Tutorial 4 Solutions

1. (*Q2.4-7*) Suppose that 10^6 points are selected independently and at random from the unit square $\{(x,y) : 0 \le x \le 1, 0 \le y \le 1\}$. Let W equal the number of points that fall in $A = \{(x,y) : x^2 + y^2 < 1\}$.

   (a) How is W distributed? Namely, what is the name of the distribution of W if it has a name? And what is the pmf of W?

   - $W \stackrel{d}{=} b(10^6, \pi/4)$.
   - *The pmf of W is* $f(w) = \binom{10^6}{w}(\pi/4)^w(1-\pi/4)^{10^6-w}, \ w = 0, 1, \cdots, 10^6$.

   (b) Give the mean, variance and standard deviation of W.

   - $\mu = E(W) = 250000\pi$ *and* $\sigma^2 = \mathrm{Var}(X) = 250000\pi(1-\pi/4)$.

   (c) What is the expected value of $W/250000$?

   - $E(W/250000) = E(W)/250000 = \pi$.

2. (*Q2.4-14*) A certain type of mint has a label weight of 20.4 grams. Suppose that the probability is 0.90 that a mint weighs more than 20.7 grams. Let X equal the number of mints that weigh more than 20.7 grams in a sample of 8 mints selected at random.

   (a) How is X distributed if we assume independence?

   - $X \stackrel{d}{=} b(8, 0.9)$.

   (b) Find $P(X = 8)$ and $P(X \le 7)$.

   - $P(X = 8) = 0.9^8 = 0.4305$.
   - $P(X \le 7) = 1 - P(X = 8) = 0.5695$.

3. (*Q2.5-1*) Define the pmf and give the values of μ and σ^2 when the moment-generating function (mgf) of X is defined by

   (a) $M(t) = \frac{1}{3} + \frac{2}{3}e^t$.

   - *X has a Bernoulli distribution* $b(1, 2/3)$.
   - *Pmf* $f(x) = (2/3)^x(1/3)^{1-x}, \ x = 0, 1$.
   - $\mu = 2/3$ *and* $\sigma^2 = 2/9$.

   (b) $M(t) = (0.25 + 0.75e^t)^{12}$.

   - *X has a binomial distribution* $b(12, 0.75)$.
   - *Pmf* $f(x) = \binom{12}{x}0.75^x0.25^{12-x}, \ x = 0, 1, 2, \cdots, 12$.
   - $\mu = np = 9$ *and* $\sigma^2 = npq = 9/4$.

   (c) $M(t) = \left(\frac{0.6e^t}{1-0.4e^t}\right)^2, \quad t < -\ln(0.4)$.

   - *X has a negative binomial distribution* $NB(2, 0.6)$.
   - *Pmf* $f(x) = \binom{x-1}{1}0.6^2 0.4^{x-2}, \ x = 2, 3, \cdots$.

- $\mu = \frac{r}{p} = \frac{2}{0.6} = \frac{10}{3}$ *and* $\sigma^2 = \frac{rq}{p^2} = \frac{2\times 0.4}{0.6^2} = \frac{20}{9}$.

4. (*Q2.5-3*) If the moment-generating function of X is

$$M(t) = \frac{2}{5}e^t + \frac{1}{5}e^{2t} + \frac{2}{5}e^{3t},$$

Find the mean, variance, and pmf of X.

- *The pmf of* X *is*

x	*1*	*2*	*3*
$f(x)$	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{2}{5}$

- $\mu = 2$, $E(X^2) = \frac{24}{5}$, *and* $\sigma^2 = \frac{24}{5} - 2^2 = \frac{4}{5}$.

5. (*Q2.5-4*). Let X equal the number of people selected at random that you must ask in order to find someone with the same birthday as yours. Assuming each day of the year is equally likely (and ignoring February 29).

(a) What probability distribution does X have? Namely, what is the pmf of X?

- X *has a geometric distribution Geo*$(p = 1/365)$.

(b) Give the mean and variance of X.

- $\mu = \frac{1}{p} = 365$ *and* $\sigma^2 = \frac{q}{p^2} = \frac{364/365}{1/365^2} = 364 \times 365 = 132860$.

(c) Find $P(X > 400)$ and $P(X < 300)$.

- $P(X > 400) = q^{400} = \left(\frac{364}{365}\right)^{400} = 0.3337$.
- $P(X < 300) = 1 - P(X > 299) = 1 - q^{299} = 1 - \left(\frac{364}{365}\right)^{299} = 0.5597$.

6. (*Q2.5-10*) Suppose that a basketball player can make a successful free throw 60% of the time. Let X equal the number of free throws made by this player until reaching a total of 10 shots,

(a) What probability distribution does X have? Namely, what is the pmf of X?

- X *has a negative binomial distribution* $NB(10, 0.6)$.
- *Pmf* $f(x) = \binom{x-1}{9}0.6^{10}0.4^{x-10}$, $x = 10, 11, \cdots$.

(b) Give the mean and variance of X.

- $\mu = E(X) = \frac{r}{p} = \frac{10}{0.6} = \frac{50}{3}$.
- $\sigma^2 = \mathrm{Var}(X) = \frac{rq}{p^2} = \frac{100}{9}$.

(c) Find $P(X = 16)$.

- $P(X = 16) = \binom{15}{9}0.6^{10}0.4^6 = 0.1240$.

7. (*Q2.5-18*). If $E(X^r) = 5^r$, $r = 1, 2, 3, \cdots$, find $M(t)$, the mgf of X, and the pmf of X.

- *Note that $E(X^r) = M^{(r)}(0)$ (the rth derivative of $M(t)$ at $t = 0$). If the expansion of $M(t)$ exists, we have*

$$\begin{aligned} M(t) &= M(0) + M'(0)\frac{t}{1!} + M''(0)\frac{t^2}{2!} + M^{(3)}(0)\frac{t^3}{3!} + \cdots \\ &= \sum_{r=0}^{\infty} E(X^r)\frac{t^r}{r!} = \sum_{r=0}^{\infty}\frac{(5t)^r}{r!} = e^{5t} \end{aligned}$$

  So X has only one possible value 5, and the pmf of X is $f(5) = P(X = 5) = 1$.

- *When a random variable X has only one possible value, we say X is a* **degenerate random variable** *and has a* **degenerate distribution***.*

The following questions are optional for MAST20006 students but prescribed for MAST90057 ones.

8. (*Q2.4-20, difficult*) A hospital obtains 40% of its flu vaccine from Company A, 50% from Company B, and 10% from Company C. From past experience it is known that 3% of the vials from A are ineffective, 2% from B are ineffective, and 5% from C are ineffective. The hospital test 5 vials from each shipment. If at least one of the five is ineffective, find the conditional probability of that shipment coming from C.

   - *Let X be the number of ineffective vials in the sample of 5. Then,*
     if the shipment is from A, X has a binomial distribution, i.e. $X|A \stackrel{d}{=} b(5, 0.03)$;
     if the shipment is from B, X has a binomial distribution, i.e. $X|B \stackrel{d}{=} b(5, 0.02)$;
     if the shipment is from C, X has a binomial distribution, i.e. $X|C \stackrel{d}{=} b(5, 0.05)$.
   - $P(X \geq 1|A) = 1 - P(X = 0|A) = 1 - (1 - 0.03)^5 = 1 - 0.97^5$
     $P(X \geq 1|B) = 1 - P(X = 0|B) = 1 - (1 - 0.02)^5 = 1 - 0.98^5$
     $P(X \geq 1|C) = 1 - P(X = 0|C) = 1 - (1 - 0.05)^5 = 1 - 0.95^5$
   - *By Bayes's Theorem,*

$$\begin{aligned} P(C|\{X \geq 1\}) &= \frac{P(C)P(X \geq 1|C)}{P(A)P(X \geq 1|A) + P(B)P(X \geq 1|B) + P(C)P(X \geq 1|C)} \\ &= \frac{0.1(1 - 0.95^5)}{0.4(1 - 0.97^5) + 0.5(1 - 0.98^5) + 0.1(1 - 0.95^5)} = 0.1779. \end{aligned}$$

9. (*Q2.5-11, challenging, not required to be done in class*). Let X equal the number of flips of a fair coin that are required to observe the same face on consecutive flips. Find the pmf of X.

   - *This experiment corresponds to a sequence of Bernoulli trials in which a success corresponds to having two consecutive heads or two consecutive tails. The probability of success at the nth trial is the probability that the result of the nth trial matches the result of the $(n - 1)$th trial, for $n \geq 2$ (X cannot take the value 1 because we need at least 2 flips to have a success). The success probability is therefore $1/2$.*

- *Therefore* $X = Y + 1$ *where* $Y \stackrel{d}{=} Geo(1/2)$ *and* $p_X(x) = p_Y(x-1) = (1/2)^{x-1}$, $x \geq 2$.

10. (*Q2.5-15, challenging, not required to be done in class*). One of four different prizes was randomly put into each box of a cereal. If a family decided to buy this cereal until it obtained at least one of each of the four different prizes, what is the expected number of boxes of cereal that must be purchased?

    - *The first box of cereal that is purchased contains the first prize. The probability of obtaining a different prize when purchasing a new box of cereal is* $3/4$*. Let* Z_2 *denote the number of boxes to purchase after the first one until one gets a prize different from the first prize. Then* $Z_2 \stackrel{d}{=} Geo(3/4)$.
    - *Using a similar argument, if* Z_3 *denotes the number of boxes to purchase after obtaining the second distinct prize until one gets a prize different from the first two prizes, then* $Z_3 \stackrel{d}{=} Geo(1/2)$.
    - *Similarly, if* Z_4 *denotes the number of boxes to purchase after obtaining the third distinct prize until one gets a prize different from the first three prizes, then* $Z_4 \stackrel{d}{=} Geo(1/4)$.
    - *So if* X *is the number of boxes of cereal that must be purchased until the family obtains at least one of each of the four different prizes, then* $X = 1 + Z_2 + Z_3 + Z_4$. *Therefore* $E[X] = 1 + (4/3) + 2 + 4 = 25/3$.