Tutorial 3 Solutions

1. Let a chip be taken at random from a bowl that contains 6 white chips, 3 red chips, and 1 blue chip. Let the random variable $X = 1$ if the outcome is a white chip; let $X = 5$ if the outcome is a red chip; and let $X = 10$ if the outcome is a blue chip.

   (a) Find the pmf of X. (Namely, find the possible values of X and then the probability for each such possible value.)

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x	1	5	10
$f(x)$	$\frac{6}{10}$	$\frac{3}{10}$	$\frac{1}{10}$

   (b) Find the expectation of X.

   - $E(X) = \sum_{x \in S_X} xf(x) = 1 \times \frac{6}{10} + 5 \times \frac{3}{10} + 10 \times \frac{1}{10} = 3.1.$

2. Let $f(x) = \frac{x}{c}$, $x = 1, 2, 3, 4$. Find the value of c so that $f(x)$ satisfies the conditions of being a pmf for a random variable X.

   - *First $f(x) > 0$ is known for any $x \in S_X = \{1, 2, 3, 4\}$. In order that $f(x)$ is a pmf, it has to satisfy that $f(1) + f(2) + f(3) + f(4) = 1$, i,.e. $\frac{1+2+3+4}{c} = 1$. Therefore, $c = 10$.*

3. Let $f(x) = (1/4)^{|x|}(1/2)^{1-|x|}$ for $x = -1, 0, 1$ and $f(x) = 0$ for other x values. Is $f(x)$ a pmf? If yes, re-express the pmf by a table.

   - *It follows from the formula that $f(-1) = 1/4$, $f(0) = 1/2$, $f(1) = 1/4$ and $f(x) = 0$ for all other x values. The sample space $S_X = \{-1, 0, 1\}$; $f(x) > 0$ for any $x \in S_X$ and the total probability $f(-1) + f(0) + f(1) = 1$. Therefore, $f(x)$ is a pmf and can be expressed by the following table:*

x	-1	0	1
$f(x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

4. Let $f(x) = (1 - x)/4$ for $x = -1, 0, 2$ and $f(x) = 0$ for other values of x. Is $f(x)$ a pmf? If yes, re-express the pmf by a table.

   - *According to the formula, $f(2) = -\frac{1}{4}$, a negative value. So $f(x)$ cannot be a pmf.*

5. (*Q2.1-7*) Let a random experiment be the cast of a pair of unbiased 6-sided dice and let X equal the smaller of the outcomes if they are different and the common value if they are equal.

   (a) With reasonable assumptions, find the pmf of X.

   - *The pmf of X is*

x	1	2	3	4	5	6
$f(x)$	$\frac{11}{36}$	$\frac{9}{36}$	$\frac{7}{36}$	$\frac{5}{36}$	$\frac{3}{36}$	$\frac{1}{36}$

   - *Alternatively, the pmf of X is $f(x) = \frac{13-2x}{36}$, $x = 1, 2, 3, 4, 5, 6$.*

(b) Let Y equal the range of the two outcomes (i.e., the absolute value of the difference of the largest and smallest outcomes). Determine the pmf of Y.

- *The pmf of Y is*

y	0	1	2	3	4	5
$g(y)$	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{8}{36}$	$\frac{6}{36}$	$\frac{4}{36}$	$\frac{2}{36}$

- *Alternatively, the pmf of Y is* $g(y) = \left(\frac{6}{36}\right)^{1-\min(1,y)} \left(\frac{12-2y}{36}\right)^{\min(1,y)}$, $y = 0, 1, 2, 3, 4, 5.$

6. (*Q2.1-11*). In a lot of 100 light bulbs, there are 5 bad bulbs. An inspector inspects 10 bulbs selected at random. Let X be the number of bad bulbs in the sample.

(a) What probability distribution does X have?

- *X has a hypergeometric distribution Hyper(N_1, N_2, n) with* $N_1 = 5$, $N_2 = 95$ *and* $n = 10$.

(b) Calculate the probability that at least one defective bulb will be found in the sample.

- $$P(X \geq 1) = 1 - P(X = 0) = 1 - \frac{\binom{5}{0}\binom{95}{10}}{\binom{100}{10}} = 1 - \frac{95 \times 94 \times \cdots \times 86}{100 \times 99 \times \cdots \times 91} = 0.416.$$

(c) Find the mean of X, i.e. $E(X)$.

- $E(X) = n\left(\frac{N_1}{N_1+N_2}\right) = 10\left(\frac{5}{100}\right) = 0.5$

(d) Find the variance of X, i.e. $\text{Var}(X)$.

- $\text{Var}(X) = n\left(\frac{N_1}{N}\right)\left(\frac{N_2}{N}\right)\left(\frac{N-n}{N-1}\right) = 10\left(\frac{5}{100}\right)\left(\frac{95}{100}\right)\left(\frac{90}{99}\right) = 0.432$

(e) Find the second moment of X, i.e. $E(X^2)$.

- *Since* $\text{Var}(X) = E(X^2) - [E(X)]^2$,
it follows that $E(X^2) = \text{Var}(X) + [E(X)]^2 = 0.432 + 0.5^2 = 0.682.$

7. Given $E(X + 4) = 10$ and $E[(X + 4)^2] = 116$, determine

(a) $\text{Var}(X + 4)$.

- $\text{Var}(X + 4) = E[(X + 4)^2] - [E(X + 4)]^2 = 116 - 10^2 = 16.$

(b) $\mu = E(X)$.

- $\mu = E(X) = E(X + 4) - 4 = 10 - 4 = 6.$

(c) $\sigma^2 = \text{Var}(X)$.

- $116 = E[(X + 4)^2] = E(X^2 + 8X + 16) = E(X^2) + 8E(X) + 16 = \text{Var}(X) + [E(X)]^2 + 8\mu + 16 = \sigma^2 + \mu^2 + 8\mu + 16 = \sigma^2 + 6^2 + 48 + 16 = \sigma^2 + 100.$ *So* $\sigma^2 = 16.$

8. A box contains 4 coloured balls: 2 black and 2 white. Balls are randomly drawn successively without replacement. If X is the number of draws until the last black ball is obtained, what are the possible values of X? Find the pmf $f(x)$ for X. (*Hint*: Define events $B_i = \{\text{the i-th draw is a black ball}\}$ and $W_j = \{\text{the j-th draw is a white ball}\}$. Then find how each outcome of X is related to B_i and W_j.)

   - *The possible values of X are 2, 3, and 4.*
   - $P(X = 2) = P(B_1 \cap B_2) = \frac{2}{4} \times \frac{1}{3} = \frac{1}{6}$.
   - $P(X = 3) = P((B_1 \cap W_2 \cap B_3) \cup (W_1 \cap B_2 \cap B_3))$
     $= P(B_1 \cap W_2 \cap B_3) + P(W_1 \cap B_2 \cap B_3) = \frac{2}{4} \times \frac{2}{3} \times \frac{1}{2} + \frac{2}{4} \times \frac{2}{3} \times \frac{1}{2} = \frac{2}{6}$.
   - $P(X = 4) = P((B_1 \cap W_2 \cap W_3 \cap B_4) \cup (W_1 \cap B_2 \cap W_3 \cap B_4) \cup (W_1 \cap W_2 \cap B_3 \cap B_4))$
     $= P(B_1 \cap W_2 \cap W_3 \cap B_4) + P(W_1 \cap B_2 \cap W_3 \cap B_4) + P(W_1 \cap W_2 \cap B_3 \cap B_4)$
     $= \frac{2}{4} \times \frac{2}{3} \times \frac{1}{2} \times \frac{1}{1} + \frac{2}{4} \times \frac{2}{3} \times \frac{1}{2} \times \frac{1}{1} + \frac{2}{4} \times \frac{1}{3} \times \frac{2}{2} \times \frac{1}{1} = \frac{3}{6}$.
   - *So the pmf of X is* $f(x) = \frac{x-1}{6}, \quad x = 2, 3, 4.$

The following questions are optional for MAST20006 students but prescribed for MAST90057 ones.

9. Let X be the number of accidents in a factory per week having pmf

$$f(x) = \frac{1}{(x+1)(x+2)}, \quad x = 0, 1, 2, \cdots .$$

   (a) Find the conditional probability of $X \geq 4$, given that $X \geq 1$. (*Hint*: Write $f(x) = \frac{1}{x+1} - \frac{1}{x+2}$.)

   - *First*

$$P(X \geq 4 | X \geq 1) = \frac{P((X \geq 4) \cap (X \geq 1))}{P(X \geq 1)} = \frac{P(X \geq 4)}{1 - P(X = 0)}$$
$$= \frac{1 - \{P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)\}}{1 - P(X = 0)}.$$

   - $P(X = 0) = \frac{1}{(0+1)(0+2)} = \frac{1}{2}$.
   - $P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) = [\frac{1}{0+1} - \frac{1}{0+2}] + [\frac{1}{1+1} - \frac{1}{1+2}] + [\frac{1}{2+1} - \frac{1}{2+2}] + [\frac{1}{3+1} - \frac{1}{3+2}] = 1 - \frac{1}{5} = \frac{4}{5}$.
   - *So* $P(X \geq 4 | X \geq 1) = \frac{1 - \frac{4}{5}}{1 - \frac{1}{2}} = \frac{2}{5}$.

   (b) (*Optional*) Does $E(X)$ exist? If yes, find it; if not, why?

   - $E(X) = \sum_{x=0}^{\infty} x \cdot \frac{1}{(x+1)(x+2)} = \sum_{x=0}^{\infty} \frac{x+1-1}{(x+1)(x+2)}$
     $= \sum_{x=0}^{\infty} \frac{1}{x+2} - \sum_{x=0}^{\infty} \frac{1}{(x+1)(x+2)} = +\infty$.
   - *So $E(X)$ does not exist.*

10. (*Q2.2-14*) Suppose that a school has 20 classes: 16 with 25 students in each, three with 100 students in each, and one with 300 students for a total of 1000 students.

(a) What is the average class size?

- average size $= \frac{16\times25+3\times100+1\times300}{20} = 50$.

(b) Select a student randomly out of the 1000 students. Let the random variable X equal the size of the class to which this student belongs. Find the pmf of X.

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x	*25*	*100*	*300*
$f(x) = P(X = x)$	$\frac{16\times25}{1000} = 0.4$	$\frac{3\times100}{1000} = 0.3$	$\frac{1\times300}{1000} = 0.3$

(c) Find $E(X)$, the expected value of X. Does this answer surprise you?

- $E(X) = 25 \times 0.4 + 100 \times 0.3 + 300 \times 0.3 = 130.$

11. (*Q2.3-19*) A warranty is written on a product worth \$10,000 so that the buyer is given \$8000 if it fails in the first year, \$6000 if it fails in the second, \$4000 if it fails in the third, \$2000 if it fails in the fourth, and zero after that. Its probability of failing in a year is 0.1; failures are independent of those of other years. What is the expected value of the warranty?

- *Let X be such that the product fails at the X-th year. Let $Y = u(X)$ be the amount of money (value of the warranty) the buyer is given.*
- *Then*

$$Y = u(X) = \begin{cases} 8000, & x = 1 \\ 6000, & x = 2 \\ 4000, & x = 3 \\ 2000, & x = 4 \\ 0, & x \geq 5. \end{cases}$$

- *It can be seen that the pmf of Y is*

y	*8000*	*6000*	*4000*	*2000*	*0*
$P(Y = y)$	*0.1*	$0.9 \cdot 0.1$	$0.9^2 \cdot 0.1$	$0.9^3 \cdot 0.1$	$1 - \{0.1 + 0.9 \cdot 0.1 + 0.9^2 \cdot 0.1 + 0.9^3 \cdot 0.1\}$

- *So* $E(u(X)) = E(Y) = 8000(0.1) + 6000(0.9 \cdot 0.1) + 4000(0.9^2 \cdot 0.1)$
  $+2000(0.9^3 \cdot 0.1) + 0 \cdot P(Y = 0) = 1809.8.$