Correlation (1): Cross correlation

- Correlation~ measure of similarity between two data sequences:

Cross correlation b/w x[n] and y[n] i.e. Correlation Coefficient

$$r_{xy}[l] = \sum_{n=-\infty}^{\infty} x[n]y[n-l], \quad l = 0, \pm1, \pm2, \ldots$$

"lag", indicates the time shift between the pair of signals.

- Note:

$$r_{yx}[l] = \sum_{n=-\infty}^{\infty} y[n]x[n-l]$$

call $m = n - l$

$$= \sum_{m=-\infty}^{\infty} y[m+l]x[m] = r_{xy}[-l]$$

Time reversed version

- Autocorrelation function is symmetric w.r.t. the vertical axis.

Drill Problem

- Determine the cross-correlation sequence $r_{xy}(l)$

x(n) = {......0. 0. 2. -1. 3. 7 .1. 2. -3. 0. 0.........}
↑

y(n) = {......0. 0. 1. -1. 2. -2. 4. 1. -2. 5. 0. 0.........}
↑

$$r_{xy}(0) = \sum_{n=-\infty}^{\infty} x(n)y(n)$$

$$r_{xy}(0) = 2 + 1 + 6 - 14 + 4 + 2 + 6 = 7$$

$$r_{xy}(1) = \sum_{n=-\infty}^{\infty} x(n)y(n-1)$$

y(n - 1) = {......0. 0. 1. -1. 2. -2. 4. 1. -2. 5.0.0..........}
↑

$$r_{xy}(1) = -1 - 3 + 14 - 2 + 8 - 3 = 13$$

Drill Problem (Cont'd)

$r_{xy}(2) = -18 \quad r_{xy}(3) = 16 \quad r_{xy}(4) = -7 \quad r_{xy}(5) = 5 \quad r_{xy}(6) = -3 \quad r_{xy}(l) = 0 \quad l \geq 7$

$r_{xy}(-1) = 0 \quad r_{xy}(-2) = 33 \quad r_{xy}(-3) = -14 \quad r_{xy}(-4) = 36 \quad r_{xy}(-5) = 19$

$r_{xy}(-6) = -9 \quad r_{xy}(-7) = 10 \quad r_{xy}(l) = 0 \quad for\, l \leq -8$

$r_{xy}(l)$ = {10. -9. 19. 36 .-14. 33. 0. 7. 13. -18. 16. -7. 5. -3}
↑

Correlation (2): Auto correlation

- Autocorrelation (AC) is correlation of signal with itself. The autocorrelation of a sequence is the correlation of a sequence with its shifted version, and this indicates how fast the signal changes.

$$r_{xx}[l] = \sum_{n=-\infty}^{\infty} x[n]y[n-l] = r_{xx}[-l]$$

Even Sequence for a real sequence x[n]

- Note:

Autocorrelation has a maximum value at l= 0

$$r_{xx}[0] = \sum_{n=-\infty}^{\infty} x^2[n] = E_x$$

Energy of sequence x[n]

$$r_{xx}[l] \leq r_{xx}[0]$$

$$x[n] \longrightarrow \boxed{x[-n]} \longrightarrow r_{xx}[n]$$

Correlation (3): Relation between Correlation and Convolution

- Correlation:

$$r_{xy}[n] = \sum_{k=-\infty}^{\infty} x[k]y[k-n]$$

- Convolution:

$$x[n] * y[n] = \sum_{k=-\infty}^{\infty} x[k]y[n-k]$$

- Hence:

$$r_{xy}[n] = x[n] * y[-n]$$

- Correlation may be calculated by convolving with time-reversed sequence

Correlation (4): Properties of Cross and Auto correlation Sequences

- Properties:

$$|r_{xy}|^2 \leq r_{xx} r_{yy}$$

$$|r_{xy}|^2 = r_{xx} r_{yy}$$

- Correlation Coefficient

$$\rho_{xy} = \frac{|r_{xy}|}{\sqrt{r_{xx} r_{yy}}}$$

$$0 \leq \rho_{xy} \leq 1$$

$\rho_{xy} \cong 0$ — x,y uncorrelated

$\rho_{xy} \cong 1$ — x,y strongly correlated

Correlation maxima

- Note: The normalized autocorrelation of x[n] is defined as

$$r_{xx}[l] \le r_{xx}[0] \quad \Rightarrow \left|\frac{r_{xx}[l]}{r_{xx}[0]}\right| \le 1$$

- Similarly, The normalized cross-correlation b/w x[n] and y[n]is defined as

$$r_{xy}[l] \le \sqrt{E_x E_y} \quad \Rightarrow \frac{r_{xy}[l]}{\sqrt{r_{xx}[0] r_{yy}[0]}} \le 1$$

- From Geometry,

$$\sqrt{\sum x_i^2}$$

$$\langle xy \rangle = \sum_i x_i y_i = |x||y|\cos\theta$$

angle between x and y

- when x//y, cos θ = 1, else cos θ < 1

Drill Problem

- Determine the normalized autocorrelation of x[n]=aⁿu[n], 0<a<1.
- For l ≥ 0 the autocorrelation

$$r_{xx}(l) = \sum_{n=l}^{\infty} x(n)x(n-l) = \sum_{n=l}^{\infty} a^n a^{n-l}$$

$$r_{xx}(l) = a^l \sum_{n=0}^{\infty} (a^2)^n = \frac{a^l}{1-a^2}$$

- For l < 0 the autocorrelation

$$r_{xx}(l) = \sum_{n=0}^{\infty} x(n)x(n-l) = \sum_{n=0}^{\infty} a^n a^{n-l}$$

$$r_{xx}(l) = a^{-l} \sum_{n=0}^{\infty} (a^2)^n = \frac{a^{-l}}{1-a^2}$$

Drill Problem (Cont'd)

- Then for all l

$$r_{xx}(l) = \frac{a^{|l|}}{1-a^2}$$

$$r_{xx}(0) = \frac{1}{1-a^2}$$

$$\rho_{xx}(l) = \frac{r_{xx}(l)}{r_{xx}(0)} = a^{|l|}$$