Lab 2 Solutions

1. (*Q1.4-10*) An urn contains 17 balls marked LOSE and 3 balls marked WIN. You and an opponent take turns selecting at random a single ball from the urn without replacement. The person who selects the third WIN ball wins the game. It does not matter who selected the first two WIN balls. Use Maple to calculate the probabilities asked in the following.

   (a) If you draw first, find the probability that you win the game on your second draw.

   - $P(\textit{You win on your second draw}) = \frac{\binom{3}{2}\binom{17}{0}}{\binom{20}{2}} \times \frac{1}{18} = \frac{1}{1140}$.

   (b) If you draw first, find the probability that your component wins the game on his second draw.

   - $P(\textit{Your component wins on his second draw}) = \frac{\binom{3}{2}\binom{17}{1}}{\binom{20}{3}} \times \frac{1}{17} = \frac{1}{380}$.

   (c) If you draw first, the probability that you win can be found from

   $$P(\text{You win if you draw first}) = \sum_{k=1}^{9} \frac{\binom{3}{2}\binom{17}{2k-2}}{\binom{20}{2k}} \times \frac{1}{20-2k} \qquad \text{(Why?)}$$

   Note: You could win on your second, third, fourth, ..., or tenth draw, not on your first.

   ```
   > binomial(3,2)*sum(binomial(17, 2*k-2)/((20-2*k)*binomial(20, 2*k)), k = 1..9)
                                                               Answer:  35/76
   ```

   (d) If you draw second, the probability that you win can be found from

   $$P(\texttt{You win if you draw second}) = \sum_{k=1}^{9} \frac{\binom{3}{2}\binom{17}{2k-1}}{\binom{20}{2k+1}} \times \frac{1}{19-2k}. \qquad \texttt{(Why?)}$$

   ```
   > binomial(3,2)*sum(binomial(17, 2*k-1)/((21-2*k-2)*binomial(20, 2*k+1)), k = 1..9)
                                                                 Answer 41/76
   ```

   (e) Based on your results in (c) and (d), would you prefer to draw first or second? Why?

   - *Draw second, as the probability of winning is larger.*

2. (*Q1.5.20*) An urn contains n balls numbered from 1 to n. A random sample of n balls is selected from the urn, one at a time. A match occurs if ball numbered i is selected on the ith draw.

   - If the draws are done *with replacement*, it can be shown that

   $$P(\text{at least one match}) = 1 - \left(1 - \frac{1}{n}\right)^n.$$

   - If the draws are done *without replacement*, it can be shown that

   $$P(\text{at least one match}) = 1 - \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right) = 1 - \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

(a) For each value of n given in the following table, use Maple to find P(at least one match) and write down the results at appropriate places of the table.

Note:

- To get the "with replacement" results, you need to use the commands `eval(..., n=...)`, `limit(..., n=infinity)`, `evalf(eval(..., n=...))` and `evalf(limit(...,n=infinity))`.
- To get the "without replacement" results, you will need to use `sum(...)` command as expected. Also use `evalf(sum(...))` command to return the result of `sum` in floating-point format.

(b) We can also use R to simulate the processes of drawing n balls with or without replacement from a set of n balls numbered from 1 to n. We can then simulate the probability of at least one match using the relevant relative frequencies.

- Follow the following to create a function `match.f` in R.

```
> match.f=function(){}
> match.f=edit(match.f)
```

  Save the following as the content of `match.f`:

```
function(n, simsize, rep=TRUE){
freq=0
for(i in 1:simsize){
sam=sample(1:n, size=n,replace=rep)
freq=freq + (sum(sam==1:n)>=1)
}
freq/simsize
}
```

- Note that `(sum(sam==1:n)>=1)` in `match.f` is for checking whether or not there is at least 1 match in `sam`.
- Now we simulate the drawing process 1000 times (`simsize=1000`) for each given n and `rep` (`rep=TRUE` indicates the "with replacement" procedure is used.) Execute the following and write down the results at appropriate places in the table.

```
> match.f(n=1,simsize=1000,rep=TRUE)
> match.f(n=3,simsize=1000,rep=TRUE)
> match.f(n=10,simsize=1000,rep=TRUE)
> match.f(n=15,simsize=1000,rep=TRUE)
> match.f(n=100,simsize=1000,rep=TRUE)
> match.f(n=10000,simsize=1000,rep=TRUE)
> match.f(n=1,simsize=1000,rep=FALSE)
> match.f(n=3,simsize=1000,rep=FALSE)
> match.f(n=10,simsize=1000,rep=FALSE)
> match.f(n=15,simsize=1000,rep=FALSE)
> match.f(n=100,simsize=1000,rep=FALSE)
```

```
> match.f(n=10000,simsize=1000,rep=FALSE)
```

n	P(at least one match)			
	with replacement		**without replacement**	
	by Maple	by R simulation	by Maple	by R simulation
1				
3				
10				
15				
100				
∞				

n	P(*at least one match*)			
	with replacement		**without replacement**	
	by Maple	*by R simulation*	*by Maple*	*by R simulation*
1	*1*	*1*	*1*	*1*
3	$\frac{19}{27} \approx 0.704$	*0.731*	$\frac{2}{3} \approx 0.667$	*0.692*
10	$\frac{6513215599}{10^{10}} \approx 0.651$	*0.665*	$\frac{28319}{44800} \approx 0.632$	*0.623*
15	$\frac{282325794823047151}{437893890380859375} \approx 0.645$	*0.634*	$\frac{590434418019}{93405312000} \approx 0.632$	*0.628*
100	≈ 0.634	*0.637*	≈ 0.632	*0.626*
∞	$1 - e^{-1} \approx 0.632$	*0.632* ($n = 10^4$)	$1 - e^{-1} \approx 0.632$	*0.634* ($n = 10^4$)