THE UNIVERSITY OF MELBOURNE

Student ID:

The University of Melbourne

Semester 1 Exam *Solutions — June, 2019*

School of Mathematics and Statistics

MAST20006 Probability for Statistics

Exam Duration: 3 Hours

Reading Time: 15 Minutes

This paper has 9 pages

Authorised materials:
This is a closed book exam.
A University approved hand-held calculator, i.e. Casio FX82 (with any suffix) may be used.

Instructions to Invigilators:
Script books shall be supplied to each student.
Students may not take this paper with them at the end of the exam.

Instructions to Students:
This paper has **10** questions. Formula sheet is given at the end of this paper.
Attempt as many questions, or parts of questions, as you can.
Questions carry marks as shown in the brackets after the questions statements.
The total number of marks available for this examination is **100**.
Working and/or reasoning must be given to obtain full credit.

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1. Voters living in the city vote for candidate A with probability 0.6, while voters living in the country vote for candidate A with probability 0.4. It is known that 80% of all voters live in the city and the rest 20% live in the country.

   (a) Compute the probability that a randomly chosen voter will vote for candidate A. [3]

   - $P(\textit{vote } A) = 0.8 \times 0.6 + 0.2 \times 0.4 = 0.56.$

   (b) Compute the probability that a person who will vote for candidate A lives in the city. [3]

   - *Thus* $P(\textit{city} \mid \textit{vote } A) = \frac{0.8\times 0.6}{0.56} = \frac{6}{7} = 0.8571.$

   (c) Four randomly surveyed voters indicated they would vote for candidate A. Let X be the number of voters among these four who live in the city. Compute the probability $P(X = 3)$. [3]

   - $X \overset{d}{=} Bin(4, 6/7).$
   - *Thus* $P(X = 3) = 4 \times (6/7)^3 \times (1/7) = 864/2401 = 0.35985.$

2. Suppose X_1 and X_2 are two independent random variables that have the following moment generating function (mgf):

$$M_1(t) = E\left(e^{tX_1}\right) = M_2(t) = E\left(e^{tX_2}\right) = \frac{0.5e^t}{1 - 0.5e^t}, \quad t < \ln 2.$$

   (a) Compute the probability $P(\min(X_1, X_2) > 2)$. [3]

   - $X_1 \overset{d}{=} X_2 \overset{d}{=} Geometric(p = 0.5)$
   - *Since* X_1 *and* X_2 *are i.i.d.,*

$$P(\min(X_1, X_2) > 2) = P(X_1 > 2)P(X_2 > 2) = ((1 - 0.5)^2)^2 = 0.0625.$$

   (b) Define $Y = X_1 + X_2$. Compute the probability $P(Y = 4)$. [3]

   - *Mgf of* Y *is* $M_Y(t) = \dfrac{(0.5e^t)^2}{(1 - 0.5e^t)^2}$, $t < \ln 2$. *Hence* $Y \overset{d}{=} NB(r{=}2, p{=}0.5)$.
   - $P(Y = 4) = \binom{3}{1} \cdot 0.5^2 \cdot 0.5^{4-2} = \frac{3}{16} = 0.1875.$

   (c) Compute the probability $P(Y > 7)$. [3]

   - *Define* $Z \overset{d}{=} Bin(7, 0.5)$. *Then* $P(Y > 7) = P(Z < 2)$.
   - *So* $P(Y > 7) = P(Z = 0) + P(Z = 1) = 0.5^7 + 7 \cdot 0.5^7 = \frac{8}{128} = \frac{1}{16} = 0.0625.$

   (d) Compute $\mathrm{Var}(0.5^Y + 80)$. [3]

   - $Var(0.5^Y + 80) = E(0.25^Y) - \left[E(0.5^Y)\right]^2 = M_Y(\ln 0.25) - [M_Y(\ln 0.5)]^2$

$$= \frac{(0.5 \times 0.25)^2}{(1 - 0.5 \times 0.25)^2} - \left(\frac{(0.5 \times 0.5)^2}{(1 - 0.5 \times 0.5)^2}\right)^2 = \frac{1}{49} - \frac{1}{81} = \frac{32}{3969} = 0.00806$$

3. Mutation in a certain gene can occur with probability 0.001 in human population. Suppose X people in a random sample of 2500 people will be observed to have this mutation. Note that X follows a binomial distribution.

   (a) Compute $P(X \leq 2)$. [2]

   - $P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2)$
     $= 0.999^{2500} + 2500 \times 0.001 \times 0.999^{2499} + \binom{2500}{2} \times 0.001^2 \times 0.999^{2498}$
     $= 0.08198 + 0.20516 + 0.25661 = 0.54375.$

   (b) A binomial distribution $b(n, p)$ can be approximated by a Poisson($\lambda = np$) distribution if p is small and n is large. Use this result to approximate the probability in part (a) by a Poisson probability. [2]

   - $X \overset{d}{\approx}$ *Poisson*$(\lambda = 2.5)$, *thus*
     $$P(X \leq 2) \approx e^{-2.5} + 2.5e^{-2.5} + \frac{2.5^2}{2!}e^{-2.5} = 6.625e^{-2.5} = 0.54381.$$

   (c) The probability in part (a) may also be approximated by a normal probability based on the central limit theorem. Give a normal approximation (using the continuity correction) to $P(X \leq 2)$. [2]

   - *By CLT,* $X \overset{d}{\approx} N(\mu = 2.5, \sigma^2 = 2.4975)$.
   - *So* $P(X \leq 2) \approx P\left(Z \leq \frac{2+0.5-2.5}{\sqrt{2.4975}}\right) = P(Z \leq 0) = \Phi(0) = 0.5.$

4. Let X_1 and X_2 be two independent Bernoulli($p = 0.5$) random variables. Define two new random variables: $Y_1 = \min(X_1, X_2)$ and $Y_2 = \max(X_1, X_2)$.

   (a) Compute the joint probability mass function (pmf) of (Y_1, Y_2). [3]

   - *Possible values of* (Y_1, Y_2)*: (0, 0), (0, 1), (1, 1). Then the pmf* $f(y_1, y_2)$*:*
   - $f(0, 0) = P(Y_1 = Y_2 = 0) = P(X_1 = X_2 = 0) = 0.25$;
   - $f(0, 1) = P(Y_1 = 0, Y_2 = 1) = P(X_1 = 0, X_2 = 1) + P(X_1 = 1, X_2 = 0) = 0.25 + 0.25 = 0.5$.
   - $f(1, 1) = P(Y_1 = Y_2 = 1) = P(X_1 = X_2 = 1) = 0.25$.

   (b) Compute $E(Y_1)$, $E(Y_2)$, $\text{Var}(Y_1)$ and $\text{Var}(Y_2)$. [4]

   - $E(Y_1) = P(Y_1 = 1) = P(X_1 = X_2 = 1) = 0.25$.
   - $E(Y_2) = P(Y_2 = 1) = 1 - P(Y_2 = 0) = 1 - P(X_1 = X_2 = 0) = 0.75$.
   - $Y_1 \overset{d}{=} b(1, 0.25)$. *So* $Var(Y_1) = 0.25 \cdot 0.75 = 3/16 = 0.1875$.
   - $Y_2 \overset{d}{=} b(1, 0.75)$. *So* $Var(Y_2) = 0.75 \cdot 0.25 = 3/16 = 0.1875$.

   (c) Compute $\text{Cov}(Y_1, Y_2)$. Are Y_1 and Y_2 independent? Why or why not? [3]

   - $Cov(Y_1, Y_2) = E(Y_1Y_2) - E(Y_1)E(Y_2) = f(1, 1) - 0.25 \cdot 0.75 = 1/16 = 0.0625$.
   - Y_1 *and* Y_2 *are not independent since they are correlated.*

5. Let X be a continuous random variable with probability density function (pdf)

$$f(x) = \begin{cases} c & \text{if } -4 < x < 0 \\ 2c & \text{if } 0 < x < 2 \\ 0 & \text{elsewhere} \end{cases}$$

where c is a constant with its value to be determined.

(a) Find the value of c and the cumulative distribution function (cdf) of X. [2]

- *Solving* $1 = \int_{-4}^{0} c dx + \int_0^2 2c dx = 4c + 4c$, *it follows that* $c = \frac{1}{8}$
- *Hence the cdf of* X *is* $F(x) = \begin{cases} 0 & \text{if } x \leq -4 \\ \frac{x+4}{8} & \text{if } -4 < x < 0 \\ \frac{x+2}{4} & \text{if } 0 \leq x < 2 \\ 1 & \text{if } x \geq 2 \end{cases}$

(b) Let X_1 and X_2 be two independent random variables each having the pdf $f(x)$ given above. Define $W = \min\{X_1, X_2\}$. Find the 75-th percentile of W. [3]

- $P(W > w) = P(\min\{X_1, X_2\} > w) = P^2(X > w)$. *The cdf of* W *is*

$$F_W(w) = 1 - (1 - F(w))^2 = \begin{cases} 0 & \text{if } w \leq -4 \\ 1 - \left[1 - \frac{w+4}{8}\right]^2 & \text{if } -4 < w < 0 \\ 1 - \left[1 - \frac{w+2}{4}\right]^2 & \text{if } 0 \leq w < 2 \\ 1 & \text{if } w \geq 2 \end{cases}$$

- *Solving* $\frac{3}{4} = F_w(\pi_{0.75})$, *it follows that* $\pi_{0.75} = 0$.

(c) Consider the transformation $Y = X^2$ of X.

i. Is this transformation one-to-one? Find the support of Y. [2]

- *This is not one-to-one. The support of* Y *is* $0 \leq y < 16$.

ii. Derive the cdf of Y. [3]

- *For* $0 \leq y \leq 4$,
  $G(y) = P(-\sqrt{y} \leq X \leq \sqrt{y}) = \int_{-\sqrt{y}}^{0} \frac{1}{8} dx + \int_0^{\sqrt{y}} \frac{2}{8} dx = \frac{3\sqrt{y}}{8}$.
- *For* $4 < y < 16$,
  $G(y) = P(-\sqrt{y} \leq X \leq 2) = \int_{-\sqrt{y}}^{0} \frac{1}{8} dx + \int_0^2 \frac{2}{8} dx = \frac{\sqrt{y}}{8} + \frac{1}{2}$.
- *So the cdf of* Y *is*

$$G(y) = \begin{cases} 0, & y < 0, \\ \frac{3}{8}\sqrt{y}, & 0 \leq y \leq 4, \\ \frac{1}{8}\sqrt{y} + \frac{1}{2}, & 4 < y < 16, \\ 1, & y \geq 16. \end{cases}$$

iii. Compute the pdf of Y. [2]

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$$g(y) = G'(y) = \begin{cases} 0, & y < 0, \\ \frac{3}{16\sqrt{y}}, & 0 \leq y \leq 4, \\ \frac{1}{16\sqrt{y}} & 4 < y < 16, \\ 0, & y \geq 16. \end{cases}$$

6. Let X_1, X_2, X_3 be independent random variables having Bernoulli($p = 0.5$), Bernoulli($p = 0.5$) and Poisson($\lambda = 0.75$) distributions, respectively.

   Define $Y_1 = X_1 + X_3$ and $Y_2 = X_2 + X_3$.

   (a) Compute the correlation coefficient ρ between Y_1 and Y_2. [2]

   - $Var(Y_1) = Var(X_1) + Var(X_3) = 0.25 + 0.75 = 1$. *Similarly,* $Var(Y_2) = 1$.
   - $Cov(Y_1, Y_2) = Var(X_3) = 0.75$. *So* $\rho = Cor(Y_1, Y_2) = \dfrac{0.75}{1} = 0.75$.

   (b) Use Chebyshev's inequality $P(|X - \mu| < k\sigma) \geq 1 - \frac{1}{k^2}$ to find a lower bound for $P(|Y_1 - 1.25| < \sqrt{3})$. [1]

   - $E(Y_1) = E(X_1) + E(X_3) = 1.25$ *and* $Var(Y_1) = 1$. *So* $\sigma = 1$
   - $P(|Y_1 - 1.25| < \sqrt{3}) = P(|Y_1 - 1.25| < \sqrt{3}\sigma) \geq 1 - \frac{1}{3} = 0.\dot{6}$.

   (c) Compute the exact value of $P(|Y_1 - 1.25| < \sqrt{3})$. [3]

   - $|Y_1 - 1.25| < \sqrt{3} \iff 0 \leq Y_1 \leq 2$. *Hence*
   - $P(|Y_1 - 1.25| < \sqrt{3}) = P(0 \leq Y_1 \leq 2)$
     $= P(X_1 = 0)P(0 \leq Y_1 \leq 2 | X_1 = 0) + P(X_1 = 1)P(0 \leq Y_1 \leq 2 | X_1 = 1)$
     $= P(X_1 = 0)P(X_3 = 0, 1, 2) + P(X_1 = 1)P(X_3 = 0, 1)$
     $= 0.5(0.75^0 + 0.75 + 0.75^2/2)e^{-0.75} + 0.5(0.75^0 + 0.75)e^{-0.75} = 0.8931$.

   (d) Define

   $$Z_1 = \begin{cases} 1 & \text{if } Y_1 = 0, \\ 0 & \text{otherwise;} \end{cases} \quad \text{and} \quad Z_2 = \begin{cases} 1 & \text{if } Y_2 = 0, \\ 0 & \text{otherwise.} \end{cases}$$

   i. Compute $P(Z_1 = 1)$ and $P(Z_2 = 1)$. [2]

   - *Both* Z_1 *and* Z_2 *are Bernoulli r.v.s with* $P(Z_1 = 1) = P(Z_2 = 1) = P(X_1 = X_3 = 0) = P(X_2 = X_3 = 0) = 0.5e^{-0.75} = 0.2362$.

   ii. Compute the joint pmf of (Z_1, Z_2). [2]

   - $P(Z_1 = 1, Z_2 = 1) = P(X_1 = X_2 = X_3 = 0) = 0.25e^{-0.75} = 0.1181$.
   - $P(Z_1 = 1, Z_2 = 0) = P(X_1 = X_3 = 0, X_2 > 0) = 0.25e^{-0.75} = 0.1181$.
   - $P(Z_1 = 0, Z_2 = 1) = P(X_1 > 0, X_2 = X_3 = 0) = 0.25e^{-0.75} = 0.1181$.
   - $P(Z_1 = 0, Z_2 = 0) = 1 - 3 \cdot 0.25e^{-0.75} = 0.6457$.

7. Suppose X and Y are continuous random variables with the joint pdf

$$f(x,y) = \begin{cases} 3 & \text{if } 0 \le x \le 1 \text{ and } 0 \le y \le x^2, \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Find the marginal pdf of X. Is the marginal pdf of X a uniform pdf? [2]

- *The marginal of X is* $f_1(x) = \int_0^{x^2} 3dy = 3x^2, \quad 0 \le x \le 1.$
- $f_1(x)$ *is not a uniform pdf.*

(b) Find the conditional pdf of Y given $X = x$, $0 \le x \le 1$. Is it a uniform pdf? [2]

- $h(y|x) = \dfrac{f(x,y)}{f_1(x)} = \dfrac{3}{3x^2} = \dfrac{1}{x^2}, \quad 0 \le y \le x^2; \ 0 \le x \le 1.$
- *So $h(y|X=x)$ is a Uniform(0, x^2) pdf.*

(c) Are X and Y independent? Why or why not? [1]

- *X and Y are not independent because $h(y|x)$ depends on x, or because the support of (X,Y) is not rectangular.*

(d) Compute the conditional expectation $E\left[2^Y \,\middle|\, X = 2^{-1/2}\right]$. [2]

- $E\left[2^Y \,\middle|\, X = 2^{-1/2}\right] = \int_0^{1/2} \dfrac{2^y}{(2^{-1/2})^2} dy = \dfrac{2\sqrt{2}-2}{\ln 2} = 1.1952.$
- *Alternatively, use the mgf of Uniform(0, 2^{-1}) to get the same answer.*

(e) Compute the probability $P(Y \ge X^3)$. [3]

- $P(Y \ge X^3) = \int_0^1 \int_{x^3}^{x^2} 3dydx = \int_0^1 3(x^2 - x^3)dx = \left\{x^3 - \frac{3}{4}x^4\right\}_0^1 = \frac{1}{4}.$

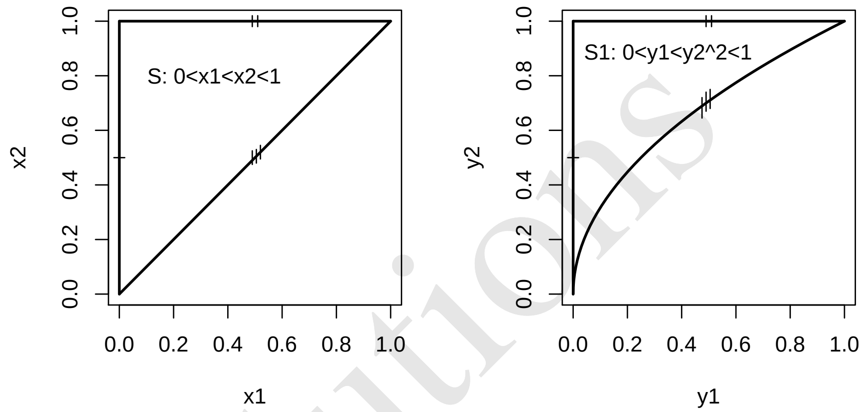
8. Consider two random variables X_1 and X_2 with the joint probability density

$$f(x_1, x_2) = \begin{cases} 2, & 0 \le x_1 \le x_2 \le 1, \\ 0 & \text{elsewhere.} \end{cases}$$

Let $Y_1 = X_1 X_2$ and $Y_2 = X_2$ be a joint transformation of (X_1, X_2).

(a) Find the support of (Y_1, Y_2) and sketch it. [3]

- *The support of* (Y_1, Y_2) *is* $\{0 \le Y_1 \le Y_2^2 \le 1\}$.



(b) Find the inverse transformation. [1]

- $X_1 = Y_1/Y_2, X_2 = Y_2$.

(c) Compute the Jacobian of the inverse transformation. [2]

- $J = \begin{vmatrix} 1/y_2 & -y_1/y_2^2 \\ 0 & 1 \end{vmatrix} = \dfrac{1}{y_2}$.

(d) Compute the joint pdf of (Y_1, Y_2). [2]

- $g(y_1, y_2) = |J| \cdot f(\dfrac{y_1}{y_2}, y_2) = \dfrac{2}{y_2}, \quad 0 \le y_1 \le y_2^2 \le 1.$

(e) Find the marginal pdf of Y_1 from the joint pdf of (Y_1, Y_2). [2]

- $g_1(y_1) = \displaystyle\int_{\sqrt{y_1}}^{1} \frac{2}{y_2} dy_2 = -2 \ln \sqrt{y_1} = -\ln y_1, \quad 0 < y_1 < 1.$

9. Let $X_1, X_2, \cdots, X_n$ be independent random variables each having the moment-generating function (mgf)

$$M(t) = \frac{8-3t}{(2-t)(4-t)}, \quad t < 2.$$

(a) Compute the mgf $M_{Y_n}(t)$ of the sum $Y_n = X_1 + X_2 + \cdots + X_n$. [2]

- $M_{Y_n}(t) = E(e^{tY_n}) = [M_{X_1}(t)]^n = \dfrac{(8-3t)^n}{(2-t)^n(4-t)^n}, \quad t < 2.$

(b) Compute the mgf $M_{\bar{Y}_n}(t)$ of the sample mean $\bar{Y}_n = \frac{Y_n}{n}$. [2]

- $M_{\bar{Y}_n}(t) = M_{Y_n}\left(\dfrac{t}{n}\right) = \dfrac{(8-\frac{3t}{n})^n}{(2-\frac{t}{n})^n(4-\frac{t}{n})^n} = \dfrac{(1-\frac{3t}{8n})^n}{(1-\frac{t}{2n})^n(1-\frac{t}{4n})^n}, \quad t < 2n.$

(c) Compute the limiting mgf $\lim_{n\to\infty} M_{\bar{Y}_n}(t)$. What distribution does the limiting mgf correspond to? What is the implication of this result? [2]

- $\lim\limits_{n\to\infty} M_{\bar{Y}_n}(t) = \lim\limits_{n\to\infty} \dfrac{(1-\frac{3t}{8n})^n}{(1-\frac{t}{2n})^n(1-\frac{t}{4n})^n} = \dfrac{e^{-3t/8}}{e^{-t/2}e^{-t/4}} = e^{3t/8}$
- *The limit is the mgf of the degenerate distribution having probability 1 at* $\frac{3}{8}$.
- *This implies that* $\bar{Y}_n \xrightarrow{p} \frac{3}{8} = E(X_1)$ *as* $n \to \infty$.
- *Note it is also a correct answer if it is based on applying the WLLN. Then the WLLN must be correctly stated, and* $E(X_1) = \frac{3}{8}$ *be proved.*

(d) Let $Z_n = \sqrt{n}(\bar{Y}_n - \dfrac{3}{8})$. Compute $M_{Z_n}(t)$, the mgf of Z_n. Then use this result to compute $\lim_{n\to\infty} M_{Z_n}(t)$. Finally explain what is the limiting distribution of Z_n as $n \to \infty$. [5]

- $$\begin{aligned} M_{Z_n}(t) &= E(e^{t\sqrt{n}(\bar{Y}_n - 3/8)}) = e^{-3t\sqrt{n}/8} M_{\bar{Y}_n}(\sqrt{n}t) \\ &= \frac{e^{-3t\sqrt{n}/8}(1-\frac{3\sqrt{n}t}{8n})^n}{(1-\frac{\sqrt{n}t}{2n})^n(1-\frac{\sqrt{n}t}{4n})^n} = \left[\frac{e^{-3t/(8\sqrt{n})}(1-\frac{3t}{8\sqrt{n}})}{1-\frac{3t}{4\sqrt{n}}+\frac{t^2}{8n}}\right]^n, \quad t < 2\sqrt{n}. \end{aligned}$$
- *By Taylor's series expansion,* $e^u \approx 1 + u + \frac{1}{2}u^2$ *when* $|u|$ *is small. Using this result, for any given* t,

$$\begin{aligned} \lim_{n\to\infty} M_{Z_n}(t) &= \lim_{n\to\infty}\left[\frac{e^{-3t/(8\sqrt{n})}(1-\frac{3t}{8\sqrt{n}})}{1-\frac{3t}{4\sqrt{n}}+\frac{t^2}{8n}}\right]^n = \lim_{n\to\infty}\left[\frac{(1-\frac{3t}{8\sqrt{n}}+\frac{9t^2}{128n})(1-\frac{3t}{8\sqrt{n}})}{1-\frac{3t}{4\sqrt{n}}+\frac{t^2}{8n}}\right]^n \\ &= \lim_{n\to\infty}\left[\frac{1-\frac{3t}{4\sqrt{n}}+\frac{27t^2}{128n}}{1-\frac{3t}{4\sqrt{n}}+\frac{t^2}{8n}}\right]^n = \lim_{n\to\infty}\left(1+\frac{11t^2}{128n}\right)^n = e^{11t^2/128} \end{aligned}$$

- *The limit mgf is that of* $N(\mu = 0, \sigma^2 = \frac{11}{64})$.
- *This implies that* $Z_n \xrightarrow{d} N(0, \frac{11}{64})$ *as* $n \to \infty$.
- *Note it is also a correct answer if it is based on applying the CLT. Then the CLT must be correctly stated, and* $Var(X_1) = \frac{11}{64}$ *be proved.*

10. A random variable X has the following mgf:

$$M(t) = \frac{e^{-t}(1-2t)}{(1-t)(1-4t)}, \quad t < \frac{1}{4}.$$

(a) Compute the value of $\mathrm{Var}(X)$. [5]

- *It can be shown that* $M(t) = e^{-t}\left[\frac{1}{3}\cdot\frac{1}{1-t} + \frac{2}{3}\cdot\frac{1}{1-4t}\right]$.
- *Thus* $X+1$ *has probability* $\frac{1}{3}$ *to follow an* $Exp(1)$ *distribution, and probability* $\frac{2}{3}$ *to follow an* $Exp(4)$ *distibution.*
- *Hence* $E((X+1)^2) = \frac{1}{3}\cdot 2\cdot 1^2 + \frac{2}{3}\cdot 2\cdot 4^2 = 22$, *and* $E(X+1) = \frac{1}{3}\cdot 1 + \frac{2}{3}\cdot 4 = 3$.
- *Therefore,* $Var(X) = Var(X+1) = 22 - 3^2 = 13$.

(b) Compute the probability $P(X^2 > 4)$. [5]

- *It can be seen that* $P(X^2 > 4) = P(X > 2 \text{ or } X < -2) = P(X+1 > 3)$.
- *Because* $X+1$ *is a mixture exponential r.v. from (a),*

$$P(X^2 > 4) = \frac{1}{3}\int_3^\infty e^{-x}dx + \frac{2}{3}\int_3^\infty \frac{1}{4}e^{-x/4}dx = \frac{1}{3}e^{-3} + \frac{2}{3}e^{-3/4} = 0.3315$$

`Total marks = 100`

End of the exam questions.
Formulas are on the next page.