MAST20006/90057 Probability for Statistics /Elements of Probability

Sample Lab Test Solutions

Name :________________________ Student Number : ________________________

Tutor's name : ________________________

For this lab-based test you are permitted to use the lecture notes/handouts, the textbook, and a calculator, and of course, the computer. You can ask your tutor for help with technical difficulties. You must hand this sheet back to your tutor before you leave the lab. You should not communicate with other students during the test. Please also do not communicate with any other students about this test after you have completed it, until the overall testing period is complete. The marks for each question are indicated in brackets `[ ]`. **The total marks are 40.**

Please always `simplify` your final results in doing the following questions.

Question 1.

Suppose there are N birds in a reserve park, all being attached ID numbers from 1,2, ..., until N respectively. A bird watcher has observed 10 birds and recorded their ID numbers during a visit to the park.

Let X be the largest ID number that the watcher has recorded. It can be shown that the pmf of X is

$$P(X=x)=\frac{x^{10}-(x-1)^{10}}{N^{10}}, \quad x=1,2,3,\cdots,N.$$

(a). Find the probability $P(X \geq 4)$. [2]

Answer: $1-\frac{59049}{N^{10}}$

(b). Find the mean $E(X)$. [2]

Answer:
$E(X)=\frac{-5+33N^9+33N^2-66N^4-55N^8+66N^{10}+66N^6}{66N^9}$

(c). Evaluate $E(X)$ when $N=100$. Keep 4 significant digits after the decimal point in your result. [2]

Answer: $E(X)=91.4008$ when $N=100$.

(d). Find the variance $\mathrm{Var}(X)$ when $N=100$. Keep 4 significant digits after the decimal point in your result. [2]

Answer: $\mathrm{Var}(X)=68.8024$.

```
> f:=(x^(10)-(x-1)^(10))/N^(10)
> sum(f,x=4..N)
> me:=sum(x*f, x=1..N)
> evalf(subs(N=100,me))
> var:=sum((x-me)^2*f, x=1..N)
> evalf(subs(N=100,var))
```

Question 2.

Let a continuous random variable X have the following pdf

$$f(x) = e^{-(x+2)}, \quad -2 < x < \infty.$$

(a). Find the cdf $F(x)$ of X. [2]

Answer:
$F(x) = 1 - e^{-x-2}, \quad -2 < x < \infty$ and $F(x) = 0, \quad x \leq -2$.

(b). Numerically evaluate the probability $P(-3 < X \leq 2)$. [2]

Answer: $P(-3 < X \leq 2) = F(2) - F(-3) = 1 - e^{-4}$.

(c). Find the mean $E(X)$. [2]

Answer: $E(X) = \int_{-2}^{\infty} e^{-x-2} dx = -1$.

(d). Find the second moment $E(X^2)$. [2]

Answer: $E(X^2) = 2$.

(e). Find the 95th percentile of of X. [2]

Answer: $\pi_{0.95} = -2 + ln(20) = 0.9957$.

(f). Find the mgf $M(t) = E[e^{tX}]$ of X for $t < 1$. [2]

Answer: $M(t) = \frac{e^{-2t}}{1-t}$.

(g). Let $Y = X^3$.

(i) Find the space of Y. [1]

Answer: $-8 \leq Y < \infty$

(ii) Find the cdf $G(y)$ of Y. [2]

Answer: $G(y) = 1 - e^{-2-y^{1/3}}, y > -8; = 0 \quad$ elsewhere.

(iii) Find the pdf $g(y)$ of Y. [2]

Answer: $g(y) = \frac{e^{-2-y^{1/3}}}{3y^{2/3}}, y > -8; = 0 \quad$ elsewhere.

```
> f:=exp(-x-2)
> F:=int(f, x=-2..x)
> int(f,x=-2..2)   #Note P(-3<X<=2)=P(-2<X<=2) here.
> int(x*f,x=-2..infinity)
> int(x^2*f, x=-2..infinity)
> solve(F=0.95,x)
```

```
> assume(t<1)
> int(exp(t*x)*f, x=-2..infinity)
> int(f,x=-2..y^(1/3))
> diff(%,y)
Alternatively,
> f:=piecewise(x>-2, exp(-x-2),0)
> f:=unapply(f,x)
> X:=RandomVariable(Distribution(PDF=f))
> CDF(X,x)
> Probability({X>-3, X<=2})
> Mean(X)
> Moment(X,2)
> Percentile(X,95);
> assume(t<1)
> MGF(X,t)
> Y:=X^3
> simplify(CDF(Y,y))
> simplify(PDF(Y,y))
```

Question 3.

Consider continuous random variables X and Y which have the following joint pdf

$$f(x,y) = 24xy, \quad x > 0,\ y > 0,\ x + y < 1.$$

(a). Sketch a graph of the support of X and Y. [1]

(b). Find the probability $P(Y > 2X)$. [2]

Answer: $= \int_0^{1/3} \int_{2x}^{1-x} 24xydydx = \frac{7}{27}$.

(c). Find the marginal pdf $f_1(x)$ of X. [2]

Answer:
$f_1(x) = \int_0^{1-x} 24xydy = 12x(1-x)^2, \quad 0 < x < 1.$

(d). Find the mean $E(X)$. [2]

Answer:
$E(X) = \int_0^1 x12x(1-x)^2dx = \frac{2}{5}$.

(e). Find the variance $\mathrm{Var}(X)$.

Answer:
$\mathrm{Var}(X) = \int_0^1 (x - \frac{2}{5})^2 12x(1-x)^2dx = \frac{1}{25}$.

(f). Find the covariance $\mathrm{Cov}(X,Y)$. [2]

> Answer:
> $\mathrm{Cov}(X,Y) = \int_0^1 \int_0^{1-y} (x - \frac{2}{5})(y - \frac{2}{5}) 24xy\,dx\,dy = -\frac{2}{75}$

(g). Find the correlation coefficient ρ between X and Y. [2]

> Answer:
> $\rho = -\frac{2}{3}$

(h). Find the conditional pdf $h(y|x)$ of Y given $X = x$. [2]

> Answer:
> $h(y|x) = \frac{24xy}{12x(1-x^2)} = \frac{2y}{(1-x)^2}, \quad 0 < y < 1.$

(i). Find the condition probability $P(Y \leq \frac{1}{3}(1 - X)|X = x)$. [2]

> Answer:
> $= \int_0^{(1-x)/3} h(y|x)dy = \frac{1}{9}.$

```
> f:=24*x*y
> int(int(f,y=2*x..(1-x)),x=0..1/3)
> f1:=int(f, y=0..(1-x))
> me:=int(x*f1, x=0..1)
> int((x-me)^2*f1,x=0..1)
> int(int((x-me)*(y-me)*f,x=0..(1-y)),y=0..1)
> (-2/75)/(1/25)
> int(f/f1, y=0..(1-x)/3)
> simplify(%)
```