Tutorial 5 Solutions

1. (*Q2.6-2*). Let X have a Poisson distribution with a variance of 3. Find $P(X = 2)$.

   - *For Poisson random variable,* $\mu = \sigma^2 = \lambda$.
     So $P(X = 2) = \frac{\lambda^2 e^{-\lambda}}{2!} = \frac{3^2 e^{-3}}{2} = 4.5e^{-3} = 0.224$.

2. (*Q2.6-5*). Flaws in a certain type of drapery material appear on the average of one in 150 square feet. If we assume the Poisson distribution, find the probability of at most one flaw in 225 square feet.

   - *Let X be the number of flaws on a piece of drapery of 225 square feet. One in 150 square feet is equivalent to* $225/150$ *in 225 square feet. Thus* $X \stackrel{d}{=} Poi(\lambda = 225/150)$.
   - $P(X \le 1) = P(X = 0) + P(X = 1) = e^{-225/150} + \frac{225}{150} e^{-225/150} = 0.5578$.

3. (*Q2.6-8*). Suppose that the probability of suffering a side effect from a certain flu vaccine is 0.005. Also suppose 1000 persons are inoculated.

   (a) Find the exact probability that at most 1 person suffers using a binomial distribution.

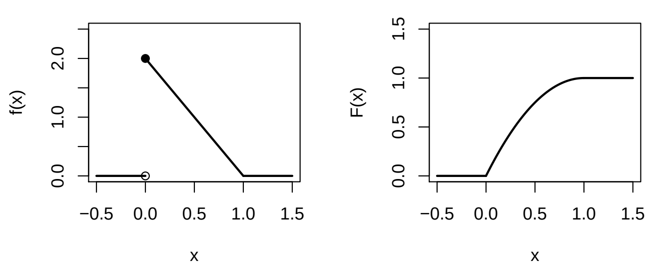
   - $P(X \le 1) = 0.995^{1000} + 1000 \times 0.005 \times 0.995^{999} = 0.040091$.

   (b) Find approximately the probability that at most 1 person suffers using a Poisson distribution.

   - $P(X \le 1) \approx e^{-1000\times0.005} + (1000 \times 0.005)^1 \times e^{-1000\times0.005} = 6e^{-5} = 0.040428$.

4. Let the random variable X have the pdf $f(x) = 2(1 - x)$, $0 \le x \le 1$, 0 elsewhere.

   (a) Sketch the graph of this pdf.



   (b) Determine and sketch the graph of the distribution function of X.

   - $F(x) = P(X \le x) = \begin{cases} 0, & x < 0 \\ 1 - (1 - x)^2, & 0 \le x < 1 \\ 1, & x \ge 1 \end{cases}$

   (c) Find

   i. $P(0 \le X \le 1/2)$,

- $P(0 \le X \le 1/2) = \int_0^{1/2} 2(1-x)dx = \frac{3}{4}$.

ii. $P(1/4 \le X \le 3/4)$,

- $P(1/4 \le X \le 3/4) = \int_{1/4}^{3/4} 2(1-x)dx = \frac{1}{2}$.

iii. $P(1/4 \le X \le 5/4)$,

- $P(1/4 \le X \le 5/4) = \int_{1/4}^{1} 2(1-x)dx = \frac{9}{16}$.

iv. $P(X = 3/4)$,

- $P(X = 3/4) = 0$.

v. $P(X \ge 3/4)$,

- $P(X \ge 3/4) = \int_{3/4}^{1} 2(1-x)dx = \frac{1}{16}$.

vi. the value of μ,

- $\mu = \int_0^1 x \cdot 2(1-x)dx = (x^2 - \frac{2}{3}x^3)|_0^1 = \frac{1}{3}$.

vii. the value of σ^2, and

- $E(X^2) = \int_0^1 x^2 \cdot 2(1-x)dx = (\frac{2}{3}x^3 - \frac{2}{4}x^4)|_0^1 = \frac{1}{6}$.
- *So* $\sigma^2 = E(X^2) - [E(X)]^2 = \frac{1}{6} - (\frac{1}{3})^2 = \frac{1}{18}$.

viii. the 36th percentile $\pi_{0.36}$ of X.

- $F(\pi_{0.36}) = 0.36$. *Hence* $1 - (1-\pi_{0.36})^2 = 0.36$ *and* $\pi_{0.36} = 0.2$.

5. The pdf of X is $f(x) = c/x^2$, $1 < x < \infty$.

(a) Find the value of c so that $f(x)$ is a well defined pdf.

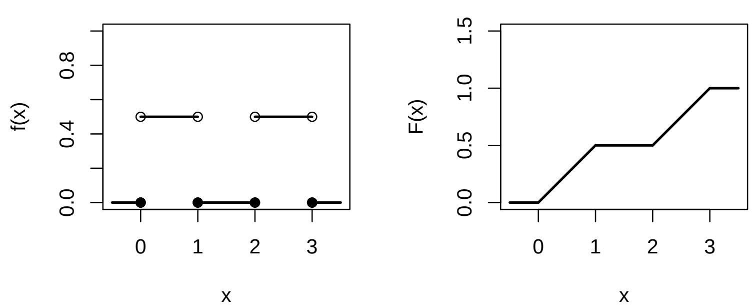
- *First, observe that* $\int_1^\infty \frac{c}{x^2}dx = c$.
- *On the other hand, for* $f(x) = c/x^2$ *to be a well defined pdf, we need* $\int_1^\infty \frac{c}{x^2}dx = 1$. *Thus* $c = 1$.

(b) Show that $E(X)$ is not finite.

- $E(X) = \int_1^\infty xf(x)dx = \int_1^\infty \frac{1}{x}dx = \lim_{b\to\infty} \int_1^b \frac{1}{x}dx = \lim_{b\to\infty} \ln b = \infty$.

6. Let $f(x) = 1/2$, $0 < x < 1$ or $2 < x < 3$, 0 elsewhere, be the pdf of X.

(a) Sketch the graph of this pdf.



(b) Define cdf of X and sketch its graph.

- $F(x) = \int_{-\infty}^{x} f(t)dt = \begin{cases} 0, & x \le 0 \\ \frac{1}{2}x, & 0 \le x \le 1 \\ \frac{1}{2}, & 1 \le x \le 2 \\ \frac{1}{2}(x-1), & 2 \le x \le 3 \\ 1, & x \ge 3 \end{cases}.$

(c) Find $q_1 = \pi_{0.25}$.

- $F(q_1) = 0.25$, *thus* $q_1 = 0.5$.

(d) Find the median $m = \pi_{0.50}$. Is it unique?

- $m =$ *any value in [1,2] because for any value* $a \in [1,2]$, $F(a) = 0.5$.

(e) Find the value of $E(X)$.

- $E(X) = \int_{-\infty}^{\infty} xf(x)dx = \int_0^1 x \cdot \frac{1}{2}dx + \int_2^3 x \cdot \frac{1}{2}dx = \frac{1}{4} + \frac{5}{4} = \frac{3}{2}$.

7. Let $F(x) = 1 - (\frac{1}{2}x^2 + x + 1)e^{-x}$, $0 < x < \infty$ be the cdf of X.

(a) Find the mgf $M(t)$ of X.

- *First find the pdf of* X: $f(x) = F'(x) = \frac{1}{2}x^2e^{-x}$, $0 < x < \infty$ *and 0 elsewhere.*
- *Then* $M(t) = E(e^{tX}) = \int_0^{\infty} e^{tx} \cdot \frac{1}{2}x^2e^{-x}dx = \int_0^{\infty} \frac{1}{2}x^2e^{(t-1)x}dx$ $= \frac{1}{(1-t)^3}\frac{1}{2}\int_0^{\infty} y^2e^{-y}dy$ *if we denote* $y = (1-t)x$ *and assume* $t < 1$.
- *By integration by parts it can be found that* $\int_0^{\infty} y^2e^{-y}dy = 2! = 2$. *Therefore* $M(t) = \frac{1}{(1-t)^3}$, $t < 1$.

(b) Find the values of μ and σ^2.

- $E(X) = \int_{-\infty}^{\infty} xf(x)dx = \int_0^{\infty} \frac{1}{2}x^3e^{-x}dx = (-\frac{1}{2}x^3e^{-x})|_0^{\infty} + \int_0^{\infty} \frac{3}{2}x^2e^{-x}dx$

  $= (-\frac{3}{2}x^2e^{-x})|_0^{\infty} + \int_0^{\infty} \frac{6}{2}xe^{-x}dx = (-\frac{6}{2}xe^{-x})|_0^{\infty} + \int_0^{\infty} \frac{6}{2}e^{-x}dx = (-3e^{-x})|_0^{\infty} = 3.$
- $E(X^2) = \int_{-\infty}^{\infty} x^2f(x)dx = \int_0^{\infty} \frac{1}{2}x^4e^{-x}dx = 12$ *by integration by parts.*
- *So* $\sigma^2 = 12 - 3^2 = 3$.

8. (*Q3.2-25*)The life X in years of a voltage regulator of a car has the pdf

$$f(x) = \frac{3x^2}{7^3}e^{-(x/7)^3}, \quad 0 < x < \infty.$$

(a) What is the probability that this regulator will last at least 7 years?

- $P(X \ge 7) = \int_7^{\infty} \frac{3x^2}{7^3}e^{-(x/7)^3}dx = -e^{-(x/7)^3}|_7^{\infty} = e^{-1}$.

(b) Given that it has lasted at least 7 years, what is the conditional probability it will last at least another 3.5 years?

- $P(X \ge 10.5|X \ge 7) = \frac{P(\{X\ge10.5\}\cap\{X\ge7\})}{P(X\ge7)} = \frac{P(X\ge10.5)}{P(X\ge7)}$ $= \frac{\int_{10.5}^{\infty} \frac{3x^2}{7^3}e^{-(x/7)^3}dx}{\int_7^{\infty} \frac{3x^2}{7^3}e^{-(x/7)^3}dx} = \frac{e^{-1.5^3}}{e^{-1}} = e^{-2.375}$.

9. (*Q2.6-4*). If X has a Poisson distribution so that $3P(X = 1) = P(X = 2)$, find $P(X = 4)$.

   - $3P(X = 1) = P(X = 2)$ *implies* $3\lambda e^{-\lambda} = \frac{\lambda^2}{2!}e^{-\lambda}$. *So* $\lambda = 6$. *Accordingly* $P(X = 4) = \frac{6^4}{4!}e^{-6} = 54e^{-6} = 0.1339$.

10. (*Q3.2-2*) A function is given as $f(x) = (3/16)x^2$, $-c < x < c$.

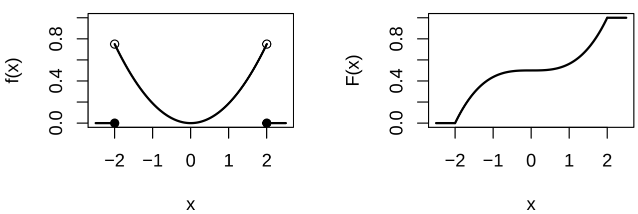
    (a) Find the constant c so that $f(x)$ is a pdf of a random variable X.

    - $1 = \int_{-c}^{c} \frac{3}{16}x^2 dx = \frac{1}{16}x^3|_{-c}^{c} = \frac{1}{8}c^3$.
    - *Thus* $c = 2$.

    (b) Find the cdf $F(x) = P(X \le x)$.

    - $F(x) = P(X \le x) = \int_{-2}^{x} \frac{3}{16}t^2 dt = \frac{1}{16}x^3 + \frac{1}{2}$, *if* $-2 < x < 2$.
    - *Thus* $F(x) = \begin{cases} 0, & x \le -2 \\ \frac{1}{16}x^3 + \frac{1}{2}, & -2 < x \le 2 \\ 1, & x > 2. \end{cases}$

    (c) Sketch graphs of the pdf $f(x)$ and the cdf $F(x)$.



11. (*Q3.2-3*) A function is given as $f(x) = 4x^c$, $0 \le x \le 1$.

    (a) Find the constant c so that $f(x)$ is a pdf of a random variable X.

    - $1 = \int_0^1 4x^c dx = \frac{4}{c+1}x^{c+1}|_0^1 = \frac{4}{c+1}$.
    - *Thus* $c = 3$.

    (b) Find the cdf $F(x) = P(X \le x)$.

    - $F(x) = P(X \le x) = \int_0^x 4t^3 dt = x^4$, *if* $0 < x < 1$.
    - *Thus*

      $$F(x) = \begin{cases} 0, & x \le 0 \\ x^4, & 0 < x \le 1 \\ 1, & x > 1. \end{cases}$$

    (c) Sketch graphs of the pdf $f(x)$ and the cdf $F(x)$.