Tutorial 6 Solutions

1. (*Q3.3-3*). Customers arrive randomly at a bank teller's window. Given that one customer arrived during a particular 10-minute period, let X equal the time within the 10 minute-period that the customer arrived. If X is $U(0, 10)$, find:

   (a) the pdf of X,

   - $$f(x) = \begin{cases} \frac{1}{10}, & 0 < x < 10, \\ 0, & elsewhere. \end{cases}$$

   (b) $P(X \geq 8)$,

   - $P(X \geq 8) = \int_8^{10} \frac{1}{10} dx = 0.2.$

   (c) $P(2 \leq X < 8)$,

   - $P(2 \leq X < 8) = \int_2^8 \frac{1}{10} dx = 0.6.$

   (d) $E(X)$, and

   - $E(X) = \int_0^{10} x \frac{1}{10} dx = \frac{10}{2} = 5.$

   (e) $\mathrm{Var}(X)$.

   - $\mathrm{Var}(X) = \int_0^{10} x^2 \frac{1}{10} dx - 5^2 = \frac{25}{3}.$

2. (*Q3.3-6*). Let X have an exponential distribution with a mean of $\theta = 20$. Compute

   (a) $P(10 < X < 30)$,

   - $P(10 < X < 30) = \int_{10}^{30} \frac{1}{20} e^{-x/20} dx = [-e^{-x/20}]_{10}^{30} = e^{-1/2} - e^{-3/2}.$

   (b) $P(X > 30)$.

   - $P(X > 30) = \int_{30}^{\infty} \frac{1}{20} e^{-x/20} dx = [-e^{-x/20}]_{30}^{\infty} = e^{-3/2}.$

   (c) $P(X > 40 | X > 10)$.

   - $P(X > 40) = \int_{40}^{\infty} \frac{1}{20} e^{-x/20} dx = e^{-2}$. *Similarly,* $P(X > 10) = e^{-1/2}$
   - *So* $P(X > 40 | X > 10) = \frac{P(X>40)}{P(X>10)} = e^{-3/2} = P(X > 30).$

   (d) What are the variance and the mgf of X?

   - $\mathrm{Var(X)} = \sigma^2 = \theta^2 = 400,\ M(t) = (1 - 20t)^{-1},\ t < 1/20.$

   (e) Find the 80th percentile of X.

   - $0.8 = \int_0^{\pi_{0.8}} \frac{1}{20} e^{-x/20} dx = 1 - e^{-\pi_{0.8}/20}.$
   - *So* $\pi_{0.8} = -20 \ln(1 - 0.8) = 20 \ln(5) = 32.19.$

3. (*Q3.3-9*). What are the pdf, the mean, and the variance of X if the mgf of X is given by the following?

   (a) $M(t) = (1 - 3t)^{-1},\ t < 1/3.$

- $X \stackrel{d}{=}$ *exponential*$(\theta = 3)$*; pdf* $f(x) = \frac{1}{3}e^{-x/3},\ x > 0$*;* $\mu = 3$*;* $\sigma^2 = 9$*.*

(b) $M(t) = \frac{3}{3-t},\ t < 3$.

- $X \stackrel{d}{=}$ *exponential*$(\theta = 1/3)$*; pdf* $f(x) = 3e^{-3x},\ x > 0$*;* $\mu = 1/3$*;* $\sigma^2 = 1/9$*.*

4. Let random variable X have the pdf

$$f(x) = \frac{e^{-x}}{(1+e^{-x})^2}, \quad -\infty < x < \infty.$$

(The distribution of such X is known as the *logistic distribution.*)

(a) Write down the cdf of X.

- $F(x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{x} \frac{e^{-t}}{(1+e^{-t})^2}dt = \frac{1}{1+e^{-x}} = \frac{e^x}{e^x+1}, \quad -\infty < x < \infty.$

(b) Find the mean and variance of X.

- $\mu = \int_{-\infty}^{\infty} x\frac{e^{-x}}{(1+e^{-x})^2}dx = 0$ *because* $x\frac{e^{-x}}{(1+e^{-x})^2}$ *is an odd function.*
- $\sigma^2 = \int_{-\infty}^{\infty} x^2\frac{e^{-x}}{(1+e^{-x})^2}dx - 0^2 = \frac{1}{3}\pi^2.$
  Without the help of Maple, it would be very difficult to calculate this integral.

(c) Find $P(3 < X < 5)$.

- $P(3 < X < 5) = F(5) - F(3) = \frac{1}{1+e^{-5}} - \frac{1}{1+e^{-3}}.$

(d) Find the 85-th percentile of X.

- *Solve* $0.85 = F(\pi_{0.85}) = \frac{1}{1+e^{-\pi_{0.85}}}$*. We get* $\pi_{0.85} = \ln(17/3)$*.*

(e) Let $Y = \frac{1}{1+e^{-X}}$. Find the cdf of Y.
Can you tell the name of the distribution of Y?

- *First the support of* Y *is* $0 < y < 1$*.*
- $G(y) = P(Y \le y) = P(\frac{1}{1+e^{-X}} < y) = P(X \le \ln(\frac{y}{1-y})) = F(\ln(\frac{y}{1-y})) = \frac{1}{1+e^{-\ln[y/(1-y)]}} = y, \quad 0 < y < 1.$
- *Therefore,* Y *has a Uniform(0,1) distribution.*

5. (*Q3.4-1*) Telephone calls enter a college switchboard at a mean rate of 2/3 call per minute according to a Poisson process. Let X denote the waiting time until the 10th call arrives.

(a) What is the pdf of X?

- $X \stackrel{d}{=}$ *Gamma*$(\theta = 3/2, \alpha = 10)$*.*
- $f(x) = \frac{1}{\Gamma(10)(3/2)^{10}}x^9e^{-2x/3}, \quad 0 \le x < \infty.$

(b) What are the mgf, mean and variance of X?

- $M(t) = (1-\theta t)^{-\alpha} = (1 - \frac{3}{2}t)^{-10}, \quad t < \frac{2}{3}.$
- $\mu = \alpha\theta = 15$ *and* $\sigma^2 = \alpha\theta^2 = 22.5$*.*

6. (*Q3.4-2*) If X has a gamma distribution with $\theta = 4$ and $\alpha = 2$, find $P(X < 5)$.

   - *First let Y be a Poisson random variable with mean $\lambda x = (1/\theta)x = (1/4)\,5 = 5/4$.*
   - *Then by the relationship between gamma and Poisson distributions,*
     $P(X < 5) = P(Y \geq \alpha) = P(Y \geq 2) = 1 - [P(Y = 0) + P(Y = 1)]$
     $= 1 - [e^{-5/4} + (5/4)e^{-5/4}] = 0.35536$

7. (*Q3.4-4*) Use the moment-generating function of a gamma distribution to show that $E(X) = \alpha\theta$ and $\mathrm{Var}(X) = \alpha\theta^2$.

   - $M(t) = (1 - \theta t)^{-\alpha}, \quad t < \theta^{-1}$.
   - *So* $M'(t) = \alpha\theta(1 - \theta t)^{-\alpha-1}$, *and* $E(X) = M'(0) = \alpha\theta$
   - *Also* $M''(t) = (\alpha + 1)\alpha\theta^2(1 - \theta t)^{-\alpha-2}$. *Thus* $M''(0) = (\alpha + 1)\alpha\theta^2$.
     Therefore $\mathrm{Var}(X) = M''(0) - [M'(0)]^2 = (\alpha + 1)\alpha\theta^2 - \alpha^2\theta^2 = \alpha\theta^2$.

8. Let X have a $\chi^2(2)$ distribution. Find constants a and b such that

   $$P(a < X < b) = 0.90, \quad \text{and} \quad P(X < a) = 0.05.$$

   - *The distribution of X is also gamma with $\theta = 2$ and $\alpha = 2/2 = 1$.*
   - *Thus the distribution of X is further exponential with $\theta = 2$.*
   - $0.05 = P(X < a) = \int_0^a \frac{1}{2}e^{-x/2}dx = 1 - e^{-a/2}$.
     So $a = -2\ln(0.95) = 0.1026$.
   - $P(X < b) = P(X < a) + P(a < X < b) = 0.95$.
     So $0.95 = P(X < b) = \int_0^b \frac{1}{2}e^{-x/2}dx = 1 - e^{-b/2}$.
     Hence $b = -2\ln(0.05) = 5.9915$.