Journal of Computer Science*, 42:123–456, 2019.