Lab 9 Solutions

1. (*Q4.5-13*) Flip $n = 8$ fair coins and remove all that came up heads. Flip the other (tails) coins and remove the heads. Continuing flipping the remaining coins until each has come up heads. We shall find the pmf of Y, the number of trials needed to finish off all coins. Let X_i equal the number of flips required to observe heads on coin i, $i = 1, 2, \cdots, 8$. Then $Y = \max\{X_1, X_2, \cdots, X_8\}$.

   (a) (*Already done in the tutorial.*) Show that $P(Y \le y) = [1 - (1/2)^y]^8$.

   - *From the definition of X_i, it is easy to see that $X_i \stackrel{d}{=} \text{Geometric}(\frac{1}{2})$. So* $P(X_i \le y) = 1 - P(X_i > y) = 1 - (1/2)^y$.
   - $X_1, \cdots, X_8$ *are independent random variables having the same distribution. Therefore,* $P(Y \le y) = P(\max\{X_1, X_2, \cdots, X_8\} \le y) = P(X_1 \le y, \cdots, X_8 \le y)$ $= [P(X_1 \le y)]^8 = [1 - (1/2)^y]^8$.

   (b) (*Already done in the tutorial.*) Show that $P(Y = y) = [1 - (1/2)^y]^8 - [1 - (1/2)^{y-1}]^8, \quad y = 1, 2, \cdots$.

   - $P(Y = y) = P(\{Y \le y\} \cap \{Y \le y - 1\}') = P(Y \le y) - P(Y \le y - 1) = [1 - (1/2)^y]^8 - [1 - (1/2)^{y-1}]^8$

   (c) Use a computer algebra system such as Maple to show that $E(Y) = 4.421$.

   ```
   >  sum(y*((1-(1/2)^y)^8-(1-(1/2)^(y-1))^8), y=1..infinity)
   ```

   (d) What happens to the expected value of Y as the number of coins is doubled. (It would take a few minutes for Maple to compute this job.)

   ```
   > sum(y*((1-(1/2)^y)^16-(1-(1/2)^(y-1))^16), y=1..infinity)
   > evalf(%)

   The answer is 5.377
   ```

2. (*Q4.5-14*) The owner of a property that is for sale is willing to accept the maximum of four independent bids (in \$ 100,000 units), which have a common pdf $f(x) = 2x$, $0 < x < 1$.

   (a) What is the pdf of the highest bid value? Also plot this pdf.

   ```
   >  with(Statistics)
   >  f := piecewise(x < 0, 0, x < 1, 2 x, 0)
   > f := unapply(f, x)
   > X1 := RandomVariable(Distribution(PDF = f))
   > X2 := RandomVariable(Distribution(PDF = f))
   > X3 := RandomVariable(Distribution(PDF = f))
   > X4 := RandomVariable(Distribution(PDF = f))
   ```

```
> Y := max(X1, X2, X3, X4)
> PDF(Y, y)
> fy:=simplify(%)
> DensityPlot(Y, range=0..1)
```

(b) What is the expected value of the highest bid? (HINT: It can take very long time or Maple does not have enough memory to compute $E(Y)$ if you use `Mean(Y)` command.)

```
> int(y*fy,y=0..1) #Answer:  =8/9
```

(c) What is the pdf of the sample mean bid value from the 4 bids? Compare the pdf of this sample mean with the pdf of an individual bid using density plots. Also compute the mean of the sample mean.

```
> Y4:= (X1+ X2+ X3+ X4)/4
> PDF(Y4, y)
> DensityPlot(Y4, range=0..1)
> DensityPlot(X1, range=0..1)
> Mean(Y4)     # Answer = 2/3
```

3. (*Q4.6-11*) Let X_1, X_2, X_3 denote a random sample of size 3 from a gamma distribution with $\alpha = 7$ and $\theta = 5$.

(a) Find the mgf of X_1 using Maple.

```
> X1 := RandomVariable(GammaDistribution(5, 7))
> X2 := RandomVariable(GammaDistribution(5, 7))
> X3 := RandomVariable(GammaDistribution(5, 7))
> MGF(X1,t)
```

(b) Find the mgf of $Y = X_1 + X_2 + X_3$ using the mgf's of X_1, X_2 and X_3. Note that the domain of $M_Y(t)$ is $t < 1/5$.

```
> Y := X1+X2+X3
> assume(t<1/5)
> MGF(Y,t)
> factor(%)
```

(c) How is Y distributed? Also plot the density of Y over the interval (0, 200).

$Y \overset{d}{=} \text{Gamma}(\theta = 5, \alpha = 21)$.

```
> DensityPlot(Y, range=0..200)
```