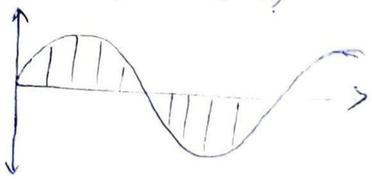
Discrete Time Periodic Signal

$x(t) \xrightarrow{\text{Sampled}} x(n)$ $\qquad$ Fundamental Period $= N = \frac{F_s}{F}$

When $t = nT_s$

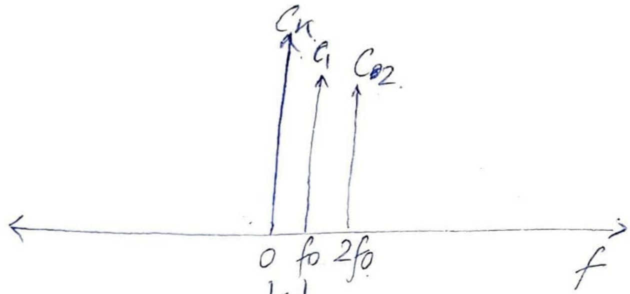
$\Rightarrow x(nT_s)$



Now, any continuous time periodic signal can be represented as continuous time fourier series.

$$x(t) = \sum_{K=-\infty}^{\infty} C_K e^{j2\pi K f_0 t}$$

$\Rightarrow$ Spectrum of CTPS ranges from $-\infty$ to $+\infty$.



$\frac{1}{T_P} = f_0 = \frac{1}{\text{Fundamental period}}$

For, spectrum of DTPS.

$x(t) = A \sin(2\pi f t)$

$x(n) = A \sin(2\pi f nT_s)$

$x(n) = A \sin(2\pi \frac{f}{f_s} n)$

$2\pi \frac{f}{f_s}$ = Discrete freq.

$\therefore$ Spectrum of DTPS ranges from $-\pi$ to π.

Without FFTshift, DTPS spectrum will range from 0 to 2π.

With fft shift, $-\pi$ to 0 will be mirror effect of 0 to π.
Without fft shift, π to 2π will be mirror effect of 0 to π.

For 2π period, fundamental period $= \frac{2\pi}{N}$

DTFS $\rightarrow$ $\boxed{x(n) = \sum_{K=0}^{N-1} C_K e^{j\frac{2\pi n K}{N}}}$ Synthesis Expression

$$\boxed{x(n) = \sum_{K=0}^{N-1} C_K e^{\frac{j2\pi Kn}{N}}}$$ ~~Anal~~ Synthesis Expression

Multiply both sides by $e^{-j\frac{2\pi ln}{N}}$

$$\sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi ln}{N}} = \sum_{n=0}^{N-1}\sum_{K=0}^{N-1} C_K e^{j\frac{2\pi nK}{N}}\, e^{-j\frac{2\pi ln}{N}}$$

$$= \sum_{K=0}^{N-1} C_K \sum_{n=0}^{N-1} e^{j\frac{2\pi(K-l)n}{N}}$$

Case I $\quad K = l$ #

$$RHS = \sum_{l=0}^{N-1} C_l \sum_{n=0}^{N-1} e^{(0)}$$

can be removed, as we are looking at only one place. $K = l$.

$$RHS = C_l N.$$

$$\boxed{\sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi ln}{N}} = C_l N}$$

Case II $\quad K \neq l$

$$RHS = \sum_{K=0}^{N-1} C_K \sum_{n=0}^{N-1} e^{j\frac{2\pi(K-l)n}{N}}$$

$$= \sum_{K=0}^{N-1} C_K \left\{ \sum_{n=0}^{N-1} \left[e^{j\frac{2\pi(K-l)}{N}} \right]^n \right\}$$

$$= \sum_{K=0}^{N-1} C_K \left[\frac{1 - e^{j2\pi(K-l)\cancel{N}/\cancel{N}}}{1 - e^{j\frac{2\pi(K-l)}{N}}} \right]$$

$$\boxed{\begin{aligned} a &= e^{j\frac{2\pi(K-l)}{N}} \\ \sum_{n=0}^{N-1} a^n &= \frac{1-a^N}{1-a} \\ a &\neq 1 \end{aligned}}$$

$$\boxed{\sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi ln}{N}} = 0}$$

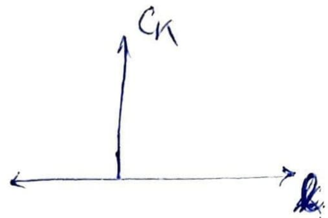
Now,

$$\boxed{C_l = \frac{1}{N} \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi ln}{N}}}$$ Analysis Expression.

$$\boxed{C_{l+N} = C_l}$$

Ex

$x[n] = \cos\left(\frac{\pi n}{3}\right)$ Find c_K.



$= \cos\left(\frac{2\pi n}{6}\right)$

$$c_l = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi l n}{N}}$$

$N = 6 \qquad K = 1$

$e^{-i\theta} = \cos\theta - i\sin\theta$

$$c_l = \frac{1}{6}\sum_{n=0}^{5} x(n)\, e^{-j\frac{2\pi l n}{6}}$$

$$c_l = \frac{1}{6}\sum_{n=0}^{5} \cos\left(\frac{2\pi n}{6}\right) e^{-j\frac{2\pi l n}{6}}$$

$$c_0 = \frac{1}{6}\left[1 + \frac{1}{2} - \frac{1}{2} - 1 - \frac{1}{2} + \frac{1}{2}\right]$$

$= 0$

$$c_1 = \frac{1}{6}\left[1 + \frac{1}{2}\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right) - \frac{1}{2}\left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right) - 1\left(\right.\right.$$

Power signal DTPS.

$$\boxed{P_x = \frac{1}{N}\sum_{n=0}^{N-1}|x(n)|^2}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, x^*(n)$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x(n)\left[\sum_{K=0}^{N-1} C_K e^{j\frac{2\pi nK}{N}}\right]^*$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x(n)\left[\sum_{K=0}^{N-1} C_K^* e^{-j\frac{2\pi nK}{N}}\right]$$

Changing order of summation

$$= \sum_{K=0}^{N-1} C_K^* \underbrace{\left[\frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi nK}{N}}\right]}_{C_K}$$

$$= \sum_{K=0}^{N-1} C_K^* C_K$$

$$= \sum_{K=0}^{N-1} |C_K|^2.$$

$$\boxed{P_x = \frac{1}{N}\sum_{n=0}^{N-1}|x(n)|^2 = \sum_{K=0}^{N-1}|C_K|^2}$$ Power density Spectrum.

Parseval's Theorem of DTFS.

$\Rightarrow$ If $N \to \infty$, $\frac{1}{N} \to 0$, then spectrum lines will come closer and closer.

Fundamental period $\to \infty$

$\downarrow$

Signal becomes aperiodic.

Discrete Time Aperiodic Signal

For aperiodic signal, spectrum is continuous.

F.T. of DTAS is $\boxed{X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}}$

For DT signal, spectrum range is $(-\pi, \pi)$ or $(0, 2\pi)$

Using F.T. expression,

Multiply both sides by $e^{j\omega m}$ and integrate over 2π period.

$$\int_{-\pi}^{\pi} X(\omega)\, e^{j\omega m}\, d\omega = \int_{-\pi}^{\pi}\left(\sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}\right) e^{j\omega m}\, d\omega.$$

Now $RHS = \int_{-\pi}^{\pi} \left(\sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \right) e^{j\omega m} d\omega$

Changing order of summation & integration by convergence condition

$$= \sum_{n=-\infty}^{\infty} x(n) \int_{-\pi}^{\pi} e^{-j\omega n} \cdot e^{j\omega m} d\omega$$

$\lim_{N \to \infty} X(\omega) = X_n(\omega)$

$$= \sum_{n=-\infty}^{\infty} x(n) \int_{-\pi}^{\pi} e^{j\omega(m-n)} d\omega.$$

Case I $\Rightarrow$ $m = n$

$$RHS = 2\pi \sum_{n=-\infty}^{\infty} x(m)$$

$$2\pi \sum_{n=-\infty}^{\infty} x(m) = \int_{-\pi}^{\pi} X(\omega) e^{j\omega m} d\omega.$$

$$x(m) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega m} d\omega.$$

Case II $\Rightarrow$ $m \neq n$.

$$RHS = \sum_{n=-\infty}^{\infty} x(n) \int_{-\pi}^{\pi} e^{j\omega(m-n)} d\omega.$$

$$= \sum_{n=-\infty}^{\infty} x(n) \left[\frac{e^{j\omega(m-n)}}{j(m-n)} \right]_{-\pi}^{\pi}$$

$$= \sum_{n=-\infty}^{\infty} x(n) \left[\frac{e^{j\pi(m-n)} - e^{-j\pi(m-n)}}{j(m-n)} \right]$$

$$RHS = \sum_{n=-\infty}^{\infty} x(n) \left[\frac{\cancel{2\sin(\pi(m-n))}}{(m-n)} \right] \to 0$$

Since m & n are integers, $m - n$ will also be integers and term will be zero, as it is multiple of π.

$$RHS = 0$$

$$\int_{-\pi}^{\pi} X(\omega) e^{j\omega m} d\omega = 0$$

$$\boxed{x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) \cdot e^{j\omega n} \, d\omega}$$ Synthesis Expression

$$\boxed{X(\omega) = \sum_{n=-\infty}^{\infty} x(n) \, e^{-j\omega n}}$$ Analysis Expression.

Conditions of Convergence

If either of these two conditions satisfies, then F.T. surely exist for discrete signals.

1.) Uniform Convergence.

$$\int_{-\pi}^{\pi} X(\omega) \, e^{j\omega n} \, d\omega = \int_{-\pi}^{\pi} \underbrace{\left[\sum_{n=-\infty}^{\infty} x(n) \, e^{-j\omega n} \right]}_{} e^{j\omega n} \, d\omega$$

$$\boxed{\lim_{N \to \infty} |X(\omega) - X_N(\omega)| = 0} \qquad X_n(\omega) = \sum_{n=-N}^{N} x(n) \, e^{-j\omega n}$$

Then only one can change order of summation.

→ Uniform convergence is guaranteed if $x(n)$ is absolutely summable.

$$\boxed{\sum_{n=-\infty}^{\infty} |x(n) \, e^{-j\omega n}| \leq \sum_{n=-\infty}^{\infty} x(n) \, \underbrace{|e^{-j\omega n}|}_{=1} \leq \infty}$$

In some cases, uniform convergence do not exist due to non-absolute summability of $x(n)$, then.

2.) Mean square Convergence.

It is just the energy of signal.

Energy of signal $$\boxed{E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2 < \infty}$$

If this condition satisfies than also F.T. exists.

$$\boxed{\lim_{N \to \infty} \int_{\pi}^{\pi} |X(\omega) - X_N(\omega)|^2 \, d\omega = 0}$$ Mean square convergence

~~Ex~~

Energy of DTAS

$$E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \sum_{n=-\infty}^{\infty} x(n) \cdot x^*(n)$$

$$= \sum_{n=-\infty}^{\infty} x(n) \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n} d\omega \right]^*$$

$$= \sum_{n=-\infty}^{\infty} x(n) \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} X^*(\omega)\, e^{-j\omega n} d\omega \right]$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X^*(\omega) \underbrace{\left[\sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \right]}_{} d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X^*(\omega)\, X(\omega)\, d\omega$$

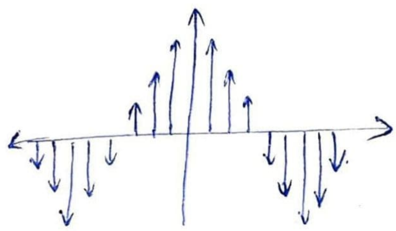
$$E_x = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2 d\omega$$

$$\boxed{E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2 d\omega}$$ Parsevals Theorem for DTAS.

Ex

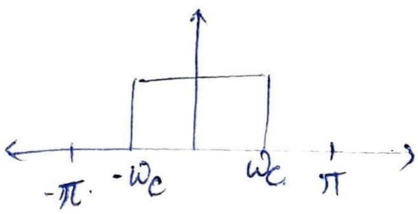
$$x(n) = \frac{\omega_c}{\pi} \qquad n = 0$$

$$= \frac{\omega_c}{\pi} \cdot \frac{\sin \omega_c n}{\omega_c n}, \quad n \neq 0$$



Since, adding all the samples step by step is not possible
~~then~~ $\therefore$ it is not absolutely not summable.

So. $$X(\omega) = \begin{cases} 1, & |\omega| \leq \omega_c \\ 0, & \text{otherwise.} \end{cases}$$



Finding I.F.T.

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} 1\, e^{j\omega n} d\omega = \frac{1}{2\pi} \left[\frac{e^{j\omega n}}{jn} \right]_{-\omega_c}^{\omega_c} = \frac{1}{2\pi} \left[\frac{e^{j\omega_c n} - e^{-j\omega_c n}}{jn} \right]$$

$$= \frac{(2\omega_c)}{2\pi} \frac{\sin(\omega_c n)}{\omega_c n} = \frac{\omega_c}{\pi} \frac{\sin(\omega_c n)}{\omega_c n}$$

$$X(\omega) = \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & \text{otherwise} \end{cases} \iff x(n) = \frac{\omega_c}{\pi} \frac{\sin(\omega_c n)}{\omega_c n}$$

Let's look at 2nd condition of convergence which says that signal should have finite energy. i.e. mean square convergence.

$$E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2 d\omega$$

$$E_x = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} 1 \, d\omega$$

$$= \frac{2\omega_c}{2\pi} = \frac{\omega_c}{\pi}$$

$$\boxed{E_x = \frac{\omega_c}{\pi}} \therefore \text{Finite.}$$

Since 2nd condition is true.
$\therefore$ F.S. converges and F.T. exists.

Discrete Time Fourier Transform

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$X(\omega) = X_R(\omega) + j X_I(\omega)$$

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, d\omega.$$

$$x(n) = x_R(n) + j x_i(n)$$

$\Rightarrow$ $\cancel{X(\omega)}$ $X(\omega) = \sum_{n=-\infty}^{\infty} x(n).\, e^{-j\omega n}$

$\hookrightarrow \cos(\omega n) - j\sin(\omega n)$

$$X_R(\omega) + j X_I(\omega) = \sum_{n=-\infty}^{\infty} \left[x_R(n) + j x_I(n)\right]\left[\cos(\omega n) - j\sin(\omega n)\right]$$

$$\boxed{\begin{aligned} X_R(\omega) &= \sum_{n=-\infty}^{\infty} \left[x_R(n).\cos(\omega n) + x_I(n)\sin(\omega n)\right] \\ X_I(\omega) &= \sum_{n=-\infty}^{\infty} \left[x_I(n)\cos(\omega n) - x_R(n)\sin(\omega n)\right] \end{aligned}}$$

$\Rightarrow$ If $x(n)$ is <u>real sequence</u>, $x(n) = x_R(n)$

$$X_R(\omega) = \sum_{n=-\infty}^{\infty} \left[x_R(n)\cos(\omega n)\right]$$

$$X_I(\omega) = \sum_{n=-\infty}^{\infty} \left[-x_R(n)\sin(\omega n)\right]$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} \left[x_R(n)\cos(\omega n) - j x_R(n)\sin(\omega n)\right]$$

$$\boxed{X_R(-\omega) = X_R(\omega)}$$

$$\boxed{X_I(-\omega) = -X_I(\omega)}$$

$$\boxed{X(-\omega) = X_R(\omega) - j X_I(\omega)}$$

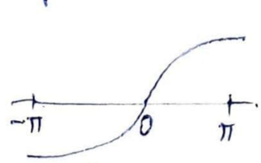
$$\boxed{|X(-\omega)| = |X(\omega)| = \sqrt{X_R^2(\omega) + X_I^2(\omega)}}$$

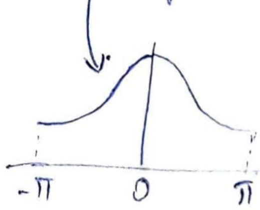
Magnitude spectrum is even symmetric.

$$\angle X(\omega) = \tan^{-1}\left[\frac{X_I(\omega)}{X_R(\omega)}\right]$$

$$\angle X(-\omega) = -\tan^{-1}\left[\frac{X_I(\omega)}{X_R(\omega)}\right]$$

$$\angle X(\omega) = -\angle X(-\omega)$$





Phase spectrum is odd symmetric.

⇒ If $x(n)$ is real and even.

$$x(n) = x_R(n)$$

$$x(-n) = x(n) \quad \text{i.e. } x_R(-n) = x_R(n)$$

$$X_R(\omega) = \sum_{n=-\infty}^{\infty} x_R(n)\cos(\omega n)$$

$$X_I(\omega) = -\sum_{n=-\infty}^{\infty} \underbrace{x_R(n)}_{\text{even symmetric}}\ \underbrace{\sin(\omega n)}_{\text{odd symmetric}}$$

$$X_I(\omega) = 0 \qquad X(\omega) = X(-\omega)$$

$$\cancel{X_R(\omega) = \Sigma} \qquad |X(\omega)| = |X(-\omega)|$$

⇒ If $x(n)$ is imaginary, $x(n) = x_I(n)$

$$X_R(\omega) = \sum_{n=-\infty}^{\infty}[x_I(n)\sin(\omega n)], \quad X_I(\omega) = \sum_{n=-\infty}^{\infty}(x_I(n)\cos(\omega n))$$

$$X(\omega) = \sum_{n=-\infty}^{\infty}[x_I(n)\sin(\omega n) + j\, x_I(n)\cos(\omega n)]$$

$$\boxed{X_R(-\omega) = -X_R(\omega)} \qquad \boxed{X(-\omega) = -X_R(\omega) + jX_I(\omega)}$$

$$\boxed{X_I(-\omega) = X_I(\omega)} \qquad |X(-\omega)| = |X(\omega)| = \sqrt{X_R^2(\omega) + X_I^2(\omega)}$$

Magnitude spectrum is even symmetric.

$$\angle X(\omega) = \tan^{-1}\left[\frac{X_I(\omega)}{X_R(\omega)}\right]$$

$$\angle X(-\omega) = -\tan^{-1}\left[\frac{X_I(\omega)}{X_R(\omega)}\right]$$

$$\angle X(\omega) = -\angle X(-\omega)$$

Phase spectrum is odd symmetric.

Convolution of two Sequence

$$x_1(n) \quad , \quad x_2(n)$$

$$x_3(n) = x_1(n) * x_2(n)$$

$$= \sum_{K=-\infty}^{\infty} x_1(K)\, x_2(n-K)$$

$$X_3(\omega) = \sum_{n=-\infty}^{\infty} x_3(n)\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left[\sum_{K=-\infty}^{\infty} x_1(K)\, x_2(n-K) \right] e^{-j\omega n} = \sum_{K=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x_1(K)\, x_2(n-K)\, e^{-j\omega n}$$

$$n-K = m \quad , \quad n = K+m$$

$$= \sum_{K=-\infty}^{\infty} x_1(K) \sum_{m=-\infty}^{\infty} x_2(m)\, e^{-j\omega(K+m)}$$

$$= \sum_{K=-\infty}^{\infty} x_1(K) \sum_{m=-\infty}^{\infty} x_2(m)\, e^{-j\omega(K+m)}$$

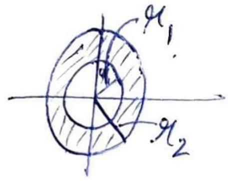
$$= \sum_{K=-\infty}^{\infty} x_1(K) \sum_{m=-\infty}^{\infty} x_2(m)\, e^{-j\omega K}\, e^{-j\omega m}$$

$$= \underbrace{\sum_{K=-\infty}^{\infty} x_1(K)\, e^{-j\omega K}}\ \underbrace{\sum_{m=-\infty}^{\infty} x_2(m)\, e^{-j\omega m}}$$

$$X_3(\omega) = X_1(\omega) \cdot X_2(\omega)$$

Relation of F.T. & Z. Transform

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n)\, Z^{-n} \qquad r_1 < |Z| < r_2$$



$$Z = r e^{j\omega}$$

$$X(r e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, r^{-n} e^{-j\omega n}$$

$$\boxed{X(Z) = X(\omega)\, r^{-n}}$$

If $r = 1$

$$X(Z) = X(\omega)$$

F.T. is z transform of a sequence $x(n)$ evaluated on a unit circle.

$\Rightarrow$ If $r > 1$, r^{-n} decreases.] This only decides R.O.C.
If $r < 1$, r^{-n} increases.]

$\Rightarrow$ If ROC contains $|z| = 1$ circle, i.e. Z transform at $Z = 1$ can be evaluated and F.T. exists for a given sequence.

$\Rightarrow$ $x(n) = a^n u(n)$, $a > 1$
$X(Z) = \frac{1}{1 - az^{-1}}$, $|z| > a$.

$\therefore$ F.T. doesn't exist for given sequence as., $a > 1$

Stability

$$\sum_{n=-\infty}^{\infty} |x(n) \, z^{-n}| < \infty \Rightarrow \sum_{n=-\infty}^{\infty} |x(n) \, r^{-n} \underbrace{e^{-j\omega n}}_{1}| < \infty$$

$$\sum_{n=-\infty}^{\infty} |x(n) \, r^{-n}| < \infty$$

If $r > 1$, $\sum_{n=-\infty}^{\infty} |x(n) \, r^{-n}| < \infty$ always.

$\therefore$ It will be stable & Z.T. exists but not F.T.