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1 Gradient free optimization algorithms

We have already studied optimization algorithms using calculus. In practice most of the functions in the real life optimization models are not differentiable. In that case, though minimum exists but the algorithms, discussed so far like Newton and its variations, can not help us to derive the minimum value. For example: min $x_1|x_2-1|^3 - x_2|x_1-1|^3$. Following algorithms are useful to handle such type problems. These do not use use gradients or derivatives. There are many other methods which are at research stage. We will study few methods only.

- Golden Section Method for single variable case
- Fibonacci Search method for single variable case
- Nelder Mead Simplex method for multi-variable case
- Coordinate search methods for multi-variable case

1.1 Golden section method

An interval $[a, b]$ is subdivided in Golden section ratio if

$$\frac{\text{length of the whole interval}}{\text{length of bigger subinterval}} = \frac{\text{length of bigger subinterval}}{\text{length of the smaller subinterval}}.$$

Consider an optimization problem in R as

$$\min_{x \in I} f(x),$$

where I is an interval and f is not necessarily a differentiable function. In golden section method, the initial interval I is divided into a sequence if subintervals I_k following Golden section rule so that the final interval of uncertainty of desired length contains the solution.

Notation

$[x_{lk}, x_{uk}]$= subinterval at k^{th} iteration.

$I_k = x_{uk} - x_{lk}$= length of the subinterval at k^{th} iteration.
I_k^L= length of the left part of k^{th} subinterval.
I_k^R= length of the right part of k^{th} subinterval.
x_{pk}= 1st trial point at k^{th} iteration ($x_{lk} \leq x_{pk} \leq x_{uk}$).
x_{qk}= 2nd trial point at k^{th} iteration. ($x_{lk} \leq x_{qk} \leq x_{uk}$)

Two trial points x_{pk} and x_{qk} are chosen in such a way that:
(i) $I_k^L = I_k^R \ \forall k$
(ii) $x_{pk+1} = x_{qk}$ and x_{qk+1} is computed after.
(or $x_{qk+1} = x_p k$ and x_{pk+1} is computed a fresh for I_{k+1}^L).
That is, only one trail point is computed at k^{th} iteration and other trial point is obtained from previous iteration so that

$$I_k = I_{k+1} + I_{k+2} \tag{1.1}$$

and

$$\frac{I_k}{I_{k+1}} = \frac{I_{k+1}}{I_{k+2}} = c \tag{1.2}$$

From (1.1) and (1.2),
$\frac{I_k}{I_{k+2}} = 1 + \frac{I_{k+1}}{I_{k+2}}$
or $\frac{I_k}{I_{k+1}} \cdot \frac{I_{k+1}}{I_{k+2}} = 1 + \frac{I_{k+1}}{I_{k+2}}$
or $c^2 = 1 + c$
Solving this we get $c = 1.618$, which is known as Golden section ratio. Note that $\frac{1}{c} = 0.618$.
In general, if the new interval of uncertainty is $[x, b]$, then

$$\frac{b-a}{b-x} = c \Rightarrow x = b - \frac{1}{c}(b-a) = b - 0.618(b-a),$$

and if the new interval of uncertainty is $[a, y]$, then

$$\frac{b-a}{y-a} = c \Rightarrow y = a + \frac{1}{c}(b-a) = b - 0.618(b-a)$$

Algorithm for Golden section method (For min problem)

1. $[a, b] = [x_{l1}, x_{u1}],\ \epsilon > 0, f$

2. $$x_{p1} = x_{u1} - 0.618(x_{u1} - x_{l1})$$
   $$x_{q1} = x_{l1} + 0.618(x_{u1} - x_{l1}$$
   set $k = 1$

3. Compute $f(x_{pk}), f(x_{qk})$. If $f(x_{pk}) \leq f(x_{qk})$, then go to next step otherwise go to step 5

4. New interval is $[x_{lk+1}, x_{uk+1}]$,
   $x_{lk+1} = x_{lk}$, $x_{uk+1} = x_{qk}$
   $$x_{pk+1} = x_{uk+1} - 0.618I_{k+1}$$
   $$x_{qk+1} = x_{pk}$$

5. Update data. New interval is $[x_{lk+1}, x_{uk+1}]$
   $x_{lk+1} = x_{pk}$, $x_{uk+1} = x_{uk}$
   $$x_{pk+1} = x_{pk}$$
   $$x_{qk+1} = x_{lk} + 0.618I_{k+1}$$

6. Test if $I_k < \epsilon$. If so, go th=o next step, otherwise set $k = k + 1$, then go to step 3.

7. Output $(x_{lk}x_{uk}),\ x_{min} \in (x_{lk}x_{uk})$

1.1.1 Fibonacci search method

Fibonacci sequence: $F_0 = 1$, $F_1 = 1$, $F_2 = F_0 + F_1 = 2$,...,$F_i = F_{i-1} + F_{i-2}$,....
Logic for Fibonacci search method is (i) $I_k^L = I_k^R$, (ii) $I_k = I_{k+1} + I_{k+2}$, (iii) $\frac{I_k}{I_{k+1}} =$

$\frac{F_{n-k+1}}{F_{n-k}}$.

$$\begin{aligned}\frac{I_1}{I_n} &= \frac{I_1}{I_2}\cdot\frac{I_2}{I_3}\cdot\ldots\frac{I_{n-1}}{I_n}\\ &= \frac{F_n}{F_{n-1}}\cdot\frac{F_{n-1}}{F_{n-2}}\cdot\ldots\frac{F_1}{F_0} = \frac{F_n}{F_0} = F_n\end{aligned}$$

In Golden section method, $\frac{I_k}{I_{k+1}} = c = 1.618$,
In Fibonacci method, $\frac{I_k}{I_{k+1}} = \frac{F_{n-k}}{F_{n-k+1}}$.
In Fibonacci method:
Step 4: $x_{pk+1} = x_{uk+1} - \frac{F_{n-k}}{F_{n-k+1}} I_{k+1}$
Step 5: $x_{lk} + \frac{F_{n-k}}{F_{n-k+1}} I_{k+1}$
If $I_1 = 20$, $\epsilon = 1.5$, then $I_n = 1.5$. So in Fibonacci method $\frac{I_1}{I_n} = 13.3 \quad \Rightarrow n = 7$.
In Golden section $\frac{I_1}{I_n} = \frac{1}{(c)^{n-1}} \Rightarrow n = 7$.

2 Assignment

Consider the following unconstrained nonlinear programming problems. Execute Golden Section and Fibonacci method to find the minimum value. Consider other parameters like ϵ as per your choice and find the number of iterations too execute. Write a program for this algorithm and find the answer. Compute atleast 5 iterations manually. See the book chapter for details.

- P_3 : $\min_{x\in[-6,6]} x^2 + 9x$
- P_4 : $\max_{x\in[1,9]} 21x - 2x^2$