Size of a DT Signal (1)

- Signal Size: depends on signal amplitude and its duration i.e. area.
- Signal Size Measure: Energy and Power. Area under the signal x[n], because it takes account not only amplitude but also duration.
- Energy must be finite i.e.: $Amplitude \to 0 \quad as \quad |n| \to \infty$
- Measured by signal energy E_x $\quad E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$
- Energy Signal: finite energy and zero average power i.e. 0<E<∞ and P= 0. Ex. $x[n] = e^{-|n|}$

Size of a DT Signal (2)

- If amplitude of x[n] does not→0 when n →∞, need to measure time average of energy i.e. power P_x instead energy (because E_x = ∞ in this case):

$$P_x = \lim_{N\to\infty} \frac{1}{2N_0+1} \sum_{n=-N_0}^{N_0} |x[n]|^2 = \lim_{N\to\infty} \frac{1}{N_0} \sum_{n=0}^{N_0-1} |x[n]|^2$$

$2N_0+1$ samples in the interval from $-N_0$ to N_0.

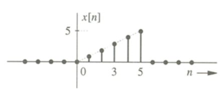
- Power signal: must have finite average power and infinite energy i.e. $0<P_x<\infty$ and E= ∞. Ex. $x[n] = \sin[n]$
- P_x is the time average (mean) of signal amplitude squared, i.e. mean squared value of x(t). i.e. RMS value of x(t) is √ P_x

Note: A DT signal can either be an energy signal or power signal but can not be both at the same time.

Example

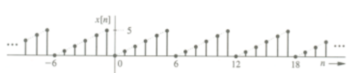
- Determine the suitable measures of the signal
  - ✓ (a) Since **signal amplitude** → 0 as |n|→ ∞., Choose Energy

$$E_x = \sum_{n=0}^{5} n^2 = 55$$



  - ✓ (b) Since amplitude does not → 0 as |n|→ ∞. However, it is periodic, and therefore its power exists. periodic signal repeats regularly each period(6 seconds in this case).

$$P_y = \frac{1}{6}\sum_{n=0}^{5} n^2 = \frac{55}{6}$$

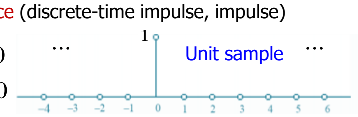


- Signal power is the square of its RMS value. Hence RMS value of the signal 1/√3

Typical Sequences and Sequences Representation: Basic sequences (1)

- Unit sample sequence (discrete-time impulse, impulse)

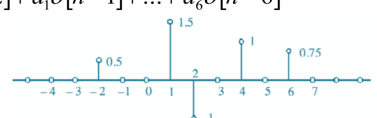
$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$



- Any sequence can be represented as a sum of scaled, delayed impulses

$$x[n] = a_{-2}\delta[n+2] + a_1\delta[n-1] + \ldots + a_6\delta[n-6]$$

- More generally

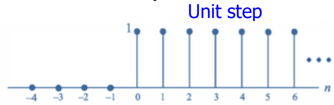


$$x[n] = \sum_{k=0}^{\infty} x[k]\delta[n-k]$$

Typical Sequences and Sequences Representation: Basic sequences (2)

- Unit step sequence

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$



- Related to the impulse by

$$u[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \ldots$$

$$u[n] = \sum_{k=-\infty}^{\infty} u[k]\delta[n-k] = \sum_{k=0}^{\infty} \delta[n-k]$$

Running Sum

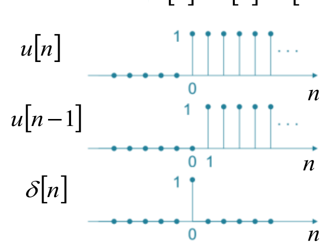
- More generally

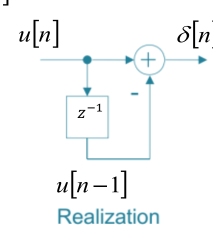
$$\therefore \delta[n] = u[n] - u[n-1]$$

Typical Sequences and Sequences Representation: Basic sequences (3)

- The unit sample is the first difference of the unit step:

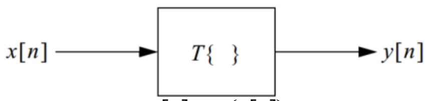
$$\delta[n] = u[n] - u[n-1]$$





Discrete-Time Systems (1)

- Discrete-time system: A device or an algorithm that performs some prescribed operation on a discrete-time signal (input or excitation) to produce another discrete-time signal (output or response)

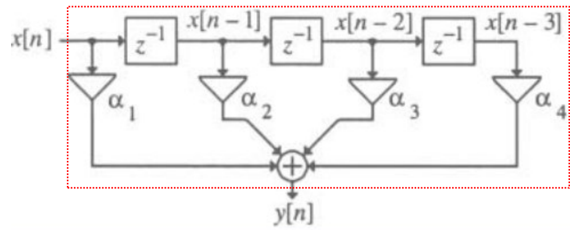


$$y[n] = T\{x[n]\}$$

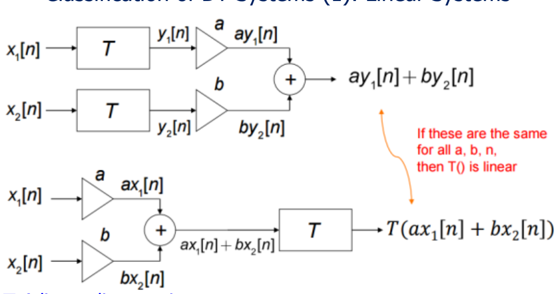
- Single-input single-output system
  - Output sequence is generated sequentially, beginning with a certain time index value n

A certain class of DT systems are
linear and time invariant (LTI) systems

DT System Example



Classification of DT Systems (1): Linear Systems



- A linear discrete-time system:

$$y[n] = 0.5\,x[n] + 0.5\,x[n-1]$$

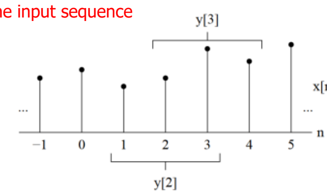
- A non-linear discrete-time system: $y[n] = (x[n])^3$

Linear DT Systems : Moving (Running) Average System OR Filter

- M-point moving-average system: Used to reduce fluctuations in the data OR Used in smoothing random variations in data.

$$y[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x[n-k]$$

- For M_1 = 1 and M_2 = 1, the input sequence DT system whose output y[n] is the average of the three most recent values of the input signal, called Moving Average System.



$\vdots$

$$y[2] = \frac{1}{3}(x[1] + x[2] + x[3])$$

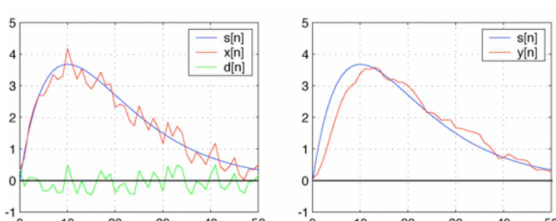
$$y[3] = \frac{1}{3}(x[2] + x[3] + x[4])$$

$\vdots$

10

Linear Systems: Moving Average Filter

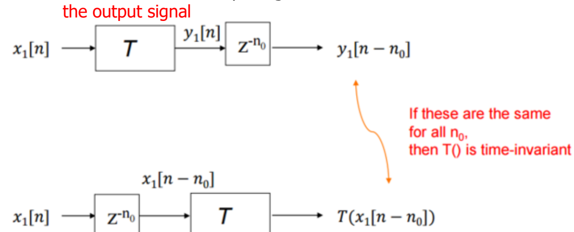
- An application: Consider x[n]=s[n]+d[n] where $s[n]=2[(0.9)^n]$ is the signal corrupted by a random noise d[n]



5-point moving average $y[n] = \frac{1}{5}\sum_{k=0}^{4} x[n-k]$

Classification of DT Systems (2): Shift-Invariant Systems

- A system is time-invariant (or shift-invariant):
  - if a time shift in the input signal results in an identical time shift in the output signal



$$\begin{aligned} &If \quad x[n] \Rightarrow y[n] \\ &Then \quad x[n] = x[n-n_0] \\ &\Rightarrow y[n] = y[n-n_0] \end{aligned}$$

Classification of DT Systems (3): Causal System

- If output depends only on past and current inputs (not future), system is called causal
- The output sequence value at the index $n=n_0$ depends only on the input sequence values for $n \le n_0$.
- Example

$$y[n]=x[n-n_d], \quad -\infty<n<\infty$$

  - Causal for $n_d<=0$
  - Non-causal for $n_d>0$

- Example: Accumulator

Linear DT Systems : Accumulator

- Accumulator

$$y[n]=\sum_{l=-\infty}^{n} x[l]$$

$$=\sum_{l=-\infty}^{n-1} x[l]+x[n]=y[n-1]+x[n]$$

- The output y[n] is the sum of the input sample x[n] and the previous output y[n−1]
- The system cumulatively adds, i.e., it accumulates all input sample values
- Input-output relation can also be written in the form

$$y[n]=\sum_{l=-\infty}^{-1} x[l]+\sum_{l=0}^{n} x[l]=y[-1]+\sum_{l=0}^{n} x[l], \quad n\ge 0$$

- The second form is used for a causal input sequence, in which case y[−1] is called the initial condition

Drill Problem

- An accumulator is excited by the sequence x[n] = nu[n]. Determine its output under the condition that:
  - It is initially relaxed [i.e., y(-1) = 0]
  - Initially, y(-1) = 1.

$$\underbrace{0+1+2+3+...+n}_{AP}$$

$$S=\frac{(n+1)}{2}\left[2a+\{(n+1)-1\}d\right]$$

$$y[n]=\sum_{k=-\infty}^{\infty} x[k]=\sum_{k=-\infty}^{-1} x[k]+\sum_{k=0}^{n} x[k]$$

$$=y[-1]+\sum_{k=0}^{n} x[k]=y[-1]+\frac{n(n+1)}{2}$$

  - It is initially relaxed [i.e., y(-1) = 0] $\quad y[n]=\frac{n(n+1)}{2}, \quad n\ge 0$
  - Initially, y(-1) = 1. $\quad y[n]=1+\frac{n(n+1)}{2}, \quad n\ge 0$

Linear Time-Invariant DT Systems (1)

- Important due to convenient representations and significant applications
- A linear system is completely characterized by its impulse response

$$y[n]=T\{x[n]\}=T\left\{\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\right\}$$

$$=\sum_{k=-\infty}^{\infty} x[k]T\{\delta[n-k]\}=\sum_{k=-\infty}^{\infty} x[k]h_k[n]$$

- Time invariance $\quad h_k[n]=\delta[n-k]$

$$y[n]=\sum_{k=-\infty}^{\infty} x[k]h[n-k]=x[n]*h[n]$$

Convolution sum

Linear Time-Invariant DT Systems (2): Computation of the Convolution Sum

$$y[n]=x[n]*h[n]=\sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

- Obtain the sequence h[n-k]
  - Reflecting h[k] about the origin to get h[-k]
  - Shifting the origin of the reflected sequence to k=n
- Multiply x[k] and h[n-k] for $-\infty<k<\infty$
- Sum the products to compute the output
- sample y[n]

Impulse and Step Response (1)

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

- Unit sample response or (unit) impulse response is the response of the system to a unit impulse

$$x[n]=\delta[n]$$

$$y[n]=x[n]*h[n]=\sum_{m=-\infty}^{\infty} h[m]x[n-m]$$

- Unit step response or step response is the output sequence when the input sequence is the unit step

$$x[n]=u[n]; \quad y[n]=s[n]$$

Impulse and Step Response (2)

- The impulse response of the system (Given by Difference equation) is obtained by putting x[n] = δ[n] and y[n] =h[n]

$$y[n]=\alpha_1 x[n]+\alpha_2 x[n-1]+\alpha_3 x[n-2]+\alpha_4 x[n-3]$$

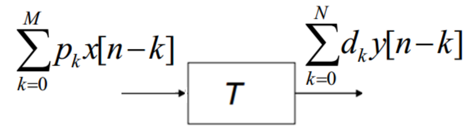
$$h[n]=\alpha_1\delta[n]+\alpha_2\delta[n-1]+\alpha_3\delta[n-2]+\alpha_4\delta[n-3]$$

- The impulse response is thus a finite-length sequence of length 4

$$\{h[n]\}=\{\underset{\uparrow}{\alpha_1}, \quad \alpha_2, \quad \alpha_3, \quad \alpha_4\}$$

Finite-Dimensional LTI Discrete-Time Systems (1)

- LTI discrete-time is characterized by a **linear constant coefficient difference equation.**



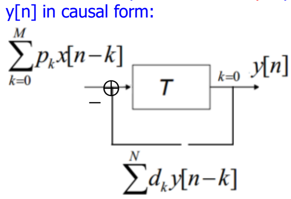
$$\sum_{k=0}^{N} d_k y[n-k]=\sum_{k=0}^{M} p_k x[n-k]$$

where x[n] and y[n] are, respectively, the input and output of the system and $\{d_k\}$ and $\{p_k\}$ are constants

- **The ORDER of the system is max{N,M}**

Finite-Dimensional LTI Discrete-Time Systems (2)

- The output can also be computed recursively by solving y[n] i.e. Rearrange for y[n] in causal form:



$$y[n]=-\sum_{k=1}^{N}\frac{d_k}{d_0}y[n-k]+\sum_{k=0}^{M}\frac{p_k}{d_0}x[n-k]$$

provided that $d_0 \neq 0$

- The output y[n] can be computed for all $n \geq n_0$, knowing the input x[n] and the initial conditions $y[n_0-1]$, $y[n_0-2]$, …, $y[n_0-N]$

Impulse Response Length Based Classification

- **FIR system: If h[n] is of finite length, i.e.,**
  h[n] = 0, for n < N_1 and n > N_2 , with $N_1 < N_2$
- The output can be computed as the finite convolution sum:

$$y[n]=x[n]*h[n]=\sum_{k=N_1}^{N_2} x[k]h[n-k]$$

Same number of operations for all n

- **IIR system: If h[n] is of infinite length, i.e.,**
- Causal IIR output $y[n]=\sum_{k=0}^{n} x[k]h[n-k]=\sum_{k=0}^{n} h[k]x[n-k]$
- Number of operations grows with n
- The limits are due to causality of an LTI system

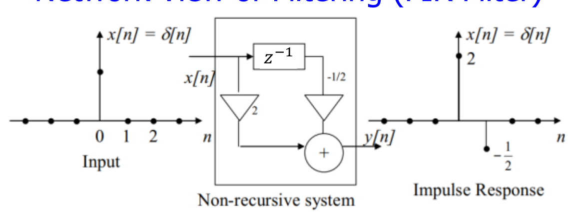
Classification of LTI Discrete-Time Systems (1)

- **LTI discrete-time are usually classified**
  - according to the length of their impulse response sequences
  - according to the method of calculation employed to determine the output samples

✓ **Impulse response classification:**
  - Finite impulse response(FIR) systems
  - Infinite impulse response (IIR) systems

✓ **Output calculation classification:**
  - Recursive systems i.e. using Feedback → IIR
  - non-recursive systems i.e. without feedback → FIR

Network View of Filtering (FIR Filter)



$x[n] = \delta[n]$ $\quad$ $y[n] = 2x[n] - 0.5x[n-1]$

$n = 0$	$y[0] = h[0] = 2x[0] - 0.5x[-1] = 2$
$n = 1$	$y[1] = h[1] = 2x[1] - 0.5x[0] = -0.5$
$n = 2$	$y[2] = h[2] = 2x[2] - 0.5x[1] = 0$
$n = n$	0

Moving (Running) Average is the example of the FIR Filter

Moving average (MA) model → All Zero system

Network View of Filtering (FIR Filter)

- Find the impulse response h[n] of the following fourth order non-recursive system.

$$y[n] = a_0[n] + a_1x[n-1] + a_2x[n-2] + a_3x[n-3] + a_4x[n-4]$$

To find h[n], put $x[n] = \delta[n]$

$n=0 \rightarrow h[0] = a_0\delta[0] + a_1\delta[-1] + a_2\delta[-2] + a_3\delta[-3] + a_4\delta[-4] = a_0$

$n=1 \rightarrow h[1] = a_0\delta[1] + a_1\delta[0] + a_2\delta[-1] + a_3\delta[-2] + a_4\delta[-3] = a_1$

$n=2 \rightarrow h[2] = a_0\delta[2] + a_1\delta[1] + a_2\delta[0] + a_3\delta[-1] + a_4\delta[-2] = a_2$

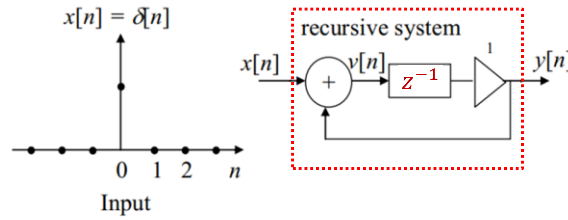
$n=3 \rightarrow h[3] = a_0\delta[3] + a_1\delta[2] + a_2\delta[1] + a_3\delta[0] + a_4\delta[-1] = a_3$

$n=4 \rightarrow h[4] = 0 + 0 + 0 + 0 + a_4\delta[0] = a_4$

$n=5 \rightarrow h[5] = 0 + 0 + 0 + 0 + a_4\delta[1] = 0$

For $n \geq 5$, h[n] = 0, since the nonzero value of δ[n] has moved out of the memory of this system.

Network View of Filtering (IIR Filter)



$$v[n] = x[n] + y[n]$$
$$y[n] = 1.v[n-1]$$

If x[n] = δ[n], calculate h[n] for n=0,1,2,...

Network View of Filtering (IIR Filter)

- Find the impulse response h[n] of the following first-order recursive system.

$$y[n] = \begin{cases} ay[n-1] + x[n] & n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

- To find h[n], Put x[n] = δ[n] and apply the zero-condition.

$$n = 0,\ y[0] = h[0] = ay[-1] + \delta[0] = 1$$

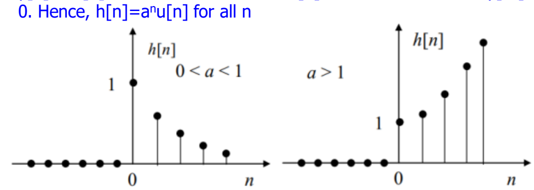
$$n = 1,\ y[1] = h[1] = ay[0] + \delta[1] = a$$

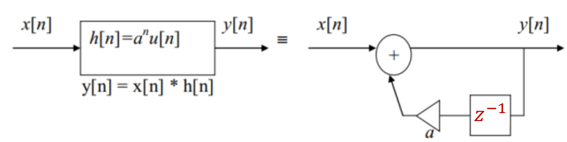
$$n = 2,\ y[2] = h[2] = ay[1] + \delta[2] = a^2$$

$$n = n,\ y[n] = h[n] = a^n \quad \text{for } n \geq 0$$

Network View of Filtering (IIR Filter)

- y[n] = h[n] = 0 for n < 0, because δ[n] is zero for n < 0 and y[-1]= 0. Hence, $h[n]=a^n u[n]$ for all n





Discrete-time systems (Cont'd): Example of IIR Discrete Time System

- **Autoregressive (AR)** model is IIR system: past values have an effect on current values

$$y[n] = x[n] - \sum_{k=0}^{N} d_k y[n-k]$$

AR model → All Pole system

- **Autoregressive moving average** (ARMA) model is IIR system:

$$y[n] = \sum_{k=0}^{M} p_k x[n-k] - \sum_{k=0}^{N} d_k y[n-k]$$

AR model → Poles and Zeros system