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Universiteit Leiden
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Opleiding Informatica

Simplicial Coalgebras

for Concurrent Regular Languages

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BACHELOR THESIS

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Abstract

This is where you write an abstract that concisely summarizes your thesis. Keep it short. No references here — exceptions do occur.

Contents

1 Introduction

In this section we give an introduction to the problem addressed in this thesis.

1.1 The situation

Sections may include subsections.

To make sure that this document renders correctly, execute these commands:

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pdflatex thesis
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Here, the `pdflatex` command may need to be executed three times in order to generate the table of contents and so on. Note that a good thesis has figures and tables; examples can be found in Figure 1 and Table 1. And every thesis has references, like [DT19].

Figure 1: Every thesis should have figures. Source: `www.marxbrothers.org`.

Column A	Column B
Point 1	Good
Point 2	Bad

Table 1: Every thesis should have tables.

Final reminder: this template is just an example, if you want you can make adjustments; also discuss with your supervisor which layout he or she likes. But the front page should be as it is now.

TODO: quite a lot!

1.2 Thesis overview

It is recommended to end the introduction with an overview of the thesis. This chapter contains the introduction; Section **??** includes the definitions; Section 4 discusses related work; Section 5 describes the experiments and their outcome; Section 6 concludes. By the way, different section titles are certainly possible.

Also, produce a nice sentence with “bachelor thesis”, LIACS and the names of the supervisors.

2 Ranked Term Graphs

2.1 Definition and Structure

Definition 1. *Ranked Term Graph A* ranked term graph *consists of:*

- *A directed acyclic graph (DAG) g,*
- *A* rank $(d_r, d_v) \in \mathbb{N}^2$*, denoting the size of the root and variable interfaces,*
- *Let $[n] \coloneqq \{1, 2, \dots, n\}$. Maps $r : [d_r] \to \mathcal{P}(g)$ and $v : [d_v] \to \mathcal{P}(g)$, where $r([d_r])$ consists of minimal vertices of g and $v([d_v])$ consists of maximal vertices called the* root *and* variable *maps.*

2.2 Monoidal Structure on Ranked Term Graphs

Definition 2. *Let $G = (g^G, (d_r^G, d_v^G), r^G, v^G)$, $F = (g^F, (d_r^F, d_v^F), r^F, v^F)$ be ranked term graphs. We define a monoidal product functor as:*

$$G \oplus F = \left(g^G \sqcup g^F, (d_r^G + d_r^F, d_v^G + d_v^F), r', v'\right)$$

Where r', v' are the concatenations of r^G and r^F, v^G and v^F respectively:

$$r'(k) = \begin{cases} r^G(k) & k \leq d_r^G \\ r^F(k - d_r^G) & d_r^G < k \leq d_r^G + d_r^F \end{cases}$$

$$v'(k) = \begin{cases} v^G(k) & k \leq d_v^G \\ v^F(k - d_v^G) & d_v^G < k \leq d_v^G + d_v^F \end{cases}$$

2.3 Composition of Morphisms

Definition 3. *Let $G = (g^G, (d_r^G, d_v^G), r^G, v^G)$, $F = (g^F, (d_r^F, d_v^F), r^F, v^F)$ be ranked term graphs such that $d_v^G = d_r^F$, their composition is defined as follows:*

$$G \circ F = (H, (d_r^G, d_v^F), r^G, v^F)$$

Where $H = g^G \sqcup g^F \sqcup \{(x, x') \in g^G \times g^F | \exists i \in [d_v^G]. x \in v^G(i) \wedge x' \in r^F(i)\}$ which is equal to the union of the two graphs with added arrows going from roots to variables that are equally numbered.

A category with ranked term graphs as morphisms, $\mathbb{N}$ as the set of objects, the disjoint union as monoidal product, and the defined composition of morphisms gives a monoidal category.

3 Simplicial Structure on Ranked Term Graphs

3.1 Parallel Composition and Identity Morphisms

Definition 4. *Let G, F be ranked term graphs of equal rank (d_r, d_v). Then:*

$$G \otimes F = (g^G \sqcup g^F, (d_r, d_v), r^G \sqcup r^F, v^G \sqcup v^F)$$

This allows for an identity element for our morphisms:

$$id_n = (\varnothing, (n, n), r, v)$$

where $r(k) = \varnothing = v(k) \forall k \in [n]$.

Let G be a ranked graph of rank (n, n). Clearly, we have:

$$id_n \otimes G = G \otimes id_n = G.$$

3.2 Face Maps: Identifying Consecutive Interfaces

Definition 5. *Let G be a ranked term graph of rank (n, n), define p_m to be the identification of the m-th and $(m+1)$-th interface points, for $m \in \{0\} \cup [n]$:*

$$p_m(G) = (g, (n-1, n-1), r', v').$$

Where we have:

$$r'(k) = \begin{cases} r(k) & k < m \\ r(m) \cup r(m+1) & k = m \\ r(k+1) & m < k < n \end{cases}$$

$$v'(k) = \begin{cases} v(k) & k < m \\ v(m) \cup v(m+1) & k = m \\ v(k+1) & m < k \leq n-1 \end{cases}$$

3.3 Degeneracy Maps: Adding Extra Interfaces

Definition 6. *Let G be a ranked graph, define a_m to be the addition of an extra interface point at position m:*

$$a_m(G) = (g, (n+1, n+1), r', v').$$

Where we have:

$$r'(k) = \begin{cases} r(k) & k \leq m \\ \varnothing & k = m+1 \\ r(k-1) & m+1 < k \leq n+1 \end{cases}$$

$$v'(k) = \begin{cases} v(k) & k < m \\ v(m) \cup v(m+1) & k = m \\ v(k+1) & m < k < n \end{cases}$$

Clearly, $p_m \circ a_m(G) = G$.

3.4 Simplicial Set of Ranked Term Graphs

Let $\mathcal{L}_n$ be the set of ranked term graphs of rank (n, n) and $\mathcal{L} = \bigcup_{n\in\mathbb{N}} \mathcal{L}_n$. We define a simplicial set $N\mathcal{L}$ as follows:

Definition 7. ***Simplicial Set of Ranked Term Graphs***

- $(N\mathcal{L})_n = \mathcal{L}_n^n$
- $d_n^0(G_1, G_2, \ldots G_n) = (p_0(G_2), \ldots p_0(G_n))$
- $d_n^k(G_1, G_2, \ldots G_n) = (p_k(G_1), \ldots p_k(G_k \otimes G_{k+1}), \ldots p_k(G_n))$
- $s_n^k(G_1, \ldots, G_n) = (a_k(G_1), \ldots, a_k(G_k), id_{n+1}, a_k(G_{k+1}), \ldots a_k(G_n))$

This setup adheres to the *simplicial identities*

1. $d_n^i \circ d_n^j = d_n^{j-1} \circ d_n^i$ if $i < j$
2. $d_n^i \circ s_n^j = s_n^{j-1} \circ d_n^i$ if $i < j$
3. $d_n^i \circ s_n^j = id$ if $i \in \{j, j+1\}$
4. $d_n^i \circ s_n^j = s_n^j \circ d_n^{i-1}$ if $i > j + 1$
5. $s_n^i \circ s_n^j = s_n^{j+1} \circ s_n^i$ if $i \leq j$

Proof. Let $s = (G_1, \ldots, G_n) \in (N\mathcal{L})_n$ we check each identity:

1. Let $i < j$, $i \neq 0$ then

   $$(d_n^i \circ d_n^j)(s) = (p_i \circ p_j(G_1), \ldots p_i \circ p_j(G_i \otimes G_{i+1}), \ldots, p_i \circ p_j(G_j \otimes G_{j+1}), \ldots p_i \circ p_j(G_n))$$

   and

   $$(d_n^{j-1} \circ d_n^i)(s) = (p_{j-1} \circ p_i(G_1), \ldots p_{j-1} \circ p_i(G_i \otimes G_{i+1}), \ldots, p_{j-1} \circ p_i(G_j \otimes G_{j+1}), \ldots p_{j-1} \circ p_i(G_n))$$

   are equal iff $p_i \circ p_j = p_{j-1} \circ p_i$ for $i < j$

□

Interfaces are simplicial sets?? Look at the calculation above. When written out all identities will come to this conclusion: $N\mathcal{L}$ is a simplicial set iff p, a adhere to the simplicial identities.

In light of this I though of the monoid:

- Let A be a countable carrier set
- $\mathcal{M} = (\mathcal{P}(A)^2, (\varnothing, \varnothing), \cup)$

This is a monoid and a single element (r, v) could represent root and target sets. Take the nerve with the standard nerve construction for monoids:

- $(N\mathcal{M})_n = \mathcal{M}^n$
- $d_0(U_1, \ldots, U_n) = (U_2, \ldots, U_n)$
- $d_k(U_1, \ldots, U_n) = (U_1, \ldots, U_k \cup U_{k+1}, \ldots, U_n)$
- $s_k(U_1, \ldots, U_n) = (U_1, \ldots, U_k, (\varnothing, \varnothing), U_{k+1}, \ldots, U_n)$

Which is known to be a simplicial set. And if you look at the definitions of the interface and p, a maps it is exactly what we have been doing. It also induces an ordering on the interface which we had to do by hand earlier.

4 Related Work

5 Experiments

6 Conclusions and Further Research

References

[DT19] B. Dylan and D. Trump. How to write a good thesis in three months. *International Journal of Computer Science*, 42:123–456, 2019.