1. Among the 16 applicants for a job, 10 have university degrees. A sample of 3 applicants are to be randomly chosen for interviews. Let X be the number of applicants in the sample who have university degrees.

   (a) Give the name to the distribution of X if it has a name. Also specify the values of all parameters involved in this distribution.

   - *X has hypergeometric distribution with parameters $N_1 = 10$, $N_2 = 6$ and $n = 3$.*

   (b) Find the probability that exactly 1 applicant in the sample has a university degree.

   - $$P(X = 1) = \frac{\binom{10}{1}\binom{6}{2}}{\binom{16}{3}} = \frac{15}{56} = 0.268.$$

   (c) Find the probability that at most 1 applicant in the sample has a university degree.

   - $$\begin{aligned} P(X \le 1) = P(X = 0) + P(X = 1) &= \frac{\binom{10}{0}\binom{6}{3}}{\binom{16}{3}} + \frac{\binom{10}{1}\binom{6}{2}}{\binom{16}{3}} \\ &= \frac{15}{56} + \frac{1}{28} = \frac{17}{56} = 0.304. \end{aligned}$$

2. A bag contains 5 coins, one of which has a head on both sides while the other 4 coins are normal. A coin is chosen at random from the bag and tossed 2 times. The number of heads obtained is a random variable, say X.

   (a) What are the possible values of X? Also tabulate the pmf of X.

   (*Hint*: The coin chosen is either normal or with head on both sides. Find the conditional probability of X for each of these two situations. Then use the law of total probability and multiplication rule to find the probability of X.)

   - *Define*

   $$\begin{aligned} A &= \{\textit{the selected coin is a normal one}\} \\ B &= \{\textit{the selected coin has head on both sides.}\} \end{aligned}$$

   Then $P(A) = 4/5$ and $P(B) = 1/5$.

   - *The possible values for X are $\{0, 1, 2\}$.*
   - *Now*

   $$\begin{aligned} P(X = 0) &= P(X = 0|A)P(A) + P(X = 0|B)P(B) = (\frac{1}{2})^2\frac{4}{5} + 0 = \frac{1}{5} \\ P(X = 1) &= P(X = 1|A)P(A) + P(X = 1|B)P(B) = 2(\frac{1}{2})^2\frac{4}{5} + 0 = \frac{2}{5} \\ P(X = 2) &= P(X = 2|A)P(A) + P(X = 2|B)P(B) = (\frac{1}{2})^2\frac{4}{5} + 1^2\frac{1}{5} = \frac{2}{5} \end{aligned}$$

   Therefore the pmf of X is

x	*0*	*1*	*2*
$f(x)$	*1/5*	*2/5*	*2/5*

(b) Calculate $E(X)$ and $\mathrm{Var}(X)$.

- $E(X) = 0 \times \frac{1}{5} + 1 \times \frac{2}{5} + 2 \times \frac{2}{5} = \frac{6}{5} = 1.2.$
- $E(X^2) = 0^2 \times \frac{1}{5} + 1^2 \times \frac{2}{5} + 2^2 \times \frac{2}{5} = 2.$
- $\mathrm{Var}(X) = E(X^2) - [E(X)]^2 = 2 - \frac{36}{25} = \frac{14}{25} = 0.56.$

3. A moment-generating function of X is given by $M(t) = 0.3e^t + 0.4e^{2t} + 0.2e^{3t} + 0.1e^{5t}$.

(a) Find the pmf of X.

-

x	*1*	*2*	*3*	*5*
$f(x)$	*0.3*	*0.4*	*0.2*	*0.1*

(b) Find the values of μ and σ^2 for X.

- $\mu = E(X) = 1 \times 0.3 + 2 \times 0.4 + 3 \times 0.2 + 5 \times 0.1 = 2.2.$
- $E(X^2) = 1^2 \times 0.3 + 2^2 \times 0.4 + 3^2 \times 0.2 + 5^2 \times 0.1 = 6.2.$
- *So* $\mathrm{Var}(X) = 6.2 - 2.2^2 = 1.36.$

(c) Calculate $P(X \geq 2)$.

- $P(X \geq 2) = 1 - P(X = 1) = 0.7.$

(d) Calculate $E(2^X)$.

- $E(2^X) = E(e^{X \ln 2}) = M(\ln 2) = 0.3e^{\ln 2} + 0.4e^{2 \ln 2} + 0.2e^{3 \ln 2} + 0.1e^{5 \ln 2} = 7.$

(e) Find the mgf of $X^2 - 1$.

- *The possible values that* $X^2 - 1$ *can take are 0,3,8 and 24.*
- *Hence its mgf is* $M_1(t) = 0.3 + 0.4e^{3t} + 0.2e^{8t} + 0.1e^{24t}$.

4. A moment-generating function of X is given by $M(t) = (0.25 + 0.75e^t)^4$.

(a) Give the name of the distribution of X (if it has a name).

- $X \stackrel{d}{=} b(4, 0.75).$

(b) Find the values of μ and σ^2 for X. (*Note:* No need to derive μ and σ^2 if you know the name of the distribution.)

- $\mu = np = 3$, $\sigma^2 = npq = 0.75.$

(c) Calculate $P(1 \leq X \leq 2)$.

- $P(1 \leq X \leq 2) = P(X = 1) + P(X = 2) = \binom{4}{1}0.75^1 0.25^3 + \binom{4}{2}0.75^2 0.25^2 = \frac{33}{128} = 0.2578125.$

5. A moment-generating function of X is given by $M(t) = \frac{0.25e^t}{1-0.75e^t}, \quad t < -\ln(0.75)$.

(a) Give the name of the distribution of X (if it has a name).

- $X \stackrel{d}{=} \textit{geometric}(0.25).$

(b) Find the values of μ and σ^2 for X. (*Note:* No need to derive μ and σ^2 if you know the name of the distribution.)

- $\mu = 1/p = 1/0.25 = 4$, $\sigma^2 = q/p^2 = 0.75/0.25^2 = 12$.

(c) Calculate $P(X \geq 2)$.

- $P(X \geq 2) = P(X > 1) = q = 0.75$.

(d) Calculate $E[(1 - 0.75e^{-1})e^{-X+1} + 0.75]$.

- $E[(1 - 0.75e)e^{-X+1} + 0.75] = (1 - 0.75e^{-1})E[e^{-X+1}] + 0.75$
  $= e(1 - 0.75e^{-1})E[e^{-X}] + 0.75 = e(1 - 0.75e^{-1})M(-1) + 0.75 = 1.$

6. An expert sharpshooter misses a target 10 percent of the time.

(a) What is the probability that she misses the target for the first time in her second shot?

- *Let X be the number of shots fired when she misses the target for the 1st time. Then X has a geometric distribution with $p = 0.1$.*
- $P(X = 2) = 0.9 \times 0.1 = 0.09$.

(b) What is the probability that she misses the target for the first time in her xth shot?

- $P(X = x) = 0.9^{x-1} \times 0.1 = 0.1(0.9)^{x-1}$.

(c) What is the probability that the first miss comes after the 3rd shot?

- $P(X > 3) = (1 - p)^3 = 0.9^3 = 0.729$.

(d) How many shots does she expect to fire to suffer the first miss?

- $E(X) = \frac{1}{p} = 10$.

(e) What is the probability that she will suffer 2 misses from 10 shots?

- *Denote Y as the number of misses in 10 shots. Then $Y \stackrel{d}{=} b(10, 0.1)$.*
- *So* $P(Y = 2) = \binom{10}{2} \times 0.1^2 \times 0.9^8 = 0.1937$.

(f) What is the probability that she will suffer at least 1 miss from the 10 shots?

- $P(Y \geq 1) = 1 - P(Y = 0) = 1 - 0.9^{10} = 0.6513$.

(g) What is the probability that she misses the target for the second time in her 10th shot?

- *Denote Z as the number of shot fired when she misses the target for the 2nd time. Then Z has a negative binomial distribution $NB(2, 0.1)$.*
- *So* $P(Z = 10) = \binom{9}{1} \times 0.1^2 \times 0.9^8 = 0.0387$.