exponential, while the open circles indicate the samples of the complex exponential that are "missing," i.e., that would be nonzero if the input were of the form $e^{j\omega n}$ for all n. The shaded dots indicate the samples of the impulse response $h[n-k]$ as a function of k for $n = 8$. In the finite-length case shown in Figure 20(a), it is clear that the output would consist only of the steady-state component for $n \geq 8$, whereas in the infinite-length case, it is clear that the "missing" samples have less and less effect as n increases, owing to the decaying nature of the impulse response.

The condition for stability is also a sufficient condition for the existence of the frequency response function. To see this, note that, in general,

$$|H(e^{j\omega})| = \left|\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}\right| \leq \sum_{k=-\infty}^{\infty} |h[k]e^{-j\omega k}| \leq \sum_{k=-\infty}^{\infty} |h[k]|,$$

so the general condition

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

ensures that $H(e^{j\omega})$ exists. It is no surprise that the condition for existence of the frequency response is the same as the condition for dominance of the steady-state solution. Indeed, a complex exponential that exists for all n can be thought of as one that is applied at $n = -\infty$. The eigenfunction property of complex exponentials depends on stability of the system, since at finite n, the transient response must have become zero, so that we only see the steady-state response $H(e^{j\omega})e^{j\omega n}$ for all finite n.

7 REPRESENTATION OF SEQUENCES BY FOURIER TRANSFORMS

One of the advantages of the frequency-response representation of an LTI system is that interpretations of system behavior such as the one we made in Example 16 often follow easily. At this point, let us return to the question of how we may find representations of the form of Eq. (111) for an arbitrary input sequence.

Many sequences can be represented by a Fourier integral of the form

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\,d\omega, \tag{130}$$

where

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}. \tag{131}$$

Equations (130) and (131) together form a Fourier representation for the sequence. Equation (130), the *inverse Fourier transform,* is a synthesis formula. That is, it represents $x[n]$ as a superposition of infinitesimally small complex sinusoids of the form

$$\frac{1}{2\pi}X(e^{j\omega})e^{j\omega n}\,d\omega,$$

with ω ranging over an interval of length 2π and with $X(e^{j\omega})$ determining the relative amount of each complex sinusoidal component. Although, in writing Eq. (130), we have chosen the range of values for ω between $-\pi$ and $+\pi$, any interval of length 2π can be used. Equation (131), the *Fourier transform,*[4] is an expression for computing $X(e^{j\omega})$ from the sequence $x[n]$, i.e., for analyzing the sequence $x[n]$ to determine how much of each frequency component is required to synthesize $x[n]$ using Eq. (130).

In general, the Fourier transform is a complex-valued function of ω. As with the frequency response, we may either express $X(e^{j\omega})$ in rectangular form as

$$X(e^{j\omega}) = X_R(e^{j\omega}) + jX_I(e^{j\omega}) \tag{132a}$$

or in polar form as

$$X(e^{j\omega}) = |X(e^{j\omega})|e^{j\angle X(e^{j\omega})}. \tag{132b}$$

With $|X(e^{j\omega})|$ representing the magnitude and $\angle X(e^{j\omega})$ the phase.

The phase $\angle X(e^{j\omega})$ is not uniquely specified by Eq. (132b), since any integer multiple of 2π may be added to $\angle X(e^{j\omega})$ at any value of ω without affecting the result of the complex exponentiation. When we specifically want to refer to the principal value, i.e., $\angle X(e^{j\omega})$ restricted to the range of values between $-\pi$ and $+\pi$, we denote this as $\mathrm{ARG}[X(e^{j\omega})]$. If we want to refer to a phase function that is a continuous function of ω for $0 < \omega < \pi$, i.e., not evaluated modulo 2π, we use the notation $\arg[X(e^{j\omega})]$.

As is clear from comparing Eqs. (104) and (131), the frequency response of an LTI system is the Fourier transform of the impulse response. The impulse response can be obtained from the frequency response by applying the inverse Fourier transform integral; i.e.,

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega. \tag{133}$$

As discussed previously, the frequency response is a periodic function of ω. Likewise, the Fourier transform is periodic in ω with period 2π. A Fourier series is commonly used to represent periodic signals, and it is worth noting that indeed, Eq. (131) is of the form of a Fourier series for the periodic function $X(e^{j\omega})$. Eq. (130), which expresses the sequence values $x[n]$ in terms of the periodic function $X(e^{j\omega})$, is of the form of the integral that would be used to obtain the coefficients in the Fourier series. Our use of Eqs. (130) and (131) focuses on the representation of the sequence $x[n]$. Nevertheless, it is useful to be aware of the equivalence between the Fourier series representation of continuous-variable periodic functions and the Fourier transform representation of discrete-time signals, since all the familiar properties of Fourier series can be applied, with appropriate interpretation of variables, to the Fourier transform representation of a sequence. (Oppenheim and Willsky (1997), McClellan, Schafer and Yoder (2003).)

Determining the class of signals that can be represented by Eq. (130) is equivalent to considering the convergence of the infinite sum in Eq. (131). That is, we are concerned with the conditions that must be satisfied by the terms in the sum in Eq. (131) such that

$$|X(e^{j\omega})| < \infty \qquad \text{for all } \omega,$$

[4]Eq. (131) is sometimes more explicitly referred to as the discrete-time Fourier transform, or DTFT, particularly when it is important to distinguish it from the continuous-time Fourier transform.

where $X(e^{j\omega})$ is the limit as $M \to \infty$ of the finite sum

$$X_M(e^{j\omega}) = \sum_{n=-M}^{M} x[n]e^{-j\omega n}. \tag{134}$$

A sufficient condition for convergence can be found as follows:

$$\begin{aligned} |X(e^{j\omega})| &= \left| \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \right| \\ &\leq \sum_{n=-\infty}^{\infty} |x[n]|\,|e^{-j\omega n}| \\ &\leq \sum_{n=-\infty}^{\infty} |x[n]| < \infty. \end{aligned}$$

Thus, if $x[n]$ is *absolutely summable,* then $X(e^{j\omega})$ exists. Furthermore, in this case, the series can be shown to converge uniformly to a continuous function of ω (Körner (1988), Kammler (2000)). Since a stable sequence is, by definition, absolutely summable, all stable sequences have Fourier transforms. It also follows, then, that any stable *system*, i.e., one having an absolutely summable impulse response, will have a finite and continuous frequency response.

Absolute summability is a sufficient condition for the existence of a Fourier transform representation. In Examples 14 and 16, we computed the Fourier transforms of the impulse response of the delay system and the moving average system. The impulse responses are absolutely summable, since they are finite in length. Clearly, any finite-length sequence is absolutely summable and thus will have a Fourier transform representation. In the context of LTI systems, any FIR system will be stable and therefore will have a finite, continuous frequency response. However, when a sequence has infinite length, we must be concerned about convergence of the infinite sum. The following example illustrates this case.

Example 17 Absolute Summability for a Suddenly-Applied Exponential

Consider $x[n] = a^n u[n]$. The Fourier transform of this sequence is

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n \\ &= \frac{1}{1 - ae^{-j\omega}} \qquad \text{if } |ae^{-j\omega}| < 1 \quad \text{or} \quad |a| < 1. \end{aligned}$$

Clearly, the condition $|a| < 1$ is the condition for the absolute summability of $x[n]$; i.e.,

$$\sum_{n=0}^{\infty} |a|^n = \frac{1}{1 - |a|} < \infty \qquad \text{if } |a| < 1. \tag{135}$$

Absolute summability is a *sufficient* condition for the existence of a Fourier transform representation, and it also guarantees uniform convergence. Some sequences are not absolutely summable, but are square summable, i.e.,

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty. \tag{136}$$

Such sequences can be represented by a Fourier transform if we are willing to relax the condition of uniform convergence of the infinite sum defining $X(e^{j\omega})$. Specifically, in this case, we have mean-square convergence; that is, with

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \tag{137a}$$

and

$$X_M(e^{j\omega}) = \sum_{n=-M}^{M} x[n]e^{-j\omega n}, \tag{137b}$$

it follows that

$$\lim_{M\to\infty} \int_{-\pi}^{\pi} |X(e^{j\omega}) - X_M(e^{j\omega})|^2 d\omega = 0. \tag{138}$$

In other words, the error $|X(e^{j\omega}) - X_M(e^{j\omega})|$ may not approach zero at each value of ω as $M \to \infty$, but the total "energy" in the error does. Example 18 illustrates this case.

Example 18 *Square-Summability for the Ideal Lowpass Filter*

In this example we determine the impulse response of the ideal lowpass filter discussed in Section 6. The frequency response is

$$H_{\text{lp}}(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c, \\ 0, & \omega_c < |\omega| \le \pi, \end{cases} \tag{139}$$

with periodicity 2π also understood. The impulse response $h_{\text{lp}}[n]$ can be found using the Fourier transform synthesis equation (130):

$$\begin{aligned} h_{\text{lp}}[n] &= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega \\ &= \frac{1}{2\pi jn} \left[e^{j\omega n} \right]_{-\omega_c}^{\omega_c} = \frac{1}{2\pi jn} (e^{j\omega_c n} - e^{-j\omega_c n}) \\ &= \frac{\sin \omega_c n}{\pi n}, \qquad -\infty < n < \infty. \end{aligned} \tag{140}$$

We note that, since $h_{\text{lp}}[n]$ is nonzero for $n < 0$, the ideal lowpass filter is noncausal. Also, $h_{\text{lp}}[n]$ is not absolutely summable. The sequence values approach zero as $n \to \infty$, but only as $1/n$. This is because $H_{\text{lp}}(e^{j\omega})$ is discontinuous at $\omega = \omega_c$. Since $h_{\text{lp}}[n]$ is not absolutely summable, the infinite sum

$$\sum_{n=-\infty}^{\infty} \frac{\sin \omega_c n}{\pi n} e^{-j\omega n}$$

does not converge uniformly for all values of ω. To obtain an intuitive feeling for this, let us consider $H_M(e^{j\omega})$ as the sum of a finite number of terms:

$$H_M(e^{j\omega}) = \sum_{n=-M}^{M} \frac{\sin \omega_c n}{\pi n} e^{-j\omega n}. \tag{141}$$

The function $H_M(e^{j\omega})$ is evaluated in Figure 21 for several values of M. Note that as M increases, the oscillatory behavior at $\omega = \omega_c$ (often referred to as the Gibbs phenomenon) is more rapid, but the size of the ripples does not decrease. In fact, it can be shown that as $M \to \infty$, the maximum amplitude of the oscillations does not approach zero, but the oscillations converge in location toward the points $\omega = \pm\omega_c$. Thus, the infinite sum does not converge uniformly to the discontinuous function $H_{\text{lp}}(e^{j\omega})$ of Eq. (139). However, $h_{\text{lp}}[n]$, as given in Eq. (140), is square summable, and correspondingly, $H_M(e^{j\omega})$ converges in the mean-square sense to $H_{\text{lp}}(e^{j\omega})$; i.e.,

$$\lim_{M\to\infty} \int_{-\pi}^{\pi} |H_{\text{lp}}(e^{j\omega}) - H_M(e^{j\omega})|^2 d\omega = 0.$$

Although the error between $H_M(e^{j\omega})$ and $H_{\text{lp}}(e^{j\omega})$ as $M \to \infty$ might seem unimportant because the two functions differ only at $\omega = \omega_c$, the behavior of finite sums such as Eq. (141) has important implications in the design of discrete-time systems for filtering.

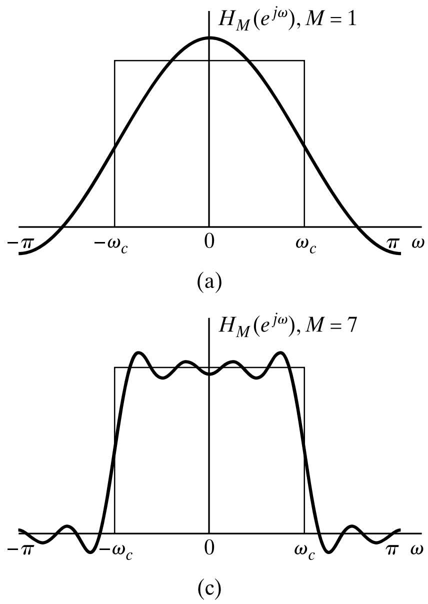

Figure 21 Convergence of the Fourier transform. The oscillatory behavior at $\omega = \omega_c$ is often called the Gibbs phenomenon.

It is sometimes useful to have a Fourier transform representation for certain sequences that are neither absolutely summable nor square summable. We illustrate several of these in the following examples.

Example 19 Fourier Transform of a Constant

Consider the sequence $x[n] = 1$ for all n. This sequence is neither absolutely summable nor square summable, and Eq. (131) does not converge in either the uniform or mean-square sense for this case. However, it is possible and useful to define the Fourier transform of the sequence $x[n]$ to be the periodic impulse train

$$X(e^{j\omega}) = \sum_{r=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi r). \tag{142}$$

The impulses in this case are functions of a continuous variable and therefore are of "infinite height, zero width, and unit area," consistent with the fact that Eq. (131) does not converge in any regular sense. (See Oppenheim and Willsky (1997) for a discussion of the definition and properties of the impulse function.) The use of Eq. (142) as a Fourier representation of the sequence $x[n] = 1$ is justified principally because formal substitution of Eq. (142) into Eq. (130) leads to the correct result. Example 20 represents a generalization of this example.

Example 20 Fourier Transform of Complex Exponential Sequences

Consider a sequence $x[n]$ whose Fourier transform is the periodic impulse train

$$X(e^{j\omega}) = \sum_{r=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi r). \tag{143}$$

We show in this example that $x[n]$ is the complex exponential sequence $e^{j\omega_0 n}$, with $-\pi < \omega_0 \le \pi$.

We can determine $x[n]$ by substituting $X(e^{j\omega})$ into the inverse Fourier transform integral of Eq. (130). Because the integration of $X(e^{j\omega})$ extends only over one period, from $-\pi < \omega < \pi$, we need include only the $r = 0$ term from Eq. (143). Consequently, we can write

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\pi\delta(\omega - \omega_0)e^{j\omega n}d\omega. \tag{144}$$

From the definition of the impulse function, it follows that

$$x[n] = e^{j\omega_0 n} \qquad \text{for any } n.$$

For $\omega_0 = 0$, this reduces to the sequence considered in Example 19.

Clearly, $x[n]$ in Example 20 is not absolutely summable, nor is it square summable, and $|X(e^{j\omega})|$ is not finite for all ω. Thus, the mathematical statement

$$\sum_{n=-\infty}^{\infty} e^{j\omega_0 n}e^{-j\omega n} = \sum_{r=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi r) \tag{145}$$

must be interpreted in the context of generalized functions (Lighthill, 1958). Using that theory, the concept of a Fourier transform representation can be extended to the class of sequences that can be expressed as a sum of discrete frequency components, such as

$$x[n] = \sum_k a_k e^{j\omega_k n}, \qquad -\infty < n < \infty. \tag{146}$$

From the result of Example 20, it follows that

$$X(e^{j\omega}) = \sum_{r=-\infty}^{\infty}\sum_k 2\pi a_k\delta(\omega - \omega_k + 2\pi r) \tag{147}$$

is a consistent Fourier transform representation of $x[n]$ in Eq. (146).

Another sequence that is neither absolutely summable nor square summable is the unit step sequence $u[n]$. Although it is not completely straightforward to show, this sequence can be represented by the following Fourier transform:

$$U(e^{j\omega}) = \frac{1}{1 - e^{-j\omega}} + \sum_{r=-\infty}^{\infty} \pi\delta(\omega + 2\pi r). \tag{148}$$

8 SYMMETRY PROPERTIES OF THE FOURIER TRANSFORM

In using Fourier transforms, it is useful to have a detailed knowledge of the way that properties of the sequence manifest themselves in the Fourier transform and vice versa. In this section and Section 9, we discuss and summarize a number of such properties.

Symmetry properties of the Fourier transform are often very useful for simplifying the solution of problems. The following discussion presents these properties. The proofs are considered in Problems 79 and 80. Before presenting the properties, however, we begin with some definitions.

A *conjugate-symmetric sequence* $x_e[n]$ is defined as a sequence for which $x_e[n] = x_e^*[-n]$, and a *conjugate-antisymmetric sequence* $x_o[n]$ is defined as a sequence for which $x_o[n] = -x_o^*[-n]$, where * denotes complex conjugation. Any sequence $x[n]$ can be expressed as a sum of a conjugate-symmetric and conjugate-antisymmetric sequence. Specifically,

$$x[n] = x_e[n] + x_o[n], \tag{149a}$$

where

$$x_e[n] = \tfrac{1}{2}(x[n] + x^*[-n]) = x_e^*[-n] \tag{149b}$$

and

$$x_o[n] = \tfrac{1}{2}(x[n] - x^*[-n]) = -x_o^*[-n]. \tag{149c}$$

Adding Eqs. (149b) and (149c) confirms that Eq. (149a) holds. A real sequence that is conjugate symmetric such that $x_e[n] = x_e[-n]$ is referred to as an *even sequence*, and a real sequence that is conjugate antisymmetric such that $x_o[n] = -x_o[-n]$ is referred to as an *odd sequence*.

A Fourier transform $X(e^{j\omega})$ can be decomposed into a sum of conjugate-symmetric and conjugate-antisymmetric functions as

$$X(e^{j\omega}) = X_e(e^{j\omega}) + X_o(e^{j\omega}), \tag{150a}$$

where

$$X_e(e^{j\omega}) = \tfrac{1}{2}[X(e^{j\omega}) + X^*(e^{-j\omega})] \tag{150b}$$

and

$$X_o(e^{j\omega}) = \tfrac{1}{2}[X(e^{j\omega}) - X^*(e^{-j\omega})]. \tag{150c}$$

By substituting $-\omega$ for ω in Eqs. (150b) and (150c), it follows that $X_e(e^{j\omega})$ is conjugate symmetric and $X_o(e^{j\omega})$ is conjugate antisymmetric; i.e.,

$$X_e(e^{j\omega}) = X_e^*(e^{-j\omega}) \tag{151a}$$

and

$$X_o(e^{j\omega}) = -X_o^*(e^{-j\omega}). \tag{151b}$$

If a real function of a continuous variable is conjugate symmetric, it is referred to as an *even function*, and a real conjugate-antisymmetric function of a continuous variable is referred to as an *odd function*.

The symmetry properties of the Fourier transform are summarized in Table 1. The first six properties apply for a general complex sequence $x[n]$ with Fourier transform $X(e^{j\omega})$. Properties 1 and 2 are considered in Problem 79. Property 3 follows from properties 1 and 2, together with the fact that the Fourier transform of the sum of two sequences is the sum of their Fourier transforms. Specifically, the Fourier transform of $\mathcal{R}e\{x[n]\} = \frac{1}{2}(x[n] + x^*[n])$ is the conjugate-symmetric part of $X(e^{j\omega})$, or $X_e(e^{j\omega})$. Similarly, $j\mathcal{I}m\{x[n]\} = \frac{1}{2}(x[n] - x^*[n])$, or equivalently, $j\mathcal{I}m\{x[n]\}$ has a Fourier transform that is the conjugate-antisymmetric component $X_o(e^{j\omega})$ corresponding to property 4. By considering the Fourier transform of $x_e[n]$ and $x_o[n]$, the conjugate-symmetric and conjugate-antisymmetric components, respectively, of $x[n]$, it can be shown that properties 5 and 6 follow.

If $x[n]$ is a real sequence, these symmetry properties become particularly straightforward and useful. Specifically, for a real sequence, the Fourier transform is conjugate symmetric; i.e., $X(e^{j\omega}) = X^*(e^{-j\omega})$ (property 7). Expressing $X(e^{j\omega})$ in terms of its real and imaginary parts as

$$X(e^{j\omega}) = X_R(e^{j\omega}) + jX_I(e^{j\omega}), \tag{152}$$

TABLE 1 SYMMETRY PROPERTIES OF THE FOURIER TRANSFORM

Sequence $x[n]$	Fourier Transform $X(e^{j\omega})$
1. $x^*[n]$	$X^*(e^{-j\omega})$
2. $x^*[-n]$	$X^*(e^{j\omega})$
3. $\mathcal{R}e\{x[n]\}$	$X_e(e^{j\omega})$ (conjugate-symmetric part of $X(e^{j\omega})$)
4. $j\mathcal{I}m\{x[n]\}$	$X_o(e^{j\omega})$ (conjugate-antisymmetric part of $X(e^{j\omega})$)
5. $x_e[n]$ (conjugate-symmetric part of $x[n]$)	$X_R(e^{j\omega}) = \mathcal{R}e\{X(e^{j\omega})\}$
6. $x_o[n]$ (conjugate-antisymmetric part of $x[n]$)	$jX_I(e^{j\omega}) = j\mathcal{I}m\{X(e^{j\omega})\}$
The following properties apply only when $x[n]$ is real:	
7. Any real $x[n]$	$X(e^{j\omega}) = X^*(e^{-j\omega})$ (Fourier transform is conjugate symmetric)
8. Any real $x[n]$	$X_R(e^{j\omega}) = X_R(e^{-j\omega})$ (real part is even)
9. Any real $x[n]$	$X_I(e^{j\omega}) = -X_I(e^{-j\omega})$ (imaginary part is odd)
10. Any real $x[n]$	$\|X(e^{j\omega})\| = \|X(e^{-j\omega})\|$ (magnitude is even)
11. Any real $x[n]$	$\angle X(e^{j\omega}) = -\angle X(e^{-j\omega})$ (phase is odd)
12. $x_e[n]$ (even part of $x[n]$)	$X_R(e^{j\omega})$
13. $x_o[n]$ (odd part of $x[n]$)	$jX_I(e^{j\omega})$

we can derive properties 8 and 9—specifically,

$$X_R(e^{j\omega}) = X_R(e^{-j\omega}) \tag{153a}$$

and

$$X_I(e^{j\omega}) = -X_I(e^{-j\omega}). \tag{153b}$$

In other words, the real part of the Fourier transform is an even function, and the imaginary part is an odd function, if the sequence is real. In a similar manner, by expressing $X(e^{j\omega})$ in polar form as

$$X(e^{j\omega}) = |X(e^{j\omega})|e^{j\angle X(e^{j\omega})}, \tag{154}$$

we can show that, for a real sequence $x[n]$, the magnitude of the Fourier transform, $|X(e^{j\omega})|$, is an even function of ω and the phase, $\angle X(e^{j\omega})$, can be chosen to be an odd function of ω (properties 10 and 11). Also, for a real sequence, the even part of $x[n]$ transforms to $X_R(e^{j\omega})$, and the odd part of $x[n]$ transforms to $jX_I(e^{j\omega})$ (properties 12 and 13).

Example 21 Illustration of Symmetry Properties

Let us return to the sequence of Example 17, where we showed that the Fourier transform of the real sequence $x[n] = a^n u[n]$ is

$$X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}} \qquad \text{if } |a| < 1. \tag{155}$$

Then, from the properties of complex numbers, it follows that

$$X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}} = X^*(e^{-j\omega}) \qquad \text{(property 7)},$$

$$X_R(e^{j\omega}) = \frac{1 - a\cos\omega}{1 + a^2 - 2a\cos\omega} = X_R(e^{-j\omega}) \qquad \text{(property 8)},$$

$$X_I(e^{j\omega}) = \frac{-a\sin\omega}{1 + a^2 - 2a\cos\omega} = -X_I(e^{-j\omega}) \qquad \text{(property 9)},$$

$$|X(e^{j\omega})| = \frac{1}{(1 + a^2 - 2a\cos\omega)^{1/2}} = |X(e^{-j\omega})| \qquad \text{(property 10)},$$

$$\angle X(e^{j\omega}) = \tan^{-1}\left(\frac{-a\sin\omega}{1 - a\cos\omega}\right) = -\angle X(e^{-j\omega}) \qquad \text{(property 11)}.$$

These functions are plotted in Figure 22 for $a > 0$, specifically, $a = 0.75$ (solid curve) and $a = 0.5$ (dashed curve). In Problem 32, we consider the corresponding plots for $a < 0$.

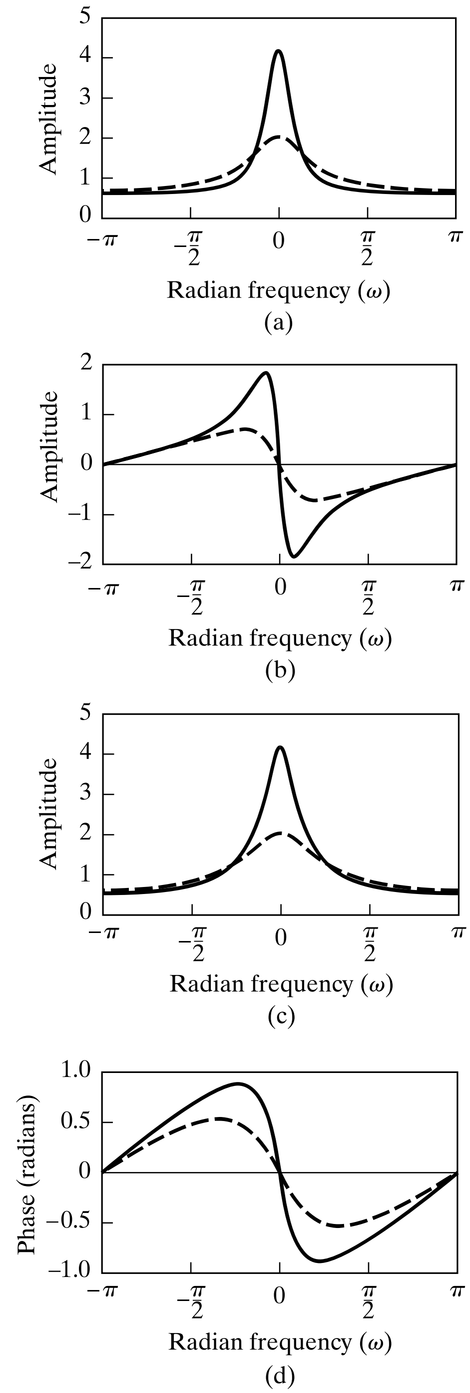

Figure 22 Frequency response for a system with impulse response $h[n] = a^n u[n]$. (a) Real part. $a > 0$; $a = 0.75$ (solid curve) and $a = 0.5$ (dashed curve). (b) Imaginary part. (c) Magnitude. $a > 0$; $a = 0.75$ (solid curve) and $a = 0.5$ (dashed curve). (d) Phase.

9 FOURIER TRANSFORM THEOREMS

In addition to the symmetry properties, a variety of theorems (presented in Sections 9.1–9.7) relate operations on the sequence to operations on the Fourier transform. We will see that these theorems are quite similar in most cases to corresponding theorems for continuous-time signals and their Fourier transforms. To facilitate the statement of the theorems, we introduce the following operator notation:

$$\begin{aligned} X(e^{j\omega}) &= \mathcal{F}\{x[n]\}, \\ x[n] &= \mathcal{F}^{-1}\{X(e^{j\omega})\}, \\ x[n] &\overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega}). \end{aligned}$$

That is, $\mathcal{F}$ denotes the operation of "taking the Fourier transform of $x[n]$," and $\mathcal{F}^{-1}$ is the inverse of that operation. Most of the theorems will be stated without proof. The proofs, which are left as exercises (Problem 81), generally involve only simple manipulations of variables of summation or integration. The theorems in this section are summarized in Table 2.

TABLE 2 FOURIER TRANSFORM THEOREMS

Sequence $x[n]$ $y[n]$	Fourier Transform $X(e^{j\omega})$ $Y(e^{j\omega})$
1. $ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
2. $x[n - n_d]$ (n_d an integer)	$e^{-j\omega n_d} X(e^{j\omega})$
3. $e^{j\omega_0 n} x[n]$	$X(e^{j(\omega-\omega_0)})$
4. $x[-n]$	$X(e^{-j\omega})$ $X^*(e^{j\omega})$ if $x[n]$ real.
5. $nx[n]$	$j\dfrac{dX(e^{j\omega})}{d\omega}$
6. $x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
7. $x[n]y[n]$	$\dfrac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
Parseval's theorem:	
8. $\sum_{n=-\infty}^{\infty} \|x[n]\|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \|X(e^{j\omega})\|^2 d\omega$	
9. $\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$	

9.1 Linearity of the Fourier Transform

If

$$x_1[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X_1(e^{j\omega})$$

and

$$x_2[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X_2(e^{j\omega}),$$

then it follows by substitution into the definition of the DTFT that

$$ax_1[n] + bx_2[n] \stackrel{\mathcal{F}}{\longleftrightarrow} aX_1(e^{j\omega}) + bX_2(e^{j\omega}). \tag{156}$$

9.2 Time Shifting and Frequency Shifting Theorem

If

$$x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega}),$$

then, for the time-shifted sequence $x[n - n_d]$, a simple transformation of the index of summation in the DTFT yields

$$x[n - n_d] \stackrel{\mathcal{F}}{\longleftrightarrow} e^{-j\omega n_d} X(e^{j\omega}). \tag{157}$$

Direct substitution proves the following result for the frequency-shifted Fourier transform:

$$e^{j\omega_0 n} x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(e^{j(\omega-\omega_0)}). \tag{158}$$

9.3 Time Reversal Theorem

If

$$x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega}),$$

then if the sequence is time reversed,

$$x[-n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(e^{-j\omega}). \tag{159}$$

If $x[n]$ is real, this theorem becomes

$$x[-n] \stackrel{\mathcal{F}}{\longleftrightarrow} X^*(e^{j\omega}). \tag{160}$$

9.4 Differentiation in Frequency Theorem

If

$$x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega}),$$

then, by differentiating the DTFT, it is seen that

$$nx[n] \stackrel{\mathcal{F}}{\longleftrightarrow} j\frac{dX(e^{j\omega})}{d\omega}. \tag{161}$$

9.5 Parseval's Theorem

If

$$x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega}),$$

then

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega. \tag{162}$$

The function $|X(e^{j\omega})|^2$ is called the *energy density spectrum*, since it determines how the energy is distributed in the frequency domain. Necessarily, the energy density spectrum is defined only for finite-energy signals. A more general form of Parseval's theorem is shown in Problem 84.

9.6 The Convolution Theorem

If

$$x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega})$$

and

$$h[n] \stackrel{\mathcal{F}}{\longleftrightarrow} H(e^{j\omega}),$$

and if

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n], \tag{163}$$

then

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}). \tag{164}$$

Thus, convolution of sequences implies multiplication of the corresponding Fourier transforms. Note that the time-shifting property is a special case of the convolution property, since

$$\delta[n-n_d] \stackrel{\mathcal{F}}{\longleftrightarrow} e^{-j\omega n_d} \tag{165}$$

and if $h[n] = \delta[n-n_d]$, then $y[n] = x[n] * \delta[n-n_d] = x[n-n_d]$. Therefore,

$$H(e^{j\omega}) = e^{-j\omega n_d} \qquad \text{and} \qquad Y(e^{j\omega}) = e^{-j\omega n_d}X(e^{j\omega}).$$

A formal derivation of the convolution theorem is easily achieved by applying the definition of the Fourier transform to $y[n]$ as expressed in Eq. (163). This theorem can also be interpreted as a direct consequence of the eigenfunction property of complex exponentials for LTI systems. Recall that $H(e^{j\omega})$ is the frequency response of the LTI system whose impulse response is $h[n]$. Also, if

$$x[n] = e^{j\omega n},$$

then

$$y[n] = H(e^{j\omega})e^{j\omega n}.$$

That is, complex exponentials are *eigenfunctions* of LTI systems, where $H(e^{j\omega})$, the Fourier transform of $h[n]$, is the eigenvalue. From the definition of integration, the Fourier transform synthesis equation corresponds to the representation of a sequence $x[n]$ as a superposition of complex exponentials of infinitesimal size; that is,

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega = \lim_{\Delta\omega\to 0}\frac{1}{2\pi}\sum_k X(e^{jk\Delta\omega})e^{jk\Delta\omega n}\Delta\omega.$$

By the eigenfunction property of linear systems and by the principle of superposition, the corresponding output will be

$$y[n] = \lim_{\Delta\omega\to 0}\frac{1}{2\pi}\sum_k H(e^{jk\Delta\omega})X(e^{jk\Delta\omega})e^{jk\Delta\omega n}\Delta\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})X(e^{j\omega})e^{j\omega n}d\omega.$$

Thus, we conclude that

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}),$$

as in Eq. (164).

9.7 The Modulation or Windowing Theorem

If

$$x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega})$$

and

$$w[n] \stackrel{\mathcal{F}}{\longleftrightarrow} W(e^{j\omega}),$$

and if

$$y[n] = x[n]w[n], \tag{166}$$

then

$$Y(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})W(e^{j(\omega-\theta)})d\theta. \tag{167}$$

Equation (167) is a periodic convolution, i.e., a convolution of two periodic functions with the limits of integration extending over only one period. The duality inherent in most Fourier transform theorems is evident when we compare the convolution and modulation theorems. However, in contrast to the continuous-time case, where this duality is complete, in the discrete-time case fundamental differences arise because the Fourier transform is a sum, whereas the inverse transform is an integral with a periodic integrand. Although for continuous time, we can state that convolution in the time domain is represented by multiplication in the frequency domain and vice versa; in discrete time, this statement must be modified somewhat. Specifically, discrete-time convolution of sequences (the convolution sum) is equivalent to multiplication of corresponding periodic Fourier transforms, and multiplication of sequences is equivalent to *periodic* convolution of corresponding Fourier transforms.

The theorems of this section and a number of fundamental Fourier transform pairs are summarized in Tables 2 and 3, respectively. One of the ways that knowledge of Fourier transform theorems and properties is useful is in determining Fourier transforms

TABLE 3 FOURIER TRANSFORM PAIRS

Sequence	Fourier Transform
1. $\delta[n]$	1
2. $\delta[n-n_0]$	$e^{-j\omega n_0}$
3. $1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega+2\pi k)$
4. $a^n u[n] \quad (\|a\| < 1)$	$\frac{1}{1-ae^{-j\omega}}$
5. $u[n]$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega+2\pi k)$
6. $(n+1)a^n u[n] \quad (\|a\| < 1)$	$\frac{1}{(1-ae^{-j\omega})^2}$
7. $\frac{r^n \sin\omega_p(n+1)}{\sin\omega_p} u[n] \quad (\|r\| < 1)$	$\frac{1}{1-2r\cos\omega_p e^{-j\omega} + r^2 e^{-j2\omega}}$
8. $\frac{\sin\omega_c n}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1, & \|\omega\| < \omega_c, \\ 0, & \omega_c < \|\omega\| \le \pi \end{cases}$
9. $x[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
10. $e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega-\omega_0+2\pi k)$
11. $\cos(\omega_0 n + \phi)$	$\sum_{k=-\infty}^{\infty} [\pi e^{j\phi}\delta(\omega-\omega_0+2\pi k) + \pi e^{-j\phi}\delta(\omega+\omega_0+2\pi k)]$

or inverse transforms. Often, by using the theorems and known transform pairs, it is possible to represent a sequence in terms of operations on other sequences for which the transform is known, thereby simplifying an otherwise difficult or tedious problem. Examples 22–25 illustrate this approach.

Example 22 Determining a Fourier Transform Using Tables 2 and 3

Suppose we wish to find the Fourier transform of the sequence $x[n] = a^n u[n-5]$. This transform can be computed by exploiting Theorems 1 and 2 of Table 2 and transform pair 4 of Table 3. Let $x_1[n] = a^n u[n]$. We start with this signal because it is the most similar signal to $x[n]$ in Table 3. The table states that

$$X_1(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}}. \tag{168}$$

To obtain $x[n]$ from $x_1[n]$, we first delay $x_1[n]$ by five samples, i.e., $x_2[n] = x_1[n-5]$. Theorem 2 of Table 2 gives the corresponding frequency-domain relationship, $X_2(e^{j\omega}) = e^{-j5\omega}X_1(e^{j\omega})$, so

$$X_2(e^{j\omega}) = \frac{e^{-j5\omega}}{1-ae^{-j\omega}}. \tag{169}$$

To get from $x_2[n]$ to the desired $x[n]$, we need only multiply by the constant a^5, i.e., $x[n] = a^5 x_2[n]$. The linearity property of the Fourier transform, Theorem 1 of Table 2, then yields the desired Fourier transform,

$$X(e^{j\omega}) = \frac{a^5 e^{-j5\omega}}{1 - ae^{-j\omega}}. \tag{170}$$

Example 23 Determining an Inverse Fourier Transform Using Tables 2 and 3

Suppose that

$$X(e^{j\omega}) = \frac{1}{(1 - ae^{-j\omega})(1 - be^{-j\omega})}. \tag{171}$$

Direct substitution of $X(e^{j\omega})$ into Eq. (130) leads to an integral that is difficult to evaluate by ordinary real integration techniques. However, using the technique of partial fraction expansion, we can expand $X(e^{j\omega})$ into the form

$$X(e^{j\omega}) = \frac{a/(a-b)}{1 - ae^{-j\omega}} - \frac{b/(a-b)}{1 - be^{-j\omega}}. \tag{172}$$

From Theorem 1 of Table 2 and transform pair 4 of Table 3, it follows that

$$x[n] = \left(\frac{a}{a-b}\right) a^n u[n] - \left(\frac{b}{a-b}\right) b^n u[n]. \tag{173}$$

Example 24 Determining the Impulse Response from the Frequency Response

The frequency response of a highpass filter with linear phase is

$$H(e^{j\omega}) = \begin{cases} e^{-j\omega n_d}, & \omega_c < |\omega| < \pi, \\ 0, & |\omega| < \omega_c, \end{cases} \tag{174}$$

where a period of 2π is understood. This frequency response can be expressed as

$$H(e^{j\omega}) = e^{-j\omega n_d}(1 - H_{\text{lp}}(e^{j\omega})) = e^{-j\omega n_d} - e^{-j\omega n_d} H_{\text{lp}}(e^{j\omega}),$$

where $H_{\text{lp}}(e^{j\omega})$ is periodic with period 2π and

$$H_{\text{lp}}(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c, \\ 0, & \omega_c < |\omega| < \pi. \end{cases}$$

Using the result of Example 18 to obtain the inverse transform of $H_{\text{lp}}(e^{j\omega})$, together with properties 1 and 2 of Table 2, we have

$$\begin{aligned} h[n] &= \delta[n - n_d] - h_{\text{lp}}[n - n_d] \\ &= \delta[n - n_d] - \frac{\sin \omega_c (n - n_d)}{\pi(n - n_d)}. \end{aligned}$$

Example 25 Determining the Impulse Response for a Difference Equation

In this example, we determine the impulse response for a stable LTI system for which the input $x[n]$ and output $y[n]$ satisfy the linear constant-coefficient difference equation

$$y[n] - \tfrac{1}{2}y[n-1] = x[n] - \tfrac{1}{4}x[n-1]. \tag{175}$$

The z-transform is more useful than the Fourier transform for dealing with difference equations. However, this example offers a hint of the utility of transform methods in the analysis of linear systems. To find the impulse response, we set $x[n] = \delta[n]$; with $h[n]$ denoting the impulse response, Eq. (175) becomes

$$h[n] - \tfrac{1}{2}h[n-1] = \delta[n] - \tfrac{1}{4}\delta[n-1]. \tag{176}$$

Applying the Fourier transform to both sides of Eq. (176) and using properties 1 and 2 of Table 2, we obtain

$$H(e^{j\omega}) - \tfrac{1}{2}e^{-j\omega}H(e^{j\omega}) = 1 - \tfrac{1}{4}e^{-j\omega}, \tag{177}$$

or

$$H(e^{j\omega}) = \frac{1 - \frac{1}{4}e^{-j\omega}}{1 - \frac{1}{2}e^{-j\omega}}. \tag{178}$$

To obtain $h[n]$, we want to determine the inverse Fourier transform of $H(e^{j\omega})$. Toward this end, we rewrite Eq. (178) as

$$H(e^{j\omega}) = \frac{1}{1 - \frac{1}{2}e^{-j\omega}} - \frac{\frac{1}{4}e^{-j\omega}}{1 - \frac{1}{2}e^{-j\omega}}. \tag{179}$$

From transform 4 of Table 3,

$$\left(\tfrac{1}{2}\right)^n u[n] \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{1 - \frac{1}{2}e^{-j\omega}}.$$

Combining this transform with property 2 of Table 2, we obtain

$$-\left(\tfrac{1}{4}\right)\left(\tfrac{1}{2}\right)^{n-1} u[n-1] \stackrel{\mathcal{F}}{\longleftrightarrow} -\frac{\frac{1}{4}e^{-j\omega}}{1 - \frac{1}{2}e^{-j\omega}}. \tag{180}$$

Based on property 1 of Table 2, then,

$$h[n] = \left(\tfrac{1}{2}\right)^n u[n] - \left(\tfrac{1}{4}\right)\left(\tfrac{1}{2}\right)^{n-1} u[n-1]. \tag{181}$$

10 DISCRETE-TIME RANDOM SIGNALS

The preceding sections have focused on mathematical representations of discrete-time signals and systems and the insights that derive from such mathematical representations. Discrete-time signals and systems have both a time-domain and a frequency-domain representation, each with an important place in the theory and design of discrete-time signal-processing systems. Until now, we have assumed that the signals are deterministic, i.e., that each value of a sequence is uniquely determined by a mathematical expression, a table of data, or a rule of some type.

In many situations, the processes that generate signals are so complex as to make precise description of a signal extremely difficult or undesirable, if not impossible. In such cases, modeling the signal as a random process is analytically useful.[5] As an example, many of the effects encountered in implementing digital signal-processing algorithms with finite register length can be represented by additive noise, i.e., a random sequence. Many mechanical systems generate acoustic or vibratory signals that can be processed to diagnose potential failure; again, signals of this type are often best modeled in terms of random signals. Speech signals to be processed for automatic recognition or bandwidth compression and music to be processed for quality enhancement are two more of many examples.

A random signal is considered to be a member of an ensemble of discrete-time signals that is characterized by a set of probability density functions. More specifically, for a particular signal at a particular time, the amplitude of the signal sample at that time is assumed to have been determined by an underlying scheme of probabilities. That is, each individual sample $x[n]$ of a particular signal is assumed to be an outcome of some underlying random variable x_n. The entire signal is represented by a collection of such random variables, one for each sample time, $-\infty < n < \infty$. This collection of random variables is referred to as a *random process*, and we assume that a particular sequence of samples $x[n]$ for $-\infty < n < \infty$ has been generated by the random process that underlies the signal. To completely describe the random process, we need to specify the individual and joint probability distributions of all the random variables.

The key to obtaining useful results from such models of signals lies in their description in terms of averages that can be computed from assumed probability laws or estimated from specific signals. While random signals are not absolutely summable or square summable and, consequently, do not directly have Fourier transforms, many (but not all) of the properties of such signals can be summarized in terms of averages such as the autocorrelation or autocovariance sequence, for which the Fourier transform often exists. As we will discuss in this section, the Fourier transform of the autocorrelation sequence has a useful interpretation in terms of the frequency distribution of the power in the signal. The use of the autocorrelation sequence and its transform has another important advantage: The effect of processing random signals with a discrete-time linear system can be conveniently described in terms of the effect of the system on the autocorrelation sequence.

In the following discussion, we assume that the reader is familiar with the basic concepts of random processes, such as averages, correlation and covariance functions, and the power spectrum. A detailed presentation of the theory of random signals can be found in a variety of excellent texts, such as Davenport (1970), and Papoulis (2002), Gray and Davidson (2004), Kay (2006) and Bertsekas and Tsitsiklis (2008).

Our primary objective in this section is to present a specific set of results that will be useful. Therefore, we focus on wide-sense stationary random signals and their representation in the context of processing with LTI systems. Although, for simplicity,

[5]It is common in the signal processing literature to use the terms "random" and "stochastic" interchangeably. We primarily refer to this class of signals as random signals or random processes.

we assume that $x[n]$ and $h[n]$ are real valued, the results can be generalized to the complex case.

Consider a stable LTI system with real impulse response $h[n]$. Let $x[n]$ be a real-valued sequence that is a sample sequence of a wide-sense stationary discrete-time random process. Then, the output of the linear system is also a sample sequence of a discrete-time random process related to the input process by the linear transformation

$$y[n] = \sum_{k=-\infty}^{\infty} h[n-k]x[k] = \sum_{k=-\infty}^{\infty} h[k]x[n-k].$$

As we have shown, since the system is stable, $y[n]$ will be bounded if $x[n]$ is bounded. We will see shortly that if the input is stationary,[6] then so is the output. The input signal may be characterized by its mean m_x and its autocorrelation function $\phi_{xx}[m]$, or we may also have additional information about 1^{st}- or even 2^{nd}-order probability distributions. In characterizing the output random process $y[n]$ we desire similar information. For many applications, it is sufficient to characterize both the input and output in terms of simple averages, such as the mean, variance, and autocorrelation. Therefore, we will derive input–output relationships for these quantities.

The means of the input and output processes are, respectively,

$$m_{x_n} = \mathcal{E}\{\mathrm{x}_n\}, \qquad m_{y_n} = \mathcal{E}\{\mathrm{y}_n\}, \tag{182}$$

where $\mathcal{E}\{\cdot\}$ denotes the expected value of a random variable. In most of our discussion, it will not be necessary to carefully distinguish between the random variables x_n and y_n and their specific values $x[n]$ and $y[n]$. This will simplify the mathematical notation significantly. For example, Eqs. (182) will alternatively be written

$$m_x[n] = \mathcal{E}\{x[n]\}, \qquad m_y[n] = \mathcal{E}\{y[n]\}. \tag{183}$$

If $x[n]$ is stationary, then $m_x[n]$ is independent of n and will be written as m_x, with similar notation for $m_y[n]$ if $y[n]$ is stationary.

The mean of the output process is

$$m_y[n] = \mathcal{E}\{y[n]\} = \sum_{k=-\infty}^{\infty} h[k]\mathcal{E}\{x[n-k]\},$$

where we have used the fact that the expected value of a sum is the sum of the expected values. Since the input is stationary, $m_x[n-k] = m_x$, and consequently,

$$m_y[n] = m_x \sum_{k=-\infty}^{\infty} h[k]. \tag{184}$$

From Eq. (184), we see that the mean of the output is also constant. An equivalent expression to Eq. (184) in terms of the frequency response is

$$m_y = H(e^{j0})m_x. \tag{185}$$

[6]We will use the term *stationary* to mean "wide-sense stationary," i.e., that $E\{x[n_1]x[n_2]\}$ for all n_1, n_2 depends only on the difference $(n_1 - n_2)$. Equivalently, the autocorrelation is only a function of the time difference $(n_1 - n_2)$.

Assuming temporarily that the output is nonstationary, the autocorrelation function of the output process for a real input is

$$\begin{aligned}\phi_{yy}[n, n+m] &= \mathcal{E}\{y[n]y[n+m]\}\\ &= \mathcal{E}\left\{\sum_{k=-\infty}^{\infty}\sum_{r=-\infty}^{\infty} h[k]h[r]x[n-k]x[n+m-r]\right\}\\ &= \sum_{k=-\infty}^{\infty} h[k]\sum_{r=-\infty}^{\infty} h[r]\mathcal{E}\{x[n-k]x[n+m-r]\}.\end{aligned}$$

Since $x[n]$ is assumed to be stationary, $\mathcal{E}\{x[n-k]x[n+m-r]\}$ depends only on the time difference $m+k-r$. Therefore,

$$\phi_{yy}[n, n+m] = \sum_{k=-\infty}^{\infty} h[k]\sum_{r=-\infty}^{\infty} h[r]\phi_{xx}[m+k-r] = \phi_{yy}[m]. \tag{186}$$

That is, the output autocorrelation sequence also depends only on the time difference m. Thus, for an LTI system having a wide-sense stationary input, the output is also wide-sense stationary.

By making the substitution $\ell = r - k$, we can express Eq. (186) as

$$\begin{aligned}\phi_{yy}[m] &= \sum_{\ell=-\infty}^{\infty} \phi_{xx}[m-\ell]\sum_{k=-\infty}^{\infty} h[k]h[\ell+k]\\ &= \sum_{\ell=-\infty}^{\infty} \phi_{xx}[m-\ell]c_{hh}[\ell],\end{aligned} \tag{187}$$

where we have defined

$$c_{hh}[\ell] = \sum_{k=-\infty}^{\infty} h[k]h[\ell+k]. \tag{188}$$

The sequence $c_{hh}[\ell]$ is referred to as the *deterministic autocorrelation sequence* or, simply, the *autocorrelation sequence of* $h[n]$. It should be emphasized that $c_{hh}[\ell]$ is the autocorrelation of an aperiodic—i.e., finite-energy—sequence and should not be confused with the autocorrelation of an infinite-energy random sequence. Indeed, it can be seen that $c_{hh}[\ell]$ is simply the discrete convolution of $h[n]$ with $h[-n]$. Equation (187), then, can be interpreted to mean that the autocorrelation of the output of a linear system is the convolution of the autocorrelation of the input with the aperiodic autocorrelation of the system impulse response.

Equation (187) suggests that Fourier transforms may be useful in characterizing the response of an LTI system to a random input. Assume, for convenience, that $m_x = 0$; i.e., the autocorrelation and autocovariance sequences are identical. Then, with $\Phi_{xx}(e^{j\omega})$, $\Phi_{yy}(e^{j\omega})$, and $C_{hh}(e^{j\omega})$ denoting the Fourier transforms of $\phi_{xx}[m]$, $\phi_{yy}[m]$, and $c_{hh}[\ell]$, respectively, from Eq. (187),

$$\Phi_{yy}(e^{j\omega}) = C_{hh}(e^{j\omega})\Phi_{xx}(e^{j\omega}). \tag{189}$$

Also, from Eq. (188),

$$\begin{aligned}C_{hh}(e^{j\omega}) &= H(e^{j\omega})H^*(e^{j\omega})\\ &= |H(e^{j\omega})|^2,\end{aligned}$$

so

$$\Phi_{yy}(e^{j\omega}) = |H(e^{j\omega})|^2 \Phi_{xx}(e^{j\omega}). \tag{190}$$

Equation (190) provides the motivation for the term *power density spectrum*. Specifically,

$$\begin{aligned} \mathcal{E}\{y^2[n]\} = \phi_{yy}[0] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{yy}(e^{j\omega})\, d\omega \\ &= \text{total average power in output.} \end{aligned} \tag{191}$$

Substituting Eq. (190) into Eq. (191), we have

$$\mathcal{E}\{y^2[n]\} = \phi_{yy}[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} |H(e^{j\omega})|^2 \Phi_{xx}(e^{j\omega})\, d\omega. \tag{192}$$

Suppose that $H(e^{j\omega})$ is an ideal bandpass filter, as shown in Figure 18(c). Since $\phi_{xx}[m]$ is a real, even sequence, its Fourier transform is also real and even, i.e.,

$$\Phi_{xx}(e^{j\omega}) = \Phi_{xx}(e^{-j\omega}).$$

Likewise, $|H(e^{j\omega})|^2$ is an even function of ω. Therefore, we can write

$$\begin{aligned} \phi_{yy}[0] &= \text{average power in output} \\ &= \frac{1}{2\pi}\int_{\omega_a}^{\omega_b} \Phi_{xx}(e^{j\omega})\, d\omega + \frac{1}{2\pi}\int_{-\omega_b}^{-\omega_a} \Phi_{xx}(e^{j\omega})\, d\omega. \end{aligned} \tag{193}$$

Thus, the area under $\Phi_{xx}(e^{j\omega})$ for $\omega_a \le |\omega| \le \omega_b$ can be taken to represent the mean-square value of the input in that frequency band. We observe that the output power must remain nonnegative, so

$$\lim_{(\omega_b - \omega_a)\to 0} \phi_{yy}[0] \ge 0.$$

This result, together with Eq. (193) and the fact that the band $\omega_a \le \omega \le \omega_b$ can be arbitrarily small, implies that

$$\Phi_{xx}(e^{j\omega}) \ge 0 \qquad \text{for all } \omega. \tag{194}$$

Hence, we note that the power density function of a real signal is real, even, and nonnegative.

Example 26 White Noise

The concept of white noise is exceedingly useful in a wide variety of contexts in the design and analysis of signal processing and communications systems. A white-noise signal is a signal for which $\phi_{xx}[m] = \sigma_x^2\delta[m]$. We assume in this example that the signal has zero mean. The power spectrum of a white-noise signal is a constant, i.e.,

$$\Phi_{xx}(e^{j\omega}) = \sigma_x^2 \qquad \text{for all } \omega.$$

The average power of a white-noise signal is therefore

$$\phi_{xx}[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{xx}(e^{j\omega})\, d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sigma_x^2\, d\omega = \sigma_x^2.$$

The concept of white noise is also useful in the representation of random signals whose power spectra are not constant with frequency. For example, a random signal $y[n]$ with power spectrum $\Phi_{yy}(e^{j\omega})$ can be assumed to be the output of an LTI system with a white-noise input. That is, we use Eq. (190) to define a system with frequency response $H(e^{j\omega})$ to satisfy the equation

$$\Phi_{yy}(e^{j\omega}) = |H(e^{j\omega})|^2\sigma_x^2,$$

where σ_x^2 is the average power of the assumed white-noise input signal. We adjust the average power of this input signal to give the correct average power for $y[n]$. For example, suppose that $h[n] = a^n u[n]$. Then,

$$H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}},$$

and we can represent all random signals whose power spectra are of the form

$$\Phi_{yy}(e^{j\omega}) = \left|\frac{1}{1 - ae^{-j\omega}}\right|^2 \sigma_x^2 = \frac{\sigma_x^2}{1 + a^2 - 2a\cos\omega}.$$

Another important result concerns the cross-correlation between the input and output of an LTI system:

$$\begin{aligned}\phi_{yx}[m] &= \mathcal{E}\{x[n]y[n+m]\}\\ &= \mathcal{E}\left\{x[n]\sum_{k=-\infty}^{\infty} h[k]x[n+m-k]\right\} \qquad (195)\\ &= \sum_{k=-\infty}^{\infty} h[k]\phi_{xx}[m-k].\end{aligned}$$

In this case, we note that the cross-correlation between input and output is the convolution of the impulse response with the input autocorrelation sequence.

The Fourier transform of Eq. (195) is

$$\Phi_{yx}(e^{j\omega}) = H(e^{j\omega})\Phi_{xx}(e^{j\omega}). \qquad (196)$$

This result has a useful application when the input is white noise, i.e., when $\phi_{xx}[m] = \sigma_x^2\delta[m]$. Substituting into Eq. (195), we note that

$$\phi_{yx}[m] = \sigma_x^2 h[m]. \qquad (197)$$

That is, for a zero-mean white-noise input, the cross-correlation between input and output of a linear system is proportional to the impulse response of the system. Similarly, the power spectrum of a white-noise input is

$$\Phi_{xx}(e^{j\omega}) = \sigma_x^2, \qquad -\pi \le \omega \le \pi. \tag{198}$$

Thus, from Eq. (196),

$$\Phi_{yx}(e^{j\omega}) = \sigma_x^2 H(e^{j\omega}). \tag{199}$$

In other words, the cross power spectrum is in this case proportional to the frequency response of the system. Equations (197) and (199) may serve as the basis for estimating the impulse response or frequency response of an LTI system if it is possible to observe the output of the system in response to a white-noise input. An example application is in the measurement of the acoustic impulse response of a room or concert hall.

11 SUMMARY

In this chapter, we have reviewed and discussed a number of basic definitions relating to discrete-time signals and systems. We considered the definition of a set of basic sequences, the definition and representation of LTI systems in terms of the convolution sum, and some implications of stability and causality. The class of systems for which the input and output satisfy a linear constant-coefficient difference equation with initial rest conditions was shown to be an important subclass of LTI systems. The recursive solution of such difference equations was discussed and the classes of FIR and IIR systems defined.

An important means for the analysis and representation of LTI systems lies in their frequency-domain representation. The response of a system to a complex exponential input was considered, leading to the definition of the frequency response. The relation between impulse response and frequency response was then interpreted as a Fourier transform pair.

We called attention to many properties of Fourier transform representations and discussed a variety of useful Fourier transform pairs. Tables 1 and 2 summarize the properties and theorems, and Table 3 contains some useful Fourier transform pairs.

The chapter concluded with an introduction to discrete-time random signals.

Problems

Basic Problems with Answers

1. For each of the following systems, determine whether the system is (1) stable, (2) causal, (3) linear, (4) time invariant, and (5) memoryless:

(a) $T(x[n]) = g[n]x[n]$ with $g[n]$ given
(b) $T(x[n]) = \sum_{k=n_0}^{n} x[k] \qquad n \neq 0$
(c) $T(x[n]) = \sum_{k=n-n_0}^{n+n_0} x[k]$
(d) $T(x[n]) = x[n - n_0]$