Fast Fourier Transform

It reduces the no. of complex multiplication & additions required.

$$X(K) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi nK}{N}}$$

For single value of K, N complex multiplications are required.

Since there are N ~~to~~ terms on R.H.S. for single K and N-1 complex additions are required

Now K ranges from $0 \to N-1$

$$\boxed{\begin{array}{l} N^2 \text{ complex multiplication} \\ N^2-N \text{ complex addition} \end{array}}$$

Let we have a sequence $x(n)$ of length N.

DFT $\Rightarrow$ $X(K) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi Kn}{N}}$

$W_N^{nK} = e^{-j\frac{2\pi nK}{N}}$] Twiddle Factor.

Now $X(K) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi Kn}{N}}$

By divide and conquer approach. we divide into multiplication of prime number.

Say $N = L.M.$

$x(n)$ is stored in a 2-D array.

Index $= (m + lM)$ (rowwise)

$$\begin{array}{c|cccc} l \backslash m & 0 & 1 & \cdots & M-1 \\ \hline 0 & x(0) & x(1) & \cdots & x(m-1) \\ 1 & x(m) & x(m+2) & \cdots & x(2m-1) \\ \vdots & & & & \\ L-1 & & & & \end{array}$$

Reading the values

Index $= (p + Lq)$ (columnwise)

$$\begin{array}{c|cccc} p \backslash q & 0 & 1 & \cdots & M-1 \\ \hline 0 & & & & \\ 1 & & & & \\ \vdots & & & & \\ L-1 & & & & \end{array}$$

$$W_N^{n(K+\frac{N}{2})} = e^{-j\frac{2\pi n(K+N/2)}{N}}$$
$$= e^{-j\frac{2\pi nK}{N}} \cdot e^{-j\frac{2\pi n N/2}{N}}$$
$$= e^{-j\frac{2\pi nK}{N}}\, e^{-j\pi n}$$
$$= e^{-j\frac{2\pi nK}{N}} (-1)$$

$$\boxed{\begin{array}{l} W_N^{n(K+\frac{N}{2})} = -W_N^{nK} \\ W_N^{n(K+N)} = W_N^{nK} \end{array}}$$ Periodicity property.

Index

Storing	Retrieving
$m + lM$ (row)	$p + Lq$ (column)
$l + Lm$ (column)	$q + Mp$ (row)

$$X(K) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi nK}{N}}$$

$$X(p,q) = \sum_{l=0}^{L-1} \sum_{m=0}^{M-1} x(l,m)\, e^{-j\frac{2\pi (l+Lm)(q+Mp)}{LM}}$$

$$X(p,q)=\sum_{l=0}^{L-1}\sum_{m=0}^{M-1} x(l,m)\, e^{-j2\pi\frac{(lq+lMp+Lmq+LMmp)}{LM}}$$

$$=\sum_{l=0}^{L-1}\sum_{m=0}^{M-1} x(l,m)\, e^{-j2\pi\frac{lq}{LM}}\, e^{-j2\pi\frac{lMp}{LM}}\, e^{-j2\pi\frac{Lmq}{LM}}\, \cancel{e^{-j2\pi\frac{LMmp}{LM}}}^{\,1}$$

$$=\sum_{l=0}^{L-1}\sum_{m=0}^{M-1} x(l,m)\, e^{-j2\pi\frac{lq}{N}}\, e^{-j2\pi\frac{lp}{L}}\, e^{-j2\pi\frac{mq}{M}}$$

$$=\underbrace{\sum_{l=0}^{L-1} \underbrace{e^{-j2\pi\frac{lq}{N}} \underbrace{\sum_{m=0}^{M-1} x(l,m)\, e^{-j2\pi\frac{mq}{M}}}_{F(l,q)=\text{ M point DFT.}}}_{\substack{G(l,q)=F(l,q)\,e^{-j2\pi\frac{lq}{N}} \\ \text{single value multiplication}}} e^{-j2\pi\frac{lp}{L}}}_{\text{L-point DFT of } G(l,q)}$$

No. of complex multiplication $= (L(1(M)))$ for single value of (p,q)

Total No. of multipli.
$1^{st} = LM(M) = LM^2$
$2^{nd} = LM(1) = LM$
$3^{rd} = LM(L) = L^2M$

For N point DFT, N^2 is complexity
After divide & conquer, (LM^2+LM+L^2M) is complexity
→ This process is called decimation and can be continued till the factors become relatively prime.

Now

$$N = \underset{\substack{\downarrow \\ 2}}{L} \cdot \underset{\substack{\downarrow \\ N/2}}{M}$$

l \ m	0	1	2	...	M-1
0	x(0)	x(2)			
1	x(1)	x(3)			

for column wise storing
$l + Lm$

$$X(K)=\sum_{n=0}^{N-1} x(n)\, e^{-j2\pi\frac{nK}{N}} = \sum_{n=0}^{N-1} x(n)\, W_N^{nK}$$

$$\boxed{W_N^{nK} = e^{-j2\pi\frac{nK}{N}}}$$

$$=\sum_{n=\text{even}} x(n)\, e^{-j2\pi\frac{nK}{N}} + \sum_{n=\text{odd}} x(n)\, e^{-j2\pi\frac{nK}{N}}$$

$$=\sum_{n=0}^{N/2-1} x(2n)\, e^{-j2\pi\frac{(2n)K}{N}} + \sum_{n=0}^{N/2-1} x(2n+1)\, e^{-j2\pi\frac{(2n+1)K}{N}}$$

$$X(K) = \sum_{n=0}^{N/2-1} x(2n)\, e^{-j\frac{2\pi nK}{N/2}} + \sum_{n=0}^{N/2-1} x(2n+1)\, e^{-j\frac{2\pi nK}{N/2}} \cdot e^{-j\frac{2\pi K}{N}}$$

Assume $x(2n) = f_1(n)$. $\quad x(2n+1) = f_2(n)$

$$X(K) = \underbrace{\sum_{n=0}^{N/2-1} f_1(n)\, e^{-j\frac{2\pi nK}{N/2}}}_{N/2 \text{ point DFT}} + \underbrace{\sum_{n=0}^{N/2-1} f_2(n)\, e^{-j\frac{2\pi nK}{N/2}}}_{N/2 \text{ point DFT}} \cdot e^{-j\frac{2\pi K}{N}}$$

$$X(K) = F_1(K) + F_2(K)\, e^{-j\frac{2\pi K}{N}} \quad \text{——— (1)}$$

$$X\left(K+\frac{N}{2}\right) = F_1\left(K+\frac{N}{2}\right) + F_2\left(K+\frac{N}{2}\right) e^{-j\frac{2\pi (K+N/2)}{N}}$$

$$= F_1(K) + F_2(K)\, e^{-j\frac{2\pi K}{N}} \cdot e^{-j\frac{2\pi N/2}{N}}$$

$$= F_1(K) + F_2(K)\, e^{-j\frac{2\pi K}{N}} (-1) \quad \text{——— (2)}$$

After $\frac{N}{2}$ points, spectrum starts repeating.

$\therefore K = 0, 1, \ldots \left(\frac{N}{2}-1\right)$

$F_1\left(K+\frac{N}{2}\right) = F_1(K)$

$F_2(K) = F_2\left(K+\frac{N}{2}\right)$

$$X(K) = F_1(K) + F_2(K)\, W_N^K \qquad \left(K = 0, 1, \ldots \frac{N}{2}-1\right)$$

$$X\left(K+\frac{N}{2}\right) = F_1(K) - F_2(K)\, W_N^K \qquad \left(K = 0, 1, \ldots \frac{N}{2}-1\right)$$

Ex. $N = 8$. $\quad N = L \cdot M = 2(4)$

$x(0)$	$x(2)$	$x(4)$	$x(6)$
$x(1)$	$x(3)$	$x(5)$	$x(7)$

$l = (0, 1)$

$m = (0, 1, 2, 3)$

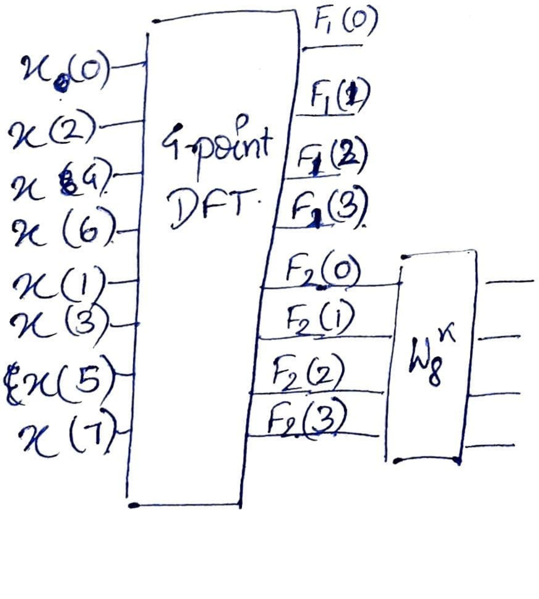
$$X(K) = F_1(K) + F_2(K)\, W_N^K$$

$$X(K) = F_1(K) + F_2(K)\, W_8^K \qquad K = 0, 1, 2, 3$$

$$X\left(K+\frac{N}{2}\right) = F_1(K) - F_2(K)\, W_8^K \qquad K = 0, 1, 2, 3.$$

$F_1(K) \rightarrow$ 4 point DFT of even sequence $f_1(n) = x(2n)$

$F_2(K) \rightarrow$ 4 point DFT of odd sequence $f_2(n) = x(2n+1)$



$F_1(0) + F_2(0)\, W_8^0 = X(0)$

$F_1(1) + F_2(1)\, W_8^1 = X(1)$

$F_1(2) + F_2(2)\, W_8^2 = X(2)$

$F_1(3) + F_2(3)\, W_8^3 = X(3)$

$F_1(0) - F_2(0)\, W_8^0 = X(4)$

$F_1(1) - F_2(1)\, W_8^1 = X(5)$

$F_1(2) - F_2(2)\, W_8^2 = X(6)$

$F_1(3) - F_2(3)\, W_8^3 = X(7)$

Advantage In Terms of Complexity

N^2 complex multiplication.

$$X(K) = \underset{\frac{N}{2} \text{ mul.}}{F_1(K)} + \underset{\frac{N}{2} \text{ mul.}}{F_2(K)}\, W_N^K$$

For 1 value of K, $\frac{N}{2}$ complex multiplication

for $\frac{N}{2}$ values of K, $\left(\frac{N}{2}\right)^2$ complex multiplication.

for $\frac{N}{2}$ values, $\frac{N}{2}$ complex multiplication

$$\text{Total} = \left(\frac{N}{2}\right)^2 + \frac{N}{2} \text{ complex multiplication.}$$

N-point DFT $\longrightarrow N^2$.

while DIT by 1st decimation $\longrightarrow 2\left(\frac{N}{2}\right)^2 + \frac{N}{2} = \frac{N^2}{2} + \frac{N}{2}$

If N is large, then $\frac{N}{2}$ can be neglected.

2nd Stage of decimation

$N = 2^V$, $\Rightarrow V = \log_2 N \rightarrow$ no. of decimation stages.

$$X(K) = F_1(K) + F_2(K)\, W_N^K$$

$$X\left(K+\frac{N}{2}\right) = F_1(K) - F_2(K)\, W_N^K$$

$V_{11}(n) = f_1(2n)$ $\quad V_{12}(n) = f_1(2n+1)$

$V_{21}(n) = f_2(2n)$ $\quad V_{22}(n) = f_2(2n+1)$

$$F_1(K) = \sum_{n=0}^{N/2-1} f_1(n)\, e^{-j\frac{2\pi nK}{N/2}}$$

$$= \sum_{n=\text{even}} f_1(2n)\, e^{-j\frac{2\pi(2n)K}{N/2}} + \sum_{n=\text{odd}} f_1(2n+1)\, e^{-j\frac{2\pi(2n+1)K}{N/2}}$$

$$= \sum_{n=0}^{N/4-1} \cancel{f_1(2n)}\, V_1(n)\, e^{-j\frac{2\pi nK}{N/4}} + \sum_{n=0}^{N/4-1} V_{12}(n)\, e^{-j\frac{2\pi nK}{N/4}} \cdot e^{-j\frac{2\pi K}{N/2}}$$

$$= V_{11}(K) + V_{12}(K)\, e^{-j\frac{2\pi K}{N/2}} \quad \text{——} \quad ①$$

$$F_1(K) = V_{11}(K) + V_{12}(K)\, W_{N/2}^K \quad \text{——} \quad ②$$

$$F_1\left(K+\frac{N}{4}\right) = V_{11}\left(K+\frac{N}{4}\right) + V_{12}\left(K+\frac{N}{4}\right) W_{N/2}^{(K+N/4)}$$

$$F_1\left(K+\frac{N}{4}\right) = V_{11}(K) + V_{12}(K)\cdot e^{-j\frac{2\pi\left(K+N/4\right)}{N/2}}$$

$$= V_{11}(K) + V_{12}(K)\cdot e^{-j\frac{2\pi K}{N/2}} \cdot e^{-j\frac{2\pi\, N/4}{N/2}}$$

$$F_1\left(K+\frac{N}{4}\right) = V_{11}(K) - V_{12}(K)\; e^{-j\frac{2\pi K}{N/2}}$$

$$F_1\left(K+\frac{N}{4}\right) = V_{11}(K) - V_{12}(K)\; W_{N/2}^{K} \qquad \text{——— (3)}$$

$$F_2(K) = V_{21}(K) + V_{22}(K)\, W_{N/2}^{K} \qquad \text{——— (4)}$$

$$F_2\left(K+\frac{N}{4}\right) = V_{21}(K) - V_{22}(K)\; W_{N/2}^{K} \qquad \text{——— (5)}$$

Using all expressions.

$$F_1(K) = V_{11}(K) + V_{12}(K)\, W_{N/2}^{K} \qquad F_2(K) = V_{21}(K) + V_{22}(K)\, W_{N/2}^{K}$$

$$F_1\left(K+\frac{N}{4}\right) = V_{11}(K) - V_{12}(K)\, W_{N/2}^{K} \qquad F_2\left(K+\frac{N}{4}\right) = V_{21}(K) + V_{22}(K)\, W_{N/2}^{K}$$

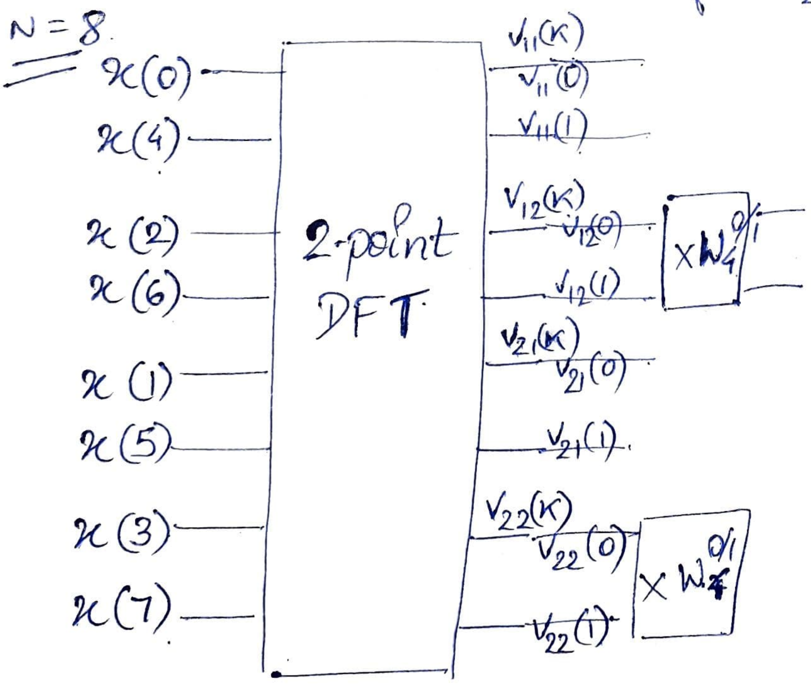
V_{11}, V_{12}, V_{21}, V_{22} all are $\frac{N}{4}$ point DFT, $K = 0, 1, \ldots \frac{N}{4}-1$

V_{11}, V_{12}, V_{21}, V_{22} $\longrightarrow \left(\frac{N}{4}\right)^2$ complex multiplication.

Overall Complexity $= 4\left(\frac{N}{4}\right)^2 + \frac{N}{2}$.

If N is large, $\frac{N}{2}$ can be neglected. $\Rightarrow \frac{N^2}{4}$.

N = 8.



$V_{11}(0) + V_{12}(0)W_4^0 = F_1(0)$

$V_{11}(1) + V_{12}(1)W_4^1 = F_1(1)$

$V_{11}(2) + V_{12}($ $= F_1(2)$

$F_1(3)$

$F_2(0)$

$F_2(1)$

$F_2(2)$

$F_2(3)$

Let $V_{11}(K) \Rightarrow$ 2 point DFT

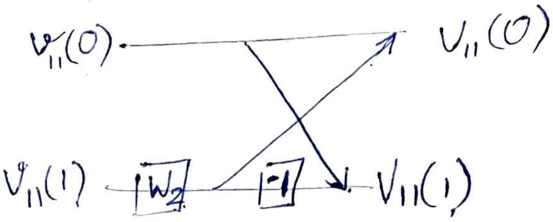
$$V_{11}(K) = \sum_{n=\text{even}} + \sum_{n=\text{odd}}$$

$$V_{11}(K) = \sum_{n=0}^{N-1} v_{11}(n) \cdot e^{-j\frac{2\pi nK}{N}}$$

$K=0 \quad V_{11}(0) = v_{11}(0) \cdot e^{-j\frac{2\pi(0)(0)}{2}} + v_{11}(1)\, e^{-j\frac{2\pi(0)(1)}{2}}$

$K=1 \quad V_{11}(1) = v_{11}(0)\, e^{-j\frac{2\pi(0)(1)}{2}} + v_{11}(1)\, e^{-j\frac{2\pi(1)(1)}{2}}$

$$V_{11}(0) = v_{11}(0) + v_{11}(1)$$
$$V_{11}(1) = v_{11}(0) - v_{11}(1)$$



Butterfly structure.

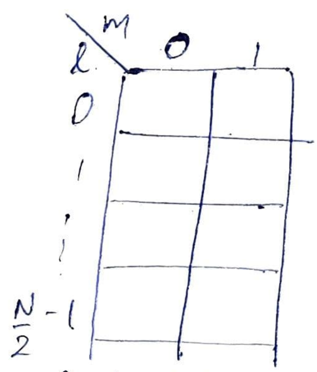
$\rightarrow$ $N = 8$, $N = 2^3$, 3 stages of decimation
Decimation in time algorithm, the input is in bit reverse order while output is in natural order.

Decimation in Frequency

$N = L \cdot M = \frac{N}{2} \cdot 2$ $\qquad x(n)$

$n = l + mL$

$X(K) \rightarrow$ even, odd.



Apply DFT on first column.

$$X(K) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi nK}{N}}$$

$$X(K) = \sum_{n=0}^{N/2-1} x(n)\, e^{-j\frac{2\pi nK}{N}} + \sum_{n=N/2}^{N-1} x(n)\, e^{-j\frac{2\pi nK}{N}}$$

$$X(K) = \sum_{n=0}^{N/2-1} x(n)\, e^{-j\frac{2\pi nK}{N}} + \sum_{n=\frac{N}{2}}^{N-1} x(n)\, e^{-j\frac{2\pi nK}{N}}$$

$m = n - \frac{N}{2} \Rightarrow n = \frac{N}{2}, m = 0$

$n = N-1, m = \frac{N}{2} - 1$

$$X(K) = \sum_{n=0}^{N/2-1} x(n)\, e^{-j\frac{2\pi nK}{N}} + \sum_{m=0}^{\frac{N}{2}-1} x\left(m+\frac{N}{2}\right) e^{-j\frac{2\pi\left(m+\frac{N}{2}\right)K}{N}}$$

$$= \quad '' \quad + \sum_{m=0}^{\frac{N}{2}-1} x\left(m+\frac{N}{2}\right) e^{-j\frac{2\pi mK}{N}} \cdot e^{-j\frac{2\pi \frac{N}{2} K}{N}}$$

$$= \quad '' \quad + \sum_{m=0}^{N/2-1} x\left(m+\frac{N}{2}\right) e^{-j\frac{2\pi mK}{N}} \cdot (-1)^K$$

$$X(K) = \sum_{n=0}^{N/2-1} x(n)\, e^{-j\frac{2\pi nK}{N}} + \sum_{m=0}^{\frac{N}{2}-1} x\left(n+\frac{N}{2}\right) e^{-j\frac{2\pi nK}{N}} (-1)^K$$

$$X(K) = \sum_{n=0}^{\frac{N}{2}-1} x(n)\, e^{-j\frac{2\pi nK}{N}} + (-1)^K \sum_{n=0}^{\frac{N}{2}-1} x\left(n+\frac{N}{2}\right) e^{-j\frac{2\pi nK}{N}}$$

Even Frequency

$$X(2K) = \sum_{n=0}^{\frac{N}{2}-1} x(n)\, e^{-j\frac{2\pi n 2K}{N}} + (-1)^{2K} \sum_{n=0}^{\frac{N}{2}-1} x\left(n+\frac{N}{2}\right) e^{-j\frac{2\pi n 2K}{N}}$$

$$\boxed{X(2K) = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) + x\left(n+\frac{N}{2}\right)\right] e^{-j\frac{2\pi nK}{N/2}}} \quad \text{—— } ①$$

Odd Frequency

$$X(2K+1) = \sum_{n=0}^{\frac{N}{2}-1} x(n)\, e^{-j\frac{2\pi n(2K+1)}{N}} + (-1)^{2K+1} \sum_{n=0}^{\frac{N}{2}-1} x\left(n+\frac{N}{2}\right) e^{-j\frac{2\pi n(2K+1)}{N}}$$

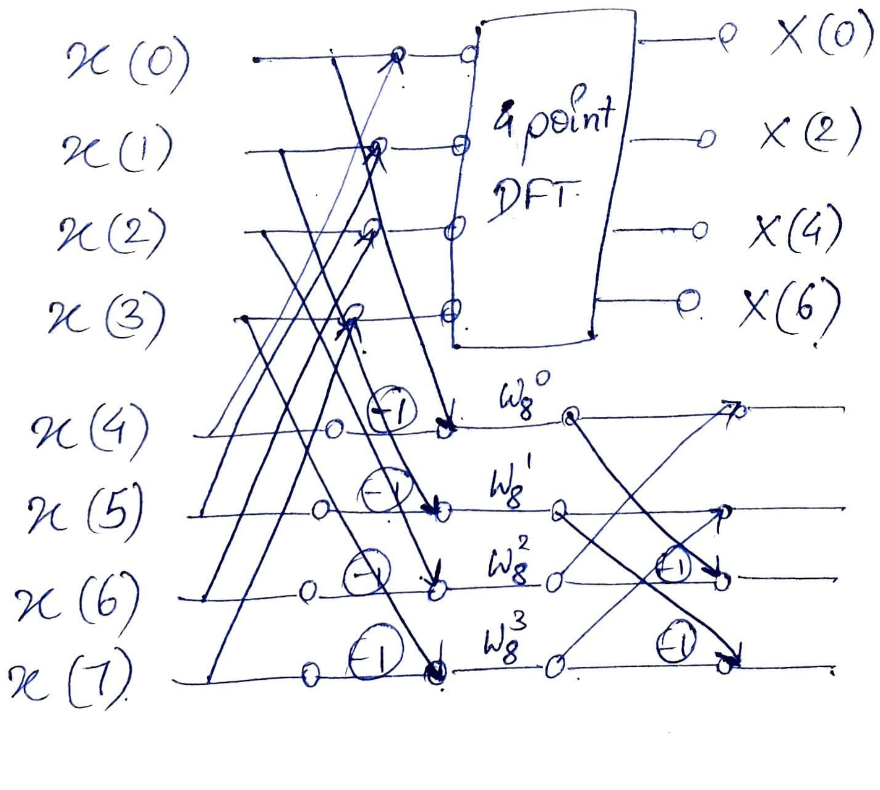
$$= \sum_{n=0}^{\frac{N}{2}-1} x(n)\, e^{-j\frac{2\pi n 2K}{N}}\, e^{-j\frac{2\pi n}{N}} - \sum_{n=0}^{\frac{N}{2}-1} x\left(n+\frac{N}{2}\right) e^{-j\frac{2\pi n 2K}{N}} \cdot e^{-j\frac{2\pi n}{N}}$$

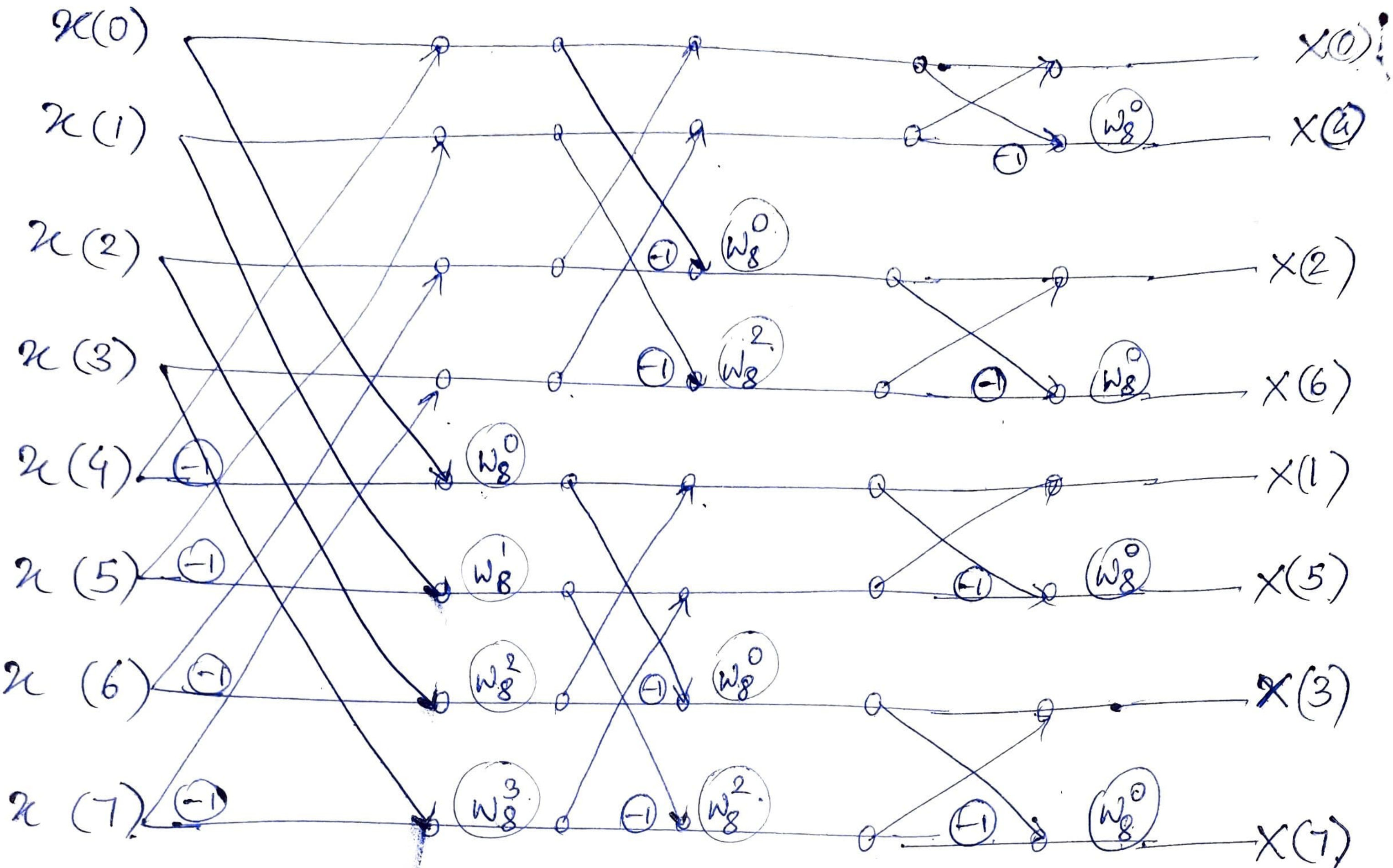
$$\boxed{X(2K+1) = \sum_{n=0}^{N/2-1} \left[x(n) - x\left(n+\frac{N}{2}\right)\right] e^{-j\frac{2\pi nK}{N/2}}\, e^{-j\frac{2\pi n}{N}}} \quad \text{—— } ②$$

Ex $N = 8$

$$X(2K) = \sum_{n=0}^{3} \left[x(n) + x(n+4)\right] e^{-j\frac{2\pi nK}{4}}$$

$$x(2K+1) = \sum_{n=0}^{3} \left[x(n) - x(n+4)\right] e^{-j\frac{2\pi nK}{4}} \cdot e^{-j\frac{2\pi n}{8}}$$





Input is in natural order

8 point dif fft

Output is bit reversed

Computational Complexity

$N = 2^v$ $\quad v = \log_2 N \Rightarrow$ Total no. of stages of decimation

No. of butterfly structure per stage $= \frac{N}{2}$

No. of complex multiplication per butterfly $= 1$

Total no. of complex multiplication $= \left(\frac{N}{2}\right) \log_2 N.$

2 complex additions per butterfly

Total no. of complex addition $= 2 \cdot \left(\frac{N}{2}\right) \log_2 N$

$= N \log_2 N.$