The Theory of Relation between Creation-Annihilation and Statistics

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(Dated: October 27, 2024)

Most of fields which has Lorentz invariance(scalar, spinor, etc.), can construct each spectra of fields by acting the creation or annihilation operators from the proper vacuum. In former discussions about 'spin-statistics theorem', the facts that some quantization condition of creation and annihilation operators, like $[\hat{a}_\mathbf{p}, \hat{a}_{\mathbf{p}'}]_+ = [\hat{a}^\dagger_\mathbf{p}, \hat{a}^\dagger_{\mathbf{p}'}]_+ = 0$, $[\hat{a}_\mathbf{p}, \hat{a}^\dagger_{\mathbf{p}'}]_+ = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')$ (Bose-Einstein), or $[\hat{a}_\mathbf{p}, \hat{a}_{\mathbf{p}'}]_- = [\hat{a}^\dagger_\mathbf{p}, \hat{a}^\dagger_{\mathbf{p}'}]_- = 0$, $[\hat{a}_\mathbf{p}, \hat{a}^\dagger_{\mathbf{p}'}]_- = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')$ (Fermi-Dirac) can explain the causality problem of each fields, also well-knowed. In the aspects of statistics of particles, only the terms of $[\hat{a}_\mathbf{p}, \hat{a}_{\mathbf{p}'}]_\pm = [\hat{a}_\mathbf{p}, \hat{a}_{\mathbf{p}'}]_\pm = 0$ can be explained statistically in the former discussions, so the problem of statistical explanation about the term of $[\hat{a}_\mathbf{p}, \hat{a}^\dagger_{\mathbf{p}'}]_\pm = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')$ remaind: Why the term $[\hat{a}_\mathbf{p}, \hat{a}^\dagger_{\mathbf{p}'}]_\pm = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')$ must be add up to each fields to set up the postulates of Bose-Einstein statistics and Fermi-Dirac statistics? In this paper, we construct the theory of explaining the physical meaning of the term $[\hat{a}_\mathbf{p}, \hat{a}^\dagger_{\mathbf{p}'}]_\pm = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')$ involving to one's statistics of particle, for the formal relativistic fields. Exactly, this term $[\hat{a}_\mathbf{p}, \hat{a}^\dagger_{\mathbf{p}'}]_\pm = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')$ supports the superposition principle of energy of spectra for 2 or more multi-particles systems.

I. INTRODUCTION

The statistics of particles represented by formal relativistic fields $\varphi^{A,B}_{a,b}(x)$ can be determined by considering its causality. In former discussions about *spin-statistic theorem*[1, 2], the postulates of Bose-Einstein statistics or Fermi-Dirac statistics for the general Lorentz invariant field[1][1] [2]

$$\varphi^{A,B}_{a,b}(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_\mathbf{p}}} \sum_\lambda \alpha a^\lambda_\mathbf{p} f^{A,B}_{a,b}(\mathbf{p},\lambda) e^{-ip\cdot x} + \int \frac{d^3p}{(2\pi)^3\sqrt{2E_\mathbf{p}}} \sum_\lambda \beta b^{\lambda\dagger}_\mathbf{p} g^{A,B}_{a,b}(\mathbf{p},\lambda) e^{ip\cdot x} \tag{1}$$

with its spectra(k-particle system) $|\mathbf{p}_1, \cdots, \mathbf{p}_k; \pm\rangle$ defined[3] from its vacuum by

$$|\mathbf{p}_1, \cdots, \mathbf{p}_k; +\rangle = \prod_{i=1}^{k} (\Gamma_{\mathbf{p}_i} a^{\lambda_i\dagger}_{\mathbf{p}_i})|0\rangle \tag{2}$$

has been supposed by:

Postulate I. (Bose-Einstein statistics) For the Bose-Einstein statistics, we set up following canonical quantization condition of operators as:

$$\begin{aligned} &[a^\lambda_\mathbf{p}, a^{\lambda'}_{\mathbf{p}'}]_+ = [b^\lambda_\mathbf{p}, b^{\lambda'}_{\mathbf{p}'}]_+ = 0, \\ &[a^\lambda_\mathbf{p}, a^{\lambda'\dagger}_{\mathbf{p}'}]_+ = [b^\lambda_\mathbf{p}, b^{\lambda'\dagger}_{\mathbf{p}'}]_+ = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')\delta^{\lambda\lambda'}. \end{aligned} \tag{3}$$

Postulate II. (Fermi-Dirac statistics) For the Fermi-Dirac statistics, we set up following canonical quantization condition of operators as:

$$\begin{aligned} &[a^\lambda_\mathbf{p}, a^{\lambda'}_{\mathbf{p}'}]_- = [b^\lambda_\mathbf{p}, b^{\lambda'}_{\mathbf{p}'}]_- = 0, \\ &[a^\lambda_\mathbf{p}, a^{\lambda'\dagger}_{\mathbf{p}'}]_- = [b^\lambda_\mathbf{p}, b^{\lambda'\dagger}_{\mathbf{p}'}]_- = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')\delta^{\lambda\lambda'}. \end{aligned} \tag{4}$$

The reason why each term $[a^\lambda_\mathbf{p}, a^{\lambda'}_{\mathbf{p}'}]_+ = [b^\lambda_\mathbf{p}, b^{\lambda'}_{\mathbf{p}'}]_+ = 0$ and $[a^\lambda_\mathbf{p}, a^{\lambda'}_{\mathbf{p}'}]_- = [b^\lambda_\mathbf{p}, b^{\lambda'}_{\mathbf{p}'}]_- = 0$ insists the *Bose-Einstein statistics* and *Fermi-Dirac statistics* is trivial. As each commutation and anti-commutation relation gives:

$$[\hat{A}, \hat{B}]_+ = 0 \qquad \text{s.t.} \qquad \hat{A}\hat{B} = \hat{B}\hat{A}, \tag{5}$$

$$[\hat{A}, \hat{B}]_- = 0 \qquad \text{s.t.} \qquad \hat{A}\hat{B} = -\hat{B}\hat{A}, \tag{6}$$

so the quantization condition $[a^\lambda_\mathbf{p}, a^{\lambda'}_{\mathbf{p}'}]_+ = [b^\lambda_\mathbf{p}, b^{\lambda'}_{\mathbf{p}'}]_+ = 0$ supports that spectra of system behaves like

$$|\cdots, \mathbf{p}_i, \cdots \mathbf{p}_j, \cdots; +\rangle = |\cdots, \mathbf{p}_j, \cdots \mathbf{p}_i, \cdots; +\rangle. \tag{7}$$

Similarly, in case of anti-commutative condition $[a^\lambda_\mathbf{p}, a^{\lambda'}_{\mathbf{p}'}]_- = [b^\lambda_\mathbf{p}, b^{\lambda'}_{\mathbf{p}'}]_- = 0$ supposed, it shows that the spectra be anti-symmetric:

$$|\cdots, \mathbf{p}_i, \cdots \mathbf{p}_j, \cdots; +\rangle = -|\cdots, \mathbf{p}_j, \cdots \mathbf{p}_i, \cdots; +\rangle. \tag{8}$$

But, the problem about physical meaning of the term $[a^\lambda_\mathbf{p}, a^{\lambda'\dagger}_{\mathbf{p}'}]_\pm = [b^\lambda_\mathbf{p}, b^{\lambda'\dagger}_{\mathbf{p}'}]_\pm = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')\delta^{\lambda\lambda'}$ still remained. Why these terms which include Dirac-delta

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[1] Here we set each operator $a^{\lambda\dagger}_\mathbf{p}, a^\lambda_\mathbf{p}$ as creation and annihilation operator of particles, $b^{\lambda\dagger}_\mathbf{p}, b^\lambda_\mathbf{p}$ as creation and annihilation operator of antiparticles.

[2] Each quantities $f^{A,B}_{a,b}(\mathbf{p},\lambda), g^{A,B}_{a,b}(\mathbf{p},\lambda)$ appeared on equation (1) is determined by the representation of field. For example, if we select complex scalar field, each quantities comes to

$$f^{A,B}_{a,b}(\mathbf{p},\lambda) = g^{A,B}_{a,b}(\mathbf{p},\lambda) = 1.$$

[3] Here we introduce function $\Gamma_{\mathbf{p}_i}$ for the Lorentz invariance of spectra $|\mathbf{p}_1, \cdots, \mathbf{p}_k; \pm\rangle$.

function must be add up to support its statistics? In this paper, we construct the theory what gives proper explanation for the statistical justification of necessary of the term $[a_{\mathbf{p}}^{\lambda}, a_{\mathbf{p}'}^{\lambda'\dagger}]_{\pm} = [b_{\mathbf{p}}^{\lambda}, b_{\mathbf{p}'}^{\lambda'\dagger}]_{\pm} = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')\delta^{\lambda\lambda'}$ based on superposition principle about energy for each *Bose-Einstein statistics* or *Fermi-Dirac statistics*, in aspects of statistics of system.

II. HAMILTONIAN APPROACH TO 1-PARTICLE SYSTEM

Here we going to have discussion about Lorentz invariant field $\varphi_{a,b}^{A,B}$ which has its hamiltonian

$$H = \sum_{\lambda} \int \frac{d^3p}{(2\pi)^3}\, E_{\mathbf{p}} \left(\tilde{\alpha}_{a,b}^{A,B} a_{\mathbf{p}}^{\lambda\dagger} a_{\mathbf{p}}^{\lambda} + \tilde{\beta}_{a,b}^{A,B} b_{\mathbf{p}}^{\lambda\dagger} b_{\mathbf{p}}^{\lambda} \right). \quad (9)$$

for the ‘empty vacuum’,

$$H = \sum_{\lambda} \int \frac{d^3p}{(2\pi)^3}\, E_{\mathbf{p}} \left(\tilde{\alpha}_{a,b}^{A,B} a_{\mathbf{p}}^{\lambda\dagger} a_{\mathbf{p}}^{\lambda} + \tilde{\beta}_{a,b}^{A,B} b_{\mathbf{p}}^{\lambda} b_{\mathbf{p}}^{\lambda\dagger} \right). \quad (10)$$

for the ‘anti-particle filled vacuum’ and U(1)-Noether charge Q as

$$Q = \sum_{\lambda} \int \frac{d^3p}{(2\pi)^3} \left(|\alpha|^2 a_{\mathbf{p}}^{\lambda\dagger} a_{\mathbf{p}}^{\lambda} - |\beta|^2 b_{\mathbf{p}}^{\lambda\dagger} b_{\mathbf{p}}^{\lambda} \right). \quad (11)$$

This hamiltonian representation using creation and annihilation operators (9) can explain the each representations of particle well. For example, in the case of real scalar field, the condition $b_{\mathbf{p}}^{\lambda\dagger} = a_{\mathbf{p}}^{\lambda}$ is derived, so that each constants $\tilde{\alpha}_{a,b}^{A,B}, \tilde{\beta}_{a,b}^{A,B}, \Omega_0$ also determined by

$$\tilde{\alpha}_{0,0}^{0,0} = \tilde{\beta}_{0,0}^{0,0} = \frac{1}{2}, \quad \pm = +. \quad (12)$$

Similar way, in case of complex scalar field and Dirac field select:

$$\begin{aligned} &\tilde{\alpha} = \tilde{\beta} = 1, \quad \pm = + \qquad \text{(complex scalar field)} \\ &\tilde{\alpha} = -\tilde{\beta} = 1, \quad \pm = - \qquad \text{(Dirac field)} \end{aligned} \quad (13)$$

Now then, we have discussions about statistics of field using this hamiltonian H and U(1)-Noether charge Q. In this section, we begin by considering the 1-particle spectra

$$|\mathbf{p};+\rangle = \Gamma_{\mathbf{p}} a_{\mathbf{p}}^{\lambda\dagger}|0\rangle. \quad (14)$$

On the discussion of 1-particle system, we can ignore the anti-particle component of hamiltonian by

$$H|\mathbf{p};+\rangle = \sum_{\lambda} \int \frac{d^3p}{(2\pi)^3}\, E_{\mathbf{p}} \tilde{\alpha}_{a,b}^{A,B} a_{\mathbf{p}}^{\lambda\dagger} a_{\mathbf{p}}^{\lambda} |\mathbf{p};+\rangle \quad (15)$$

because only particle generating system yields $\tilde{\beta} = 0$ on field representation (1), whatever each vacuum we take. Thus, the action of hamiltonian on 1-particle spectra is

$$\begin{aligned} H|\mathbf{p};+\rangle &= \sum_{\lambda} \tilde{\alpha}_{a,b}^{A,B} \int \frac{d^3k}{(2\pi)^3}\, E_{\mathbf{k}} a_{\mathbf{k}}^{\lambda\dagger} a_{\mathbf{k}}^{\lambda} |\mathbf{p};+\rangle \\ &= \sum_{\lambda} \tilde{\alpha}_{a,b}^{A,B} \int \frac{d^3k}{(2\pi)^3}\, E_{\mathbf{k}} a_{\mathbf{k}}^{\lambda\dagger} a_{\mathbf{k}}^{\lambda} \left(\Gamma_{\mathbf{p}} a_{\mathbf{p}}^{\bar{\lambda}\dagger}|0\rangle \right) \\ &= \sum_{\lambda} \tilde{\alpha}_{a,b}^{A,B} \int \frac{d^3k}{(2\pi)^3}\, E_{\mathbf{k}} \Gamma_{\mathbf{p}} a_{\mathbf{k}}^{\lambda\dagger} \left[a_{\mathbf{k}}^{\lambda}, a_{\mathbf{p}}^{\bar{\lambda}\dagger} \right]_{\pm} |0\rangle \\ &\quad \pm \sum_{\lambda} \tilde{\alpha}_{a,b}^{A,B} \int \frac{d^3k}{(2\pi)^3}\, E_{\mathbf{k}} \Gamma_{\mathbf{p}} a_{\mathbf{k}}^{\lambda\dagger} a_{\mathbf{p}}^{\bar{\lambda}\dagger} a_{\mathbf{p}}^{\lambda} |0\rangle \\ &= \sum_{\lambda} \tilde{\alpha}_{a,b}^{A,B} \int \frac{d^3k}{(2\pi)^3}\, E_{\mathbf{k}} \Gamma_{\mathbf{p}} a_{\mathbf{k}}^{\lambda\dagger} \left[a_{\mathbf{k}}^{\lambda}, a_{\mathbf{p}}^{\bar{\lambda}\dagger} \right]_{\pm} |0\rangle. \end{aligned} \quad (16)$$

The hamiltonian H gives eigenstate to 1-particle spectra $H|\mathbf{p};+\rangle = E_{\mathbf{p}}|\mathbf{p};+\rangle$, we obtain

$$\begin{aligned} H|\mathbf{p};+\rangle &= \sum_{\lambda} \tilde{\alpha}_{a,b}^{A,B} \int \frac{d^3k}{(2\pi)^3}\, E_{\mathbf{k}} \Gamma_{\mathbf{p}} a_{\mathbf{k}}^{\lambda\dagger} \left[a_{\mathbf{k}}^{\lambda}, a_{\mathbf{p}}^{\bar{\lambda}\dagger} \right]_{\pm} |0\rangle. \\ &= E_{\mathbf{p}} \left(\Gamma_{\mathbf{p}} a_{\mathbf{p}}^{\bar{\lambda}\dagger}|0\rangle \right), \end{aligned} \quad (17)$$

so it implies

$$\left[a_{\mathbf{k}}^{\lambda}, a_{\mathbf{p}}^{\bar{\lambda}\dagger} \right]_{\pm} = \frac{(2\pi)^3}{\tilde{\alpha}_{a,b}^{A,B}} \delta^3(\mathbf{k}-\mathbf{p}) \delta^{\lambda\bar{\lambda}}. \quad (18)$$

Hereby we shall confirm that the commutator or anti-commutator relation (18) must be add up to system to support eigenstate hamiltonian. Sign convention on commutator or anti-commutator (18) can be determined by considering k-particle($k \geq 2$) system and its statistics.

III. HAMILTONIAN APPROACH TO 2-PARTICLE SYSTEM

We shall begin by considering the 2-particle spectra:

$$|\mathbf{p}_1, \mathbf{p}_2;+\rangle = \Gamma_{\mathbf{p}_1}\Gamma_{\mathbf{p}_2} a_{\mathbf{p}_1}^{\lambda_1\dagger} a_{\mathbf{p}_2}^{\lambda_2\dagger}|\mathbf{0}\rangle \quad (19)$$

with physical quantities of each particle $(\mathbf{p}_1, \lambda_1), (\mathbf{p}_2, \lambda_2)$. Since we are considering 2-particle spectra consisting only of particles, let us put $\tilde{\beta}_{a,b}^{A,B} = 0$ on field representation (1). Thus, the hamiltonian of this 2-particle spectra satisfies

$$H|\mathbf{p}_1, \mathbf{p}_2;+\rangle = \sum_{\lambda} \int \frac{d^3p}{(2\pi)^3}\, E_{\mathbf{p}} \tilde{\alpha}_{a,b}^{A,B} a_{\mathbf{p}}^{\lambda\dagger} a_{\mathbf{p}}^{\lambda} |\mathbf{p}_1, \mathbf{p}_2;+\rangle. \quad (20)$$

We shall calculate the action on 2-particle spectra $H|\mathbf{p}_1, \mathbf{p}_2;+\rangle$ as

$$
\begin{aligned}
H|\mathbf{p}_1,\mathbf{p}_2;+\rangle &= \sum_\lambda \int \frac{d^3k}{(2\pi)^3}\ E_{\mathbf{k}}\tilde{\alpha}^{A,B}_{a,b} a^{\lambda\,\dagger}_{\mathbf{k}} a^\lambda_{\mathbf{k}} |\mathbf{p}_1,\mathbf{p}_2;+\rangle = \tilde{\alpha}^{A,B}_{a,b} \sum_\lambda \int \frac{d^3k}{(2\pi)^3}\ E_{\mathbf{k}} a^{\lambda\,\dagger}_{\mathbf{k}} a^\lambda_{\mathbf{k}} \left(\Gamma_{\mathbf{p}_1}\Gamma_{\mathbf{p}_2} a^{\lambda_1\,\dagger}_{\mathbf{p}_1} a^{\lambda_2\,\dagger}_{\mathbf{p}_2}|\mathbf{0}\rangle\right) \\
&= \tilde{\alpha}^{A,B}_{a,b} \sum_\lambda \int \frac{d^3k}{(2\pi)^3}\ \Gamma_{\mathbf{p}_1}\Gamma_{\mathbf{p}_2} E_{\mathbf{k}} a^{\lambda\,\dagger}_{\mathbf{k}} \left(\left[a^\lambda_{\mathbf{k}}, a^{\lambda_1\,\dagger}_{\mathbf{p}_1}\right]_\pm \pm a^{\lambda_1\,\dagger}_{\mathbf{p}_1} a^\lambda_{\mathbf{k}}\right) a^{\lambda_2\,\dagger}_{\mathbf{p}_2}|\mathbf{0}\rangle \\
&= E_{\mathbf{p}_1}\left(\Gamma_{\mathbf{p}_1}\Gamma_{\mathbf{p}_2} a^{\lambda_1\,\dagger}_{\mathbf{p}_1} a^{\lambda_2\,\dagger}_{\mathbf{p}_2}|\mathbf{0}\rangle\right) \pm \tilde{\alpha}^{A,B}_{a,b} \sum_\lambda \int \frac{d^3k}{(2\pi)^3}\ \Gamma_{\mathbf{p}_1}\Gamma_{\mathbf{p}_2} E_{\mathbf{k}} a^{\lambda\,\dagger}_{\mathbf{k}} a^{\lambda_1\,\dagger}_{\mathbf{p}_1} \left([a^\lambda_{\mathbf{k}}, a^{\lambda_2\,\dagger}_{\mathbf{p}_2}]_\pm \pm a^{\lambda_2\,\dagger}_{\mathbf{p}_2} a^\lambda_{\mathbf{k}}\right)|\mathbf{0}\rangle \\
&= E_{\mathbf{p}_1}\left(\Gamma_{\mathbf{p}_1}\Gamma_{\mathbf{p}_2} a^{\lambda_1\,\dagger}_{\mathbf{p}_1} a^{\lambda_2\,\dagger}_{\mathbf{p}_2}|\mathbf{0}\rangle\right) \pm E_{\mathbf{p}_1}\left(\Gamma_{\mathbf{p}_1}\Gamma_{\mathbf{p}_2} a^{\lambda_2\,\dagger}_{\mathbf{p}_2} a^{\lambda_1\,\dagger}_{\mathbf{p}_1}|\mathbf{0}\rangle\right).
\end{aligned} \tag{21}
$$

where we used the relation (18) on third line. Since either term of $\left[a^\lambda_{\mathbf{k}}, a^{\bar{\lambda}\,\dagger}_{\mathbf{p}}\right]_+$ or $\left[a^\lambda_{\mathbf{k}}, a^{\bar{\lambda}\,\dagger}_{\mathbf{p}}\right]_-$ satisfies

$$\left[a^\lambda_{\mathbf{k}}, a^{\bar{\lambda}\,\dagger}_{\mathbf{p}}\right]_+ = \frac{(2\pi)^3}{\tilde{\alpha}^{A,B}_{a,b}}\delta^3(\mathbf{k}-\mathbf{p})\delta^{\lambda\bar{\lambda}}, \tag{22}$$

or

$$\left[a^\lambda_{\mathbf{k}}, a^{\bar{\lambda}\,\dagger}_{\mathbf{p}}\right]_- = \frac{(2\pi)^3}{\tilde{\alpha}^{A,B}_{a,b}}\delta^3(\mathbf{k}-\mathbf{p})\delta^{\lambda\bar{\lambda}}, \tag{23}$$

hereby we take proper sign $\pm$ which satisfies relation (18). The proper sign $\pm$ can be determined by considering the statistics of 2-particle spectra $|\mathbf{p}_1,\mathbf{p}_2;+\rangle$. Hereby we shall consider following two cases with spectra $|\mathbf{p}_1,\mathbf{p}_2;+\rangle_{BE}$ which shows *Bose-Einstein statistics* and $|\mathbf{p}_1,\mathbf{p}_2;+\rangle_{FD}$ which shows *Fermi-Dirac statistics*:

Case I. (Bosonic Spectra) The bosonic 2-particle spectra is symmetric($|\mathbf{p}_1,\mathbf{p}_2;+\rangle_{BE} = |\mathbf{p}_2,\mathbf{p}_1;+\rangle_{BE}$), so we obtain[4]

$$\left[a^{\lambda_1\,\dagger}_{\mathbf{p}_1}, a^{\lambda_2\,\dagger}_{\mathbf{p}_2}\right]_+ = a^{\lambda_1\,\dagger}_{\mathbf{p}_1} a^{\lambda_2\,\dagger}_{\mathbf{p}_2} - a^{\lambda_2\,\dagger}_{\mathbf{p}_2}, a^{\lambda_1\,\dagger}_{\mathbf{p}_1} = 0. \tag{24}$$

We shall evaluate the statistics for identical particles. Under the *Bose-Einstein statistics*, calculation (21) reduces to

$$H|\mathbf{p}_1,\mathbf{p}_2;+\rangle_{BE} = \left(E_{\mathbf{p}_1} \pm E_{\mathbf{p}_2}\right)|\mathbf{p}_1,\mathbf{p}_2;+\rangle_{BE}. \tag{25}$$

The superposition principle on eigenstate of energy yields $H|\mathbf{p}_1,\mathbf{p}_2;+\rangle_{BE} = (E_{\mathbf{p}_1} + E_{\mathbf{p}_2})|\mathbf{p}_1,\mathbf{p}_2;+\rangle_{BE}$, so the sign $\pm = +$ only permitted on results (21). Hence, we obtain

$$\left[a^\lambda_{\mathbf{k}}, a^{\bar{\lambda}\,\dagger}_{\mathbf{p}}\right]_+ = \frac{(2\pi)^3}{\tilde{\alpha}^{A,B}_{a,b}}\delta^3(\mathbf{k}-\mathbf{p})\delta^{\lambda\bar{\lambda}} \tag{26}$$

for the *Bose-Einstein statistics.*

Case II. (Fermionic Spectra) The fermionic 2-particle spectra is antisymmetric, so we obtain

$$\left[a^{\lambda_1\,\dagger}_{\mathbf{p}_1}, a^{\lambda_2\,\dagger}_{\mathbf{p}_2}\right]_- = a^{\lambda_1\,\dagger}_{\mathbf{p}_1} a^{\lambda_2\,\dagger}_{\mathbf{p}_2} + a^{\lambda_2\,\dagger}_{\mathbf{p}_2}, a^{\lambda_1\,\dagger}_{\mathbf{p}_1} = 0. \tag{27}$$

Having similar approach to (25), considering about the statistic of particles for the *Fermi-Dirac statistics*, calculation (21) also comes to

$$H|\mathbf{p}_1,\mathbf{p}_2;+\rangle_{FD} = \left(E_{\mathbf{p}_1} \mp E_{\mathbf{p}_2}\right)|\mathbf{p}_1,\mathbf{p}_2;+\rangle_{FD}. \tag{28}$$

where double-sign corresponds for (21) and (28). The superposition principle on eigenstate of energy yields $H|\mathbf{p}_1,\mathbf{p};+\rangle_{BE} = (E_{\mathbf{p}_1} + E_{\mathbf{p}_2})|\mathbf{p}_1,\mathbf{p};+\rangle_{BE}$, so the sign $\pm = -$ only permitted on results (21). Hence, we obtain

$$\left[a^\lambda_{\mathbf{k}}, a^{\bar{\lambda}\,\dagger}_{\mathbf{p}}\right]_- = \frac{(2\pi)^3}{\tilde{\alpha}^{A,B}_{a,b}}\delta^3(\mathbf{k}-\mathbf{p})\delta^{\lambda\bar{\lambda}} \tag{29}$$

for the *Fermi-Dirac statistics.*

These commutator or anti-commutator relations hold the superposition principles on eigenstate of energy, thus the term (29) must be added to hold bosonic or fermionic statistics(If we do not so, only putting (27) can not hold the eigenstate of energy for 2-particle system).

IV. HAMILTONIAN APPROACH TO MULTI-PARTICLE SYSTEM

Let us begin by considering the k-particle spectra:

$$|\mathbf{p}_1,\cdots,\mathbf{p}_k;+\rangle = \prod_{i=1}^{k}\left(\Gamma_{\mathbf{p}_i} a^{\lambda_i\,\dagger}_{\mathbf{p}_i}\right) \tag{30}$$

with physical quantities of each particle $(\mathbf{p}_i,\lambda_i)$ for $i = 1,2,\cdots,k$

[4] Also, relation (27) gives

$$\left[a^{\lambda_1}_{\mathbf{p}_1}, a^{\lambda_2}_{\mathbf{p}_2}\right]_+ = -\left(\left[a^{\lambda_1\,\dagger}_{\mathbf{p}_1}, a^{\lambda_2\,\dagger}_{\mathbf{p}_2}\right]_+\right)^\dagger = 0.$$

Appendix A: Analysis about Suitability of Hamiltonian (9)

We shall evaluate the suitability of hamiltonian representation (9) for the field $\varphi_{a,b}^{A,B}$. Remind the hamiltonian must make eigenstate with spectra of k-particle system $|\mathbf{p}_1,\cdots,\mathbf{p}_k;+\rangle$, so that hamiltonian of system must be constructed with terms

$$a_{\mathbf{p}}^{\lambda\dagger}a_{\mathbf{p}}^{\lambda},\quad a_{\mathbf{p}}^{\lambda}a_{\mathbf{p}}^{\lambda\dagger},\quad b_{\mathbf{p}}^{\lambda\dagger}b_{\mathbf{p}}^{\lambda},\quad b_{\mathbf{p}}^{\lambda}b_{\mathbf{p}}^{\lambda\dagger}. \tag{A1}$$

This means the hamiltonian must be written by in form

$$H=\sum_{\lambda}\int\frac{d^3p}{(2\pi)^3}\left(A_{a,b}^{A,B}(\mathbf{p},\lambda)a_{\mathbf{p}}^{\lambda\dagger}a_{\mathbf{p}}^{\lambda}+B_{a,b}^{A,B}(\mathbf{p},\lambda)a_{\mathbf{p}}^{\lambda}a_{\mathbf{p}}^{\lambda\dagger}\right.$$
$$\left.+C_{a,b}^{A,B}(\mathbf{p},\lambda)b_{\mathbf{p}}^{\lambda\dagger}b_{\mathbf{p}}^{\lambda}+D_{a,b}^{A,B}(\mathbf{p},\lambda)b_{\mathbf{p}}^{\lambda}b_{\mathbf{p}}^{\lambda\dagger}\right). \tag{A2}$$

Hereby we consider following two cases of vacuums; the empty vacuum($a_{\mathbf{p}}^{\lambda}|0\rangle=b_{\mathbf{p}}^{\lambda}|0\rangle=0$) and fermionic filled vacuum($|0\rangle=B_{\mathbf{p}}^{\lambda\dagger}|\tilde{0}\rangle$):

I. The Empty Vacuum In case of vacuum set as 'empty space'($a_{\mathbf{p}}^{\lambda}|0\rangle=b_{\mathbf{p}}^{\lambda}|0\rangle=0$), the vacuum states must have zero energy. So the $H|0\rangle=0$ only gives

$$B_{a,b}^{A,B}(\mathbf{p},\lambda)=D_{a,b}^{A,B}(\mathbf{p},\lambda)=0. \tag{A3}$$

Then, the hamiltonian comes to

$$H=\sum_{\lambda}\int\frac{d^3p}{(2\pi)^3}\left(A_{a,b}^{A,B}(\mathbf{p},\lambda)a_{\mathbf{p}}^{\lambda\dagger}a_{\mathbf{p}}^{\lambda}+C_{a,b}^{A,B}(\mathbf{p},\lambda)b_{\mathbf{p}}^{\lambda\dagger}b_{\mathbf{p}}^{\lambda}\right). \tag{A4}$$

Here by we only consider the free field $\varphi_{a,b}^{A,B}$, each λ has same contribution to hamiltonian. Thus, we have

$$\begin{aligned}A_{a,b}^{A,B}(\mathbf{p},\lambda)&=A_{a,b}^{A,B}(\mathbf{p})=\tilde{\alpha}_{a,b}^{A,B}E_{\mathbf{p}},\\ C_{a,b}^{A,B}(\mathbf{p},\lambda)&=C_{a,b}^{A,B}(\mathbf{p})=\tilde{\beta}_{a,b}^{A,B}E_{\mathbf{p}}.\end{aligned} \tag{A5}$$

Determination of each coefficient $A_{a,b}^{A,B}(\mathbf{p},\lambda),C_{a,b}^{A,B}(\mathbf{p},\lambda)$ on (A11) is supported by eigenstate of k-particles spectra $H|\mathbf{p}_1,\cdots,\mathbf{p}_k;+\rangle=(\sum_i E_{\mathbf{p}_i})|\mathbf{p}_1,\cdots,\mathbf{p}_k;+\rangle$ and the operators on (A1) generating the eigenstates of spectra:

$$\begin{aligned}a_{\mathbf{p}}^{\lambda\dagger}a_{\mathbf{p}}^{\lambda}|\mathbf{p}_1,\cdots,\mathbf{p}_k;+\rangle&\sim|\mathbf{p}_1,\cdots,\mathbf{p}_k;+\rangle,\\ b_{\mathbf{p}}^{\lambda\dagger}b_{\mathbf{p}}^{\lambda}|\mathbf{p}_1,\cdots,\mathbf{p}_k;+\rangle&\sim|\mathbf{p}_1,\cdots,\mathbf{p}_k;+\rangle.\end{aligned} \tag{A6}$$

That is we want

$$H=\sum_{\lambda}\int\frac{d^3p}{(2\pi)^3}\ E_{\mathbf{p}}\left(\tilde{\alpha}_{a,b}^{A,B}a_{\mathbf{p}}^{\lambda\dagger}a_{\mathbf{p}}^{\lambda}+\tilde{\beta}_{a,b}^{A,B}b_{\mathbf{p}}^{\lambda\dagger}b_{\mathbf{p}}^{\lambda}\right). \tag{A7}$$

II. The Fermionic Filled Vacuum In case of fermionic particles, the fermionic filled vacuum we defined as $a_{\mathbf{p}}^{\lambda}|0\rangle=b_{\mathbf{p}}^{\lambda}|0\rangle=0$ with fermionic antiparticle-filled vacuum $|0\rangle$[5] can also rewritten by introducing operator $B_{\mathbf{p}}^{\lambda\dagger}=b_{\mathbf{p}}^{\lambda}$ and really emptied vacuum $|\tilde{0}\rangle$ (where it satisfies $a_{\mathbf{p}}^{\lambda}|\tilde{0}\rangle=B_{\mathbf{p}}^{\lambda}|\tilde{0}\rangle=0$)[6]:

$$b_{\mathbf{p}}^{\lambda}\left(B_{\mathbf{p}}^{\lambda\dagger}|\tilde{0}\rangle\right)=\left(b_{\mathbf{p}}^{\lambda}\right)^2|\tilde{0}\rangle=0. \tag{A8}$$

Then, let us put vacuum set above as $|0\rangle=B_{\mathbf{p}}^{\lambda\dagger}|\tilde{0}\rangle$, here we can analysis spectra using this new defined 'real emptied vacuum' $|\tilde{0}\rangle$. Thus, from the similar approach to *empty vacuum*, coefficients satisfies

$$B_{a,b}^{A,B}(\mathbf{p},\lambda)=C_{a,b}^{A,B}(\mathbf{p},\lambda)=0. \tag{A9}$$

Then, the hamiltonian comes to

$$H=\sum_{\lambda}\int\frac{d^3p}{(2\pi)^3}\left(A_{a,b}^{A,B}(\mathbf{p},\lambda)a_{\mathbf{p}}^{\lambda\dagger}a_{\mathbf{p}}^{\lambda}+D_{a,b}^{A,B}(\mathbf{p},\lambda)B_{\mathbf{p}}^{\lambda\dagger}B_{\mathbf{p}}^{\lambda}\right). \tag{A10}$$

Here by we only consider the free field $\varphi_{a,b}^{A,B}$, each λ has same contribution to hamiltonian. Thus, we have

$$\begin{aligned}A_{a,b}^{A,B}(\mathbf{p},\lambda)&=A_{a,b}^{A,B}(\mathbf{p})=\tilde{\alpha}_{a,b}^{A,B}E_{\mathbf{p}},\\ C_{a,b}^{A,B}(\mathbf{p},\lambda)&=C_{a,b}^{A,B}(\mathbf{p})=\tilde{\beta}_{a,b}^{A,B}E_{\mathbf{p}}\end{aligned} \tag{A11}$$

by using similar method to 'empty vacuum'. This yeilds

$$H=\sum_{\lambda}\int\frac{d^3p}{(2\pi)^3}\ E_{\mathbf{p}}\left(\tilde{\alpha}_{a,b}^{A,B}a_{\mathbf{p}}^{\lambda\dagger}a_{\mathbf{p}}^{\lambda}+\tilde{\beta}_{a,b}^{A,B}B_{\mathbf{p}}^{\lambda\dagger}B_{\mathbf{p}}^{\lambda}\right). \tag{A12}$$

Recovering $b_{\mathbf{p}}^{\lambda}=B_{\mathbf{p}}^{\lambda\dagger}$, that is we want

$$H=\sum_{\lambda}\int\frac{d^3p}{(2\pi)^3}\ E_{\mathbf{p}}\left(\tilde{\alpha}_{a,b}^{A,B}a_{\mathbf{p}}^{\lambda\dagger}a_{\mathbf{p}}^{\lambda}+\tilde{\beta}_{a,b}^{A,B}b_{\mathbf{p}}^{\lambda}b_{\mathbf{p}}^{\lambda\dagger}\right). \tag{A13}$$

From the results of (A7), (A13), the hamiltonian representation (9), (10) are suitable for field $\varphi_{a,b}^{A,B}(x)$.

[5] As vacuum $|0\rangle$ is not emptied, so generally we have $H|0\rangle\neq 0$.

[6] Following relation (A8) is supported by fermionic statistics of creation and aniihilation operators:

$$[b_\lambda,b_\lambda]_-=2\left(b_\lambda\right)^2=0.$$

[1] S. Weinberg, The Quantum Theory of Fields — Volume I: Foundations (Cambridge University Press, Cambridge, 1995)

[2] R. Ferro-Hernández, J. Olmos, E. Peinado, C. A. Vaquera-Araujo, *Quantization of second-order fermions*, *Phys. Rev. D* **109** (2024)