Modern Electrodynamics I

Osaka University Core Courses in Physics

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ABSTRACT: Report of Lorentz invariance in Maxwell field.

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Contents

1 Introduction

1.1 Notation and convention

In these notes, we use natural unit system:

$$c = \mu_0 = \epsilon_0 = k_B = 1.$$

Also we use the sign convention for the Lorentzian spacetime:

$$g_{\mu\nu} = \mathrm{diag}(-, +, +, \cdots, +)$$

1.2 References

- Tatsuma Nishioka, "Quantum Field Theory I," lecture notes at Osaka University.
- Makoto Sakamoto, "Qunatum Field Theory focus on the invariance and free fields," Shokabo Press, 2014.
- L.D. Landau, E.M. Lifshitz, "The Classical Theory of Fields," Pergamon Press, 1951.

2 Construction of Lorentzian spacetime

Let x^μ be the coordinate four-vector in Cartesian system:

$$x^\mu = (x^0, x^1, x^2, x^3) = (t, \mathbf{x}) \tag{2.1}$$

The distance between x^μ and the origin is given by

$$x^2 \equiv g_{\mu\nu} x^\mu x^\nu = -t^2 + \mathbf{x}^2, \tag{2.2}$$

and the transformation between inertial frames($\mathcal{S} \mapsto \mathcal{S}'$):

$$x'^\nu = \frac{\partial x'^\nu}{\partial x^\mu} x^\mu = \Lambda^\nu_\mu x^\mu. \tag{2.3}$$

2.1 SO(3) transformation

By requesting the Mincowski distance invariance[1]$(x'^2 = x^2)$, we get

$$x'^2 = x'_\nu x'^\nu = (\Lambda^\sigma_\nu \Lambda^\omega_\tau \Lambda^\tau_\sigma x_\omega)(\Lambda^\nu_\rho x^\rho) = (\Lambda^\omega_\tau \Lambda^\tau_\sigma \Lambda^\sigma_\nu \Lambda^\nu_\rho) x_\omega x^\rho = x^2. \tag{2.4}$$

In here, we use the transformation of vector x'_ν like:

$$x'_\nu = g'_{\nu\rho} x'^\rho = g'_{\nu\rho}(\Lambda^\rho_\sigma x^\sigma) = g'_{\nu\rho} g^{\sigma\omega} \Lambda^\rho_\sigma x_\omega = g'_{\nu\rho} g'^{\mu\tau} \Lambda^\sigma_\mu \Lambda^\omega_\tau \Lambda^\rho_\sigma x_\omega, \tag{2.5}$$

from (2.4) and (2.5), we get orthogonality of SO(3) transformation.

$$\Lambda^\mu_\nu \Lambda^\nu_\rho = \delta^\mu_\rho \ , \quad x'_\nu = \Lambda^\mu_\nu x_\mu \tag{2.6}$$

and the Mincowski distance invariance requests:

$$x'^2 = x'_\nu x'^\nu = (g'_{\nu\rho} x'^\rho)(g'^{\nu\sigma} x'_\sigma) = g'_{\nu\rho} g'^{\nu\sigma} \Lambda^\rho_\mu \Lambda^\gamma_\sigma x^\mu x_\gamma = (\Lambda^\rho_\mu \Lambda^\gamma_\rho) x^\mu x_\gamma = x^2. \tag{2.7}$$

Here we also get contravariant representation of (2.6):

$$\Lambda^\rho_\mu \Lambda^\gamma_\rho = \delta^\gamma_\mu. \tag{2.8}$$

From the result of (2.6) and (2.8), here we get invariance follow:

SO(3) invariance(Lorentzian spacetime)

$$\Lambda^\mu_\nu \Lambda^\nu_\rho = \delta^\mu_\rho \ , \quad \Lambda^\nu_\mu \Lambda^\rho_\nu = \delta^\rho_\mu. \tag{2.9}$$

[1] under the SO(n) transformation between inertial frame, Mincowski distance invariance also be satisfied.

Here we define the Einstein convention on derivate like:

$$\partial'_\mu \equiv \frac{\partial}{\partial x'^\mu} \ , \quad \partial_\mu \equiv \frac{\partial}{\partial x^\mu}. \tag{2.10}$$

where we notate each ∂_μ and ∂^μ with Mincowski metric tensor $g_{\mu\nu}$ and the chain rule follow.

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \partial_\mu (g_{\nu\rho} x^\rho) \frac{\partial}{\partial x_\nu} = g_{\nu\rho} \delta^\rho_\mu \partial^\nu = g_{\mu\nu} \partial^\nu \tag{2.11}$$

Now that we calculate the derivate operator ∂'_μ and ∂_μ between the inertial frame $\mathcal{S}$ and $\mathcal{S}'$ also given by

$$\partial'_\mu = \frac{\partial x'^\nu}{\partial x^\mu} \partial_\nu = \Lambda^\nu_\mu \partial_\nu, \tag{2.12}$$

else[2],

$$\partial'^\mu = \frac{\partial x'_\nu}{\partial x_\mu} \partial^\nu = \Lambda^\mu_\nu \partial^\nu. \tag{2.13}$$

From the calculation (2.11) and (2.12), we also get invariance of derivate following:

SO(3) derivative invariance

The transformation on SO(3) group also be calculated by

$$\partial'_\mu = \Lambda^\nu_\mu \partial_\nu \ , \quad \partial'^\mu = \Lambda^\mu_\nu \partial^\nu. \tag{2.14}$$

which has its own invariance:

$$\partial'_\mu \partial'^\mu = (\Lambda^\nu_\mu \partial_\nu)(\Lambda^\mu_\nu \partial^\nu) = (\Lambda^\nu_\mu \Lambda^\mu_\nu) \partial_\nu \partial^\nu = \partial_\nu \partial^\nu. \tag{2.15}$$

[2]Here, we use the SO(3) trnasformation of vector x'_ν (2.5) like:

$$x'_\nu = \frac{\partial x'_\nu}{\partial x_\rho} x_\rho = g'_{\nu\rho} g'^{\mu\tau} \Lambda^\sigma_\mu \Lambda^\omega_\tau \Lambda^\rho_\sigma x_\omega = \Lambda^\rho_\nu x_\rho.$$

3 Classical Maxwell Fields

In this section, we consider SO(3) transformation under the vector field $A_\mu(x)$.

$$x^\mu \;\mapsto\; x'^\mu = \Lambda^\mu_\nu x^\nu, \tag{3.1}$$

else, on the convariant representation:

$$x_\mu \;\mapsto\; x'_\mu = \Lambda^\nu_\mu x_\nu. \tag{3.2}$$

Now we forget all of things about Maxwell's equation.

3.1 Scalar fields and vector fileds

Adding the SO(3) transformation on inertial frame, the scalar field $\phi(x)$ transforms:

$$\phi(x) \;\mapsto\; \phi'(x) = \phi(\Lambda^{-1}x). \tag{3.3}$$

Else, to find SO(3) transformation on vector field $A_\mu(x)$, here we use the invariance on four-vector dot product.

$$x'_\mu A'^\mu(x) = (\Lambda^\nu_\mu x_\nu)A'^\mu(x) = x_\nu(\Lambda^\nu_\mu A'^\mu(x)) = x_\nu A^\nu(x) \tag{3.4}$$

so that using the relation (3.4), we get SO(3) transformation on vector field:

$$A_\mu(x) \;\mapsto\; A'_\mu(x') = \Lambda^\nu_\mu A_\nu(\Lambda^{-1}x). \tag{3.5}$$

3.2 The Lagrangian density of Maxwell field

Before we constructing the Lagrangian density of Maxwell field $\mathcal{L}_M$, hereby we suppose rules which construct the Maxwell field:

Summary(Maxwell field)

The Maxwell field satisfies following rules:

- Superposition principle
- Lorentzian invariance
- Gauge invariance

Now that the Lagrangian density of Maxwell field can be constructed by:

$$S_M = S_p + S_{pf} + S_f. \tag{3.6}$$

The superposition principle holds the liniarity of Maxwell action in (3.6) where we define as the action of electromagnetic field S_f, the action of interacting between electromagnetic field and particle S_{pf}, the action of particle S_p.

Action of particle

To construct the action of particle S_p, we consider the free fields on the inertial frame of particle. Here we know the Lagrangian density $\mathcal{L}_p$ of free particle becomes constant on the inertial frame[3] of particle:

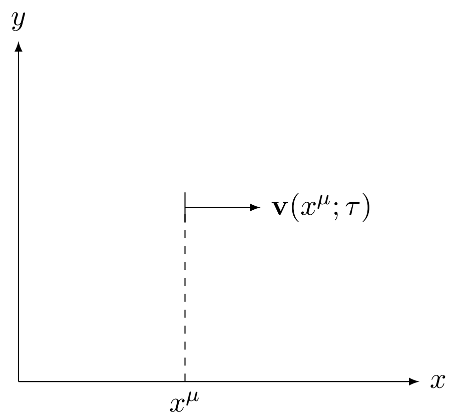

Figure 1. The convariance Lagrangian density of free particle

So that we notate the Lagrangian density of free particle on the inertial frame with its proper time τ.

$$S_p = \int d\tau\ \mathcal{L}_p = \int d\tau\ m. \tag{3.7}$$

Here, we can check its Lorentzian invariance in (3.7).

Action of interaction

Here we define the transformation on Lorentz group like:

$$\tilde{x}^\mu = \frac{\partial \tilde{x}^\mu}{\partial x^\nu} x^\nu = \gamma^\mu_\nu x^\nu. \tag{3.8}$$

Else, the Lorentz transformation of convariant vector also notated by

$$\tilde{x}_\mu = \tilde{g}_{\mu\nu}\tilde{x}^\nu = \tilde{g}_{\mu\nu}(\gamma^\nu_\rho x^\rho) = \tilde{g}_{\mu\nu}\tilde{g}^{\omega\tau}\gamma^\rho_\omega\gamma^\sigma_\tau\gamma^\nu_\rho x_\sigma = \gamma^\sigma_\mu x_\sigma. \tag{3.9}$$

Hereby using the relation (3.8) and (3.9), we get the property of transformation on Lorentz group following:

[3]On the proper time τ.

Lorentz invariance

The Lorentz transformation satisfies following invariances:

$$\gamma^{\rho}_{\mu}\gamma^{\nu}_{\rho} = \delta^{\nu}_{\mu} \ , \quad \gamma^{\mu}_{\rho}\gamma^{\rho}_{\nu} = \delta^{\mu}_{\nu}, \tag{3.10}$$

each derivative goes to

$$\tilde{\partial}_{\mu} = \gamma^{\nu}_{\mu}\partial_{\nu} \ , \quad \tilde{\partial}^{\mu} = \gamma^{\mu}_{\nu}\partial^{\nu}. \tag{3.11}$$

Then, hereby we can notate the action of interaction between electromagnetic field and particle which holds the Lorentzian invariance as:

$$S_{pf} = \int (-\tilde{q}A_{\mu}\ dx^{\mu}) = \int_{\Omega} \sqrt{-g}\ d^4x\ \left(-\varrho A_{\mu}\frac{dx^{\mu}}{dt}\right). \tag{3.12}$$

Also, here we define following Lorentz transformation[4] and four-current.

$$d^4x = \frac{\sqrt{-\tilde{g}}}{\sqrt{-g}}\ dt\ d^3\tilde{x} \ , \quad J^{\mu} = \frac{\varrho}{\gamma}\frac{dx^{\mu}}{d\tau} = \begin{pmatrix} \varrho \\ \varrho\mathbf{v} \end{pmatrix} \tag{3.13}$$

so that we get the conservation rule[5] of four-rule from (3.13):

$$\sqrt{-g}\ d^4x = \sqrt{-\tilde{g}}\ d^4\tilde{x}. \tag{3.14}$$

Now that using the four-parametre in (3.13), we can rewrite the action of interaction following.

$$S_{pf} = -\int_{\Omega} \sqrt{-g}\ d^4x\ A_{\mu}J^{\mu} \tag{3.15}$$

Also, we can check the Lorentzian invariance on (3.15) like:

$$-\int_{\Omega} d^4x\tilde{A}_{\mu}\tilde{J}^{\mu} = -\int_{\Omega} d^4x\ \gamma^{\nu}_{\mu}\gamma^{\mu}_{\rho}A_{\nu}J^{\rho} = -\int_{\Omega} d^4x\ A_{\nu}J^{\nu}. \tag{3.16}$$

[4]The transformation of each 4-vector volume $d^4x \ \mapsto \ d^4\tilde{x} = d\tau\ d^3\tilde{x}$ becomes:

$$d^4x = \det\left(\frac{\partial x}{\partial \tilde{x}}\right)\ d^4\tilde{x}.$$

Else, the transformation of metric tensor $g_{\mu\nu}$ to $\tilde{g}_{\rho\sigma}$ also given by

$$g_{\mu\nu} = \gamma^{\rho}_{\mu}\gamma^{\sigma}_{\nu}\tilde{g}_{\rho\sigma} \ , \quad \text{i.e. } \sqrt{-g} = \det\left(\frac{\partial \tilde{x}}{\partial x}\right)\sqrt{-\tilde{g}}.$$

[5]Also, here we set the inertial frame of proper time as $\sqrt{-\tilde{g}} = 1$.

Action of field

To constructing the Action of electromagnetic field S_f, we request following rules:

- The independence of coordinates system:

$$S_f[A_\mu] = \int_\Omega \sqrt{-g}\ d^4x\ \mathcal{L}_f[A_\mu]. \tag{3.17}$$

- The Gauge invariance:

$$S_f[A_\mu] = S_f[A_\mu + \partial_\mu \Gamma]. \tag{3.18}$$

Even the action in (3.17) has the integral of d^4x, we can check that the action S_f, on the space of $\int_\Omega \sqrt{-g}\ d^4x$ has the independence of coordinates system.

$$\int_\Omega \sqrt{-g}\ d^4x = \int_{\tilde{\Omega}} \sqrt{-\tilde{g}}\ d^4\tilde{x} \tag{3.19}$$

From the rules of (3.17) and (3.18), we construct the field action like:

$$S_f = \int_\Omega \sqrt{-g}\ d^4x\ \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right)\ , \tag{3.20}$$

here we define the **electromagnetic tensor** $F_{\mu\nu}$ following.

$$F_{\mu\nu}[A_\mu] \equiv \partial_\mu A_\nu - \partial_\nu A_\mu\ , \quad F^{\mu\nu} = g^{\mu\rho}g^{\nu\sigma}F_{\rho\sigma} \tag{3.21}$$

So that we can check the Gauge invariance of field action S_f.

$$F_{\mu\nu}[A_\mu + \partial_\mu\Gamma] = \partial_\mu(A_\nu + \partial_\nu\Gamma) - \partial_\nu(A_\mu + \partial_\mu\Gamma) = F_{\mu\nu}[A_\mu] \tag{3.22}$$

3.3 The Maxwell equation

Now that, making some linear connection of each actions on (3.7), (3.15) and (3.20), we can construct the total action of Maxwell field following:

$$S_M = \int d\tau\ m - \int_\Omega \sqrt{-g}\ d^4x\ A_\mu J^\mu + \int_\Omega \sqrt{-g}\ d^4x\ \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right). \tag{3.23}$$

Here we argue the Least action principles on $\delta S[A_\mu]$:

$$\begin{aligned}
\delta S[A_\mu] &= -\int_\Omega \sqrt{-g}\ d^4x\ J^\mu(\delta A_\mu) + \int_\Omega \sqrt{-g}\ d^4x\ \left(-\frac{1}{2}(\delta A_{\nu,\mu} - \delta A_{\mu,\nu})F^{\mu\nu}\right) \\
&= -\int_\Omega d^4x\ J^\mu\sqrt{-g}(\delta A_\mu) + \sqrt{-g}F^{\mu\nu}(\delta A_{\nu,\mu}) \\
&= -\int_\Omega d^4x\ \left(\sqrt{-g}J^\nu - \partial_\mu(\sqrt{-g}F^{\mu\nu})\right)\delta(A_\nu) = 0.
\end{aligned} \tag{3.24}$$

From the Least action principle in (3.24), we finally get Maxwell's equation following:

Maxwell's equation

On the curved spacetime:

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}F^{\mu\nu}) = J^\nu. \tag{3.25}$$

Specially, on the orthogonal space time:

$$\partial_\mu F^{\mu\nu} = J^\nu. \tag{3.26}$$

So that we get the Maxwell's equation on the curved spacetime as general form. Using equation (3.25), we verify the Maxwell's equation we know. Here we remember the four-vector potenials again:

$$A^\mu = \begin{pmatrix} \varphi \\ \mathbf{A} \end{pmatrix} \quad , \quad J^\mu = \begin{pmatrix} \varrho \\ \varrho\mathbf{v} \end{pmatrix}. \tag{3.27}$$

Hereby we select the metric tensor[6] for spherical coordinates.

$$g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & g_{rr} & & \\ & & g_{\theta\theta} & \\ & & & g_{\phi\phi} \end{pmatrix} \tag{3.28}$$

Then, we get Gauss' law and Ampere-Maxwell law in form of spherical coordinates:

1. *Gauss' law*

$$\boldsymbol{\nabla} \cdot \left(\boldsymbol{\nabla}\varphi + \frac{\partial \mathbf{A}}{\partial t} \right) = \varrho. \tag{3.29}$$

Here we use the convariant representation:

$$\frac{1}{\sqrt{-g}} g_{\mu\rho}\partial^\rho(\sqrt{-g}g^{\mu\sigma}g^{\nu\omega}F_{\sigma\omega}) = g^{\nu\tau}J_\tau. \tag{3.30}$$

2. *Ampere-Maxwell law*

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}) = \mathbf{J} + \frac{\partial}{\partial t}\left(\boldsymbol{\nabla}\varphi + \frac{\partial \mathbf{A}}{\partial t} \right). \tag{3.31}$$

[6] Here we define:

$$g_{rr} = 1 \ , \quad g_{\theta\theta} = r^2 \ , \quad g_{\phi\phi} = r^2 \sin^2\theta.$$

Else, the definition of electromagnetic field yeilds:

$$F_{\mu\nu,\rho} + F_{\nu\rho,\mu} + F_{\rho\mu,\nu} = 0. \tag{3.32}$$

Hereby the identity (3.32) shows Faraday's law and Gauss' law in magnetic fields.

3. Faraday's law

$$\boldsymbol{\nabla} \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = \mathbf{0}. \tag{3.33}$$

4. Gauss' law in magnetic fields

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0. \tag{3.34}$$

4 Conservation law and invariance

In section 3.3, we calssified that Maxwell's equation can be rewritten by following froms in curved spacetime:

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}F^{\mu\nu}) = J^\nu \ , \qquad \frac{1}{\sqrt{-g}} g_{\mu\rho}\partial^\rho(\sqrt{-g}g^{\mu\sigma}g^{\nu\omega}F_{\sigma\omega}) = g^{\nu\tau}J_\tau. \tag{4.1}$$

In this section, we use Maxwell's equation in form of (4.1).

4.1 Equation of continuity

The Maxwell's equation on cuvered spacetime, it yeilds the **equation of continuity** on electric current:

$$\partial_\nu(\sqrt{-g}J^\nu) = \partial_\nu\partial_\mu(\sqrt{-g}F^{\mu\nu}) = -\partial_\mu\partial_\nu(\sqrt{-g}F^{\mu\nu}) = 0. \tag{4.2}$$

So that we notate the equation of continuity following.

$$\partial_\nu j^\nu = 0 \ , \qquad j^\nu \equiv \sqrt{-g}J^\nu \tag{4.3}$$

The conservation law of electric charge also can be notated by each coordinates like:

Equation of continuity

The Maxwell's equation yeilds the conservation of electric charges:

$$\partial_\nu(\sqrt{-g}J^\nu) = 0. \tag{4.4}$$

Especially, in the spherical coordinates gives conservation law following.

$$-\frac{\partial \varrho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 J_r) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}(\sin\theta J_\theta) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 J_\phi}{\partial \phi^2} = 0 \tag{4.5}$$

4.2 Invariance of conservation laws

Let us consider the transformation on SO(3) group and Lorentz group. Each transforms the four-current like:

$$J^\mu(x) \;\mapsto\; J'^\mu(x') = \Lambda^\mu_\nu J^\nu(\Lambda^{-1}x)\ , \qquad J^\mu(x) \;\mapsto\; J'^\mu(x') = \gamma^\mu_\nu J^\nu(\gamma^{-1}x). \tag{4.6}$$

Then, each transformation on (4.6) gives invariance follow.

$$\partial'_\mu A'^\mu(x') = \Lambda^\rho_\mu \Lambda^\mu_\nu \partial_\rho A^\nu(\Lambda^{-1}x) = \partial_\nu A^\nu(\Lambda^{-1}x) = 0 \tag{4.7}$$

Else, transformation on Lorentz group acts:

$$\tilde{\partial}_\mu \tilde{A}^\mu(x') = \gamma^\rho_\mu \gamma^\mu_\nu \partial_\rho A^\nu(\Lambda^{-1}x) = \partial_\nu A^\nu(\Lambda^{-1}x) = 0. \tag{4.8}$$

From the equation development (4.7) and (4.8), we get the invariance conservation law of electric charges.

4.3 Invariance of Maxwell field

In this section, we argue about the invariance of Maxwell field. Here we consider the SO(3) transformation of Maxwell's equation.

$$x'^\mu = \Lambda^\mu_\nu x^\nu\ , \quad x'_\mu = \Lambda^\nu_\mu x_\nu \tag{4.9}$$

The transformation (4.9) on SO(3) group acts on:

$$\varrho(x) \;\mapsto\; \varrho'(x') = \varrho(\Lambda^{-1}x), \tag{4.10}$$

else, the four-current changes:

$$J^\mu(x) \;\mapsto\; J'^\mu(x') = \Lambda^\mu_\nu J^\nu(\Lambda^{-1}x). \tag{4.11}$$

Then we construct the transformation of Maxwell's equation on curved spacetime following.

$$F'_{\mu\nu} = \partial'_\mu A'_\nu - \partial'_\nu A'_\mu = \Lambda^\rho_\mu \Lambda^\sigma_\nu(\partial_\rho A_\sigma - \partial_\sigma A_\rho) = \Lambda^\rho_\mu \Lambda^\sigma_\nu F_{\rho\sigma} \tag{4.12}$$

So that the Maxwell's equation on curved spacetime transforms to:

$$\begin{aligned} \partial'_\mu(\sqrt{-g}F'^{\mu\nu}) - \sqrt{-g}J'^\nu &= (\Lambda^\omega_\mu \partial_\omega)(\sqrt{-g}\Lambda^\mu_\rho \Lambda^\nu_\sigma F^{\rho\sigma}) - \sqrt{-g}\Lambda^\nu_\sigma J^\sigma \\ &= \Lambda^\nu_\sigma \left(\partial_\rho(\sqrt{-g}F^{\rho\sigma}) - \sqrt{-g}J^\sigma\right) = 0, \end{aligned} \tag{4.13}$$

here we verify the invariance of Maxwell's equation by:

Invariance on Maxwell's equation

The Maxwell's equation has following invariance:

$$\frac{1}{\sqrt{-g}}\partial'_\mu(\sqrt{-g}F'^{\mu\nu}) = J'^\nu \quad \mapsto \quad \frac{1}{\sqrt{-g}}\partial_\rho(\sqrt{-g}F^{\rho\sigma}) = J^\sigma. \tag{4.14}$$

From transformation (4.14) we conclude that Maxwell's equation has invariance on any SO(3) group.

4.4 The parity operator

In this section, we select the metric tensor $g_{\mu\nu}$ on Cartesian coordinates:

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{pmatrix}, \tag{4.15}$$

Also, let the parity operator Λ as

$$\Lambda(x \to x') = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}. \tag{4.16}$$

From the transformation (4.16), the four-current transforms following:

$$J'^{\mu}(x') = \Lambda^{\mu}_{\nu} J^{\nu}(\Lambda^{-1}x) = \begin{pmatrix} \varrho \\ -\mathbf{J} \end{pmatrix}, \tag{4.17}$$

from the result of (4.14), we know that the Maxwell's equation holds its invariance. So that the transformation of electromagnetic tensor yeilds:

$$F'^{\mu\nu} = \Lambda^{\mu}_{\rho}\Lambda^{\nu}_{\sigma}F^{\rho\sigma} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}. \tag{4.18}$$

Using the invariance of Maxwell's equation, the electromagnetic tensor on space inversion can be notated by following forms.

$$F'^{\mu\nu} = g^{\mu\rho}g^{\nu\sigma}(\partial_\rho A_\sigma - \partial_\sigma A_\rho) = \begin{pmatrix} 0 & E'_x & E'_y & E'_z \\ -E'_x & 0 & B'_z & -B'_y \\ -E'_y & -B'_z & 0 & B'_x \\ -E'_z & B'_y & -B'_x & 0 \end{pmatrix} \tag{4.19}$$

By comparing (4.18) and (4.19), the elctromagnetic fields transforms by:

$$\mathbf{E}'(\mathbf{r}', t) = -\mathbf{E}(\mathbf{r}, t)\ , \quad \mathbf{B}'(\mathbf{r}', t) = \mathbf{B}(\mathbf{r}, t). \tag{4.20}$$

So that the parity operator Λ acts on Maxwell's equation following.

Parity of Maxwell field

The parity operator acts on Maxwell field by

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix} \xrightarrow{\text{parity}} F'^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} . \tag{4.21}$$

Also, transformation of electromagnetic tensor $F^{\mu\nu} \mapsto F'^{\mu\nu}$ yields:

$$\mathbf{E}(\mathbf{r},t) \mapsto \mathbf{E}'(\mathbf{r}',t) = -\mathbf{E}(\mathbf{r},t) \ , \quad \mathbf{B}(\mathbf{r},t) \mapsto \mathbf{B}'(\mathbf{r}',t) = \mathbf{B}(\mathbf{r},t). \tag{4.22}$$

The invariance of Maxwell's equation can be verified by

$$\partial'_\mu = \Lambda^\nu_\mu \partial_\nu \mapsto \partial'_\mu = \left(\frac{\partial}{\partial t}, -\boldsymbol{\nabla}\right), \tag{4.23}$$

so that the transformation (4.23) holds each Maxwell's equation on coordinate x'.

1. Gauss' law and Faraday's law

$$\boldsymbol{\nabla}' \cdot (-\mathbf{E}) = \varrho \ , \quad \boldsymbol{\nabla}' \times (-\mathbf{E}) + \frac{\partial \mathbf{B}}{\partial t} = \mathbf{0} \ , \tag{4.24}$$

2. Gauss' law in magnetic field and Amepre-Maxwell law

$$\boldsymbol{\nabla}' \cdot \mathbf{B} = 0 \ , \quad \boldsymbol{\nabla}' \times \mathbf{B} = -\mathbf{J} + \frac{\partial}{\partial t}(-\mathbf{E}). \tag{4.25}$$

Remind that the transfomration Λ gives $\boldsymbol{\nabla} \mapsto \boldsymbol{\nabla}' = -\boldsymbol{\nabla}$.

4.5 One axis parity

By using same method of section 4.4, it is also possible for considering one axis parity:

$$x^\mu \mapsto x'^\mu = \Lambda^\mu_\nu x^\nu = \begin{pmatrix} t \\ -x \\ y \\ z \end{pmatrix}, \tag{4.26}$$

else, the convariance representation:

$$x_\mu \mapsto x'_\mu = \Lambda^\nu_\mu x_\nu = \begin{pmatrix} -t \\ -x \\ y \\ z \end{pmatrix}. \tag{4.27}$$

Now we put the one axis parity operator $\overline{\Lambda}$ under metric tensor $g_{\mu\nu}$ (4.15) following:

$$\overline{\Lambda}(x \to x') = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & +1 & \\ & & & +1 \end{pmatrix}. \tag{4.28}$$

Here we know each transformations of four-current J^{μ} and electromagnetic tensor $F^{\mu\nu}$ in matrix representations.

1. The four-current

$$J' = \Lambda J = \begin{pmatrix} \varrho \\ -J_x \\ J_y \\ J_z \end{pmatrix}, \tag{4.29}$$

2. The electromagnetic tensor

$$F' = \Lambda^t F \Lambda = \begin{pmatrix} 0 & -E_x & E_y & E_z \\ E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & B_x \\ -E_z & -B_y & -B_x & 0 \end{pmatrix}. \tag{4.30}$$

So that the parity operator $\overline{\Lambda}$ acts on Maxwell's equation following.

Invariance of Maxwell field(one axis parity)

The parity operator acts on Maxwell field by

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix} \xrightarrow{\text{parity}} F'^{\mu\nu} = \begin{pmatrix} 0 & -E_x & E_y & E_z \\ E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & B_x \\ -E_z & -B_y & -B_x & 0 \end{pmatrix}. \tag{4.31}$$

Also, transformation of electromagnetic tensor $F^{\mu\nu} \mapsto F'^{\mu\nu}$ yields:

$$\mathbf{E}(\mathbf{r}, t): \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} \mapsto \begin{pmatrix} -E_x \\ E_y \\ E_z \end{pmatrix}, \quad \mathbf{B}(\mathbf{r}, t): \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} \mapsto \begin{pmatrix} B_x \\ -B_y \\ -B_z \end{pmatrix}. \tag{4.32}$$