Tutorial 9 Solutions

1. We have learned the following in lectures:

    - (Example 20, slide 4.58) If X_1 and X_2 are independent gamma random variables with parameters (θ, α) and (θ, β) respectively, then the random variable $\frac{X_1}{X_1+X_2}$ has a beta(α, β) distribution.
    - (Example 21, slide 4.62) If X_1 and X_2 are independent chi-square random variables with degrees of freedom being r_1 and r_2 respectively, then the random variable $\frac{X_1/r_1}{X_2/r_2}$ has an $F(r_1, r_2)$ distribution.

    Using these results, name the distributions of the following random variables defined from X and Y, where X and Y are independent random variables both having an exponential $(\theta = 2)$ distribution.

    (a) $U = \frac{X}{X+Y}$

    (b) $V = \frac{Y}{X+Y}$

    (c) $W = \frac{X}{Y}$

    (d) $Z = \frac{Y}{X}$.

    - *The exponential($\theta = 2$) distribution is the same as the gamma($\theta = 2, \alpha = 1$) distribution, which is also a $\chi^2(2)$ distribution.*
    - *Therefore, $U \stackrel{d}{=} V \stackrel{d}{=} \text{beta}(1,1) \equiv U(0,1)$, and $W \stackrel{d}{=} Z \stackrel{d}{=} F(2,2)$.*

2. (*Q4.5-2*). Let X_1 and X_2 be independent random variables with respective binomial distributions $b(3, 1/2)$ and $b(5, 1/2)$. Determine

    (a) $P(X_1 = 2, X_2 = 4)$.

    - *Because X_1 and X_2 are independent,*

    $$\begin{aligned} P(X_1 = 2, X_2 = 4) &= P(X_1 = 2)P(X_2 = 4) \\ &= \left[\binom{3}{2}0.5^2(1-0.5)^1\right] \cdot \left[\binom{5}{4}0.5^4(1-0.5)^1\right] = \frac{15}{2^8} = \frac{15}{256}. \end{aligned}$$

    (b) $P(X_1 + X_2 = 7)$.

    - *There are two ways to do this part.*
    - *Method 1: $X_1 + X_2$ is the number of 'successes' in 3+5 independent Bernoulli trials having probability of 'success' being 1/2. Hence $X_1 + X_2$ has a $b(8, 1/2)$ distribution. Therefore*

    $$P(X_1 + X_2 = 7) = \binom{8}{7}0.5^7(1-0.5)^1 = \frac{8}{2^8} = \frac{1}{32}.$$

- *Method 2:* $\{X_1 + X_2 = 7\}$ *can occur in two mutually exclusive ways:* $\{X_1 = 3, X_2 = 4\}$ *or* $\{X_1 = 2, X_2 = 5\}$. *Therefore*

$$\begin{aligned} P(X_1 + X_2 = 7) &= P(X_1 = 3, X_2 = 4) + P(X_1 = 2, X_2 = 5) \\ &= P(X_1 = 3)P(X_2 = 4) + P(X_1 = 2)P(X_2 = 5) = \frac{5}{2^8} + \frac{3}{2^8} = \frac{1}{32}. \end{aligned}$$

3. (*Q4.5-3*). Let X_1 and X_2 be independent random variables with probability density functions $f_1(x_1) = 2x_1$, $0 < x_1 < 1$, and $f_2(x_2) = 4x_2^3$, $0 < x_2 < 1$, respectively. Compute

   (a) $P(0.5 < X_1 < 1.0 \text{ and } 0.4 < X_2 < 0.8)$.

   - $$\begin{aligned} &P(0.5 < X_1 < 1.0 \text{ and } 0.4 < X_2 < 0.8) \\ &\quad = P(0.5 < X_1 < 1)P(0.4 < X_2 < 0.8) \\ &\quad = \left(\int_{0.5}^{1} 2x_1 dx_1\right)\left(\int_{0.4}^{0.8} 4x_2^3 dx_2\right) = (1 - 0.5^2)(0.8^4 - 0.4^4) = 0.288. \end{aligned}$$

   (b) $E(X_1^2 X_2^3)$.

   - $E(X_1^2X_2^3) = E(X_1^2)E(X_2^3) = \left(\int_0^1 x_1^2 \cdot 2x_1 dx_1\right)\left(\int_0^1 x_2^3 \cdot 4x_2^3 dx_2\right) = \frac{2}{4} \cdot \frac{4}{7} = \frac{2}{7}$.

4. (*Q4.5-13*) Flip $n = 8$ fair coins and remove all that came up heads. Flip the other (tails) coins and remove the heads. Continuing flipping the remaining coins until each has come up heads. We shall find the pmf of Y, the number of trials needed to finish off all coins. Let X_i equal the number of flips required to observe heads on coin i, $i = 1, 2, \cdots, 8$. Then $Y = \max\{X_1, X_2, \cdots, X_8\}$.

   (a) Show that $P(Y \le y) = [1 - (1/2)^y]^8, \quad y = 1, 2, \cdots$.

   - *From the definition of* X_i, *it is easy to see that* $X_i \stackrel{d}{=} \text{Geometric}(\frac{1}{2})$. *So* $P(X_i \le y) = 1 - P(X_i > y) = 1 - (1/2)^y$.
   - $X_1, \cdots, X_8$ *are independent random variables having the same distribution. Therefore,*
     $P(Y \le y) = P(\max\{X_1, X_2, \cdots, X_8\} \le y) = P(X_1 \le y, \cdots, X_8 \le y)$
     $= [P(X_1 \le y)]^8 = [1 - (1/2)^y]^8$.

   (b) Show that $P(Y = y) = [1 - (1/2)^y]^8 - [1 - (1/2)^{y-1}]^8, \quad y = 1, 2, \cdots$.

   - $P(Y = y) = P(\{Y \le y\} \cap \{Y \le y - 1\}') = P(Y \le y) - P(Y \le y - 1) = [1 - (1/2)^y]^8 - [1 - (1/2)^{y-1}]^8$

   (c) (*To be done in the lab.*) Use a computer algebra system such as Maple to show that $E(Y) = 4.421$.

   - $E(Y) = \sum_{y=1}^{\infty} yP(Y = y) = \sum_{y=1}^{\infty} y\{[1 - (1/2)^y]^8 - [1 - (1/2)^{y-1}]^8\}$.

   (d) (*To be done in the lab.*) What happens to the expected value of Y as the number of coins is doubled.

- $E(Y) = \sum_{y=1}^{\infty} yP(Y = y) = \sum_{y=1}^{\infty} y\{[1 - (1/2)^y]^{16} - [1 - (1/2)^{y-1}]^{16}\}$.

5. (*Q4.5-14*) The owner of a property that is for sale is willing to accept the maximum of four independent bids (in \$ 100,000 units), which have a common pdf $f(x) = 2x,\ 0 < x < 1$. What is the expected value of the highest bid?

   - *Let $X_1, \cdots, X_4$ be the individual bid values, and Y the highest bid value. Then $Y = \max\{X_1, X_2, X_3, X_4\}$, and the support of Y is $0 < y < 1$.*
   - *For $0 < y < 1$, the cdf of Y is*
     $G(y) = P(Y \le y) = P(\max\{X_1, X_2, X_3, X_4\} \le y)$
     $= P(X_1 \le y)P(X_2 \le y)P(X_3 \le y)P(X_4 \le y) = [P(X_1 \le y)]^4 = [\int_0^y 2xdx]^4 = y^8$.
   - *The pdf of Y is $g(y) = G'(y) = 8y^7, \quad 0 < y < 1$.*
   - $E(Y) = \int_0^1 y \cdot 8y^7 dy = \frac{8}{9}$.
     So the highest bid value in dollars is \$(8/9)(100,000).

6. (*Q4.6-11*) Let X_1, X_2, X_3 denote a random sample of size 3 from a gamma distribution with $\alpha = 7$ and $\theta = 5$.

   (a) Find the mgf of $Y = X_1 + X_2 + X_3$ using the mgf's of X_1, X_2 and X_3.

   - $M_Y(t) = E(e^{t(X_1+X_2+X_3)}) = [M_{X_1}(t)]^3 = [(1-5t)^{-7}]^3 = (1-5t)^{-21}, \quad t < 1/21$.

   (b) How is Y distributed?

   - $Y \stackrel{d}{=} \text{Gamma}(\alpha = 21, \theta = 5)$.

7. (*Q4.6-14*) Let X and Y, with respective pmf's $f(x)$ and $g(y)$, be independent discrete random variables, each of whose support is a subset of the nonnegative integers 0, 1, 2, ... Show that the pmf of $W = X + Y$ is given by the **convolution formula**

$$h(w) = \sum_{x=0}^{w} f(x)g(w - x), \quad w = 0, 1, 2, \ldots.$$

   HINT: Argue that $h(w) = P(W = w)$ is the probability of the $w+1$ mutually exclusive events $(x, y = w - x), x = 0, 1, \cdots, w$.

   - *The possible pairs of values that (X, Y) can take so that $W = X + Y = w$ are $(X{=}0, Y{=}w)$, $(X{=}1, Y{=}w-1), \cdots, (X{=}k, Y{=}w-k), \cdots, (X{=}w, y{=}0)$, for any $w = 0, 1, 2, \cdots$.*
   - *Hence* $h(w) = P(W = w) = P(X{=}0, Y{=}w) + P(X{=}1, Y{=}w-1) + \cdots + P(X{=}w, Y = 0) = \sum_{x=0}^{w} P(X = x, Y = w - x) = \sum_{x=0}^{w} P(X = x)P(Y = w - x) = \sum_{x=0}^{w} f(x)g(w - x)$. *QED*

8. (*Q4.7-1*). If X is a random variable with mean 33 and variance 16, use Chebyshev's inequality to find

(a) A lower bound for $P(23 < X < 43)$.

- $P(23 < X < 43) = P(|X - 33| < 10) = P(|X - 33| < 2.5 \cdot 4) \geq 1 - \frac{1}{2.5^2} = \frac{21}{25} = 0.84$

(b) An upper bound for $P(|X - 33| \geq 14)$.

- $P(|X - 33| \geq 14) = P(|X - 33| \geq \frac{14}{4} \cdot 4) \leq \frac{1}{(14/4)^2} = \frac{4}{49} = 0.082$.

9. (*Q4.5-18*) Each of eight bearings in a bearing assembly has a diameter (in mm) that has the pdf $f(x) = 10x^9,\ 0 < x < 1$. Assuming independence, find the distribution function and the pdf of the maximum diameter, say Y, of these eight and compute $P(0.9999 < Y < 1)$.

   - *The support of* Y *is* $0 < y < 1$.
   - *For any* $0 < Y < 1$, *the cdf of* Y *is* $G(y) = P(Y < y) = P(\max\{X_1, \cdots, X_8\} < y) = [P(X_1 \leq y)]^8 = [\int_0^y 10x^9 dx]^8 = y^{80}$.
   - *The pdf of* Y *is then* $g(y) = 80y^{79}, \quad 0 < y < 1$.
   - *Hence* $P(0.9999 < Y < 1) = G(1) - G(0.9999) = 1^{80} - 0.9999^{80} = 0.007968$

10. (*Q4.7-6*) Let $\bar{X}$ be the mean of a random sample of size $n = 15$ from a distribution with mean $\mu = 80$ and variance $\sigma^2 = 60$. Use Chebyshev's inequality to find a lower bound for $P(75 < \bar{X} < 85)$.

    - $\mu_{\bar{X}} = \mu = 80$ *and* $\sigma^2_{\bar{X}} = \frac{\sigma^2}{n} = 4$.
    - *Applying Chebyshev's inequality to* $\bar{X}$ *we have* $P(|\bar{X} - \mu_{\bar{X}}| < k\sigma_{\bar{X}}) \geq 1 - \frac{1}{k^2}$.
    - *Therefore,* $P(75 < \bar{X} < 85) = P(|\bar{X} - 80| < 5) = P(|\bar{X} - 80| < 2.5 \cdot 2) \geq 1 - \frac{1}{2.5^2} = \frac{21}{25} = 0.84$.

11. (*Q4.7-7*). The characteristics of the empirical distribution of test scores of 900 students are sample mean $\bar{x} = 83$ and sample variance $s^2 = 36$, respectively. At least how many students received test scores between 71 and 95?

    - *Let* X *be the test score for a student randomly selected from the class. Applying Chebyshev's inequality to the empirical distribution of* X, *we have* $P(71 < X < 95) = P(|X - 83| < 12) = P(|X - \bar{x}| < 2s) \geq 1 - \frac{1}{2^2} = 0.75$.
    - *So there are at least* $900 \times 0.75 = 675$ *students whose test scores are between 71 and 95.*