Quantum Theory of Gauge Fields I

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Chapter 1

Quantum Electrodynamics for Scattring

1.1 Propagators for Photons

In this section, we prepare the gauge field for some discussions of photons with its action integral:

$$S = \int d^4x \ \mathcal{L}_G = \int d^4x \ \left(-\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} \right). \tag{1.1}$$

Another convention for $A_\mu \to eA_\mu$, the action integral also denoted by

$$\mathcal{L}_G \quad \to \quad -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \tag{1.2}$$

So, we have the equation of motion for photons followed from the Hamilton principle on free fields($J^\mu = eQ\bar{\psi}\gamma^\mu\psi = 0$)[1]:

$$\partial_\mu F^{\mu\nu} = \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = (\eta^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu)A_\mu = 0. \tag{1.3}$$

Using the equation of motion, now we introduce the representation of photons $A_\mu(x)$. For this task, hereby we consider its Fourier transformation:

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^3 2k_0} \sum_\lambda \epsilon_\mu^{(\lambda)\star}(\mathbf{k}) a_\lambda(\mathbf{k}) e^{-ikx} + \epsilon_\mu^{(\lambda)}(\mathbf{k}) a_\lambda^\dagger(\mathbf{k}) e^{ikx} \tag{1.4}$$

with its polarization λ. Since both of Lagrangian density $\mathcal{L}_G$ and action integral S are gauge invariant, we take gauge fixing without loss of generality:

[1] If we take the fermion field as Dirac field(electron), $Q = -1$.

1.1.1 The Lorentz Gauge

Hereby we take Lorentz gauge $\partial^\mu A_\mu = 0$ on gauge field. Now it satisfies

$$k^\mu \epsilon_\mu^{(\lambda)}(\mathbf{k}) = 0 \tag{1.5}$$

for expansion (1.4). So, we can construct the space V:

$$V := \left\{ \epsilon_\mu^{(\lambda)}(\mathbf{k}) | k^\mu \epsilon_\mu^{(\lambda)}(\mathbf{k}) = 0 \right\}. \tag{1.6}$$

Since this space obtain 4-basis ε_μ^i $(i = 0, 1, 2, 3)$, any 4-polarization vectors $\epsilon_\mu^{(\lambda)}(\mathbf{k})$ can be denoted by

$$\epsilon_\mu^{(\lambda)}(\mathbf{k}) = \sum_{i=0}^{3} \xi_i^{(\lambda)}(\mathbf{k}) \varepsilon_\mu^i \tag{1.7}$$

where we select each basis as

$$\varepsilon_\mu^0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \varepsilon_\mu^1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \varepsilon_\mu^2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \varepsilon_\mu^3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{1.8}$$

Then, we can easily confirm that the polarization sums of these basis ε^i by calculating following quantities:

$$\varepsilon_\mu^\lambda [\varepsilon^{\lambda' \star}]^\nu = \eta^{\lambda\lambda'} \delta_\mu^\nu, \qquad \varepsilon_\mu^\lambda [\varepsilon^{\lambda' \star}]_\nu = \eta^{\lambda\lambda'} \eta_{\nu\sigma} \delta_\mu^\sigma = \eta^{\lambda\lambda'} \eta_{\mu\nu}. \tag{1.9}$$

So, we finally get the sums of 4-polarization vectors as

$$\sum_{\lambda=\lambda'} \eta_{\lambda\lambda'} \varepsilon_\mu^\lambda [\varepsilon^{\lambda' \star}]_\nu = \eta_{\mu\nu}. \tag{1.10}$$

Hence, the expansion of photons $A_\mu(x)$ now then also expressed by under its basis ϵ_μ^i. Feeding (1.10) to Fourier expansion (1.4), $A_\mu(x)$ gives

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^3 2k_0} \sum_\lambda \varepsilon_\mu^{\lambda \star} b_\lambda(\mathbf{k}) e^{-ikx} + \varepsilon_\mu^\lambda b_\lambda^\dagger(\mathbf{k}) e^{ikx} \tag{1.11}$$

where we denote new creation-annihilation operators:

$$b_\lambda(\mathbf{k}) = \sum_\sigma \xi_\lambda^{(\sigma)\star}(\mathbf{k}) a_\sigma(\mathbf{k}). \tag{1.12}$$

1.1.2 The R_ξ Gauge Theory and Propagator

In this section, let confirm more gauge fixing theories on massless spin-1 fields(photons). The R_ξ gauge theory is started from modified massless spin-1 Lagrangian density:

$$\mathcal{L}_{R_\xi} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2. \tag{1.13}$$

This new Lagrangian density also gives equation of motion as

$$\left[\eta^{\mu\nu}\partial^2 - \left(1 - \frac{1}{\xi}\right)\partial^\nu\partial^\mu\right]A_\mu(x) = P^{\mu\nu}A_\mu(x) = 0. \tag{1.14}$$

Now we shall consider its green function $\Delta_{\mu\nu}(x-y)$ which satisfies

$$P^{\mu\nu}\Delta_{\mu\sigma}(x-y) = \delta^\nu_\sigma\delta^4(x-y). \tag{1.15}$$

To solve this green equation for massless spin-1 gauge field, hereby we take the Fourier transformation of $\Delta_{\mu\sigma}(x-y)$. So,

$$\Delta_{\mu\sigma}(x-y) = \int \frac{d^4k}{(2\pi)^4}\,\tilde{\Delta}_{\mu\sigma}(k)e^{ik(x-y)}, \tag{1.16}$$

the green equation we want is also denoted by

$$\left[\eta^{\mu\nu}k^2 - \left(1 - \frac{1}{\xi}\right)k^\mu k^\nu\right]\tilde{\Delta}_{\mu\sigma}(k) = \tilde{P}^{\mu\nu}\tilde{\Delta}_{\mu\sigma}(k) = \delta^\nu_\sigma. \tag{1.17}$$

Then, the Fourier transformation of green function $\tilde{\Delta}_{\mu\sigma}(k)$ can be derived by considering the inverse-operator $Q_{\mu\sigma}$ ($\tilde{P}^{\mu\nu}Q_{\mu\sigma} = \delta^\nu_\sigma$). Now if we suppose $Q_{\mu\sigma} = D_1\eta_{\mu\sigma} + D_2k_\mu k_\sigma$, we shall calculate:

$$\begin{aligned}\tilde{P}^{\mu\nu}Q_{\mu\sigma}(k) &= \left[\eta^{\mu\nu}k^2 - \left(1 - \frac{1}{\xi}\right)k^\mu k^\nu\right](D_1\eta_{\mu\sigma} + D_2k_\mu k_\sigma)\\ &= D_1k^2\delta^\nu_\sigma + D_2k^2(k_\sigma k^\nu) - (D_1 + D_2k^2)\left(1 - \frac{1}{\xi}\right)(k_\sigma k^\nu).\end{aligned} \tag{1.18}$$

Then, equating calculation on (1.49) and $\tilde{P}^{\mu\nu}Q_{\mu\sigma} = \delta^\nu_\sigma$, this comes to

$$\delta^\nu_\sigma = D_1k^2\delta^\nu_\sigma + \left[D_2k^2 - (D_1 + D_2k^2)\left(1 - \frac{1}{\xi}\right)\right]k_\sigma k^\nu, \tag{1.19}$$

which leads relations of coefficients by

$$D_1 k^2 = 1, \qquad D_2 k^2 - (D_1 + D_2 k^2)\left(1 - \frac{1}{\xi}\right) = 0. \tag{1.20}$$

So, each equation of (1.20) yeilds:

$$D_1 = \frac{1}{k^2}, \qquad D_2 = -\frac{1-\xi}{k^2 \cdot k^2}. \tag{1.21}$$

Now then we can only determine the inverse-operator $Q_{\mu\sigma}$ using D_1, D_2 from (1.21). Also, the relation $Q_{\rho\nu}\tilde{P}^{\mu\nu}\tilde{\Delta}_{\mu\sigma}(k) = \tilde{\Delta}_{\rho\sigma}(k) = Q_{\rho\sigma}$ leads Fourier transformation of green function $\tilde{\Delta}_{\mu\nu}(k)$ as

$$\tilde{\Delta}_{\mu\nu}(k) = Q_{\mu\nu} = \frac{1}{k^2}\left(\eta_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right), \tag{1.22}$$

which gives the representation of green function $\Delta_{\mu\nu}(x-y)$ from (1.16):

$$\Delta_{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2}\left(\eta_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right) e^{ik(x-y)}. \tag{1.23}$$

Now we determined the mathematical green functions for photons, but this results can not explain the Huygens' principles. For this task, let us put propagator for photons $G_{\mu\nu}(x-y)$ as:

$$G_{\mu\nu}(x-y) := \Delta_{\mu\nu}(x-y)\theta(x^0-y^0) + \Delta_{\mu\nu}(y-x)\theta(y^0-x^0). \tag{1.24}$$

Now then propagator $G_{\mu\nu}(x-y)$ exactly satisfies Huygens' principles, with its equation $P^{\mu\nu}G_{\mu\sigma}(x-y) = -i\delta^\nu_\sigma\delta^4(x-y)$. Hence, (1.57) yeilds

$$\begin{aligned}\frac{1}{i}G_{\mu\nu}(x-y) &= \theta(x^0-y^0)\int \frac{d^4k}{(2\pi)^4}\,\frac{-i}{k^2}\left(\eta_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right) e^{ik(x-y)} + (x \leftrightarrow y)\\ &= \int \frac{d^4k}{(2\pi)^4}\,\frac{-i}{k^2 - i\epsilon}\left(\eta_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right) e^{ik(x-y)}\end{aligned} \tag{1.25}$$

from Residue theorem. This is the **Propagator for Photons** on R_ξ gauge theories. Let us cut-off the calculation of propagator, now then consider other approach for propagator for photons (1.25). Using time-ordering products, we shall also have

$$\frac{1}{i}G_{\mu\nu}(x-y) = \langle 0|\mathrm{T}[A_\mu(x)A_\nu(y)]|0\rangle. \tag{1.26}$$

So, the propagator for photons now comes to

$$
\begin{aligned}
\frac{1}{i}G_{\mu\nu}(x-y) = & \int d^3\tilde{k}d^3\tilde{p}\sum_{\lambda,\lambda'}[b_{\lambda'}(\mathbf{p}), b^\dagger_\lambda(\mathbf{k})]\varepsilon^{\lambda'\star}_\mu \varepsilon^\lambda_\nu e^{i(px-ky)}\theta(x^0-y^0) \\
& + \int d^3\tilde{k}d^3\tilde{p}\sum_{\lambda,\lambda'}[b_{\lambda'}(\mathbf{p}), b^\dagger_\lambda(\mathbf{k})]\varepsilon^{\lambda\star}_\nu \varepsilon^{\lambda'}_\mu e^{-i(px-ky)}\theta(y^0-x^0) \\
= & \int \frac{d^3k}{(2\pi)^3 2k_0}\theta(x^0-y^0)\sum_{\lambda=\lambda'}\eta_{\lambda\lambda'}\varepsilon^{\lambda'\star}_\mu \varepsilon^\lambda_\nu e^{ik(x-y)} \\
& + \int \frac{d^3k}{(2\pi)^3 2k_0}\theta(y^0-x^0)\sum_{\lambda=\lambda'}\eta_{\lambda\lambda'}\varepsilon^{\lambda'\star}_\nu \varepsilon^\lambda_\mu e^{-ik(x-y)}
\end{aligned}
\tag{1.27}
$$

where we put $d^4\tilde{k} = d^3k/[(2\pi)^3 2k_0]$. The canonical quantization condition for photons $[b_{\lambda'}(\mathbf{p}), b^\dagger_\lambda(\mathbf{k})] = \eta_{\lambda\lambda'}(2\pi)^3 2k_0\delta^3(p-k)$ are supposed. For the massless spin-1, this takes norm of 4-momentum as $k^2 = 0$ without loss of generality. So, now we take $k_0 = |\mathbf{k}|$, the Residue theorem yields:

$$
\begin{aligned}
\frac{1}{i}G_{\mu\nu}(x-y) = & \int \frac{d^3k}{(2\pi)^3 2|\mathbf{k}|}\theta(x^0-y^0)\sum_{\lambda=\lambda'}\eta_{\lambda\lambda'}\varepsilon^{\lambda'\star}_\mu \varepsilon^\lambda_\nu e^{ik(x-y)} \\
& + \int \frac{d^3k}{(2\pi)^3 2|\mathbf{k}|}\theta(y^0-x^0)\sum_{\lambda=\lambda'}\eta_{\lambda\lambda'}\varepsilon^{\lambda'\star}_\nu \varepsilon^\lambda_\mu e^{-ik(x-y)} \\
= & \int \frac{d^4k}{(2\pi)^4}\ \frac{-i}{k^2-i\epsilon}e^{ik(x-y)}\left(\sum_\lambda \eta_{\lambda\lambda}\varepsilon^\lambda_\mu \varepsilon^{\lambda\star}_\nu\right).
\end{aligned}
\tag{1.28}
$$

Comparing (1.25) and (1.61), we shall take the sums of 4-polarization vectors for R_ξ gauges we want as

$$
\sum_\lambda \eta_{\lambda\lambda}\varepsilon^\lambda_\mu \varepsilon^{\lambda\star}_\nu = \eta_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}. \tag{1.29}
$$

In this R_ξ gauge theories, we can take the mapping of each gauge below by selecting proper ξs:

Definition 1.1.1 (t' Hooft-Feynman Gauge)
If we select $\xi = 1$, it is easy for calculating propagators and S-matrices. This gauge fixing now gives

$$\sum_\lambda \eta_{\lambda\lambda}\varepsilon^\lambda_\mu \varepsilon^{\lambda\star}_\nu = \eta_{\mu\nu}, \qquad \frac{1}{i}G_{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4}\,\frac{-i\eta_{\mu\nu}}{k^2 - i\epsilon}e^{ik(x-y)}. \tag{1.30}$$

This is really Lorentz gauge, from the from of Lagrangian density we introduced on (1.46) when $\xi = 1$, this goes to

$$\mathcal{L}_{R_{\xi=1}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial_\mu A^\mu)^2, \qquad \eta^{\mu\nu}A_\mu(x) = 0, \tag{1.31}$$

so now we also confirm the results of (1.10) on section 1.2.1.

Definition 1.1.2 (Landau Gauge)
If we select $\xi = 0$, we shall take:

$$\sum_\lambda \eta_{\lambda\lambda}\varepsilon^\lambda_\mu \varepsilon^{\lambda\star}_\nu = \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}, \tag{1.32}$$

so the propagator comes to

$$\frac{1}{i}G_{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4}\,\frac{-i}{k^2 - i\epsilon}\left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)e^{ik(x-y)}. \tag{1.33}$$

1.1.3 Propagators on Feynman Gauge

For further discussions about Quantum Electrodynamics, now we take the propagator of photons following, on t' Hooft-Feynman gauge:

$$S_{\mu\nu}(x-y) := \langle 0|\mathrm{T}[A_\mu(x)A_\nu(y)]|0\rangle = \int \frac{d^4k}{(2\pi)^4}\,\frac{-i\eta_{\mu\nu}}{k^2 - i\epsilon}e^{ik(x-y)}. \tag{1.34}$$

This called **Propagator on Feynman Gauge**, usually used on calculation of S-matrices.

1.2 Propagators for Fermions

In this section, we prepare the Dirac field for some discussions of spin-1/2 particles using Weyl spinors from the representation theory of Lorentz groups. For this task, starting from Lorentz algebra, the Lie generator of Lorentz group gives the algebra of J^μ and K^μ as

$$[J^\mu, J^\nu] = i\epsilon^{\mu\nu\sigma}J^\sigma, \qquad [J^\mu, K^\nu] = i\epsilon^{\mu\nu\sigma}K^\sigma, \qquad [K^\mu, K^\nu] = -i\epsilon^{\mu\nu\sigma}J^\sigma. \tag{1.35}$$

Since, this Lie algebra of J^μ, K^μ are not separated, so now we denote the **spin operators** of A^μ, B^μ as

$$A^\mu = \frac{1}{2}(J^\mu + iK^\mu), \qquad\qquad B^\mu = \frac{1}{2}(J^\mu - iK^\mu). \tag{1.36}$$

So, the Lie algebra of spin operators now goes to

$$[A^\mu, A^\nu] = i\epsilon^{\mu\nu\sigma}A^\sigma, \qquad [B^\mu, B^\nu] = i\epsilon^{\mu\nu\sigma}B^\sigma, \qquad [A^\mu, B^\nu] = 0, \tag{1.37}$$

then, we shall confirm that two algebra of A^μs and B^μs are separated. In group notation, $\mathcal{O}_+^\uparrow = SU(2) \oplus SU(2)$. From the algebra of (1.37), each spin operators $\mathbf{A} = \sum_i (A^i)^2$, A^3 and $\mathbf{B} = \sum_i (B^i)^2$, B^3 have simultaneous eigenstate [2] $\Phi_{a,b}^{A,B}$. From the representation of SU(2), now we obtain

$$\begin{aligned} \mathbf{A}\Phi_{a,b}^{A,B} &= A(A+1)\Phi_{a,b}^{A,B}, \quad A^3\Phi_{a,b}^{A,B} = a\Phi_{a,b}^{A,B} \quad (a = -A, -A+1, \cdots, A), \\ \mathbf{B}\Phi_{a,b}^{A,B} &= B(B+1)\Phi_{a,b}^{A,B}, \quad B^3\Phi_{a,b}^{A,B} = b\Phi_{a,b}^{A,B} \quad (b = -B, -B+1, \cdots, B). \end{aligned} \tag{1.38}$$

Hence, we shall categorize field $\Phi_{a,b}^{A,B}$ by selecting each spin A and B.

1.2.1 Right-handed Weyl Spinor

The representation of each spin operators A^μs and B^μs are only determined by spin A, B. Since we can derive the representation of Lie algebra of J^μs and K^μs from spin operators, we shall have representation of Lorentz groups from spin A, B:

$$D(\Lambda) := \exp\left(i\xi_\mu J^\mu + i\eta_\mu K^\mu\right) = \exp\left[i\xi_\mu(A^\mu + B^\mu) + \eta_\mu(A^\mu - B^\mu)\right], \tag{1.39}$$

$$\Phi_{a,b}^{A,B} \quad \to \quad \Phi_{a,b}'^{A,B} = D(\Lambda)\Phi_{a,b}^{A,B}. \tag{1.40}$$

[2] $[\mathbf{A}, A^3] = [\mathbf{A}, B^3] = 0$.

Let us construct the representation of $(A, B) = (0, 1/2)$. The representation theory of SU(2) groups now yeilds

$$\xi^-(x) = \Phi^{0,\frac{1}{2}}_{0,-\frac{1}{2}}(x), \qquad \xi^+(x) = \Phi^{0,\frac{1}{2}}_{0,\frac{1}{2}}(x). \tag{1.41}$$

This is the quantities called **Right-handed Weyl Spinor**[3]. Now then we denote basis of right-handed Weyl spinor as

$$\xi^+(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \phi_+(x), \qquad \xi^-(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \phi_-(x) \tag{1.42}$$

for representations. If we take right-handed Weyl spinor as 2-component spinors, the representation of spin operators now comes to 2×2 matrices. Then, the equation of eigenstates on (1.38) gives:

$$B^3 \xi^\pm(x) = \pm\frac{1}{2}\xi^\pm(x), \qquad \left(a = \pm\frac{1}{2}\right) \tag{1.43}$$

which leads following matrices relations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \to \quad a = \frac{1}{2},\ c = 0, \tag{1.44}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \to \quad b = 0,\ d = -\frac{1}{2}. \tag{1.45}$$

Feeding (1.44) and (1.45) to each eigenstate in form of (1.42), the representation of B^3 on this basis gives

$$B^3 = \frac{\sigma^3}{2}. \tag{1.46}$$

But each A^μ, B^μ are the Lie generators of SU(2) Lie group, then now we have only possible pairs for A^μ, B^μ as

$$A^\mu = 0, \qquad B^\mu = \frac{\sigma^\mu}{2} \tag{1.47}$$

from the representation theory of SU(2). Hence, from the definition of spin operators A^μ, B^μ on (1.36), results of (1.47) now recover the representation of Lie generator J^μs and K^μs:

$$J^\mu = A^\mu + B^\mu = \frac{\sigma^\mu}{2}, \qquad K^\mu = -i(A^\mu - B^\mu) = i\frac{\sigma^\mu}{2}. \tag{1.48}$$

[3]This is new Lorentz invariant quantities.

for this right-handed Weyl spinor $\xi(x)$. Feeding these Lie generators J^μ, K^μ to the representation of Lorentz groups (1.39), now we shall take Lorentz transformation for right-handed Weyl spinors[4] by

$$\xi(x) \quad \to \quad \xi'(x) = \exp\left(-\frac{\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}}{2}\eta\right)\xi(x). \tag{1.49}$$

This is only for generated by rapidity[5], without any spatial rotations($\xi_\mu = 0$). This new transformation formula is the **Lorentz transformation of Right-handed Weyl Spinors** introduced by representation theories.

1.2.2 Left-handed Weyl Spinor

Let us construct the representation of $(A, B) = (1/2, 0)$. The representation theory of SU(2) groups now yeilds

$$\eta^-(x) = \Phi^{\frac{1}{2},0}_{-\frac{1}{2},0}(x), \qquad \eta^+(x) = \Phi^{\frac{1}{2},0}_{\frac{1}{2},0}(x) \tag{1.50}$$

This is the quantities called **Left-handed Weyl Spinor**. Now then we denote basis of left-handed Weyl spinor as

$$\eta^+(x) = \begin{pmatrix}1\\0\end{pmatrix}\psi_+(x), \qquad \eta^-(x) = \begin{pmatrix}0\\1\end{pmatrix}\psi_-(x). \tag{1.51}$$

If we take same approach to section 1.2.1 of right-handed Weyl spinors, the spin operators A^μ, B^μ take

$$A^\mu = \frac{\sigma^\mu}{2}, \qquad B^\mu = 0. \tag{1.52}$$

So, from the representation of (1.52), now we also recover the representation of Lie generator J^μs and K^μs:

$$J^\mu = A^\mu + B^\mu = \frac{\sigma^\mu}{2}, \qquad K^\mu = -i(A^\mu - B^\mu) = -i\frac{\sigma^\mu}{2}. \tag{1.53}$$

[4]After this section, we take right-handed Weyl spinors as 2-component spinors as

$$\xi(x) = c_+\xi^+(x) + c_-\xi^-(x) = \begin{pmatrix}c_+\phi_+(x)\\ c_-\phi_-(x).\end{pmatrix}$$

This is linear combination of basis on (1.42)s.

[5]Lorentz boosts.

Take care of opposite sign for K^μs to right-handed Weyl spinor's. Feeding these Lie generators J^μ, K^μ to the representation of Lorentz groups (1.39), now we shall take Lorentz transformation for left-handed Weyl spinors by

$$\eta(x) \quad \rightarrow \quad \eta'(x) = \exp\left(\frac{\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}}{2}\eta\right)\eta(x). \tag{1.54}$$

This is only for generated by rapidity[6], without any spatial rotations($\xi_\mu = 0$). This new transformation formula is the **Lorentz transformation of Left-handed Weyl Spinors** introduced by representation theories.

1.2.3 Dirac spinors

In this section we shall have disccusion for Dirac spinors. Let us introduce the Dirac spinors with right-handed and left-handed Weyl spinors:

$$\psi(x) = \begin{pmatrix}\xi(x)\\ \eta(x)\end{pmatrix}. \tag{1.55}$$

Now we want to the Lorentz transformation of Dirac spinors $S(\Lambda)$. From the definition of Dirac spinors (1.55), we shall take

$$\psi'(x) \quad \rightarrow \quad S(\Lambda)\psi(x); \qquad \begin{pmatrix}\xi'(x)\\ \eta'(x)\end{pmatrix} = \underbrace{\begin{pmatrix}\exp\left(-\frac{\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}}{2}\eta\right) & 0\\ 0 & \exp\left(\frac{\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}}{2}\eta\right)\end{pmatrix}}_{S(\Lambda)}\begin{pmatrix}\xi(x)\\ \eta(x)\end{pmatrix} \tag{1.56}$$

This definition of Dirac spinors(or bispinors) (1.55) which can separated by right-handed and left-handed Weyls spinors, is called the notation of **Chiral representation**. So, the Lorentz transformation $S(\Lambda)$ for Chiral representation now comes to

$$\begin{aligned} S(\Lambda) &= \begin{pmatrix}\exp\left(-\frac{\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}}{2}\eta\right) & 0\\ 0 & \exp\left(\frac{\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}}{2}\eta\right)\end{pmatrix} \\ &= \lim_{N\rightarrow\infty}\begin{pmatrix}\hat{\mathbf{1}}_2 - \frac{\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}}{2N}\eta & 0\\ 0 & \hat{\mathbf{1}}_2 + \frac{\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}}{2N}\eta\end{pmatrix}^N \\ &= \lim_{N\rightarrow\infty}\left(\hat{\mathbf{1}}_4 - \frac{\eta}{2N}\hat{\mathbf{n}}\cdot\boldsymbol{\Gamma}\right)^N = \exp\left(-\frac{\boldsymbol{\Gamma}\cdot\hat{\mathbf{n}}}{2}\eta\right). \end{aligned} \tag{1.57}$$

[6]Lorentz boosts.

Here we denote matrix $\mathbf{\Gamma}$ with the Pauli matrices $\boldsymbol{\sigma} = \sigma^i$ $(i = 1, 2, 3)$as:

$$\mathbf{\Gamma} := \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & -\boldsymbol{\sigma} \end{pmatrix}, \qquad \boldsymbol{\eta} = \hat{\mathbf{n}}\eta. \tag{1.58}$$

So, the Tayler expansion of (1.57) now gives

$$\begin{aligned} S(\Lambda) &= \exp\left(-\frac{\mathbf{\Gamma}\cdot\hat{\mathbf{n}}}{2}\eta\right) = \hat{\mathbf{1}}_4 \cosh\frac{\eta}{2} - (\mathbf{\Gamma}\cdot\hat{\mathbf{n}})\sinh\frac{\eta}{2} \\ &= \begin{pmatrix} \cosh\frac{\eta}{2} - \boldsymbol{\sigma}\cdot\hat{\mathbf{n}}\sinh\frac{\eta}{2} & 0 \\ 0 & \cosh\frac{\eta}{2} + \boldsymbol{\sigma}\cdot\hat{\mathbf{n}}\sinh\frac{\eta}{2} \end{pmatrix}. \end{aligned} \tag{1.59}$$

Since the Wick rotation of rapidity η[7] leads $\tanh\eta = \beta$, we shall calculate the rapidity on representation (1.59) as

$$\tanh\frac{\eta}{2} = \frac{\sinh\eta}{\cosh\eta + 1} = \frac{\gamma\beta}{\gamma+1} = \frac{p}{E+m} \qquad (E = \gamma m,\ p = |\mathbf{p}|), \tag{1.60}$$

$$\cosh\frac{\eta}{2} = \frac{E+m}{\sqrt{(E+m)^2 - p^2}} = \sqrt{\frac{E+m}{2m}}, \qquad \sinh\frac{\eta}{2} = \frac{p}{\sqrt{2m(E+m)}} \tag{1.61}$$

on the mass-shall[8]. Then, the representation of Lorentz transformation $S(\lambda)$ of Dirac spinors using (1.60) and (1.61):

$$S(\Lambda) = \frac{1}{\sqrt{2m(E+m)}} \begin{pmatrix} E + m - \boldsymbol{\sigma}\cdot\mathbf{p} & 0 \\ 0 & E + m + \boldsymbol{\sigma}\cdot\mathbf{p} \end{pmatrix} \tag{1.62}$$

The is the spinor representation which called **Chiral representation**.

1.2.4 Free Spinor Fields

In this section, we introduce the positive-energy Dirac spinor $u(\mathbf{p}, s_i)$s. If we select the rest frame, we denote the representation of spinor $u(\mathbf{0}, s_i)$s which are spin up and down, using right-handed and left-handed Weyl spinors:

$$u(\mathbf{0}, s_i = +) = \sqrt{m}\begin{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix}, \qquad u(\mathbf{0}, s_i = -) = \sqrt{m}\begin{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} \tag{1.63}$$

[7] Hence, $t \to \tau = -it$.
[8] $E^2 = \mathbf{p}^2 + m^2$.

and its energy eigenstate on rest frame

$$\hat{H}(\mathbf{0})u(\mathbf{0}, s_i = \pm) = mu(\mathbf{0}, s_i = \pm); \qquad m\gamma^0 u(\mathbf{0}, s_i = \pm) = mu(\mathbf{0}, s_i = \pm), \tag{1.64}$$

where we put

$$\gamma^0 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \qquad E^0 = m \quad \text{(rest frame)}. \tag{1.65}$$

Take care of the term of $\sqrt{m}$ on (1.63) only added for its normalization and its eigenstate of $\hat{S}_z = \sigma^3/2$ on spin state of $s_i = +$. These are

$$S_z \xi(x, s_i = +) = \frac{1}{2}\xi(x, s_i = +), \quad S_z \eta(x, s_i = +) = \frac{1}{2}\eta(x, s_i = +), \tag{1.66}$$

$$\rightarrow \qquad \xi(x, s_i = +) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \phi_+(x), \qquad \eta(x, s_i = +) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \psi_+(x). \tag{1.67}$$

Also, we have some disccusions for spin state $s_i = -$ with same approach. These results give the matrix representation of **positive-energy spinors** in free space as:

$$u(\mathbf{0}, s_i) = \sqrt{m} \begin{pmatrix} \chi \\ \chi \end{pmatrix}, \qquad \chi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{1.68}$$

Using same approach, we also take some discussions for **negative-energy spinors** [9] and its Lorentz transformations. Feeding negative-energy Hamiltonian eigenstate as (1.64) to Weyl spinor(left or right handed) formation, we shall denote the negative-energy Weyl spinor representation $v(\mathbf{0}, s_i)$ as

$$v(\mathbf{0}, s_i) = \sqrt{m} \begin{pmatrix} \chi \\ -\chi \end{pmatrix}, \qquad \chi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{1.69}$$

The 2-component spinors χ is only determined by its spin states. If we take new Dirac spinors $v(\mathbf{0}, s_i)$ on (1.69), these really gives the energy eigenstate:

$$\hat{H}(\mathbf{0})v(\mathbf{0}, s_i = \pm) = m\gamma^0 v(\mathbf{0}, s_i = \pm) = -mv(\mathbf{0}, s_i = \pm) \tag{1.70}$$

which seen to take negative-energy[10]. We shall derived the representation of spinor in a proper frame for vacuum(in positive and negative energy). Now we also have a inverse Lorentz transformation (1.62) from here, so we find each spinors of $u(\mathbf{p}, si), v(\mathbf{p}, s_i)$ followed:

[9] Here we introduce the negative-energy Dirac spinors for the casualty of fields.

[10] Futher discussions, see 'R. P. Feynman, *The Theory of Positrons*'.

1. Positive-energy Dirac spinors

$$u(\mathbf{p},s_i)=S^{-1}(\Lambda)u(\mathbf{0},s_i)=\frac{1}{\sqrt{2(E+m)}}\begin{pmatrix}E+m+\boldsymbol{\sigma}\cdot\mathbf{p} & 0\\ 0 & E+m-\boldsymbol{\sigma}\cdot\mathbf{p}\end{pmatrix}\begin{pmatrix}\chi\\ \chi\end{pmatrix}$$
$$=\frac{1}{\sqrt{2(E+m)}}\begin{pmatrix}m & E+\boldsymbol{\sigma}\cdot\mathbf{p}\\ E-\boldsymbol{\sigma}\cdot\mathbf{p} & m\end{pmatrix}\begin{pmatrix}\chi\\ \chi\end{pmatrix}=\frac{\not{p}+m}{\sqrt{2(E+m)}}\begin{pmatrix}\chi\\ \chi\end{pmatrix}, \tag{1.71}$$

2. Negative-energy Dirac spinors

$$v(\mathbf{p},s_i)=S^{-1}(\Lambda)v(\mathbf{0},s_i)=\frac{1}{\sqrt{2(E+m)}}\begin{pmatrix}E+m+\boldsymbol{\sigma}\cdot\mathbf{p} & 0\\ 0 & E+m-\boldsymbol{\sigma}\cdot\mathbf{p}\end{pmatrix}\begin{pmatrix}\chi\\ -\chi\end{pmatrix}$$
$$=\frac{1}{\sqrt{2(E+m)}}\begin{pmatrix}m & -E-\boldsymbol{\sigma}\cdot\mathbf{p}\\ -E+\boldsymbol{\sigma}\cdot\mathbf{p} & m\end{pmatrix}\begin{pmatrix}\chi\\ -\chi\end{pmatrix}=\frac{-\not{p}+m}{\sqrt{2(E+m)}}\begin{pmatrix}\chi\\ -\chi\end{pmatrix}. \tag{1.72}$$

Then, we shall denote the representation of Lorentz group for each Dirac spinors using Dirac-Feynman slash:

$$u(\mathbf{p},s_i)=\frac{\not{p}+m}{\sqrt{2m(E+m)}}u(\mathbf{0},s_i)\ , \qquad v(\mathbf{p},s_i)=\frac{-\not{p}+m}{\sqrt{2m(E+m)}}v(\mathbf{0},s_i) \tag{1.73}$$

1.2.5 Dirac Propagators

Let us introduce the path integrals for free Dirac spinors with spinor sources $\eta(x),\bar{\eta}(x)$:

$$Z[\eta,\bar{\eta}]\sim\int\mathcal{D}\bar{\psi}\mathcal{D}\psi\ \exp\left(i\int d^4x\ \bar{\psi}(i\not{\partial}-m+i\epsilon)\psi+\bar{\psi}\eta+\bar{\eta}\psi\right). \tag{1.74}$$

where we suppose all of these physical variables are Grassmann variables. Thus, by normalizing this path integrals $Z[0]=1$, we shall have more calculations using the theory of functional determinants, so the path integral (1.74) now goes to

$$Z[\eta,\bar{\eta}]\sim\det\left(i\not{\partial}-m+i\epsilon\right)\exp\left(i\int d^4xd^4y\ \bar{\eta}(x)S(x-y)\eta(y)\right), \tag{1.75}$$

where we denote the propagators for Dirac fields:

$$\begin{aligned} S(x-y) &:= \int \frac{d^4k}{(2\pi)^4} \frac{1}{-\not{k} - m + i\epsilon} e^{ik\cdot(x-y)} \\ &= \int \frac{d^4k}{(2\pi)^4} \frac{-\not{k} + m}{k^2 - m^2 - i\epsilon} e^{ik\cdot(x-y)}. \end{aligned} \tag{1.76}$$

Even, we dropped the functional determinant $\det\left(i\not{\partial} + m\right)$ in path integral (1.74) as constant:

$$\det\left(-i\not{\partial} + m\right) \sim \int \mathcal{D}\bar{c}\mathcal{D}c \ \exp\left(i \int d^4x \ i(\not{\partial}\bar{c})c - m\bar{c}c\right) = Z. \tag{1.77}$$

1.3 Propagators for QED

Let us introduce the formal path integral $Z[J, \eta, \bar{\eta}]$ as:

$$Z[J, \eta, \bar{\eta}] \sim \int \mathcal{D}A_\mu \mathcal{D}\bar{\psi}\mathcal{D}\psi \ \exp\left(i \int d^4x \ \mathcal{L}_f + \mathcal{L}_G + J^\mu A_\mu + \bar{\psi}\eta + \bar{\eta}\psi\right). \tag{1.78}$$

where we denote each of Lagrangian density of fermion field and Gauge field using covariant derivative $D_\mu = \partial_\mu + ieA_\mu$:

$$\mathcal{L}_f := \bar{\psi}(-i\not{D} + m)\psi, \qquad \mathcal{L}_G := -\frac{1}{4}F^2_{\mu\nu}(x). \tag{1.79}$$

Taking R_ξ Gauge, we shall give gauge fixing through the Faddeev-Popov method using Gauge fixing functional $G = \partial^\mu A_\mu(x) - \omega(x)$. Hence, under the R_ξ Gauge fixing, we shall have the path integral of QED in form of

$$Z^{FP}[J, \eta, \bar{\eta}] \sim \int \mathcal{D}A_\mu \mathcal{D}\bar{\psi}\mathcal{D}\psi \ \exp\left(i \int d^4x \ \mathcal{L}_f + \tilde{\mathcal{L}}_G + J^\mu A_\mu + \bar{\psi}\eta + \bar{\eta}\psi\right) \tag{1.80}$$

where we take $\tilde{\mathcal{L}}_G$ followed[11]:

$$\tilde{\mathcal{L}}_G = \mathcal{L}_G + \mathcal{L}_{gf} = -\frac{1}{4}F^2_{\mu\nu}(x) - \frac{1}{2\xi}[\partial^\mu A_\mu(x)]^2. \tag{1.81}$$

[11] In abelian gauge theory, we can drop the component of Faddeev-Popov ghost Lagrangian.

Also, we can calculate the path integral for QED using the relation of

$$
\begin{aligned}
Z^{FP}[J,\eta,\bar{\eta}] &\sim \int \mathcal{D}A_\mu \mathcal{D}\bar{\psi}\mathcal{D}\psi \ \exp\left(i\int d^4x \ \mathcal{L}_f^0 + \tilde{\mathcal{L}}_G + eA_\mu(x)\bar{\psi}(x)\gamma^\mu\psi(x)\right) \\
&\qquad \times \exp\left(i\int d^4x \ J^\mu A_\mu + \bar{\psi}\eta + \bar{\eta}\psi\right) \\
&= \exp\left[ie\int d^4x \ \left(\frac{1}{i}\frac{\delta}{\delta J^\mu(x)}\right)\left(-\frac{1}{i}\frac{\delta}{\delta \eta_\alpha(x)}\right)(\gamma^\mu)_{\alpha\beta}\left(\frac{1}{i}\frac{\delta}{\delta \bar{\eta}_\beta(x)}\right)\right] \\
&\qquad \times Z_0[J,\eta,\bar{\eta}]
\end{aligned}
\tag{1.82}
$$

where

$$
\begin{aligned}
Z_0[J,\eta,\bar{\eta}] &:= \int \mathcal{D}A_\mu \mathcal{D}\bar{\psi}\mathcal{D}\psi \ \exp\left(i\int d^4x \ \mathcal{L}_f^0 + \tilde{\mathcal{L}}_G + J^\mu A_\mu + \bar{\psi}\eta + \bar{\eta}\psi\right) \\
&= \int \mathcal{D}A_\mu \mathcal{D}\bar{\psi}\mathcal{D}\psi \ \exp\left(i\int d^4x \ -\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2\xi}[\partial^\mu A_\mu]^2 + J^\mu A_\mu\right) \\
&\qquad \times \exp\left(i\int d^4x \ \bar{\psi}(i\partial\!\!\!/ + m)\psi + \bar{\psi}\eta + \bar{\eta}\psi\right) \\
&= \exp\left(i\int d^4x d^4y \ \bar{\eta}(x)S(x-y)\eta(y)\right) \\
&\qquad \times \exp\left(\frac{i}{2}\int d^4x d^4y \ J^\mu(x)\Delta_{\mu\nu}(x-y)J^\nu(y)\right)
\end{aligned}
\tag{1.83}
$$

for

$$
\begin{aligned}
S(x-y) &= \int \frac{d^4k}{(2\pi)^4}\frac{-k\!\!\!/ + m}{k^2 - m^2 - i\epsilon}e^{ik\cdot(x-y)}, \\
\Delta_{\mu\nu}(x-y) &= \int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2 - i\epsilon}\left(g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right)e^{ik\cdot(x-y)}.
\end{aligned}
\tag{1.84}
$$

In this R_ξ gauge fixing we give variable gauge fixings by selecting proper ξs. For example, for $\xi = 1$, we shall take Feynman-t' Hooft gauge.

1.3.1 Time-ordered products from Path integrals

From the results of LSZ reduction rules, the 4-point correlation function $\langle 0|\mathrm{T}\left[\psi(x_1)\psi(x_2)\bar{\psi}(y_1)\bar{\psi}(y_2)\right]|0\rangle$ is needed for evaluating the $e^-e^- \to e^-e^-$ scattering. For this task, we shall expand the path integral in (1.82) as:

$$Z^{FP}[J,\eta,\bar{\eta}] = \sum_{P=0}^{\infty}\frac{1}{P!}\left[ie\int d^4x\ \left(\frac{1}{i}\frac{\delta}{\delta J^\mu(x)}\right)\left(-\frac{1}{i}\frac{\delta}{\delta\eta_\alpha(x)}\right)(\gamma^\mu)_{\alpha\beta}\left(\frac{1}{i}\frac{\delta}{\delta\bar{\eta}_\beta(x)}\right)\right]^P$$
$$\times\sum_{V=0}^{\infty}\frac{1}{V!}\left[i\int d^4xd^4y\ \bar{\eta}(x)S(x-y)\eta(y)\right]^V$$
$$\times\sum_{N=0}^{\infty}\frac{1}{N!}\left[\frac{i}{2}\int d^4xd^4y\ J^\mu(x)\Delta_{\mu\nu}(x-y)J^\nu(y)\right]^N. \tag{1.85}$$

Then, the terms which makes 4-point correlation functions are the one only remained each two of fermion fields η and $\bar{\eta}$. Thus, now we shall consider the terms of satisfying $V-P=2$:

1. $P=2, V=4.$

Then, path integral expands by

$$Z^{FP}_{(2,4)}[J,\eta,\bar{\eta}] = \frac{1}{2!}\left[ie\int d^4x\ \left(\frac{1}{i}\frac{\delta}{\delta J^\mu(x)}\right)\left(-\frac{1}{i}\frac{\delta}{\delta\eta_\alpha(x)}\right)(\gamma^\mu)_{\alpha\beta}\left(\frac{1}{i}\frac{\delta}{\delta\bar{\eta}_\beta(x)}\right)\right]^2$$
$$\times\frac{1}{4!}\left[i\int d^4xd^4y\ \bar{\eta}(x)S(x-y)\eta(y)\right]^4$$
$$\times\sum_{N=0}^{\infty}\frac{1}{N!}\left[\frac{i}{2}\int d^4xd^4y\ J^\mu(x)\Delta_{\mu\nu}(x-y)J^\nu(y)\right]^N$$
$$= -\frac{ie^2}{2!}\int d^4x_1d^4x_2d^4wd^4zd^4y_1d^4y_2$$
$$\bar{\eta}(x_1)S(x_1-w)\gamma^\mu S(w-y_1)\eta(y_1)$$
$$\times\Delta_{\mu\nu}(w-z)\bar{\eta}(x_2)S(x_2-z)\gamma^\nu S(z-y_2)\eta(y_2)$$
$$+(y_1\leftrightarrow y_2)+O(J). \tag{1.86}$$

So, the 4-point correlation function goes to($S = 4! \times 2!$):

$$\begin{aligned}\langle 0|\mathrm{T}\left[\cdots\right]|0\rangle = \frac{e^2}{i}\int d^4w d^4z & \\ & S(x_1 - w)\gamma^\mu S(w - y_1)\Delta_{\mu\nu}(w - z)S(x_2 - z)\gamma^\nu S(z - y_2) \\ & \qquad - (y_1 \leftrightarrow y_2).\end{aligned} \tag{1.87}$$

1.3.2 LSZ reduction rules for Spinors

In this section, we shall construct the LSZ reduction rules for spinor scattering process. Starting from plane wave representation of free Dirac fields with helicity:

$$\Psi(x) = \sum_{s=\pm}\int \frac{d^3k}{(2\pi)^3 2k^0}\ \left[b_s(\mathbf{k})u_s(\mathbf{k})e^{-ik\cdot x} + d_s^\dagger v_s(\mathbf{k})e^{ik\cdot x}\right], \tag{1.88}$$

and its Dirac adjoint:

$$\bar{\Psi}(x) = \sum_{s=\pm}\int \frac{d^3k}{(2\pi)^3 2k^0}\ \left[b_s^\dagger(\mathbf{k})\bar{u}_s(\mathbf{k})e^{ik\cdot x} + d_s\bar{v}_s(\mathbf{k})e^{-ik\cdot x}\right]. \tag{1.89}$$

In this representation of free Dirac field as a asymptotic field on scattering process, the creation-annihilation operators of particle or antiparticle $b_s(\mathbf{k}), b_s^\dagger(\mathbf{k})$s also can be written in form of

$$\begin{aligned} b_s(\mathbf{k}) &= \int d^3x\ e^{ik\cdot x}\bar{u}_s(\mathbf{k})\gamma^0\Psi(x), \\ b_s^\dagger(\mathbf{k}) &= \int d^3x\ e^{-ik\cdot x}\bar{\Psi}(x)\gamma^0 u_s(\mathbf{k}),\end{aligned} \tag{1.90}$$

using the relation of Nilpotency[12]

$$\begin{aligned}\bar{u}_{s'}(\mathbf{k})\gamma^0 v_s(-\mathbf{k}) &= \frac{1}{2m}\bar{u}_{s'}(\mathbf{p})\left\{\gamma^0, \not{p} + m - \not{p}\right\}v_s(\mathbf{p}) \\ &= -\frac{p_\mu}{2m}\bar{u}_{s'}(\mathbf{p})\left\{\gamma^0, \gamma^\mu\right\}v_s(\mathbf{p}) = -\frac{p_\mu g^{\mu 0}}{m}\bar{u}_{s'}(\mathbf{p})v_s(\mathbf{p}) = 0,\end{aligned} \tag{1.91}$$

[12]Take care of $\gamma^0 S^\dagger(\Lambda)\gamma^0 = S^{-1}(\Lambda)$.

and orthogonality of

$$\bar{u}_{s'}(\mathbf{k})\gamma^0 u_s(\mathbf{k}) = \frac{1}{2m}\bar{u}_{s'}(\mathbf{p})\left\{\gamma^0, -\not{p} + m + \not{p}\right\} u_s(\mathbf{p})$$
$$= \frac{p_\mu}{2m}\bar{u}_{s'}(\mathbf{p})\left\{\gamma^0, \gamma^\mu\right\} u_s(\mathbf{p}) = \frac{p_\mu g^{\mu 0}}{m}\bar{u}_{s'}(\mathbf{p})u_s(\mathbf{p}) = 2E\delta_{ss'}. \tag{1.92}$$

Similarly,

$$\begin{aligned} d_s^\dagger(\mathbf{k}) &= \int d^3x\; e^{-ik\cdot x}\bar{v}_s(\mathbf{k})\gamma^0\Psi(x), \\ d_s(\mathbf{k}) &= \int d^3x\; e^{ik\cdot x}\bar{\Psi}(x)\gamma^0 v_s(\mathbf{k}). \end{aligned} \tag{1.93}$$

Then, let us calculate S-matrix using representation of creation-annihilations $b_s, b_s^\dagger, d_s, d_s^\dagger$ on (1.90), (1.93). For example, computing $e^-e^- \to e^-e^-$ scattering process, the S-matrix is given by

$$S_{fi} = \langle \mathbf{p}', \mathbf{k}'|\mathbf{p}, \mathbf{k}\rangle = \langle 0|\mathrm{T}\left[b_{\mathbf{p}'}(+\infty)b_{\mathbf{k}'}(+\infty)b_{\mathbf{p}}^\dagger(-\infty)b_{\mathbf{k}}^\dagger(-\infty)\right]|0\rangle. \tag{1.94}$$

Also, the time-evolution of particle creating operator $b_s^\dagger$s[13] are developed by

$$\begin{aligned} b_{\mathbf{k}}^\dagger(+\infty) - b_{\mathbf{k}}^\dagger(-\infty) &= \int dt\; \partial_0 b_{\mathbf{k}}^\dagger(t) \\ &= \int d^4x\; \partial_0\left(e^{-ik\cdot x}\bar{\Psi}(x)\gamma^0 u_s(\mathbf{k})\right) \\ &= \int d^4x\; \bar{\Psi}(x)\left[\gamma^0\overleftarrow{\partial_0} - i\gamma^0 k_0\right] u_s(\mathbf{k})e^{-ik\cdot x} \\ &= \int d^4x\; \bar{\Psi}(x)\left[\gamma^0\overleftarrow{\partial_0} - i\gamma^0 k_0 - i(-\not{k} + m)\right] u_s(\mathbf{k})e^{-ik\cdot x} \\ &= \int d^4x\; \bar{\Psi}(x)\left[\gamma^0\overleftarrow{\partial_0} - \gamma^i\overrightarrow{\partial_i} - im\right] u_s(\mathbf{k})e^{-ik\cdot x} \\ &= i\int d^4x\; \bar{\Psi}(x)\left[-i\overleftarrow{\not{\partial}} - m\right] u_s(\mathbf{k})e^{-ik\cdot x}. \end{aligned} \tag{1.95}$$

[13] Here we simply denote $b_{\mathbf{k}}^\dagger(t) := b_s(\mathbf{k})$ which is annihilation operator of particle, on time t.

Similarly, we also have

$$b_{\mathbf{k}}(+\infty) - b_{\mathbf{k}}(-\infty) = \frac{1}{i}\int d^4x\ e^{ik\cdot x}\bar{u}_s(\mathbf{k})\left[i\partial\!\!\!/ - m\right]\Psi(x). \tag{1.96}$$

Thus, we finally calculate the S-matrix on (1.94) feeding upper time-evolution relations and time ordered product:

Theorem 1.3.1 (LSZ reduction formula for Spinor Scattering)

$$\begin{aligned}
S_{fi} = \langle 0|\Bigg[&\left(b_{\mathbf{p}'}(-\infty) - i\int d^4x\ e^{ik\cdot x}\bar{u}_s(\mathbf{p}')\left[i\partial\!\!\!/ - m\right]\Psi(x)\right)\\
&\cdots\left(b^\dagger_{\mathbf{k}}(+\infty) - i\int d^4x\ \bar{\Psi}(x)\left[-i\overleftarrow{\partial\!\!\!/} - m\right]u_s(\mathbf{k})e^{-ik\cdot x}\right)\Bigg]|0\rangle\\
= \frac{1}{i^4}\int &d^4x_1 d^4x_2 d^4y_1 d^4y_2\\
&\times e^{ip'\cdot x_1}\bar{u}_{s_1'}(\mathbf{p}')\left[i\partial\!\!\!/_{x_1} - m\right]\\
&\times e^{ik'\cdot x_2}\bar{u}_{s_2'}(\mathbf{k}')\left[i\partial\!\!\!/_{x_2} - m\right]\\
&\times \langle 0|\mathrm{T}\left[\Psi(x_2)\Psi(x_1)\bar{\Psi}(y_1)\bar{\Psi}(y_2)\right]|0\rangle\\
&\times \left[-i\overleftarrow{\partial\!\!\!/_{y_1}} - m\right]u_{s_1}(\mathbf{p})e^{-ip\cdot y_1}\\
&\times \left[-i\overleftarrow{\partial\!\!\!/_{y_2}} - m\right]u_{s_2}(\mathbf{k})e^{-ik\cdot y_2}.
\end{aligned} \tag{1.97}$$

This called **Lehmann-Symanzik-Zimmermann reduction formula**, is the relation of time ordered product of fields and S-matrix.

1.3.3 Scattering Matrices for Spinor QED

In this section, we shall exactly calculate S-matrix on spinor scattering. Let us compute $e^-e^- \to e^-e^-$ scattering process. Now we shall directly calculate the S-matrix from the results of calculation for 4-point correlation function (1.87) and LSZ reduction formula (1.97):

$$\begin{aligned}
S^{(1)}_{fi} &= \frac{e^2}{i^5}\int d^4x_1 d^4x_2 d^4y_1 d^4y_2 d^4w d^4z \\
&\qquad\qquad \times e^{ip'\cdot x_1}\bar{u}_{s'_1}(\mathbf{p}')\left[i\partial\!\!\!/_{x_1}-m\right]e^{ik'\cdot x_2}\bar{u}_{s'_2}(\mathbf{k}')\left[i\partial\!\!\!/_{x_2}-m\right] \\
&\qquad\qquad \times S(x_1-w)\gamma^\mu S(w-y_1)\Delta_{\mu\nu}(w-z)S(x_2-z)\gamma^\nu S(z-y_2) \\
&\qquad\qquad \times \left[-i\overleftarrow{\partial\!\!\!/}_{y_1}-m\right]u_{s_1}(\mathbf{p})e^{-ip\cdot y_1}\left[-i\overleftarrow{\partial\!\!\!/}_{y_2}-m\right]u_{s_2}(\mathbf{k})e^{-ik\cdot y_2} \\
&= \frac{e^2}{i}\int d^4x_1 d^4x_2 d^4y_1 d^4y_2 d^4w d^4z\; e^{i(p'\cdot x_1+k'\cdot x_2-p\cdot y_1-k\cdot y_2)} \\
&\qquad \times \delta^4(x_1-w)\delta^4(y_1-w)\delta^4(x_2-z)\delta^4(y_2-z) \\
&\qquad \times \bar{u}_{s'_1}(\mathbf{p}')\gamma^\mu u_{s_1}(\mathbf{p})\left[\int\frac{d^4\bar{k}}{(2\pi)^4}\,\frac{e^{i\bar{k}\cdot(w-z)}}{\bar{k}^2-i\epsilon}G_{\mu\nu}(\bar{k})\right]\bar{u}_{s'_2}(\mathbf{k}')\gamma^\nu u_{s_2}(\mathbf{k}) \\
&= \frac{e^2}{i}\int\frac{d^4\bar{k}}{(2\pi)^4}d^4w d^4z\; e^{i(\bar{k}+p'-p)w}e^{i(k'-k-\bar{k})z} \\
&\qquad \times \bar{u}_{s'_1}(\mathbf{p}')\gamma^\mu u_{s_1}(\mathbf{p})\left[\frac{1}{\bar{k}^2-i\epsilon}G_{\mu\nu}(\bar{k})\right]\bar{u}_{s'_2}(\mathbf{k}')\gamma^\nu u_{s_2}(\mathbf{k}) \\
&= -(2\pi)^4 ie^2\int d^4\bar{k}\;\delta^4(\bar{k}+p'-p)\delta^4(\bar{k}-k'+k) \\
&\qquad\qquad \times \bar{u}_{s'_1}(\mathbf{p}')\gamma^\mu u_{s_1}(\mathbf{p})\frac{G_{\mu\nu}(\bar{k})}{\bar{k}^2-i\epsilon}\bar{u}_{s'_2}(\mathbf{k}')\gamma^\nu u_{s_2}(\mathbf{k}) \\
&= -(2\pi)^4 ie^2\delta^4(p+k-p'-k') \\
&\qquad\qquad \times \bar{u}_{s'_1}(\mathbf{p}')\gamma^\mu u_{s_1}(\mathbf{p})\frac{G_{\mu\nu}(p-p')}{(p-p')^2}\bar{u}_{s'_2}(\mathbf{k}')\gamma^\nu u_{s_2}(\mathbf{k}).
\end{aligned} \tag{1.98}$$

where we denote $G_{\mu\nu}(\bar{k})$ as:

$$G_{\mu\nu}(\bar{k}) := g_{\mu\nu} - (1-\xi)\frac{\bar{k}_\mu \bar{k}_\nu}{\bar{k}^2}. \tag{1.99}$$

Also, we separated the result of time ordered products to each terms which are connected by summation. Similarly,

$$\begin{aligned}
S^{(2)}_{fi} = \frac{e^2}{i^3}\int & d^4x_1 d^4x_2 d^4y_1 d^4y_2 d^4w d^4z \\
& \times e^{ip'\cdot x_1}\bar{u}_{s'_1}(\mathbf{p}')\left[i\not{\partial}_{x_1} - m\right] e^{ik'\cdot x_2}\bar{u}_{s'_2}(\mathbf{k}')\left[i\not{\partial}_{x_2} - m\right] \\
& \times S(x_1-w)\gamma^\mu S(w-y_2)\Delta_{\mu\nu}(w-z)S(x_2-z)\gamma^\nu S(z-y_1) \\
& \times \left[-i\overleftarrow{\not{\partial}_{y_1}} - m\right] u_{s_1}(\mathbf{p})e^{-ip\cdot y_1}\left[-i\overleftarrow{\not{\partial}_{y_2}} - m\right] u_{s_2}(\mathbf{k})e^{-ik\cdot y_2} \\
= ie^2\int & d^4x_1 d^4x_2 d^4y_1 d^4y_2 d^4w d^4z \; e^{i(p'\cdot x_1 + k'\cdot x_2 - p\cdot y_1 - k\cdot y_2)} \\
& \times \delta^4(x_1-w)\delta^4(y_2-w)\delta^4(x_2-z)\delta^4(y_1-z) \\
& \times \bar{u}_{s'_1}(\mathbf{p}')\gamma^\mu u_{s_2}(\mathbf{k})\left[\int \frac{d^4\bar{k}}{(2\pi)^4}\,\frac{e^{i\bar{k}\cdot(w-z)}}{\bar{k}^2 - i\epsilon}G_{\mu\nu}(\bar{k})\right]\bar{u}_{s'_2}(\mathbf{k}')\gamma^\nu u_{s_1}(\mathbf{p}) \\
= ie^2\int & \frac{d^4\bar{k}}{(2\pi)^4} d^4w d^4z \; e^{i(\bar{k}+p'-k)w}e^{i(k'-p-\bar{k})z} \\
& \times \bar{u}_{s'_1}(\mathbf{p}')\gamma^\mu u_{s_2}(\mathbf{k})\left[\frac{1}{\bar{k}^2 - i\epsilon}G_{\mu\nu}(\bar{k})\right]\bar{u}_{s'_2}(\mathbf{k}')\gamma^\nu u_{s_1}(\mathbf{p}) \\
= (2\pi)^4 ie^2\int & d^4\bar{k}\; \delta^4(\bar{k}+p'-k)\delta^4(\bar{k}-k'+p) \\
& \times \bar{u}_{s'_1}(\mathbf{p}')\gamma^\mu u_{s_2}(\mathbf{k})\frac{G_{\mu\nu}(\bar{k})}{\bar{k}^2 - i\epsilon}\bar{u}_{s'_2}(\mathbf{k}')\gamma^\nu u_{s_1}(\mathbf{p}) \\
= (2\pi)^4 & ie^2\delta^4(p+k-p'-k') \\
& \times \bar{u}_{s'_1}(\mathbf{p}')\gamma^\mu u_{s_2}(\mathbf{k})\frac{G_{\mu\nu}(k-p')}{(k-p')^2}\bar{u}_{s'_2}(\mathbf{k}')\gamma^\nu u_{s_1}(\mathbf{p}).
\end{aligned} \tag{1.100}$$

So, the S-matrix now goes to

$$
\begin{aligned}
S_{fi} = -(2\pi)^4 ie^2\delta^4(p+k-p'-k') \\
\times \Bigg[\bar{u}_{s_1'}(\mathbf{p}')\gamma^\mu u_{s_1}(\mathbf{p})\frac{G_{\mu\nu}(p-p')}{(p-p')^2}\bar{u}_{s_2'}(\mathbf{k}')\gamma^\nu u_{s_2}(\mathbf{k}) \\
-\bar{u}_{s_1'}(\mathbf{p}')\gamma^\mu u_{s_2}(\mathbf{k})\frac{G_{\mu\nu}(k-p')}{(k-p')^2}\bar{u}_{s_2'}(\mathbf{k}')\gamma^\nu u_{s_1}(\mathbf{p})\Bigg].
\end{aligned} \tag{1.101}
$$

on tree level.

1.3.4 LSZ reduction rules for Compton scattering

In this section we shall construct LSZ reduction rules for Compton scattering. Let us start from the plane wave representation of free gauge field:

$$
A_\mu(x) = \sum_\lambda \int \frac{d^3k}{(2\pi)^3 2k^0}\left[\varepsilon_\mu^{\lambda *}(\mathbf{k})a_\lambda(\mathbf{k})e^{-ik\cdot x} + \varepsilon_\mu^\lambda(\mathbf{k})a_\lambda^\dagger(\mathbf{k})e^{ik\cdot x}\right], \tag{1.102}
$$

under the R_ξ gauge theories:

$$
\mathcal{L}_G = -\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2\xi}[\partial^\mu A_\mu(x)]^2, \qquad \sum_\lambda \varepsilon_\mu^\lambda(\mathbf{k})\varepsilon_\nu^{\lambda *}(\mathbf{k}) = g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}. \tag{1.103}
$$

Then, taking similar approach to section 1.3.2, the orthogonality of polarization vector $\varepsilon_\mu^\lambda(\mathbf{k})[\varepsilon^{\lambda' *}(\mathbf{k})]^\mu = \delta^{\lambda\lambda'}$ now leads following expansion of creation-annihilation operators for photons $a_\lambda^\dagger(\mathbf{k}), a_\lambda(\mathbf{k})$ in form of

$$
\begin{aligned}
a_\lambda^\dagger(\mathbf{k}) &= -i\int d^3x\ \varepsilon_\mu^{\lambda *}(\mathbf{k})A^\mu(x)\overleftrightarrow{\partial_0}e^{-ik\cdot x}, \\
a_\lambda(\mathbf{k}) &= -i\int d^3x\ e^{ikx}\overleftrightarrow{\partial_0}A^\mu(x)\varepsilon_\mu^\lambda(\mathbf{k}),
\end{aligned} \tag{1.104}
$$

where we denote operator $g\overleftrightarrow{\partial_0}f := g(\partial_0 f) - (\partial_0 g)f$. Then, computing Compton scattering process($e^-\gamma \to e^-\gamma$), now we shall write the S-matrix by

$$
S_{fi} = \langle \mathbf{p}', \mathbf{k}'|\mathbf{p}, \mathbf{k}\rangle = \langle 0|\mathrm{T}\left[b_{\mathbf{p}'}(+\infty)a_{\mathbf{k}'}(+\infty)b_{\mathbf{p}}^\dagger(-\infty)a_{\mathbf{k}}^\dagger(-\infty)\right]|0\rangle. \tag{1.105}
$$

Here we supposed Compton scattering is progressed by initial state of electron(momentum $\mathbf{p}$) and photon(momentum $\mathbf{k}$) to final state of $\mathbf{p}'$ and $\mathbf{k}'$.

So, the time-evolution of photon creating operator $a^\dagger_\lambda$s are developed by

$$
\begin{aligned}
a^\dagger_{\mathbf{k}}(+\infty) - a^\dagger_{\mathbf{k}}(-\infty) &= \int dt\ \partial_0 a^\dagger_{\mathbf{k}}(t) \\
&= -i \int d^4x\ \partial_0 \left(\varepsilon^{\lambda *}_\mu(\mathbf{k}) A^\mu(x) \overleftrightarrow{\partial_0}\, e^{-ik\cdot x} \right) \\
&= -i \int d^4x\ \partial_0 \left(\varepsilon^{\lambda *}_\mu(\mathbf{k}) A^\mu(x) \left(-\overleftarrow{\partial_0} - ik_0 \right) e^{-ik\cdot x} \right) \\
&= -i \int d^4x\ \varepsilon^{\lambda *}_\mu(\mathbf{k}) A^\mu(x) \left(-\overleftarrow{\partial}^2_0 - (k_0)^2 \right) e^{-ik\cdot x} \\
&= -i \int d^4x\ A^\mu(x) \left(-\overleftarrow{\partial}^2_0 - (k_0)^2 + k^2 \right) \varepsilon^{\lambda *}_\mu(\mathbf{k}) e^{-ik\cdot x} \\
&= -i \int d^4x\ A^\mu(x) \left(-\overleftarrow{\partial}^2_0 + \overrightarrow{\partial}^2_0 - \overrightarrow{\partial}^2 \right) \varepsilon^{\lambda *}_\mu(\mathbf{k}) e^{-ik\cdot x} \\
&= -i \int d^4x\ A^\mu(x) \left(-\overleftarrow{\partial}^2 \right) \varepsilon^{\lambda *}_\mu(\mathbf{k}) e^{-ik\cdot x}.
\end{aligned}
\tag{1.106}
$$

Here we inserted following Nilpotent term on 5th line:

$$
-k^2 \varepsilon^\lambda_\mu(\mathbf{k}) = 0, \tag{1.107}
$$

which is supported by massless Klein-Gordon equation $\partial^2 A^\mu(x) = 0$. Similarly, we also have

$$
a_{\mathbf{k}}(+\infty) - a_{\mathbf{k}}(-\infty) = +i \int d^4x\ e^{ik\cdot x} \varepsilon^\lambda_\mu(\mathbf{k}) \left(-\overrightarrow{\partial}^2 \right) A^\mu(x). \tag{1.108}
$$

Also, for the calculation of S-matrix we shall recall the expansion of $b_{\mathbf{p}}, b^\dagger_{\mathbf{p}}$s:

$$
\begin{aligned}
b^\dagger_{\mathbf{k}}(+\infty) - b^\dagger_{\mathbf{k}}(-\infty) &= i \int d^4x\ \bar{\Psi}(x) \left[-i\overleftarrow{\not{\partial}} - m \right] u_s(\mathbf{k}) e^{-ik\cdot x}, \\
b_{\mathbf{k}}(+\infty) - b_{\mathbf{k}}(-\infty) &= \frac{1}{i} \int d^4x\ e^{ik\cdot x} \bar{u}_s(\mathbf{k}) \left[i\not{\partial} - m \right] \Psi(x).
\end{aligned}
\tag{1.109}
$$

Feeding relations of (1.106), (1.108) and (1.109), now we shall calculate S-matrix for Compton scattering($e^-\gamma \to e^-\gamma$) followed:

Theorem 1.3.2 (LSZ reduction tormula for Compton Scattering)

$$S_{fi} = \langle 0| \Bigg[\left(b_{\mathbf{p}'}(-\infty) - i \int d^4x \; e^{ik\cdot x} \bar{u}_s(\mathbf{p}') \left[i\partial\!\!\!/ - m \right] \Psi(x) \right)$$

$$\cdots \left(b^\dagger_{\mathbf{p}}(+\infty) - i \int d^4x \; \bar{\Psi}(x) \left[-i \overleftarrow{\partial\!\!\!/} - m \right] u_s(\mathbf{p}) e^{-ik\cdot x} \right) \Bigg] |0\rangle$$

$$= \frac{1}{i^4} \int d^4x_1 d^4x_2 d^4y_1 d^4y_2$$

$$\times e^{ip'\cdot x_1} \bar{u}_{s_1'}(\mathbf{p}') \left[i\partial\!\!\!/_{x_1} - m \right]$$

$$\times e^{ik'\cdot x_2} \varepsilon^{\lambda'}_{\mu}(\mathbf{k}') \left[-\partial^2_{x_2} \right]$$

$$\times \langle 0| \mathrm{T} \left[\Psi(x_1) A^\mu(x_2) \bar{\Psi}(y_1) A^\nu(y_2) \right] |0\rangle$$

$$\times \left[-i\overleftarrow{\partial\!\!\!/_{y_1}} - m \right] u_{s_1}(\mathbf{p}) e^{-ip\cdot y_1}$$

$$\times \left[-\overleftarrow{\partial^2_{y_2}} \right] \varepsilon^{\lambda *}_{\nu}(\mathbf{k}) e^{-ik\cdot y_2}. \tag{1.110}$$

This called **LSZ reduction formula for Compton scattering**, is the relation of time ordered product of fields and S-matrix.

1.3.5 Scattering Matrices for Compton scattering

From the result of LSZ reduction formula for Compton scattering, calculation of 4-point correlation function $\langle 0|\mathrm{T}\left[\Psi(x_1)A^\mu(x_2)\bar{\Psi}(y_1)A^\nu(y_2)\right]|0\rangle$ is required. So, let us consider $V-P=1$ and $2N-P=2$ on path integral (1.85):

1. $P=2, V=3, N=2.$

Then, path integral expands by

$$
\begin{aligned}
Z^{FP}_{(2,3)}[J,\eta,\bar{\eta}] &= \frac{1}{2!}\left[ie\int d^4x\ \left(\frac{1}{i}\frac{\delta}{\delta J^\mu(x)}\right)\left(-\frac{1}{i}\frac{\delta}{\delta\eta_\alpha(x)}\right)(\gamma^\mu)_{\alpha\beta}\left(\frac{1}{i}\frac{\delta}{\delta\bar{\eta}_\beta(x)}\right)\right]^2 \\
&\quad\times\frac{1}{3!}\left[i\int d^4xd^4y\ \bar{\eta}(x)S(x-y)\eta(y)\right]^3 \\
&\quad\times\frac{1}{2!}\left[\frac{i}{2}\int d^4xd^4y\ J^\mu(x)\Delta_{\mu\nu}(x-y)J^\nu(y)\right]^2 \\
&= -\frac{ie^2}{2!}\int d^4x_1d^4x_2d^4wd^4zd^4y_1d^4y_2 \\
&\quad\bar{\eta}(x_1)S(x_1-w)\gamma^\mu\Delta_{\mu\rho}(w-y_1)J^\rho(y_1) \\
&\quad\times S(w-z)J^\sigma(x_2)\Delta_{\nu\sigma}(x_2-z)\gamma^\nu S(z-y_2)\eta(y_2) \\
&\quad+(x_2\leftrightarrow y_1)+O(J).
\end{aligned}
\tag{1.111}
$$

So, the 4-point correlation function goes to($S=3!\times 2!\times 2!$):

$$
\begin{aligned}
\langle 0|\mathrm{T}\left[\cdots\right]|0\rangle &= \left(\frac{1}{i}\frac{\delta}{\delta\bar{\eta}_\alpha(x_1)}\right)\left(\frac{1}{i}\frac{\delta}{\delta J_\mu(x_2)}\right)\left(-\frac{1}{i}\frac{\delta}{\delta\eta_\alpha(y_1)}\right) \\
&\quad\times\left(\frac{1}{i}\frac{\delta}{\delta J_\mu(y_2)}\right)Z^{FP}_{(2,3)}[J,\eta,\bar{\eta}]\bigg|_{J,\eta,\bar{\eta}=0} \\
&= \frac{e^2}{i}\int d^4wd^4z \\
&\quad S(x_1-w)\gamma_\rho\Delta^{\mu\rho}(w-x_2)S(w-z)\Delta^{\nu\sigma}(y_2-z)\gamma_\sigma S(z-y_1) \\
&\quad+(x_2\leftrightarrow y_2).
\end{aligned}
\tag{1.112}
$$

Now we shall directly calculate the S-matrix from the results of calculation for 4-point correlation function (1.112) and LSZ reduction formula (1.110):

$$
\begin{aligned}
S_{fi}^{(1)} = \frac{e^2}{i^5} \int & d^4x_1 d^4x_2 d^4y_1 d^4y_2 d^4w d^4z \\
& \times e^{ip'\cdot x_1} \bar{u}_{s_1'}(\mathbf{p}') \left[i\partial\!\!\!/_{x_1} - m \right] e^{ik'\cdot x_2} \varepsilon_\mu^{\lambda'}(\mathbf{k}') \left[-\partial_{x_2}^2 \right] \\
& \times S(x_1 - w)\gamma_\rho \Delta^{\mu\rho}(w - x_2) S(w - z) \Delta^{\nu\sigma}(y_2 - z)\gamma_\sigma S(z - y_1) \\
& \times \left[-i\overleftarrow{\partial\!\!\!/}_{y_1} - m \right] u_{s_1}(\mathbf{p}) e^{-ip\cdot y_1} \left[-\overleftarrow{\partial}_{y_2}^2 \right] \varepsilon_\nu^{\lambda^*}(\mathbf{k}) e^{-ik\cdot y_2} \\
= \frac{e^2}{i} \int & d^4x_1 d^4x_2 d^4y_1 d^4y_2 d^4w d^4z \; e^{i(p\cdot y_1 + k\cdot y_2 - p'\cdot x_1 - k'\cdot x_2)} \\
& \times \delta^4(x_1 - w)\delta^4(x_2 - w)\delta^4(y_2 - z)\delta^4(y_1 - z) G^{\mu\rho} G^{\nu\sigma} \\
& \times \varepsilon_\mu^{\lambda'}(\mathbf{k}') \bar{u}_{s_1'}(\mathbf{p}') \gamma_\rho \left[\int \frac{d^4\bar{k}}{(2\pi)^4} \frac{e^{i\bar{k}\cdot(w-z)}}{-\bar{k}\!\!\!/ - m + i\epsilon} \right] \gamma_\sigma u_{s_1}(\mathbf{p}) \varepsilon_\nu^{\lambda^*}(\mathbf{k}) \\
= \frac{e^2}{i} \int & \frac{d^4\bar{k}}{(2\pi)^4} d^4w d^4z \; e^{i(\bar{k} - p' - k')w} e^{i(p + k - \bar{k})z} G^{\mu\rho} G^{\nu\sigma} \\
& \times \varepsilon_\mu^{\lambda'}(\mathbf{k}') \bar{u}_{s_1'}(\mathbf{p}') \gamma_\rho \frac{1}{-\bar{k}\!\!\!/ - m + i\epsilon} \gamma_\sigma u_{s_1}(\mathbf{p}) \varepsilon_\nu^{\lambda^*}(\mathbf{k}) \\
= -(2\pi)^4 ie^2 \int & d^4\bar{k} \; \delta^4(\bar{k} - p' - k')\delta^4(\bar{k} - p - k) G^{\mu\rho} G^{\nu\sigma} \\
& \times \varepsilon_\mu^{\lambda'}(\mathbf{k}') \bar{u}_{s_1'}(\mathbf{p}') \gamma_\rho \frac{1}{-\bar{k}\!\!\!/ - m + i\epsilon} \gamma_\sigma u_{s_1}(\mathbf{p}) \varepsilon_\nu^{\lambda^*}(\mathbf{k}) \\
= -(2\pi)^4 & ie^2 \delta^4(p + k - p' - k') G^{\mu\rho} G^{\nu\sigma} \\
& \times \varepsilon_\mu^{\lambda'}(\mathbf{k}') \bar{u}_{s_1'}(\mathbf{p}') \gamma_\rho \frac{1}{-(p\!\!\!/' + k\!\!\!/') - m + i\epsilon} \gamma_\sigma u_{s_1}(\mathbf{p}) \varepsilon_\nu^{\lambda^*}(\mathbf{k}).
\end{aligned}
\tag{1.113}
$$

On **t' Hooft-Feynman Gauge** fixing, this can be written more simply:

$$S_{fi}^{(1)} = -(2\pi)^4 ie^2 \delta^4(p + k - p' - k') \\ \times \varepsilon_\mu^{\lambda'}(\mathbf{k}') \bar{u}_{s_1'}(\mathbf{p}') \gamma^\mu \frac{1}{-(\not{p} + \not{k}) - m + i\epsilon} \gamma^\nu u_{s_1}(\mathbf{p}) \varepsilon_\nu^{\lambda^*}(\mathbf{k}). \tag{1.114}$$

Also, we separated the result of time ordered products to each terms which are connected by summation. Similarly,

$$
\begin{aligned}
S_{fi}^{(2)} = \frac{e^2}{i^5} \int & d^4x_1 d^4x_2 d^4y_1 d^4y_2 d^4w d^4z \\
& \times e^{ip'\cdot x_1} \bar{u}_{s_1'}(\mathbf{p}') \left[i\partial\!\!\!/_{x_1} - m \right] e^{ik'\cdot x_2} \varepsilon_\mu^{\lambda'}(\mathbf{k}') \left[-\partial_{x_2}^2 \right] \\
& \times S(x_1 - w)\gamma_\rho \Delta^{\mu\rho}(w - y_2) S(w - z) \Delta^{\nu\sigma}(x_2 - z)\gamma_\sigma S(z - y_1) \\
& \times \left[-i\overleftarrow{\partial\!\!\!/_{y_1}} - m \right] u_{s_1}(\mathbf{p}) e^{-ip\cdot y_1} \left[-\overleftarrow{\partial_{y_2}^2} \right] \varepsilon_\nu^{\lambda^*}(\mathbf{k}) e^{-ik\cdot y_2} \\
= \frac{e^2}{i} \int & d^4x_1 d^4x_2 d^4y_1 d^4y_2 d^4w d^4z \; e^{i(p\cdot y_1 + k\cdot y_2 - p'\cdot x_1 - k'\cdot x_2)} \\
& \times \delta^4(x_1 - w)\delta^4(y_2 - w)\delta^4(x_2 - z)\delta^4(y_1 - z) G^{\mu\rho} G^{\nu\sigma} \\
& \times \varepsilon_\mu^{\lambda'}(\mathbf{k}) \bar{u}_{s_1'}(\mathbf{p}')\gamma_\rho \left[\int \frac{d^4\bar{k}}{(2\pi)^4} \frac{e^{i\bar{k}\cdot(w-z)}}{-\bar{k}\!\!\!/ - m + i\epsilon} \right] \gamma_\sigma u_{s_1}(\mathbf{p}) \varepsilon_\nu^{\lambda^*}(\mathbf{k}') \\
= \frac{e^2}{i} \int & \frac{d^4\bar{k}}{(2\pi)^4} d^4w d^4z \; e^{i(\bar{k} - p' + k)w} e^{i(p - k' - \bar{k})z} G^{\mu\rho} G^{\nu\sigma} \\
& \times \varepsilon_\mu^{\lambda'}(\mathbf{k}) \bar{u}_{s_1'}(\mathbf{p}')\gamma_\rho \frac{1}{-\bar{k}\!\!\!/ - m + i\epsilon} \gamma_\sigma u_{s_1}(\mathbf{p}) \varepsilon_\nu^{\lambda^*}(\mathbf{k}') \\
= -(2\pi)^4 i e^2 \int & d^4\bar{k} \; \delta^4(\bar{k} - p' + k)\delta^4(\bar{k} - p + k') G^{\mu\rho} G^{\nu\sigma} \\
& \times \varepsilon_\mu^{\lambda'}(\mathbf{k}) \bar{u}_{s_1'}(\mathbf{p}')\gamma_\rho \frac{1}{-\bar{k}\!\!\!/ - m + i\epsilon} \gamma_\sigma u_{s_1}(\mathbf{p}) \varepsilon_\nu^{\lambda^*}(\mathbf{k}') \\
= -(2\pi)^4 i e^2 & \delta^4(p + k - p' - k') G^{\mu\rho} G^{\nu\sigma} \\
& \times \varepsilon_\mu^{\lambda'}(\mathbf{k}) \bar{u}_{s_1'}(\mathbf{p}')\gamma_\rho \frac{1}{-(p\!\!\!/' - k\!\!\!/) - m + i\epsilon} \gamma_\sigma u_{s_1}(\mathbf{p}) \varepsilon_\nu^{\lambda^*}(\mathbf{k}').
\end{aligned}
\tag{1.115}
$$

On **t' Hooft-Feynman Gauge** fixing, this can be written more simply:

$$S_{fi}^{(2)} = -(2\pi)^4 ie^2 \delta^4(p+k-p'-k') \times \varepsilon_\mu^{\lambda'}(\mathbf{k})\bar{u}_{s_1'}(\mathbf{p}')\gamma^\mu \frac{1}{-(\not{p}-\not{k}')-m+i\epsilon}\gamma^\nu u_{s_1}(\mathbf{p})\varepsilon_\nu^{\lambda^*}(\mathbf{k}'). \tag{1.116}$$

So, he S-matrix now goes to

$$S_{fi} = -(2\pi)^4 ie^2 \delta^4(p+k-p'-k') G^{\mu\rho}G^{\nu\sigma} \times \Bigg[\varepsilon_\mu^{\lambda'}(\mathbf{k}')\bar{u}_{s_1'}(\mathbf{p}')\gamma_\rho \frac{1}{-(\not{p}+\not{k})-m+i\epsilon}\gamma_\sigma u_{s_1}(\mathbf{p})\varepsilon_\nu^{\lambda^*}(\mathbf{k}) + \varepsilon_\mu^{\lambda'}(\mathbf{k})\bar{u}_{s_1'}(\mathbf{p}')\gamma_\rho \frac{1}{-(\not{p}-\not{k}')-m+i\epsilon}\gamma_\sigma u_{s_1}(\mathbf{p})\varepsilon_\nu^{\lambda^*}(\mathbf{k}')\Bigg]. \tag{1.117}$$

Chapter 2

Yang-Mills Gauge Theories

2.1 Extension of Gauge Theories

In this section, we shall construct a non-abelian gauge theory, also which called Yang-Mills gauge theories. Let us consider a extension of U(1) gauge field which we dealt on previous sections:

- Global U(1) symmetry for complex scalar field gives

$$\phi(x) \ \rightarrow \ e^{ie\theta}\phi(x). \tag{2.1}$$

- Introducing gauge field $A_\mu(x)$:
  one can extend U(1) symmetry to position dependent U(1) symmetry

$$\phi(x) \ \rightarrow \ e^{ie\theta(x)}\phi(x), \tag{2.2}$$

  by introducing covariant derivative $D_\mu := \partial_\mu - ieA_\mu(x)$. The local gauge symmetry yields matter fields(scalar fields, Dirac fields, etc.) interacts with gauge field.

So, we shall consider a extension of gauge symmetry of non-abelian Lie groups of SO(N), SU(N). Now then, our gauge theories also can be extended by

$$\phi_i(x) = U_{ij}\phi_j(x), \qquad U_{ij} \in \mathrm{SO}(N), \mathrm{SU}(N) \tag{2.3}$$

for global gauge symmetry. By introducing gauge field, one can extend the system to position dependent SO(N), SU(N) system:

$$\phi_i(x) = U_{ij}(x)\phi_j(x), \qquad U_{ij}(x) \in \mathrm{SO}(N), \mathrm{SU}(N) \tag{2.4}$$

.

2.1.1 Group Representation

Let us consider Lie group $G(\mathrm{SO}(N), \mathrm{SU}(N))$ of $n = \dim G$ for each Lie groups we extended:

$$\mathrm{SO}(N): \ n = \frac{N(N+1)}{2}, \qquad \mathrm{SU}(N): \ n = N^2 - 1. \tag{2.5}$$

The element of Lie group G now selected by

$$g = \exp\left(i\sum_{a=1}^{n}\theta^a T^a\right) \tag{2.6}$$

where we denote T^a as a Lie generator[1], θ^a as proper parametre. Since product of group element, we have

$$\begin{aligned} g\cdot h &= \exp\left(i\sum_{a=1}^{n}\theta^a T^a\right)\exp\left(i\sum_{b=1}^{n}\eta^b T^b\right) \\ &= \exp\left(i\sum_{a,b=1}^{n}(\theta^a+\eta^a)T^a + \frac{\theta^a\eta^b}{2}[T^a,T^b] + \mathcal{O}(\theta,\eta)\right) \in G \end{aligned} \tag{2.7}$$

for $g, h \in G$. Here we used *Campbell-Backer-Hausdorff formula* on second line. From the calculation of (2.7), now we finally derives a relation of Lie bracket with Lie generators, which is called Lie algebra:

Theorem 2.1.1 (Lie Algebra)

$$[T^a, T^b] = i\sum_{c=1}^{n} f^{ab}{}_c T^c, \qquad f^{ab}{}_c = -f^{ba}{}_c. \tag{2.8}$$

Also, here we put a normalization condition to Lie generator T^as by

$$\mathrm{tr}(T^aT^b) = \frac{\delta^{ab}}{2}. \tag{2.9}$$

Here we shall derived the upper normalization condition by considering the term of

$$\mathrm{tr}\left(T^a[T^a,T^b]\right) = \underbrace{\mathrm{tr}\left(T^aT^aT^b\right) - \mathrm{tr}\left(T^aT^bT^a\right)}_{=0} = f^{abc}\mathrm{tr}\left(T^aT^c\right), \tag{2.10}$$

which leads $\mathrm{tr}(T^aT^c) \sim \delta^{ac}$.

[1] So, it must be selected by hermitian $N\times N$ matrix.

2.1.2 Extension of Gauge Field

Next, we shall extend the gauge symmetry to non-abelian groups of SO(N) or SU(N). Let us extend Dirac field $\Psi(x)$ with N-fields[2]:

$$\Psi(x) = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_N \end{pmatrix}, \qquad \mathcal{L}_f^a = (\bar{\psi}_a)_\alpha (i\partial\!\!\!/^{\alpha\beta} - m)(\psi_a)_\beta, \tag{2.11}$$

with spinor index α, β and Dirac field index a. So, the Lagrangian density of our extended Dirac field now denoted by

$$\mathcal{L}_f = \bar{\Psi}_{a\alpha} \left[i\partial_\mu (\gamma^\mu)^{\alpha\beta} - m\delta^{\alpha\beta} \right] \Psi^a{}_\beta. \tag{2.12}$$

Remind that each index of spinor α, β and Dirac fields a are separated. Now then, our gauge transformation of matter field, in Dirac fermion case, one can extend:

Definition 2.1.1 (non-Abelian Gauge transf. of Fermion)
For a global gauge transformation,

$$\Psi^a(x) \;\to\; U^a{}_b \Psi^b(x), \qquad \bar{\Psi}_a(x) \;\to\; \bar{\Psi}_b(x) \left(U^{-1}\right)^b{}_a. \tag{2.13}$$

Introducing gauge field $A_\mu(x)$, one can extend $G(\mathrm{SO}(N), \mathrm{SU}(N))$ symmetry to position dependent $G(\mathrm{SO}(N), \mathrm{SU}(N))$ symmetry:

$$\Psi^a(x) \;\to\; U^a{}_b(x) \Psi^b(x), \qquad \bar{\Psi}_a(x) \;\to\; \bar{\Psi}_b(x) \left(U^{-1}\right)^b{}_a(x). \tag{2.14}$$

Exactly, the representation of non-abelian gauge transformation on Lie group G takes:

$$U^a{}_b(x) = \exp\left(ig \sum_{a=1}^{N} \theta^a(x) T^a \right)^a{}_b \in G. \tag{2.15}$$

Now we shall examine local gauge invariance of Dirac fields, so let us consider a gauge transformation on free Dirac field (2.12). One can see mass term is gauge invariant:

$$\bar{\Psi}_{a\alpha} \delta^{\alpha\beta} \Psi^a{}_\beta \;\to \bar{\Psi}_{b\alpha} \delta^{\alpha\beta} \left(U^{-1}\right)^b{}_a(x) U^a{}_c(x) \Psi^c{}_\beta = \bar{\Psi}_{b\alpha} \delta^{\alpha\beta} \Psi^b{}_\beta. \tag{2.16}$$

[2]Also, see chapter 5 of 九後太一郎, **ゲージ場の量子論 I** (1989). Remind that the freedom of each Dirac fields N is come from the scale of our taken gauge symmetry SO(N), SU(N).

However, kinetic term is not invariant:

$$
\begin{aligned}
\bar{\Psi}_{a\alpha}\not{\partial}^{\alpha\beta}\Psi^{a}{}_{\beta} \;\to\; & \bar{\Psi}_{b\alpha}\left(U^{-1}\right)^{b}{}_{a}(x)\not{\partial}^{\alpha\beta}\left[U^{a}{}_{c}(x)\Psi^{c}{}_{\beta}\right] \\
&= \bar{\Psi}_{b\alpha}\not{\partial}^{\alpha\beta}\Psi^{b}{}_{\beta} + \bar{\Psi}_{b\alpha}\left(U^{-1}\right)^{b}{}_{a}(x)\left[\not{\partial}^{\alpha\beta}U^{a}{}_{c}(x)\right]\Psi^{c}{}_{\beta}.
\end{aligned}
\tag{2.17}
$$

For a local gauge invariance, the extra piece must be canceled. In order to cancel the extra piece, one introduce the covariant derivative D_μ:

$$
a^{\mu}D_{\mu}\Psi := \lim_{h\to 0}\frac{1}{h}\left[\Psi(x+ha) - \Psi_{\parallel}(x)\right] \tag{2.18}
$$

where we take the parallel transport of Dirac field on position x by

$$
\Psi_{\parallel}(x) = R(x+ha, x)\Psi(x). \tag{2.19}
$$

Then, we shall consider a gauge transformation of covariant derivative of Dirac field (2.18), in order to cancel the extra piece on gauge transformation of kinetic term (2.17), one can take its gauge transformation by

$$
D_{\mu}\Psi(x) \;\to U(x)D_{\mu}\Psi(x), \qquad R(y,x) \;\to\; U(y)R(y,x)U^{-1}(x). \tag{2.20}
$$

Geometrically[3], this yields:

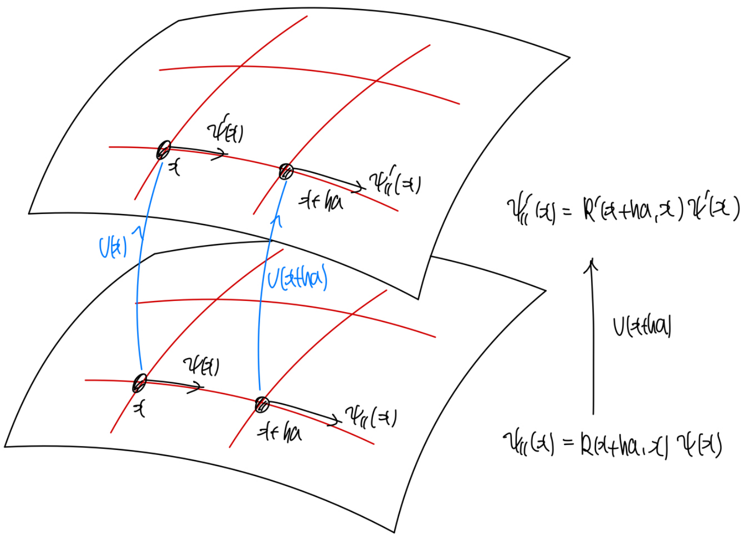


Figure 2.1: Geometry of Gauge transformation.

[3]Suggestion from *Haruki Tomita*, Dept. of Phys., University of Tokyo(B4).

Geometrically, the gauge transformation is equivalent to shifting of our surface which is selected by gauge fixing[4], the parallel transport can be taken $\Psi_{\parallel}(x) = R(x+ha,x)\Psi(x)$ for lower surface and $\Psi_{\parallel}(x) = R(x+ha,x)\Psi(x)$ for shifted upper surface. Returning to gauge transformation of parallel transport function $R(y,x)$, direct route of $\Psi_{\parallel}(x) \ \to \ \Psi'_{\parallel}(x)$ now gives:

$$\begin{aligned}\Psi'_{\parallel}(x) &= R'(x+ha,x)\Psi'(x) = U(x+ha)R(x+ha,x)\Psi(x) \\ &= U(x+ha)R(x+ha,x)U^{-1}(x)\Psi'(x),\end{aligned} \tag{2.21}$$

which directly yields gauge transformation of (2.20). Reminding $U(x) \in G$, one can take $R(y,x) \in G$ for well-defined covariant derivative. Then, we impose the parallel transport function $R(x+h,x)$ by

$$\begin{aligned}R(x+ha,x) &= \exp\left(igha^{\mu}\sum_{a=1}^{N} A_{\mu}^{a}(x)T^{a}\right) \\ &= \mathbf{1} + igha^{\mu}\sum_{a=1}^{N} A_{\mu}^{a}(x)T^{a} + \mathcal{O}(ha).\end{aligned} \tag{2.22}$$

Now the field $\sum_{a=1}^{N} A_{\mu}^{a}(x)T^{a}$ on expansion of (2.22) is the extended gauge field, one is a element of group G. Feeding the expansion of (2.22) to (2.18), finally we determine:

Theorem 2.1.2 (Representation of Covariant Derivative)

$$\begin{aligned}a^{\mu}D_{\mu}\Psi &= \lim_{h\to 0}\frac{1}{h}\left(\Psi(x+ha) - \Psi(x) - igha^{\mu}\sum_{a=1}^{N} A_{\mu}^{a}(x)T^{a}\Psi + \mathcal{O}(ha)\right) \\ &= a^{\mu}\left(\partial_{\mu} - ig\sum_{a=1}^{N} A_{\mu}^{a}(x)T^{a}\right)\Psi.\end{aligned} \tag{2.23}$$

So, our covariant derivative now takes

$$D_{\mu} = \partial_{\mu} - ig\sum_{a=1}^{N} A_{\mu}^{a}(x)T^{a}. \tag{2.24}$$

[4]So, geometrically, gauge fixing yields the selection of surface for our gauge theory.

Next, we shall find the gauge transformation of gauge field $A_\mu := A^a_\mu(x)T_a$. Feeding (2.22) to (2.21), gauge transformation of $R(x) \rightarrow R'(x)$ gives

$$\begin{aligned}
R'(x+ha,x) &= U(x+ha)R(x+ha,x)U^{-1}(x) \\
&= \mathbf{1} + igha^\mu U(x)\left(\sum_{a=1}^{N} A^a_\mu(x)T^a - \frac{i}{g}\overleftarrow{\partial_\mu}\right)U^{-1}(x) + \mathcal{O}'(ha) \\
&= \mathbf{1} + igha^\mu A'^a_\mu(x)T_a + \mathcal{O}(ha),
\end{aligned} \tag{2.25}$$

where we expand

$$U(x+ha) = \left(\mathbf{1} + ha^\mu \frac{\partial}{\partial x^\mu} + \mathcal{O}(ha)\right)U(x). \tag{2.26}$$

Our gauge transform of parallel transport function $R(x+ha,x)$ now directly gives the gauge transformation of gauge field $A_\mu(x)$:

Theorem 2.1.3 (Gauge transf. of Gauge Field)

$$A_\mu(x) \ \rightarrow \ U(x)A_\mu(x)U^{-1}(x) + \frac{i}{g}U(x)\partial_\mu U^{-1}(x). \tag{2.27}$$

This approach to extension of gauge theories gives more reasonable explanations for why also gauge field extended to $A_\mu(x) \in G$. Furthermore, the matter field(Dirac fields) and gauge field must interact each other for the local gauge invariance, one can have its gauge transformation as (2.14) for matter field and (2.27) for gauge field.

The Field Strength

Now we shall construct the Lagrangian for our extended non-abelian gauge theories on Lie group G. First, the gauge transformation of gauge field exactly[5] goes to

$$A_\mu(x) \ \rightarrow \ A_\mu(x) + ig[A_\mu(x),\theta(x)] - \partial_\mu\theta(x), \tag{2.28}$$

[5] This calculation is supported by *Campvell-Backer-Hausdorff formula.*

or equivalently,

$$\begin{aligned} A^a_\mu(x)T_a &\to A^a_\mu(x)T_a + igA^a_\mu(x)\theta^b(x)[T_a, T_b] - \partial_\mu\theta^a(x)T_a \\ &= A^a_\mu(x)T_a - gA^a_\mu(x)\theta^b(x){f_{ab}}^c T_c - \partial_\mu\theta^a T_a \\ &= \left[A^a_\mu(x) - \left(\delta^a_c\partial_\mu + gA^b_\mu(x){f_{bc}}^a\right)\theta^c\right]T_a, \end{aligned} \tag{2.29}$$

that is also showing the gauge transformation of components of $A^a_\mu(x)$ by

$$A^a_\mu(x) \;\to\; A^a_\mu(x) - {(D_\mu)^a}_c\theta^c(x), \qquad {(D_\mu)^a}_c := \delta^a_c\partial_\mu + gA^b_\mu(x){f_{bc}}^a. \tag{2.30}$$

Next, we shall consider a *field strength* which is defined as a curvature tensor on fixed gauge(Geometrically, a fixed surface):

Definition 2.1.2 (Field Strength for non-Abelian Gauge Field)

$$F_{\mu\nu} := \frac{i}{g}[D_\mu, D_\nu] = \partial_{[\mu}A_{\nu]} - ig[A_\mu(x), A_\nu(x)]. \tag{2.31}$$

So, our field strength $F_{\mu\nu}$ have gauge transformation of

$$F_{\mu\nu} \;\to U(x)F_{\mu\nu}U^{-1}(x) \tag{2.32}$$

from (2.31). That is also supported by geometry of gauge transformations:

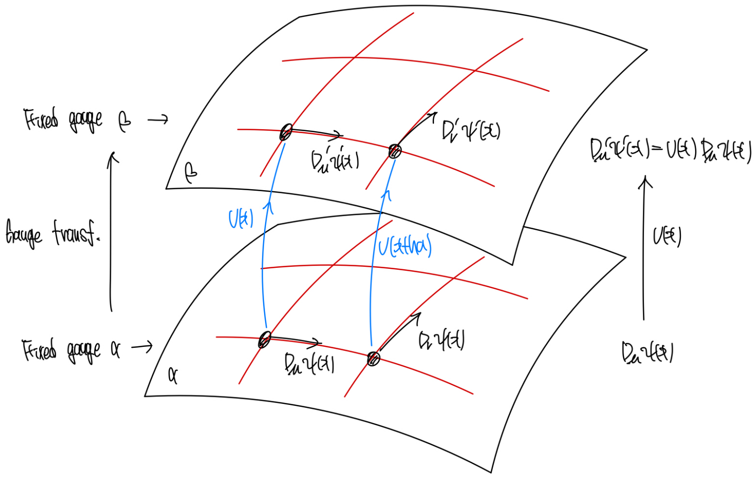


Figure 2.2: Geometry of Gauge transformation.

Geometrical interpretation of gauge transformation yields our gauge transformation runs to

$$D'_\mu \Psi'(x) \ \rightarrow \ U(x) D_\mu \Psi(x) = U(x) D_\mu U^{-1}(x) \Psi'(x), \tag{2.33}$$

that is action of shifting the surface selected by gauge fixing($\alpha \rightarrow \beta$). Now then, derived relation of (2.33) holds gauge transformation of covariant derivative $D_\mu \ \rightarrow \ U(x) D_\mu U^{-1}(x)$, which directly leads (2.32).

Lagrangian for non-Abelian Gauge Theories

Finally, we shall determine the Lagrangian[6] for non-abelian gauge theories under the supposing of Lorentz invariance and gauge invariance[7]. Since the term of $\mathrm{tr}(F^{\mu\nu}F_{\mu\nu})$ is Lorentz invariant and gauge invariant:

$$\mathrm{tr}\left(F^{\mu\nu}F_{\mu\nu}\right) \ \rightarrow \ \mathrm{tr}\left(U(x)F^{\mu\nu}U^{-1}(x)U(x)F_{\mu\nu}U^{-1}(x)\right) = \mathrm{tr}\left(F^{\mu\nu}F_{\mu\nu}\right), \tag{2.34}$$

one can determine the YM Lagrangian for non-abelian gauge theories:

Definition 2.1.3 (Yang-Mills Lagrangian density)

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{2}\mathrm{tr}\left(F^{\mu\nu}F_{\mu\nu}\right) \underset{(2.31)}{=} -\frac{1}{2}F^{a\mu\nu}F^{b}_{\mu\nu}\mathrm{tr}\left(T_a T_b\right) = -\frac{1}{4}F^{a\mu\nu}F_{a\mu\nu}. \tag{2.35}$$

Similarly, the action for non-abelian gauge theories can be extended to:

Definition 2.1.4 (Yang-Mills Action)

$$S_{\mathrm{YM}} = \int d^4x \ -\frac{1}{2}\mathrm{tr}\left(F^{\mu\nu}F_{\mu\nu}\right). \tag{2.36}$$

[6]That is also called '*Yang-Mills* Lagrangian'.
[7]More strictly, we need renormalizability.

2.2 Yang-Mills Path integrals

In this section, we shall construct the path integral representation for non-abelian gauge theories(also known as Yang-Mills gauge theories). For a exact calculation, we shall take R_ξ-gauge fixing on our non-abelian gauge field, using *Faddeev-Popov* determinant and *Faddeev-Popov ghost.* On this gauge fixing, one can perform the path integral for YM field, with external sources:

Definition 2.2.1 (Path integral for Yang-Mills Field)

$$Z[J^a] \sim \int \mathcal{D}A_\mu \mathcal{D}\bar{c}\mathcal{D}c \ \exp\left(i \int d^4x \ \mathcal{L}_{\text{YM}} + \mathcal{L}_{gf} + \mathcal{L}_{gh}^{FP} + \Phi \cdot J \right), \tag{2.37}$$

where we put

$$\mathcal{L}_{gf} = -\frac{1}{2\xi}[\partial^\mu A^e_\mu(x)]^2, \quad \mathcal{L}_{gh}^{FP} = \bar{c}^a(x)\partial^\mu (D_\mu)_{ab} c^b(x), \tag{2.38}$$

and external sources for generating functional:

$$\Phi \cdot J = \sum_{a=1}^{N} A^a_\mu J^\mu_a + \bar{c}^a \Xi^{\bar{c}}_a + \bar{\Xi}^c_a c^a \tag{2.39}$$

Remind that $\bar{c}_a(x)$ and $c^b(x)$ are grassmann odd. To extend our scattering theory of QED to YM theory, here also the time-ordered products with *colour freedom index* of each field(exactly, $\langle 0|\mathrm{T}[A^a_\mu(x)A^b_\nu(y)]|0\rangle$, etc.) can be importantly involved in scattering matrices, similar to theory of QED. So, we shall look for $\langle 0|\mathrm{T}[A^a_\mu(x)A^b_\nu(y)]|0\rangle$ and $\langle 0|\mathrm{T}[\Psi^a(x)\bar{\Psi}^b(y)]|0\rangle$, for this task, let us expand each of Lagrangian density by:

$$\begin{aligned}
\mathcal{L}_{\text{YM}} &= -\frac{1}{2}\mathrm{tr}\left(F^{\mu\nu}F_{\mu\nu}\right) \\
&= -\frac{1}{4}\left(\partial^{[\mu}A^{e\nu]} + g f_{ab}{}^e A^{a\mu}A^{b\nu}\right)\left(\partial_{[\mu}A_{e\nu]} + g f_{cde} A^c_\mu A^d_\nu\right) \\
&= -\frac{1}{2}\partial^\mu A^{e\nu}\partial_\mu A_{e\nu} + \frac{1}{2}\partial^\nu A^{e\mu}\partial_\mu A_{e\nu} \\
&\qquad - g f_{abe} A^{a\mu}A^{b\nu}\partial_\mu A^e_\nu - \frac{1}{4} g^2 f_{ab}{}^e f_{cde} A^{a\mu}A^{b\nu}A^c_\mu A^d_\nu,
\end{aligned} \tag{2.40}$$

Similarly, we have

$$\mathcal{L}_{gf} = \frac{1}{2\xi} A^e_\mu \partial^\mu \partial^\nu A_{e\nu}, \qquad \mathcal{L}^{FP}_{gh} = \bar{c}_a \partial^2 c^a + \underbrace{g A^a_\mu (\partial^\mu \bar{c}^b) f_{abc} c^c}_{\text{Gauge-ghost interaction}} \tag{2.41}$$

Now we are ready for running over exact path integral of (2.37).

2.2.1 Propagators for Gluons and Vertex Functions

Our path integral for YM gauge theories(also, we ignore any other field like fermions) on (2.37) is not easy to take direct calculation. We see that introducing of a new generating functionals of $Z_0[J]$ which eliminates[8] the interaction terms of fields sometimes gives useful trick for calculation of our path integrals involving interaction terms, on QED section. Similarly, here one can also take a new generating functionals of $Z_0[J^a]$, so we have:

Definition 2.2.2 (Generating Functionals)
First, we shall introduce:

$$\begin{aligned} Z_0[J^a] \sim & \int \mathcal{D}A_\mu \mathcal{D}\bar{c}\mathcal{D}c \\ & \times \exp\left(i \int d^4x \; \tilde{\mathcal{L}}_{\text{YM}} + \mathcal{L}_{gf} + \tilde{\mathcal{L}}^{FP}_{gh} + \Phi \cdot J \right), \end{aligned} \tag{2.42}$$

where we denote

$$\tilde{\mathcal{L}}_{\text{YM}} = -\frac{1}{2} \partial^{[\mu} A^{e\nu]}(x) \partial_\mu A_{e\nu}(x), \qquad \tilde{\mathcal{L}}^{FP}_{gh} = \bar{c}_a(x) \partial^2 c^a(x). \tag{2.43}$$

Our path integral of (2.37) now goes to

$$\begin{aligned} Z[J^a] = & \exp\left[i \int d^4x \; -g f_{abe} \left(\frac{1}{i} \frac{\delta}{\delta J_{a\mu}(x)} \right) \left(\frac{1}{i} \frac{\delta}{\delta J_{b\nu}(x)} \right) \partial_\mu \left(\frac{1}{i} \frac{\delta}{\delta J^\nu_e(x)} \right) \right] \\ & \times \exp\left[i \int d^4x \; -\frac{1}{4} g^2 f_{ab}{}^e f_{cde} \left(\frac{1}{i} \frac{\delta}{\delta J_{a\mu}(x)} \right) \cdots \left(\frac{1}{i} \frac{\delta}{\delta J^\nu_d(x)} \right) \right] \\ & \times \exp\left[i \int d^4x \; \left(\frac{1}{i} \frac{\delta}{\delta J^\mu_a(x)} \right) \left(i\partial^\mu \frac{\delta}{\delta \Xi^{\bar{c}}_b(x)} \right) g f_{abc} \left(\frac{1}{i} \frac{\delta}{\delta \bar{\Xi}^c_c(x)} \right) \right] Z_0[J^a]. \end{aligned} \tag{2.44}$$

[8] Also, it is design to excluding any other interaction of gauge-gauge field.

Now then, our propagator of gauge field $\mathcal{P} = \langle 0|\mathrm{T}[A^a_\mu(x)A^b_\nu(y)]|0\rangle$ can be conveniently expands by

$$
\begin{aligned}
\mathcal{P} &= \left(\frac{1}{i}\frac{\delta}{\delta J^\mu_a(x)}\right)\left(\frac{1}{i}\frac{\delta}{\delta J^\nu_b(y)}\right) Z[J^a]\Bigg|_{J^a=0} \\
&= \left(\frac{1}{i}\frac{\delta}{\delta J^\mu_a(x)}\right)\left(\frac{1}{i}\frac{\delta}{\delta J^\nu_b(y)}\right) \\
&\quad \times \sum_{P=0}^{\infty}\frac{1}{P!}\left[i\int d^4x' \; -gf_{abe}\left(\frac{1}{i}\frac{\delta}{\delta J_{a\rho}(x')}\right)\left(\frac{1}{i}\frac{\delta}{\delta J_{b\sigma}(x')}\right)\partial_\rho\left(\frac{1}{i}\frac{\delta}{\delta J^\sigma_e(x')}\right)\right]^P \\
&\quad \times \sum_{V=0}^{\infty}\frac{1}{V!}\left[i\int d^4x' \; -\frac{1}{4}g^2 f_{ab}{}^{e}f_{cde}\left(\frac{1}{i}\frac{\delta}{\delta J_{a\rho}(x')}\right)\cdots\left(\frac{1}{i}\frac{\delta}{\delta J^\sigma_d(x')}\right)\right]^V \\
&\quad \times \sum_{N=0}^{\infty}\frac{1}{N!}\left[i\int d^4x' \; \left(\frac{1}{i}\frac{\delta}{\delta J^\rho_a(x')}\right)\left(i\partial^\rho\frac{\delta}{\delta \Xi^{\bar{c}}_b(x')}\right)gf_{abc}\left(\frac{1}{i}\frac{\delta}{\delta \bar{\Xi}^c_c(x')}\right)\right]^N \\
&\quad \times Z_0[J^a]\Bigg|_{J^a=0}.
\end{aligned}
\tag{2.45}
$$

Also, our new generating functional of $Z_0[J^a]$ can be calculated directly, which gives perturbational solution for propagator of gauge field. Thus,

$$
\begin{aligned}
Z_0[J^a] &= \int \mathcal{D}A_\mu \; \exp\left(i\int d^4x \; -\frac{1}{2}\partial^{[\mu}A^{e\nu]}\partial_\mu A_{e\nu} + \frac{1}{2\xi}A^e_\mu\partial^\mu\partial^\nu A_{e\nu} + A^e_\mu J^\mu_e\right) \\
&\qquad \times \int \mathcal{D}\bar{c}\mathcal{D}c \; \exp\left(i\int d^4x \; \bar{c}_a\partial^2 c^a + \bar{c}^a\Xi^{\bar{c}}_a + \bar{\Xi}^c_a c^a\right) \\
&= \int \mathcal{D}A_\mu \; \exp\left[\frac{i}{2}\int d^4x \; A^e_\mu\delta_{ef}\left(g^{\mu\nu}\partial^2 - \left(1-\frac{1}{\xi}\right)\partial^\mu\partial^\nu\right)A^f_\nu\right] \\
&\qquad \times \int \mathcal{D}\bar{c}\mathcal{D}c \; \exp\left(i\int d^4x \; \bar{c}^a\delta_{ab}\partial^2 c^b + \bar{c}^a\Xi^{\bar{c}}_a + \bar{\Xi}^c_a c^a\right) \\
&= \exp\left(\frac{i}{2}\int d^4x d^4y \; J^\mu_e(x)\Delta^{ef}_{\mu\nu}(x-y)J^\nu_f(y)\right) \\
&\qquad \times \exp\left(i\int d^4x d^4y \; \bar{\Xi}^c_a(x)\Delta^{ab}(x-y)\Xi^{\bar{c}}_b(y)\right)
\end{aligned}
\tag{2.46}
$$

where we take

$$\Delta^{ef}_{\mu\nu}(x-y) := \int \frac{d^4k}{(2\pi)^4}\, \frac{\delta^{ef} G_{\mu\nu}(k)}{k^2 - i\epsilon} e^{ik\cdot(x-y)}, \tag{2.47}$$

$$\Delta^{ab}(x-y) := \int \frac{d^4k}{(2\pi)^4}\, \frac{\delta^{ab}}{k^2 - i\epsilon} e^{ik\cdot(x-y)}. \tag{2.48}$$

Here we used the result of appendix **A** on the gauge field component of second line, also result of grassmann integration on ghost related path integral component:

$$\begin{aligned}\int \prod_{a,b} d^n\bar{c}^a d^n c^b \ \exp\left(\bar{c}^a_i M^{ij}_{ab} c^b_j + \bar{c}^a_i \Xi^{\bar{c}i}_a + \bar{\Xi}^{cj}_b c^b_j\right) \\ = (\det M) \exp\left(-\bar{\Xi}^{ci}_a (M^{-1})^{ab}_{ij} \Xi^{\bar{c}j}_b\right)\end{aligned} \tag{2.49}$$

on limit $i, j \to \boldsymbol{x}$. Remind that here we denote:

$$\prod_{a,b} d^n\bar{c}^a d^n c^b = (d\bar{c}^{a_N}_n dc^{b_N}_n \cdots d\bar{c}^{a_N}_1 dc^{b_N}_1) \cdots (d\bar{c}^{a_1}_n dc^{b_1}_n \cdots d\bar{c}^{a_1}_1 dc^{b_1}_1). \tag{2.50}$$

Then, now we are ready to running for exact calculation of propagator (2.45). On this calculation, we can count each terms remaining on $J^a = 0$ with coupling constant g, now we consider:

$$\begin{aligned}\mathcal{P} = &\left(\frac{1}{i}\frac{\delta}{\delta J^\mu_a(x)}\right)\left(\frac{1}{i}\frac{\delta}{\delta J^\nu_b(y)}\right) \\ &\times \sum_{P=0}^{\infty} \frac{1}{P!}\left[i\int d^4x' \, -g f_{abe}\left(\frac{1}{i}\frac{\delta}{\delta J_{a\rho}(x')}\right)\left(\frac{1}{i}\frac{\delta}{\delta J_{b\sigma}(x')}\right)\partial_\rho\left(\frac{1}{i}\frac{\delta}{\delta J^\sigma_e(x')}\right)\right]^P \\ &\times \sum_{V=0}^{\infty} \frac{1}{V!}\left[i\int d^4x' \ -\frac{1}{4}g^2 f_{ab}{}^e f_{cde}\left(\frac{1}{i}\frac{\delta}{\delta J_{a\rho}(x')}\right)\cdots\left(\frac{1}{i}\frac{\delta}{\delta J^\sigma_d(x')}\right)\right]^V \\ &\times \sum_{N=0}^{\infty} \frac{1}{N!}\left[i\int d^4x' \ \left(\frac{1}{i}\frac{\delta}{\delta J^\rho_a(x')}\right)\left(i\partial^\rho\frac{\delta}{\delta \Xi^{\bar{c}}_b(x')}\right) g f_{abc}\left(\frac{1}{i}\frac{\delta}{\delta \bar{\Xi}^c_c(x')}\right)\right]^N \\ &\times \sum_{M=0}^{\infty} \frac{1}{M!}\left(\frac{i}{2}\int d^4z d^4w \ J^\rho_p(z)\Delta^{pq}_{\rho\sigma}(z-w)J^\sigma_q(w)\right)^M \\ &\times \sum_{L=0}^{\infty} \frac{1}{L!}\left(i\int d^4z d^4w \ \bar{\Xi}^c_p(z)\Delta^{pq}(z-w)\Xi^{\bar{c}}_q(w)\right)^L.\end{aligned} \tag{2.51}$$

First-order Propagator of Gauge Field

Let us select $P = V = N = L = 0, M = 1$ which gives the term of g^0 order. Now our propagator of (2.51) now goes to

$$\begin{aligned}\mathcal{P}^{[g]=0} &= \left(\frac{1}{i}\frac{\delta}{\delta J_a^\mu(x)}\right)\left(\frac{1}{i}\frac{\delta}{\delta J_b^\nu(y)}\right)\left(\frac{i}{2}\int d^4z d^4w \; J_p^\rho(z)\Delta_{\rho\sigma}^{pq}(z-w)J_q^\sigma(w)\right)\\ &= \Delta_{\mu\nu}^{ab}(x-y) = \int \frac{d^4k}{(2\pi)^4}\,\frac{\delta^{ab}G_{\mu\nu}(k)}{k^2 - i\epsilon}e^{ik\cdot(x-y)}.\end{aligned} \tag{2.52}$$

This is the g^0-contributed free propagator for gauge field, which is conveniently used for expending our Feynman rules[9] for YM theory later. Remind that the coupling constant of g is not related to this propagator(i.e., propagator component of g^0 order).

Second-order Propagator of Gauge Field

Next, let us consider a propagator component of g^2 order. One can be selected by set of $(P, V, N, M, L) = (2, 0, 0, 4, 0), (0, 0, 2, 2, 2), (0, 2, 0, 5, 0)$, so we shall calculate 3 terms followed:

1. $(P, V, N, M, L) = (2, 0, 0, 4, 0)$.

Then, we choose the propagator component to be

$$\begin{aligned}\mathcal{P}^{[g]=2}_{(2,0,0,4,0)} = &\left(\frac{1}{i}\frac{\delta}{\delta J_a^\mu(x)}\right)\left(\frac{1}{i}\frac{\delta}{\delta J_b^\nu(y)}\right)\\ &\times \frac{1}{2!}\left[i\int d^4x' \; -g f_{cde}\left(\frac{1}{i}\frac{\delta}{\delta J_c^\rho(x')}\right)\left(\frac{1}{i}\frac{\delta}{\delta J_{d\sigma}(x')}\right)\partial^\rho\left(\frac{1}{i}\frac{\delta}{\delta J_e^\sigma(x')}\right)\right]^2\\ &\times \frac{1}{4!}\left(\frac{i}{2}\int d^4z d^4w \; J_p^\rho(z)\Delta_{\rho\sigma}^{pq}(z-w)J_q^\sigma(w)\right)^4.\end{aligned} \tag{2.53}$$

One can be clearly separated by two components followed. That are *functional derivative* component of $(\frac{1}{i}\frac{\delta}{\delta J_a^\mu(x)})(\frac{1}{i}\frac{\delta}{\delta J_b^\nu(y)})$ and *path integral* component of $Z[J^a]$. Now we shall proceeds our calculation following to these each parts.

[9] On Chapter 1, we derived the Feynman rules for QED. In this section, we shall discuss about expansion of Feynman rules for YM theory.

Our target now transports to direct calculation of $Z^{[g]=2}_{(2,0,0,4,0)}$ followed:

$$
\begin{aligned}
Z^{[g]=2}_{(2,0,0,4,0)} = \frac{1}{2!}&\left[i\int d^4x' \ -\frac{1}{2}gf_{cde}\left(\frac{1}{i}\frac{\delta}{\delta J^\rho_c(x')}\right)\left(\frac{1}{i}\frac{\delta}{\delta J_{d\sigma}(x')}\right)\partial^\rho\left(\frac{1}{i}\frac{\delta}{\delta J^\sigma_e(x')}\right)\right]^2 \\
&\times\frac{1}{4!}\left(\frac{i}{2}\int d^4zd^4w\ J^\omega_p(z)\Delta^{pq}_{\omega\lambda}(z-w)J^\lambda_q(w)\right)^4 \\
&= \frac{1}{2!4!}[\cdots]\left[i\int d^4x' \ -gf_{cde}\left(\frac{1}{i}\frac{\delta}{\delta J^\rho_c(x')}\right)\left(\frac{1}{i}\frac{\delta}{\delta J_{d\sigma}(x')}\right)\right] \\
&\times\left(4\int d^4w\ \partial^\rho\Delta^{eq}_{\sigma\omega}(x'-w)J^\omega_q(w)\right)(\cdots)^3 \\
&= \frac{1}{2!4!}[\cdots]\left[i\int d^4x' \ -gf_{cde}\left(\frac{1}{i}\frac{\delta}{\delta J^\rho_c(x')}\right)\right] \\
&\times\left[\frac{4}{i}g^{\sigma\sigma_1}\partial^\rho\Delta^{ed}_{\sigma\sigma_1}(x'-x')(\cdots)^3\right. \\
&\qquad +4\cdot 3\cdot\left(\int d^4w_1\ \partial^\rho\Delta^{eq}_{\sigma\omega}(x'-w_1)J^\omega_q(w_1)\right) \\
&\qquad\qquad \left.\times\left(\int d^4w_2\ g^{\sigma\sigma_1}\Delta^{dq}_{\sigma_1\omega}(x'-w_2)J^\omega_q(w_2)\right)(\cdots)^2\right] \\
&= \frac{1}{2!4!}[\cdots]\left(i\int d^4x' \ -gf_{cde}\right)(\cdots)^2 \\
&\times\left[\frac{4\cdot 3}{i}g^{\sigma\sigma_1}\partial^\rho\Delta^{ed}_{\sigma\sigma_1}(x'-x')\int d^4w\ \Delta^{cq}_{\rho\omega}(x'-w)J^\omega_q(w)\right. \\
&\qquad +\frac{4\cdot 3}{i}\left(\partial^\rho\Delta^{ec}_{\sigma\rho}(x'-x')\int d^4w_2\ g^{\sigma\sigma_1}\Delta^{dq}_{\sigma_1\omega}(x'-w_2)J^\omega_q(w_2)\right) \\
&\qquad +\frac{4\cdot 3}{i}\left(g^{\sigma\sigma_1}\Delta^{dc}_{\sigma_1\rho}(x'-x')\int d^4w_1\ \partial^\rho\Delta^{eq}_{\sigma\omega}(x'-w_2)J^\omega_q(w_2)\right) \\
&\qquad \left.+\frac{4\cdot 3}{i}(\cdots\partial^\rho\cdots)(\cdots g^{\sigma\sigma_1}\cdots)\left(\int d^4w_3\ \Delta^{cq}_{\rho\omega}(x'-w_2)J^\omega_q(w_3)\right)\right].
\end{aligned}
\tag{2.54}
$$

That is extremely complicate, we shall represent it with following diagramme. One can be represented by diagramme of $[\cdots]$ on (2.54):

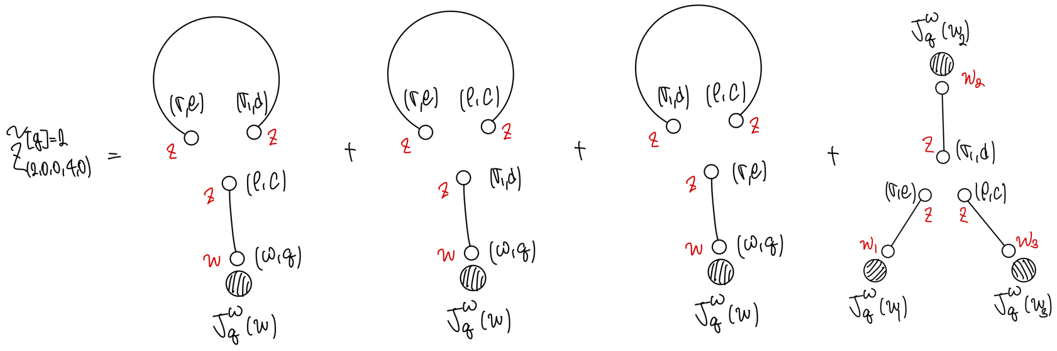


Figure 2.3: Diagramme Representation of (2.54).

where $Z^{[g]=2}_{(2,0,0,4,0)} = \frac{1}{2!4!}[\cdots]\left(i\int d^4x' \; -gf_{cde}\right)(\cdots)^2\,\tilde{Z}^{[g]=2}_{(2,0,0,4,0)}$. Remind that ∂^ρ is connected to propagator which has index of σ. The normal line represents propagator of gauge field. Taking similar calculation, we have

$$
\begin{aligned}
Z^{[g]=2}_{(2,0,0,4,0)} = &\int d^4xd^4yd^4zd^4w\; J^\mu_a(x)\partial^\lambda\Delta^{ai}_{\mu\omega}(x-w)(-gf_{ijk}g^{\omega\tau})\Delta^{jc}_{\tau\rho}(w-z)\\
&\qquad\times\partial^\rho\Delta^{ed}_{\sigma\sigma_1}(z-z)(-gf_{cde}g^{\sigma\sigma_1})\Delta^{kb}_{\lambda\nu}(w-y)J^\nu_b(y)\\
&+(17\text{ permutations of }(\omega,i),(\tau,j),(\lambda,k)\text{ and }(\rho,c),(\sigma_1,d),(\sigma,e)).\\
&+\int d^4xd^4yd^4zd^4w\; J^\mu_a(x)\partial^\lambda\Delta^{ai}_{\mu\omega}(x-w)(-gf_{ijk}g^{\omega\tau})\Delta^{jc}_{\tau\rho}(w-z)\\
&\qquad\times\Delta^{dk}_{\sigma_1\lambda}(z-w)(-gf_{cde}g^{\sigma\sigma_1})\partial^\rho\Delta^{eb}_{\sigma\nu}(z-y)J^\nu_b(y)\\
&+(35\text{ permutations of }(\omega,i),(\tau,j),(\lambda,k)\text{ and }(\rho,c),(\sigma_1,d),(\sigma,e)).
\end{aligned}
\tag{2.55}
$$

where ∂^λ also connected to propagator with index ω. Now then, the diagramme representation of Figure 2.4 now shows possibility of extension of Feynman rules for YM theory. Our purpose is to find 3-vertex function of $\mathbf{V}^{\rho\sigma_1\sigma}_{cde}$, that represents our path integral component (2.55) to be

$$
\begin{aligned}
Z^{[g]=2,\;(1)}_{(2,0,0,4,0)} = &\int \frac{d^4k}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{d^4r}{(2\pi)^4}\,(2\pi)^8\delta^4(k-p-r)\delta^4(p)\\
&\times\tilde{J}^\mu_a(-k)\tilde{\Delta}^{ai}_{\mu\omega}(k)\mathbf{V}^{\omega\tau\lambda}_{ijk}\tilde{\Delta}^{jc}_{\tau\rho}(p)\tilde{\Delta}^{ed}_{\sigma\sigma_1}(q)\mathbf{V}^{\rho\sigma_1\sigma}_{cde}\tilde{\Delta}^{kb}_{\lambda\nu}(r)\tilde{J}^\nu_b(r),
\end{aligned}
\tag{2.56}
$$

regarding to first three diagrammes[10],

$$Z^{[g]=2,\ (2)}_{(2,0,0,4,0)} = \int \frac{d^4k}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{d^4r}{(2\pi)^4}\ (2\pi)^8\delta^4(k+q-p)\delta^4(p-q-r)$$

$$\times\ \tilde{J}^{\mu}_{a}(-k)\tilde{\Delta}^{ai}_{\mu\omega}(k)\mathbf{V}^{\omega\tau\lambda}_{ijk}\tilde{\Delta}^{jc}_{\tau\rho}(p)\tilde{\Delta}^{dk}_{\sigma_1\lambda}(q)\mathbf{V}^{\rho\sigma\sigma_1}_{cde}\tilde{\Delta}^{eb}_{\sigma\nu}(r)\tilde{J}^{\nu}_{b}(r) \tag{2.57}$$

regarding to last diagramme. Also on diagramme representation, that is

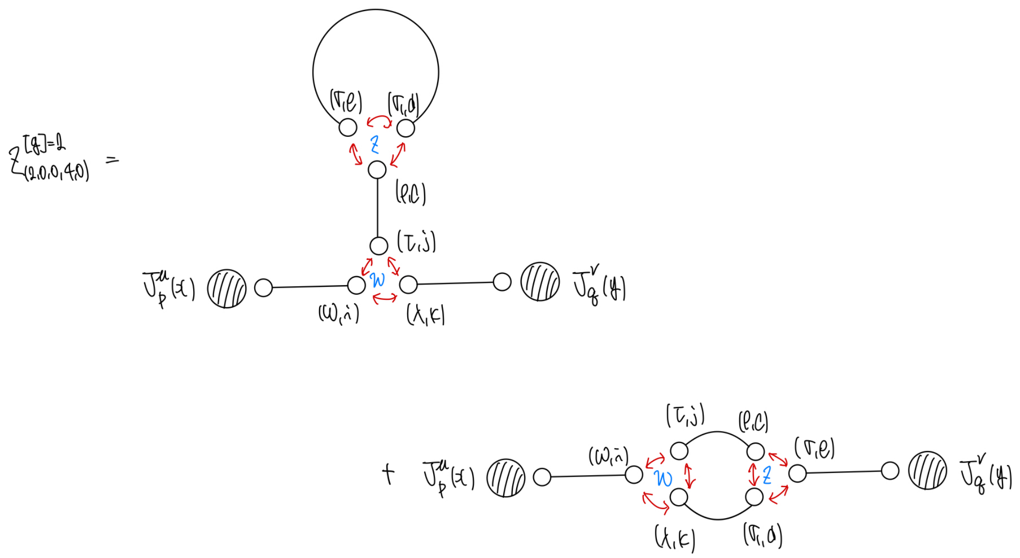


Figure 2.4: Diagramme Representation of (2.56) and (2.57).

Let us consider first three diagramme involving path integral component with a fixed permutations for $(\omega,i),(\tau,j),(\lambda,k)$. Here we shall take Fourier transformation of propagators and external sources,

$$J^{\mu}_{a}(x) = \int \frac{d^4k}{(2\pi)^4}\ \tilde{J}^{\mu}_{a}(k)e^{ik\cdot x}, \qquad \Delta^{ab}_{\rho\sigma}(x-y) = \int \frac{d^4k}{(2\pi)^4}\ \tilde{\Delta}^{ab}_{\rho\sigma}(k)e^{ik\cdot(x-y)}, \tag{2.58}$$

one can be denoted in form of (2.57)[11]. So, we shall determine our extended vertex function of $\mathbf{V}^{\omega\tau\lambda}_{ijk}$s by considering Fourier transformation of last diagramme (2.57). Hence, feeding Fourier transformation of external sources

[10] Also, under the momentum space.

[11] In QFT, formula (2.57) also called *1-loop correction* or *1-loop expansion.*

and propagators (2.58) to 1-loop diagramme expansion (2.57), we have

$$
\begin{aligned}
Z^{[g]=2,\ (2)^f}_{(2,0,0,4,0)} = & \int d^4x d^4y d^4z d^4w \; J^\mu_a(x)\partial^\lambda \Delta^{ai}_{\mu\omega}(x-w)(-gf_{ijk}g^{\omega\tau})\Delta^{jc}_{\tau\rho}(w-z) \\
& \times \Delta^{dk}_{\sigma_1\lambda}(z-w)(-gf_{cde}g^{\sigma\sigma_1})\partial^\rho \Delta^{eb}_{\sigma\nu}(z-y)J^\nu_b(y) \\
& + \int d^4x d^4y d^4z d^4w \; J^\mu_a(x)\partial^\lambda \Delta^{ai}_{\mu\omega}(x-w)(-gf_{ijk}g^{\omega\tau})\Delta^{jd}_{\tau\sigma_1}(w-z) \\
& \times \Delta^{ck}_{\rho\lambda}(z-w)(-gf_{cde}g^{\sigma\sigma_1})\partial^\rho \Delta^{eb}_{\sigma\nu}(z-y)J^\nu_b(y) \\
& + (4 \text{ permutations of } (\rho, c), (\sigma_1, d), (\sigma, e)) \\
= & \int \frac{d^4k}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{d^4r}{(2\pi)^4} \; (2\pi)^8\delta^4(k+q-p)\delta^4(p-q-r) \\
& \times \tilde{J}^\mu_a(-k)ik^\lambda \tilde{\Delta}^{ai}_{\mu\omega}(k)(-gf_{ijk}g^{\omega\tau})\tilde{\Delta}^{jc}_{\tau\rho}(p) \\
& \times \tilde{\Delta}^{dk}_{\sigma_1\lambda}(q)(-gf_{cde}g^{\sigma\sigma_1})ir^\rho \tilde{\Delta}^{eb}_{\sigma\nu}(r)\tilde{J}^\nu_b(r) \\
& + \int \frac{d^4k}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{d^4r}{(2\pi)^4} \; (2\pi)^8\delta^4(k+q-p)\delta^4(p-q-r) \\
& \times \tilde{J}^\mu_a(-k)ik^\lambda \tilde{\Delta}^{ai}_{\mu\omega}(k)(-gf_{ijk}g^{\omega\tau})\tilde{\Delta}^{jd}_{\tau\sigma_1}(p) \\
& \times \tilde{\Delta}^{ck}_{\rho\lambda}(q)(-gf_{cde}g^{\sigma\sigma_1})ir^\rho \tilde{\Delta}^{eb}_{\sigma\nu}(r)\tilde{J}^\nu_b(r) \\
& + (4 \text{ permutations of } (\rho, c), (\sigma_1, d), (\sigma, e)) \\
= & \int \frac{d^4k}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{d^4r}{(2\pi)^4} \; (2\pi)^8\delta^4(k+q-p)\delta^4(p-q-r) \\
& \times \tilde{J}^\mu_a(-k)ik^\lambda \tilde{\Delta}^{ai}_{\mu\omega}(k)(-gf_{ijk}g^{\omega\tau})\tilde{\Delta}^{jc}_{\tau\rho}(p) \\
& \times \tilde{\Delta}^{dk}_{\sigma_1\lambda}(q)\left[(-gf_{cde}g^{\sigma\sigma_1})ir^\rho + (-gf_{dce}g^{\sigma\rho})ir^{\sigma_1}\right]\tilde{\Delta}^{eb}_{\sigma\nu}(r)\tilde{J}^\nu_b(r) \\
& + (4 \text{ permutations of } (\rho, c), (\sigma_1, d), (\sigma, e)).
\end{aligned}
\tag{2.59}
$$

The notation of $(2)^f$ means that now we are considering a specific pair of $(\omega, i), (\tau, j), (\lambda, k)$, so here the summation from other 5 permutations of

$(\omega, i), (\tau, j), (\lambda, k)$ ignored. Similarly, if we denote all of terms on (2.59), one can determine the vertex function of $\mathbf{V}^{\rho\sigma_1\sigma}_{cde}$. That is,

$$
\begin{aligned}
Z^{[g]=2,\ (2)^f}_{(2,0,0,4,0)} = & \int \frac{d^4k}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{d^4r}{(2\pi)^4}\ (2\pi)^8\delta^4(k+q-p)\delta^4(p-q-r) \\
& \times \tilde{J}^\mu_a(-k)ik^\lambda\tilde{\Delta}^{ai}_{\mu\omega}(k)(-gf_{ijk}g^{\omega\tau})\tilde{\Delta}^{jc}_{\tau\rho}(p) \\
& \times \tilde{\Delta}^{dk}_{\sigma_1\lambda}(q)\left[(-gf_{cde}g^{\sigma\sigma_1})ir^\rho + (-gf_{dce}g^{\sigma\rho})ir^{\sigma_1}\right]\tilde{\Delta}^{eb}_{\sigma\nu}(r)\tilde{J}^\nu_b(r) \\
& + \int \frac{d^4k}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{d^4r}{(2\pi)^4}\ (2\pi)^8\delta^4(k+q-p)\delta^4(p-q-r) \\
& \times \tilde{J}^\mu_a(-k)ik^\lambda\tilde{\Delta}^{ai}_{\mu\omega}(k)(-gf_{ijk}g^{\omega\tau})\tilde{\Delta}^{je}_{\tau\sigma}(p) \\
& \times \tilde{\Delta}^{dk}_{\sigma_1\lambda}(q)\left[(-gf_{cde}g^{\sigma\sigma_1})ip^\rho + (-gf_{ced}g^{\sigma\sigma_1})iq^\rho\right]\tilde{\Delta}^{cb}_{\rho\nu}(r)\tilde{J}^\nu_b(r) \\
& + \int \frac{d^4k}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{d^4r}{(2\pi)^4}\ (2\pi)^8\delta^4(k+q-p)\delta^4(p-q-r) \\
& \times \tilde{J}^\mu_a(-k)ik^\lambda\tilde{\Delta}^{ai}_{\mu\omega}(k)(-gf_{ijk}g^{\omega\tau})\tilde{\Delta}^{jc}_{\tau\rho}(p) \\
& \times \tilde{\Delta}^{ek}_{\sigma\lambda}(q)\left[(-gf_{cde}g^{\sigma\sigma_1})iq^\rho + (-gf_{edc}g^{\rho\sigma_1})ip^\sigma\right]\tilde{\Delta}^{db}_{\sigma_1\nu}(r)\tilde{J}^\nu_b(r) \\
= & \int \frac{d^4k}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{d^4r}{(2\pi)^4}\ (2\pi)^8\delta^4(k+q-p)\delta^4(p-q-r) \\
& \times \tilde{J}^\mu_a(-k)ik^\lambda\tilde{\Delta}^{ai}_{\mu\omega}(k)(-gf_{ijk}g^{\omega\tau})\tilde{\Delta}^{jc}_{\tau\rho}(p) \\
& \times \tilde{\Delta}^{dk}_{\sigma_1\lambda}(q) \\
& \times \underbrace{(-igf_{cde})\left[(r-q)^\rho g^{\sigma\sigma_1} + (p-r)^{\sigma_1}g^{\rho\sigma} + (q-p)^\sigma g^{\rho\sigma_1}\right]}_{=\mathbf{V}^{\rho\sigma_1\sigma}_{cde}(p,q,r)} \\
& \times \tilde{\Delta}^{eb}_{\sigma\nu}(r)\tilde{J}^\nu_b(r).
\end{aligned}
\tag{2.60}
$$

So, now we finally determine our vertex function $\mathbf{V}^{\rho\sigma_1\sigma}_{cde}$ which is expended for YM theory.

Consequently, calculation (2.60) can be compared with our expectation (2.57), one can be treated by a 'vertex point':

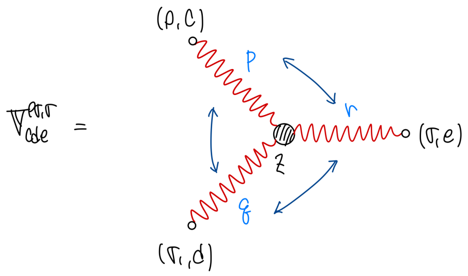


Figure 2.5: Vertex Diagramme based on (2.60).

The three-gluon vertex function can be determined by following rules, without loss of generality:

Theorem 2.2.1 (3-Vertex Function for YM thoery)

$$i\mathbf{V}^{\rho\sigma_1\sigma}_{cde} = gf_{cde}\left[(r-q)^{\rho}g^{\sigma\sigma_1} + (p-r)^{\sigma_1}g^{\rho\sigma} + (q-p)^{\sigma}g^{\rho\sigma_1}\right], \tag{2.61}$$

Similarly, other 3-vertex for coordinate w on 1-loop diagramme[12],

$$i\mathbf{V}^{\omega\tau\lambda}_{ijk} = gf_{ijk}\left[(p-q)^{\omega}g^{\tau\lambda} + (q-k)^{\tau}g^{\omega\lambda} + (k-p)^{\lambda}g^{\omega\tau}\right]. \tag{2.62}$$

These 3-vertex functions now then, one can be also treated by summation running to the exchange of each gluon lines(propagators),

$$\begin{aligned} i\mathbf{V}^{\rho\sigma_1\sigma}_{cde} &= \sum_{(l,\alpha_i,k_i)\in(p,\rho,c)\times(q,\sigma_1,d)\times(r,\sigma,e)} gf_{k_1k_2k_3}l^{\alpha_1}g^{\alpha_2\alpha_3} \\ &= gf_{cde}r^{\rho}g^{\sigma_1\sigma} + gf_{ced}q^{\rho}g^{\sigma\sigma_1} + gf_{dec}p^{\sigma_1}g^{\sigma\rho} + gf_{dce}r^{\sigma_1}g^{\rho\sigma} \\ &\qquad + gf_{ecd}q^{\sigma}g^{\rho\sigma_1} + gf_{edc}p^{\sigma}g^{\sigma_1\rho} \\ &= gf_{cde}\left[(r-q)^{\rho}g^{\sigma\sigma_1} + (p-r)^{\sigma_1}g^{\rho\sigma} + (q-p)^{\sigma}g^{\rho\sigma_1}\right]. \end{aligned} \tag{2.63}$$

Now then, these result of 3-vertex function $\mathbf{V}^{\rho\sigma_1\sigma}_{cde}$s also vanish the path integral components of first three diagrammes. That is, the 3-vertex function

[12] The path integral component for 3rd diagramme on Figure 2.4.

of each vertex $\mathbf{V}^{\omega\tau\lambda}_{ijk}, \mathbf{V}^{\rho\sigma_1\sigma}_{cde}$(see the first diagramme on Figure 2.4) and delta function of $\delta^4(p)$ works together(also, see (2.56)), so we have[13]

$$
\begin{aligned}
Z^{[g]=2,\ (1)}_{(2,0,0,4,0)} &= \int \frac{d^4k}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{d^4r}{(2\pi)^4}\ (2\pi)^8\delta^4(k-p-r)\delta^4(p) \\
&\quad \times \tilde{J}^{\mu}_a(-k)\tilde{\Delta}^{ai}_{\mu\omega}(k)\mathbf{V}^{\omega\tau\lambda}_{ijk}\tilde{\Delta}^{jc}_{\tau\rho}(p)\tilde{\Delta}^{ed}_{\sigma\sigma_1}(q)\mathbf{V}^{\rho\sigma_1\sigma}_{cde}\tilde{\Delta}^{kb}_{\lambda\nu}(r)\tilde{J}^{\nu}_b(r) \\
&= \int \frac{d^4k}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{d^4r}{(2\pi)^4}\ (2\pi)^8\delta^4(k-p-r)\delta^4(p) \\
&\quad \times \tilde{J}^{\mu}_a(-k)\tilde{\Delta}^{ai}_{\mu\omega}(k) \\
&\quad \times (-igf_{ijk})\left[(p-r)^{\omega}g^{\tau\lambda}+(r-k)^{\tau}g^{\omega\lambda}+(k-p)^{\lambda}g^{\omega\tau}\right] \\
&\quad \times \tilde{\Delta}^{jc}_{\tau\rho}(p)\tilde{\Delta}^{ed}_{\sigma\sigma_1}(q) \\
&\quad \times (-igf_{cde})\left[p^{\rho}g^{\sigma\sigma_1}-p^{\rho}g^{\sigma_1\sigma}\right] \\
&\quad \times \tilde{\Delta}^{kb}_{\lambda\nu}(r)\tilde{J}^{\nu}_b(r) = 0.
\end{aligned}
\tag{2.64}
$$

From the derivative identity of $\partial^{\rho}\Delta^{ed}_{\sigma\sigma_1}(z-z)=0$, remind that the gluon line with momentum p now vanished for vertex ρ,σ_1 for vertex label rotation $(\rho\leftrightarrow\sigma\leftrightarrow\sigma_1)$ on path integral calculation(see (2.55) and Figure 2.4), our calculation of (2.64) can be easily confirmed. On Diagramme representation, that is:

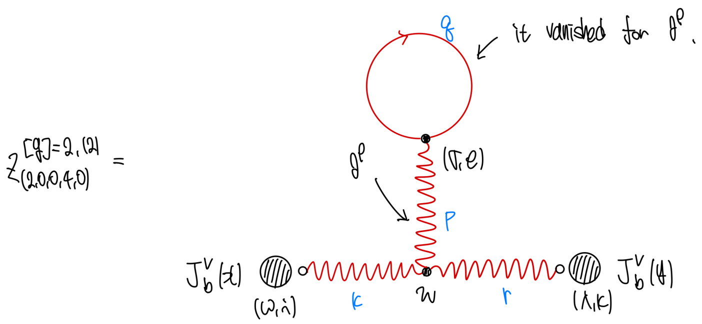


Figure 2.6: Diagramme representation for $Z^{[g]=2,\ (1)}_{(2,0,0,4,0)}$.

[13] Conveniently, we call this diagrammes by '*Bubble diagramme*'.

Now then, our calculation for $(P,V,N,M,L)=(2,0,0,4,0)$ comes to

$$\begin{aligned}
Z^{[g]=2}_{(2,0,0,4,0)} &= \int \frac{d^4k}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{d^4r}{(2\pi)^4}\,(2\pi)^8\delta^4(k+q-p)\delta^4(p-q-r)\\
&\qquad\times \tilde{J}^{\mu}_{a}(-k)\tilde{\Delta}^{ai}_{\mu\omega}(k)\mathbf{V}^{\omega\tau\lambda}_{ijk}\tilde{\Delta}^{jc}_{\tau\rho}(p)\tilde{\Delta}^{dk}_{\sigma_1\lambda}(q)\mathbf{V}^{\rho\sigma\sigma_1}_{cde}\tilde{\Delta}^{eb}_{\sigma\nu}(r)\tilde{J}^{\nu}_{b}(r)\\
&= \int \frac{d^4k}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{d^4r}{(2\pi)^4}\,(2\pi)^8\delta^4(k+q-p)\delta^4(p-q-r)\\
&\qquad\times \tilde{J}^{\mu}_{a}(-k)\tilde{\Delta}^{ai}_{\mu\omega}(k)\\
&\qquad\times(-igf_{ijk})\left[(p-q)^{\omega}g^{\tau\lambda}+(q-k)^{\tau}g^{\omega\lambda}+(k-p)^{\lambda}g^{\omega\tau}\right]\\
&\qquad\times\tilde{\Delta}^{jc}_{\tau\rho}(p)\tilde{\Delta}^{dk}_{\sigma_1\lambda}(q)\\
&\qquad\times(-igf_{cde})\left[(r-q)^{\rho}g^{\sigma\sigma_1}+(p-r)^{\sigma_1}g^{\rho\sigma}+(q-p)^{\sigma}g^{\rho\sigma_1}\right]\\
&\qquad\times\tilde{\Delta}^{eb}_{\sigma\nu}(r)\tilde{J}^{\nu}_{b}(r).
\end{aligned} \tag{2.65}$$

On Diagramme representation, that is:

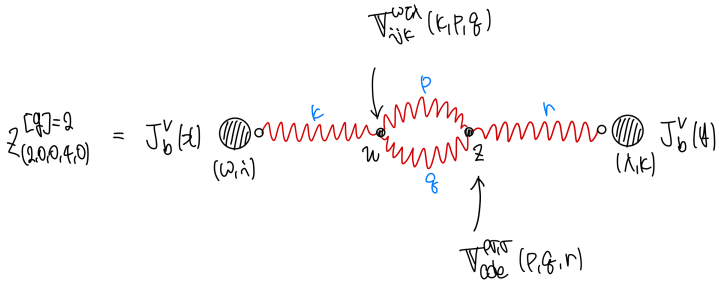


Figure 2.7: Diagramme representation for $Z^{[g]}_{(2,0,0,4,0)}$.

2.3 BRST Symmetry

Appendix A

Path Integral on Faddeev-Popov Method

A.1 Propagator for Gauge fields

In this section we shall have discussions of path integrals for gauge fields. Let path integral within its external sources by

$$Z[J] \sim \int \mathcal{D}A_\mu \ \exp\left(i \int d^4x \ -\frac{1}{4}F^2_{\mu\nu}(x) - \frac{1}{2\xi}[\partial^\mu A_\mu(x)]^2 + J^\mu A_\mu \right) \quad \text{(A.1)}$$

on R_ξ gauge theory, from the results of Faddeev-Popov gauge fixing. Then, now we shall calculate this path integral using proper Fourier transformation. For this task, we shall compute following action integral $S^{FP}[J]$ as

$$\begin{aligned}
S^{FP}[0] &= \int d^4x \ \left(-\frac{1}{4}F^2_{\mu\nu}(x) - \frac{1}{2\xi}[\partial^\mu A_\mu(x)]^2 \right) \\
&= \int d^4x \ \left(-\frac{1}{2}F^{\mu\nu}(x)\partial_\mu A_\nu(x) + \frac{1}{2\xi}A_\mu(x)\partial^\mu\partial^\nu A_\nu(x) \right) \\
&= \int d^4x \ \left(-\frac{1}{2}\left(\partial^\mu A^\nu(x) - \partial^\nu A^\mu(x)\right)\partial_\mu A_\nu(x) + \frac{1}{2\xi}A_\mu(x)\partial^\mu\partial^\nu A_\nu(x) \right) \\
&= \frac{1}{2}\int d^4x \ A_\mu(x)\left(g^{\mu\nu}\partial^2 - \left(1 - \frac{1}{\xi}\right)\partial^\mu\partial^\nu \right) A_\nu(x)
\end{aligned} \quad \text{(A.2)}$$

for $J^\mu = 0$.

Then, now we shall feed Fourier transformation of Gauge field:

$$A_\mu(x) = \int \frac{d^4k}{(2\pi)^4}\ \tilde{A}_\mu(k)e^{ik\cdot x} \tag{A.3}$$

to path integral (A.1), so relation (A.2) yeilds

$$\begin{aligned} Z^{FP}[J] \sim \int \mathcal{D}A_\mu & \\ &\times \exp\left(\frac{i}{2}\int \frac{d^4k}{(2\pi)^4}\ - \tilde{A}_\mu(-k)k^2\tilde{P}^{\mu\nu}(k)\tilde{A}_\nu(k)\right) \\ &\qquad\times \exp\left(\frac{i}{2}\int \frac{d^4k}{(2\pi)^4}\ \tilde{J}^\mu(-k)\tilde{A}_\mu(k) + \tilde{A}_\mu(-k)\tilde{J}^\mu(k)\right). \end{aligned} \tag{A.4}$$

For further calculation of (A.4), here we shall take change of integral variable (where $\tilde{R}_{\mu\nu}(k)$ is designed to satisfying $\tilde{R}_{\mu\rho}\tilde{P}^{\mu\nu} = \delta^\nu_\rho$) of

$$\begin{aligned} \tilde{\chi}_\mu(k) &= \tilde{A}_\mu(k) + \frac{1}{k^2}\tilde{R}_{\mu\nu}(k)\tilde{J}^\nu(k) \\ &= \tilde{A}_\mu(k) + \frac{1}{k^2}\left(g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right)\tilde{J}^\nu(k) \end{aligned} \tag{A.5}$$

on given path integral. On this change of integral variables, given path integral now goes to

$$S^{FP}[J] = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\ \left(-\chi_\mu(-k)k^2\tilde{P}^{\mu\nu}(k)\chi_\nu(k) + \tilde{J}^\mu(-k)\frac{\tilde{R}_{\mu\nu}(k)}{k^2}\tilde{J}^\nu(k)\right). \tag{A.6}$$

So, our path integral for R_ξ gauge theory now comes to

$$\begin{aligned} Z^{FP}[J] \sim \int \det\left(\frac{\delta A_\mu}{\delta\chi_\nu}\right)\mathcal{D}\chi_\mu & \\ &\times \exp\left(\frac{i}{2}\int \frac{d^4k}{(2\pi)^4}\ - \chi_\mu(-k)k^2\tilde{P}^{\mu\nu}(k)\chi_\nu(k)\right) \\ &\qquad\times \exp\left(\frac{i}{2}\int \frac{d^4k}{(2\pi)^4}\ \tilde{J}^\mu(-k)\frac{\tilde{R}_{\mu\nu}(k)}{k^2}\tilde{J}^\nu(k)\right). \end{aligned} \tag{A.7}$$

From $\det\left(\frac{\delta A_\mu}{\delta \chi_\nu}\right) = 1$, we shall calculate integral of functional $\chi_\mu(k)$ first, our path integral finally becomes

Theorem A.1.1 (Faddeev-Popov Path integral)

$$
\begin{aligned}
Z^{FP}[J] &= \exp\left(\frac{i}{2}\int \frac{d^4k}{(2\pi)^4}\ \tilde{J}^\mu(-k)\frac{\tilde{R}_{\mu\nu}(k)}{k^2}\tilde{J}^\nu(k)\right) \\
&= \exp\left(\frac{i}{2}\int d^4x d^4y\ J^\mu(x)\Delta_{\mu\nu}(x-y)J^\nu(y)\right),
\end{aligned}
\tag{A.8}
$$

where we take

$$
\Delta_{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4}\ \frac{1}{k^2 - i\epsilon}\left(g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right)e^{ik\cdot(x-y)}. \tag{A.9}
$$

So, we shall calculate the 2-point correlation function from the result of path integral (A.8):

$$
\begin{aligned}
G_{\mu\nu}(x-y) &:= \langle 0|\mathrm{T}\left[A_\mu(x)A_\nu(y)\right]|0\rangle \\
&= \left(\frac{1}{i}\frac{\delta}{\delta J^\mu(x)}\right)\left(\frac{1}{i}\frac{\delta}{\delta J^\mu(x)}\right)Z^{FP}[J]\bigg|_{J=0} \\
&= \Delta_{\mu\nu}(x-y).
\end{aligned}
\tag{A.10}
$$

This is **Propagator for Gauge field** on R_ξ thoeries.

Appendix B

Spinor and Gamma matrices Technology

B.1 Dirac equation

Let us construct Dirac equations for momentum space. We shall start from Lorentz transformation of positive-energy spinor $u_s(\mathbf{p})$ and negative-energy spinor $v_s(\mathbf{p})$:

$$u_s(\mathbf{p}) = \frac{\not{p} + m}{\sqrt{2m(E+m)}} u_s(\mathbf{0}), \qquad v_s(\mathbf{p}) = \frac{-\not{p} + m}{\sqrt{2m(E+m)}} v_s(\mathbf{0}). \tag{B.1}$$

Then, we shall find Nilpotent identity for positive-energy spinors. From the Clifford algebra of gamma matrices $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \times \mathbf{1}_4$ and Lie Algebra[1] of gamma matrices $[\gamma^\mu, \gamma^\nu] = -4iS^{\mu\nu}$ which leads

$$p_\mu p_\nu [\gamma^\mu, \gamma^\nu] = p_\mu p_\nu [\gamma^\nu, \gamma^\mu] = 0, \tag{B.2}$$

we shall find Nilpotent relation of

$$\begin{aligned}
(-\not{p} + m) u_s(\mathbf{p}) &= \frac{(-\not{p} + m)(\not{p} + m)}{\sqrt{2m(E+m)}} u_s(\mathbf{0}) = \frac{\not{p}\not{p} - m^2}{\sqrt{2m(E+m)}} u_s(\mathbf{0}) \\
&= \frac{\frac{1}{2} p_\mu p_\nu \left(\{\gamma^\mu, \gamma^\nu\} + [\gamma^\mu, \gamma^\nu]\right) - m^2}{\sqrt{2m(E+m)}} u_s(\mathbf{0}) \\
&= \frac{p^2 - m^2}{\sqrt{2m(E+m)}} u_s(\mathbf{0}) = 0.
\end{aligned} \tag{B.3}$$

[1]Here we define $S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$.

for positive-energy spinors. Taking same calculations, we also have the Nilpotent relation of

$$
\begin{aligned}
(\not{p}+m)v_s(\mathbf{p}) &= \frac{(\not{p}+m)(-\not{p}+m)}{\sqrt{2m(E+m)}}v_s(\mathbf{0}) = \frac{\not{p}\not{p}-m^2}{\sqrt{2m(E+m)}}v_s(\mathbf{0}) \\
&= \frac{\frac{1}{2}p_\mu p_\nu\left(\{\gamma^\mu,\gamma^\nu\}+[\gamma^\mu,\gamma^\nu]\right)-m^2}{\sqrt{2m(E+m)}}v_s(\mathbf{0}) \\
&= \frac{p^2-m^2}{\sqrt{2m(E+m)}}v_s(\mathbf{0}) = 0.
\end{aligned}
\tag{B.4}
$$

for negative-energy spinors. Now the result of (B.4) also takes

$$
(-\not{p}+m)v_s(-\mathbf{p}) = (-\not{p}+m)\frac{\not{p}+m}{\sqrt{2m(E+m)}}v_s(\mathbf{0}) = 0, \tag{B.5}
$$

so now we shall take plane wave expansion of a general solution of differential equation $(-\not{p}+m)f_{\mathbf{p}} = 0$ by

$$
f_{\mathbf{p}}(0) = \sum_{s=\pm}\frac{b_s(\mathbf{p})}{2E_{\mathbf{p}}}u_s(\mathbf{p}) + \frac{d_s^\dagger(-\mathbf{p})}{2E_{-\mathbf{p}}}v_s(-\mathbf{p}). \tag{B.6}
$$

Then, we shall consider the expansion which gives pullback of momentum space to position space on Dirac fields by its Fourier transforms:

$$
\Psi(x) = \sum_{s=\pm}\int\frac{d^3k}{(2\pi)^3 2k^0}\left[b_s(\mathbf{k})u_s(\mathbf{k})e^{-ik\cdot x} + d_s^\dagger(\mathbf{k})v_s(\mathbf{k})e^{ik\cdot x}\right]. \tag{B.7}
$$

Here we recall the Lorenz transformation of creation-annihilation operators on scalar fields, that is necessary and sufficient condition of these operators:

$$
\begin{aligned}
U^{-1}(\Lambda)\varphi(x)U(\Lambda) &= \int\frac{d^3k}{(2\pi)^3 2k^0}\left[U^{-1}(\Lambda)a_{\mathbf{k}}U(\Lambda)e^{-ik\cdot x} + U^{-1}(\Lambda)a_{\mathbf{k}}^\dagger U(\Lambda)e^{-ik\cdot x}\right] \\
&= \int\frac{d^3k}{(2\pi)^3 2k^0}\left[a_{\mathbf{k}}e^{-ik\cdot(\Lambda^{-1}x)} + a_{\mathbf{k}}^\dagger e^{ik\cdot(\Lambda^{-1}x)}\right],
\end{aligned}
\tag{B.8}
$$

so it gives Lorentz transformation rule of these operators by

Theorem B.1.1 (Lorentz transf. of creation operator)

$$U^{-1}(\Lambda, b) a_{\mathbf{k}} U(\Lambda, b) = a_{\mathbf{k}_{\Lambda^{-1}}} e^{-i(\Lambda^{-1}k)\cdot b} \tag{B.9}$$

on Poincaré transformation $U(\Lambda) \to U(\Lambda, b)$. In this expansion we took change of integration variable $k \to \Lambda^{-1}k$ by

$$\begin{aligned} U^{-1}(\Lambda, b)\varphi(x)U(\Lambda, b) &= \int \frac{d^3(\Lambda^{-1}k)}{(2\pi)^3 2(\Lambda^{-1}k)^0} \; 2\mathrm{Re}\left[a_{\mathbf{k}_{\Lambda^{-1}}} e^{-i(\Lambda^{-1}k)\cdot(\Lambda^{-1}x+b)}\right] \\ &= \frac{1}{\det\Lambda} \int \frac{d^3k}{(2\pi)^3 2k^0} \; 2\mathrm{Re}\left[e^{-ik\cdot x} a_{\mathbf{k}_{\Lambda^{-1}}} e^{-i(\Lambda^{-1}k)\cdot b}\right] \end{aligned} \tag{B.10}$$

where we denote[2] $\mathbf{k}_{\Lambda^{-1}}$ as 3-vector component of $\Lambda^{-1}k$. Now then the each of creation and annihilation operators $b_s(\mathbf{k}), d_s(\mathbf{k})$ also have Lorentz transformation of (B.9), so we shall have

$$b_s(\mathbf{k}, t) = b_s(\mathbf{k})e^{-ik\cdot x}, \qquad d_s(\mathbf{k}, t) = d_s(\mathbf{k})e^{-ik\cdot x}. \tag{B.11}$$

by taking $\Lambda = \mathbf{1}$ and b arbitrary[3]. So, now we shall derived the plane wave expansion of free Dirac fields, Now that the Nilpotent identity yields:

$$\begin{aligned} 0 &= \sum_{s=\pm} \int \frac{d^3k}{(2\pi)^3 2k^0} \; \left[b_s(\mathbf{k})(-\not{k} + m)u_s(\mathbf{k})e^{-ik\cdot x} + d_s^\dagger(\mathbf{k})(\not{k} + m)v_s(\mathbf{k})e^{ik\cdot x}\right] \\ &= (-i\not{\partial} + m)\Psi(x) \end{aligned} \tag{B.12}$$

that is called **Dirac equation**. Additionally, using the Lorentz transformation rule of creation-annihilation operators, we also construct the Lorentz transformation rule of one-particle system:

Theorem B.1.2 (Lorentz transf. of one-particle state)

$$U(\Lambda)|\mathbf{k}\rangle = U^{-1}(\Lambda^{-1}) a_{\mathbf{k}} U(\Lambda^{-1}) U(\Lambda)|0\rangle = |\mathbf{k}_\Lambda\rangle. \tag{B.13}$$

[2] Also, remind that $\det\Lambda = 1$ for proper-orthochronous Lorentz group

[3] For further discussions, see S. Weinberg, *The Quantum Field Theory*, Vol. **1** (1995).

B.2 Spin sums Spinor technology

In this section, we formally discuss about spin sums and orthogonality of spinor $u_s(\mathbf{p})$ and $v_s(\mathbf{p})$. From the definition of these spinors which is designed as positive and negative energy spinor, following relations of orthogonality on rest frame easily confirmed by

$$\bar{u}_{s'}(\mathbf{0})u_s(\mathbf{0}) = \bar{v}_{s'}(\mathbf{0})v_s(\mathbf{0}) = 2m\delta_{ss'}, \qquad \bar{u}_{s'}(\mathbf{0})v_s(\mathbf{0}) = \bar{v}_{s'}(\mathbf{0})u_s(\mathbf{0}) = 0. \tag{B.14}$$

Having same approach, we also have spin sums on rest frame:

$$\sum_{s=\pm} u_s(\mathbf{0})\bar{u}_s(\mathbf{0}) = m\gamma^0 + m, \qquad \sum_{s=\pm} v_s(\mathbf{0})\bar{v}_s(\mathbf{0}) = m\gamma^0 - m. \tag{B.15}$$

So, taking proper Lorentz transforms, we shall calculate following spin sums and orthogonality relations out of rest frame:

Orthogonality of spinors

Feeding Lorentz transformations of each spinors (B.1) to orthogonality relation on rest frame (B.14), we shall calculate

$$\begin{aligned}
\bar{u}_{s'}(\mathbf{p})u_s(\mathbf{p}) &= \bar{u}_{s'}(\mathbf{0})\gamma^0 \left(\frac{\not{p}+m}{\sqrt{2m(E+m)}}\right)^\dagger \gamma^0 \frac{\not{p}+m}{\sqrt{2m(E+m)}} u_s(\mathbf{0}) \\
&= \frac{1}{2m(E+m)}\bar{u}_{s'}(\mathbf{0})(\not{p}\not{p} + m^2 + 2m\not{p})u_s(\mathbf{0}) \\
&= \frac{1}{2m(E+m)}\bar{u}_{s'}(\mathbf{0})(p^2 + m^2 + 2m\not{p})u_s(\mathbf{0}) \\
&= \frac{1}{E+m}u^\dagger_{s'}(\mathbf{0})\left[\gamma^0(\not{p}+m)\gamma^0\right]\gamma^0 u_s(\mathbf{0}) \\
&= \frac{1}{E+m}\bar{u}_{s'}(\mathbf{0})\left[(-\not{p}+m) + 2E\gamma^0\right]u_s(\mathbf{0}) \\
&= \frac{1}{2(E+m)}\bar{u}_{s'}(\mathbf{0})\left(\not{p}+m-\not{p}+m+2E\right)u_s(\mathbf{0}) \\
&= \bar{u}_{s'}(\mathbf{0})u_s(\mathbf{0}) = 2m\delta_{ss'}.
\end{aligned} \tag{B.16}$$

Take care of exchanging symmetry of left-right Weyl spinor components on positive spinors for rest frame $\gamma^0 u_s(\mathbf{0}) = u_s(\mathbf{0})$.

In 5th line of this calculation, we used adjoint representation of gamma matrices $(\not{p}+m)^\dagger = \gamma^0(\not{p}+m)\gamma^0$ and following relation of

$$(\not{p}+m)^\dagger = \left(-E\gamma^0 + \sum_i \mathbf{p}\cdot\boldsymbol{\gamma} + 2E\gamma^0\right) = -\not{p} + m + 2E\gamma^0. \tag{B.17}$$

Similarly, we also calculate orthogonality relation of negative energy spinors:

$$\begin{aligned}
\bar{v}_{s'}(\mathbf{p})v_s(\mathbf{p}) &= \bar{v}_{s'}(\mathbf{0})\gamma^0\left(\frac{-\not{p}+m}{\sqrt{2m(E+m)}}\right)^\dagger\gamma^0\frac{-\not{p}+m}{\sqrt{2m(E+m)}}v_s(\mathbf{0}) \\
&= \frac{1}{2m(E+m)}\bar{v}_{s'}(\mathbf{0})(\not{p}\not{p}+m^2-2m\not{p})v_s(\mathbf{0}) \\
&= \frac{1}{2m(E+m)}\bar{v}_{s'}(\mathbf{0})(p^2+m^2-2m\not{p})v_s(\mathbf{0}) \\
&= \frac{1}{E+m}v^\dagger_{s'}(\mathbf{0})\left[\gamma^0(-\not{p}+m)\gamma^0\right]\gamma^0 v_s(\mathbf{0}) \\
&= \frac{1}{E+m}\bar{v}_{s'}(\mathbf{0})\left[(\not{p}+m)-2E\gamma^0\right]v_s(\mathbf{0}) \\
&= \frac{1}{2(E+m)}\bar{v}_{s'}(\mathbf{0})\left(-\not{p}+m+\not{p}+m+2E\right)v_s(\mathbf{0}) \\
&= \bar{v}_{s'}(\mathbf{0})v_s(\mathbf{0}) = 2m\delta_{ss'}.
\end{aligned} \tag{B.18}$$

Take care of exchanging antisymmetry of left-right Weyl spinor components on positive spinors for rest frame $\gamma^0 v_s(\mathbf{0}) = -v_s(\mathbf{0})$. Also, from the results of upper calculations, we shall have following results:

Theorem B.2.1 (Spinor Technology I)

$$\begin{aligned}
\bar{u}_{s'}(\mathbf{p})\gamma^\mu u_s(\mathbf{p}) &= \frac{1}{2m}\bar{u}_{s'}(\mathbf{p})\left\{\gamma^\mu, -\not{p}+m+\not{p}\right\}u_s(\mathbf{p}) \\
&= \frac{p_\nu}{2m}\bar{u}_{s'}(\mathbf{p})\left\{\gamma^\mu,\gamma^\nu\right\}u_s(\mathbf{p}) = p^\mu\delta_{ss'}.
\end{aligned} \tag{B.19}$$

Theorem B.2.2 (Spinor Technology II)

$$\begin{aligned}\bar{v}_{s'}(\mathbf{p})\gamma^\mu v_s(\mathbf{p}) &= \frac{1}{2m}\bar{v}_{s'}(\mathbf{p})\left\{\gamma^\mu, \not{p}+m-\not{p}\right\} v_s(\mathbf{p}) \\ &= -\frac{p_\nu}{2m}\bar{u}_{s'}(\mathbf{p})\left\{\gamma^\mu,\gamma^\nu\right\} u_s(\mathbf{p}) = -p^\mu\delta_{ss'}.\end{aligned} \tag{B.20}$$

These additional results are called *Gordon identity*[4], putting $\mu = 0$, it turns back to relation of (1.91) and (1.92). Additionally we also have:

Theorem B.2.3 (Gamma Matrices Technology I)

$$\begin{aligned}(\not{p}+m)\gamma^0(\not{p}+m) &= \gamma^0(-\not{p}+m+2E\gamma^0)(\not{p}+m) \\ &= \gamma^0(-p^2+m^2)+2E(\not{p}+m) = 2E(\not{p}+m),\end{aligned} \tag{B.21}$$

and as we calculated above,

$$(\not{p}+m)(\not{p}+m) = p^2+m^2+2m\not{p} = 2m(\not{p}+m). \tag{B.22}$$

Now then taking similar calculation, we also have spin sums of spinors:

Spin sums

Next, we shall calculate spin sums of each spinors $u_s(\mathbf{p})$ and $v_s(\mathbf{p})$. Feeding Lorentz transformations of each spinors (B.1) to spin sum relation on rest frame (B.15), we shall compute

$$\begin{aligned}\sum_{s=\pm} u_s(\mathbf{p})\bar{u}_s(\mathbf{p}) &= \sum_{s=\pm}\frac{\not{p}+m}{\sqrt{2m(E+m)}}u_s(\mathbf{0})\bar{u}_s(\mathbf{0})\gamma^0\left(\frac{\not{p}+m}{\sqrt{2m(E+m)}}\right)^\dagger\gamma^0 \\ &= \frac{1}{2(E+m)}(\not{p}+m)(\gamma^0+1)(\not{p}+m) \\ &= \frac{1}{2(E+m)}\left[2E(\not{p}+m)+2m(\not{p}+m)\right] = \not{p}+m.\end{aligned} \tag{B.23}$$

[4]Take care of $p^0 < 0$ on spinor $v_s(\mathbf{p})$.

Similarly, we also compute[5]

$$\begin{aligned}\sum_{s=\pm} v_s(\mathbf{p})\bar{v}_s(\mathbf{p}) &= \sum_{s=\pm} \frac{-\not{p}+m}{\sqrt{2m(E+m)}} v_s(\mathbf{0})\bar{v}_s(\mathbf{0})\gamma^0 \left(\frac{-\not{p}+m}{\sqrt{2m(E+m)}}\right)^\dagger \gamma^0 \\ &= \frac{1}{2(E+m)}(-\not{p}+m)(\gamma^0-1)(-\not{p}+m) \\ &= \frac{1}{2(E+m)}\left[2E(\not{p}-m)+2m(\not{p}-m)\right] = \not{p}-m. \end{aligned} \tag{B.24}$$

These results above are the relation called **Spin sums for Spinors**, which is used to sum over the spin states for evaluating polarized fermions.

Traces for Gamma matrices

Also, let us discuss about traces for gamma matrices, which are usually used for calculation of scattering amplitude[6] $\mathcal{T}$ or renormalizations. From the property of traces, following traces are easily confirmed:

Theorem B.2.4 (Traces for Gamma matrices I)

$$\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\right] = \frac{1}{2}\mathrm{Tr}\left[\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu\right] = g^{\mu\nu}\mathrm{Tr}(\mathbf{1}_4) = 4g^{\mu\nu}. \tag{B.25}$$

So, similarly we also calculate:

Theorem B.2.5 (Nilpotent identity for Traces)

$$\begin{aligned}\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\gamma^\rho\right] &= \mathrm{Tr}\left[\gamma^5(\gamma^5\gamma^\mu\gamma^5)(\gamma^5\gamma^\nu\gamma^5)(\gamma^5\gamma^\rho\gamma^5)\gamma^5\right] \\ &= \mathrm{Tr}\left[(-\gamma^\mu)(-\gamma^\nu)(-\gamma^\rho)(\gamma^5)^2\right] = 0.\end{aligned} \tag{B.26}$$

[5] In this calculation, we use modified relation of (B.21) by

$$\begin{aligned}(-\not{p}+m)\gamma^0(-\not{p}+m) &= \gamma^0(\not{p}+m-2E\gamma^0)(-\not{p}+m) \\ &= \gamma^0(-p^2+m^2)+2E(\not{p}-m) = 2E(\not{p}-m).\end{aligned}$$

[6] That is defined by $S_{fi} = (2\pi)^4 i\delta^4(p_{\mathrm{in}}-p_{\mathrm{out}})\mathcal{T}$.

Take cere of results of calculation for gamma matrices:

$$\gamma^5\gamma^\mu\gamma^5 = \begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & -\mathbf{1}_2 \end{pmatrix} \begin{pmatrix} 0 & -\sigma^\mu \\ \sigma^\mu & 0 \end{pmatrix} \begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & -\mathbf{1}_2 \end{pmatrix} = -\gamma^\mu. \tag{B.27}$$

Also, uppere relations also supports more generalized form of

$$\mathrm{Tr}\left[\gamma^{\mu_1}\cdots\gamma^{\mu_{2k-1}}\right] = \mathrm{Tr}\left[(-\gamma^{\mu_1})\cdots(-\gamma^{\mu_{2k-1}})\right] = 0. \tag{B.28}$$

B.3 Canonical theories on Dirac fields

In this section, we shall consider canonical theories for Dirac fields(Lagrangian, Hamiltonian and other Lorentz invariant quantities). As we discussed on former sections, results of Lorentz groups now yields spinors are more basic quantities which are Lorentz invariance, so we can denotes scalars, 4-vectors, even tensors as a expansion of spinors. First, let us consider a **spinor representation of scalars** followed. That is easily confirmed from the representation of $S(\Lambda)$, we calculated on (1.62):

$$\begin{aligned} U^{-1}(\Lambda)\bar{\Psi}(x)\Psi(x)U(\Lambda) &= \bar{\Psi}(\Lambda^{-1}x)\gamma^0 S^\dagger(\Lambda)\gamma^0 S(\Lambda)\Psi(\Lambda^{-1}x) \\ &= \bar{\Psi}(\Lambda^{-1}x)S^{-1}(\Lambda)S(\Lambda)\Psi(\Lambda^{-1}x) = \bar{\Psi}(\Lambda^{-1}x)\Psi(\Lambda^{-1}x). \end{aligned} \tag{B.29}$$

where we calculate

$$\begin{aligned} \gamma^0 S^\dagger(\Lambda)\gamma^0 &= \frac{1}{\sqrt{2m(E+m)}}\begin{pmatrix} 0 & \mathbf{1}_2 \\ \mathbf{1}_2 & 0 \end{pmatrix} \\ &\quad \times \begin{pmatrix} E+m-\boldsymbol{\sigma}\cdot\mathbf{p} & 0 \\ 0 & E+m+\boldsymbol{\sigma}\cdot\mathbf{p} \end{pmatrix}\begin{pmatrix} 0 & \mathbf{1}_2 \\ \mathbf{1}_2 & 0 \end{pmatrix} \\ &= \frac{1}{\sqrt{2m(E+m)}}\begin{pmatrix} 0 & \mathbf{1}_2 \\ \mathbf{1}_2 & 0 \end{pmatrix}\begin{pmatrix} 0 & E+m-\boldsymbol{\sigma}\cdot\mathbf{p} \\ E+m+\boldsymbol{\sigma}\cdot\mathbf{p} & 0 \end{pmatrix} \\ &= \frac{1}{\sqrt{2m(E+m)}}\begin{pmatrix} E+m+\boldsymbol{\sigma}\cdot\mathbf{p} & 0 \\ 0 & E+m-\boldsymbol{\sigma}\cdot\mathbf{p} \end{pmatrix} = S^{-1}(\Lambda). \end{aligned} \tag{B.30}$$

But, this approach is only useful for proof of scalars. Even on 4-vectors, exactly it is difficult to prove the quantity[7] $\bar{\Psi}(x)\gamma^\mu\Psi(x)$ transforms by 4-vector like Lorentz transformation:

$$\begin{aligned} U^{-1}(\Lambda)\bar{\Psi}(x)\gamma^\mu\Psi(x)U(\Lambda) &= \bar{\Psi}(\Lambda^{-1})\gamma^0 S^\dagger(\Lambda)\gamma^0\gamma^\mu S(\Lambda)\Psi(\Lambda^{-1}x) \\ &= \bar{\Psi}(\Lambda^{-1})S^{-1}(\Lambda)\gamma^\mu S(\Lambda)\Psi(x). \end{aligned} \tag{B.31}$$

The upper approach of proof under the direct calculation using exact forms of $S(\Lambda)$ is not proper on 4-vectors. Hence, these approach above is not a generalized method, so we need to develop other approach for proof of other Lorentz invariant quantities(4-vectors, tensors).

[7]This quantity of $\bar{\Psi}(x)\gamma^\mu\Psi(x)$ is know as 4-vectors.

B.3.1 Lorentz invariance on Spinors

So, we shall develop a new approach which gives projection to momentum space. Let us consider momentum space expansion of Dirac fields:

$$\Psi(x) = \sum_{s=\pm} \int \frac{d^3k}{(2\pi)^3 2k^0} \left[b_s(\mathbf{k}) u_s(\mathbf{k}) e^{-ik\cdot x} + d_s^\dagger(\mathbf{k}) v_s(\mathbf{k}) e^{ik\cdot x}\right]. \tag{B.32}$$

Then, we shall re-evaluate scalar $\bar{\Psi}(x)\Psi(x)$ under the momentum space expansion of Dirac fields above:

Scalar representation on Spinor

From the expansion of Dirac fields on (B.32), we have:

Definition B.3.1 (Scalar representation on Dirac fields)

$$\int d^4x\ \bar{\Psi}(x)\Psi(x) = 2\pi \sum_{s=\pm} \int \frac{d^3k}{(2\pi)^3 2k^0}\ 2m \left[b_s^\dagger(\mathbf{k}) b_s(\mathbf{k}) + d_s(\mathbf{k}) d_s^\dagger(\mathbf{k})\right] \tag{B.33}$$

where we used delta function expansion of

$$\frac{1}{2p^0}\delta(p^0 - k^0) = \delta\left[(p^0)^2 - (k^0)^2\right]\Big|_{p^0,k^0>0} = \delta\left(|\mathbf{p}|^2 - |\mathbf{k}|^2\right)\Big|_{p^0,k^0>0} \tag{B.34}$$

and orthogonality of spinors (B.16). So, feeding Lorentz transformation rule of creation-annihilation operators of (B.9), the Lorentz transformation of quantity (B.33) now goes to

$$\begin{aligned}\int d^4x\ U^{-1}(\Lambda)\bar{\Psi}(x)\Psi(x)U(\Lambda) &= 2\pi \sum_{s=\pm} \int \frac{d^3k}{(2\pi)^3 2k^0} \\ &\quad \times 2m \left[b_s^\dagger(\mathbf{k}_{\Lambda^{-1}}) b_s(\mathbf{k}_{\Lambda^{-1}}) + d_s(\mathbf{k}_{\Lambda^{-1}}) d_s^\dagger(\mathbf{k}_{\Lambda^{-1}})\right] \\ &= 2\pi \sum_{s=\pm} \int \frac{d^3(\Lambda k)}{(2\pi)^3 2(\Lambda k)^0} \\ &\quad \times 2m \left[b_s^\dagger(\mathbf{k}) b_s(\mathbf{k}) + d_s(\mathbf{k}) d_s^\dagger(\mathbf{k})\right] \\ &= \int d^4x\ \bar{\Psi}(\Lambda^{-1}x)\Psi(\Lambda^{-1}x).\end{aligned} \tag{B.35}$$

Take care of Lorentz invariance[8] of $\frac{d^3k}{(2\pi)^2 2k^0}$ and $\det\Lambda = 1$. Now then, here by we confirmed that the quantity of (B.33) has scalar-like Lorentz transformation, from the result above.

4-Vector representation on Spinor

Also, we shall take similar method for 4-vector representation on spinors of $\Psi(x)\gamma^\mu\Psi(x)$. From the expansion of Dirac fields on (B.32), we have:

Definition B.3.2 (4-Vector representation on Dirac fields)

$$\int d^4x\ \bar{\Psi}(x)\gamma^\mu\Psi(x) = 2\pi\sum_{s=\pm}\int\frac{d^3k}{(2\pi)^3 2k^0}\ 2p^\mu\left[b_s^\dagger(\mathbf{k})b_s(\mathbf{k}) - d_s(\mathbf{k})d_s^\dagger(\mathbf{k})\right] \tag{B.36}$$

where we used *Gordon identity* derived on (B.19) and (B.20). So, feeding Lorentz transformation rule of creation-annihilation operators of (B.9), now we shall compute:

$$\begin{aligned}\int d^4x\ U^{-1}(\Lambda)\bar{\Psi}(x)\gamma^\mu\Psi(x)U(\Lambda) &= 2\pi\sum_{s=\pm}\int\frac{d^3k}{(2\pi)^3 2k^0}\\ &\quad\times 2p^\mu\left[b_s^\dagger(\mathbf{k}_{\Lambda^{-1}})b_s(\mathbf{k}_{\Lambda^{-1}}) - d_s(\mathbf{k}_{\Lambda^{-1}})d_s^\dagger(\mathbf{k}_{\Lambda^{-1}})\right]\\ &= 2\pi\sum_{s=\pm}\int\frac{d^3(\Lambda k)}{(2\pi)^3 2(\Lambda k)^0}\\ &\quad\times 2\Lambda^\mu{}_\nu p^\nu\left[b_s^\dagger(\mathbf{k})b_s(\mathbf{k}) - d_s(\mathbf{k})d_s^\dagger(\mathbf{k})\right]\\ &= \Lambda^\mu{}_\nu\int d^4x\ \bar{\Psi}(\Lambda^{-1}x)\gamma^\nu\Psi(\Lambda^{-1}x).\end{aligned} \tag{B.37}$$

So, now the quantity of (B.36) transforms by 4-Vector-like Lorentz transformation.

[8] Remind that Lorentz invariant measure only holds on-shell 0-component.

B.3.2 Lagrangian and Hamiltonian for Dirac fields

Appendix C

Formalism of Path integrals for Fermions

C.1 Field operators and Eigenstate

Let us start from the plane wave expansion of Dirac fields (B.7). Additionally, we shall rename Dirac field $\Psi(x)$ and its conjugate momentum $\Pi(x) = i\Psi^\dagger(x)$ to **Field operator** with new notation $\hat{\Psi}(x)$ and $\hat{\Pi}(x)$, belonging to Fock space of creation-annihilation operators, introduce field state $|\Psi\rangle$ and $|\Pi\rangle$which satisfies following eiegenstates:

Definition C.1.1 (Field operators and Eigenstates)

$$\hat{\Psi}(x)|\Psi'\rangle = \Psi'(x)|\Psi'\rangle, \qquad \hat{\Pi}(x)|\Pi'\rangle = \Pi'(x)|\Pi'\rangle, \tag{C.1}$$

and its conjugation:

$$\langle\Psi'|\hat{\Psi}(x) = \langle\Psi'|\Psi'(x), \qquad \langle\Pi'|\hat{\Pi}(x) = \langle\Pi'|\Pi'(x). \tag{C.2}$$

Take care of both of field operator $\hat{\Psi}(x)$, $\hat{\Pi}(x)$ and eigenfield $|\Psi\rangle$, $|\Pi\rangle$ are grassmann variables, from the result of spin-statistics theorem. In field theory, the Dirac field operator $\hat{\Psi}(x)$ expands by

$$\hat{\Psi}(x) = \sum_{s=\pm}\int \frac{d^3k}{(2\pi)^3 2k^0}\left[b_s(\mathbf{k})u_s(\mathbf{k})e^{-ik\cdot x} + d_s^\dagger(\mathbf{k})v_s(\mathbf{k})e^{ik\cdot x}\right], \tag{C.3}$$

and conjugate momentum $\hat{\Pi}(x)$ also expands by

$$\hat{\Pi}(x) = i \sum_{s=\pm} \int \frac{d^3k}{(2\pi)^3 2k^0} \left[d_s(\mathbf{k}) v_s^\dagger(\mathbf{k}) e^{-ik\cdot x} + b_s^\dagger(\mathbf{k}) u_s^\dagger(\mathbf{k}) e^{ik\cdot x} \right] \tag{C.4}$$

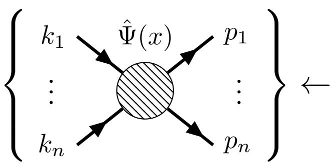
on Schrödinger picture. Now then we shall construct canonical quantization from the second quantization for creation-annihilation operators:

$$\begin{aligned}
\left\{ \hat{\Psi}_\alpha(x), \hat{\Pi}_\beta(y) \right\} &= i \sum_{s,s'=\pm} \int \frac{d^3k}{(2\pi)^3 2k^0} \int \frac{d^3p}{(2\pi)^3 2p^0} \\
&\quad \times \Bigg(\left\{ b_s(\mathbf{k}), b_{s'}^\dagger(\mathbf{p}) \right\} \left[\bar{u}_{s'}(\mathbf{p}) \gamma^0 u_s(\mathbf{k}) \right]_{\alpha\beta} e^{-i(k\cdot x - p\cdot y)} \\
&\qquad + \left\{ d_s(\mathbf{k}), d_{s'}^\dagger(\mathbf{p}) \right\} \left[\bar{v}_{s'}(\mathbf{p}) \gamma^0 v_s(\mathbf{k}) \right]_{\alpha\beta} e^{i(k\cdot x - p\cdot y)} \Bigg) \\
&= i \sum_{s=\pm} \int \frac{d^3k}{(2\pi)^3 2k^0} \\
&\quad \times \left(\left[\bar{u}_s(\mathbf{k}) \gamma^0 u_s(\mathbf{k}) \right]_{\alpha\beta} e^{-ik\cdot(x-y)} + \left[\bar{v}_s(\mathbf{k}) \gamma^0 v_s(\mathbf{k}) \right]_{\alpha\beta} e^{ik\cdot(x-y)} \right) \\
&= i\delta^3(\mathbf{x} - \mathbf{y}) \times (\mathbf{1}_4)_{\alpha\beta}
\end{aligned} \tag{C.5}$$

where we denoted[1] the second quantization for creation-annihilation operators of b-type and d-type particles by

$$\left\{ b_s(\mathbf{k}), b_{s'}^\dagger(\mathbf{p}) \right\} = \left\{ d_s(\mathbf{k}), d_{s'}^\dagger(\mathbf{p}) \right\} = (2\pi)^3 2k^0 \delta^3(\mathbf{k} - \mathbf{p}) \delta_{ss'}, \tag{C.6}$$

and also remind that $x^0 = y^0$. In quantum field theory, we shall always dealt about a second quantized fields, so our Dirac fields now satisfies canonical quantization condition of (C.5). Then, now we shall consider a exact scattering process:



$$\leftarrow \qquad \lim_{t\to\pm\infty} \hat{\Psi}(x) = \sqrt{Z}\hat{\Psi}_{\text{in/out}}(x). \tag{C.7}$$

[1]Also, we used the result of *Gordon identity* of (B.19) and (B.20).

for spinors with interaction Dirac field operator $\hat{\Psi}(x)$ and asymptotic Dirac field operators $\hat{\Psi}_{\mathrm{in/out}}(x)$ which satisfies asymptotic relation above. Reminding asymptotic relation of field, let us construct asymptotic relation of eigenstates $|\Psi\rangle$ and $|\Pi\rangle$:

Asymptotic representation of Eigenstate

Let us consider following quantity[2] with *Campbell-Backer-Hausdorff formula*:

$$\exp\left(i\int d^3y\ \hat{\Pi}_\beta(y)\Psi_\beta(y)\right)\hat{\Psi}_\alpha(x)\exp\left(-i\int d^3y\ \hat{\Pi}_\beta(y)\Psi_\beta(y)\right)$$
$$= \hat{\Psi}_\alpha(x) - i\int d^3y\ \left\{\hat{\Psi}_\alpha(x), \hat{\Pi}_\beta(y)\right\}\Psi(x) = \hat{\Psi}_\alpha(x) + \Psi_\beta(x)\times(\mathbf{1}_4)_{\alpha\beta}. \tag{C.8}$$

Then, this relation holds following asymptotic representation of eigenstate of $|\Psi\rangle$ $(t\to-\infty)$ and $\langle\Psi|$ $(t\to\infty)$, with antiparticle filled vacuum $|0\rangle$:

Theorem C.1.1 (Asymptotic representation of Eigenstate I)

$$|\Psi\rangle = \lim_{t\to-\infty}\exp\left(-i\int d^3y\ \hat{\Pi}_\beta(y)\Psi_\beta(y)\right)|0\rangle, \tag{C.9}$$

and its conjugation,

$$\langle\Psi| = \lim_{t\to\infty}\langle 0|\exp\left(i\int d^3y\ \hat{\Pi}_\beta(y)\Psi_\beta(y)\right). \tag{C.10}$$

Take care of existence of limit $t\to\pm\infty$ and antiparticle filled vacuum $|0\rangle$ is designed by $b_s(\mathbf{k})|0\rangle = d_s^\dagger(\mathbf{k})|0\rangle = 0$. These asymptotic representations above are easily confirmed by following expansion of eigenstate:

$$\hat{\Psi}(x)|\Psi'\rangle = \lim_{t\to-\infty}\exp\left(-i\int d^3y\ \hat{\Pi}(y)\Psi'(y)\right)$$
$$\times\left[\exp\left(i\int d^3y\ \hat{\Pi}(y)\Psi'(y)\right)\hat{\Psi}(x)\exp\left(-i\int d^3y\ \hat{\Pi}(y)\Psi'(y)\right)\right]$$
$$= \lim_{t\to-\infty}\left(\hat{\Psi}(x)+\Psi'(x)\right)|0\rangle = \Psi'(x)|\Psi'\rangle. \tag{C.11}$$

[2] After this section, we shall take simple notation of $[\hat{\Psi}(x), \hat{\Pi}(y)] := [\hat{\Psi}_\alpha(x), \hat{\Pi}_\beta(y)]$.

where we calculate

$$\lim_{t\to-\infty} \hat{\Psi}(x)|0\rangle = \sqrt{Z}\sum_{s=\pm}\int \frac{d^3k}{(2\pi)^3 2k^0} \left[b_s^{\text{in}} u_s(\mathbf{k})e^{-ik\cdot x} + d_s^{\text{in}\dagger} v_s(\mathbf{k})e^{ik\cdot x}\right]|0\rangle = 0 \tag{C.12}$$

with renormalization constant Z. Asymptotic representation of (C.14) also can be confirmed by similar method. Also, beginning with similar approach, this also gives:

Theorem C.1.2 (Asymptotic representation of Eigenstate II)

$$|\Pi\rangle = \lim_{t\to-\infty} \exp\left(i\int d^3y\ \hat{\Psi}_\beta(y)\Pi_\beta(y)\right)|0\rangle, \tag{C.13}$$

and its conjugation,

$$\langle\Pi| = \lim_{t\to\infty}\langle 0|\exp\left(-i\int d^3y\ \hat{\Psi}_\beta(y)\Pi_\beta(y)\right). \tag{C.14}$$

C.2 Path integral for Fermions

In this section, we would like to construct the path integral representation for Fermions. So, finally we shall derive well-known results of path integral for fermions:

$$\langle 0_+|0_-\rangle = \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi\ \exp\left(i\int d^4x\ \bar{\Psi}\left(-i\partial\!\!\!/ + m - i\epsilon\right)\Psi\right). \tag{C.15}$$

In this section, we denote $|0_\pm\rangle$ as a asymptotic vacuum[3] on $t\to\pm\infty$ and measure of path integrals by

$$\mathcal{D}\Psi = \prod_{\boldsymbol{x}} d\Psi(x). \tag{C.16}$$

Before we can do so, the discussion of orthogonality of eigenstates is needed. Now we shall look for orthogonality of these eigenstates, construct a grassmannian expansion on next section:

[3]Here we suppose that our vacuum has asymptotic completeness and uniqueness.

C.2.1 Orthogonality of Eigenstates

Let us take eigenstates of $|\Psi\rangle$ and $|\Psi'\rangle$. Then, from the asymptotic representation on *Definition* (C.1.1), the orthogonality of eigenstates is easily confirmed:

Theorem C.2.1 (Orthogonality of Eigenstates)

$$\langle\Psi'|\Psi\rangle = \prod_{\boldsymbol{x}} C_{\boldsymbol{x}} \delta\left(\Psi'(x) - \Psi(x)\right). \tag{C.17}$$

which is supported by

$$0 = \left[\Psi'(x) - \Psi(x)\right] \langle\Psi'|\Psi\rangle. \tag{C.18}$$

Next, we shall take normalization of eigenstates by $C_{\boldsymbol{x}} = 1$, consider a expansion of delta functional of $\delta\left(\Psi'(x) - \Psi(x)\right)$ on (C.17).

Fermionic Delta Functionals

In bosonic field theory, the expansion of delta functional easily derived by taking limit of $i \to \boldsymbol{x}$ on delta function expansion for discrete integration variables:

$$\int \prod_i \frac{dk_i}{2\pi} \exp\left(\sum_i k_i \xi_i\right) \quad \underset{i\to\boldsymbol{x}}{\longrightarrow} \quad \int \mathcal{D}k \; \exp\left(\int d^4x \; 2\pi k(x)\xi(x)\right). \tag{C.19}$$

However, in fermionic field theory, it can not do so. From the anticommutable property of grassmann variables, delta functional now expands by:

Theorem C.2.2 (Fermionic Delta Functionals)

$$\prod_{\boldsymbol{x}} \delta\left(\Psi'(x) - \Psi(x)\right) = \prod_{\boldsymbol{x}} \Psi'(x) - \Psi(x). \tag{C.20}$$

To prove this, here we starts from the expansion of grassmannian delta function for discrete integration variables:

$$\delta(\psi_i - \psi_i') = a + b\left(\psi_i - \psi_i'\right). \tag{C.21}$$

Feeding this expansion to delta function identity of $\int d\psi_i\, \delta(\psi_i - \psi_i')f(\psi_i)$, we can determine the coefficient of grassmannian variables a, b. So,

$$\int d\psi_i\ aA + aB\psi_i + b(\psi_i - \psi_i')A - b\psi_i' B\psi_i = aB + (A + B\psi_i')b \tag{C.22}$$

where $f(\psi_i) = A + B\psi_i$. Hence, result of (C.22) yields $a = 0, b = 1$, now the expansion of fermionic delta functional on *Theorem* (C.2.2) is confirmed.

C.2.2 Fermionic Projection Operator

In this section, we shall construct projection operator for Dirac fields, like a operator $\int dx\ |x\rangle\langle x|$ on QM. Before we can do so, let us consider $\langle\Pi|\Psi\rangle$. From the *Theorem* (C.1.1), specifically we calculate:

Theorem C.2.3 (Fourier Functional for Dirac Field)

$$\begin{aligned}\langle\Pi|\Psi\rangle &= \lim_{t\to-\infty}\langle\Pi|\exp\left(-i\int d^3y\ \hat{\Pi}(y)\Psi(y)\right)|0\rangle \\ &= \exp\left(-i\int d^3y\ \Pi(y)\Psi(y)\right)\langle\Pi|0\rangle \\ &= \exp\left(-i\int d^3y\ i\bar{\Psi}(y)\gamma^0\Psi(y)\right)\end{aligned} \tag{C.23}$$

where

$$\langle\Pi|0\rangle = \langle 0|\left(\sum_{n=0}^{\infty}\frac{(-i\sqrt{Z})^n}{n!}\int d^{3n}y\ \hat{\Pi}^n_{\text{out}}(y)\Psi^n(y)\right)|0\rangle = \langle 0|0\rangle = 1. \tag{C.24}$$

The relation between two eigenfields of $\Psi(x)$ and $\Pi(x)$ can be derived by considering conjugate of $\hat{\Psi}(x)|\Psi\rangle$. Specifically,

$$-i\langle\Psi|\hat{\Pi}(x) = \langle\Psi|\Psi^\dagger(x) \tag{C.25}$$

Hence, reminding that we defined $|\Pi\rangle$ as a eigenstate of field operator $\hat{\Pi}(x)$, now we have eigenfield relation of $\Pi(x) = i\Psi^\dagger(x)$ and eigenstate relation of $|\Psi\rangle$ = linear combination of $|\Pi\rangle$s.

So, from the relations of (C.17), (C.20) and (C.23), we shall conclude:

Theorem C.2.4 (Completeness of Eigenstate I)

$$\begin{aligned}\langle\Psi'|\Psi\rangle &= \gamma^0 \int \mathcal{D}\bar{\Psi}\ \exp\left(-i \int d^3y\ i\bar{\Psi}(y)\gamma^0 \left(\Psi(y) - \Psi'(y)\right)\right) \\ &= \gamma^0 \int \mathcal{D}\bar{\Psi}\ \langle\Psi'|\Pi\rangle\langle\Pi|\Psi\rangle.\end{aligned} \tag{C.26}$$

which is supported by limit $i \to \boldsymbol{x}$ of following grassmann integration for discrete integration variables:

$$\begin{aligned}\int d^n\bar{\psi}\ \exp\left(\sum_i \bar{\psi}_i\xi_i\right) &= \int d^n\bar{\psi}\ \prod_i \left(1 + \bar{\psi}_i\xi_i\right) = \prod_i \xi_i \\ \xrightarrow[i\to\boldsymbol{x}]{} \quad &\int \mathcal{D}\bar{\psi}\ \exp\left(\int d^4x\ \bar{\psi}(x)\xi(x)\right) = \prod_{\boldsymbol{x}} \xi(x)\end{aligned} \tag{C.27}$$

for $d^n\bar{\psi} = d\bar{\psi}_1 \cdots d\bar{\psi}_n$. Similarly,

Theorem C.2.5 (Completeness of Eigenstate II)

$$\langle\Pi'|\Pi\rangle = \int \mathcal{D}\Psi\ \langle\Pi'|\Psi\rangle\langle\Psi|\Pi\rangle. \tag{C.28}$$

C.2.3 Path integral representation for Dirac Field

Consider a vacuum norm of

$$\langle 0_+|0_-\rangle = \lim_{t_i,t_f\to\pm\infty}\langle 0|e^{-iHt_f}e^{iHt_i}|0\rangle \tag{C.29}$$

where we denote $|0\rangle$ for vacuum at $t=0$. However, since the term of e^{iHt_i} and e^{-iHt_f} diverge on limit $t_i, t_f \to \pm\infty$, so our hamiltonian must be redefined[4] by $H \to (1-i\epsilon)H$. So, we have:

$$\langle 0_+|0_-\rangle = \lim_{t_i,t_f\to\pm\infty}\langle 0|e^{-i(1-i\epsilon)Ht_f}e^{i(1-i\epsilon)Ht_i}|0\rangle \tag{C.30}$$

which is effective on asymptotic limit of $t_i, t_f \to \pm\infty$. Now then we shall take partition of $\delta t_+ = t_f/N, \delta t_- = t_i/N$, now our $\langle 0_+|0_-\rangle$ goes to

$$\begin{aligned}
\langle 0_+|0_-\rangle &= \lim_{\delta t\to 0}\int \mathcal{D}\Psi_1\cdots\mathcal{D}\Psi_N\ \langle 0|\Psi_1\rangle\langle\Psi_1|e^{-i(1-i\epsilon)H\delta t_+}|\Psi_2\rangle \\
&\qquad\times\langle\Psi_2|e^{-i(1-i\epsilon)H\delta t_+}|\Psi_3\rangle\times\cdots\times\langle\Psi_{2N}|0\rangle \\
&= \lim_{\delta t\to 0}(\gamma^0)^{2N}\int\prod_{i,j}\mathcal{D}\Psi_i\mathcal{D}\bar{\Psi}_j\ \langle 0|\Psi_1\rangle\langle\Psi_1|\Pi_1\rangle \\
&\qquad\times\exp\left(-i\delta t_+\int d^3x\ \bar{\Psi}_1\left(i\boldsymbol{\gamma}\cdot\boldsymbol{\partial}+m-i\epsilon\right)\Psi_1\right)\langle\Pi_1|\Psi_2\rangle\times\cdots \\
&\qquad\times\langle\Pi_{2N-1}|e^{-i(1-i\epsilon)H\delta t_-}|\Psi_{2N}\rangle\langle\Psi_{2N}|0\rangle \\
&= \lim_{\delta t\to 0}(\gamma^0)^{2N}\int\prod_{i,j}\mathcal{D}\Psi_i\mathcal{D}\bar{\Psi}_j\ \langle 0|\Psi_1\rangle \\
&\qquad\times\exp\left(-i\sum_i\delta t_i\int d^3x\ \bar{\Psi}_i\left(i\boldsymbol{\gamma}\cdot\boldsymbol{\partial}+m-i\epsilon\right)\Psi_i\right) \\
&\qquad\times\exp\left(i\sum_i\delta t_i\int d^3x\ \bar{\Psi}_i\gamma^0\frac{\Psi_i-\Psi_{i+1}}{\delta t_i}\right)\langle\Psi_{2N}|0\rangle
\end{aligned} \tag{C.31}$$

[4]Specifically, this modification is called *Wick rotation*.

where we inserted each of projection operator on *Theorem* (C.2.4) and *Theorem* (C.2.5). Taking $\langle 0|\Psi_i\rangle = 1$, now our $\langle 0_+|0_-\rangle$ goes to

$$\langle 0_+|0_-\rangle = \lim_{\delta t\to 0}\int \prod_{i,j}\mathcal{D}\Psi_i\mathcal{D}\bar{\Psi}_j\ \exp\left(i\sum_i \delta t_i \int d^3x\ \bar{\Psi}_i\left(i\partial\!\!\!/ - m + i\epsilon\right)\Psi_i\right) \tag{C.32}$$

with our ordering of measure

$$\begin{aligned}\prod_{i,j}\mathcal{D}\Psi_i\mathcal{D}\bar{\Psi}_j &= \mathcal{D}\Psi_1\mathcal{D}\bar{\Psi}_1\cdots\mathcal{D}\Psi_{2N}\mathcal{D}\bar{\Psi}_{2N} \\ &= \mathcal{D}\bar{\Psi}_1\mathcal{D}\Psi_1\cdots\mathcal{D}\bar{\Psi}_{2N}\mathcal{D}\Psi_{2N}.\end{aligned} \tag{C.33}$$

So, finally we take

Theorem C.2.6 (Path integral representation for Dirac Field)

$$\langle 0_+|0_-\rangle = \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi\ \exp\left(i\int d^4x\ \bar{\Psi}\left(i\partial\!\!\!/ - m + i\epsilon\right)\Psi\right) \tag{C.34}$$

where

$$\mathcal{D}\bar{\Psi}\mathcal{D}\Psi = \prod_{x^\mu} d\bar{\Psi}(x)d\Psi(x). \tag{C.35}$$

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