THE UNIVERSITY OF MELBOURNE

Student ID:

The University of Melbourne

Semester 1 Exam *Solutions* — *June, 2017*

School of Mathematics and Statistics

MAST20006 Probability for Statistics

Exam Duration: 3 Hours

Reading Time: 15 Minutes

This paper has 9 pages

Authorised materials:
This is a closed book exam.
A University approved hand-held calculator, i.e. Casio FX82 (any suffix) or Casio FX100 (any suffix), may be used.

Instructions to Invigilators:
Script books shall be supplied to each student.
Students may not take this paper with them at the end of the exam.

Instructions to Students:
This paper has **10** questions. Formula sheet is given at the end of this paper.
Attempt as many questions, or parts of questions, as you can.
Questions carry marks as shown in the brackets after the questions statements.
The total number of marks available for this examination is **100**.
Working and/or reasoning must be given to obtain full credit.

1. Medical case histories indicate that different illnesses may produce identical symptoms. Suppose that a particular set of symptoms, denoted H, occurs only when any one of three illnesses, I_1, I_2, or I_3, occurs. Assume that the simultaneous occurrence of more than one of these illnesses is impossible and that

$$P(I_1) = 0.01, \quad P(I_2) = 0.005, \quad P(I_3) = 0.02.$$

The probabilities of developing the set of symptoms H, given each of these illnesses, are known to be

$$P(H|I_1) = 0.90, \quad P(H|I_2) = 0.95, \quad P(H|I_3) = 0.75.$$

(a) Compute the probability that a person chosen at random has the symptoms H. [3]

- *Let I_0 be the event $\{$not I_1, I_2, or $I_3\}$. Then $P(H|I_0) = 0$ according to the information given. So I_0 will be ignored in the following.* (***No mark will be deducted for ignoring*** I_0)
- $P(H) = 0.01 \times 0.90 + 0.005 \times 0.95 + 0.02 \times 0.75 = 0.02875.$

(b) Assuming that an ill person exhibits the symptoms H, what is the probability that the person has illness I_1? [3]

- $P(I_1|H) = \dfrac{P(H|I_1)P(I_1)}{P(H)} = \dfrac{0.90 \times 0.01}{0.02875} = 0.313.$

(c) Suppose three people exhibit the symptoms H, what is the probability that exactly one of them has illness I_1? [3]

- $P(\text{one has } I_1 \mid \text{all three exhibit } H) = \binom{3}{1} \times 0.313 \times (1 - 0.313)^2 = 0.4431789.$

2. The employees of a firm that manufactures insulation are being tested for indications of asbestos in their lungs. Suppose 20% of the employees will have positive indications of asbestos in their lungs. Let X be the number of employees to be tested to get the first positive indication. Let Y be the number of employees to be tested to get 3 positive indications.

(a) Name the probability distribution and specify the value of any parameter(s) for each of the two random variables X and Y. [2]

- $X \overset{d}{=}$ *Geometric*$(p = 0.2)$, $\quad Y \overset{d}{=}$ *Negative binomial*$(r = 3, p = 0.2)$.

(b) What is the probability that at least 4 employees need be tested to get the first positive indication? [2]

- $P(X \geq 4) = \sum_{k=4}^{\infty} 0.2 \times 0.8^{k-1} = 1 - 0.2(1 + 0.8 + 0.8^2) = 0.8^3 = 0.512.$

(c) What is the probability that exactly 10 employees need be tested to get 3 positive indications? [2]

- $P(Y = 10) = \binom{9}{2} \times 0.2^3 \times 0.8^7 = 0.0604.$

(d) Compute $P(Y > 15)$. [3]

- *Let $Z \overset{d}{=} b(15, 0.2)$. Then using the relation between $b(n,p)$ and $Nb(r,p)$,*
- $P(Y > 15) = P(Z \leq 2) = 0.8^{15} + 15 \cdot 0.2 \cdot 0.8^{14} + \binom{15}{2} \cdot 0.2^2 \cdot 0.8^{13} = 0.03518437 + 0.1319414 + 0.2308974 = 0.3980232.$

(e) Compute $\mathrm{Var}(0.5^X)$. [3]

- *The mgf of X is* $M(t) = \dfrac{0.2e^t}{1-0.8e^t}$.
-

$$Var(0.5^X) = E(0.25^X) - \left[E(0.5^X)\right]^2 = M(\ln 0.25) - M(\ln 0.5)^2$$
$$= \frac{0.2 \times 0.25}{1-0.8\times 0.25} - \left(\frac{0.2\times 0.5}{1-0.8\times 0.5}\right)^2 = \frac{1}{16} - \frac{1}{6^2} = \frac{5}{144} = 0.03472$$

3. Insulin-dependent diabetes (IDD) is a common chronic disorder in children. Suppose a region in Australia has an incidence of 30 IDD cases per 100,000 per year. Let X be the number of IDD cases in 1000 children randomly selected from this region.

(a) Compute $P(X \le 1)$. [2]

- $X \stackrel{d}{=} binomial(n = 1000, p = 0.0003)$.
- $P(X \le 1) = P(X = 0) + P(X = 1)$
  $= (1 - 0.0003)^{1000} + 1000 \times 0.0003 \times 0.9997^{999} = 0.963087.$

(b) A binomial distribution $b(n, p)$ can be approximated by the Poisson($\lambda = np$) distribution if p is small and n large. Use this result to approximate the probability in part (a) by a Poisson probability. [2]

- $X \stackrel{d}{\approx} Poisson(\lambda = 0.3)$, *thus* $P(X \le 1) \approx e^{-0.3} + 0.3e^{-0.3} = 0.9630637.$

(c) The probability in part (a) may also be approximated by a normal probability based on the central limit theorem. Give a normal approximation (using the continuity correction, and expressed in terms of the standard normal cdf $\Phi(\cdot)$ only) to $P(X \le 1)$. [2]

- *By CLT,* $X \stackrel{d}{\approx} N(\mu = 0.3, \sigma^2 = 0.29991)$.
- *So* $P(X \le 1) \approx P\Big(Z \le \frac{1+0.5-0.3}{\sqrt{0.29991}}\Big) = P(Z \le 2.191219) = \Phi(2.191219)$.
  (=0.985782, a poor approximation; but not required for this question.)

4. A drawer contains 5 different and distinguishable pairs of socks (a total of 10 socks). A person (perhaps in the dark) randomly selects 4 socks. Let X be the number of matched pairs in the 4 selected socks.

(a) Compute the probability mass function (pmf) of X. [4]

- $$\begin{aligned} P(X=0) &= \frac{\binom{5}{4}\binom{2}{1}\binom{2}{1}\binom{2}{1}\binom{2}{1}\binom{2}{0}}{\binom{10}{4}} = \frac{8}{21} \\ P(X=1) &= \frac{\binom{5}{1}\binom{4}{2}\binom{2}{2}\binom{2}{1}\binom{2}{1}\binom{2}{0}\binom{2}{0}}{\binom{10}{4}} = \frac{12}{21} \\ P(X=2) &= \frac{\binom{5}{2}\binom{2}{2}\binom{2}{2}\binom{2}{0}\binom{2}{0}\binom{2}{0}}{\binom{10}{4}} = \frac{1}{21} \end{aligned}$$

(b) Compute $E(X)$ and $\mathrm{Var}(X)$. [2]

- $E(X) = \frac{2}{3}$, $\quad E(X^2) = \frac{16}{21} = 0.7619$, $\quad Var(X) = \frac{16}{21} - \frac{4}{9} = \frac{20}{63} = 0.3175.$

5. Let X be a continuous random variable with probability density function (pdf)

$$f(x) = \frac{2}{9}(x+2), \quad -2 < x < 1.$$

(a) Compute the cumulative distribution function (cdf) of X. [2]

- *For* $-2 < x < 1$, $P(X \le x) = \int_{-2}^{x} \frac{2}{9}(t+2)dt = \frac{1}{9}(t+2)^2 \mid_{-2}^{x} = \frac{(x+2)^2}{9}$.
- *Hence the cdf of* X *is* $F(x) = \begin{cases} 0 & \text{if } x \le -2 \\ \frac{(x+2)^2}{9} & \text{if } -2 < x < 1 \\ 1 & \text{if } x \ge 1 \end{cases}$

(b) Let X_1 and X_2 be two independent random variables each having the pdf $f(x)$ given above. Define $W = \min\{X_1, X_2\}$. Find the 75-th percentile of W. [3]

- $P(W > w) = P(\min\{X_1, X_2\} > w) = P(X_1 > w)P(X_2 > w)$. *Using this property the cdf of* W *can be found to be*

$$F_W(w) = \begin{cases} 0 & \text{if } w \le -2 \\ 1 - \left[1 - \frac{(w+2)^2}{9}\right]^2 & \text{if } -2 < w < 1 \\ 1 & \text{if } w \ge 1 \end{cases}$$

- *Solving* $\frac{3}{4} = 1 - \left[1 - \frac{(w+2)^2}{9}\right]^2$ *gives* $\pi_{0.75} = \sqrt{4.5} - 2 = 0.12132$.

(c) Consider the transformation $Y = X^2$ of X.

i. Is this transformation one-to-one? Find the support of Y. [2]

- *This is not one-to-one. The support of* Y *is* $0 \le y < 4$.

ii. Derive the cdf of Y. [3]

- *For* $0 \le y \le 1$,
  $G(y) = P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y})$
  $= \int_{-\sqrt{y}}^{\sqrt{y}} \frac{2}{9}(x+2)dx = \frac{1}{9}[(2+\sqrt{y})^2 - (2-\sqrt{y})^2] = \frac{8}{9}\sqrt{y}$.
- *For* $1 < y < 4$,
  $G(y) = P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le 1)$
  $= \int_{-\sqrt{y}}^{1} \frac{2}{9}(x+2)dx = 1 - \frac{1}{9}(2-\sqrt{y})^2 = \frac{1}{9}(5 + 4\sqrt{y} - y)$.
- *So the cdf of* Y *is*

$$G(y) = \begin{cases} 0, & y < 0, \\ \frac{8}{9}\sqrt{y}, & 0 \le y \le 1, \\ \frac{1}{9}(5 + 4\sqrt{y} - y) = \frac{1}{9}(5-\sqrt{y})(1+\sqrt{y}), & 1 < y < 4, \\ 1, & y \ge 4. \end{cases}$$

iii. Derive the pdf of Y. [2]

- $g(y) = G'(y) = \begin{cases} 0, & y < 0, \\ \frac{4}{9\sqrt{y}}, & 0 \le y \le 1, \\ \frac{2}{9\sqrt{y}} - \frac{1}{9} & 1 < y < 4, \\ 0, & y \ge 4. \end{cases}$

6. Let X_1, X_2, X_3 be independent Poisson($\lambda = 1$) random variables. Define $Y_1 = X_1 + X_3$ and $Y_2 = X_2 + X_3$.

   (a) Compute the value of $\text{Cov}(Y_1, Y_2)$. [2]

   - $\text{Cov}(Y_1, Y_2) = \text{Cov}(X_1 + X_3, X_2 + X_3) = \text{Cov}(X_1, X_2) + \text{Cov}(X_1, X_3)$ $+\text{Cov}(X_3, X_2) + \text{Cov}(X_3, X_3) = \text{Var}(X_3) = \lambda = 1.$

   (b) Use Chebyshev's inequality $P(|X - \mu| < k\sigma) \geq 1 - \frac{1}{k^2}$ to find a lower bound for $P(|Y_1 - 2| < 2)$. [1]

   - $E(Y_1) = 2E(X_1) = 2\lambda = 2$ *and* $Var(Y_1) = 2\,Var(X_1) = 2\lambda = 2.$
   - $P(|Y_1 - 2| < 2) = P(|Y_1 - 2| < \sqrt{2}\sqrt{2}) \geq 1 - \frac{1}{\sqrt{2}^2} = 0.5.$

   (c) Compute the exact value of $P(|Y_1 - 2| < 2)$. [2]

   - Y_1 *has a Pois(2) distribution because* X_1 *and* X_2 *are iid Pois(1). Thus*

$$P(|Y_1 - 2| < 2) = P(0 < Y_1 < 4) = P(1 \leq Y_1 \leq 3)$$
$$= P(Y_1 = 1, 2, 3) = \frac{16}{3}e^{-2} = 0.7218.$$

   (d) Define

$$Z_1 = \begin{cases} 1 & \text{if } Y_1 = 0, \\ 0 & \text{otherwise;} \end{cases} \quad \text{and} \quad Z_2 = \begin{cases} 1 & \text{if } Y_2 = 0, \\ 0 & \text{otherwise.} \end{cases}$$

   i. Compute the marginal pmf (probability mass function) of Z_1 and Z_2 respectively. [2]

   - *Both* Z_1 *and* Z_2 *are Bernoulli r.v.s with* $P(Z_1 = 1) = P(Z_2 = 1) = P(X_1 = X_3 = 0) = P(X_2 = X_3 = 0) = e^{-2}.$

   ii. Compute the joint pmf (joint probability mass function) of (Z_1, Z_2). [3]

   - $P(Z_1 = 1, Z_2 = 1) = P(X_1 = X_2 = X_3 = 0) = e^{-3}.$
   - $P(Z_1 = 1, Z_2 = 0) = P(X_1 = X_3 = 0, X_2 > 0) = e^{-2}(1 - e^{-1}).$
   - $P(Z_1 = 0, Z_2 = 1) = P(X_1 > 0, X_2 = X_3 = 0) = (1 - e^{-1})e^{-2}.$
   - $P(Z_1 = 0, Z_2 = 0) = 1 - e^{-3} - 2e^{-2}(1 - e^{-1}) = 1 - 2e^{-2} + e^{-3}.$

   iii. Compute the correlation coefficient between Z_1 and Z_2. [3]

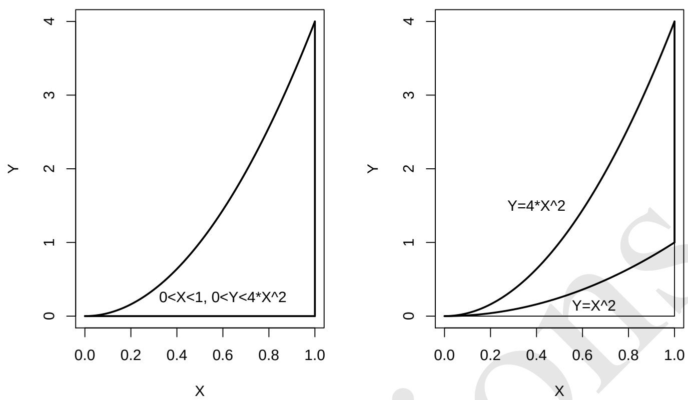
   - $E(Z_1) = P(Z_1 = 1) = e^{-2}$
   - $Var(Z_1) = P(Z_1 = 1)P(Z_1 = 0) = e^{-2}(1 - e^{-2}).$
   - Z_1 *and* Z_2 *have the same distribution. So* $E(Z_2) = e^{-2}$ *and* $Var(Z_2) = e^{-2}(1 - e^{-2}).$
   - $E(Z_1 Z_2) = P(Z_1 = Z_2 = 1) = e^{-3}.$
   - *So* $Cov(Z_1, Z_2) = e^{-3} - e^{-4} = e^{-3}(1 - e^{-1}).$
   - *Therefore,*

$$\rho(Z_1, Z_2) = \frac{Cov(Z_1, Z_2)}{\sqrt{Var(Z_1)}\sqrt{Var(Z_2)}} = \frac{e^{-3}(1 - e^{-1})}{e^{-2}(1 - e^{-2})} = \frac{e^{-1}}{1 + e^{-1}} = 0.2689.$$

7. The marginal distribution of a random variable X is uniform $U(0,1)$. The conditional distribution of another random variable Y, given $X = x$, is uniform $U(0, 4x^2)$.

(a) Compute the joint pdf of X and Y. Sketch the region where $f(x,y) > 0$. [2]

- $f(x,y) = f_1(x)h(y|x) = 1 \times \dfrac{1}{4x^2} = \dfrac{1}{4x^2}, \quad 0 < x < 1, \quad 0 < y < 4x^2.$



(b) Compute $f_2(y)$, the marginal pdf of Y. [2]

- $f_2(y) = \displaystyle\int_{\frac{1}{2}\sqrt{y}}^{1} \frac{1}{4x^2}dx = -\left.\frac{1}{4x}\right|_{\frac{1}{2}\sqrt{y}}^{1} = \frac{1}{2\sqrt{y}} - \frac{1}{4}, \quad 0 < y < 4.$

(c) Compute $g(x|y)$, the conditional pdf of X, given $Y = y$. [2]

- $g(x|y) = \dfrac{f(x,y)}{f_2(y)} = \dfrac{1/4x^2}{\frac{1}{2\sqrt{y}} - \frac{1}{4}} = \dfrac{\sqrt{y}}{(2-\sqrt{y})x^2}, \quad \dfrac{\sqrt{y}}{2} < x < 1.$

(d) Compute the probability $P(X < \frac{3}{4} \mid Y = \frac{1}{4})$. [2]

- $P(X < \dfrac{3}{4} \mid Y = \dfrac{1}{4}) = \displaystyle\int_{1/4}^{3/4} \frac{1/2}{(2-1/2)x^2}dx = -\left.\frac{1}{3x}\right|_{1/4}^{3/4} = \frac{4}{3} - \frac{4}{9} = \frac{8}{9}.$

(e) Compute the probability $P(X < \sqrt{Y})$. [3]

- $P(X < \sqrt{Y}) = \displaystyle\int_0^1 \left(\int_{x^2}^{4x^2} \frac{1}{4x^2}dy \right) dx = \int_0^1 \frac{3}{4}dx = \frac{3}{4}.$

- *Or* $P(X < \sqrt{Y}) = \displaystyle\int_0^1 \left(\int_{\frac{1}{2}\sqrt{y}}^{\sqrt{y}} \frac{1}{4x^2}dx \right) dy + \int_1^4 \left(\int_{\frac{1}{2}\sqrt{y}}^{1} \frac{1}{4x^2}dx \right) dy$

$$= \int_0^1 \frac{1}{4\sqrt{y}}dy + \int_1^4 \left(\frac{1}{2\sqrt{y}} - \frac{1}{4} \right) dy = \frac{1}{2} + \left(2 - 1 - \frac{3}{4} \right) = \frac{3}{4}.$$

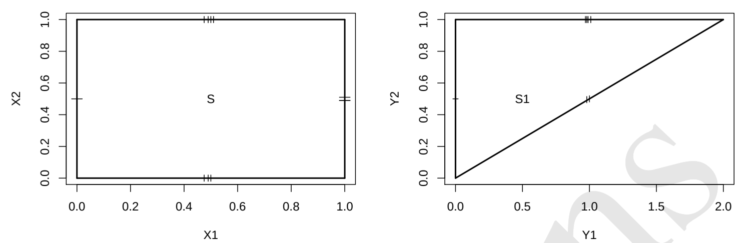
8. Consider two random variables X_1 and X_2 with the joint pdf

$$f(x_1, x_2) = \begin{cases} 4x_1x_2, & 0 < x_1 < 1, \quad 0 < x_2 < 1, \\ 0 & \text{elsewhere.} \end{cases}$$

Let $Y_1 = 2X_1X_2$ and $Y_2 = X_2$ be a joint transformation of (X_1, X_2).

(a) Find the support of (Y_1, Y_2) and sketch it. [3]

- *The support of* (Y_1, Y_2) *is* $\{0 < Y_1 < 2Y_2 < 2\}$.



(b) Find the inverse transformation. [2]

- $X_1 = \dfrac{Y_1}{2Y_2}, \quad X_2 = Y_2.$

(c) Compute the Jacobian of the inverse transformation. [2]

- $J = \begin{vmatrix} 1/2y_2 & -y_1/2y_2^2 \\ 0 & 1 \end{vmatrix} = \dfrac{1}{2y_2}.$

(d) Compute the joint pdf of (Y_1, Y_2). [2]

- $g(y_1, y_2) = |J| \cdot f(\dfrac{y_1}{2y_2}, y_2) = \dfrac{1}{2y_2} \cdot 4\left(\dfrac{y_1}{2y_2}\right) y_2 = \dfrac{y_1}{y_2}, \quad 0 < y_1 < 2y_2 < 2.$

(e) Compute the marginal pdf of Y_1 from the joint pdf of (Y_1, Y_2). [3]

- $g_1(y_1) = \displaystyle\int_{\frac{1}{2}y_1}^{1} \frac{y_1}{y_2} dy_2 = -y_1 \ln(\frac{1}{2}y_1) = y_1(\ln 2 - \ln y_1), \quad 0 < y_1 < 2.$

9. Let $X_1, X_2, \cdots, X_n$ be independent random variables each having the moment-generating function (mgf)

$$M(t) = \frac{1-3t}{(1-2t)(1-4t)}, \quad t < \frac{1}{4}.$$

(a) Compute the mgf $M_{Y_n}(t)$ of the sum $Y_n = X_1 + X_2 + \cdots + X_n$. [2]

- $M_{Y_n}(t) = E(e^{tY_n}) = [M_{X_1}(t)]^n = \dfrac{(1-3t)^n}{(1-2t)^n(1-4t)^n}, \quad t < \dfrac{1}{4}.$

(b) Compute the mgf $M_{\bar{Y}_n}(t)$ of the sample mean $\bar{Y}_n = \frac{Y_n}{n}$. [2]

- $M_{\bar{Y}_n}(t) = E(e^{t\bar{Y}_n}) = M_{Y_n}\left(\dfrac{t}{n}\right) = \dfrac{(1-\frac{3t}{n})^n}{(1-\frac{2t}{n})^n(1-\frac{4t}{n})^n}, \quad t < \dfrac{n}{4}.$

(c) Compute the limiting mgf $\lim_{n\to\infty} M_{\bar{Y}_n}(t)$. What distribution does the limiting mgf correspond to? What is the implication of this result? [2]

- *For any given t,* $\displaystyle\lim_{n\to\infty} M_{\bar{Y}_n}(t) = \lim_{n\to\infty} \frac{(1-\frac{3t}{n})^n}{(1-\frac{2t}{n})^n(1-\frac{4t}{n})^n} = \frac{e^{-3t}}{e^{-2t}e^{-4t}} = e^{3t}$
- *The limit is the mgf of a degenerate distribution having probability 1 at 3.*
- *This implies that* $\bar{Y}_n \xrightarrow{p} 3 = E(X_1)$ *as* $n \to \infty$.
- *Note it is also a correct answer if it is based on applying the WLLN. Then the WLLN must be correctly stated, and* $E(X_1) = 3$ *be proved.*

(d) Let $Z_n = \sqrt{n}(\bar{Y}_n - 3)$. Compute $M_{Z_n}(t)$, the mgf of Z_n. Then using the result to compute $\lim_{n\to\infty} M_{Z_n}(t)$. Finally explain what is the limiting distribution of Z_n as $n \to \infty$. [5]

- $M_{Z_n}(t) = E(e^{t\sqrt{n}(\bar{Y}_n - 3)}) = e^{-3t\sqrt{n}} M_{\bar{Y}_n}(\sqrt{n}t)$

$$= \frac{e^{-3t\sqrt{n}}(1-\frac{3\sqrt{n}t}{n})^n}{(1-\frac{2\sqrt{n}t}{n})^n(1-\frac{4\sqrt{n}t}{n})^n} = \left[\frac{e^{-3t/\sqrt{n}}(1-\frac{3t}{\sqrt{n}})}{1-\frac{6t}{\sqrt{n}}+\frac{8t^2}{n}}\right]^n, \quad t < \frac{\sqrt{n}}{4}.$$

- *By Taylor's series expansion,* $e^u \approx 1 + u + \frac{1}{2}u^2$ *when* $|u|$ *is sufficiently small. Using this result, for any given* t,

$$\begin{aligned}\lim_{n\to\infty} M_{Z_n}(t) &= \lim_{n\to\infty}\left[\frac{e^{-3t/\sqrt{n}}(1-\frac{3t}{\sqrt{n}})}{1-\frac{6t}{\sqrt{n}}+\frac{8t^2}{n}}\right]^n = \lim_{n\to\infty}\left[\frac{(1-\frac{3t}{\sqrt{n}}+\frac{9t^2}{2n})(1-\frac{3t}{\sqrt{n}})}{1-\frac{6t}{\sqrt{n}}+\frac{8t^2}{n}}\right]^n \\ &= \lim_{n\to\infty}\left[\frac{1-\frac{6t}{\sqrt{n}}+\frac{27t^2}{2n}}{1-\frac{6t}{\sqrt{n}}+\frac{8t^2}{n}}\right]^n = \lim_{n\to\infty}\left(1+\frac{11t^2}{2n}\right)^n = e^{11t^2/2}\end{aligned}$$

- *The limit mgf is that of* $N(\mu = 0, \sigma^2 = 11)$.
- *This implies that* $Z_n \xrightarrow{d} N(0, 11)$ *as* $n \to \infty$.
- *Note it is also a correct answer if it is based on applying the CLT. Then the CLT must be correctly stated, and* $Var(X_1) = 11$ *be proved.*

10. A random variable X has the following mgf

$$M(t) = \frac{1}{(1-2t)(1-4t)}, \quad t < \frac{1}{4}.$$

(a) Compute the value of $E(X)$. [2]

- *It can be shown that* $M(t) = \dfrac{2}{1-4t} - \dfrac{1}{1-2t} = \dfrac{1}{1-6t+8t^2}$
- $M'(t) = \dfrac{8}{(1-4t)^2} - \dfrac{2}{(1-2t)^2} = \dfrac{6-16t}{(1-6t+8t^2)^2}$.
- *Thus* $E(X) = M'(0) = 8 - 2 = 6$.

(b) Compute the probability $P(X > 4)$. [6]

- *From the mgf of an exponential distribution it can be seen that* $X = X_1 + X_2$, *where* $X_1 \stackrel{d}{=} Exp(2)$, $X_2 \stackrel{d}{=} Exp(4)$, *and* X_1 *and* X_2 *are independent of each other.*
- *The joint pdf of* (X_1, X_2) *is* $g(x_1, x_2) = \dfrac{1}{2}e^{-x_1/2} \cdot \dfrac{1}{4}e^{-x_2/4}; \quad x_1 > 0, x_2 > 0.$
- *Now define* $Y = X_2$ *together with* $X = X_1 + X_2$, *thus* (X, Y) *is a random vector transformed from* (X_1, X_2). *Hence, the joint pdf of* (X, Y) *is*

$$f(x, y) = g(x-y, y)|J| = \frac{1}{8}e^{-\frac{1}{4}(2x-y)} \cdot 1, \quad x > y > 0$$

- *The pdf of* X *is*

$$f_1(x) = \int_0^x \frac{1}{8}e^{-\frac{1}{4}(2x-y)}dy = \frac{1}{2}e^{-\frac{1}{4}x} - \frac{1}{2}e^{-\frac{1}{2}x}, \quad x > 0.$$

- *Therefore*

$$\begin{aligned} P(X > 4) &= \int_4^\infty \left(\frac{1}{2}e^{-\frac{1}{4}x} - \frac{1}{2}e^{-\frac{1}{2}x}\right) dx \\ &= \left(-2e^{-\frac{1}{4}x} + e^{-\frac{1}{2}x}\right)\Big|_4^\infty = 2e^{-1} - e^{-2} = 0.6004236 \end{aligned}$$

Total marks = 100

End of the exam questions.
Formulas are on the next page.