Tutorial 8 Solutions

1. (*Q4.2.-7*). Let the joint pmf of X and Y be

$$f(x,y) = \frac{1}{4}, \quad (x,y) \in S = \{(0,0),(1,1),(1,-1),(2.0)\}.$$

(a) Represent the joint pmf by a table.

X \ Y	-1	0	1	$f_X(x)$
0		$\frac{1}{4}$		$\frac{1}{4}$
1	$\frac{1}{4}$		$\frac{1}{4}$	$\frac{1}{2}$
2		$\frac{1}{4}$		$\frac{1}{4}$
$f_Y(y)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	

(b) Are X and Y independent?

- *No, because the space of X and Y is not rectangular. We also see that $f(x,y) \neq f_X(x)f_Y(y)$.*

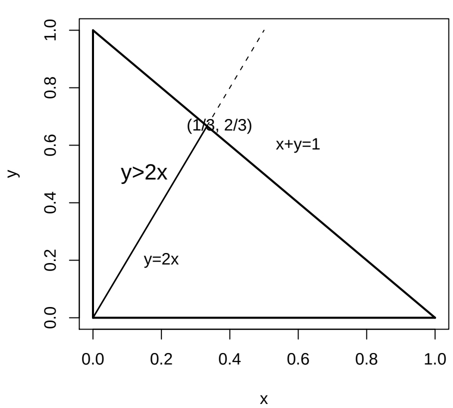
(c) Calculate $\mathrm{Cov}(X,Y)$ and ρ.

- *From the marginal pmf's in the table in (a),* $\mu_X = 0 \times \frac{1}{4} + 1 \times \frac{1}{2} + 2 \times \frac{1}{4} = 1$.
  $\mu_Y = (-1) \times \frac{1}{4} + 0 \times \frac{1}{2} + 1 \times \frac{1}{4} = 0$.
  $\sigma_X^2 = (0-1)^2 \times \frac{1}{4} + (1-1)^2 \times \frac{1}{2} + (2-1)^2 \times \frac{1}{4} = \frac{1}{2}$.
  $\sigma_Y^2 = (-1)^2 \times \frac{1}{4} + 0^2 \times \frac{1}{2} + 1^2 \times \frac{1}{4} = \frac{1}{2}$.
- $E(XY) = 0 \times 0 \times \frac{1}{4} + 1 \times (-1) \times \frac{1}{4} + 1 \times 1 \times \frac{1}{4} + 2 \times 0 \times \frac{1}{4} = 0$.
- *So* $\mathrm{Cov}(X,Y) = E(XY) - \mu_X\mu_Y = 0 - 1 \times 0 = 0$.
  And $\rho = \frac{\mathrm{Cov}(X,Y)}{\sigma_X\sigma_Y} = 0$.
- *This is another example of dependent variables having zero correlation coefficient.*

2. Consider continuous random variables X and Y which have the following joint pdf

$$f(x,y) = 24xy, \quad x > 0,\ y > 0,\ x + y < 1.$$

(a) Sketch a graph of the support of X and Y.



(b) Find the probability $P(Y > 2X)$.

- $P(Y > 2X) = \int_0^{1/3} \int_{2x}^{1-x} 24xydydx = \frac{7}{27}$.

(c) Find the marginal pdf $f_1(x)$ of X.

- $f_1(x) = \int_0^{1-x} 24xydy = 12x(1-x)^2, \quad 0 < x < 1.$

(d) Find the mean $E(X)$.

- $E(X) = \int_0^1 x12x(1-x)^2dx = \frac{2}{5}$.

(e) Find the variance $\mathrm{Var}(X)$.

- $\mathrm{Var}(X) = \int_0^1 (x - \frac{2}{5})^2 12x(1-x)^2dx = \frac{1}{25}$.

(f) Find the covariance $\mathrm{Cov}(X, Y)$.

- *Note that the numeric characteristics of Y are the same as those of X since the two random variables have a symmetrical role.*
- $\mathrm{Cov}(X, Y) = \int_0^1 \int_0^{1-y} (x - \frac{2}{5})(y - \frac{2}{5})24xydxdy = -\frac{2}{75}$

(g) Find the correlation coefficient ρ between X and Y.

- $\rho = \frac{-2/75}{\sqrt{1/25}\sqrt{1/25}} = -\frac{2}{3}$

(h) Find the conditional pdf $h(y|x)$ of Y given $X = x$.

- $h(y|x) = \frac{24xy}{12x(1-x)^2} = \frac{2y}{(1-x)^2}, \quad 0 < y < 1 - x.$

(i) Find the condition probability $P(Y \leq \frac{1}{3}(1 - X)|X = x)$.

- $P(Y \leq \frac{1}{3}(1-X)|X = x) = \int_0^{(1-x)/3} h(y|x)dy = \frac{1}{9}$.

(j) Find the conditional mean $E(Y|X = x)$.

- $E(Y|x) = \int_{-\infty}^{\infty} y \cdot h(y|x)dy = \int_0^{1-x} y\frac{2y}{(1-x)^2}dy = \frac{2(1-x)}{3}$.

3. Show that $\mathrm{Cov}(aX + b, cX + d) = ac\mathrm{Var}(X)$, where X is a random variable and a, b, c are deterministic constants.

- $\mathrm{Cov}(aX + b, cX + d) = E[(aX + b - E[aX + b])(cX + d - E[cX + d])]$
  $= E[(aX - E[aX])(cX - E[cX])] = E[ac(X - E[X])(X - E[X])]$
  $= acE[(X - E[X])^2] = ac\mathrm{Var}(X)$.

4. (*Q4.3-10*). Let the pmf of X be $f_1(x) = \frac{1}{10}$, $x = 0, 1, 2, \cdots, 9$, and the conditional pmf of Y given $X = x$ be $h(y|x) = \frac{1}{10-x}$, $y = x, x+1, \cdots, 9$. Find

(a) the joint pmf $f(x, y)$ of X and Y.

- $f(x, y) = f_1(x)h(y|x) = \frac{1}{10(10-x)}, \quad x = 0, 1, 2, \cdots, 9, y = x, x + 1, \cdots, 9.$

(b) The marginal pmf $f_2(y)$ of Y.

- $f_2(y) = \sum_{x=0}^{y} \frac{1}{10(10-x)}, \quad y = 0, 1, 2, \cdots, 9.$

(c) $E(Y|x)$.

- $E(Y|x) = \sum_{y=x}^{9} y \cdot h(y|x) = \sum_{y=x}^{9} \frac{y}{10-x} = \frac{(x+9)(9-x+1)}{2(10-x)} = \frac{x+9}{2}$.

5. (*Q4.3-17*) The marginal distribution of X is $U(0,1)$. The conditional distribution of Y, given $X = x$, is $U(0, e^x)$.

(a) Determine $h(y|x)$, the conditional pdf of Y, given $X = x$.

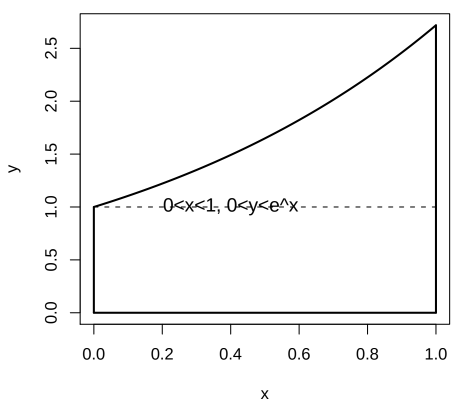
- $h(y|x) = \frac{1}{e^x} = e^{-x}, \quad 0 < y < e^x$.

(b) Find $E(Y|x)$.

- $E(Y|x) = \frac{e^x}{2}, \quad 0 < x < 1$.

(c) Find the joint pdf of X and Y. Sketch the region where $f(x,y) > 0$.

- $f(x,y) = f_1(x)h(y|x) = 1 \times e^{-x} = e^{-x}, \quad 0 < x < 1, 0 < y < e^x$.



(d) Find $f_2(y)$, the marginal pdf of Y.

- $f_2(y) = \int_{-\infty}^{\infty} f(x,y)dx = \begin{cases} \int_0^1 e^{-x}dx = 1 - e^{-1}, & \textit{if}\, 0 < y \le 1 \\ \int_{\ln(y)}^1 e^{-x}dx = y^{-1} - e^{-1}, & \textit{if}\, 1 < y < e \end{cases}$

(e) Find $g(x|y)$, the conditional pdf of X, given $Y = y$.

- $g(x|y) = \frac{f(x,y)}{f_2(y)} = \begin{cases} \frac{e^{-x}}{1-e^{-1}}, & 0 < x < 1, & \textit{if}\, 0 < y \le 1 \\ \frac{e^{-x}}{y^{-1}-e^{-1}}, & \ln(y) < x < 1, & \textit{if}\, 1 < y < e \end{cases}$.

6. (*Q4.4-1*). Let X_1 and X_2 denote two independent random variables, each with a $\chi^2(2)$ distribution. Find the joint pdf of $Y_1 = X_1$ and $Y_2 = X_2 + X_1$. Here note that the support of Y_1 and Y_2 is $0 < y_1 < y_2 < \infty$. Also find the marginal pdf of each of Y_1 and Y_2. Are Y_1 and Y_2 independent?

- *The joint pdf of X_1 and X_2 is*

$$f(x_1, x_2) = \frac{1}{2}e^{-x_1/2} \cdot \frac{1}{2}e^{-x_2/2} = \frac{1}{4}e^{-(x_1+x_2)/2}, \quad 0 < x_1 < \infty, 0 < x_2 < \infty.$$

- *The inverse transformation is* $X_1 = Y_1, X_2 = Y_2 - Y_1$. *So the Jacobian is*

$$J = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1.$$

- *The support of* Y_1 *and* Y_2 *is* $0 < y_1 < \infty, 0 < y_2 < \infty$ *and* $y_1 < y_2$.
- *The joint pdf of* Y_1 *and* Y_2 *is then*

$$g(y_1, y_2) = |J| \cdot f(y_1, y_2 - y_1) = \frac{1}{4}e^{-y_2/2}.$$

- *The marginal pdf of* Y_1 *is*
  $g_1(y_1) = \int_{-\infty}^{\infty} g(y_1, y_2)dy_2 = \int_{y_1}^{\infty} \frac{1}{4}e^{-y_2/2}dy_2 = \frac{1}{2}e^{-y_1/2}, \quad 0 < y_1 < \infty.$
  So $Y_1 \stackrel{d}{=} \exp(2)$.
- *The marginal pdf of* Y_2 *is*
  $g_2(y_2) = \int_{-\infty}^{\infty} g(y_1, y_2)dy_1 = \int_0^{y_2} \frac{1}{4}e^{-y_2/2}dy_1 = \frac{1}{4}y_2e^{-y_2/2}, \quad 0 < y_2 < \infty.$
- *It can be seen that the marginal pdf of* Y_2 *is a* $\chi^2(4)$ *pdf.*
- Y_1 *and* Y_2 *are not independent because* $g(y_1, y_2) \neq g_1(y_1)g_2(y_2)$.