Tutorial 7 Solutions

1. (*Q3.5-1*). Let X have the pdf $f(x) = 4x^3,\ 0 < x < 1$. Find the pdf of $Y = X^2$.

   - *The support of Y is $(0,1)$. And $X = \sqrt{Y} \equiv v(Y)$.*
   - *The transformation $Y = X^2$ is increasing over the support of X. So we can apply the change-of-variable technique to find the pdf of Y:*
     $f_Y(y) = f(v(y))|v'(y)| = 4(\sqrt{y})^3|\frac{1}{2}y^{-1/2}| = 2y, \quad 0 < y < 1.$

2. (*Q3.5-10*). Let X have the uniform distribution $U(-1,3)$. Find the pdf of $Y = X^2$.

   - *The support of Y is $[0,9)$. The transformation $Y = X^2$ is not one-to-one transformation. So it's better to use the distribution-function technique to find the cdf $F_Y(y)$ then pdf $f_Y(y)$ of Y.*
   - *When $0 \le y < 1$, $F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y}) = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{3-(-1)} dx = \frac{\sqrt{y}}{2}$.*
   - *When $1 \le y < 9$, $F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(-1 \le X \le \sqrt{y}) = \int_{-1}^{\sqrt{y}} \frac{1}{3-(-1)} dx = \frac{\sqrt{y}+1}{4}$.*
   - *So the pdf* $f_Y(y) = F'_Y(y) = \begin{cases} \frac{1}{4\sqrt{y}}, & 0 \le y < 1, \\ \frac{1}{8\sqrt{y}}, & 1 \le y < 9. \end{cases}$

3. (*Q3.5-11*). The pdf of X is $f(x) = \theta x^{\theta-1}, \quad 0 < x < 1,\ 0 < \theta < \infty$. Let $Y = -2\theta \ln X$. How is Y distributed?

   - *$Y = -2\theta \ln X$ is a decreasing function over $(0,1)$. $X = e^{-\frac{Y}{2\theta}} \equiv v(Y)$. And $v'(y) = -\frac{1}{2\theta}e^{-\frac{y}{2\theta}}$.*
   - *The support of Y is $(0,+\infty)$.*
   - *The pdf of Y: $f_Y(y) = \theta(e^{-\frac{y}{2\theta}})^{\theta-1}| - \frac{1}{2\theta}e^{-\frac{y}{2\theta}}| = \frac{1}{2}e^{-\frac{y}{2\theta}(\theta-1)-\frac{y}{2\theta}} = \frac{1}{2}e^{-y/2}, \quad y > 0$.*
   - *One can see that Y has an exponential($\theta = 2$) distribution.*

4. (*Q4.1-1*) Let the joint pmf of X and Y be defined by

$$f(x,y) = \frac{x+y}{32}, \quad x = 1,2,\ y = 1,2,3,4.$$

   (a) Find the marginal pmf of X.

   - $f_X(x) = \sum_{y=1}^{4} \frac{x+y}{32} = \frac{x+1}{32} + \frac{x+2}{32} + \frac{x+3}{32} + \frac{x+4}{32} = \frac{4x+10}{32}, \quad x = 1,2.$

   (b) Find the marginal pmf of Y.

   - $f_Y(y) = \sum_{x=1}^{2} \frac{x+y}{32} = \frac{1+y}{32} + \frac{2+y}{32} = \frac{3+2y}{32}, \quad y = 1,2,3,4.$

   (c) Calculate $P(X > Y)$.

   - $P(X > Y) = P(\{X = 2, Y = 1\}) = \frac{2+1}{32} = \frac{3}{32}.$

(d) Calculate $P(Y = 2X)$.

- $P(Y = 2X) = P(\{X = 1, Y = 2\} \cup \{X = 2, Y = 4\}) = \frac{1+2}{32} + \frac{2+4}{32} = \frac{9}{32}$.

(e) Calculate $P(X + Y = 3)$.

- $P(X + Y = 3) = P(\{X = 1, Y = 2\} \cup \{X = 2, Y = 1\}) = \frac{1+2}{32} + \frac{2+1}{32} = \frac{6}{32}$.

(f) Calculate $P(X \le 3 - Y)$.

- $P(X \le 3 - Y) = P(X + Y \le 3) = P(\{X = 1, Y = 1\} \cup \{X = 1, Y = 2\} \cup \{X = 2, Y = 1\}) = \frac{1+1}{32} + \frac{1+2}{32} + \frac{2+1}{32} = \frac{8}{32} = \frac{1}{4}$.

(g) Are X and Y independent?

- *No, because* $f(x, y) \neq f_X(x)f_Y(y)$.

(h) Find $E(X)$.

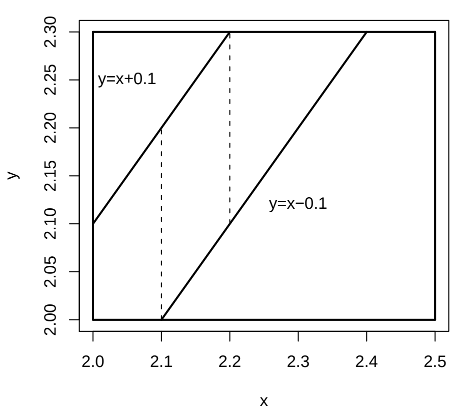
- $E(X) = \sum_{x=1}^{2} x\frac{4x+10}{32} = \frac{4+10}{32} + \frac{36}{32} = \frac{50}{32} = \frac{25}{16}$.

(i) Find $E(X + Y)$.

- $E(X + Y) = \sum_{x=1}^{2} \sum_{y=1}^{4} (x + y)\frac{x+y}{32}$
$= \frac{(1+1)^2+(1+2)^2+(1+3)^2+(1+4)^2+(2+1)^2+(2+2)^2+(2+3)^2+(2+4)^2}{32} = \frac{140}{32}$.

5. (*Q4.1-5*) Two construction companies make bids of X and Y (in \$100,000's) on a remodeling project. The joint pmf of X and Y is uniform on the space $2 < x < 2.5$, $2 < y < 2.3$. If X and Y are within 0.1 of each other, the companies will be asked to rebid; otherwise the lower bidder will be awarded the contract. What is the probability that they will be asked to rebid?

- $P(|X - Y| < 0.1) = P(X - 0.1 < Y < X + 0.1)$
$= \int_{2.0}^{2.1}\int_{2.0}^{x+0.1} \frac{1}{(2.5-2)(2.3-2)} dydx + \int_{2.1}^{2.2}\int_{x-0.1}^{x+0.1} \frac{1}{(2.5-2)(2.3-2)} dydx$
$+ \int_{2.2}^{2.4}\int_{x-0.1}^{2.3} \frac{1}{(2.5-2)(2.3-2)} dydx$
$= \frac{(x-1.9)^2}{0.3}\Big|_{2.0}^{2.1} + \frac{0.2\times0.1}{0.15} - \frac{(x-2.4)^2}{0.3}\Big|_{2.2}^{2.4} = \frac{11}{30}$.
- *In above calculation, note that* $2 < X < 2.5$ *and* $2 < Y < 2.3$ *are implicitly required.*
- *Because the joint pdf is uniform, this probability can also be obtained by calculating the proportion of the relevant area on the support of* X *and* Y.



6. (*Q4.1-9*) Let $f(x,y) = 2e^{-x-y}$, $0 \le x \le y < \infty$, be the joint pdf of X and Y.

   (a) Find the marginal pdf $f_X(x)$ of X.

   - $f_X(x) = \int_x^\infty 2e^{-x-y}dy = 2e^{-2x}, \quad 0 \le x < \infty.$

   (b) Find the marginal pdf $f_Y(y)$ of Y.

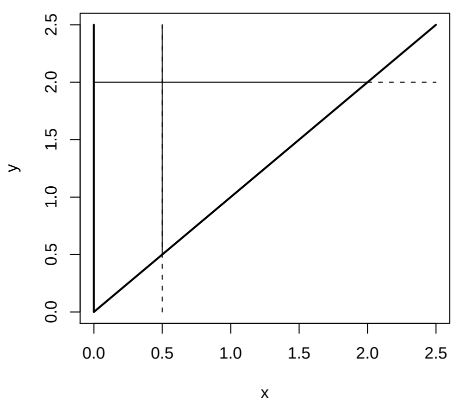
   - $f_Y(y) = \int_0^y 2e^{-x-y}dx = 2e^{-y}(1-e^{-y}), \quad 0 \le y < \infty.$

   (c) Compute $E(X)$ and $E(e^{-X-2Y})$.

   - $E(X) = \int_{-\infty}^\infty x f_X(x)dx = \int_0^\infty x \cdot 2e^{-2x}dx = \frac{1}{2}$. *Or*
     $E(X) = \int_{-\infty}^\infty \int_{-\infty}^\infty x f(x,y)dydx = \int_0^\infty \int_x^\infty x \cdot 2e^{-x-y}dydx = \frac{1}{2}$.
   - $E(e^{-X-2Y}) = \int_0^\infty \int_x^\infty e^{-x-2y} \cdot 2e^{-x-y}dydx$
     $= \int_0^\infty \int_x^\infty 2e^{-2x-3y}dydx = \int_0^\infty \frac{2}{3}e^{5x}dx = \frac{2}{15}$.

   (d) Compute $P(X > \frac{1}{2})$.

   - $P(X > \frac{1}{2}) = \int_{1/2}^\infty f_X(x)dx = \int_{1/2}^\infty 2e^{-2x}dx = e^{-1}$.



   (e) Compute $P(X > \frac{1}{2}, Y > 2)$.

   - $P(X > \frac{1}{2}, Y > 2) = \int_2^\infty \int_{1/2}^y f(x,y)dxdy = \int_2^\infty \int_{1/2}^y 2e^{-x-y}dxdy$
     $= \int_2^\infty (2e^{-\frac{1}{2}-y} - 2e^{-2y})dy = [-2e^{-\frac{1}{2}-y} + e^{-2y}]_2^\infty = 2e^{-2.5} - e^{-4}$.

   (f) Compute $P(Y > 2 | X > \frac{1}{2})$.

   - $P(Y > 2 | X > \frac{1}{2}) = \frac{P(Y>2, X>\frac{1}{2})}{P(X>\frac{1}{2})} = \frac{2e^{-2.5} - e^{-4}}{e^{-1}} = 2e^{-1.5} - e^{-3}$.