Finite Length Orthogonal Transform (FLOT) :-

$$X(k) = \sum_{n=0}^{N-1} x[n]\,\phi_k^*[n] \qquad 0 \le k \le N-1$$

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\,\phi_k(n) \qquad 0 \le n \le N-1$$

Involves:
orthogonal basis
$\phi_k(n)$; $0 \le k \le N-1$
is subjected to.

$$\frac{1}{N}\sum_{n=0}^{N-1} \phi_i(n)\,\phi_j^*[n] = \begin{cases} 1, & i=j \\ 0, & i \ne j \end{cases}$$

First, Foremost Example of FLOT is DFT

where $\phi_k(n) = e^{\frac{j2\pi}{N}kn}$

$$x[n] \xleftrightarrow{\text{DFT}} X(k)$$

Issues with DFT :-

1. $X(k) \in \mathbb{C}$ even for $x[n] \in \mathbb{R}$ for $n \in \{0,1,\dots,N-1\}$
2. Redundancy in DFT Transform coefficients for $x[n] \in \mathbb{R}$, $n \in \{0,1,2,\dots,N-1\}$

$$\boxed{X(k) = X^*(N-k).}$$

for N - even $\Rightarrow$ $X(0) = \sum_{n=0}^{N-1} x[n]$

$$X\left(\frac{N}{2}\right) = \sum_{n=0}^{N-1} (-1)^n x[n]$$

real & Distinct.

$$\boxed{\begin{gathered} X[k] = X^*(N-k) \\ 1 \le k \le \frac{N}{2}-1 \end{gathered}}$$

$X(1) = X^*(N-1)$

$X(2) = X^*(N-2)$

$\vdots$

$X\left(\frac{N}{2}-1\right) = X^*\left(\frac{N}{2}+1\right)$

"$\frac{N}{2}-1$" samples are in Conjugate pair i.e. redundant ones.

Similarly for N - Odd:

$X(0) = \sum_{n=0}^{N-1} x(n) \in \mathbb{R}$

$\boxed{X(k) = X^*(N-k); \quad 1 \le k \le \frac{N-1}{2}}$ $\Rightarrow$ $\frac{N-1}{2}$ samples are in Conj Pair / Redundant

(3) For $x[n] \in \mathbb{R}$, $n \in \{0, 1, \dots, N-1\}$

$\Downarrow$

$X(k)$ does not have Good Energy Compaction properties as that of already developed ~~...~~ others FLOTs like DCT, DST, Haar Transform, Hadamard Transform

Suggested Reading $\rightarrow$ DSP by S.K. Mitra

Discrete Cosine Transform $\rightarrow$ $\phi_k(n) = \cos(\quad k \cdot n)$

$\rightarrow X_{DCT}(k) \in \mathbb{R}$ if $x[n] \in \mathbb{R}$, $n \in \{0, 1, 2, \dots, N-1\}$
$k \in \{0, 1, 2, \dots, N-1\}$

$\Downarrow$

$\rightarrow$ No redundancy due to conjugation symmetry.

$\rightarrow$ Good energy compaction properties.

Development Rough Idea to have $X_{DCT}(k) \in \mathbb{R}$ for $x(n) \in \mathbb{R}$ without Redundancy $\rightarrow$

$x[n] \in \mathbb{R} \Rightarrow x[n] = x^*[n] \xleftrightarrow[N\text{-point}]{DFT} X(k) = X^*((-k)_N)$
$\hookrightarrow$ Real

$X(k) = X^*(N-k) \Rightarrow \boxed{X(N-k) = X^*(k)}$

$x[n]$ is cyclo-symmetric $\Rightarrow x[n] = x[(-n)_N] \xleftrightarrow[N\text{-point}]{DFT} \boxed{X(k) = X((-k)_N) = X(N-k).}$

when $x[n]$ is Real & even $\Rightarrow x[n] = x^*[n] = x[(-n)_N] \xleftrightarrow[N\text{-point}]{DFT} \boxed{X(k) = X^*(N-k) = X(N-k)}$

$X^*(N-k) \Rightarrow X^*(k)$

$\Downarrow$

$X(k) = X^*(k) \Rightarrow X(k)$ is Real

& $X(k) = X(N-k) \Rightarrow X(k)$ is Even.

$\Downarrow$

$\boxed{X(k) = X^*(k) = X(N-k)}$

$X(0) = X(0)$
$X(1) = X(N-1)$
$X(2) = X(N-2)$
$X(3) = X(N-3)$
$\vdots$
$X\left(\frac{N}{2}-1\right) = X\left(\frac{N}{2}+1\right)$

$\frac{N}{2}$-points DCT $\Leftarrow$ Keep only unique "$N/2$" points

Development of DCT in a General manner

To incorporate above ideas intelligently

DCT $\Rightarrow$ extends N-length $x[n] \in \mathbb{R}$ to length, $L > N$ such that $y_e[n] = y_e[\langle n \rangle_L]$ is periodic version $\tilde{y}_e(n) \in \mathbb{R}$ is an cyclo (even) symmetric sequence with periodicity $L > N$ instead of 'N'

Extension is the process primarily done to remove the redundancy using/exploiting cyclosymmetry (even-symmetry).

$x[n] \xrightarrow{\text{Zero padding}} x_e[n] \xrightarrow{\text{Interpolation (symmetric)}} y_e[n] = y_e[\langle -n \rangle_L] \xleftrightarrow[\text{DFT}]{L\text{-pt}} (\ )$

Zero-padded L-length sequence

extract amplitude of first 'N' points

$\Downarrow$

Take DFT with L-points instead of 'N'-points.

$\Downarrow$

At the end; extract only the amplitude part of first N-points of the L-point DFT

$\equiv$ N-point DCT

$\because$ Phase part of each k will be Linear & directly computed using 'N' only

Moreover, for Periodic version $\Rightarrow$ Phase become Zero.

Various Extensions of $x[n]$ to $y_e[n] = y_e(\langle -n \rangle_N)$

e.g. $x[n] = \{\underset{\uparrow}{a}, b, c, d\} \Rightarrow N = 4$

Type-I $\rightarrow$ $y_{e_1}[n] = \{\underset{\uparrow}{a}, b, c, d, c, b\} \Rightarrow \boxed{L = 6 = 2N-2}$ $\rightarrow \tilde{y}_{e_1}(n) \rightarrow$ DCT-1

$n=0$, $n=N-1$ Points of Symmetry

Type-II $\rightarrow$ $y_{e_2}[n] = \{\underset{\uparrow}{a}, b, c, d, d, b, c, a\} \Rightarrow \boxed{L = 8 = 2N}$ $\Rightarrow \tilde{y}_{e_2}(n) \Rightarrow$ DCT-2

$n = -\frac{1}{2}$, $n = N - \frac{1}{2}$ Points of Symmetry.

Type-III $\rightarrow$ $y_{e_3}[n] = \{\underset{\uparrow}{a}, b, c, d, 0, -d, -c, -b, -a, -b, -c, -d, 0, a, b, c\}$

$n=0$, $n=8=2N$ Points of Symmetry

$\boxed{L = 16 = 4N}$ $\rightarrow \tilde{y}_{e_3}[n] \rightarrow$ DCT-3

Type-IV $\rightarrow$ $y_{e_4}[n] = \{\underset{\uparrow}{a}, b, c, d, -d, -c, -b, -a, -a, -b, -c, -d, d, c, b, a\}$

$\boxed{n = -\frac{1}{2}}$, $\boxed{n = 2N - \frac{1}{2}}$ Points of Symmetry.

$\boxed{L = 16 = 4N.}$ $\rightarrow \tilde{y}_{e_4}[n] \Rightarrow$ DCT-4

& so-on.

& so-on.

Type-VIII $\rightarrow$ $\Rightarrow$ Till-Date; Total '8' types of cyclo-symmetric extensions possible; which results in "DCT-1, DCT-2, DCT-3, ------- DCT-8".

Note :- On the otherhand; ~~there~~ are '8' possible cyclo-anti-symmetric (odd-symmetric) extensions $\Rightarrow$ Discrete Sine Transforms

DST-1, DST-2, ---, DST-8

Note :- Type-2 DCT has been employed in a no. of international standards for image and video compression, such as JPEG, MPEG, and H.261, because of its better energy compaction property. So we consider here only DCT-2

$\Downarrow$

$\boxed{\text{Even Symmetrical DCT}}$

Type-2 DCT (DCT-2) :-

~~x[n]~~ Let $x[n]$ be an 'N' length sequence defined for $n \in \{0,1,2,\dots,N-1\}$

$\Downarrow$

$$x_e[n] = \begin{cases} x[n] & 0 \le n \le N-1 \\ 0 & N \le n \le 2N-1 \end{cases}$$

In general

$$x_e[n] = \begin{cases} x[n] & 0 \le n \le N-1 \\ 0 & N \le n \le L-1 \end{cases}$$

$$y_e[n] = x_e[L-N]$$

Type-2 symmetric sequence, $y(n) = x_e[n] + x_e[2N-1-n]$

$\Downarrow$

$$y(n) = y(2N-1-n)$$

$$W_N = e^{\frac{-j2\pi}{N}}$$

2N-point DFT

$\Downarrow$

$$Y(k) = \sum_{n=0}^{2N-1} y(n) W_{2N}^{kn}, \quad 0 \le k \le 2N-1$$

$$= \sum_{n=0}^{N-1} y[n] W_{2N}^{kn} + \sum_{n=N}^{2N-1} y[n] W_{2N}^{kn}$$

$$= \sum_{n=0}^{N-1} x[n] W_{2N}^{kn} + \sum_{n=N}^{2N-1} x[2N-1-n] W_{2N}^{kn}, \quad 0 \le k \le 2N-1$$

$\Downarrow$

change of variable

$2N-1-n = m \rightarrow n$

$$Y(k) = \sum_{n=0}^{N-1} x[n]\, W_{2N}^{kn} + \sum_{n=0}^{N-1} x[n]\, W_{2N}^{k(2N-1-n)}$$

$$= W_{2N}^{-\frac{k}{2}} \sum_{n=0}^{N-1} x[n] \left(W_{2N}^{kn} \cdot W_{2N}^{k/2} + W_{2N}^{-kn}\, W_{2N}^{-k/2} \right)$$

$$= W_{2N}^{-\frac{k}{2}} \sum_{n=0}^{N-1} 2\, x[n] \left\{ \frac{e^{-j\frac{2\pi}{2N}\cdot k\frac{(2n+1)}{2}} + e^{j\frac{2\pi}{2N}\cdot k\frac{(2n+1)}{2}}}{2} \right\}.$$

$$\boxed{Y(k) = W_{2N}^{-k/2} \sum_{n=0}^{N-1} 2\, x[n] \cos\left(\frac{\pi k}{N} \frac{(2n+1)}{2} \right), \quad 0 \le k \le 2N-1}$$

$W_{2N}^{-\frac{k}{2}} = e^{+j\frac{2\pi}{2N}\cdot\frac{k}{2}} = e^{j\frac{\pi k}{2N}} \Rightarrow W_{2N}^{-\frac{k}{2}} = e^{j\frac{\pi k}{2N}}$ ⇒ Linear phase factor parameterized on 'N'.

$$x[n] \xleftrightarrow{DCT} \boxed{X_{DCT}(k) = \sum_{n=0}^{N-1} 2\, x[n] \cos\left(\frac{\pi k (2n+1)}{2N} \right), \quad 0 \le k \le N-1}$$

$x[n] \in \mathbb{R} \xleftrightarrow{DCT} X_{DCT}(k) \in \mathbb{R}$

$n \in \{0, 1, 2, \ldots, N-1\}$. $\quad k = \{0, 1, 2, \ldots, N-1\}$

Inverse DCT :-

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} \alpha(k)\, X_{DCT}(k) \cos\left(\frac{\pi k (2n+1)}{2N} \right), \quad 0 \le n \le N-1$$

where $\alpha(k) = \begin{cases} 1/2 & k = 0 \\ 1 & 1 \le k \le N-1 \end{cases}$

Note :-

$$\frac{1}{N} \sum_{n=0}^{N-1} \cos\left(\frac{\pi k (2n+1)}{2N} \right) \cos\left(\frac{\pi m (2n+1)}{2N} \right) = \begin{cases} 1 & k = m = 0 \\ 1/2 & k = m \neq 0 \\ 0 & k \neq m \end{cases}$$

$\Downarrow$

Basis sequences $\cos\left(\frac{\pi k (2n+1)}{2N}\right)$ are orthogonal to each other.

~~Derivation~~ Verification of $x[n]$ as IDCT of $X_{DCT}(k)$:-

$$X_{DCT}(k) = \sum_{n=0}^{N-1} 2\, x[n] \cos\left(\frac{\pi k(2n+1)}{2N}\right) \qquad 0 \le k \le N-1$$

from IDCT eq^n

$$= 2\sum_{n=0}^{N-1} \left(\frac{1}{N}\sum_{l=0}^{N-1} \alpha(l)\, X_{DCT}(l) \cos\left(\frac{\pi l(2n+1)}{2N}\right)\right)\cos\left(\frac{\pi k(2n+1)}{2N}\right)$$

$$= 2\sum_{l=0}^{N-1} \alpha(l)\, X_{DCT}(l) \underbrace{\left(\frac{1}{N}\sum_{n=0}^{N-1} \cos\left(\frac{\pi l(2n+1)}{2N}\right)\cos\left(\frac{\pi k(2n+1)}{2N}\right)\right)}_{}$$

$$\begin{cases} 1 & k = l = 0 \\ 1/2 & k = l \neq 0 \\ 0 & k \neq l \end{cases}$$

$$X_{DCT}\{k\} = \begin{cases} 2\,\alpha(k)\, X_{DCT}(k) & k=0 \\ \alpha(k)\, X_{DCT}(k) & 0 \le k \le N-1 \end{cases}$$

$$= X_{DCT}(k) \qquad 0 \le k \le N-1$$

~~Computation eqn of DCT~~

Relationship of 2N-point DFT & N-point DCT :-

$$Y(k) = \begin{cases} W_{2N}^{-\frac{k}{2}}\, X_{DCT}(k) & 0 \le k \le N-1 \\ 0 & k = N \\ -W_{2N}^{-\frac{k}{2}}\, X_{DCT}(2N-k) & N+1 \le k \le 2N-1 \end{cases}$$

2-N point IDFT

$$y[n] = \frac{X_{DCT}(0)}{2N} + \frac{1}{N}\sum_{k=1}^{N-1} X_{DCT}(k) \cos\left(\frac{\pi k(2n+1)}{2N}\right), \qquad 0 \le n \le 2N-1$$

length-N IDCT of N-pt. $X_{DCT}(k)$ is given by

$$\boxed{x[n] = y[n] \quad , \quad 0 \le n \le N-1}$$

DCT Properties →

N-length sequence in TD	N-length DCT in FD
$g[n]$	$G_{DCT}(k)$
$h[n]$	$H_{DCT}(k)$
Linearity: $\alpha_1 g[n] + \alpha_2 h[n]$	$\alpha_1 G_{DCT}(k) + \alpha_2 H_{DCT}(k)$
Conjugation: $g^*[n]$	$G^*_{DCT}(k)$

Energy Preservation or Parseval's Energy Theorem

$$\sum_{k=0}^{N-1} |g(n)|^2 = \frac{1}{2N} \sum_{k=0}^{N-1} \alpha(k) |G_{DCT}(k)|^2$$

where $\alpha(k) = \begin{cases} 1/2 & k=0 \\ 1 & 1 \le k \le N-1 \end{cases}$

Energy Compaction →

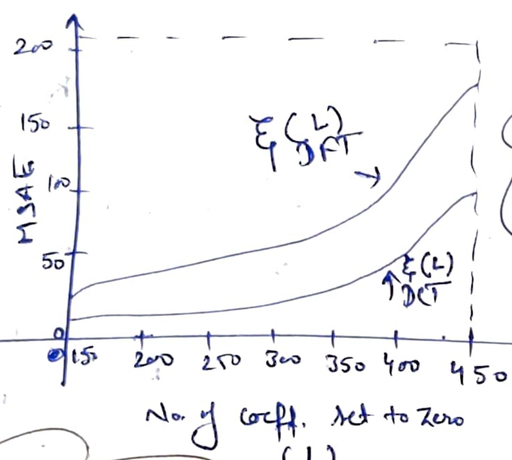
- Dominant samples of $|G_{DCT}(k)|$ with high values are usually in the LF range; i.e. near to $k=0$ & contain most of the energy
- $|G_{DCT}(k)| \to 0$ i.e. $|G_{DCT}(k)|$ with very small values tend to be in the HF range; i.e near to $k \to N-1$.
- Inverse Transform of the modified $G_{DCT}(k)$ with zero-valued HF samples is thus an approximation of $g[n]$.

If L samples of the $G_{DCT}(k)$ with indices in the range $\mathcal{R}$ are set to zero with, $L << N$ and if $g^{(M)}(n)$ denotes the inverse of the modified DCT, then the "Measure of energy compaction" property caused by the removal of 'L' samples is given by the Mean-Square approximation error (MSAE)

$$\boxed{\xi_{DCT}[L] = \frac{1}{N}\sum_{n=0}^{N-1} \left| g[n] - g_{DCT}^{(M)}(n) \right|^2}$$

Note→ DCT has significantly better energy compaction property than DFT.

∵ DCT smoothens the given $x[n]$ by extending it symmetrically



Zero-th 1D signal is used as the row of 512 X 512 image called "Goldhill" shown in Fig. 1.19 (a) of the DSP book by Sanjit K. Mitra.

which causes Smoothness

Applications of DCT due to better Energy Compaction

↳ in Signal compression

↳ image compression

↳ speech compression

zero-th order smoothening: $\delta(t) \overset{F}{\longleftrightarrow} 1$

first order smoothening: $\longleftrightarrow \zeta \frac{\sin(\omega\zeta)}{\omega}$

second order smoothening: $\longleftrightarrow \zeta^2 \frac{\sin^2(\omega\zeta)}{\omega^2}$

third order smoothening: $\longleftrightarrow \propto \frac{1}{\omega^3}$

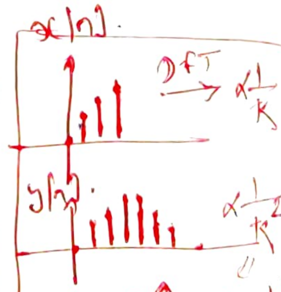
& so on

$x[n] \longleftrightarrow \boxed{X(\omega) \propto \frac{1}{\omega^q}}$

↓

Smoothening of order 'q'

$\omega \uparrow \Rightarrow |X(\omega)| \downarrow$



As $K \uparrow \Rightarrow |DCT| \downarrow$ faster as compared to DFT