A Machine Learning Approach to Inverse Source Problems

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2019/2020

Outline

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- Machine Learning Section
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- Ellipsoid Radii
- Voxels
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Motivation

- Solving the Inverse Scattering Problem is a non-linear and ill-posed problem.
- Mathematical Techniques then have to be careful when solving to avoid blowing up a solution - causing slow solving.
- Inverse Problems are predicting parameters from data - machine learning performs very well at.
- Train a ML model on lots of data, that would be very quick to evaluate and solve even if training is long.
- Possibly gain some data-led insight into properties of the inverse source problem.

Theoretical Background

Helmholtz Equation

$$\triangle u + k^2 u = 0, \text{ in } \mathbb{R}^3, \tag{1}$$

$$u(x) = e^{ikx\cdot d} + u^s(x), \tag{2}$$

$$u = 0 \text{ on } \partial D, \tag{3}$$

$$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0, \tag{4}$$

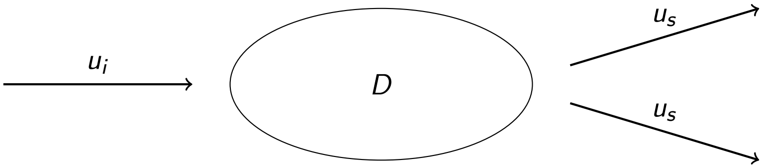

Figure: Inverse Scattering Illustration

Green's Representation

$$u(x) = \int_{\delta D} \left(\frac{\partial u}{\partial v}(y)\Phi(x,y) - \frac{\partial \Phi}{\partial v} u(x,y) \right) ds(y),\ x \in D. \tag{5}$$

$$u(x) = \int_{\partial D} \left(u(y) \frac{\partial \Phi(x,y)}{\partial v(y)} - \frac{\partial u}{\partial v}(y)\Phi(x,y) \right) ds(y),\ x \in \mathbb{R}^3 \setminus \bar{D} \tag{6}$$

Existence of Far Field Mapping

Theorem

Every radiating solution u to the Helmholtz equation has the same asymptotic behaviour as an outgoing spherical wave

$$u(x) = \frac{e^{ik|x|}}{|x|}\left(u_\infty(\hat{x}) + O\left(\frac{1}{|x|}\right)\right), |x| \to \infty$$

where

$$u_\infty(\hat{x}) = \frac{1}{4\pi}\int_{\delta D}\left(u(y)\frac{\partial e^{-ik\hat{x}\cdot y}}{\partial \nu} - \frac{\partial u}{\partial \nu}e^{-ik\hat{x}\cdot y}\right) ds(y), \hat{x} \in \mathbb{S}^2.$$

$Au = u_\infty$, where A is the far-field operator.

Characteristics of Far Field Map

- Analytic
- Injective

Theorem (Rellich's Lemma)

Assume D to be an open complement of an unbounded domain. Let $u \in C^2(\mathbb{R}^3)$ that satisfies

$$\lim_{r \to \infty} \int_{|x|=r} |u(x)|^2 ds = 0,$$

then $u = 0$ in $\mathbb{R}^3 \setminus \bar{D}$

- Ill-posed Inverse

Machine Learning Background

Introduction to Machine Learning

"The field of study that just gives computers the ability to learn without being explicitly programmed" - Arthur Samuel, 1959.

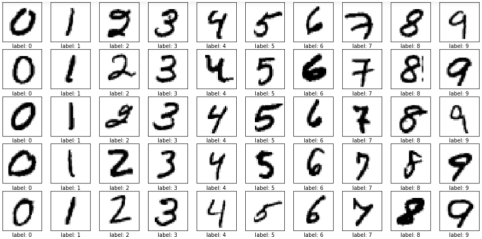

Figure: A Sample of Handwritten Digits from the MNIST [1] dataset

Machine Learning Models

- Logistic Regression
- Decision Trees
- Random Forest

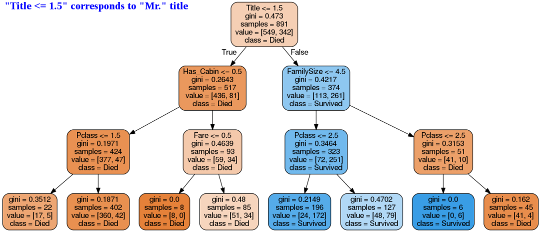

Figure: A decision tree trained on the predicting the survivors of the Titanic dataset [2]

Introduction to Deep Learning

- Stacked Linear Perceptrons
- Non-Linear Activation Functions
- Learning from gradient based methods
- Training is expensive but evaluation is quick
- Able to achieve super-human performance on some tasks.

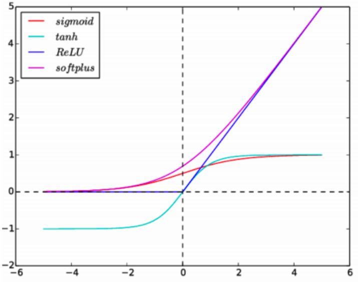

Figure: Different Activation Functions [3]

Data Generation

- Data generated using bempp-cl Python Library.
- Bempp is an open-source computational boundary element platform to solve electrostatic, acoustic and electromagnetic problems.
- Data was therefore very clean and ideal to use.
- Possible extensions to real-world problems using transfer learning.
- Data generation was expensive.

Data Examples

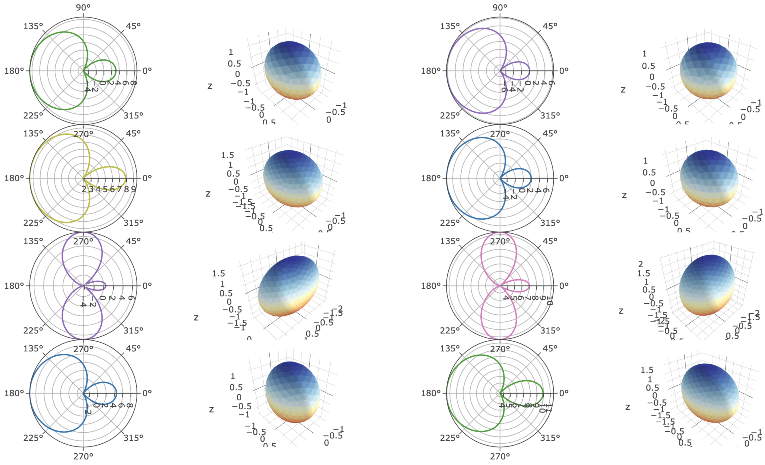

Figure: Far Field Patterns and there corresponding scattering object

Results

Ellipsoid Regression

- From far field pattern, regress 3 radii defining an ellipsoid.
- Training set of 2500, Validation set of 2500, Test set of 5000.

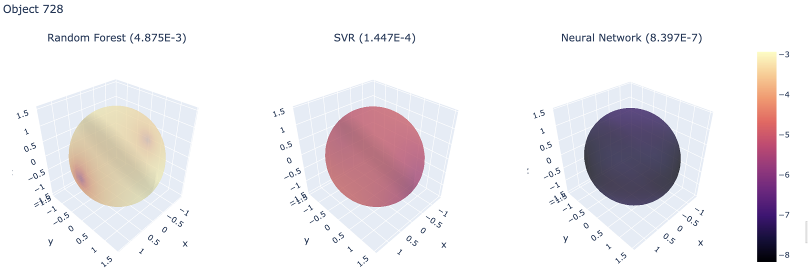

Figure: An illustration of the log error for each point of the defined ellipsoid, colour bar indicates log error at each point

Ellipsoid Regression

Model	Training Set MSE	Validation Set MSE	Test Set MSE	Test Set Eval. Time (s)
Random Forest	2.06e-04	1.26e-03	1.46e-03	0.564
SVR	6.16e-05	6.48e-04	7.08e-04	10.2
NN	3.49e-06	5.26e-06	5.52e-06	0.293

Table: Model Results for Ellipsoid Radii Regression

Dataset Size Reduction

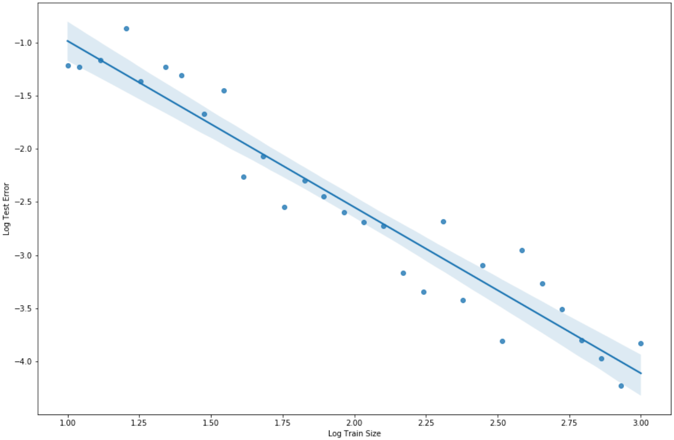

Figure: Log Training Size vs Log Test Error, with linear interpolation

Voxels

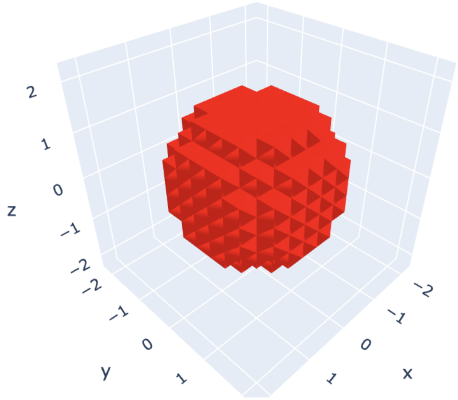

Figure: A voxel representation of an ellipsoid

Precision and Recall

- True/False - Represents where the model correctly predicted.
- Positive/Negative - What the model predicts as.
- Precision is defined as $\frac{TP}{TP+FP}$
- Recall is defined as $\frac{TP}{TP+FN}$

Ellipsoid Voxel Results

Table: Results of Logistic Regression Model On Ellipsoid Voxels Task

Metric	Train	Validation	Test
Accuracy	0.7076	0.7157	0.7106
Precision	0.1595	0.1559	0.1582
Recall	0.1596	0.1549	0.1581

Model Results

Table: Results of Neural Network Model On Ellipsoid Voxels Task

Metric	Train	Validation	Test
Accuracy	0.9992	0.9977	0.9977
Precision	0.9974	0.9926	0.9927
Recall	0.9981	0.9938	0.9939

Cuboids and Ellipsoids

Metric	Train	Validation	Test
Accuracy	0.829	0.829	0.829
Precision	0.094	0.094	0.094
Recall	0.093	0.095	0.095

Table: Results of Logistic Regression Model on Cuboids and Ellipsoids

Metric	Train	Validation	Test
Accuracy	0.999	0.998	0.998
Precision	0.996	0.993	0.992
Recall	0.995	0.991	0.990

Table: Results of Neural Network Model on Cuboids and Ellipsoids

Illustrations of model predictions

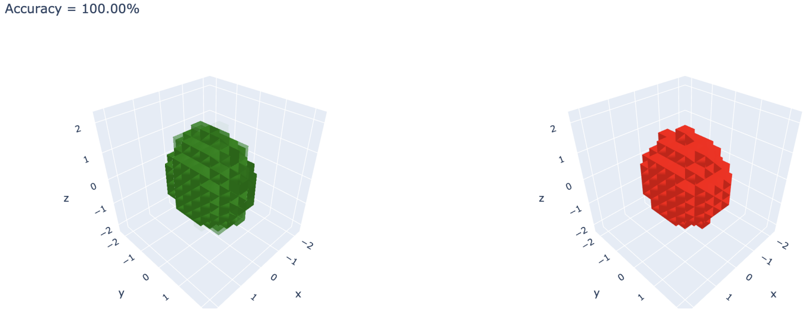

Figure: Neural Network trained on cuboids and ellipsoids prediction on an ellipsoid (Green represents prediction, Red represents actual)

Illustrations of model predictions

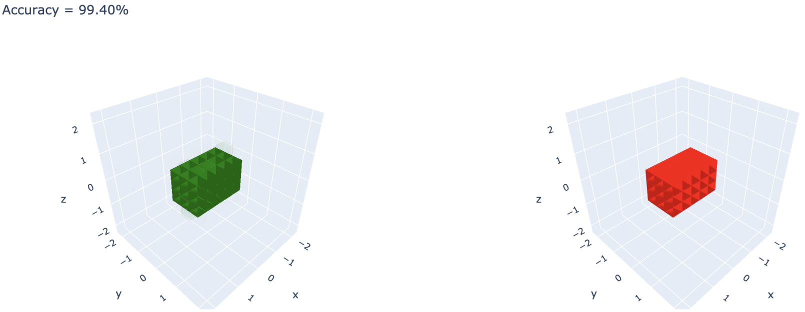

Figure: Neural Network trained on cuboids and ellipsoids prediction on a cuboid (Green represents prediction, Red represents actual)

Multiple Scatterers

- Significantly harder task traditionally.
- Data Generation was 8 times slower.

On a reduced dataset of 1450 training points, achieved following results:

Metric	Train	Validation	Test
Accuracy	0.986	0.980	0.980
Precision	0.890	0.835	0.834
Recall	0.901	0.854	0.852

Table: Results of Neural Network Model for 2 Scattering Ellipsoids

Voxel Representation

Figure: Neural Network trained on two scatterers prediction (Green represents prediction, Red represents actual)

Conclusion

- Machine Learning is very capable.
- Confident in a very capable and versatile model with sufficient data.
- Standard Linear Models are inefficent.
- Shallow Neural Network seems a strong model to go off.

Future Work

- Developing a model on a wider variety of shapes.
- Extensions into electromagnetic scattering.
- Using sound hard objects as scattering objects.
- Experimenting on the effects of the wave number.
- Attempting to find faster/more accurate machine learning models.

Bibliography

[1] Yann LeCun and Corinna Cortes. "MNIST handwritten digit database". In: (2010). URL: http://yann.lecun.com/exdb/mnist/.

[2] Dmilla. *Introduction to Decision Trees (Titanic dataset)*. Mar. 2017. URL: https://www.kaggle.com/dmilla/introduction-to-decision-trees-titanic-dataset.

[3] Will L. Hamilton. *Applied Machine Learning, Lecture 15: Back-propagation*. URL: https://www.cs.mcgill.ca/~wlh/comp551/slides/14-backprop.pdf.