THE UNIVERSITY OF MELBOURNE

Student ID:

# The University of Melbourne

## Semester 1 Exam *Solutions* — *June, 2018*

## School of Mathematics and Statistics

## MAST20006 Probability for Statistics

**Exam Duration: 3 Hours**

**Reading Time: 15 Minutes**

**This paper has 9 pages**



**Authorised materials:**
This is a closed book exam.
A University approved hand-held calculator, i.e. Casio FX82 (any suffix) or Casio FX100 (any suffix), may be used.

**Instructions to Invigilators:**
Script books shall be supplied to each student.
Students may not take this paper with them at the end of the exam.

**Instructions to Students:**
This paper has **10** questions. Formula sheet is given at the end of this paper.
Attempt as many questions, or parts of questions, as you can.
Questions carry marks as shown in the brackets after the questions statements.
The total number of marks available for this examination is **100**.
Working and/or reasoning must be given to obtain full credit.

1. A certain human cancer have incidence rate of 1 in 500. If a person does have the disease, with probability 0.945 the current diagnostic test on him or her will return a true positive result. If a person does not have the disease, the diagnostic test will return a false positive result with probability 0.005.

   (a) Compute the probability that a randomly chosen person will test positive. [3]

   - $P(\textit{positive test}) = \frac{1}{500} \times 0.945 + \frac{499}{500} \times 0.005 = 0.00688.$

   (b) Compute the probability that a person with a positive test result has the cancer. [3]

   - *Thus* $P(\textit{cancer} \mid \textit{positive test}) = \frac{\frac{1}{500} \times 0.945}{0.00688} = \frac{0.00189}{0.00688} = 0.2747.$

   (c) Four people are found to have positive test results. Let $X$ be the number of people among these four that actually have the cancer. Compute the probability $P(X = 1)$. [3]

   - $X \stackrel{d}{=} Bin(4, 0.2747).$
   - *Thus* $P(X = 1) = 4 \times 0.2747 \times (1 - 0.2747)^3 = 0.4192.$

2. Engines manufactured on an assembly line have a 5% defect rate. Now engines are randomly selected from this assembly line one at a time and tested. Let $X$ be such that the first defective engine is observed when the $X$th one was tested. Let $Y$ be such that the third defective is observed when the $Y$th engine was tested.

   (a) Name the probability distribution and specify the value of any parameter(s) for each of the two random variables $X$ and $Y$. [2]

   - $X \stackrel{d}{=} \textit{Geometric}(p = 0.05), \quad Y \stackrel{d}{=} \textit{Negative binomial}(r = 3, p = 0.05).$

   (b) What is the probability that $X \geq 10$? [2]

   - $P(X \geq 10) = \sum_{k=10}^{\infty} 0.05 \times 0.95^{k-1} = 0.95^9 = 0.6302.$

   (c) What is the probability that $Y = 30$ exactly? [2]

   - $P(Y = 30) = \binom{29}{2} \times 0.05^3 \times 0.95^{27} = 0.0127.$

   (d) Compute the probability that more than 80 engines need to be tested to get three defective ones. [3]

   - *Let $Z$ be the number of defective engines found after testing 80 engines. Then* $Z \stackrel{d}{=} b(80, 0.05)$. *By the relation between* $b(n,p)$ *and* $Nb(r,p)$, $P(Y > 80) = P(Z \leq 2) = 0.95^{80} + 80 \cdot 0.05 \cdot 0.95^{79} + \binom{80}{2} \cdot 0.05^2 \cdot 0.95^{78} = 0.0165 + 0.0695 + 0.1446 = 0.2306.$

   (e) Compute $\mathrm{Var}(0.8^X + 80)$. [3]

   - *The mgf of $X$ is* $M(t) = \dfrac{0.05e^t}{1 - 0.95e^t}$ *with* $t < -\ln 0.95 = 0.05129.$
   - $Var(0.8^X + 80) = E(0.64^X) - \left[E(0.8^X)\right]^2 = M(\ln 0.64) - M(\ln 0.8)^2$
     $= \dfrac{0.05 \times 0.64}{1 - 0.95 \times 0.64} - \left(\dfrac{0.05 \times 0.8}{1 - 0.95 \times 0.8}\right)^2 = 0.08163265 - 0.1666667^2$
     $= \frac{4}{49} - \frac{1}{36} = \frac{95}{1764} = 0.0538549$

3. The chance of a set of triplets is 1 in 1000 births in a community. Let $X$ be the number of triplets to be observed from 2500 births. Note that $X$ follows a binomial distribution.

   (a) Find $P(X \leq 2)$. [2]

   - $P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$
     $= 0.999^{2500} + 2500 \times 0.001 \times 0.999^{2499} + \binom{2500}{2} \times 0.001^2 \times 0.999^{2498}$
     $= 0.08198 + 0.20516 + 0.25661 = 0.54375.$

   (b) A binomial distribution $b(n, p)$ can be approximated by a Poisson($\lambda = np$) distribution if $p$ is small and $n$ is large. Use this result to approximate the probability in part (a) by a Poisson probability. [2]

   - $X \overset{d}{\approx} \mathit{Poisson}(\lambda = 2.5)$, *thus*
     $$P(X \leq 2) \approx e^{-2.5} + 2.5e^{-2.5} + \frac{2.5^2}{2!}e^{-2.5} = 6.625e^{-2.5} = 0.54381.$$

   (c) The probability in part (a) may also be approximated by a normal probability based on the central limit theorem. Give a normal approximation (using the continuity correction) to $P(X \leq 2)$. [2]

   - *By CLT,* $X \overset{d}{\approx} N(\mu = 2.5, \sigma^2 = 2.4975)$.
   - *So* $P(X \leq 2) \approx P\left(Z \leq \frac{2+0.5-2.5}{\sqrt{2.4975}}\right) = P(Z \leq 0) = \Phi(0) = 0.5.$

4. Imagine an experiment in which you flip two fair coins A and B and remove all that come up heads. Flip again the remaining coin(s), and remove the heads if any heads turn up. Continue flipping the remaining coins until all the coins are removed. Let $X_A$ be the number of times A will have been flipped, and $X_B$ the number of times B will have been flipped in this experiment. Also define $Y = \max\{X_A, X_B\}$.

   (a) What is the name of the probability distribution that $X_A$ and $X_B$ follow? [1]

   - *Both* $X_A$ *and* $X_B$ *follow a Geometric* ($p = 0.5$) *distribution.*

   (b) Derive the cumulative distribution function (cdf) of $Y$. [2]

   - $F(y) = P(Y \leq y) = P(X_A \leq y)P(X_B \leq y) = (1 - 0.5^y)^2, \ y = 1, 2, \cdots.$

   (c) Derive the probability mass function (pmf) of $Y$. [3]

   - $f(y) = F(y) - F(y-1) = (1-0.5^y)^2 - (1-0.5^{y-1})^2$
     $= (0.5^{y-1} - 0.5^y)(2 - 0.5^y - 0.5^{y-1}) = 0.5^y(2 - 3 \cdot 0.5^y) = 2 \cdot 0.5^y - 3 \cdot 0.5^{2y},$
     $y = 1, 2, \cdots.$

5. Let $X$ be a continuous random variable with pdf (probability density function)

$$f(x) = \frac{1}{30}(x+6), \quad -4 < x < 2.$$

(a) Find the cdf (cumulative distribution function) of $X$. [2]

- *For* $-4 < x < 2$, $P(X \le x) = \int_{-4}^{x} \frac{1}{30}(t+6)dt = \frac{1}{60}(t+6)^2 \mid_{-4}^{x} = \frac{(x+6)^2-4}{60}$.
- *Hence the cdf of* $X$ *is* $F(x) = \begin{cases} 0 & \text{if } x \le -4 \\ \frac{(x+6)^2-4}{60} & \text{if } -4 < x < 2 \\ 1 & \text{if } x \ge 2 \end{cases}$

(b) Let $X_1$ and $X_2$ be two independent random variables each having the pdf $f(x)$ given above. Define $W = \min\{X_1, X_2\}$. Find the 75-th percentile of $W$. [3]

- $P(W > w) = P(\min\{X_1, X_2\} > w) = P(X_1 > w)P(X_2 > w)$. *Using this property the cdf of* $W$ *can be found to be*

$$F_W(w) = \begin{cases} 0 & \text{if } w \le -4 \\ 1 - \left[1 - \frac{(w+6)^2-4}{60}\right]^2 & \text{if } -4 < w < 2 \\ 1 & \text{if } w \ge 2 \end{cases}$$

- *Solve* $\frac{3}{4} = F_w(\pi_{0.75}) = 1 - \left[1 - \frac{1}{60}(\pi_{0.75}+6)^2 + \frac{4}{60}\right]^2$, *and it follows* $\pi_{0.75} = \sqrt{34} - 6 = -0.1690481$.

(c) Consider the transformation $Y = X^2$ of $X$.

i. Is this transformation one-to-one? Find the support of $Y$. [2]

- *This is not one-to-one. The support of* $Y$ *is* $0 \le y < 16$.

ii. Derive the cdf of $Y$. [3]

- *For* $0 \le y \le 4$,
  $G(y) = P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y})$
  $= \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{30}(x+6)dx = \frac{1}{60}[(6+\sqrt{y})^2 - (6-\sqrt{y})^2] = \frac{2}{5}\sqrt{y}$.
- *For* $4 < y < 16$,
  $G(y) = P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le 2)$
  $= \int_{-\sqrt{y}}^{2} \frac{1}{30}(x+6)dx = \frac{64}{60} - \frac{1}{60}(6-\sqrt{y})^2 = \frac{1}{5}\sqrt{y} - \frac{1}{60}y + \frac{7}{15}$.
- *So the cdf of* $Y$ *is*

$$G(y) = \begin{cases} 0, & y < 0, \\ \frac{2}{5}\sqrt{y}, & 0 \le y \le 4, \\ \frac{1}{5}\sqrt{y} - \frac{1}{60}y + \frac{7}{15}, & 4 < y < 16, \\ 1, & y \ge 16. \end{cases}$$

iii. Find the pdf of $Y$. [2]

- $$g(y) = G'(y) = \begin{cases} 0, & y < 0, \\ \frac{1}{5\sqrt{y}}, & 0 \le y \le 4, \\ \frac{1}{10\sqrt{y}} - \frac{1}{60} & 4 < y < 16, \\ 0, & y \ge 16. \end{cases}$$

6. Let $X_1, X_2, X_3$ be independent Poisson($\lambda = 1$) random variables.
Define $Y_1 = X_1 + X_3$ and $Y_2 = X_2 + X_3$.

(a) Compute the value of $\mathrm{Cov}(2Y_1 + 3, 4Y_2 + 5)$. [2]

- $Cov(2Y_1 + 3, 4Y_2 + 5) = 8\,Cov(Y_1, Y_2) = 8\,Cov(X_1 + X_3, X_2 + X_3) = 8\,Cov(X_3, X_3) = 8\,Var(X_3) = 8\lambda = 8.$

(b) Use Chebyshev's inequality $P(|X - \mu| < k\sigma) \geq 1 - \frac{1}{k^2}$ to find a lower bound for $P(|Y_1 - 2| < 2)$. [1]

- $E(Y_1) = 2E(X_1) = 2\lambda = 2$ *and* $Var(Y_1) = 2\,Var(X_1) = 2\lambda = 2.$
- $P(|Y_1 - 2| < 2) = P(|Y_1 - 2| < \sqrt{2}\sqrt{2}) \geq 1 - \frac{1}{\sqrt{2}^2} = 0.5.$

(c) Compute the exact value of $P(|Y_1 - 2| < 2)$. [2]

- $Y_1$ *has a Pois(2) distribution because* $X_1$ *and* $X_2$ *are iid Pois(1). Thus*

$$\begin{aligned} P(|Y_1 - 2| < 2) &= P(0 < Y_1 < 4) = P(1 \leq Y_1 \leq 3) \\ &= P(Y_1 = 1, 2, 3) = \frac{16}{3}e^{-2} = 0.7218. \end{aligned}$$

(d) Define

$$Z_1 = \begin{cases} 1 & \text{if } Y_1 = 0, \\ 0 & \text{otherwise;} \end{cases} \quad \text{and} \quad Z_2 = \begin{cases} 1 & \text{if } Y_2 = 0, \\ 0 & \text{otherwise.} \end{cases}$$

i. Compute $P(Z_1 = 1)$ and $P(Z_2 = 1)$ respectively. [2]

- *Both* $Z_1$ *and* $Z_2$ *are Bernoulli r.v.s with* $P(Z_1 = 1) = P(Z_2 = 1) = P(X_1 = X_3 = 0) = P(X_2 = X_3 = 0) = e^{-2}.$

ii. Compute the joint pmf (joint probability mass function) of $(Z_1, Z_2)$. [3]

- $P(Z_1 = 1, Z_2 = 1) = P(X_1 = X_2 = X_3 = 0) = e^{-3}.$
- $P(Z_1 = 1, Z_2 = 0) = P(X_1 = X_3 = 0, X_2 > 0) = e^{-2}(1 - e^{-1}).$
- $P(Z_1 = 0, Z_2 = 1) = P(X_1 > 0, X_2 = X_3 = 0) = (1 - e^{-1})e^{-2}.$
- $P(Z_1 = 0, Z_2 = 0) = 1 - e^{-3} - 2e^{-2}(1 - e^{-1}) = 1 - 2e^{-2} + e^{-3}.$

iii. Compute the correlation coefficient between $Z_1$ and $Z_2$. [3]

- $E(Z_1) = P(Z_1 = 1) = e^{-2}$
- $Var(Z_1) = P(Z_1 = 1)P(Z_1 = 0) = e^{-2}(1 - e^{-2}).$
- $Z_1$ *and* $Z_2$ *have the same distribution. So*
  $E(Z_2) = e^{-2}$ *and* $Var(Z_2) = e^{-2}(1 - e^{-2}).$
- $E(Z_1 Z_2) = P(Z_1 = Z_2 = 1) = e^{-3}.$
- *So* $Cov(Z_1, Z_2) = e^{-3} - e^{-4} = e^{-3}(1 - e^{-1}).$
- *Therefore,*

$$\rho(Z_1, Z_2) = \frac{Cov(Z_1, Z_2)}{\sqrt{Var(Z_1)}\sqrt{Var(Z_2)}} = \frac{e^{-3}(1 - e^{-1})}{e^{-2}(1 - e^{-2})} = \frac{e^{-1}}{1 + e^{-1}} \overset{or}{=} 0.2689.$$

7. Suppose $X$ and $Y$ are continuous random variables with the joint pdf

$$f(x,y) = \begin{cases} \frac{3}{2} & \text{if } 0 \le x \le 1 \text{ and } x^2 \le y \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the marginal pdf of $X$. Is the marginal pdf of $X$ a uniform pdf? [2]

- *The marginal of $X$ is* $f_1(x) = \int_{x^2}^{1} \frac{3}{2} dy = \frac{3}{2}(1-x^2), \quad 0 \le x \le 1.$
- $f_1(x)$ *is not a uniform pdf.*

(b) Find the conditional pdf of $Y$ given $X = x$, $0 \le x \le 1$. Is it a uniform pdf? [2]

- $h(y|x) = \frac{f(x,y)}{f_1(x)} = \frac{\frac{3}{2}}{\frac{3}{2}(1-x^2)} = \frac{1}{1-x^2}, \quad x^2 \le y \le 1; \;\; 0 \le x \le 1.$
- *So* $h(y|X=x)$ *is a Uniform*($x^2$, 1) *pdf.*

(c) Are $X$ and $Y$ independent? Explain. [2]

- *$X$ and $Y$ are not independent because $h(y|x)$ depends on $x$, or because the support of $(X,Y)$ is not rectangular.*

(d) Compute the probability $P(Y \le X | X = \frac{1}{2})$. [2]

- $P(Y \le X|X = \frac{1}{2}) = P(Y \le \frac{1}{2}|X = \frac{1}{2}) = \int_{1/4}^{1/2} h(y|x = \frac{1}{2})dy$
  $= \int_{1/4}^{1/2} \frac{1}{1-(1/2)^2} dy = \frac{1}{3}.$

(e) Compute the probability $P(Y \le X)$. [3]

- $P(Y \le X) = \int_0^1 \int_{x^2}^{x} \frac{3}{2} dy dx = \int_0^1 \frac{3}{2}(x - x^2)dx = \left\{\frac{3}{4}x^2 - \frac{1}{2}x^3\right\}_0^1 = \frac{1}{4}.$

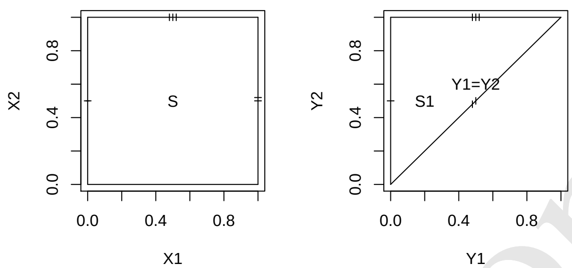
8. Consider two random variables $X_1$ and $X_2$ with the joint probability density

$$f(x_1, x_2) = \begin{cases} 12x_1x_2(1-x_2), & 0 < x_1 < 1, \;\; 0 < x_2 < 1, \\ 0 & \text{elsewhere.} \end{cases}$$

Let $Y_1 = X_1X_2$ and $Y_2 = X_2$ be a joint transformation of $(X_1, X_2)$.

(a) Find the support of $(Y_1, Y_2)$ and sketch it. [3]

- *The support of* $(Y_1, Y_2)$ *is* $\{0 < Y_1 < Y_2 < 1\}$.




(b) Find the inverse transformation. [2]

- $X_1 = Y_1/Y_2, X_2 = Y_2.$

(c) Compute the Jacobian of the inverse transformation. [2]

- $J = \begin{vmatrix} 1/y_2 & -y_1/y_2^2 \\ 0 & 1 \end{vmatrix} = \dfrac{1}{y_2}.$

(d) Compute the joint pdf of $(Y_1, Y_2)$. [2]

- $g(y_1, y_2) = |J| \cdot f(\dfrac{y_1}{y_2}, y_2) = \dfrac{1}{y_2} \cdot 12\dfrac{y_1}{y_2}y_2(1-y_2) = \dfrac{12y_1(1-y_2)}{y_2}, \quad 0 < y_1 < y_2 < 1.$

(e) Find the marginal pdf of $Y_1$ from the joint pdf of $(Y_1, Y_2)$. [2]

- $g_1(y_1) = \displaystyle\int_{y_1}^{1} \frac{12y_1(1-y_2)}{y_2} dy_2 = 12y_1^2 - 12y_1 \ln(y_1) - 12y_1, \quad 0 < y_1 < 1.$

9. Let $X_1, X_2, \cdots, X_n$ be independent random variables each having the moment-generating function (mgf)

$$M(t) = \frac{1-2t}{(1-t)(1-3t)}, \quad t < \frac{1}{3}.$$

(a) Compute the mgf $M_{Y_n}(t)$ of the sum $Y_n = X_1 + X_2 + \cdots + X_n$. [2]

- $M_{Y_n}(t) = E(e^{tY_n}) = [M_{X_1}(t)]^n = \dfrac{(1-2t)^n}{(1-t)^n(1-3t)^n}, \quad t < \dfrac{1}{3}.$

(b) Compute the mgf $M_{\bar{Y}_n}(t)$ of the sample mean $\bar{Y}_n = \frac{Y_n}{n}$. [2]

- $M_{\bar{Y}_n}(t) = E(e^{t\bar{Y}_n}) = M_{Y_n}\left(\dfrac{t}{n}\right) = \dfrac{(1-\frac{2t}{n})^n}{(1-\frac{t}{n})^n(1-\frac{3t}{n})^n}, \quad t < \dfrac{n}{3}.$

(c) Compute the limiting mgf $\lim_{n\to\infty} M_{\bar{Y}_n}(t)$. What distribution does the limiting mgf correspond to? What is the implication of this result? [2]

- *For any given* $t$, $\displaystyle\lim_{n\to\infty} M_{\bar{Y}_n}(t) = \lim_{n\to\infty} \frac{(1-\frac{2t}{n})^n}{(1-\frac{t}{n})^n(1-\frac{3t}{n})^n} = \frac{e^{-2t}}{e^{-t}e^{-3t}} = e^{2t}$
- *The limit is the mgf of a degenerate distribution having probability 1 at 2.*
- *This implies that* $\bar{Y}_n \xrightarrow{p} 2 = E(X_1)$ *as* $n \to \infty$.
- *Note it is also a correct answer if it is based on applying the WLLN. Then the WLLN must be correctly stated, and* $E(X_1) = 3$ *be proved.*

(d) Let $Z_n = \sqrt{n}(\bar{Y}_n - 2)$. Compute $M_{Z_n}(t)$, the mgf of $Z_n$. Then use this result to compute $\lim_{n\to\infty} M_{Z_n}(t)$. Finally explain what is the limiting distribution of $Z_n$ as $n \to \infty$. [5]

- $M_{Z_n}(t) = E(e^{t\sqrt{n}(\bar{Y}_n - 2)}) = e^{-2t\sqrt{n}} M_{\bar{Y}_n}(\sqrt{n}t)$

$$= \frac{e^{-2t\sqrt{n}}(1-\frac{2\sqrt{n}t}{n})^n}{(1-\frac{\sqrt{n}t}{n})^n(1-\frac{3\sqrt{n}t}{n})^n} = \left[\frac{e^{-2t/\sqrt{n}}(1-\frac{2t}{\sqrt{n}})}{1-\frac{4t}{\sqrt{n}}+\frac{3t^2}{n}}\right]^n, \quad t < \frac{\sqrt{n}}{3}.$$

- *By Taylor's series expansion,* $e^u \approx 1 + u + \frac{1}{2}u^2$ *when* $|u|$ *is sufficiently small. Using this result, for any given* $t$,

$$\begin{aligned}
\lim_{n\to\infty} M_{Z_n}(t) &= \lim_{n\to\infty}\left[\frac{e^{-2t/\sqrt{n}}(1-\frac{2t}{\sqrt{n}})}{1-\frac{4t}{\sqrt{n}}+\frac{3t^2}{n}}\right]^n = \lim_{n\to\infty}\left[\frac{(1-\frac{2t}{\sqrt{n}}+\frac{4t^2}{2n})(1-\frac{2t}{\sqrt{n}})}{1-\frac{4t}{\sqrt{n}}+\frac{3t^2}{n}}\right]^n \\
&= \lim_{n\to\infty}\left[\frac{1-\frac{4t}{\sqrt{n}}+\frac{6t^2}{n}}{1-\frac{4t}{\sqrt{n}}+\frac{3t^2}{n}}\right]^n = \lim_{n\to\infty}\left(1+\frac{3t^2}{n}\right)^n = e^{3t^2}
\end{aligned}$$

- *The limit mgf is that of* $N(\mu = 0, \sigma^2 = 6)$.
- *This implies that* $Z_n \xrightarrow{d} N(0, 6)$ *as* $n \to \infty$.
- *Note it is also a correct answer if it is based on applying the CLT. Then the CLT must be correctly stated, and* $Var(X_1) = 6$ *be proved.*

10. A random variable $X$ has the following mgf:

$$M(t) = \frac{1}{2}\left[\frac{2}{1-4t} - \frac{1}{1-2t}\right] + \frac{1}{2(1-t)}, \quad t < \frac{1}{4}.$$

(a) Compute the value of $E(X)$. [2]

- $M'(t) = \dfrac{4}{(1-4t)^2} - \dfrac{1}{(1-2t)^2} + \dfrac{1}{2(1-t)^2}$.
- *Thus* $E(X) = M'(0) = 4 - 1 + 0.5 = 3.5$.

(b) Compute the probability $P(X > 4)$. [7]

- *It can be seen that* $\frac{2}{1-4t} - \frac{1}{1-2t} = \frac{1}{(1-4t)(1-2t)}$ *is the mgf of random variable* $W = X_1 + X_2$, *where* $X_1 \overset{d}{=} Exp(2)$, $X_2 \overset{d}{=} Exp(4)$, *and* $X_1$ *and* $X_2$ *are independent of each other. Joint pdf of* $(X_1, X_2)$ *is* $g(x_1, x_2) = \frac{1}{8}e^{-\frac{x_1}{2}-\frac{x_2}{4}}$, $x_1 > 0, x_2 > 0$.
- *Hence the given* $M(t)$ *corresponds to a random variable which has probability 0.5 to follow the distribution of* $W$ *and another probability 0.5 to follow the* $Exp(1)$ *distribution.*
- *Now define* $V = X_2$ *together with* $W = X_1 + X_2$, *thus* $(W, V)$ *is a random vector transformed from* $(X_1, X_2)$. *Hence, the joint pdf of* $(W, V)$ *is*

$$f(w, v) = g(w - v, v)|J| = \frac{1}{8}e^{-\frac{1}{4}(2w-v)} \cdot 1, \quad w > v > 0$$

- *The pdf of* $W$ *is*

$$f_1(w) = \int_0^w \frac{1}{8}e^{-\frac{1}{4}(2w-v)}dv = \frac{1}{2}e^{-\frac{1}{4}w} - \frac{1}{2}e^{-\frac{1}{2}w}, \quad w > 0.$$

- *Let* $U$ *be an* $Exp(1)$ *random variable. Then*

$$\begin{aligned} P(X > 4) &= \frac{1}{2}P(U > 4) + \frac{1}{2}P(W > 4) \\ &= \frac{1}{2}\int_4^\infty e^{-u}du + \frac{1}{2}\int_4^\infty \left(\frac{1}{2}e^{-\frac{1}{4}w} - \frac{1}{2}e^{-\frac{1}{2}w}\right)dw \\ &= \frac{1}{2}e^{-4} + \frac{1}{2}\left(-2e^{-\frac{1}{4}x} + e^{-\frac{1}{2}x}\right)\Big|_4^\infty \\ &= \frac{1}{2}e^{-4} + \frac{1}{2}(2e^{-1} - e^{-2}) \overset{or}{=} 0.3093696 \end{aligned}$$

Total marks = 100

**End of the exam questions.**
**Formulas are on the next page.**