Tutorial 10 Solutions

The theory related to some questions in this tutorial might not have been covered in the lectures before the tutorial takes place. The main results necessary to solve these questions are summarised below.

- Normal approximation for the binomial distribution (with continuity correction): let $X \stackrel{d}{=} b(n,p)$ and let $Z \stackrel{d}{=} N(0,1)$. Then, when n is large enough,
$$P(a \le X \le b) = P(a - 0.5 \le X \le b + 0.5) \approx P\left(\frac{a - 0.5 - np}{\sqrt{np(1-p)}} \le Z \le \frac{b + 0.5 - np}{\sqrt{np(1-p)}}\right).$$
- Normal approximation for the Poisson distribution (with continuity correction): let $X \stackrel{d}{=} Poisson(\lambda)$, and let $Z \stackrel{d}{=} N(0,1)$. Then, when λ is large enough,
$$P(a \le X \le b) = P(a - 0.5 \le X \le b + 0.5) \approx P\left(\frac{a - 0.5 - \lambda}{\sqrt{\lambda}} \le Z \le \frac{b + 0.5 - \lambda}{\sqrt{\lambda}}\right).$$
- If a sequence of moment-generating functions approaches a certain mgf, say $M(t)$, then the limit of the corresponding distributions must be the distribution corresponding to $M(t)$.

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1. Suppose $X_1 \stackrel{d}{=} N(\mu = 3, \sigma^2 = 4)$, $X_2 \stackrel{d}{=} N(3,4)$, and X_1 and X_2 are independent.
   (a) Write down the mgf of X_1 and X_2.
   - $M_{X_1}(t) = M_{X_2}(t) = e^{3t+2t^2}$

   (b) Let $Y = 5X_1 - 2X_2 + 6$. Find the mgf of Y.
   - $E_Y(t) = E(e^{t(5X_1 - 2X_2 + 6)}) = e^{6t}E(e^{5tX})E(e^{-2tX_2})$
     $= e^{6t}M_{X_1}(5t)M_{X_2}(-2t) = e^{6t}e^{15t+50t^2}e^{-6t+8t^2} = e^{15t+58t^2}$

   (c) Name the distribution of Y and give the values of the associated parameters.
   - *Therefore* $Y \stackrel{d}{=} N(15, 116)$.

2. (*Q5.2-26*) The serum zinc level X in micrograms per deciliter for males between ages 15 and 17 has a distribution which is approximately normal with $\mu = 90$ and $\sigma = 15$. Compute the conditional probability $P(X > 120|X > 105)$.
   - $P(X > 120|X > 105) = \frac{P(X>120)}{P(X>105)} = \frac{1-\Phi(2)}{1-\Phi(1)} = \frac{0.0288}{0.1587} = 0.1437.$

3. (*Q5.3-5*) Let $Z_1, Z_2, \cdots, Z_7$ be a random sample from the standard normal distribution $N(0,1)$. Let $W = Z_1^2 + Z_2^2 + \cdots + Z_7^2$. Name the distribution of W with associated parameter value. Justify your answer.

- $W \stackrel{d}{=} \chi^2(7)$. *This is because a squared standard normal random variable has a $\chi^2(1)$ distribution, and the sum of independent chi-square random variables is chi-square with the degrees of freedom being the sum of the individual degrees of freedom of the random variables.*

4. If $X_1, X_2, \cdots, X_{16}$ is a random sample of size $n = 16$ from the normal distribution $N(50, 100)$.

   (a) What is the distribution of $\frac{1}{100}\sum_{i=1}^{16}(X_i - 50)^2$?

   - $\chi^2(16)$

   (b) What is the distribution of $\frac{1}{100}\sum_{i=1}^{16}(X_i - \bar{X})^2$?

   - $\chi^2(15)$

   (c) What is the distribution of $\bar{X}$?

   - $N(50, \frac{100}{16} = 6.25)$.

   (d) What is the distribution of $T = \frac{\bar{X}-50}{\sqrt{(16\cdot 15)^{-1}\sum_{i=1}^{16}(X_i-\bar{X})^2}}$?

   - $t(15)$.

5. (*Q5.4-1*) Let $\bar{X}$ be the mean of a random sample of size 12 from the uniform distribution on the interval (0,1) (which has the mean 1/2 and variance 1/12). Approximate the probability $P(1/2 \le \bar{X} \le 2/3)$ using the central limit theorem. (*The last step of calculating the standard normal probability can be down by using Maple or looking up the normal probability table*).

   - *By CLT,* $\bar{X} \stackrel{d}{\approx} N(\frac{1}{2}, \frac{1}{144})$.
   - *So* $P(\frac{1}{2} \le \bar{X} \le \frac{2}{3}) \approx P(\frac{1/2-1/2}{1/12} \le Z \le \frac{2/3-1/2}{1/12})$
     $= P(0 \le Z \le 2) = \Phi(2) - \Phi(0) = 0.9772 - 0.4 = 0.4772.$

6. (*Q5.5-1*) Let the distribution of Y be $b(25, 1/2)$. Find the probability $P(10 \le Y \le 12)$ in two ways: using the binomial pmf formula, and using the normal approximation with continuity correction.

   - $P(10 \le Y \le 12) = \sum_{y=10}^{12}\binom{25}{y}0.5^{25} = 0.3852.$
   - *By CLT,* $Y \stackrel{d}{\approx} N(12.5, 6.25)$.
     So $P(10 \le Y \le 12) = P(9.5 \le Y \le 12.5) \approx P(\frac{9.5-12.5}{2.5} \le Z \le \frac{12.5-12.5}{2.5})$
     $= P(-1.2 \le Z \le 0) = \Phi(0) - \Phi(-1.2) = 0.5 - 0.1151 = 0.3849.$

7. (*Q5.5-22*) The number X of flaws on a certain tape of length one yard follows a Poisson distribution with mean 0.3. We examine $n = 100$ such tapes and count the total number Y of flaws.

   (a) Assuming independence, what is the distribution of Y?

- $Y \stackrel{d}{=} \text{Poisson}(30)$. *By CLT* $Y \stackrel{d}{\approx} N(30, 30)$.

(b) Approximate $P(Y \le 25)$ using normal distribution with continuity correction.

- $P(Y \le 25) = P(Y \le 25.5) \approx P(Z \le \frac{25.5-30}{\sqrt{30}}) = P(Z \le -0.8216) = \Phi(-0.8216) = 0.2057$.

8. (*Q5.7-3*) Let S^2 be the sample variance of a random sample of size n from $N(\mu, \sigma^2)$. We already learned that $\frac{n-1}{\sigma^2}S^2$ has a $\chi^2(n-1)$ distribution. Show that the limit, as $n \to \infty$, of the moment-generating function of S^2 is $e^{\sigma^2 t}$. Thus, in the limit, the distribution of S^2 is degenerate with probability 1 at σ^2.

- *Let* $W = \frac{n-1}{\sigma^2}S^2$. *Then* $S^2 = \frac{\sigma^2}{n-1}W$.
- *The mgf of* W *is* $M_W(t) = (1-2t)^{-(n-1)/2},\ t < 1/2$.
- *So the mgf of* S^2 *is*
  $M_{S^2}(t) = M_W(\frac{\sigma^2 t}{n-1}) = (1 - \frac{2\sigma^2 t}{n-1})^{-(n-1)/2} = \left[(1 - \frac{2\sigma^2 t}{n-1})^{-(n-1)/2\sigma^2 t}\right]^{\sigma^2 t}, \quad t < \frac{n-1}{2\sigma^2}$.
- *Therefore,* $\lim_{n\to\infty} M_{S^2}(t) = e^{\sigma^2 t}$.

9. (*Q5.4-2*) Let $Y = X_1 + X_2 + \cdots + X_{15}$ be the sum of a random sample of size 15 from the distribution whose pdf is $f(x) = (3/2)x^2,\ -1 < x < 1$. Approximate $P(-0.3 \le Y \le 1.5)$ using the central limit theorem.

- $E(X_1) = \int_{-1}^{1}(3/2)x^3 dx = 0$. $\text{Var}(X_1) = \int_{-1}^{1}(x-0)^2(3/2)x^2 dx = \frac{3}{5}$.
- *By CLT* $Y \stackrel{d}{\approx} N(15 \cdot 0, 15 \cdot \frac{3}{5}) = N(0, 9)$.
- *So* $P(-0.3 \le Y \le 1.5) \approx P(\frac{-0.3-0}{3} \le Z \le \frac{1.5-0}{3}) = P(-0.1 \le Z \le 0.5) = 0.2313$.