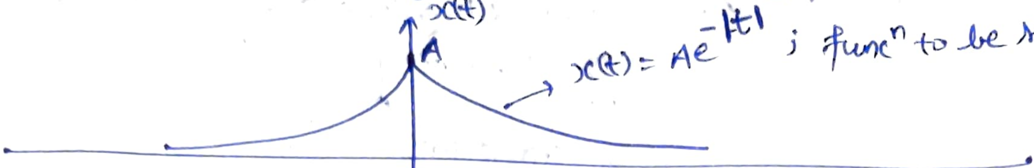
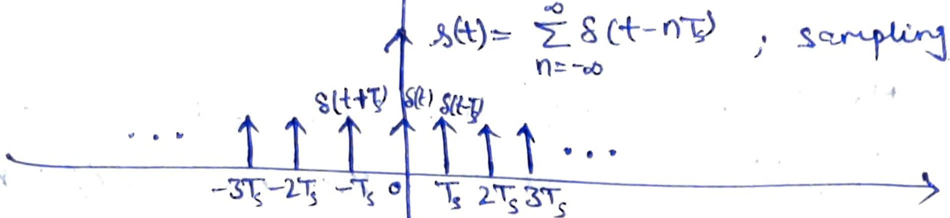
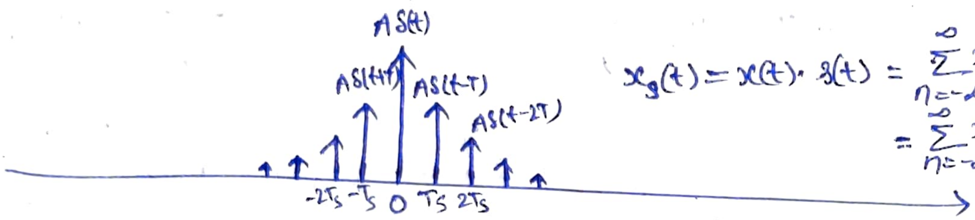
DTFT

→ The DTFT gives us the well defined spectrum or Fourier Representation of a non-periodic DT signal. (like CTFT).

→ To Derive the DTFT, let us start with an aperiodic CT signal, $x(t)$ and obtain a DT signal $x_s(t)$ from it through a process of sampling







$$x_s(t) = x(t)\cdot s(t) = \sum_{n=-\infty}^{\infty} x(t)\cdot\delta(t-nT_s) = \sum_{n=-\infty}^{\infty} x(nT_s)\,\delta(t-nT_s)$$

$$x_s(t) = x(t)\cdot s(t) = \sum_{n=-\infty}^{\infty} \overset{t=nT_s}{x(t)}\,\delta(t-nT_s) = \sum_{n=-\infty}^{\infty} x(nT_s)\cdot\delta(t-nT_s)$$

Taking the Fourier Transform on both sides.

$$X_s(\omega) = FT\{x_s(t)\} = \int_{-\infty}^{\infty} x_s(t)\,e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{\infty}\left[\sum_{n=-\infty}^{\infty} x(nT_s)\,\delta(t-nT_s)\right]\cdot e^{-j\omega t}\,dt$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s)\left\{\int_{-\infty}^{\infty}\delta(t-nT_s)\cdot \overset{t=nT_s}{e^{-j\omega t}}\,dt\right\}$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s)\left\{\int_{-\infty}^{\infty}\delta(t-nT_s)\cdot e^{-j\omega nT_s}\,dt\right\}$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s)\cdot e^{-j\omega nT_s}\left\{\int_{-\infty}^{\infty}\delta(t-nT_s)\,dt\right\}$$

$X(e^{j\omega})$ ← To distinguish with CTFT

$$\boxed{X_s(\omega) = \sum_{n=-\infty}^{\infty} x(nT_s)\cdot e^{-j\omega nT_s}}$$ → n taking only integer values

In above representation if T_s normalized to unity

$$\boxed{X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}} \rightarrow (1)$$

Inverse DTFT :-

$x(t)$ —/— $x_s(t) \xleftrightarrow{F}$ $\boxed{X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(nT_s)\cdot e^{-j\omega n T_s}}$

$$= \sum_{n=-\infty}^{\infty} x(t)\cdot \delta(t-nT_s)$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t-nT_s)$$

Note :- Above $X(e^{j\omega})$ is periodic in ω with a period of $\frac{2\pi}{T_s} = \omega_s$. Check $X_s(\omega + K\omega_s) = X(e^{j(\omega + K\omega_s)})$

$$\Downarrow$$

$$\left(T_s = \frac{2\pi}{\omega_s} = \frac{1}{f_s}\right)$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s)\cdot e^{-j(\omega+K\omega_s)nT_s}$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s)\cdot e^{-j\omega n T_s}\cdot e^{-jK\omega_s n T_s}$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s)\cdot e^{-j\omega n T_s}\cdot e^{-j\frac{2\pi}{T_s}KnT_s}$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s)\, e^{-j\omega n T_s}\cdot \underbrace{e^{-j2\pi Kn}}_{1}$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s)\cdot e^{-j\omega n T_s} \quad \text{As } n, K \text{ integer}$$

$$\Rightarrow \left| X_s(\omega + K\omega_s) = X_s(\omega) = X(e^{j\omega}) \right.$$

Derivation of Inverse DTFT :-

As $X(e^{j\omega})$ in (1) is periodic with period $\omega = 2\pi$; then eqn (1) can be treated as its FS representation in freq domain with period "2π" and basis func^n $e^{-j\omega n}$; $n \in \mathbb{Z}$ and $x[n]$ as its FS coefficients which are evaluated as follows.

$$\langle X(e^{j\omega}), e^{-j\omega l}\rangle = \int_{-\pi}^{\pi} X(e^{j\omega})\cdot \overbrace{e^{j\omega l}}^{conj} d\omega$$

inner product of $X(e^{j\omega})$ with its basis func^n

$$= \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}\cdot e^{j\omega l}\, d\omega = \sum_{n=-\infty}^{\infty} x[n] \int_{-\pi}^{\pi} e^{-j\omega(n-l)} d\omega$$

$$= \sum_{\substack{n=-\infty \\ n\neq l}}^{\infty} x[n] \underbrace{\int_{-\pi}^{\pi} e^{-j\omega(n-l)} d\omega}_{0} + x(l)\int_{-\pi}^{\pi} 1\cdot d\omega. =$$

$$= \sum_{\substack{n=-\infty \\ n\neq l}}^{\infty} \left.\frac{e^{-j\omega(n-l)}}{-j(n-l)}\right|_{-\pi}^{\pi} + x(l)\cdot 2\pi = 2\pi x(l) + \sum_{\substack{n=-\infty \\ n\neq l}}^{\infty} \frac{e^{+j\pi(n-l)} - e^{-j\pi(n-l)}}{j(n-l)}$$

$$= 2\pi x(\ell) + \frac{a}{n-\ell}\sum_{\substack{n=-\infty \\ n\neq \ell}}^{\infty} \sin((n-\ell)\pi) \to 0$$

$n-\ell \in \mathbb{Z}$

$\sin((n-\ell)\pi) = 0 \ \forall\ n \neq \ell$

$$= 2\pi\, x(\ell)$$

$$x(\ell) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega \ell}\, d\omega.$$

DTFT

TD ; $x[n]$		FD, $X(e^{j\omega})$.
1. Discrete	$\longleftrightarrow$	Periodic.
2. Aperiodic	$\longleftrightarrow$	Continuous

Summary.

$x(n) \xleftrightarrow{F} X(\omega) = X(e^{j\omega}) \to$ periodic with 2π irrespective of $x(n)$

DTFT : $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\omega n} = X(z)\Big|_{z=e^{j\omega}}$; i.e. $r=1$; $|z|=1$

$X(z) = ZT\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$; where $z = re^{j\omega}$

IDTFT : $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega}) \cdot e^{j\omega n}\, d\omega.$

Que→ Find DTFT of $x_1[n] = a^n u[n]$; $|a|<1$

Ans. $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^n u[n] \cdot e^{-j\omega n} = \sum_{n=0}^{\infty} a^n e^{-j\omega n}$

$= 1 + a e^{-j\omega} + a^2 e^{-j2\omega} + a^3 e^{-j3\omega} + \ldots$

$$\boxed{X_1(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}}} \quad (1)$$

$\to S_\infty^{GP} = \frac{\text{First term}}{1 - \text{Common Ratio (C.R.)}}$, iff $|C.R.|<1$

$|ae^{-j\omega}|<1 \Rightarrow |a||e^{-j\omega}| < 1 \Rightarrow |a|<1$

Alternatively

$a^n u[n] \xleftrightarrow{Z} \frac{1}{1-az^{-1}}$ $\quad |z|>|a|$

where $|a|<1$

$\Rightarrow$ Unit circle inside ROC.

$\Downarrow$

DTFT defined & evaluated if $|z|=1$; $z=e^{j\omega}$

$a^n u[n] \xleftrightarrow[z=e^{j\omega}]{DTFT} \frac{1}{1-ae^{-j\omega}}$

Que→ Find the DTFT of $x_2[n] = -a^n u[-n-1]$; $|a|>1$

M-1 $X(e^{j\omega}) = -\{a^{-1}e^{j\omega} + a^{-2}e^{j2\omega} + a^{-3}e^{j3\omega} + \ldots\} = -\frac{a^{-1}e^{j\omega}}{1-a^{-1}e^{j\omega}} = \frac{1}{a^{-1}e^{j\omega}-1}$

M-2

$x[n] \longleftrightarrow X(z) = \frac{1}{\ldots}$ $|z|<|a|$; unit circle inside ROC

$\boxed{X(e^{j\omega}) = \frac{1}{\ldots}}$ (2)

$$= 2\pi x(\ell) + \frac{1}{n-\ell}\sum_{\substack{n=-\infty \\ n\neq \ell}}^{\infty} \sin((n-\ell)\pi) \to 0$$

$n-\ell \in \mathbb{Z}$

$\Downarrow$

$\sin((n-\ell)\pi) = 0 \ \forall\ n \neq \ell$

$$= 2\pi\, x(\ell)$$

$$x(\ell) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega \ell}\, d\omega.$$

DTFT

TD ; $x[n]$		FD. $X(e^{j\omega})$.
1. Discrete	$\longleftrightarrow$	Periodic.
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$\boxed{\text{Summary.}}$

$X(z) = ZT\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$; where $z = re^{j\omega}$.

$x(n) \xleftrightarrow{F} X(\omega) = X(e^{j\omega}) \to$ periodic with 2π irrespective of $x(n)$

DTFT : $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\cdot e^{-j\omega n} = X(z)\Big|_{z=e^{j\omega}}$; i.e. $r=1$ or $|z|=1$

IDTFT : $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\cdot e^{j\omega n}\, d\omega.$

Que → Find DTFT of $x_1[n] = a^n u[n]$; $|a|<1$

Ans.

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^n u[n]\, e^{-j\omega n} = \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$

$$= 1 + ae^{-j\omega} + a^2 e^{-j2\omega} + a^3 e^{-j3\omega} + \dots$$

$$\boxed{X_1(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}}} \quad (1) \Longrightarrow S_\infty^{GP} = \frac{\text{First term}}{1 - \text{Common Ratio (C.R.)}}\ ;\ \text{iff } |C.R.|<1$$

$|ae^{-j\omega}| < 1 \Rightarrow |a||e^{-j\omega}| < 1 \Rightarrow |a|<1$

Alternatively

$$a^n u[n] \xleftrightarrow{z} \frac{1}{1-az^{-1}} \quad |z|>|a|$$

$$a^n u[n] \xleftrightarrow[z=e^{j\omega}]{DTFT} \frac{1}{1-ae^{-j\omega}}$$

where $|a|<1$

$\Rightarrow$ Unit circle Inside RoC.

$\Downarrow$

DTFT defined & evaluated if $|z|=1$; $z=e^{j\omega}$

Que → Find the DTFT of $x_2[n] = -a^n u[-n-1]$; $|a|>1$

M-1

$$X(e^{j\omega}) = -\{a^{-1}e^{j\omega} + a^{-2}e^{+j2\omega} + a^{-3}e^{j3\omega} + \dots\} = -\frac{a^{-1}e^{j\omega}}{1-a^{-1}e^{j\omega}} = \frac{1}{a^{-1}e^{-j\omega}-1}$$

$$\boxed{X_2(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}}} \quad (2)$$

M-2

$x[n] \longleftrightarrow X(z) = \frac{1}{1-az^{-1}}$ $|z|<|a|$; Unit circle inside RoC

$\Rightarrow \boxed{X_2(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}}}$

Note → Expression are looking same in eqⁿ ① & ② but they are not same because they are evaluated for the two mutually exclusive set of value of 'a'.

Note → FT is particular case of LT or ZT i.e. evaluated at some some specific value of $\sigma = 0$ or some specific value of $|z|=1$. Hence not need to mention ROC. where LT & ZT are general case hence required to mention ROC.

Note → DTFT is also unique of the given signal.

Que → Find DTFT of $f_1(n) = x(-n) = a^{-n}u[-n]$; $|a|<1$

$$F_1(e^{j\omega}) = F_1(\omega) = 1 + a^{+1}e^{j\omega} + a^{+2}e^{j2\omega} + a^{+3}e^{j3\omega} + \dots$$

$$= \frac{1}{1-a^{+1}e^{j\omega}} = \frac{1}{1-ae^{j\omega}}$$

$$= X(e^{-j\omega}) = X(-\omega)$$

Time Reversal Property →

$$x[n] \longleftrightarrow X(\omega) = X(e^{j\omega})$$

$$x[-n] \longleftrightarrow X(-\omega) = X(e^{-j\omega})$$

Time Reversal implies Frequency reversal

Que → Find DTFT of $f_2[n] = a^{-n}u[-n-1]$; $|a|<1$

$$F_2(\omega) = F_2(e^{j\omega}) = ae^{j\omega} + a^2e^{j2\omega} + a^3e^{j3\omega} + \dots$$

$$= \frac{ae^{j\omega}}{1-ae^{j\omega}} \; ; \; |a|<1$$

Que → Find DTFT of $f_3[n] = a^{|n|}$; $|a|<1$

Note $f_3[n] = a^n.u[n] + a^{-n}u[-n-1]$

$$\updownarrow$$

$$F_3(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}} + \frac{ae^{j\omega}}{1-ae^{j\omega}}$$

$$= \frac{1-\cancel{ae^{j\omega}}+\cancel{ae^{j\omega}}-a^2}{1-ae^{-j\omega}-ae^{j\omega}+a^2} = \frac{1-a^2}{1-2a\cos\omega+a^2}$$

Linearity Property :-

$$f_1[n] \longleftrightarrow F_1(e^{j\omega})$$

$$f_2[n] \longleftrightarrow F_2(e^{j\omega})$$

$$a_1 f_1[n] + a_2 f_2[n] \longleftrightarrow a_1 F_1(e^{j\omega}) + a_2 F_2(e^{j\omega}).$$

Existence & Convergence Concepts :-

The DTFT $X(e^{j\omega})$ of a non-periodic DT signal $x[n]$ is said to exist if

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} \quad \text{is}$$

converges ; i.e. if $|X(e^{j\omega})| < \infty$.

From the R.H.S. of the above eq^n it is clear that the summation converges uniformly $\forall\, \omega$; if

$$\boxed{\sum_{n=-\infty}^{\infty} |x(n)| < \infty} \Rightarrow \text{Sufficient condition for existence of FT } X(e^{j\omega}) \text{ of } x(n).$$

i.e. if $x(n)$ is absolutely summable

We note that this is the DT counterpart of the first Dirichlet condition for the CTFT. The last two conditions do not apply because of the DT nature of $x[n]$.

Note :- IDFT $\quad x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$

~~Above integral will converge $\Rightarrow$ $x[n]$ finite iff $X(e^{j\omega})$ is bounded $\forall\, \omega$.~~

To have bounded $x[n]$. $\quad |x[n]| < \infty \;\forall n \Rightarrow \int_{-\pi}^{\pi} |X(e^{j\omega})|\, d\omega < \infty$

Conjugate Property :-

$$x[n] \xleftrightarrow{F} X(\omega) \text{ or } X(e^{j\omega})$$

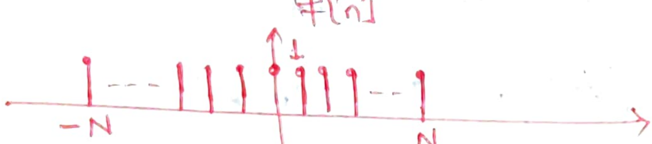
$$x^*[n] \longleftrightarrow X^*(-\omega) \text{ or } X^*(e^{j(-\omega)})$$

Note :-

$x[n]$ real ; $x[n] = x^*[n] \Rightarrow X(\omega) = X^*(-\omega) \Rightarrow X(\omega)$ is Conj. Symmetric

$x[n] \rightarrow$ purely imag. ; $x[n] = -x^*[n] \Rightarrow X(\omega) = -X^*(-\omega) \Rightarrow X(\omega)$ is Anti symmetric

Question Find the DTFT of $f[n] = u[n+N] - u[n-N]$; i.e. a DT Gate pulse or rectangular pulse $\text{rect}\left[\frac{n}{2N+1}\right]$



Ans

$$F(\omega) = \sum_{n=-N}^{N} 1 \cdot e^{-j\omega n}$$

change the variable $n + N = m$

$$= \sum_{m=0}^{2N} 1 \cdot e^{-j\omega(m-N)}$$

$$= e^{j\omega N} \sum_{m=0}^{2N} e^{-j\omega m}$$

$$= e^{j\omega N} \left[\underbrace{1 + e^{-j\omega} + e^{-j2\omega} + \dots + e^{-j2N\omega}}_{\text{G.P. with No. of term } 2N+1,\ a=1,\ r=e^{-j\omega}} \right]$$

$\Rightarrow S_{2N+1} = a\left[\frac{1-r^{2N+1}}{1-r}\right]$

$$= e^{j\omega N} \cdot \frac{1 \cdot \left[1 - e^{-j\omega(2N+1)}\right]}{1 - e^{-j\omega}}$$

Note :- ① If $x[n]$ is even

$\Rightarrow x[n] = x[-n] \Rightarrow \boxed{X(\omega) = X(-\omega)} \Rightarrow$ Even DTFT

② If $x[n]$ is odd

$x[n] = -x[-n] \Rightarrow \boxed{X(\omega) = -X(-\omega)} \Rightarrow$ odd DTFT

$$= \cancel{e^{j\omega N}} \cdot \frac{\cancel{e^{-j\omega\frac{2N+1}{2}}}}{\cancel{e^{-j\omega/2}}} \left\{ \frac{e^{j\omega\frac{2N+1}{2}} - e^{-j\omega\frac{2N+1}{2}}}{e^{j\omega/2} - e^{-j\omega/2}} \right\}$$

$$= \frac{\cancel{2j} \sin \omega\left(\frac{2N+1}{2}\right)}{\cancel{2j} \sin \omega/2} = \frac{\sin\left\{\omega\left(\frac{2N+1}{2}\right)\right\}}{\sin(\omega/2)}.$$

$$\boxed{F(\omega) = \frac{\sin\left[\omega \cdot \left(\frac{2N+1}{2}\right)\right]}{\sin\left(\omega \cdot \frac{1}{2}\right)}}$$

$\left(\frac{2N+1}{2}\right) \rightarrow \frac{\text{No. of samples}}{2}$

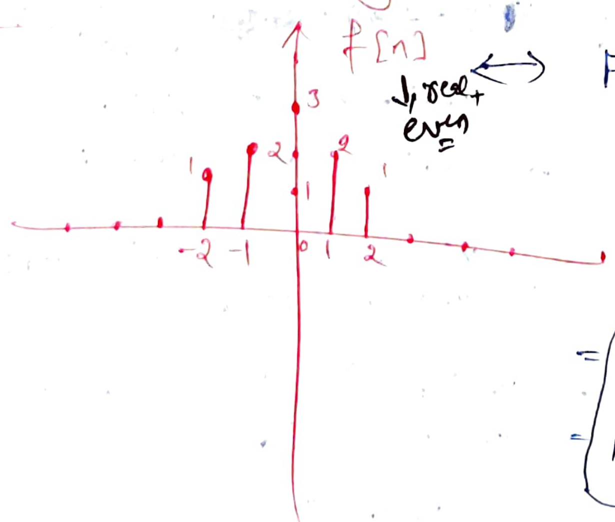
M.M.IMP

$$\boxed{F(\omega) = \frac{\sin\left\{\omega\left(N+\frac{1}{2}\right)\right\}}{\sin\left(\omega \cdot \frac{1}{2}\right)}}$$

(conjugate symm $F(\omega)$ + even. $F(\omega)$) ; Purely real & $F(\omega)$

$\Rightarrow$ For any gate pulse from $-N$ to N or $G_{2N+1}(n)$ or $\text{rect}\left(\frac{n}{2N+1}\right)$ have DTFT is ratio of two sine func's always.

Q. Find the DTFT of Discrete time triangular pulse



↓, real + even $\longleftrightarrow$

$$F(\omega) = \sum_{n=-\infty}^{\infty} f[n] e^{-j\omega n}$$

$n = -2, -1, 0, 2, 1$

$$= 1 \cdot e^{-j\omega(-2)} + 2e^{-j\omega(-1)} + 3e^{-j\omega(0)}$$
$$+ 1 \cdot e^{-j\omega(2)} + 2e^{-j\omega(1)}$$

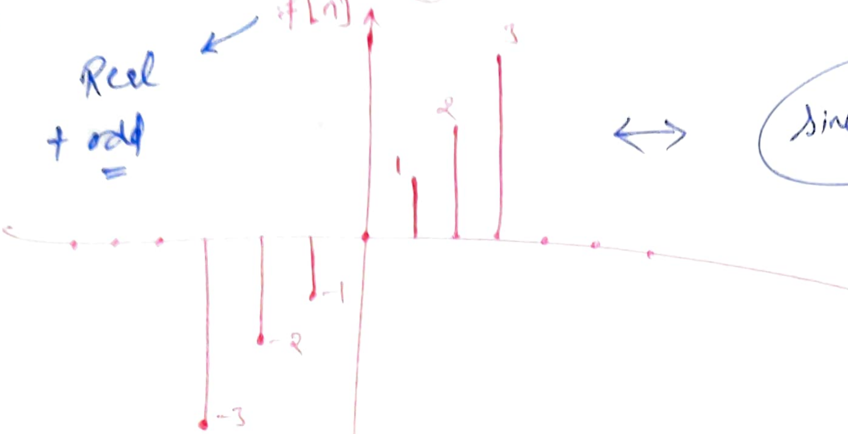
$$F(\omega) = 2\cos 2\omega + 4\cos\omega + 3$$

real & even + conj symm

9. Find the DTFT of DT sawtooth pulse.

DTS

Real + odd



$\leftrightarrow$ sine signals

$$F(\omega) = \sum_{n=-3}^{3} f[n] e^{-j\omega n} = e^{-j\omega} + 2e^{-j2\omega} + 3e^{-j3\omega} - e^{j\omega} - 2e^{+j2\omega} - 3e^{j\omega 3}$$

$$= -\left\{ e^{j\omega} - e^{-j\omega} + 2e^{j2\omega} - 2e^{-j2\omega} + 3e^{j3\omega} - 3e^{-j3\omega} \right\}$$

$$= -\left\{ 2j\sin\omega + 4j\sin(2\omega) + 6j\sin(3\omega) \right\}$$

Purely imag. $X(\omega)$ $\leftarrow$ $\begin{cases} \text{Conj. Symm. } X(\omega) \\ + \\ \text{odd } X(\omega) \end{cases}$

$$\boxed{F(\omega) = -2j\left\{ \sin(\omega) + 2\sin(2\omega) + 3\sin(3\omega) \right\}}$$

Symmetry Property of DTFT

1. If $x[n]$ is real $\Rightarrow$ $X^*(\omega) = X(-\omega)$ $\Rightarrow$ $X(\omega)$ have Conjugate symmetry.
2. If $x[n]$ is purely imag. $\Rightarrow$ $X^*(\omega) = -X(-\omega)$; $\Rightarrow$ $X(\omega)$ have Conjugate Antisymmetry.
3. If $x[n]$ is even $\Rightarrow$ $X(-\omega) = X(\omega)$ $\Rightarrow$ $X(\omega)$ is even
4. If $x[n]$ is odd $\Rightarrow$ $X(-\omega) = -X(\omega)$ $\Rightarrow$ $X(\omega)$ is odd.
5. If $x[n]$ is real + even $\Rightarrow$ $\left.\begin{array}{l} X(-\omega) = X^*(\omega) \\ X(-\omega) = X(\omega) \end{array}\right\}$ $\Rightarrow$ $X^*(\omega) = X(\omega)$ $\Rightarrow$ $X(\omega)$ is purely real along with Conjugate symmetric & even.
6. If $x[n]$ is real + odd $\Rightarrow$ $\left.\begin{array}{l} X(-\omega) = X^*(\omega) \\ X(-\omega) = -X(\omega) \end{array}\right\}$ $\Rightarrow$ $X^*(\omega) = -X(\omega)$ $\Rightarrow$ $X(\omega)$ is purely Imag. along with conj. symm. & evenness.
7. If $x[n]$ is purely imag. + odd $\Rightarrow$ $\left.\begin{array}{l} X(-\omega) = -X^*(\omega) \\ X(-\omega) = -X(\omega) \end{array}\right\}$ $\Rightarrow$ $X^*(\omega) = X(\omega)$ $\Rightarrow$ $X(\omega)$ is purely real along with antisymmetry & oddness.
8. If $x[n]$ is purely imag. + even $\Rightarrow$ $\left.\begin{array}{l} X(-\omega) = -X^*(\omega) \\ X(-\omega) = X(\omega) \end{array}\right\}$ $\Rightarrow$ $X^*(\omega) = -X(\omega)$ $\Rightarrow$ $X(\omega)$ is purely imag. along with antisymm. & evenness

Note :- Like CTFT, DTFT is also of 3 types well defined DTFT i.e $\sum_{n=-\infty}^{\infty} |f(n)| < \infty$; limitedly defined DTFT i.e at least one value of ω b/w the period of 2π where it is not defined. & the last one is Not defined DTFT where f[n] is not absolutely summable & not follows Dirichlets condition (or continuously increasing magnitude).

$\left(\frac{1}{2}\right)^n u[n]$

$-2^n u[-n-1]$

→ Well defined

→ we should use formula

⟹ use linearity property of well defined func^n combination.

⟹ Never use combination of power signal combination to get FT of well-defined f(n).

$2^n u[n]$

$-\left(\frac{1}{2}\right)^n u[-n-1]$

→ Not defined

$1 \rightarrow 2\pi\delta(\omega)$

limitedly defined, we can't use formula.

$e^{j\omega_0 n} \rightarrow 2\pi\,\delta(\omega-\omega_0)$

$e^{-j\omega_0 n} \rightarrow 2\pi\,\delta(\omega+\omega_0)$

$\cos\omega_0 n \rightarrow \pi\{\delta(\omega+\omega_0) + \delta(\omega+\omega_0)\}$

$\sin\omega_0 n \rightarrow \frac{\pi}{j}\{\delta(\omega+\omega_0) - \delta(\omega+\omega_0)\}$.

in one period 2π

Tool	Time Domain	Freq^{c} Domain
CTFS	Continuous & periodic	Discrete & aperiodic
CTFT	Continuous & aperiodic	Continuous & aperiodic
DTFT	Discrete & aperiodic	Continuous & periodic
DTFS	Discrete & periodic	Discrete & periodic

Only Here Duality Property exists.

Multiplication or [Modulation or Windowing] Property of DTFT :-

$$x_1[n] \longleftrightarrow X_1(e^{j\omega})$$

$$x_2[n] \longleftrightarrow X_2(e^{j\omega})$$

then

$$x_1[n]\, x_2[n] \longleftrightarrow \frac{1}{2\pi}\left[X_1(e^{j\omega}) \overset{2\pi}{\circledast} X_2(e^{j\omega}) \right]$$

Periodic convolution with period 2π

$$= \frac{1}{2\pi}\left[\int_{2\pi} X_1(e^{j\theta}) \cdot X_2\left(e^{j(\omega-\theta)}\right) d\theta \right]$$

Correlation Property of FT :-

CT FT

both energy signal $\Leftarrow$ $x_1(\tau) \longleftrightarrow X_1(\omega)$

$x_2^*(-\tau) \longleftrightarrow X_2^*(-\omega)$

$$\boxed{\begin{aligned}&\phi_{x_1 x_2}(\tau) \\ &= x_1(\tau) \circledast x_2^*(-\tau) \longleftrightarrow X_1(\omega)\, X_2^*(-\omega) \\ &\qquad\qquad \downarrow \\ &\qquad\qquad X_1(\omega)\, X_2^*(\omega)\end{aligned}}$$

cross correlation $\Rightarrow$

$\phi_x(\tau) = x_1(\tau) \circledast x_1^*(-\tau) \longleftrightarrow X_1(\omega) . X_1^*(-\omega)$

$\uparrow$ Auto-correllation function.

$= X_1(\omega) . X_1^*(\omega)$

$= |X_1(\omega)|^2$

$=$ ESD of $x(t)$

$\circledast \Rightarrow$ Convolution operation

x_1

DTFT

let $x_1[n]$ & $x_2[n]$ be two DT signals which has finite energy each.

then

$x_1[n] \longleftrightarrow X_1(e^{j\omega})$

$x_2[n] \longleftrightarrow X_2(e^{j\omega})$

$$\phi_{x_1 x_2}(k) = \sum_{n=-\infty}^{\infty} x_1(n)\, x_2^*(n-k)$$

$$\phi_{x_1 x_2}(k) = \sum_{n=-\infty}^{\infty} x_1(n)\, x_2^*(n-k)$$

$= x_1(k) \circledast x_2^*(-k) \longleftrightarrow X_1(e^{j\omega})\, X_2^*(e^{j(-\omega)})$

Convolution operation

$\Psi_{x_1 x_2}(\omega) = X_1(e^{j\omega}) . X_2^*(e^{j\omega})$

$\Psi_{x_1 x_2}(\omega) = \text{DTFT}\{\phi_{x_1 x_2}(n)\}$

If

$x_1[n] = x_2[n] = x[n]$.

$\phi_x(k) = x(k) \circledast x^*(-k) \longleftrightarrow X(e^{j\omega})\, X^*(e^{j(-\omega)})$

$= X(e^{j\omega})\, X^*(e^{j\omega})$

$= |X(e^{j\omega})|^2$.

$$\boxed{\text{ACF of } x(n) \longleftrightarrow \text{ESD of } x[n].}$$

$\Downarrow$ Wiener Khintchine Theorem

Note :- For Power signals. $x[n]$ → anaylsis we use truncated version of it.

$$x_{2N}(n) = \begin{cases} x(n) & -N < n < N \\ 0 & \text{o.w.} \end{cases}$$

$$\boxed{P_x = \lim_{N\to\infty} \frac{E_{x_{2N}}}{2N}} \qquad \boxed{R_x(\tau) = \lim_{N\to\infty} \frac{\phi_{x_{2N}}(\tau)}{2N}} \qquad \boxed{S_x(\omega) = \lim_{N\to\infty} \frac{\Psi_{x_{2N}}(\omega)}{2N}}$$

Signal Transmission Through LTI System :-

$$x[n] \xrightarrow{} \boxed{\begin{matrix} h[n] \\ \updownarrow F \\ H(e^{j\omega}) \end{matrix}} \xrightarrow{} y[n] = x[n] \otimes h[n]$$

$x[n] \overset{F}{\updownarrow} X(e^{j\omega})$

$$y[n] \overset{F}{\updownarrow} Y(e^{j\omega}) = X(e^{j\omega}) \cdot H(e^{j\omega})$$

$$\Downarrow$$

$$\boxed{H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}}$$

$$Y(e^{j\omega}) = |Y(e^{j\omega})| \angle Y(e^{j\omega}) = |Y(e^{j\omega})| e^{j\angle Y(e^{j\omega})} = |X(e^{j\omega})| e^{j\angle X(e^{j\omega})} \cdot |H(e^{j\omega})| e^{j\angle H(e^{j\omega})}$$

$$|Y(e^{j\omega})| e^{j\angle Y(e^{j\omega})} = |X(e^{j\omega})| \, |H(e^{j\omega})| \, e^{j[\angle X(e^{j\omega}) + \angle H(e^{j\omega})]}$$

$$\Rightarrow |Y(e^{j\omega})| = |X(e^{j\omega})| \, |H(e^{j\omega})|$$

If $x[n]$, is energy signal

I/P ESD $\Psi_i(\omega)$ = $|X(e^{j\omega})|^2$

O/P ESD $\Psi_o(\omega) = |Y(e^{j\omega})|^2 = |X(e^{j\omega})|^2 \, |H(e^{j\omega})|^2$

O/P ESD → $\boxed{\Psi_o(\omega) = |H(e^{j\omega})|^2 \, \Psi_i(\omega)}$ ← I/P ESD

Similarly if $x[n]$ is power signal

$$x[n] \equiv x_{2N}[n] \longrightarrow \boxed{\begin{matrix} h[n] \\ H(e^{j\omega}) \end{matrix}} \longrightarrow y[n],\ y_{2N}(n)$$

$Y_{2N}(e^{j\omega}) = X_{2N}(e^{j\omega}) H(e^{j\omega})$

$x_{2N}[n] \xrightarrow{PSD}$ $\boxed{S_x(\omega) = \lim_{N\to\infty} \frac{\Psi_{x_{2N}}(\omega)}{2N} = \lim_{N\to\infty} \frac{|X_{2N}(e^{j\omega})|^2}{2N}}$

I/P PSD $S_x(\omega)$ or $S_i(\omega) = \lim_{N\to\infty} \frac{|X_{2N}(e^{j\omega})|^2}{2N}$

O/P PSD; $S_y(\omega)$ or $S_o(\omega) = \lim_{N\to\infty} \frac{|Y_{2N}(e^{j\omega})|^2}{2N}$

$$= \frac{|X_{2N}(e^{j\omega})|^2}{2N} |H(e^{j\omega})|^2$$

M. Imp

O/P PSD → $\boxed{S_o(\omega) = S_y(\omega) = S_x(\omega) \cdot |H(e^{j\omega})|^2 = S_i(\omega) \times |H(e^{j\omega})|^2}$ ← I/P PSD

Note :-

Tools	Time Domain		Freq^c Domain
CTFS	Continuous & Periodic	duals	Discrete & Aperiodic
CTFT	Continuous & Aperiodic	Self Dual	Continuous & Aperiodic
DTFT	Discrete & Aperiodic	duals	Continuous & Periodic
DTFS	Discrete & Periodic	Self Dual	Discrete & Periodic

Note :- Due to Self Dual characteristic of CTFT & DTFS ; there is duality property for these two systems or tools only. No corresponding Duality exists in DTFT & CTFS.

——— *** ENJOY LEARNING *** ———

Note → Types of CTFT $(X(j\omega))$ Depending upon $x[t]$.

$|X(j\omega)| < \infty;\ \forall\ \omega$

Well-defined FT; if $\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$

e.g.

$\delta(t) \xleftrightarrow{F} 1$

$e^{-2t}u(t) \xleftrightarrow{F} \frac{1}{2+j\omega}$

$e^{2t}u(-t) \xleftrightarrow{F} \frac{1}{2-j\omega}$

$e^{-2|t|} \xleftrightarrow{F} \frac{4}{4+\omega^2}$

$e^{-2t}u(t) - e^{2t}u(-t) \leftrightarrow \frac{-2j\omega}{4+\omega^2}$

Using Duality

Note

1) Well-defined FT can be evaluated easily using $X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$

2) $X(j\omega) = LT\{x(t)\}$

$LT\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-st}\, dt$, $s = j\omega$.

where $s = \sigma + j\omega$

$|X(j\omega)| = \infty$ for few ω's;

Limitedly Defined FT;

if i) $\int_{-\infty}^{\infty} |x(t)|\, dt \rightarrow \infty$ — $x(t)$ is not absolutely summable

& ii) $|x(t)| < \infty;\ \forall t$

↳ but $x(t)$ is bounded.

e.g.

$1 \xleftrightarrow{F} 2\pi\,\delta(\omega)$

$e^{j\omega_0 kt} \xleftrightarrow{F} 2\pi\,\delta(\omega - k\omega_0);\ k \in \mathbb{Z}$

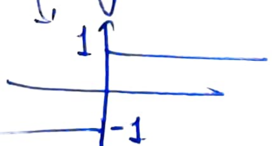
$$\boxed{\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \xleftrightarrow{F} 2\pi \sum_{k=-\infty}^{\infty} c_k \delta(\omega - k\omega_0)}$$

↳ F.T. of CT periodic signals.

$\cos(\omega_0 t) \xleftrightarrow{F} \pi\left(\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right)$

$\sin(\omega_0 t) \xleftrightarrow{F} \frac{\pi}{j}\left(\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right)$

$\mathrm{sgn}(t) \xleftrightarrow{F} \frac{2}{j\omega}$



Imp. Note → ① Limitedly Defined F.T. Can't be evaluated using $X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$

$X(j\omega)$ is Not-defined, i.e. **Undefined FT**

⇓

if i) $\int_{-\infty}^{\infty} |x(t)|\, dt \rightarrow \infty$

ii) $|x(t)| \rightarrow \infty$ for few 't'

↳ Unbounded signals.

e.g.

i) $r(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases}$

ii) $e^{2t}u(t)$

Note → Undefined F.T. can't be evaluated by any how.