MAST20006 Probability for Statistics /MAST90057 Elements of Probability
Assignment 4, Semester 1 2020
Due time: 4pm, Friday May 22.

Name:

Student ID:

- Note this assignment is being handled using a similar process to that now planned for the final exam so you can start to become familiar with it.

  To complete this assignment, you need to write your solutions into the blank answer spaces following each question in this assignment PDF.

  If you have a printer (or can access one), then you must print out the assignment template and handwrite your solutions into the answer spaces.

  If you do not have a printer but you can figure out how to annotate a PDF using an iPad/Android tablet/Graphics tablet or using Adobe Acrobat, then annotate your answers directly onto the assignment PDF and save a copy for submission.

  Failing both of these methods, you may handwrite your answers as normal on blank paper and then scan for submission (but note that you will thereby miss valuable practice for the exam process). In that case, however, your document should have the same length as the assignment template otherwise Gradescope will reject your submission. So you will need to add as many blank pages as necessary to reach that criterion.

  Scan your assignment to a PDF file using your mobile phone then upload by going to the Assignments menu on Canvas and submit the PDF to the GradeScope tool by first selecting your PDF file and then clicking on 'Upload PDF'.

- A **poor presentation penalty** of 10% will apply unless your submitted assignment meets all of the following requirements:

  - it is a single pdf with all pages in correct template order and the correct way up, and with any blank pages with additional working added only at the end of the template pages;
  - has all pages clearly readable;
  - has all pages cropped to the A4 borders of the original page and is imaged from directly above to avoid excessive 'keystoning'

  These requirements are easy to meet if you use a scanning app on your phone and take some care with your submission - please review it before submitting to double check you have satisfied all of the above requirements.

- The **strict** submission deadline is **4pm Melbourne time on Friday 22 May**. You have longer than of the normal one week to complete this assignment. Consequently late assignments will **NOT** be accepted. We recommend you submit at least a day before the due date to avoid any technical delays. If there are extenuating, eg medical, circumstances, contact the Tutorial Coordinator.
- There are 6 questions, of which 2 randomly chosen questions will be marked. Note you are expected to submit answers to **all** questions, otherwise **a mark penalty will apply**.
- Working and reasoning **must** be given to obtain full credit. Give clear and concise explanations. Clarity, neatness, and style count.

* * *

1. A random variable X has a pdf given by

$$f(x) = \frac{2}{9}(x+2), \quad -2 < x < 1.$$

Let $Y = X^2$ be a transformation of X.

(a) Is this transformation one-to-one? Find the support of Y.

Solution:

It is not one-to-one for $X \in (-1, 1)$.

It is one-to-one for $X \in (-2, -1]$.

The support of Y is $S_Y = [0, 4)$.

(b) Derive the cdf of Y.

Solution:

For $y \in [0, 1]$,

$$\begin{aligned} P(Y \leq y) &= P(X^2 \leq y) \\ &= P(-\sqrt{y} \leq X \leq \sqrt{y}) \\ &= \int_{-\sqrt{y}}^{\sqrt{y}} \frac{2}{9}(x+2)dx \\ &= \frac{8}{9}\sqrt{y}. \end{aligned}$$

For $y \in (1, 4)$,

$$\begin{aligned} P(Y \leq y) &= P(X^2 \leq y) \\ &= P(-\sqrt{y} \leq X \leq 1) \\ &= \int_{-\sqrt{y}}^{1} \frac{2}{9}(x+2)dx \\ &= \frac{1}{9}(-y+5+4\sqrt{y}). \end{aligned}$$

So, the cdf $F_Y(y) = \begin{cases} 0, & y < 0, \\ \frac{8}{9}\sqrt{y}, & 0 \leq y \leq 1, \\ \frac{1}{9}(-y+5+4\sqrt{y}), & 1 < y < 4, \\ 1, & y \geq 4. \end{cases}$

(c) Find the pdf of Y.

Solution:

The cdf is not differentiable at $y = 0$ and $y = 1$.
For the other values of y, we have

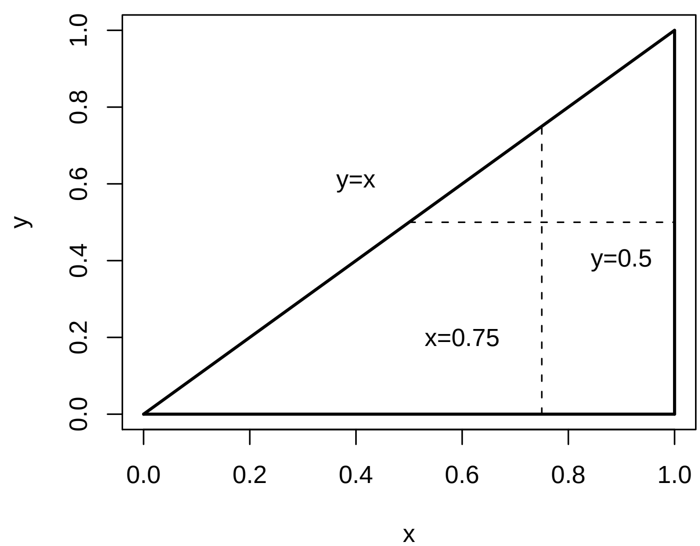
$$f_Y(y) = F'_Y(y) = \begin{cases} \frac{4}{9\sqrt{y}}, & 0 < y < 1, \\ -\frac{1}{9} + \frac{2}{9\sqrt{y}}, & 1 < y < 4, \\ 0, & \text{elsewhere.} \end{cases}$$

2. Let X and Y be continuous random variables having the joint pdf

$$f(x, y) = 8xy, \quad 0 \le y \le x \le 1.$$

(a) Sketch the graph of the support of X and Y.

Solution:



(b) Find $f_1(x)$, the marginal pdf of X.

Solution:

$$\begin{aligned} f_1(x) &= \int_0^x 8xy\, dy \\ &= 4x^3, \quad 0 \le x \le 1. \end{aligned}$$

(c) Find $f_2(y)$, the marginal pdf of Y.

Solution:

$$
\begin{aligned}
f_2(y) &= \int_y^1 8xy\,dx \\
&= 4y(1-y^2), \quad 0 \le y \le 1.
\end{aligned}
$$

(d) Compute $\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \mathrm{Cov}(X, Y)$, and ρ.

Solution:

- $\mu_X = \int_0^1 x\, 4x^3\, dx = \frac{4}{5}$.

- $\mu_Y = \int_0^1 y\, 4y(1 - y^2)\, dy = \frac{8}{15}$.

- $E(X^2) = \int_0^1 x^2\, 4x^3\, dx = \frac{2}{3}$.

  So $\sigma_X^2 = \frac{2}{3} - \left(\frac{4}{5}\right)^2 = \frac{2}{75}$.

- $E(Y^2) = \int_0^1 y^2\, 4y(1 - y^2)\, dy = \frac{1}{3}$.

  So $\sigma_Y^2 = \frac{1}{3} - \left(\frac{8}{15}\right)^2 = \frac{11}{225}$.

- $E(XY) = \int_0^1 \int_0^x xy\, 8xy\, dy dx = \int_0^1 \frac{8}{3} x^5 dx = \frac{8}{18}$.

  So $\mathrm{Cov}(X, Y) = \frac{8}{18} - \frac{4}{5} \times \frac{8}{15} = \frac{4}{225}$.

- $\rho = \dfrac{4/225}{\sqrt{2/75}\sqrt{11/225}} = \frac{2\sqrt{66}}{33} = 0.492366$.

(e) Find $g(x|y = \frac{1}{2})$, the conditional pdf of X given $Y = \frac{1}{2}$.

Solution:

$$\begin{aligned} g(x|y = 1/2) &= \frac{f(x,y)}{f_2(y)}\Big|_{y=\frac{1}{2}} \\ &= \frac{8xy}{4y(1-y^2)}\Big|_{y=\frac{1}{2}} \\ &= \frac{8}{3}x, \quad \frac{1}{2} \le x \le 1. \end{aligned}$$

(f) Find $P(X < \frac{3}{4}|Y = \frac{1}{2})$.

Solution:

$$\begin{aligned} P(X < 3/4|Y = 1/2) &= \int_{1/2}^{3/4} g(x|y = 1/2)dx \\ &= \int_{1/2}^{3/4} \frac{8}{3}x dx \\ &= \frac{5}{12}. \end{aligned}$$

(g) Find $P(X < \frac{3}{4}|Y < \frac{1}{2})$.

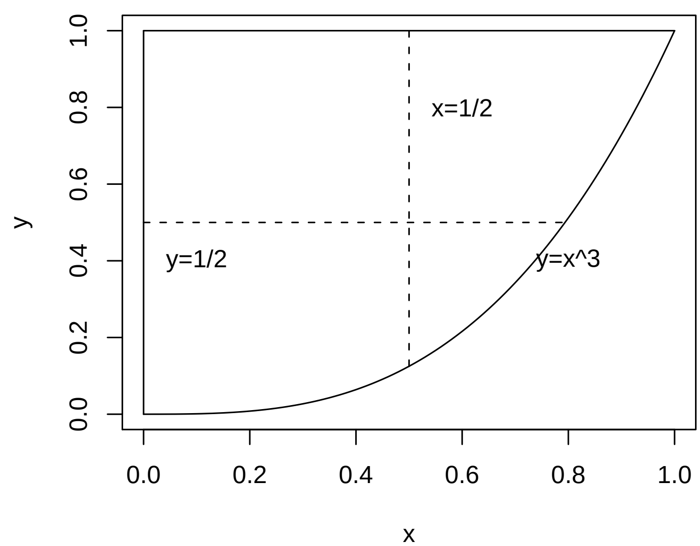
Solution:

$$\begin{aligned} P(X < 3/4|Y < 1/2) &= \frac{\int_0^{1/2}\int_y^{3/4} 8xy\,dxdy}{\int_0^{1/2}\int_y^{1} 8xy\,dxdy} \\ &= \frac{7/32}{7/16} \\ &= \frac{1}{2}. \end{aligned}$$

3. Let $f(x, y) = \frac{4}{3}$, $x^3 \le y \le 1, 0 \le x \le 1$, be the joint pdf of X and Y.

   (a) Sketch the graph of the support of X and Y.

Solution:



(b) Find $P(X \leq \frac{1}{2})$.

Solution:

$$\begin{aligned} P(X \leq 1/2) &= \int_0^{1/2} \int_{x^3}^{1} \frac{4}{3}\, dy dx \\ &= \int_0^{1/2} \frac{4}{3}(1 - x^3) dx \\ &= \frac{31}{48}. \end{aligned}$$

(c) Find $P(X \geq \frac{1}{2}, Y \geq \frac{1}{2})$.

Solution:

$$\begin{aligned} P(X \geq 1/2, Y \geq 1/2) &= \int_{1/2}^{1} \int_{1/2}^{y^{1/3}} \frac{4}{3} dx dy \\ &= \int_{1/2}^{1} (\frac{4}{3} y^{1/3} - \frac{2}{3}) dy \\ &= \frac{2}{3} - 2^{-4/3} \\ &= 0.2698. \end{aligned}$$

(d) Are X and Y independent? Why or why not?

Solution:

X and Y are dependent because their support is non-rectangular.

4. Let X and Y have the joint pmf defined by $f(0,0) = f(1,2) = 0.2$, $f(0,1) = f(1,1) = 0.3$.

   (a) Represent the joint pmf by a table, and give the marginal pmf's of X and Y in the "margins" of the table.

Solution:

	y			
x	0	1	2	$f_X(x)$
0	0.2	0.3	0	**0.5**
1	0	0.3	0.2	**0.5**
$f_Y(y)$	**0.2**	**0.6**	**0.2**	

(b) Compute $\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \mathrm{Cov}(X, Y)$, and ρ.

Solution:

- $\mu_X = 0 \times 0.5 + 1 \times 0.5 = 0.5.$

- $\mu_Y = 0 \times 0.2 + 1 \times 0.6 + 2 \times 0.2 = 1.$

- $E(X^2) = 0^2 \times 0.5 + 1^2 \times 0.5 = 0.5.$

  So $\sigma_X^2 = 0.5 - 0.5^2 = 0.25.$

- $E(Y^2) = 0^2 \times 0.2 + 1^2 \times 0.6 + 2^2 \times 0.2 = 1.4.$

  So $\sigma_Y^2 = 1.4 - 1^2 = 0.4.$

- $E(XY) = 0\times0\times0.2+0\times1\times0.3+1\times1\times0.3+1\times2\times0.2 = 0.7.$

  So $\mathrm{Cov}(X, Y) = E(XY) - E(X)E(Y) = 0.7 - 0.5 \times 1 = 0.2.$

- $\rho = \frac{0.2}{\sqrt{0.25}\sqrt{0.4}} = 0.6325.$

(c) Tabulate the conditional pmf of Y given $X = 0$.

Solution:

$h(y|x=0) = \frac{f(0,y)}{f_X(0)}$.

So, we have

y	0	1	2
$h(y\|x=0)$	0.4	0.6	0

(d) Tabulate the conditional pmf of X given $Y = 2$.

Solution:

$g(x|y=2) = \frac{f(x,2)}{f_Y(2)}$.

So we have

x	0	1
$g(x\|y=2)$	0	1

5. Let X have a uniform distribution on the interval $(0, 1)$. Given $X = x$, let Y have a (conditional) uniform distribution on the interval $(0, x/2)$.

   (a) Write down the conditional pdf of Y given that $X = x$. Make sure to include the domain.

Solution:

$$h(y|x) = \frac{2}{x}, \quad 0 < y < x/2,\ 0 < x < 1.$$

   (b) Find $E(Y|x)$ and $E(Y)$.

Solution:

We have

$$E(Y|x) = \int_0^{x/2} y\,\frac{2}{x}\,dy = x/4.$$

So,

$$E(Y) = E[E(Y|X)] = E[X/4] = \frac{1}{8}.$$

(c) Determine $f(x, y)$, the joint pdf of X and Y.

Solution:

$$\begin{aligned} f(x,y) &= f_X(x)h(y|x) \\ &= 1 \times \frac{2}{x} \\ &= \frac{2}{x}, \quad 0 < y < x/2,\ 0 < x < 1. \end{aligned}$$

(d) Find $f_Y(y)$, the marginal pdf of Y.

Solution:

$$\begin{aligned} f_Y(y) &= \int_{2y}^{1} \frac{2}{x} dx \\ &= -2\ln(2y), \quad 0 < y < 1/2. \end{aligned}$$

(e) Find $g(x|y)$, the conditional pdf of X given $Y = y$.

Solution:

$$\begin{aligned} g(x|y) &= \frac{f(x,y)}{f_Y(y)} \\ &= \frac{\frac{2}{x}}{-2\ln(2y)} \\ &= -\frac{1}{x\ln(2y)}, \quad 2y < x < 1,\ 0 < y < 1/2. \end{aligned}$$

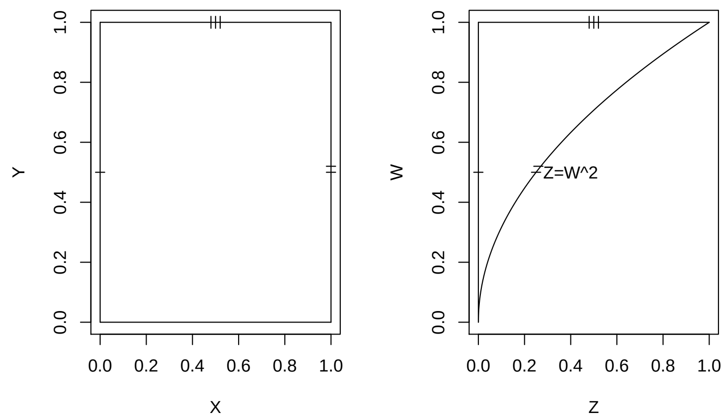
6. Consider two random variables X and Y with the joint probability density

$$f(x,y) = \begin{cases} 12xy(1-y), & 0 < x < 1, 0 < y < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Let $Z = XY^2$ and $W = Y$ be a joint transformation of (X, Y).

(a) Sketch the graph of the support of (Z, W) and describe it mathematically.

Solution:



The support of (Z, W) is $\{0 < Z < 1, \ \ 0 < W < 1, \ \ Z < W^2\}$.

(b) Find the inverse transformation.

Solution:

$$X = \frac{Z}{W^2} \qquad \text{and} \qquad Y = W.$$

(c) Find the Jacobian of the inverse transformation.

Solution:

$$J = \begin{vmatrix} 1/w^2 & -2z/w^3 \\ 0 & 1 \end{vmatrix} = \frac{1}{w^2}.$$

(d) Find the joint pdf of (Z, W).

Solution:

The joint pdf of Z and W is

$$\begin{aligned} g(z,w) &= |J| f(z/w^2, w) \\ &= \frac{1}{w^2} \cdot 12\frac{z}{w^2} w(1-w) \\ &= \frac{12z(1-w)}{w^3}, \end{aligned}$$

where $0 < z < 1, 0 < w < 1, z < w^2$.

(e) Find the pdf of $Z = XY^2$ from the joint pdf of (Z, W).

Solution:

The marginal pdf of Z is

$$\begin{aligned} g_Z(z) &= \int_{\sqrt{z}}^{1} g(z, w)dw \\ &= \int_{\sqrt{z}}^{1} \frac{12z(1-w)}{w^3}dw \\ &= 6z + 6 - 12\sqrt{z}, \quad 0 < z < 1. \end{aligned}$$