Background of Fourier Analysis

Let us consider a 2-D Euclidean space as V.S.

Given by.

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \quad \text{s.t.} \quad \|\vec{x}\| = \sqrt{x_1^2 + x_2^2} < \infty ; \quad \langle \vec{x}, \vec{y} \rangle = \vec{y}^T \vec{x}$$

2 → dimension of the Given V.S.

$$\vec{x} = a_1 \vec{u}_1 + a_2 \vec{u}_2 = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \qquad \vec{u}_1 \perp \vec{u}_2$$

Coordinate of $\vec{x}$ along $\vec{u}_1$ & $\vec{u}_2$
or
Coefficient of the linear combination.

$\{\vec{u}_1, \vec{u}_2\} \Rightarrow$ Orthogonal basis set.

↳ spans complete $\mathbb{R}^2$

i.e. every $\vec{x} \in \mathbb{R}^2$ can be rep. as $\vec{x} \triangleq \sum_{k=1}^{2} a_k \vec{u}_k$

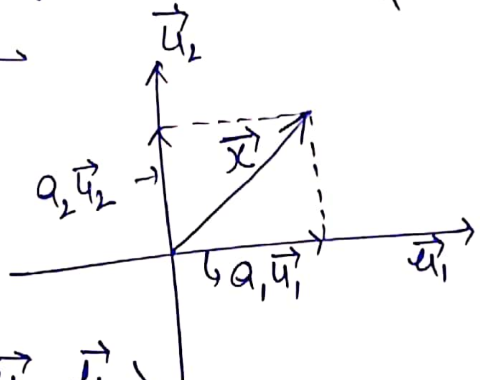
$a_k \in \mathbb{R}$.

Typically. let $\vec{u}_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$

$\vec{u}_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

$\vec{x} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$

Derivation of a_1 & a_2 :-



1. $\langle \vec{x}, \vec{u}_1 \rangle =$

$$= \langle (a_1 \vec{u}_1 + a_2 \vec{u}_2), \vec{u}_1 \rangle$$

$$= a_1 \langle \vec{u}_1, \vec{u}_1 \rangle + a_2 \langle \vec{u}_2, \vec{u}_1 \rangle$$

$\langle \vec{u}_2, \vec{u}_1 \rangle = 0 \;\because\; \vec{u}_2 \perp \vec{u}_1 = \vec{u}_1^T \vec{u}_2 = [3 \; 0] \begin{bmatrix} 0 \\ 2 \end{bmatrix} = 0$

$$\Downarrow$$

$$\langle \vec{x}, \vec{u}_1 \rangle = a_1 \|\vec{u}_1\|^2 + 0$$

$$a_1 = \frac{\langle \vec{x}, \vec{u}_1 \rangle}{\|\vec{u}_1\|^2} = \frac{\vec{u}_1^T \vec{x}}{\vec{u}_1^T \vec{u}_1} = \frac{[3 \; 0] \begin{bmatrix} 6 \\ 4 \end{bmatrix}}{[3 \; 0] \begin{bmatrix} 3 \\ 0 \end{bmatrix}} = \frac{18}{9} = 2$$

Similarly.

2. $$a_2 = \frac{\langle \vec{x}, \vec{u}_2 \rangle}{\langle \vec{u}_2, \vec{u}_2 \rangle} = \frac{\langle \vec{x}, \vec{u}_2 \rangle}{\|\vec{u}_2\|^2} = \frac{[0 \; 2] \begin{bmatrix} 6 \\ 4 \end{bmatrix}}{[0 \; 2] \begin{bmatrix} 0 \\ 2 \end{bmatrix}} = \frac{8}{4} = 2$$

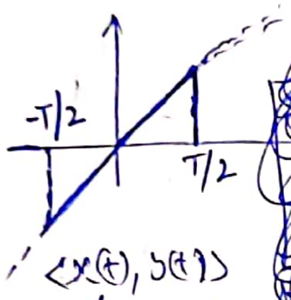
$$\boxed{\vec{x} = a_1 \vec{u}_1 + a_2 \vec{u}_2 = 2\vec{u}_1 + 2\vec{u}_2}$$

e.g. considered here.
$$\begin{bmatrix} 6 \\ 4 \end{bmatrix} = 2 \begin{bmatrix} 3 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

Analogously; complex Exponential Fourier Series → EFS considers a signal space where $x(t) \in \mathbb{C}$ defines over $-\frac{T}{2} < t < \frac{T}{2}$

$\mathbb{C} \to$ Complex number

s.t. $\|x(t)\|^2 = \langle x(t), x(t) \rangle = \int_{-T/2}^{T/2} x^*(t)\, x(t)\, dt$



-T/2

T/2

$$\boxed{\underbrace{\|x(t)\|^2}_{l_2 \text{ norm}} = \int_{-T/2}^{T/2} |x(t)|^2\, dt \triangleq E_x < \infty .}$$

$\langle x(t), y(t) \rangle$ ← Inner-product of $x(t)$ with $y(t)$ ⟹

$$\langle x(t), y(t) \rangle = \int_{-T/2}^{T/2} y^*(t) \cdot x(t)\, dt = \int_{-T/2}^{T/2} x(t)\, y^*(t)\, dt$$

Such signals can be represented by the linear combination of the following orthogonal basis set.

1.) $\{ e^{jk\omega_0 t}$; $k \in \mathbb{Z}$; $k \in \{\dots,-2,-1,0,1,2,\dots\} \}$

2.) $\{ 1, \cos(k\omega_0 t), \sin(k\omega_0 t)$; $k \in \mathbb{Z}^+$; $k \in \{1,2,3,\dots\} \}$.

where $\omega_0 = \frac{2\pi}{T}$

Hence

EFS

$$\boxed{x(t) \triangleq \sum_{k=-\infty}^{\infty} C_k \cdot e^{jk\omega_0 t} \; ; \; -\frac{T}{2} \le t \le \frac{T}{2}} \qquad \boxed{C_k = \frac{\langle x(t), e^{jk\omega_0 t} \rangle}{\| e^{jk\omega_0 t} \|^2} \; ; \; k \in \mathbb{Z}}$$

⇓ Synthesis Eq^n ⇓ Analysis Eq^n.

$$\| e^{jk\omega_0 t} \|^2 = \int_{-T/2}^{T/2} | e^{jk\omega_0 t} |^2\, dt = \int_{-T/2}^{T/2} 1\, dt = T$$

EFS Coefficients. ⟹

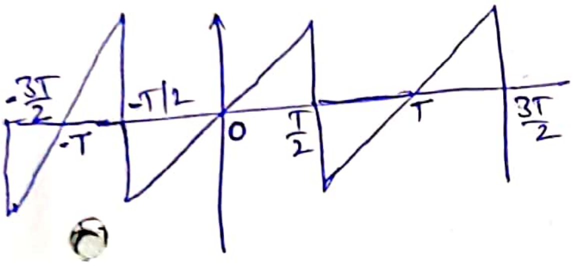
$$\boxed{C_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t) \cdot e^{-jk\omega_0 t}\, dt .}$$

$\Rightarrow x(t) = x(t+\ell T);\ \ell \in \mathbb{Z}$

Moreover, if $x(t)$ is periodic with fundamental time-period, T; i.e.

then $$\boxed{x(t) \doteq \sum C_K e^{jK\omega_0 t} \ ; \ \forall\ t \in \mathbb{R}}$$

where $$\boxed{C_K \triangleq \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jK\omega_0 t}\, dt}$$



Similarly, for $x(t) = x(t+T)$ s.t. $\int_{-T/2}^{T/2} |x(t)|^2 < \infty$.

TFS / RFS $\Rightarrow$ $x(t) \triangleq a_0 \cdot 1 + \sum_{K=1}^{\infty} a_K \cos(K\omega_0 t) + \sum_{K=1}^{\infty} b_K \sin(K\omega_0 t)$ $\forall\ t$

trigonometric or Real FS

considers only real freq^c i.e. only +ve freqs.

$$a_0 = \langle x(t), 1 \rangle / \|1\|^2$$

$$= \frac{\int_{-T/2}^{T/2} x(t) \cdot 1 \cdot dt}{\int_{-T/2}^{T/2} |1|\, dt} = \frac{\int_{-T/2}^{T/2} x(t)\, dt}{\int_{-T/2}^{T/2} 1\, dt}$$

$$\boxed{a_0 = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, dt}$$

$$a_K = \frac{\langle x(t), \cos(K\omega_0 t) \rangle}{\|\cos(K\omega_0 t)\|^2} =$$

$$\|\cos(K\omega_0 t)\|^2 = \int_{-T/2}^{T/2} \cos^2(K\omega_0 t)\, dt = \int_{-T/2}^{T/2} \left(\frac{1}{2} + \frac{\cos(2K\omega_0 t)}{2}\right) dt$$

$$= \int_{-T/2}^{T/2} \frac{1}{2}\, dt + \int_{-T/2}^{T/2} \cos(2K\omega_0 t)\, dt = \frac{1}{2}T + \left.\frac{\sin\left(2K\omega_0 \cdot \frac{t}{2}\right)}{2 \cdot 2K\omega_0}\right|_{-T/2}^{T/2}$$

$$a_k = \frac{T}{2} + \frac{\cos\left(2k\omega_0 \cdot \frac{T}{2}\right) - \cos\left(-2k\omega_0 \frac{T}{2}\right)}{4k\omega_0}$$

$$= \frac{T}{2} + \frac{\cos\left(k \cdot \frac{2\pi}{T} \times T\right) - \cos\left(-k \frac{2\pi}{T} \times T\right)}{4k\omega_0}$$

$$\|\cos(k\omega_0 t)\|^2 = \frac{T}{2} + \frac{1 - 1}{4k\omega_0} \Rightarrow \boxed{\|\cos(k\omega_0 t)\|^2 = \frac{T}{2}}$$

$$\Downarrow$$

$$a_k = \frac{\langle x(t), \cos(k\omega_0 t)\rangle}{\|\cos(k\omega_0 t)\|^2}$$

$$\boxed{a_k = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos(k\omega_0 t)\, dt}$$

Similarly

$$\boxed{b_k = \frac{\langle x(t), \sin(k\omega_0 t)\rangle}{\underbrace{\|\sin(k\omega_0 t)\|^2}_{T/2}} = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin(k\omega_0 t)\, dt}$$

Relationship b/w EFS & TFS $\Rightarrow$

Scanned with CamScanner

Relationship b/w EFS & TFS.

EFS Coefficient of $x(t)$ → $\boxed{c_k = \frac{a_k}{2} - j\frac{b_k}{2}}$ → (1)

↳ May be real or complex valued

$\Downarrow \; k \to -k$

$$\boxed{c_{-k} = \frac{a_{-k}}{2} - j\frac{b_{-k}}{2}}$$

⇒ Holds true even if $x(t)$ is complex valued.

EFS

Let $x(t) = x(t+kT)$;

$$x(t) \leftrightarrow \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \; ; \quad \omega_0 = \frac{2\pi}{T}$$

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t}\, dt \; ; \quad k \in \mathbb{Z}$$

TFS

$$x(t) \leftrightarrow a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + \sum_{k=1}^{\infty} b_k \sin(k\omega_0 t)$$

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, dt; \quad b_k = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin(k\omega_0 t)\, dt$$

$$a_k = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos(k\omega_0 t)\, dt$$

∴ but, $a_{-k} = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos(-k\omega_0 t)\, dt$

$= \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos(k\omega_0 t)\, dt$

$\boxed{a_{-k} = a_k}$ ⇒ 'a_k' always an even function of 'k'

Similarly

$\boxed{b_{-k} = -b_k}$ ⇒ 'b_k' is always an odd func^n of 'k'.

$$\boxed{c_{-k} = \frac{a_k}{2} + j\frac{b_k}{2}} \to (2)$$

Eq^n (1), (2), (3) & (4) are true irrespective of $x(t)$ is real or complex. Furthermore (1) & (2)

$\Downarrow$

$\boxed{a_k = c_k + c_{-k}}$ → (3)

$\boxed{b_k = \frac{c_{-k} - c_k}{j}}$ → (4)

BUT

$\boxed{c_{-k} = c_k^*}$ ⇒ iff a_k & b_k are real.

$\Downarrow$

$x(t)$ is real.

So.

Conclusion: $\boxed{x(t) = x^*(t)}$ i.e. Real $\xleftrightarrow{CTFS}$ $\boxed{c_k = c_{-k}^*}$

$\Downarrow$

Conjugate Symmetric for real $x(t)$.

Lec. - 2

①

Do Not have any symmetry unlike for real $x(t)$

Spectrum.

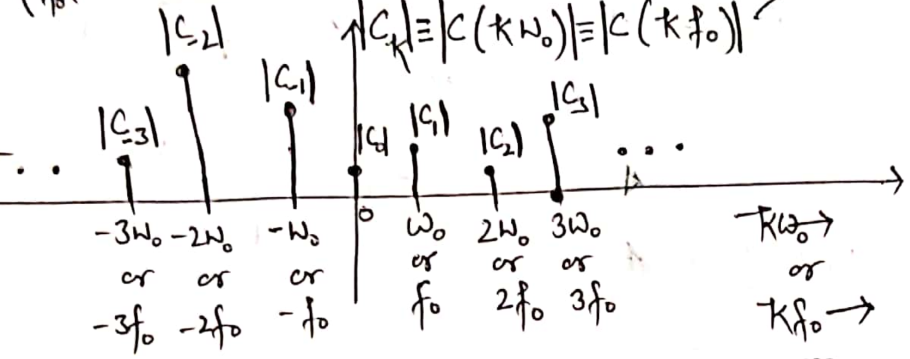
$C_k \in \mathbb{C}$ (Complex no. set) $\forall k$ even for real $x(t)$.

$c(k\omega_0) \equiv c(kf_0)$ $\quad C_k = |C_k| e^{j \angle C_k}$; $\forall k$

$\omega_0 = \frac{2\pi}{T} = 2\pi f_0$

$f_0 = \frac{1}{T}$.

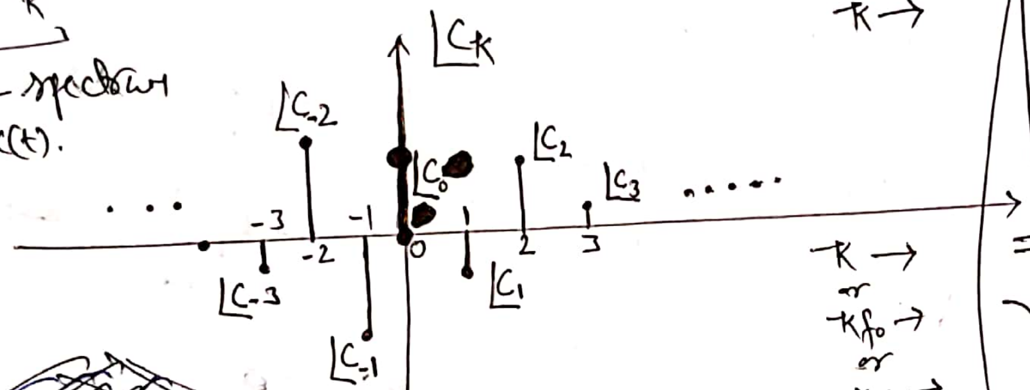
$|C_k|$ vs. k
↳ Magnitude spectrum of $x(t)$



~~[illegible]~~

Complex spectrum for complex-valued $x(t)$.

$\angle C_k$ vs. k
↳ Phase spectrum of $x(t)$.



⟹ Line spectrum of CT periodic signal.

Why Complex
↳ ∵ it considers -ve frequencies; which does not exist in practical

Few Imp. Properties of CT EFS :-

Proofs also imp.
⇓
You need to do yourself (DYS).

1. Conjugation Property :-

$$x(t) \xleftrightarrow{\text{CT-EFS}} C_k$$

$$x^*(t) \longleftrightarrow C^*_{-k}$$

Conj. Symmetric (CS)

1.1 If $x(t)$ is purely real ⇒ $x(t) = x^*(t) \longleftrightarrow \boxed{C_k = C^*_{-k}}$

1.2 If $x(t)$ is purely imag. ⇒ $x(t) = -x^*(t)$ $\forall t$ $\longleftrightarrow \boxed{C_k = -C^*_{-k}}$ Conj Anti-Symmetric (CAS)

1.3. If $x(t)$ is neither purely real & nor purely imag. $\Rightarrow$ C_k is neither CS nor CAS.

$\Downarrow$

C_k is non-symmetric.

For Case 1.1:

When $x(t)$ is Purely Real

$\Downarrow$

$$\boxed{C_{-k} = C_k^*}$$

$\Downarrow$

$$C_k = \frac{a_k}{2} - j\frac{b_k}{2}$$

$$C_{-k} = \frac{a_k}{2} + j\frac{b_k}{2} = C_k^*$$

$$|C_k| = \frac{\sqrt{|a_k|^2 + |b_k|^2}}{2}$$

$$\angle C_k = -\tan^{-1}\left(\frac{b_k}{a_k}\right)$$

$$|C_{-k}| = \frac{\sqrt{a_k^2 + b_k^2}}{2}$$

$$\angle C_{-k} = \tan^{-1}\left(\frac{b_k}{a_k}\right)$$

$|C_k| = |C_{-k}| \Rightarrow$ Even Symmetric

$\angle C_k = -\angle C_{-k} \Rightarrow$ Odd Symmetric

For Case 1.2

When $x(t)$ is purely imag. $\Rightarrow$ a_k & b_k are Purely imag.

$\Downarrow$

$$\boxed{C_k = -C_{-k}^*}$$

$$C_k = \frac{a_k}{2} - j\frac{b_k}{2}$$

$$C_{-k} = \frac{a_k}{2} + j\frac{b_k}{2} = -C_k^* = -\frac{a_k}{2} - j\frac{b_k}{2}$$

$$\boxed{|C_k| = \frac{\sqrt{|a_k|^2 + |b_k|^2}}{2} = |C_{-k}|}$$

Even Symm.

$$\boxed{\angle C_k = +\tan^{-1}\left(\frac{a_k}{b_k}\right) = -\angle C_{-k}}$$

Odd Symmetric

i.e. Either $x(t)$ is purely Real or Purely Imag. $\Rightarrow$ $|C_k|$ is even func^n of 'k'. $\angle C_k$ is odd func^n of 'k'.

signal is Real valued ← ← 'f₀' or 'ω₀' are also not unreal; i.e. +ve only.

(3)

Real Spectrum. (Real FS spectrum)

Assuming $x(t)$ is purely real

then $C_k = |C_k| e^{j\angle C_k} \xRightarrow{\text{s.t.}} \boxed{|C_k| = |C_{-k}|}$ & $\boxed{\angle C_k = -\angle C_{-k}}$

TFS ⇒

$$\boxed{x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + \sum_{k=1}^{\infty} b_k \sin(k\omega_0 t)} \Rightarrow \text{TFS}$$

As. $a\cos\theta + b\sin\theta = r\cos(\theta - \phi)$

$$= \underbrace{r\cos\phi}_{a}\cos\theta + \underbrace{r\sin\phi}_{b}\sin\theta$$

$r = \sqrt{a^2+b^2}$, $\phi = \tan^{-1}\frac{b}{a}$

So Assuming

P1 ⟶ $a_k\cos(k\omega_0 t) + b_k\sin(k\omega_0 t) = r_k\cos(k\omega_0 t - \phi_k)$.

$$\boxed{\begin{aligned} a_k &= r_k\cos(\phi_k) \\ b_k &= r_k\sin(\phi_k)\end{aligned}} \Rightarrow \text{where } \boxed{\begin{aligned} r_k &= \sqrt{a_k^2 + b_k^2} = 2|C_k| \\ \phi_k &= \tan^{-1}\left(\frac{b_k}{a_k}\right) = -\angle C_k\end{aligned}}$$

Further for $k=0$

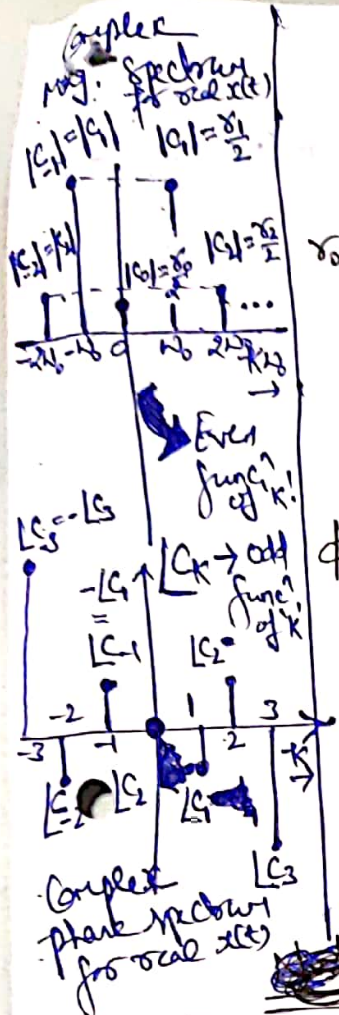
P2 ⟶ $\boxed{\begin{aligned}&\text{Assuming} \\ &r_0 = a_0\end{aligned}}$ via $b_0 = 0$ ⇒ $\boxed{\begin{aligned} r_0 &= \sqrt{a_0^2 + b_0^2} = a_0 \\ \phi_0 &= \tan^{-1}\left(\frac{b_0}{a_0}\right) = 0\end{aligned}}$

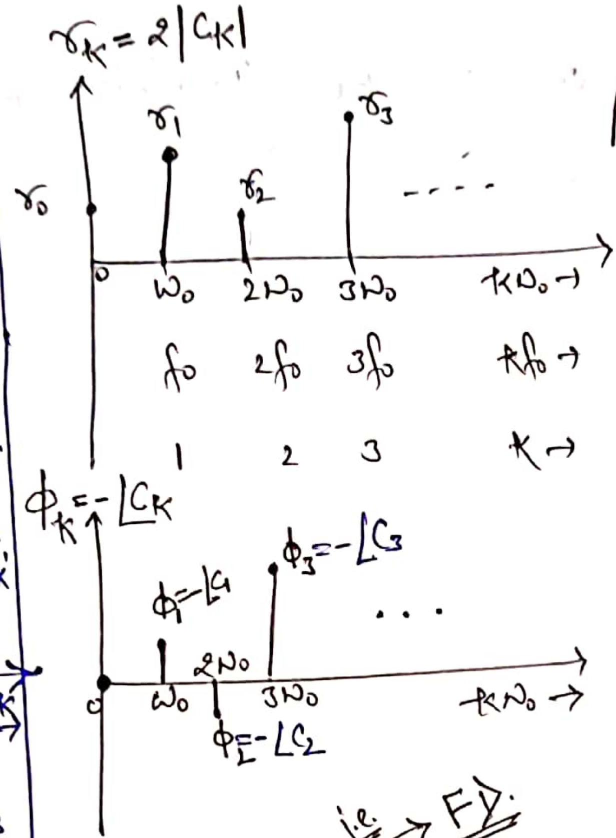
TFS can be re-written as

$$\boxed{x(t) = \sum_{k=0}^{\infty} r_k\cos(k\omega_0 t - \phi_k)} \Rightarrow \text{Harmonic / Amplitude-Angle form of CTFS}$$

r_k → Amplitude at $k\omega_0$, ϕ_k → Phase at $k\omega_0$ where $k\omega_0$ → harmonics.

Scanned with CamScanner





Real Amplitude/Magnitude Spectrum for real x(t)

Real Angle/Phase spectrum for real x(t).

Real Spectrum as 'ω' is considered to be +ve only; which is the real fact in practical.

i.e. → FD.

Conclusion →

(1) Spectrum (corresponding to CTFS) of Continuous in TD ; Periodic Signal is always Discrete & Aperiodic in FD

(2) Real magnitude spectrum of the real-valued signal reduces to complex magnitude spectrum by reducing/dividing the amplitude/magnitude at -ve & +ve freq^c equally in complex magnitude spectrum.

(3) Key Points to Note while looking at Complex Spectrum :-

(3.1) whenever, the freq^c dependent info^n (e.g. 'Phase' and 'Bandwidth') needs to be addressed; only +ve freq^c will be taken care.

(3.2) whenever, the magnitude dependent info^n e.g. Energy/Power needs to be addressed/find ; Both -ve & +ve freq^c will be taken care in the magnitude spectrum.

Lec-3.

Recommended Reading : Signals & Systems by A.V. Oppenheim

Existence & Convergence Conditions for CTFS of a Continuous periodic signal :- (Developed by P.L. Dirichlet).

Dirichlet Conditions :- guarantees that $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$, $\forall t$ except at isolated values of 't' for which $x(t)$ is discontinuous.

At these values of 't', the infinite series, $\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$ converges to the average of the values on either side of the discontinuity.

are only the sufficient conditions (i.e. not necessary) to have EFS of continuous periodic signals or of periodic signals with finite no. of finite-size discontinuity.

Exception: $x(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT) \leftrightarrow c_k = \frac{1}{T}$

Condition-1 (Existence Condition of CTFS $\Rightarrow |c_k| < \infty \ \forall k$).

$$\boxed{\int_T |x(t)|\, dt < \infty}$$

$$|c_k| = \left| \frac{1}{T} \int_T x(t)\, e^{-jk\omega_0 t}\, dt \right| \leq \frac{1}{T} \int_T |x(t)|\, \underbrace{|e^{-jk\omega_0 t}|}_{1}\, dt$$

$$\boxed{|c_k| \leq \frac{1}{T} \int_T |x(t)|\, dt}$$

$x(t) = \frac{1}{t}$; $0 < t \leq 1$ violates Condition-1 $\Downarrow$ $c_k = ??$

So if $\int_T |x(t)|\, dt < \infty \Rightarrow |c_k| < \infty \ \forall k$.

Condition-2 :- $x(t)$ must have finite no. of maxima & minima during any ~~sig~~. single period of the signal.

$x(t) = \sin\left(\frac{2\pi}{t}\right)$, $0 < t \leq 1$ meets Condition 1 but not Condition-2.

Condition-3 :- In any finite interval of time, there are only a finite no. of discontinuities. Furthermore, each of these discontinuities is of finite size.

Conclusions :-

Case 1 For periodic signals that has no discontinuities

e.g. $e^{j\omega_0 t}$, $\sin(\omega_0 t)$, $\cos(\omega_0 t)$ etc. ; $\sum_{K=-\infty}^{\infty} C_K e^{jK\omega_0 t} = x(t) \; \forall \; t$.

Case 2 For a periodic signal with a finite no. of finite-sized discontinuities in each period; e.g. periodic square wave,

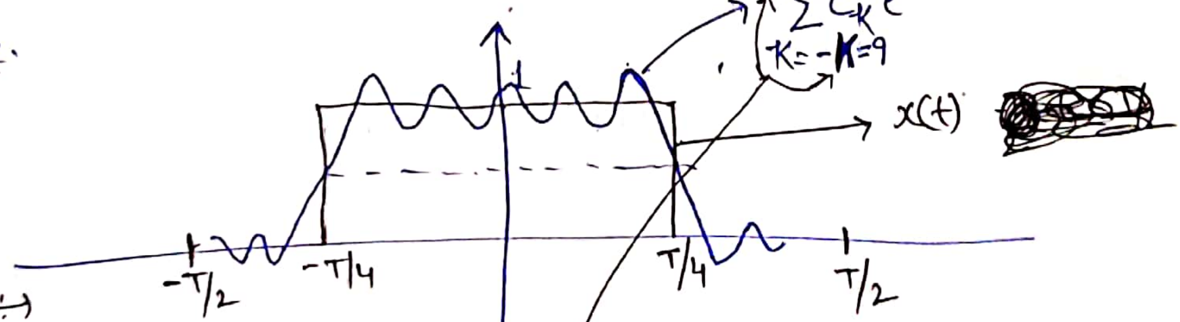
$$\boxed{\begin{array}{l} \sum_{K=-\infty}^{\infty} C_K e^{jK\omega_0 t} = x(t) \quad \forall \text{ 't' except 't' where discontinuities are located.} \\ \text{lets say} \\ \sum_{K=-\infty}^{\infty} C_K e^{jK\omega_0 t} = \dfrac{x(t^+) + x(t^-)}{2} \quad \text{'t' where discontinuities are located} \end{array}}$$

Furthermore; the error signal, $e(t) = x(t) - \sum_{K=-\infty}^{\infty} C_K e^{jK\omega_0 t}$ have zero energy over one 'T'.

in this case ; i.e. $\boxed{\int_T |e(t)|^2 \, dt = 0.}$

Gibbs Phenomenon occurs for Case-2 signals :-

i.e.



Conclusion :-

Even for large values of 'K' an overshoot occurs at discontinuity but the ripples goes lower & lower as $K \to \infty$.

5 Parseval's Relation (CTFS)

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$P_x = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, f^*(t)\, dt$

$= \frac{1}{T}\int_{-T/2}^{T/2} f(t) \left(\sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}\right)^* dt$

$= \frac{1}{T}\int_{-T/2}^{T/2} f(t) \left(\sum_{n=-\infty}^{\infty} C_n^* e^{-jn\omega_0 t}\right) dt$

$= \sum_{n=-\infty}^{\infty} C_n^* \cdot \frac{1}{T}\int_{-T/2}^{T/2} f(t) \cdot e^{-jn\omega_0 t}\, dt$

$= \sum_{n=-\infty}^{\infty} C_n^* \, C_n$

$$\boxed{P_x = \sum_{n=-\infty}^{\infty} |C_n|^2}$$

$= C_0^2 + \sum_{n=1}^{\infty} |C_n|^2 + \sum_{n=-1}^{-\infty} |C_n|^2$

$= C_0^2 + \sum_{n=1}^{\infty} |C_n|^2 + \sum_{n=1}^{\infty} |C_{-n}|^2$

$= a_0^2 + \sum \left(\frac{a_n^2 + b_n^2}{2}\right)$

But $C_n = \frac{a_n}{2} - j\frac{b_n}{2}$

$C_0 = a_0 \Rightarrow \boxed{|a_0|^2 = |C_0|^2} \Rightarrow$ DC Power

$C_{-n} = \frac{a_n}{2} + j\frac{b_n}{2}$

$|C_n|^2 = \frac{a_n^2 + b_n^2}{4}$

$|C_{-n}|^2 = \frac{a_n^2 + b_n^2}{4}$

$$\boxed{|C_n|^2 + |C_{-n}|^2 = \frac{a_n^2}{2} + \frac{b_n^2}{2}}$$ $\Rightarrow$ AC Power corresponding to $n\omega_0$ harmonic

replace in eq^n shown on last page

$P = C_0^2 + \sum_{n=1}^{\infty} \left\{ |C_n|^2 + |C_{-n}|^2 \right\}$

$|C_1|^2 + |C_2|^2 + |C_3|^2 + \dots$; $|C_{-1}|^2 + |C_{-2}|^2 + \dots$

$= C_0^2 + \sum_{n=1}^{\infty} |C_n|^2 + \sum_{n=1}^{\infty} |C_{-n}|^2$

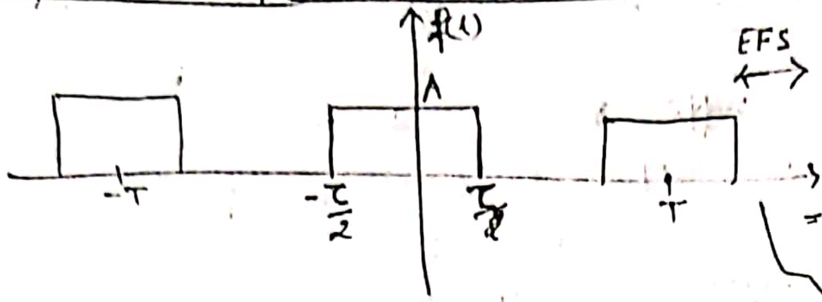
$$\boxed{P_{f(t)} = \sum_{n=-\infty}^{\infty} |C_n|^2}$$ $\Rightarrow$ Parseval's Power Relation for periodic power signal.

$$\boxed{\frac{1}{T}\int_{-T/2}^{T/2} |f(t)|^2\, dt = \sum_{n=-\infty}^{\infty} |C_n|^2}$$ $\Rightarrow$ Average Power of Periodic Signal

$= a_0^2 + \sum_{n=1}^{\infty} \frac{a_n^2 + b_n^2}{2}$

$|C_n|^2 \rightarrow$ power spectrum or Power Spectral density (P.S.D.) of the given CT periodic signal

Transformation from CTFS to CTFT :→



$$\xleftrightarrow{EFS} \quad C_K = \frac{1}{T}\int_T f(t)\, e^{-jK2\pi f_0 t}\, dt$$

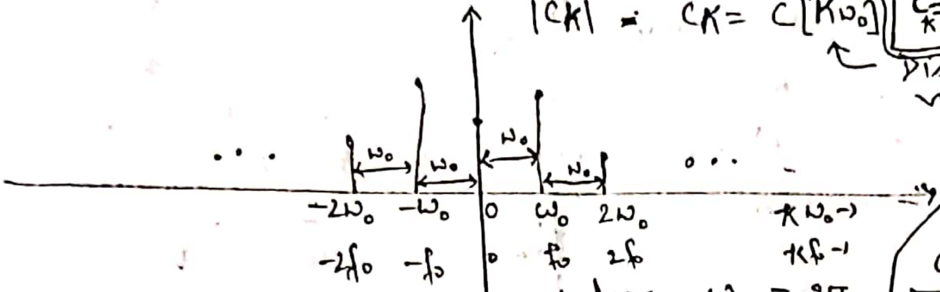
$$= \frac{1}{T}\int_{-\tau/2}^{\tau/2} A e^{-jK2\pi f_0 t}\, dt$$

$$= \frac{A}{T}\left[\frac{e^{-jK2\pi f_0 t}}{-j2\pi f_0 K}\Big|_{-\tau/2}^{\tau/2}\right]$$

$$\boxed{C_K = \frac{A\tau}{T}\frac{\sin(\pi K f_0 \tau)}{\pi K f_0 \tau}} \; ; \; Kf_0$$

$$= \frac{A\tau}{T}\,\text{sinc}(\pi K f_0 \tau)$$

$|C_K|$; $C_K = C[K\omega_0]$ ← Discrete variable



where $\omega_0 = \frac{2\pi}{T}$

$C_0 = \frac{0}{0}$

can't apply LH ∵ $K \in Z$

$$\boxed{C_0 = \frac{1}{T}\int_T x(t)\, dt = \frac{A\tau}{T}}$$

$T \uparrow \longrightarrow \omega_0 = \frac{2\pi}{T} \downarrow$

samples in FS spectrum comes closer to eachother.

$T \to \infty \longrightarrow \omega_0 = \frac{2\pi}{T} \to 0 \equiv \delta\omega$ or $d\omega$

$K\omega_0 \to \omega$ ← Continuous variable.

$$C[K\omega_0] = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{-jK\omega_0 t}\, dt$$

$$T\,C[K\omega_0] = \int_{-T/2}^{T/2} f(t)\cdot e^{-jn\omega_0 t}\, dt$$

Let $T\,C[K\omega_0] = F[K\omega_0]$

$$\boxed{F[K\omega_0] = \int_{-T/2}^{T/2} f(t)\, e^{-jK\omega_0 t}\, dt.}$$

$$f(t) = \sum_{K=-\infty}^{\infty} C_K \cdot e^{jK\omega_0 t}$$

$$= \sum_{K=-\infty}^{\infty} C[K\omega_0]\, e^{jK\omega_0 t} = \sum_{n=-\infty}^{\infty} \frac{F[K\omega_0]}{T} e^{jK\omega_0 t}$$

$$\boxed{f(t) = \sum_{n=-\infty}^{\infty} F[K\omega_0]\, e^{jK\omega_0 t} \cdot \frac{\omega_0}{2\pi}}$$

Scanned with CamScanner

As $T \to \infty$, $\omega_0 \to 0$ or $d\omega$, $n\omega_0 \to \omega$; $\Sigma \to \int$

$f(t)_{\text{periodic}} \xrightarrow{T=\infty}$ Aperiodic $f(t)$ & Discrete Frequency Spectrum $\to$ Continuous Frequency Spectrum.

<u>Analysis Eq^n</u>

$$\boxed{F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} \cdot dt}$$

$F[K\omega_0] \to F(\omega)$. Fourier Transform

<u>Synthesis Eq^n</u>

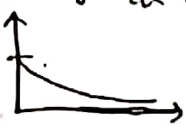
$$\boxed{f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega}$$

Inverse Fourier Transform

<u>Note</u> →

Fourier Series existed only for Periodic signals but Fourier Transform will exist for both Aperiodic as well as periodic signal but FT of periodic signals are Limitedly defined i.e. FT=∞ atleast at one freq^c.

Q1 → Find FT of $e^{-at} \cdot u(t)$, $a > 0$



A → $F(\omega) = \int_{-\infty}^{\infty} e^{-at} u(t) e^{-j\omega t} dt$

$$= \int_{0}^{\infty} e^{-at} e^{-j\omega t} dt$$

$$= \int_{0}^{\infty} e^{-(a+j\omega)t} dt$$

$$= \left. \frac{e^{-(a+j\omega)t}}{-(a+j\omega)} \right|_0^{\infty}$$

$$\boxed{F(\omega) = \frac{1}{a+j\omega}} \Rightarrow F(\omega) = \frac{1}{a+j\omega} \frac{(a-j\omega)}{(a-j\omega)} = \frac{a-j\omega}{a^2+\omega^2}$$

$$F(\omega) = \frac{a}{a^2+\omega^2} - \frac{j\omega}{a^2+\omega^2}$$

$\text{Re}[F(\omega)] = \frac{a}{a^2+\omega^2} \Rightarrow$ Even func^n of ω.

Scanned with CamScanner

$$\text{Im}\{F(\omega)\} = \frac{-\omega}{a^2+\omega^2} \Rightarrow \text{odd func}^n \text{ of } \omega.$$

Hence $F(\omega) \Rightarrow$ Conjugate Symmetric

i.e. $F(-\omega) = F^*(\omega)$

Real $f(t) \xleftrightarrow{FT} \boxed{F^*(-\omega) = F(\omega).}$

Note:-

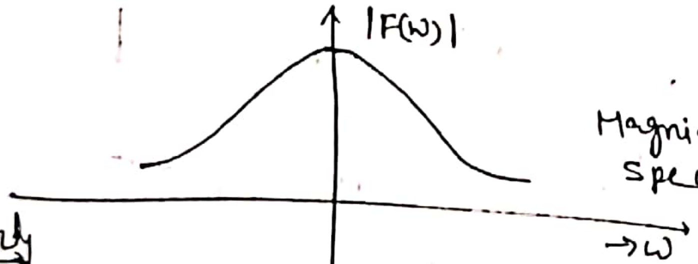
If $f(t)$ is real valued, $F(\omega)$ must be Conjugate symmetric in frequency; $\boxed{|F(\omega)| \to \text{even} \ \& \ \angle F(\omega) \to \text{odd}}$

Polar form, $F(\omega)$, Conj. Symm. $\Rightarrow \boxed{\text{Re}[F(\omega)] \to \text{Even} \ \& \ \text{Im}\{F(\omega)\} \to \text{odd}}$

$F(\omega) \to$ Frequency Response

$$\boxed{\text{Frequency Response} \leftarrow F(\omega) = \underbrace{|F(\omega)|}_{\text{Magnitude Response}} \ \underbrace{\angle F(\omega)}_{\text{phase Response}}}$$

$$|F(\omega)| = \frac{1}{\sqrt{a^2+\omega^2}} \ ; \ \angle F(\omega) = -\tan^{-1}\frac{\omega}{a}$$



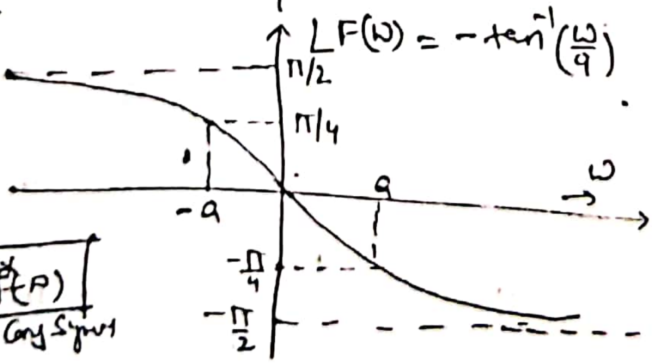
Magnitude spectrum $\Rightarrow$ Even ~~odd~~ func^n

Conjugate Property

of FT :-

$f(t) \leftrightarrow F(\omega)$

$f^*(t) \leftrightarrow F^*(-\omega)$



Phase spectrum $\Rightarrow$ odd func^n.

Case-1

$f(t) \to$ real

$f(t) = f^*(t) \leftrightarrow \boxed{F(\omega) = F^*(-\omega)}$ ↳ Conj Symm

Case-2

$f(t) \to$ Purely Imag.

$f(t) = -f^*(t) \leftrightarrow \boxed{F(\omega) = -F^*(-\omega)}$ ↳ Conj Anti Symm.

referred to as the Dirichlet conditions, require that:

1. $x(t)$ be absolutely integrable; that is,

$$\int_{-\infty}^{+\infty} |x(t)|dt < \infty. \tag{4.13}$$

2. $x(t)$ have a finite number of maxima and minima within any finite interval.
3. $x(t)$ have a finite number of discontinuities within any finite interval. Futhermore, each of these discontinuities must be finite.

Scanned with CamScanner

Parseval's Energy Relation (with CTFT)

Spectral Density Property

~~f(t) → real-valued & an energy signal.~~

$$E = \int_{-\infty}^{\infty} |f(t)|^2 \, dt$$

$f(t) \longleftrightarrow F(\omega)$

$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$

$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$

$F^*(\omega) = \int_{-\infty}^{\infty} f^*(t) e^{j\omega t} dt$

$$= \int_{-\infty}^{\infty} f(t) \cdot f^*(t) \, dt$$

conjugate

$$= \int_{-\infty}^{\infty} f(t) \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega \right)^* dt$$

$$= \int_{-\infty}^{\infty} f(t) \left(\int_{-\infty}^{\infty} \frac{1}{2\pi} F^*(\omega) e^{-j\omega t} d\omega \right) dt$$

Changing the order of integration

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(\omega) \left(\int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \right) d\omega$$

$C = a + jb$

$C^* = a - jb$

$C \cdot C^* = a^2 + b^2 = |C|^2$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(\omega) \, F(\omega) \, d\omega$$

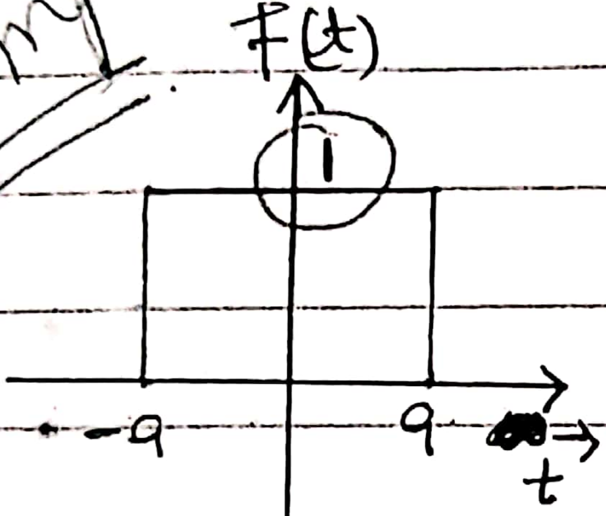
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \, F^*(\omega) \, d\omega$$

$$\boxed{E = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 \, d\omega}$$

→ Rayleigh Energy theorem or Parseval's energy theorem.

$\psi(\omega)$: Energy spectral Density of $f(t)$.

Imp.



$$\xleftrightarrow{F} \frac{2a\sin(\omega a)}{\omega a} = \frac{2}{\omega}\sin(a\omega)$$

$$G_{2a}(t) \longleftrightarrow \frac{2a\sin(\omega a)}{\omega a} = 2a\,Sa(\omega a) = \frac{2}{\omega}\sin(a\omega)$$

Duality Property

$$x(t) \xleftrightarrow{F} X(\omega)$$

$$X(t) \xleftrightarrow{F} 2\pi x(-\omega)$$

e.g. $e^{-2t}u(t) \xleftrightarrow{F} \frac{1}{2+j\omega}$

$\omega = t$ $\quad t = -\omega$

$\frac{1}{2+jt} \xleftrightarrow{F} 2\pi e^{2\omega} u(-\omega).$

$\delta(t) \xleftrightarrow{F} 1$

$1 \longleftrightarrow 2\pi\delta(-\omega)$

$= 2\pi\delta(\omega)$

Scanned with CamScanner