Only ***two questions*** *(the same 2 for all students, and decided by random selection) will be marked by your tutor. The other questions are mandatory with solutions to be given later. Two marks will be automatically deducted if you do not provide evidence on your submitted work that you have attempted all the questions. The total marks for this assignment is 20.*

1. Let $P(A) = P(B) = \frac{3}{4}$ and $P(A \cap B) = \frac{1}{2}$. Find the following:

   (a) $P(B')$.

   - $P(B') = 1 - P(B) = \frac{1}{4}$.

   (b) $P(A' \cap B')$.

   - $P(A' \cap B') = P((A \cup B)') = 1 - P(A \cup B) = 1 - [P(A) + P(B) - P(A \cap B)] = 1 - [\frac{3}{4} + \frac{3}{4} - \frac{1}{2}] = 0$.

   (c) $P(B \cap A')$.

   - *Note that* $B = (B \cap A) \cup (B \cap A')$, *and* $B \cap A$ *and* $B \cap A'$ *are mutually exclusive. So* $P(B) = P(B \cap A) + P(B \cap A')$ *and*
   $P(B \cap A') = P(B) - P(B \cap A) = \frac{3}{4} - \frac{1}{2} = \frac{1}{4}$.

   (d) $P(A \cup B')$.

   - *Since* $A \cup B' = A \cup (A' \cap B')$ *and* A *and* $A' \cap B'$ *are mutually exclusive, it follows that* $P(A \cup B') = P(A) + P(A' \cap B') = \frac{3}{4} + 0 = \frac{3}{4}$ *by (b).*

   The following is an alternative derivation.

   - *First* $B = (A \cap B) \cup (A' \cap B)$, *and* $A \cap B$ *and* $A' \cap B$ *are mutually exclusive. So* $P(B) = P(A \cap B) + P(A' \cap B)$ *and*
   $P(A' \cap B) = P(B) - P(A \cap B) = \frac{3}{4} - \frac{1}{2} = \frac{1}{4}$.
   Therefore, $P(A \cup B') = P((A' \cap B)') = 1 - P(A' \cap B) = 1 - \frac{1}{4} = \frac{3}{4}$.

2. A hockey team plays two games on a certain day. The probability that it wins the first game is 0.7; the probability that it wins the second game is 0.6. Also whether or not the team wins one game is **independent** of whether or not it wins the other game. Find the probability that

   (a) the team wins at least one game. [4]

   - *Define* $A = \{\textit{It wins the first game}\}$ *and* $B = \{\textit{It wins the second game}\}$. *Then* $P(\textit{win at least one game}) =$
   - $P(A \cup B) = P(A) + P(B) - P(A \cap B) = P(A) + P(B) - P(A)P(B) = 0.7 + 0.6 - 0.7 \times 0.6 = 0.88$.

   (b) the team wins exactly one game. [3]

   - $P(\textit{win exactly one game}) = P(A \cup B) - P(A \cap B) = 0.88 - 0.7 \cdot 0.6 = 0.46$.
   Alternatively,
   - $P(A \cap B') = P(A)P(B') = 0.7(1 - 0.6) = 0.28$.
   - $P(A' \cap B) = P(A')P(B) = (1 - 0.7)0.6 = 0.18$.

- *So* $P(\textit{win exactly one race}) = P((A \cap B') \cup (B \cap A')) = 0.28 + 0.18 = 0.46$

(c) the team wins neither games. [3]

- $P(\textit{win neither game}) = P(A' \cap B') = P(A')P(B') = (1-0.7)(1-0.6) = 0.12.$

3. *Birthday paradox.* Consider a group of 3 students. Each student has a birthday that can be any one of the days numbered 1, 2, 3, ..., 365.

(a) What is the probability that **none of them** have the same birthday with each other?

- *Let A be the event that none of the three people have the same birthday. Thus A can be expressed as a set of outcomes where the 3 birthday numbers for each outcome are all different from each other.*
- *Therefore* $P(A) = \frac{365\times364\times363}{365\times365\times365} = 0.9917858.$

(b) What is the probability that **some of them** have the same birthday with each other?

- *Here we are interested in* $P(A')$.
- *Therefore* $P(A') = 1 - P(A) = 0.0082142.$

(c) What is the probability that **all of them** have the same birthday with each other?

- *Let B be the event that all of the three people have the same birthday. Thus B can be expressed as a set of outcomes where the 3 birthday numbers for each outcome are all equal to each other.*
- *Therefore* $P(B) = \frac{365}{365\times365\times365} = \frac{1}{365^2} = 7.5 \cdot 10^{-6}.$

4. Binary digits 0 and 1 are transmitted over a communication channel. If a 1 is sent, it will be received as a 1 with probability 0.9 and as a 0 with probability 0.1; if a 0 is sent, it will be received as a 0 with probability 0.7 and as a 1 with probability 0.3. If the probability that a 0 is sent is the same as the probability that a 1 is sent, what is

(a) the conditional probability that a 1 was sent given that a 1 was received?

- *Using Bayes's theorem*

$$\begin{aligned} P(\textit{1 sent} \mid \textit{1 received}) &= \frac{P(\textit{1 sent})P(\textit{1 received} \mid \textit{1 sent})}{P(\textit{1 received})} \\ &= \frac{0.5 \times 0.9}{0.5 \times 0.9 + 0.5 \times 0.3} = \frac{3}{4} = 0.75. \end{aligned}$$

(b) the conditional probability that a 0 was sent given that a 0 was received?

- *Again using Bayes's theorem*

$$\begin{aligned} P(\textit{0 sent} \mid \textit{0 received}) &= \frac{P(\textit{0 sent})P(\textit{0 received} \mid \textit{0 sent})}{P(\textit{0 received})} \\ &= \frac{0.5 \times 0.7}{0.5 \times 0.7 + 0.5 \times 0.1} = \frac{7}{8} = 0.875. \end{aligned}$$

5. It is known from experience that in a certain industry 60 percent of all labor-management disputes are over wages, 15 percent are over working conditions, and 25 percent are over fringe issues. Also, 45 percent of the disputes over wages are resolved without strikes, 70 percent of the disputes over working conditions are resolved without strikes, and 40 percent of the disputes over fringe issues are resolved without strikes. What is the probability that a labor-management dispute in this industry will be resolved without a strike? (*Hint:* Apply the law of total probability.)

   - *Let A be the event that a labor-management dispute will be resolved without a strike, B_1 be the event that the dispute is over wages, B_2 be that over working conditions, and B_3 be that over fringe issues. Then*
     $P(B_1) = 0.6, \quad P(B_2) = 0.15, \quad P(B_3) = 0.25$
     $P(A|B_1) = 0.45, \quad P(A|B_2) = 0.7, \quad P(A|B_3) = 0.4.$
     Note that B_1, B_2 and B_3 are mutually exclusive and exhaustive events.
   - *By the law of total probability*
     $P(A) = P(A \cap B_1) + P(A \cap B_2) + P(A \cap B_3)$
     $= P(B_1)P(A|B_1) + P(B_2)P(A|B_2) + P(B_3)P(A|B_3)$
     $= 0.6 \cdot 0.45 + 0.15 \cdot 0.7 + 0.25 \cdot 0.4 = 0.475.$

6. In a certain community, 8 percent of all adults over 50 years old have diabetes. If a health service in this community correctly diagnoses 95 percent of all persons with diabetes as having the disease and incorrectly diagnoses 2 percent of all persons without diabetes as having the disease, find the probabilities that

   (a) the community health service will diagnose an adult over 50 as having diabetes; [5]

   - *Let D be the event that a person over 50 has diabetes, C be the event that the community health service will diagnose an adult over 50 as having diabetes. Then*
     $P(D) = 0.08, \quad P(D') = 1 - 0.08 = 0.92, \quad P(C|D) = 0.95, \quad P(C|D') = 0.02.$
   - *Using the law of total probability and the definition of conditional probability*
     $P(C) = P(C \cap D) + P(C \cap D') = P(D)P(C|D) + P(D')P(C|D')$
     $= 0.08 \cdot 0.95 + 0.92 \cdot 0.02 = 0.0944.$

   (b) a person over 50 diagnosed by the health service as having diabetes actually has the disease. [5]

   - *Using Bayes's theorem*
     $P(D|C) = \frac{P(D \cap C)}{P(C)} = \frac{P(D)P(C|D)}{P(C)} = \frac{0.08 \cdot 0.95}{0.0944} = \frac{0.76}{0.0944} = 0.805.$

$\boxed{\texttt{Total marks = 20}}$