z-Transform

- z-Transform: used to analyze a DT signal in the frequency domain.
- z-Transform: two-sided, bilateral z-transform of a DT signal x[n]

$$X(z)=Z\{x[n]\}=\sum_{n=-\infty}^{\infty}x[n]z^{-n}$$

$$x[n]\overset{ZT}{\Leftrightarrow}X(z)$$

For non-causal signals, the summation should start from minus infinity, and the ZT so defined is called two-sided ZT.

where z is the complex variable **z = e$^{j\Omega}$.**

z $^{-1}$ corresponds to a delay of one unit in the signal.

- One-sided, unilateral z-transform: One-sided z-transform is meaningful for causal signals

$$X(z)=\sum_{n=0}^{\infty}x[n]z^{-n}$$

- NOTE: ZT converges (or existed) is the sum of above eq. existed.

Inverse z-Transform

- Inverse z-Transform:

$$x[n]=Z^{-1}\{X(z)\}=\frac{1}{2\pi j}\oint X(z)z^{n-1}dz$$

Integration in counter clock wise (CCW) direction around a closed path in the complex z-plane (known as contour integral).

- Where the integration is performed in a counter-clockwise direction around a closed contour in the ROC of X(z) and encircling the origin.
  - ✓ Such contour integral is difficult to evaluate (but could be done using Cauchy's residue theorem), therefore we often use other techniques to obtain the IZT.
- If X(z) is a rational function of z, i.e. a ratio of polynomials, it is not necessary to evaluate this by integral.
- Partial fraction is used to evaluate IZT.

Relation between ZT and DTFT

- The z-transform of sequence x[n] is

$$X(z)=\sum_{n=-\infty}^{\infty}x[n]z^{-n}$$

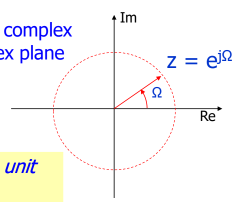
- Put z = e$^{j\Omega}$.

$$X(\Omega)=\sum_{n=-\infty}^{\infty}x[n]e^{-j\Omega n}$$

- **z-plane:** Representation of complex number z as a location in a complex plane

$$X(\Omega)=X(z)\big|_{z=e^{-j\Omega}}$$

$$-\pi\le\Omega<\pi \quad \Leftrightarrow \quad unit\ circle$$



DTFT is to *evaluate z-transform on a unit circle.*

ROC

- The **z transform does not converge** for all sequences or for all values of z
- The set of values of z for which the **z transform converges** is called **region of convergence**
- ROC: for a discrete time signal x[n] is defined as a continuous region in z plane where the ZT converges (or existed).
- ROC determination: represent ZT as a rational function.

$$X(z)=\frac{P(z)}{Q(z)}$$

where P(z) and Q(z) are polynomials in *z.*

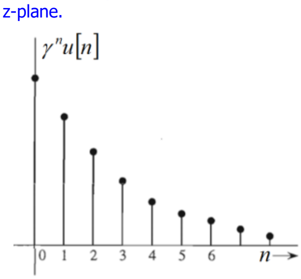
- The roots of the eq. P(z) = 0 correspond to the 'zeros' of X(z)
- The roots of the eq. Q(z) = 0 correspond to the 'poles' of X(z)

- The ROC of the Z-transform depends on the convergence of the polynomials P(z) and Q(z).

Drill Problem: Right side Sequence

- Determine the ZT of causal signal $x[n]=\gamma^n u[n]$

where γ is a constant. Depicts the ROC and the location of poles and zeros in the z-plane.



Drill Problem: Right side Sequence (1)

$$x[n]=\gamma^n u[n]$$

$$X(z)=\sum_{n=-\infty}^{\infty}\gamma^n u[n]z^{-n}$$

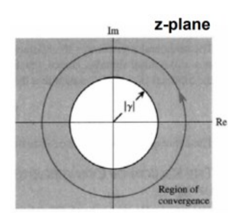
$$=\sum_{n=0}^{\infty}\gamma^n z^{-n}$$

$$=\sum_{n=0}^{\infty}(\gamma z^{-1})^n$$

$$X(z)=\sum_{n=0}^{\infty}(\gamma z^{-1})^n=\frac{1}{1-\gamma z^{-1}}=\frac{z}{z-\gamma}$$

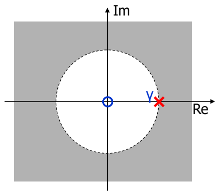
$$|\gamma z^{-1}|<1$$

$$\Rightarrow |z|>|\gamma|$$



Gray region: ROC
i.e. entire region outside the circle

Drill Problem: Right side Sequence (2)

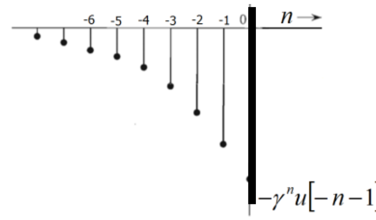


ROC is bounded by the pole and is the exterior of a circle.

Drill Problem: Left side Sequence

- Determine the ZT of causal signal $x[n] = -\gamma^n u[-n-1]$

where γ is a constant. Depicts the ROC and the location of poles and zeros in the z-plane.



Drill Problem: Left side Sequence (1)

$$x[n] = -\gamma^n u[-n-1]$$

$$X(z) = -\sum_{n=-\infty}^{\infty} \gamma^n u[-n-1] z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} \gamma^n z^{-n}$$

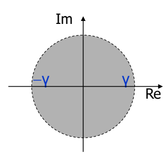
$$= -\sum_{n=1}^{\infty} \gamma^{-n} z^{n}$$

$$= 1 - \sum_{n=0}^{\infty} \gamma^{-n} z^{n}$$

$$X(z) = 1 - \sum_{n=0}^{\infty} (\gamma^{-1} z)^n = 1 - \frac{1}{1-\gamma^{-1} z} = \frac{z}{z-\gamma}$$

$$\left|\gamma^{-1} z\right| < 1$$

$$\Rightarrow |z| < |\gamma|$$



Gray region: ROC
i.e. entire region inside the circle

NOTE: ZT of causal and anti-causal signals are identical except their ROC.

Drill Problem: A Two Sided Sequence

$$x[n] = \underbrace{(-\tfrac{1}{3})^n u[n]}_{x_1[n]} - \underbrace{(\tfrac{1}{2})^n u[-n-1]}_{x_2[n]}$$

$$x_1[n] = \sum_{n=0}^{\infty}\left(-\frac{1}{3}z^{-1}\right)^n = \frac{\left(-\frac{1}{3}z^{-1}\right)^0 - \left(-\frac{1}{3}z^{-1}\right)^\infty}{1+\frac{1}{3}z^{-1}} = \frac{1}{1+\frac{1}{3}z^{-1}}$$

$$x_2[n] = \sum_{n=-\infty}^{-1}\left(\frac{1}{2}z^{-1}\right)^n = \frac{\left(\frac{1}{2}z^{-1}\right)^{-\infty} - \left(\frac{1}{2}z^{-1}\right)^0}{1-\frac{1}{2}z^{-1}} = \frac{-1}{1-\frac{1}{2}z^{-1}}$$

$$X(z) = \frac{1}{1+\frac{1}{3}z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}} = \frac{2z\left(z-\frac{1}{12}\right)}{\left(z+\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}$$

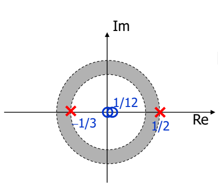
Drill Problem: A Two Sided Sequence

$$ROC: \left|\frac{1}{3}z^{-1}\right| < 1$$

$$\frac{1}{3} < |z|$$

$$ROC: \left|\frac{1}{2}z^{-1}\right| > 1$$

$$\frac{1}{2} > |z|$$



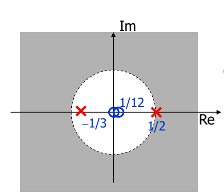
ROC is bounded by poles and is a ring.

ROC does not include any pole.

Drill Problem: Sum of Two Right Sided Sequences (1)

$$x[n] = (\tfrac{1}{2})^n u[n] + (-\tfrac{1}{3})^n u[n]$$

$$X(z) = \frac{z}{z-\frac{1}{2}} + \frac{z}{z+\frac{1}{3}} = \frac{2z(z-\frac{1}{12})}{(z-\frac{1}{2})(z+\frac{1}{3})}$$



ROC is bounded by poles and is the exterior of a circle.

ROC does not include any pole.

z-transforms of δ[n] and u[n]

- By definition:

$$X[z] = \sum_{n=0}^{\infty} x[n] z^{-n} = x[0] + \frac{x[1]}{z} + \frac{x[2]}{z^2} + \frac{x[3]}{z^3} + \cdots$$

- Since $x[n] = \delta[n]$, $x[0] = 1$ and $x[2] = x[3] = x[4] = \cdots = 0$.

$$\boxed{\delta[n] \Longleftrightarrow 1 \qquad \text{for all } z}$$

ROC is entire z-plane including z=0 and z=∞

- Also, for $x[n] = u[n]$, $x[0] = x[1] = x[3] = \cdots = 1$.
- Therefore: $X[z] = 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots$

$$= \frac{1}{1 - \frac{1}{z}} \qquad \left|\frac{1}{z}\right| < 1 \qquad = \frac{z}{z-1} \qquad |z| > 1$$

$$\boxed{u[n] \Longleftrightarrow \frac{z}{z-1} \qquad |z| > 1}$$

z-transforms of cos(βn)u[n]

- Since $\cos \beta n = (e^{j\beta n} + e^{-j\beta n})/2$
- From previous slide, we know

$$\gamma^n u[n] \Longleftrightarrow \frac{z}{z-\gamma} \qquad |z| > |\gamma|$$

- Hence

$$\boxed{e^{\pm j\beta n} u[n] \Longleftrightarrow \frac{z}{z - e^{\pm j\beta}} \qquad |z| > |e^{\pm j\beta}| = 1}$$

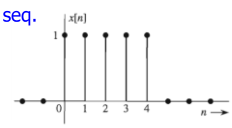
- Therefore:

$$X[z] = \frac{1}{2}\left[\frac{z}{z - e^{j\beta}} + \frac{z}{z - e^{-j\beta}}\right]$$

$$= \frac{z(z - \cos \beta)}{z^2 - 2z\cos\beta + 1} \qquad |z| > 1$$

ZT of 5 impulses i.e. Finite Length Sequence

- Find the z-tranform of finite duration seq.
- By definition



$$X[z] = 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4}$$

- Now, equation for sum of a power series:

$$\sum_{k=0}^{n} r^k = \frac{r^{n+1} - 1}{r - 1}$$

- Let $r = z^{-1}$ and $n = 4$

$$X[z] = \frac{z^{-5} - 1}{z^{-1} - 1}$$

$$= \frac{z}{z-1}(1 - z^{-5})$$

Inverse z-Transform – Real Unique Poles

- Find the inverse z-transform of: $X[z] = \frac{8z - 19}{(z-2)(z-3)}$
- Step 1: Divide both sides by z: $\frac{X[z]}{z} = \frac{8z - 19}{z(z-2)(z-3)}$
- Step 2: Perform partial fraction: $\frac{X[z]}{z} = \frac{(-19/6)}{z} + \frac{(3/2)}{z-2} + \frac{(5/3)}{z-3}$
- Step 3: Multiply both sides by z: $X[z] = -\frac{19}{6} + \frac{3}{2}\left(\frac{z}{z-2}\right) + \frac{5}{3}\left(\frac{z}{z-3}\right)$

$$x[n] = -\tfrac{19}{6}\delta[n] + \left[\tfrac{3}{2}(2)^n + \tfrac{5}{3}(3)^n\right] u[n]$$

Inverse z-Transform – Repeat Peal Poles (1)

- Find the inverse z-transform of: $X[z] = \frac{z(2z^2 - 11z + 12)}{(z-1)(z-2)^3}$
- Divide both sides by z and expand:

$$\frac{X[z]}{z} = \frac{2z^2 - 11z + 12}{(z-1)(z-2)^3} = \frac{k}{z-1} + \frac{a_0}{(z-2)^3} + \frac{a_1}{(z-2)^2} + \frac{a_2}{(z-2)}$$

- Use covering method to find k and a_0:

$$k = \left.\frac{2z^2 - 11z + 12}{(z-1)(z-2)^3}\right|_{z=1} = -3 \qquad a_0 = \left.\frac{2z^2 - 11z + 12}{(z-1)(z-2)^3}\right|_{z=2} = -2$$

- We get: $\frac{X[z]}{z} = \frac{2z^2 - 11z + 12}{(z-1)(z-2)^3} = \frac{-3}{z-1} - \frac{2}{(z-2)^3} + \frac{a_1}{(z-2)^2} + \frac{a_2}{(z-2)}$
- To find a_2, multiply both sides by z and let z→∞:

$$0 = -3 - 0 + 0 + a_2 \implies a_2 = 3$$

Inverse z-Transform – Repeat Real Poles (2)

- To find a_1, let z = 0: $\frac{12}{8} = 3 + \frac{1}{4} + \frac{a_1}{4} - \frac{3}{2} \implies a_1 = -1$
- Therefore, we find: $\frac{X[z]}{z} = \frac{-3}{z-1} - \frac{2}{(z-2)^3} - \frac{1}{(z-2)^2} + \frac{3}{z-2}$

$$X[z] = -3\frac{z}{z-1} - 2\frac{z}{(z-2)^3} - \frac{z}{(z-2)^2} + 3\frac{z}{z-2}$$

- Use pairs :

6	$\gamma^n u[n] \Longleftrightarrow \frac{z}{z-\gamma}$
10	$\frac{n(n-1)(n-2)\cdots(n-m+1)}{\gamma^m m!}\gamma^n u[n] \Longleftrightarrow \frac{z}{(z-\gamma)^{m+1}}$

$$x[n] = \left[-3 - 2\frac{n(n-1)}{8}(2)^n - \frac{n}{2}(2)^n + 3(2)^n\right] u[n]$$

$$= -\left[3 + \frac{1}{4}(n^2 + n - 12)2^n\right] u[n]$$

Find inverse z-Transform – Complex Poles (1)

- Find inverse z-transform of: $X[z] = \frac{2z(3z+17)}{(z-1)(z^2-6z+25)}$

$$= \frac{2z(3z+17)}{(z-1)(z-3-j4)(z-3+j4)}$$

- Whenever we encounter complex pole, we need to use a special partial fraction method (called quadratic factors):

$$\frac{X[z]}{z} = \frac{2(3z+17)}{(z-1)(z^2-6z+25)} = \frac{2}{z-1} + \frac{Az+B}{z^2-6z+25}$$

- Now multiply both sides by z, and let z→∞:

$$0 = 2 + A \Longrightarrow A = -2$$

- We get: $\frac{2(3z+17)}{(z-1)(z^2-6z+25)} = \frac{2}{z-1} + \frac{-2z+B}{z^2-6z+25}$

Find Inverse z-Transform – Complex Poles (2)

- To find B, we let z=0: $\frac{-34}{25} = -2 + \frac{B}{25} \Longrightarrow B = 16$
- Now, we have X[z] in a convenient form:

$$\frac{X[z]}{z} = \frac{2}{z-1} + \frac{-2z+16}{z^2-6z+25} \Longrightarrow X[z] = \frac{2z}{z-1} + \frac{z(-2z+16)}{z^2-6z+25}$$

- Use table pair (11), we identify A = -2, B = 16, and a = -3.

$$r = \sqrt{\frac{A^2|\gamma|^2 + B^2 - 2ABa}{|\gamma|^2 - a^2}} \quad \theta = \tan^{-1}\left(\frac{Aa - B}{A\sqrt{|\gamma|^2 - a^2}}\right) \quad \beta = \cos^{-1}\left(\frac{-a}{|\gamma|}\right)$$

$$r = \sqrt{\frac{100 + 256 - 192}{25 - 9}} = 3.2, \quad \beta = \cos^{-1}\left(\frac{3}{5}\right) = 0.927 \text{ rad}$$

$$\theta = \tan^{-1}\left(\frac{-10}{-8}\right) = -2.246 \text{ rad} \qquad \boxed{r|\gamma|^n \cos(\beta n + \theta)u[n] \Longleftrightarrow \frac{z(Az+B)}{z^2 + 2az + |\gamma|^2}}$$

- Therefore: $x[n] = [2 + 3.2(5)^n \cos(0.927n - 2.246)]u[n]$

Find Inverse z-Transform – Complex Poles (2)

$$X[z] = \frac{2z(3z+17)}{(z-1)(z^2-6z+25)}$$

$$= \frac{2z(3z+17)}{(z-1)(z-3-j4)(z-3+j4)}$$

$$\frac{X[z]}{z} = \frac{2(3z+17)}{(z-1)(z-3-j4)(z-3+j4)} = \frac{k_1}{z-1} + \frac{k_2}{z-3-j4} + \frac{k_2^*}{z-3+j4}$$

$$\frac{X[z]}{z} = \frac{2}{z-1} + \frac{-1-j1.25}{z-3-j4} + \frac{-1+j1.25}{z-3+j4}$$

$$X[z] = \frac{2z}{z-1} + \frac{(-1-j1.25)z}{z-3-j4} + \frac{(-1+j1.25)z}{z-3+j4}$$

Inverse z-Transform – Long Division

$$X[z] = \frac{z^2(7z-2)}{(z-0.2)(z-0.5)(z-1)} = \frac{7z^3 - 2z^2}{z^3 - 1.7z^2 + 0.8z - 0.1}$$

- Perform long division:

$$\begin{array}{r} 7 + 9.9z^{-1} + 11.23z^{-2} + 11.87z^{-3} + \cdots \\ z^3 - 1.7z^2 + 0.8z - 0.1 \,\big)\, 7z^3 - 2z^2 \qquad\qquad\qquad\qquad \\ \underline{7z^3 - 11.9z^2 + \ 5.60z - \ 0.7} \qquad\quad \\ 9.9z^2 - \ 5.60z + \ 0.7 \qquad\quad \\ \underline{9.9z^2 - 16.83z + \ 7.92 - 0.99z^{-1}} \\ 11.23z - \ 7.22 + 0.99z^{-1} \\ \underline{11.23z - 19.09 + 8.98z^{-1}} \\ 11.87 - 7.99z^{-1} \end{array}$$

- Thus:

$$X[z] = \frac{z^2(7z-2)}{(z-0.2)(z-0.5)(z-1)} = 7 + 9.9z^{-1} + 11.23z^{-2} + 11.87z^{-3} + \cdots$$

- Therefore: $x[0] = 7, \ x[1] = 9.9, \ x[2] = 11.23, \ x[3] = 11.87, \ldots$

ZT Properties: Linearity

$$\text{if} \quad x_1[n] \Leftrightarrow X_1(z), \qquad z \in R_1$$

$$\text{and} \quad \text{if } x_2[n] \Leftrightarrow X_2(z), \qquad z \in R_2$$

$$\text{Then} \quad Z\{ax_1[n] + bx_2[n]\} = aX_1(z) + bX_2(z), \qquad \underbrace{z \in R_1 \cap R_2}_{\text{Overlay of the above two ROC's}}$$

for all a and b.

Example

- Compute the ZT of a sampled signal

$$x[n] = \delta[n+1] + 3\delta[n] + 6\delta[n-3] - \delta[n-4]$$

- Apply Linearity Property

$$X[z] = z + 3 + 6z^{-3} - z^{-4}$$

$$x[n] = \{1, \ \underset{\uparrow}{3}, \ 0, \ 0, \ 6, \ -1\}$$

- Sequence representation: A signal sequence at time origin (n=0) is indicated by symbol ↑. If the sequence is not indicated by ↑, then it is understood that Ist (left most) point in sequence is at origin.

ZT Properties: Shift Property (1)

- Right Shift (Delay):

$$if \quad x[n]u[n] \Leftrightarrow X(z), \qquad z \in R_x$$

$$Then \quad x[n-n_0]u[n-n_0] \Leftrightarrow z^{-n_0} X(z) \qquad z \in R_x$$

ZT Properties: Shift Property (2)

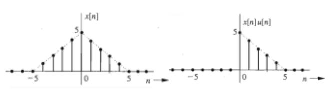
- Left Shift (Advance):

$$if \quad x[n]u[n] \Leftrightarrow X(z), \qquad z \in R_x$$

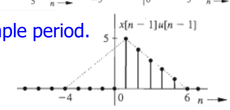
$$Then \quad x[n+n_0]u[n+n_0] \Leftrightarrow z^{n_0} X(z) \qquad z \in R_x$$

Example : Shift Property of z-Transform

- If $\quad x[n]u[n] \Longleftrightarrow X[z]$
- Then $\quad x[n-1]u[n-1] \Longleftrightarrow \frac{1}{z}X[z]$



- which is delay causal signal by 1 sample period.



Multiplication by an Exponential Sequence: Scaling Property

$$If \quad x[n]u[n] \Leftrightarrow X(z), \qquad R_{x-} < |z| < R_{x+}$$

$$Then \quad \gamma^n x[n]u[n] \Leftrightarrow X(\gamma^{-1}z) \qquad z \in |\gamma| \cdot R_x$$

for any γ, real or complex

$$\begin{aligned} Z\{a^n x(n)\} &= \sum_{n=-\infty}^{\infty} x(n) a^n z^{-n} \\ &= \sum_{n=-\infty}^{\infty} x(n) \left(\frac{z}{a}\right)^{-n} \\ &= X\left(\frac{z}{a}\right) \end{aligned}$$

Multiply by n property: Differentiation of X(z)

$$If \quad x[n]u[n] \Leftrightarrow X(z), \qquad z \in R_x$$

$$Then \quad nx[n]u[n] \Leftrightarrow -z\frac{dX(z)}{dz} \qquad z \in R_x$$

$$\begin{aligned} \frac{dX(z)}{dz} &= \frac{d}{dz}\sum_{n=-\infty}^{\infty} x(n) z^{-n} \\ &= \sum_{n=-\infty}^{\infty} x(n) \frac{d}{dz}(z^{-n}) \\ &= \sum_{n=-\infty}^{\infty} x(n)(-n) z^{-n-1} \\ &= -z^{-1}\sum_{n=-\infty}^{\infty} nx(n) z^{-n} = -z^{-1} Z\{nx(n)\} \end{aligned}$$

Multiply by n property: Differentiation of X(z): Example

If $X(z) = \log(1 + az^{-1}), |z| > 0$, determine $x(n)$.

$$\begin{aligned} nx(n) &\overset{z}{\longleftrightarrow} -z\frac{dX(z)}{dz} \\ -z\frac{dX(z)}{dz} &= -z\frac{d\log(1+az^{-1})}{dz} = \frac{az^{-1}}{1+az^{-1}} \\ \Rightarrow \frac{az^{-1}}{1+az^{-1}} &= \underbrace{\left(az^{-1}\right)}_{\text{Const. \& T-Shift}} \underbrace{\left(\frac{1}{1+az^{-1}}\right)}_{(-a)^n u(n)} \\ \Rightarrow nx(n) &= a(-a)^{n-1}u(n-1) \\ \Rightarrow x(n) &= -\frac{(-a)^n}{n}u(n-1) \end{aligned}$$

Multiply by n property: Differentiation of X(z): Example

If $X(z) = \frac{az^{-1}}{(1-az^{-1})^2}$, $|z| > a$, determine $x(n)$

$$a^n u(n) \overset{z}{\longleftrightarrow} \frac{1}{1-az^{-1}}, \quad \text{ROC: } |z| > a$$

$$\Rightarrow na^n u(n) \overset{z}{\longleftrightarrow} -z\frac{d}{dz}\left(\frac{1}{1-az^{-1}}\right) = \frac{az^{-1}}{(1-az^{-1})^2}, \quad \text{ROC: } |z| > |a|$$

$$\Rightarrow x(n) = na^n u(n)$$

Time Reversal Property

$$If \quad x[n] \Leftrightarrow X(z), \qquad z \in R_x$$

$$Then \quad x[-n] \Leftrightarrow X(z^{-1}) \qquad z \in 1/R_x$$

$$Z\{x(-n)\} = \sum_{n=-\infty}^{\infty} x(-n)z^{-n} = \sum_{m=-\infty}^{\infty} x(m)z^{m} = \sum_{m=-\infty}^{\infty} x(m)(z^{-1})^{-m} = X(z^{-1})$$

Initial Value Theorem

- For a causal sequence i.e. if x[0] is finite, then $\lim_{z\to\infty} X(z)$ is finite.
- If x(n) is causal [i.e., x(n) = 0 for n < 0], then

$$If \quad x[n] = 0, \qquad \text{for } n < 0$$

$$Then \quad x[0] = \lim_{z\to\infty} X(z)$$

- IVT helps to finds a **DC gain of a signal** of a signal.

$$X(z) = \sum_{n=0}^{\infty} x(n)z^{-n} = x(0) + \underbrace{x(1)z^{-1} + x(2)z^{-2} + \cdots}_{\to 0 \text{ as } z\to\infty}$$

$$\lim_{z\to\infty} X(z) = x(0)$$

Final Value Theorem

$$If \quad x[n] \Leftrightarrow X(z),$$

$$Then \quad \lim_{n\to\infty} x[n] = x(\infty) = \lim_{z\to 1}(1-z^{-1})X(z)$$

- i.e. the limit exist if all the poles of $(1-z^{-1})$ X(z) lie inside the unit circle, i.e. all the poles of $(1-z^{-1})$ X(z) have magnitude less than one.
- FVT helps to find a final steady state value of a signal.

Convolution of Sequences

$$If \quad x_1[n] \Leftrightarrow X_1(z), \qquad z \in R_1$$

$$and \quad if\ x_2[n] \Leftrightarrow X_2(z), \qquad z \in R_2$$

$$Then \quad x_1[n] * x_2[n] \Leftrightarrow X_1(z)X_2(z) \qquad z \in R_1 \cap R_2$$

- i.e. convolution in the time-domain is the same as multiplication in the z-domain.

- Proof:

$$x_1[n] * x_2[n] = \sum_{k=-\infty}^{\infty} x_1(k)x_2(n-k)$$

$$Z[x_1(n) * x_2(n)] = \sum_{n=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty} x_1(k)x_2(n-k)\right)z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} x_1(k)\sum_{n=-\infty}^{\infty} x_2(n-k)z^{-n} = \sum_{k=-\infty}^{\infty} x_1(k)z^{-k}\sum_{n=-\infty}^{\infty} x_2(n)z^{-n}$$

$$= X_1(z)X_2(z)$$

Convolution of Sequences: Example

$$x_1(n) = \alpha^n u(n)$$
$$x_2(n) = \beta^n u(-n)$$
$$y(n) = x_1(n) * x_2(n). \quad \text{Find } Y(z)$$

$$x_1(n) = \alpha^n u(n) \Longrightarrow X_1(z) = \frac{1}{1-\alpha z^{-1}}$$

$$x_2(n) = \beta^n u(-n) = \left(\frac{1}{\beta}\right)^{-n} u(-n)$$

$$Z\left[\left(\frac{1}{\beta}\right)^n u(n)\right] = \frac{1}{1-\left(\frac{1}{\beta}\right)z^{-1}} \quad \text{[time reversal]}$$

$$\Longrightarrow X_2(z) = \frac{1}{1-\beta^{-1}z} = \frac{-\beta z^{-1}}{1-\beta z^{-1}}$$

Autocorrelation of Sequences

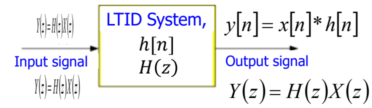
If $x[n] \Leftrightarrow X(z)$,

Then $\boldsymbol{R_{xx}(l) = x[n] * x[-n] \Leftrightarrow X(z)X(z^{-1}) = R_{xx}(z)}$

❑ Calculate the autocorrelation $R_{xx}(z)$ of the sequence x[n]

$$x[n] = b^n[u(-n-1)]$$

LTID System Response and System Function



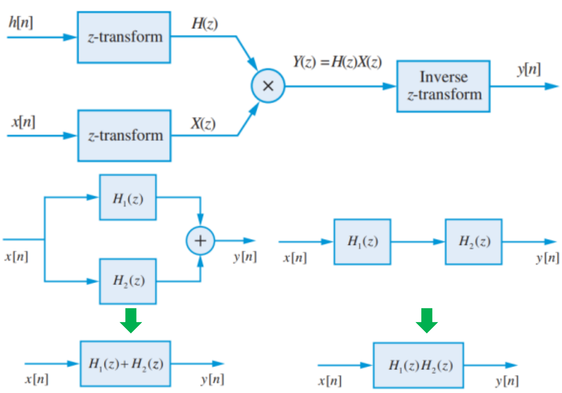
❑ If h[n] is the impulse response of a LTID system, then the system response y[n] to an input x[n] is x[n] * h[n].

❑ Assuming causality, and that h[n] ↔ H(z) and x[n] ↔ X(z) then

$$Y(z) = H(z)X(z)$$

❑ The response y[n] is the zero-state response of the LTID system to the input x[n]. It follows that the transfer function H(z):

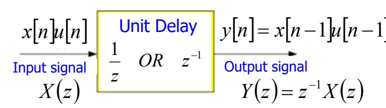
$$Y(z) = H(z)X(z)$$



(a) parallel connection, and (b) cascade connection.

System Function of a Unit Delay

❑ The z-transform can be considered as a unit delay operator.



❑ Similarly, the filter:

$$y[n] = x[n] - x[n-1]$$

can be viewed as the operator:

$$Y(z) = X(z) - z^{-1}X(z) = (1 - z^{-1})X(z)$$

Important Transfer Functions

There are two LTI systems whose transfer functions we are particularly interested in:

1) A **Finite Impulse Response (FIR)** digital filter
   - The impulse response is of finite length
   - The realization (output) is non-recursive
   - The transfer function is a polynomial in z^{-1}

2) An **Infinite Impulse Response (IIR)** digital filter
   - The impulse response is of infinite length
   - The realization (output) is recursive
   - The transfer function is a rational function in z^{-1}

FIR Digital Filters

❑ **FIR filter: impulse response is defined for $N_1 \le n \le N_2$** and thus $h[n] = 0; \quad n < N_1, \quad$ and $\quad n > N_2$

❑ Output $y[n] = \sum_{k=N_1}^{N_2} h[k]\,x[n-k]$

$$Y(z) = \underbrace{\left(\sum_{n=N_1}^{N_2} h[n]z^{-n}\right)}_{H(z)} X(z) = H(z)X(z)$$

$$\Rightarrow H(z) = \sum_{n=N_1}^{N_2} h[n]z^{-n}$$

The system function of a non recursive filter can be expressed as a numerator polynomial.

LTID System Response and System Function

- If H is a set of difference equations, then H(z) is simple to find
- Suppose H is defined by the difference equations

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

$$Y(z) = -\sum_{k=1}^{N} a_k Y(z) z^{-k} + \sum_{k=0}^{M} b_k X(z) z^{-k}$$

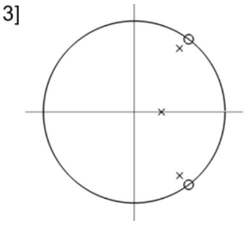
$$\Rightarrow Y(z)\left(1+\sum_{k=1}^{N} a_k z^{-k}\right) = X(z)\sum_{k=0}^{M} b_k z^{-k}$$

$$\Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1+\sum_{k=1}^{N} a_k z^{-k}}$$

Example: Causal IIR Filter

- Transfer function of IIR filter given by difference equation

$$y[n] = x[n-1] - 1.2x[n-2] + x[n-3] + 1.3y[n-1] - 1.04y[n-2] + 0.222y[n-3]$$



$$H(z) = \frac{z^2 - 1.2z + 1}{z^3 - 1.3z^2 - 1.04z - 0.222} = \frac{\big(z-(0.6+j0.8)\big)\big(z-(0.6-j0.8)\big)}{(z-0.3)\big(z-(0.5+j0.7)\big)\big(z-(0.5-j0.7)\big)}$$

LTID System Response and System Function

- Determine the system function for the difference equation system

$$y(n) = -y(n-1) + 0.25y(n-2) + x(n) + 0.5x(n-1)$$

$$Y(z) = -Y(z)z^{-1} + 0.25Y(z)z^{-2} + X(z) + 0.5X(z)z^{-1}$$

$$\Rightarrow Y(z)[1+z^{-1}-0.25z^{-2}] = X(z)[1+0.5z^{-1}]$$

$$\Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{1+0.5z^{-1}}{1+z^{-1}-0.25z^{-2}}$$

- System functions can usually be determined in one step from the difference equations
  Feedforward terms ⇒ numerator; Feedback terms ⇒ denominator

Impulse Response, Step Response, and System Response (1)

- The impulse response h(n) of the DSP system H(z) can be obtained by solving its difference equation using a unit impulse input δ(n).

$$X(z) = Z\{\delta(n)\} = 1,$$

$$h(n) = Z^{-1}\{H(z)X(z)\} = Z^{-1}\{H(z)\}$$

- For a step input, we can determine step response assuming the zero ICs.

$$X(z) = Z[\,u(n)] = \frac{z}{z-1},$$

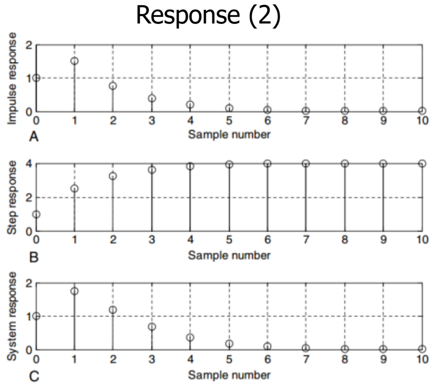
$$y(n) = Z^{-1}\left\{H(z)\frac{z}{z-1}\right\}$$

- z-transform of the general system response

$$Y(z) = H(z)X(z)$$

$$y(n) = Z^{-1}\{Y(z)\}$$

Impulse Response, Step Response, and System Response (2)



Frequency Response $H(e^{j\omega})$

- If the region of convergence, ROC, for H(z) includes the unit circle, the transfer function is related to the frequency response $H(e^{j\omega})$ as

$$H\left(e^{j\omega}\right) = H(z)\Big|_{z=e^{j\omega}}$$

- If **ROC contains the unit circle,** then $H(e^{j\omega})$ converges uniformly → **LTI system is stable**
- z-transform is important in general for digital filter design since it allows relatively easy characterizations and manipulations
- The Fourier transformation of the impulse response of an LTI system is called the frequency response of the system.

Evaluation of $H(e^{j\omega})$

$$H(e^{j\omega}) = H_{re}(e^{j\omega}) + jH_{im}(e^{j\omega}) = \left|H(e^{j\omega})\right| e^{j\arg[H(e^{j\omega})]}$$

- For a real coefficient transfer function

$$\left|H(e^{j\omega})\right|^2 = H(e^{j\omega})H^*(e^{j\omega}) = H(e^{j\omega})H(e^{-j\omega})$$

$$= H(z)H^*(z^{-1})\Big|_{z=e^{j\omega}}$$

- The values of the frequency response can be obtained by evaluating the z-transform on the unit circle in the z-plane, i.e., $H(e^{j\omega})$ is $H(z)$ at $z = e^{j\omega}$

Frequency Response Calculation

$H[z] = 1/(z+3)$

$$\begin{aligned} |H[e^{j\omega}]| &= \frac{1}{|e^{j\omega}+3|} \\ &= \frac{1}{|\cos\omega + 3 + j\sin\omega|} \\ &= \frac{1}{\sqrt{\cos^2\omega + 6\cos\omega + 9 + \sin^2\omega}} \\ &= \frac{1}{\sqrt{10+6\cos\omega}}. \end{aligned}$$

$$\angle H[e^{j\omega}] = -\tan^{-1}\left(\frac{\sin\omega}{\cos\omega + 3}\right)$$

Frequency Response Calculation

- Find the response of the discrete-time system, given by the following difference equation $y[n] = x[n] - y[n-2]$ to input $x[n] = cos\left(\frac{\pi n}{4}\right)$?

$$\mathbf{Y(z)} = \mathbf{X(z)} - \mathbf{z}^{-2}\,\mathbf{X(z)} = [1 - \mathbf{z}^{-2}]\,\mathbf{X(z)}.$$

The transfer function is

$$\mathbf{H(z)} = \frac{\mathbf{Y(z)}}{\mathbf{X(z)}} = \frac{1}{1-\mathbf{z}^{-2}}\frac{\mathbf{z}^2}{\mathbf{z}^2} = \frac{\mathbf{z}^2}{\mathbf{z}^2-1}.$$

Substituting $\mathbf{z} = e^{j\Omega}$ gives the frequency response function

$$\mathbf{H}(e^{j\Omega}) = \frac{e^{j2\Omega}}{e^{j2\Omega}-1}.$$

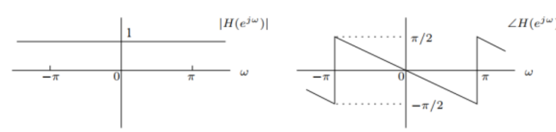
Substituting $\Omega = \frac{\pi}{4}$ gives

$$\mathbf{H}(e^{j\pi/4}) = \frac{e^{j\pi/2}}{e^{j\pi/2}-1} = \frac{j}{j-1} = \frac{1e^{j\pi/2}}{\sqrt{2}\,e^{j\pi/4}} = 0.707e^{j\pi/4}.$$

The response of the system to $x[n]$ is $\boxed{0.707\cos\left(\frac{\pi}{4}n + \frac{\pi}{4}\right).}$

Frequency Response Calculation

- A discrete-time LTI system has the following magnitude and phase response:
- Determine the output if the input is the signal $x[n] = e^{j\frac{5\pi n}{2}}$
- Determine and sketch the output if the input is the signal $x[n] = \cos\left(\frac{5\pi n}{2}\right)$.

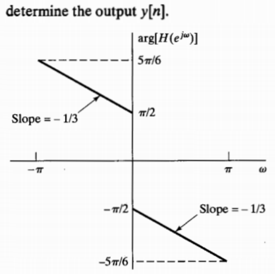


Frequency Response Calculation

Consider an LTI system with $|H(e^{j\omega})| = 1$, and let $\arg[H(e^{j\omega})]$ be as shown in Figure P2.33-1. If the input is

$$x[n] = \cos\left(\frac{3\pi}{2}n + \frac{\pi}{4}\right),$$

determine the output $y[n]$.



Frequency Response Calculation

Consider an LTI system with frequency response

$$H(e^{j\omega}) = \frac{1-e^{-j2\omega}}{1+\frac{1}{2}e^{-j4\omega}}, \qquad -\pi < \omega \le \pi.$$

Determine the output $y[n]$ for all n if the input $x[n]$ for all n is

$$x[n] = \sin\left(\frac{\pi n}{4}\right).$$

Stability Condition

- LTI system is BIBO stable if h[n] is absolutely summable, that is

$$S=\sum_{n=-\infty}^{\infty}|h[n]|<\infty$$

- For the transfer function, we have a bound

$$|H(z)|=\left|\sum_{n=-\infty}^{\infty}h[n]z^{-n}\right|\le\sum_{n=-\infty}^{\infty}\left|h[n]z^{-n}\right|=\sum_{n=-\infty}^{\infty}|h[n]|\left|z^{-n}\right|$$

which implies existence of DTFT since now

$$\left|H(e^{j\omega})\right|\le\sum_{n=-\infty}^{\infty}|h[n]|\left|e^{-j\omega n}\right|<\infty$$

absolutely summable h[n]

- Thus, BIBO stable system has always ROC that contains the unit circle

Stability Condition and Pole Locations

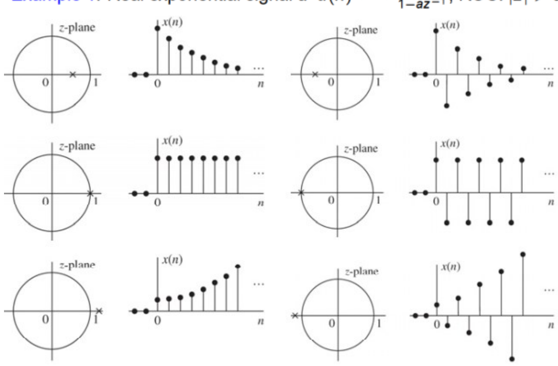
- The reverse is also true, i.e., if ROC contains the unit circle the system is BIBO stable

LTI system is BIBO stable if and only if the transfer function has a ROC that contains the unit circle

- Consequence of the above is that:
  - Causal FIR filter with bounded impulse response is always stable since all poles are at the origin
  - Causal IIR filter is stable if all poles are inside the unit circle (and for anti-causal IIR filter, outside of it)

Pole Location & TD Behavior (Causal Signals)

Example 1: Real exponential signal $a^n u(n) \overset{z}{\longleftrightarrow} \frac{1}{1-az^{-1}}$, ROC: $|z|>a$



Rational z-Transforms

- If X(z) is a rational function, then

$$X(z)=\frac{A(z)}{B(z)}=\frac{b_0+b_1z^{-1}+\cdots+b_Mz^{-M}}{a_0+a_1z^{-1}+\cdots+a_Nz^{-N}}$$

- If a_0, b_0 = 0, then negative powers z are avoided by factoring out b_0z^{-M} and a_0z^{-N}:

$$X(z)=\underbrace{\left(\frac{b_0z^{-M}}{a_0z^{-N}}\right)}_{\frac{b_0}{a_0}z^{N-M}}\underbrace{\frac{z^M+\frac{b_1}{b_0}z^{M-1}+\cdots+\frac{b_M}{b_0}}{z^N+\frac{a_1}{a_0}z^{N-1}+\cdots+\frac{a_N}{a_0}}}_{\text{can be factored}}$$

$$X(z)=\frac{b_0}{a_0}z^{N-M}\frac{(z-z_1)(z-z_2)\cdots(z-z_M)}{(z-p_1)(z-p_2)\cdots(z-p_N)}$$

$$=Gz^{N-M}\frac{\prod_{k=1}^{M}(z-z_k)}{\prod_{k=1}^{N}(z-p_k)}$$

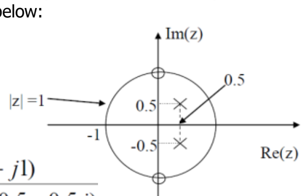
$$X(z)|_{z=z_k}=0,\quad k=1,2,\ldots,M \implies z_k \text{ are zeros of } X(z)$$

$$X(z)|_{z=p_k}=\infty,\quad k=1,2,\ldots,N \implies p_k \text{ are poles of } X(z)$$

Interpretation: X(z) = 0 at the zeros and X(z) = ∞ at the poles

Drill Problem

- Determine the transfer function H(z) of a discrete-time system with the pole-zero diagram shown below:



$$H(z)=\frac{K(z-j1)(z+j1)}{(z-0.5-0.5j)(z-0.5+0.5j)}$$

$$=\frac{K(1+z^{-2})}{1-z^{-1}-0.5z^{-2}}$$