Tutorial 2 Solutions

1. Let $P(A) = 0.4$ and $P(A \cup B) = 0.6$.

   (a) What is the value of $P(B)$ if A and B are mutually exclusive?

   - *If A and B are mutually exclusive, then $P(A \cup B) = P(A) + P(B)$. Since $P(A) = 0.4$ and $P(A \cup B) = 0.6$, it follows that $P(B) = 0.2$.*

   (b) What is the value of $P(B)$ if A and B are independent?

   - *If A and B are independent, then $P(A \cap B) = P(A)P(B)$. So $0.6 = P(A \cup B) = P(A) + P(B) - P(A)P(B) = 0.4 + P(B) - 0.4P(B)$. It follows that $P(B) = 1/3$.*

2. A box contains 5 green balls, 3 black balls, and 7 red balls. Two balls are selected at random without replacement from the box. What is the probability that:

   (a) both balls are red?

   - *Method 1:* $P(\textit{both are red}) = P(R_1R_2) = \frac{\binom{7}{2}}{\binom{5+3+7}{2}} = \frac{1}{5}$.
   - *Method 2: By the multiplication rule,*
     $P(R_1R_2) = P(R_1)P(R_2|R_1) = \frac{7}{5+3+7} \times \frac{7-1}{5+3+7-1} = \frac{1}{5}$.

   (b) both balls are of the same colour?

   - *Method 1:* $P(\textit{both are same color}) = P(G_1G_2) + P(B_1B_2) + P(R_1R_2) = \frac{\binom{5}{2}}{\binom{5+3+7}{2}} + \frac{\binom{3}{2}}{\binom{5+3+7}{2}} + \frac{\binom{7}{2}}{\binom{5+3+7}{2}} = \frac{34}{105}$.
   - *Method 2:* $P(\textit{both are same color}) = P(G_1)P(G_2|G_1) + P(B_1)P(B_2|B_1) + P(R_1)P(R_2|R_1) = \frac{5}{5+3+7} \times \frac{5-1}{5+3+7-1} + \frac{3}{5+3+7} \times \frac{3-1}{5+3+7-1} + \frac{7}{5+3+7} \times \frac{7-1}{5+3+7-1} = \frac{2}{21} + \frac{1}{35} + \frac{1}{5} = \frac{34}{105}$.

   (c) one ball is red and the other is black?

   - *Method 1:* $P(R_1B_2 \textit{ or } B_1R_2) = \frac{\binom{5}{0}\binom{3}{1}\binom{7}{1}}{\binom{5+3+7}{2}} = \frac{1}{5}$.
   - *Method 2:* $P(R_1B_2 \textit{ or } B_1R_2) = P(R_1)P(B_2|R_1) + P(B_1)P(R_2|B_1) = \frac{7}{5+3+7} \times \frac{3}{5+3+7-1} + \frac{3}{5+3+7} \times \frac{7}{5+3+7-1} = \frac{1}{5}$.

   Try to find the above probabilities using two ways: one through a classical probability model and counting formulas, and the other through *conditional probabilities/multiplication rule.*

3. The probability that a marksman hits a target is 0.9 on any given shot, and repeated shots are independent. He has two pistols; one contains two bullets and the other contains only one bullet. He selects a pistol at random and shoots at the target until the pistol is empty. What is the probability of hitting the target exactly one time?

   - *Denote* $A_1 = \{\textit{choose the first pistol}\}$, $A_2 = \{\textit{choose the second pistol}\}$, *and* $B = \{\textit{hit the target exactly one time}\}$.

- *Then by the law of total probability and multiplication rule*
  $P(B) = P(A_1)P(B|A_1) + P(A_2)P(B|A_2)$
  $= 0.5 \times (0.9 \times 0.1 + 0.1 \times 0.9) + 0.5 \times 0.9 = 0.54.$

4. (*Q1.4-3 in the textbook*) Let A_1 and A_2 be the events that a person is left eye dominant and right eye dominant, respectively. When a person folds his/her hands, let B_1 and B_2 be the events that their left thumb and right thumb, respectively, are on top. A survey in one statistics class yielded the following table.

	B_1	B_2	Totals
A_1	5	7	12
A_2	14	9	23
Totals	19	16	35

If a student is selected randomly, find the following probabilities:

(a) $P(A_1 \cap B_1)$,
   - $P(A_1 \cap B_1) = \frac{5}{35} = \frac{1}{7}$.

(b) $P(A_1 \cup B_1)$,
   - $P(A_1 \cup B_1) = P(A_1) + P(B_1) - P(A_1 \cap B_1) = \frac{12}{35} + \frac{19}{35} - \frac{5}{35} = \frac{26}{35}$.

(c) $P(A_1|B_1)$,
   - $P(A_1|B_1) = \frac{5}{19}$.

(d) $P(B_2|A_2)$.
   - $P(B_2|A_2) = \frac{9}{23}$.

5. (*Q1.5-7*) Each of three football players will attempt to kick a field goal from the 25-yard line. Let A_i denote the event that the field goal is made by player i, $i = 1, 2, 3$. Assume that A_1, A_2 and A_3 are mutually independent and that $P(A_1) = 0.5$, $P(A_2) = 0.7$, $P(A_3) = 0.6$.

(a) Compute the probability that exactly one player is successful.
   - $P(\textit{Exactly one kicks a goal})$
     $= P(A_1 \cap A_2' \cap A_3') + P(A_1' \cap A_2 \cap A_3') + P(A_1' \cap A_2' \cap A_3)$
     $= 0.5(1 - 0.7)(1 - 0.6) + (1 - 0.5)0.7(1 - 0.6) + (1 - 0.5)(1 - 0.7)0.6 = 0.29.$

(b) Compute the probability that exactly two players make a field goal (i.e., one misses).
   - $P(\textit{Exactly two make a goal})$
     $= P(A_1 \cap A_2 \cap A_3') + P(A_1' \cap A_2 \cap A_3) + P(A_1 \cap A_2' \cap A_3)$
     $= 0.5 \times 0.7 \times (1 - 0.6) + (1 - 0.5) \times 0.7 \times 0.6 + 0.5 \times (1 - 0.7) \times 0.6 = 0.44.$

6. Lie detectors are controversial instruments, barred from use as evidence in many courts. Nonetheless, many employers use lie detector screening as part of their hiring process in the hope that they can avoid hiring people who might be dishonest. There has been some research, but no agreement, about the reliability of polygraph tests. Based on this research, suppose that a polygraph can detect 65% of lies, but incorrectly identifies 15% of true statements as lies.

   A certain company believes that 95% of its job applicants are trustworthy. The company gives everyone a polygraph test, asking "Have you ever stolen anything from your place of work?" Naturally, all the applicants answer "No", but the polygraph identifies some of those answers as lies, making the person ineligible for a job. What is the probability that a job applicant rejected under suspicion of dishonesty was actually trustworthy?

   - *Let* $A = \{$*An applicant chosen at random is trustworthy*$\}$, $B = \{$*detected by polygraph as lies*$\}$.
   - *Then* $P(A) = 0.95$, $P(B|A) = 0.15$ *and* $P(B|A') = 0.65$.
   - *And by Bayes's theorem* $P(\textit{trustworthy}|\textit{detected as lying}) = P(A|B)$ $= \frac{P(A)P(B|A)}{P(A)P(B|A)+P(A')P(B|A')} = \frac{0.95\times0.15}{0.95\times0.15+0.05\times0.65} = 0.8143.$

The following questions are optional for MAST20006 students but prescribed for MAST90057 ones.

7. If events A and B are independent, show that A' and B are independent.

   - *First* $B = (A \cap B) \cup (A' \cap B)$, *and* $A \cap B$ *and* $A' \cap B$ *are mutually exclusive.*
   - *Thus* $P(B) = P(A \cap B) + P(A' \cap B) = P(A)P(B) + P(A' \cap B)$ *because* A *and* B *are independent.*
   - *Hence* $P(A' \cap B) = P(B) - P(A)P(B) = (1 - P(A))P(B) = P(A')P(B)$, *suggesting that* A' *and* B *are independent.*

8. (*Q1.4-7*) Suppose that the genes for eye colour for a certain male fruit fly are (R, W) and the genes for eye colour for the mating female fruit fly are (R, W), where R and W represent red and white, respectively. Their offspring receive one gene for eye colour from each parent.

   (a) Define the sample space for the genes for eye colour for the offspring.

      - $S = \{(R_m, R_f), (R_m, W_f), (W_m, R_f), (W_m, W_f)\}$.

   (b) Assume that each of the four possible outcomes has equal probability. If an offspring ends up with either two red genes or one red and one white gene for eye colour, its eyes will look red. Given that an offspring's eyes look red, what is the conditional probability that it has two red genes for eye colour?

- $P((R_m, R_f)|\{(R_m, R_f), (R_m, W_f), (W_m, R_f)\}) = \frac{P((R_m,R_f))}{P(\{(R_m,R_f),(R_m,W_f),(W_m,R_f)\})} = 0.25/0.75 = 1/3.$

9. (*Q1.4-9*) An urn contains four coloured balls: two orange and two blue. Two balls are selected at random without replacement, and you are told that at least one of them is orange. What is the probability that the other ball is also orange?

    - $P(\textit{at least one orange}) = 1 - P(\textit{no orange}) = 1 - \frac{2}{4} \times \frac{1}{3} = \frac{5}{6}$. *So*
    - $P(\textit{both orange}|\textit{at least one orange}) = \frac{P(\textit{both orange})}{P(\textit{at least one orange})} = \frac{\frac{2}{4}\times\frac{1}{3}}{\frac{5}{6}} = \frac{1}{5}.$

10. (*Q1.6-6*) A life insurance company issues standard, preferred, and ultra-preferred policies. Of the company's policyholders of a certain age, 60% are standard with a probability of 0.01 of dying in the next year, 30% preferred with a probability of 0.008 of dying in the next year, and 10% are ultra-preferred with a probability of 0.007 of dying in the next year. A policyholder of that age dies in the next year. What are the conditional probabilities of the deceased being standard, preferred, and ultra-preferred?

    - $P(\textit{standard}) = P(S) = 0.6$, $P(P) = 0.3$, $P(UP) = 0.1$. *And* $P(D|S) = 0.01$, $P(D|P) = 0.008$, $P(D|UP) = 0.007$.
    - *Thus by the law of total probability and multiplication rule* $P(D) = P(S)P(D|S) + P(P)P(D|P) + P(UP)P(D|UP) = 0.6 \times 0.01 + 0.3 \times 0.008 + 0.1 \times 0.007 = 0.0091.$
    - *Hence by Bayes's theorem*
    - $P(S|D) = \frac{P(S)P(D|S)}{P(D)} = \frac{0.6\times0.01}{0.0091} = 0.6593407.$
    - $P(P|D) = \frac{P(P)P(D|P)}{P(D)} = \frac{0.3\times0.008}{0.0091} = 0.2637363.$
    - $P(UP|D) = \frac{P(UP)P(D|UP)}{P(D)} = \frac{0.1\times0.007}{0.0091} = 0.07692308.$