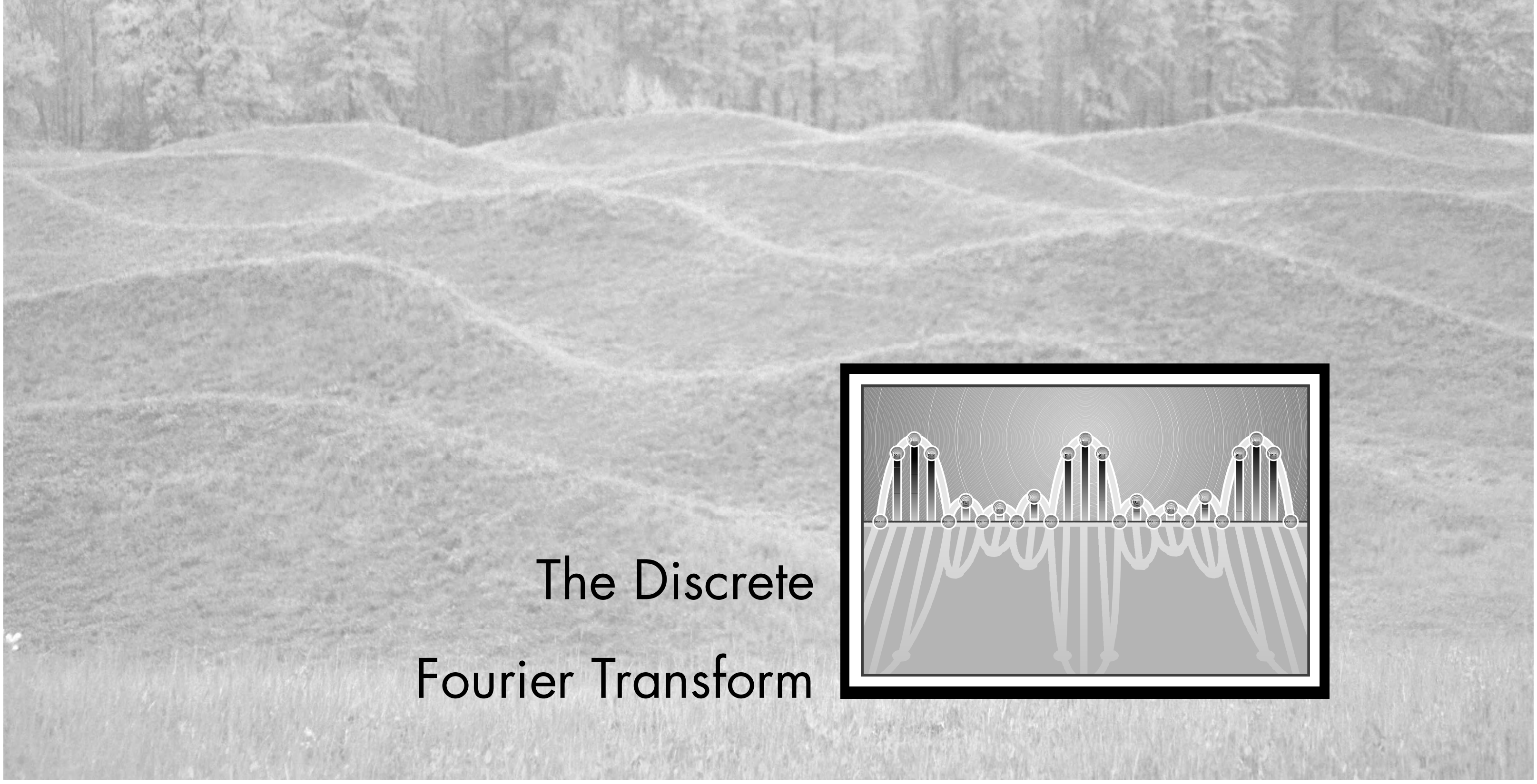


The Discrete Fourier Transform

0 INTRODUCTION

You should already be familiar with the representation of sequences and LTI systems in terms of the discrete-time Fourier and z-transforms, respectively. For finite-duration sequences, there is an alternative discrete-time Fourier representation, referred to as the *discrete Fourier transform* (DFT). The DFT is itself a sequence rather than a function of a continuous variable, and it corresponds to samples, equally spaced in frequency, of the DTFT of the signal. In addition to its theoretical importance as a Fourier representation of sequences, the DFT plays a central role in the implementation of a variety of digital signal-processing algorithms. This is because efficient algorithms exist for the computation of the DFT.

Although several points of view can be taken toward the derivation and interpretation of the DFT representation of a finite-duration sequence, we have chosen to base our presentation on the relationship between periodic sequences and finite-length sequences. We begin by considering the Fourier series representation of periodic sequences. Although this representation is important in its own right, we are most often interested in the application of Fourier series results to the representation of finite-length sequences. We accomplish this by constructing a periodic sequence for which each period is identical to the finite-length sequence. The Fourier series representation of the periodic sequence then corresponds to the DFT of the finite-length sequence. Thus, our approach is to define the Fourier series representation for periodic sequences and to study the properties of such representations. Then, we repeat essentially the same derivations, assuming that the sequence to be represented is a finite-length sequence.

From Chapter 8 of *Discrete-Time Signal Processing*, Third Edition, Alan V. Oppenheim, Ronald W. Schafer.

This approach to the DFT emphasizes the fundamental inherent periodicity of the DFT representation and ensures that this periodicity is not overlooked in applications of the DFT.

1 REPRESENTATION OF PERIODIC SEQUENCES: THE DISCRETE FOURIER SERIES

Consider a sequence $\tilde{x}[n]$ that is periodic[1] with period N, so that $\tilde{x}[n] = \tilde{x}[n + rN]$ for any integer values of n and r. As with continuous-time periodic signals, such a sequence can be represented by a Fourier series corresponding to a sum of harmonically related complex exponential sequences, i.e., complex exponentials with frequencies that are integer multiples of the fundamental frequency $(2\pi/N)$ associated with the periodic sequence $\tilde{x}[n]$. These periodic complex exponentials are of the form

$$e_k[n] = e^{j(2\pi/N)kn} = e_k[n + rN], \tag{1}$$

where k is any integer, and the Fourier series representation then has the form[2]

$$\tilde{x}[n] = \frac{1}{N}\sum_{k} \tilde{X}[k]e^{j(2\pi/N)kn}. \tag{2}$$

The Fourier series representation of a continuous-time periodic signal generally requires infinitely many harmonically related complex exponentials, whereas the Fourier series for any discrete-time signal with period N requires only N harmonically related complex exponentials. To see this, note that the harmonically related complex exponentials $e_k[n]$ in Eq. (1) are identical for values of k separated by N; i.e., $e_0[n] = e_N[n]$, $e_1[n] = e_{N+1}[n]$, and, in general,

$$e_{k+\ell N}[n] = e^{j(2\pi/N)(k+\ell N)n} = e^{j(2\pi/N)kn}e^{j2\pi\ell n} = e^{j(2\pi/N)kn} = e_k[n], \tag{3}$$

where ℓ is any integer. Consequently, the set of N periodic complex exponentials $e_0[n]$, $e_1[n], \ldots, e_{N-1}[n]$ defines all the distinct periodic complex exponentials with frequencies that are integer multiples of $(2\pi/N)$. Thus, the Fourier series representation of a periodic sequence $\tilde{x}[n]$ need contain only N of these complex exponentials. For notational convenience, we choose k in the range of 0 to $N - 1$; hence, Eq. (2) has the form

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k]e^{j(2\pi/N)kn}. \tag{4}$$

However, choosing k to range over any full period of $\tilde{X}[k]$ would be equally valid.

To obtain the sequence of Fourier series coefficients $\tilde{X}[k]$ from the periodic sequence $\tilde{x}[n]$, we exploit the orthogonality of the set of complex exponential sequences.

[1]Henceforth, we will use the tilde (˜) to denote periodic sequences whenever it is important to clearly distinguish between periodic and aperiodic sequences.

[2]The multiplicative constant $1/N$ is included in Eq. (2) for convenience. It could also be absorbed into the definition of $\tilde{X}[k]$.

After multiplying both sides of Eq. (4) by $e^{-j(2\pi/N)rn}$ and summing from $n = 0$ to $n = N-1$, we obtain

$$\sum_{n=0}^{N-1} \tilde{x}[n]e^{-j(2\pi/N)rn} = \sum_{n=0}^{N-1} \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k]e^{j(2\pi/N)(k-r)n}. \tag{5}$$

After interchanging the order of summation on the right-hand side, Eq. (5) becomes

$$\sum_{n=0}^{N-1} \tilde{x}[n]e^{-j(2\pi/N)rn} = \sum_{k=0}^{N-1} \tilde{X}[k] \left[\frac{1}{N} \sum_{n=0}^{N-1} e^{j(2\pi/N)(k-r)n} \right]. \tag{6}$$

The following identity expresses the orthogonality of the complex exponentials:

$$\frac{1}{N} \sum_{n=0}^{N-1} e^{j(2\pi/N)(k-r)n} = \begin{cases} 1, & k - r = mN, \quad m \text{ an integer}, \\ 0, & \text{otherwise}. \end{cases} \tag{7}$$

This identity can easily be proved (see Problem 54), and when it is applied to the summation in brackets in Eq. (6), the result is

$$\sum_{n=0}^{N-1} \tilde{x}[n]e^{-j(2\pi/N)rn} = \tilde{X}[r]. \tag{8}$$

Thus, the Fourier series coefficients $\tilde{X}[k]$ in Eq. (4) are obtained from $\tilde{x}[n]$ by the relation

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n]e^{-j(2\pi/N)kn}. \tag{9}$$

Note that the sequence $\tilde{X}[k]$ defined in Eq. (9) is also periodic with period N if Eq. (9) is evaluated outside the range $0 \le k \le N-1$; i.e., $\tilde{X}[0] = \tilde{X}[N]$, $\tilde{X}[1] = \tilde{X}[N+1]$, and, more generally,

$$\begin{aligned} \tilde{X}[k+N] &= \sum_{n=0}^{N-1} \tilde{x}[n]e^{-j(2\pi/N)(k+N)n} \\ &= \left(\sum_{n=0}^{N-1} \tilde{x}[n]e^{-j(2\pi/N)kn} \right) e^{-j2\pi n} = \tilde{X}[k], \end{aligned}$$

for any integer k.

The Fourier series coefficients can be interpreted to be a sequence of finite length, given by Eq. (9) for $k = 0, \ldots, (N-1)$, and zero otherwise, or as a periodic sequence defined for all k by Eq. (9). Clearly, both of these interpretations are acceptable, since in Eq. (4) we use only the values of $\tilde{X}[k]$ for $0 \le k \le (N-1)$. An advantage to interpreting the Fourier series coefficients $\tilde{X}[k]$ as a periodic sequence is that there is then a duality between the time and frequency domains for the Fourier series representation of periodic sequences. Equations (9) and (4) together are an analysis–synthesis pair and will be referred to as the *discrete Fourier series* (DFS) representation of a periodic sequence.

For convenience in notation, these equations are often written in terms of the complex quantity

$$W_N = e^{-j(2\pi/N)}. \tag{10}$$

With this notation, the DFS analysis–synthesis pair is expressed as follows:

$$\text{Analysis equation:} \quad \tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn}. \tag{11}$$

$$\text{Synthesis equation:} \quad \tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-kn}. \tag{12}$$

In both of these equations, $\tilde{X}[k]$ and $\tilde{x}[n]$ are periodic sequences. We will sometimes find it convenient to use the notation

$$\tilde{x}[n] \stackrel{\mathcal{DFS}}{\longleftrightarrow} \tilde{X}[k] \tag{13}$$

to signify the relationships of Eqs. (11) and (12). The following examples illustrate the use of those equations.

Example 1 DFS of a Periodic Impulse Train

We consider the periodic impulse train

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} \delta[n - rN] = \begin{cases} 1, & n = rN, \quad r \text{ any integer}, \\ 0, & \text{otherwise}. \end{cases} \tag{14}$$

Since $\tilde{x}[n] = \delta[n]$ for $0 \le n \le N-1$, the DFS coefficients are found, using Eq. (11), to be

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \delta[n] W_N^{kn} = W_N^0 = 1. \tag{15}$$

In this case, $\tilde{X}[k] = 1$ for all k. Thus, substituting Eq. (15) into Eq. (12) leads to the representation

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} \delta[n - rN] = \frac{1}{N} \sum_{k=0}^{N-1} W_N^{-kn} = \frac{1}{N} \sum_{k=0}^{N-1} e^{j(2\pi/N)kn}. \tag{16}$$

Example 1 produced a useful representation of a periodic impulse train in terms of a sum of complex exponentials, wherein all the complex exponentials have the same magnitude and phase and add to unity at integer multiples of N and to zero for all other integers. If we look closely at Eqs. (11) and (12), we see that the two equations are very similar, differing only in a constant multiplier and the sign of the exponents. This duality between the periodic sequence $\tilde{x}[n]$ and its DFS coefficients $\tilde{X}[k]$ is illustrated in the following example.

Example 2 Duality in the DFS

In this example, the DFS coefficients are a periodic impulse train:

$$\tilde{Y}[k] = \sum_{r=-\infty}^{\infty} N\delta[k - rN].$$

Substituting $\tilde{Y}[k]$ into Eq. (12) gives

$$\tilde{y}[n] = \frac{1}{N}\sum_{k=0}^{N-1} N\delta[k]W_N^{-kn} = W_N^{-0} = 1.$$

In this case, $\tilde{y}[n] = 1$ for all n. Comparing this result with the results for $\tilde{x}[n]$ and $\tilde{X}[k]$ of Example 1, we see that $\tilde{Y}[k] = N\tilde{x}[k]$ and $\tilde{y}[n] = \tilde{X}[n]$. In Section 2.3, we will show that this example is a special case of a more general duality property.

If the sequence $\tilde{x}[n]$ is equal to unity over only part of one period, we can also obtain a closed-form expression for the DFS coefficients. This is illustrated by the following example.

Example 3 The DFS of a Periodic Rectangular Pulse Train

For this example, $\tilde{x}[n]$ is the sequence shown in Figure 1, whose period is $N = 10$. From Eq. (11),

$$\tilde{X}[k] = \sum_{n=0}^{4} W_{10}^{kn} = \sum_{n=0}^{4} e^{-j(2\pi/10)kn}. \tag{17}$$

This finite sum has the closed form

$$\tilde{X}[k] = \frac{1 - W_{10}^{5k}}{1 - W_{10}^{k}} = e^{-j(4\pi k/10)}\frac{\sin(\pi k/2)}{\sin(\pi k/10)}. \tag{18}$$

The magnitude and phase of the periodic sequence $\tilde{X}[k]$ are shown in Figure 2.

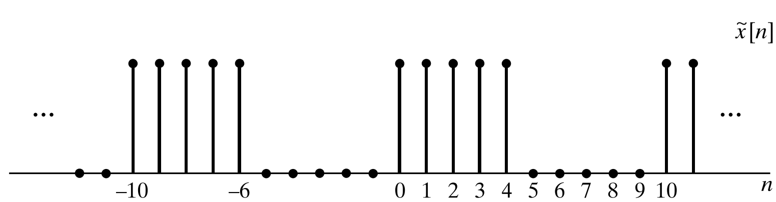


Figure 1 Periodic sequence with period $N = 10$ for which the Fourier series representation is to be computed.

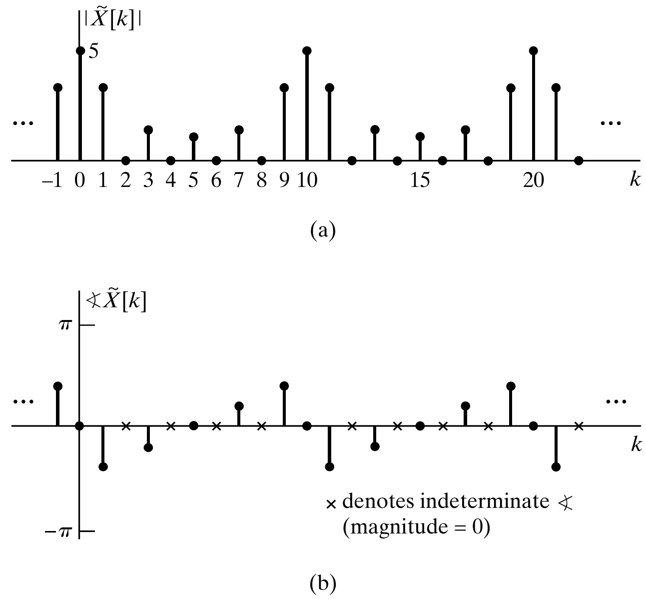


Figure 2 Magnitude and phase of the Fourier series coefficients of the sequence of Figure 1.

We have shown that any periodic sequence can be represented as a sum of complex exponential sequences. The key results are summarized in Eqs. (11) and (12). As we will see, these relationships are the basis for the DFT, which focuses on finite-length sequences. Before discussing the DFT, however, we will consider some of the basic properties of the DFS representation of periodic sequences in Section 2, and then, in Section 3, we will show how we can use the DFS representation to obtain a DTFT representation of periodic signals.

2 PROPERTIES OF THE DFS

Just as with Fourier series and Fourier and Laplace transforms for continuous-time signals, and with discrete-time Fourier and z-transforms for nonperiodic sequences, certain properties of the DFS are of fundamental importance to its successful use in signal-processing problems. In this section, we summarize these important properties. It is not surprising that many of the basic properties are analogous to properties of the z-transform and DTFT. However, we will be careful to point out where the periodicity of both $\tilde{x}[n]$ and $\tilde{X}[k]$ results in some important distinctions. Furthermore, an exact duality exists between the time and frequency domains in the DFS representation that does not exist in the DTFT and z-transform representation of sequences.

2.1 Linearity

Consider two periodic sequences $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$, both with period N, such that

$$\tilde{x}_1[n] \overset{\mathcal{DFS}}{\longleftrightarrow} \tilde{X}_1[k], \tag{19a}$$

and

$$\tilde{x}_2[n] \overset{\mathcal{DFS}}{\longleftrightarrow} \tilde{X}_2[k]. \tag{19b}$$

Then

$$a\tilde{x}_1[n] + b\tilde{x}_2[n] \overset{\mathcal{DFS}}{\longleftrightarrow} a\tilde{X}_1[k] + b\tilde{X}_2[k]. \tag{20}$$

This linearity property follows immediately from the form of Eqs. (11) and (12).

2.2 Shift of a Sequence

If a periodic sequence $\tilde{x}[n]$ has Fourier coefficients $\tilde{X}[k]$, then $\tilde{x}[n-m]$ is a shifted version of $\tilde{x}[n]$, and

$$\tilde{x}[n-m] \overset{\mathcal{DFS}}{\longleftrightarrow} W_N^{km}\tilde{X}[k]. \tag{21}$$

The proof of this property is considered in Problem 55. Note that any shift that is greater than or equal to the period (i.e., $m \geq N$) cannot be distinguished in the time domain from a shorter shift m_1 such that $m = m_1 + m_2N$, where m_1 and m_2 are integers and $0 \leq m_1 \leq N-1$. (Another way of stating this is that $m_1 = m$ modulo N or, equivalently, m_1 is the remainder when m is divided by N.) It is easily shown that with this representation of m, $W_N^{km} = W_N^{km_1}$; i.e., as it must be, the ambiguity of the shift in the time domain is also manifest in the frequency-domain representation.

Because the sequence of Fourier series coefficients of a periodic sequence is a periodic sequence, a similar result applies to a shift in the Fourier coefficients by an integer ℓ. Specifically,

$$W_N^{-n\ell}\tilde{x}[n] \overset{\mathcal{DFS}}{\longleftrightarrow} \tilde{X}[k-\ell]. \tag{22}$$

Note the difference in the sign of the exponents in Eqs. (21) and (22).

2.3 Duality

Because of the strong similarity between the Fourier analysis and synthesis equations in continuous time, there is a duality between the time domain and frequency domain. However, for the DTFT of aperiodic signals, no similar duality exists, since aperiodic signals and their Fourier transforms are very different kinds of functions: Aperiodic discrete-time signals are, of course, aperiodic sequences, whereas their DTFTs are always periodic functions of a continuous frequency variable.

From Eqs. (11) and (12), we see that the DFS analysis and synthesis equations differ only in a factor of $1/N$ and in the sign of the exponent of W_N. Furthermore, a periodic sequence and its DFS coefficients are the same kinds of functions; they are both periodic sequences. Specifically, taking account of the factor $1/N$ and the difference in sign in the exponent between Eqs. (11) and (12), it follows from Eq. (12) that

$$N\tilde{x}[-n] = \sum_{k=0}^{N-1} \tilde{X}[k]W_N^{kn} \tag{23}$$

or, interchanging the roles of n and k in Eq. (23),

$$N\tilde{x}[-k] = \sum_{n=0}^{N-1} \tilde{X}[n]W_N^{nk}. \tag{24}$$

We see that Eq. (24) is similar to Eq. (11). In other words, the sequence of DFS coefficients of the periodic sequence $\tilde{X}[n]$ is $N\tilde{x}[-k]$, i.e., the original periodic sequence in reverse order and multiplied by N. This duality property is summarized as follows: If

$$\tilde{x}[n] \overset{\mathcal{DFS}}{\longleftrightarrow} \tilde{X}[k], \tag{25a}$$

then

$$\tilde{X}[n] \overset{\mathcal{DFS}}{\longleftrightarrow} N\tilde{x}[-k]. \tag{25b}$$

2.4 Symmetry Properties

The Fourier transform of an aperiodic sequence has a number of useful symmetry properties. The same basic properties also hold for the DFS representation of a periodic sequence. The derivation of these properties is left as an exercise. (See Problem 56.) The resulting properties are summarized for reference as properties 9–17 in Table 1 in Section 2.6.

2.5 Periodic Convolution

Let $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$ be two periodic sequences, each with period N and with DFS coefficients denoted by $\tilde{X}_1[k]$ and $\tilde{X}_2[k]$, respectively. If we form the product

$$\tilde{X}_3[k] = \tilde{X}_1[k]\tilde{X}_2[k], \tag{26}$$

then the periodic sequence $\tilde{x}_3[n]$ with Fourier series coefficients $\tilde{X}_3[k]$ is

$$\tilde{x}_3[n] = \sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m]. \tag{27}$$

This result is not surprising, since our previous experience with transforms suggests that multiplication of frequency-domain functions corresponds to convolution of time-domain functions and Eq. (27) looks very much like a convolution sum. Equation (27) involves the summation of values of the product of $\tilde{x}_1[m]$ with $\tilde{x}_2[n-m]$, which is a time-reversed and time-shifted version of $\tilde{x}_2[m]$, just as in aperiodic discrete convolution. However, the sequences in Eq. (27) are all periodic with period N, and the summation is over only one period. A convolution in the form of Eq. (27) is referred to as a *periodic convolution*. Just as with aperiodic convolution, periodic convolution is commutative; i.e.,

$$\tilde{x}_3[n] = \sum_{m=0}^{N-1} \tilde{x}_2[m]\tilde{x}_1[n-m]. \tag{28}$$

To demonstrate that $\tilde{X}_3[k]$, given by Eq. (26), is the sequence of Fourier coefficients corresponding to $\tilde{x}_3[n]$ given by Eq. (27), let us first apply Eq. (11), the DFS analysis equation, to Eq. (27) to obtain

$$\tilde{X}_3[k] = \sum_{n=0}^{N-1} \left(\sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m] \right) W_N^{kn}, \tag{29}$$

which, after we interchange the order of summation, becomes

$$\tilde{X}_3[k] = \sum_{m=0}^{N-1} \tilde{x}_1[m] \left(\sum_{n=0}^{N-1} \tilde{x}_2[n-m] W_N^{kn} \right). \tag{30}$$

The inner sum on the index n is the DFS for the shifted sequence $\tilde{x}_2[n-m]$. Therefore, from the shifting property of Section 2.2, we obtain

$$\sum_{n=0}^{N-1} \tilde{x}_2[n-m] W_N^{kn} = W_N^{km} \tilde{X}_2[k],$$

which can be substituted into Eq. (30) to yield

$$\tilde{X}_3[k] = \sum_{m=0}^{N-1} \tilde{x}_1[m] W_N^{km} \tilde{X}_2[k] = \left(\sum_{m=0}^{N-1} \tilde{x}_1[m] W_N^{km} \right) \tilde{X}_2[k] = \tilde{X}_1[k]\tilde{X}_2[k]. \tag{31}$$

In summary,

$$\sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m] \overset{\mathcal{DFS}}{\longleftrightarrow} \tilde{X}_1[k]\tilde{X}_2[k]. \tag{32}$$

The periodic convolution of periodic sequences thus corresponds to multiplication of the corresponding periodic sequences of Fourier series coefficients.

Since periodic convolutions are somewhat different from aperiodic convolutions, it is worthwhile to consider the mechanics of evaluating Eq. (27). First, note that Eq. (27) calls for the product of sequences $\tilde{x}_1[m]$ and $\tilde{x}_2[n-m] = \tilde{x}_2[-(m-n)]$ viewed as functions of m with n fixed. This is the same as for an aperiodic convolution, but with the following two major differences:

1. The sum is over the finite interval $0 \le m \le N-1$.
2. The values of $\tilde{x}_2[n-m]$ in the interval $0 \le m \le N-1$ repeat periodically for m outside of that interval.

These details are illustrated by the following example.

Example 4 Periodic Convolution

An illustration of the procedure for forming the periodic convolution of two periodic sequences corresponding to Eq. (27) is given in Figure 3, wherein we have illustrated the sequences $\tilde{x}_2[m]$, $\tilde{x}_1[m]$, $\tilde{x}_2[-m]$, $\tilde{x}_2[1-m] = \tilde{x}_2[-(m-1)]$, and $\tilde{x}_2[2-m] = \tilde{x}_2[-(m-2)]$. To evaluate $\tilde{x}_3[n]$ in Eq. (27) for $n = 2$, for example, we multiply $\tilde{x}_1[m]$ by $\tilde{x}_2[2-m]$ and then sum the product terms $\tilde{x}_1[m]\tilde{x}_2[2-m]$ for $0 \le m \le N-1$, obtaining $\tilde{x}_3[2]$. As n changes, the sequence $\tilde{x}_2[n-m]$ shifts appropriately, and Eq. (27) is evaluated for each value of $0 \le n \le N-1$. Note that as the sequence $\tilde{x}_2[n-m]$ shifts to the right or left, values that leave the interval between the dotted lines at one end reappear at the other end because of the periodicity. Because of the periodicity of $\tilde{x}_3[n]$, there is no need to continue to evaluate Eq. (27) outside the interval $0 \le n \le N-1$.

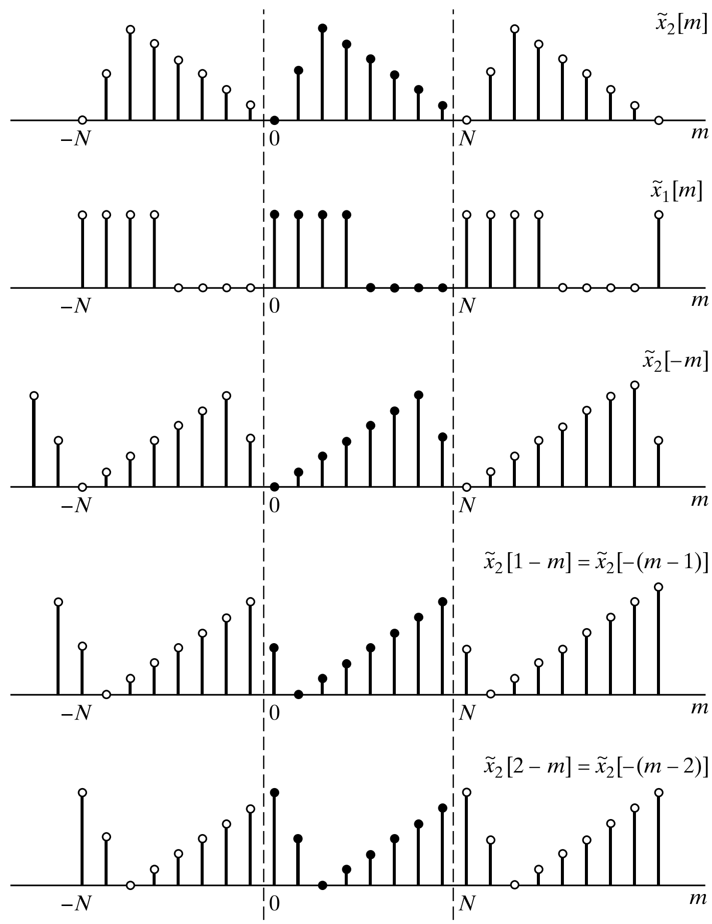


Figure 3 Procedure for forming the periodic convolution of two periodic sequences.

The duality theorem in Section 2.3 suggests that if the roles of time and frequency are interchanged, we will obtain a result almost identical to the previous result. That is, the periodic sequence

$$\tilde{x}_3[n] = \tilde{x}_1[n]\tilde{x}_2[n], \tag{33}$$

where $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$ are periodic sequences, each with period N, has the DFS coefficients given by

$$\tilde{X}_3[k] = \frac{1}{N}\sum_{\ell=0}^{N-1} \tilde{X}_1[\ell]\tilde{X}_2[k-\ell], \tag{34}$$

corresponding to $1/N$ times the periodic convolution of $\tilde{X}_1[k]$ and $\tilde{X}_2[k]$. This result can also be verified by substituting $\tilde{X}_3[k]$, given by Eq. (34), into the Fourier series relation of Eq. (12) to obtain $\tilde{x}_3[n]$.

2.6 Summary of Properties of the DFS Representation of Periodic Sequences

The properties of the DFS representation discussed in this section are summarized in Table 1.

3 THE FOURIER TRANSFORM OF PERIODIC SIGNALS

Uniform convergence of the Fourier transform of a sequence requires that the sequence be absolutely summable, and mean-square convergence requires that the sequence be square summable. Periodic sequences satisfy neither condition. However, sequences that can be expressed as a sum of complex exponentials can be considered to have a Fourier transform representation as a train of impulses. Similarly, it is often useful to incorporate the DFS representation of periodic signals within the framework of the discrete-time Fourier transform. This can be done by interpreting the discrete-time Fourier transform of a periodic signal to be an impulse train in the frequency domain with the impulse values proportional to the DFS coefficients for the sequence. Specifically, if $\tilde{x}[n]$ is periodic with period N and the corresponding DFS coefficients are $\tilde{X}[k]$, then the Fourier transform of $\tilde{x}[n]$ is defined to be the impulse train

$$\tilde{X}(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{N}\tilde{X}[k]\delta\left(\omega - \frac{2\pi k}{N}\right). \tag{35}$$

Note that $\tilde{X}(e^{j\omega})$ has the necessary periodicity with period 2π since $\tilde{X}[k]$ is periodic with period N, and the impulses are spaced at integer multiples of $2\pi/N$, where N is an

TABLE 1 SUMMARY OF PROPERTIES OF THE DFS

Periodic Sequence (Period N)	DFS Coefficients (Period N)
1. $\tilde{x}[n]$	$\tilde{X}[k]$ periodic with period N
2. $\tilde{x}_1[n], \tilde{x}_2[n]$	$\tilde{X}_1[k], \tilde{X}_2[k]$ periodic with period N
3. $a\tilde{x}_1[n] + b\tilde{x}_2[n]$	$a\tilde{X}_1[k] + b\tilde{X}_2[k]$
4. $\tilde{X}[n]$	$N\tilde{x}[-k]$
5. $\tilde{x}[n-m]$	$W_N^{km}\tilde{X}[k]$
6. $W_N^{-\ell n}\tilde{x}[n]$	$\tilde{X}[k-\ell]$
7. $\sum_{m=0}^{N-1}\tilde{x}_1[m]\tilde{x}_2[n-m]$ (periodic convolution)	$\tilde{X}_1[k]\tilde{X}_2[k]$
8. $\tilde{x}_1[n]\tilde{x}_2[n]$	$\frac{1}{N}\sum_{\ell=0}^{N-1}\tilde{X}_1[\ell]\tilde{X}_2[k-\ell]$ (periodic convolution)
9. $\tilde{x}^*[n]$	$\tilde{X}^*[-k]$
10. $\tilde{x}^*[-n]$	$\tilde{X}^*[k]$
11. $\mathcal{R}e\{\tilde{x}[n]\}$	$\tilde{X}_e[k] = \frac{1}{2}(\tilde{X}[k] + \tilde{X}^*[-k])$
12. $j\mathcal{I}m\{\tilde{x}[n]\}$	$\tilde{X}_o[k] = \frac{1}{2}(\tilde{X}[k] - \tilde{X}^*[-k])$
13. $\tilde{x}_e[n] = \frac{1}{2}(\tilde{x}[n] + \tilde{x}^*[-n])$	$\mathcal{R}e\{\tilde{X}[k]\}$
14. $\tilde{x}_o[n] = \frac{1}{2}(\tilde{x}[n] - \tilde{x}^*[-n])$	$j\mathcal{I}m\{\tilde{X}[k]\}$
Properties 15–17 apply only when $x[n]$ is real.	
15. Symmetry properties for $\tilde{x}[n]$ real.	$\begin{cases} \tilde{X}[k] = \tilde{X}^*[-k] \\ \mathcal{R}e\{\tilde{X}[k]\} = \mathcal{R}e\{\tilde{X}[-k]\} \\ \mathcal{I}m\{\tilde{X}[k]\} = -\mathcal{I}m\{\tilde{X}[-k]\} \\ \lvert\tilde{X}[k]\rvert = \lvert\tilde{X}[-k]\rvert \\ \angle\tilde{X}[k] = -\angle\tilde{X}[-k] \end{cases}$
16. $\tilde{x}_e[n] = \frac{1}{2}(\tilde{x}[n] + \tilde{x}[-n])$	$\mathcal{R}e\{\tilde{X}[k]\}$
17. $\tilde{x}_0[n] = \frac{1}{2}(\tilde{x}[n] - \tilde{x}[-n])$	$j\mathcal{I}m\{\tilde{X}[k]\}$

integer. To show that $\tilde{X}(e^{j\omega})$ as defined in Eq. (35) is a Fourier transform representation of the periodic sequence $\tilde{x}[n]$, we substitute Eq. (35) into an inverse Fourier transform; i.e.,

$$\frac{1}{2\pi}\int_{0-\epsilon}^{2\pi-\epsilon}\tilde{X}(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{0-\epsilon}^{2\pi-\epsilon}\sum_{k=-\infty}^{\infty}\frac{2\pi}{N}\tilde{X}[k]\delta\left(\omega - \frac{2\pi k}{N}\right)e^{j\omega n}d\omega, \tag{36}$$

where ϵ satisfies the inequality $0 < \epsilon < (2\pi/N)$. Recall that in evaluating the inverse Fourier transform, we can integrate over any interval of length 2π, since the integrand $\tilde{X}(e^{j\omega})e^{j\omega n}$ is periodic with period 2π. In Eq. (36) the integration limits are denoted $0-\epsilon$ and $2\pi - \epsilon$, which means that the integration is from just before $\omega = 0$ to just before $\omega = 2\pi$. These limits are convenient, because they include the impulse at $\omega = 0$ and exclude the impulse at $\omega = 2\pi$.[3] Interchanging the order of integration and summation leads to

$$\begin{aligned}\frac{1}{2\pi}\int_{0-\epsilon}^{2\pi-\epsilon} \tilde{X}(e^{j\omega})e^{j\omega n}d\omega &= \frac{1}{N}\sum_{k=-\infty}^{\infty}\tilde{X}[k]\int_{0-\epsilon}^{2\pi-\epsilon}\delta\left(\omega - \frac{2\pi k}{N}\right)e^{j\omega n}d\omega \\ &= \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]e^{j(2\pi/N)kn}.\end{aligned} \tag{37}$$

The final form of Eq. (37) results because only the impulses corresponding to $k = 0, 1, \ldots, (N-1)$ are included in the interval between $\omega = 0 - \epsilon$ and $\omega = 2\pi - \epsilon$.

Comparing Eq. (37) and Eq. (12), we see that the final right-hand side of Eq. (37) is exactly equal to the Fourier series representation for $\tilde{x}[n]$, as specified by Eq. (12). Consequently, the inverse Fourier transform of the impulse train in Eq. (35) is the periodic signal $\tilde{x}[n]$, as desired.

Although the Fourier transform of a periodic sequence does not converge in the normal sense, the introduction of impulses permits us to include periodic sequences formally within the framework of Fourier transform analysis. Although the DFS representation is adequate for most purposes, the Fourier transform representation of Eq. (35) sometimes leads to simpler or more compact expressions and simplified analysis.

Example 5 The Fourier Transform of a Periodic Discrete-Time Impulse Train

Consider the periodic discrete-time impulse train

$$\tilde{p}[n] = \sum_{r=-\infty}^{\infty}\delta[n - rN], \tag{38}$$

which is the same as the periodic sequence $\tilde{x}[n]$ considered in Example 1. From the results of that example, it follows that

$$\tilde{P}[k] = 1, \qquad \text{for all } k. \tag{39}$$

Therefore, the DTFT of $\tilde{p}[n]$ is

$$\tilde{P}(e^{j\omega}) = \sum_{k=-\infty}^{\infty}\frac{2\pi}{N}\delta\left(\omega - \frac{2\pi k}{N}\right). \tag{40}$$

The result of Example 5 is the basis for a useful interpretation of the relation between a periodic signal and a finite-length signal. Consider a finite-length signal $x[n]$ such that $x[n] = 0$ except in the interval $0 \le n \le N-1$, and consider the convolution

[3]The limits 0 to 2π would present a problem since the impulses at both 0 and 2π would require special handling.

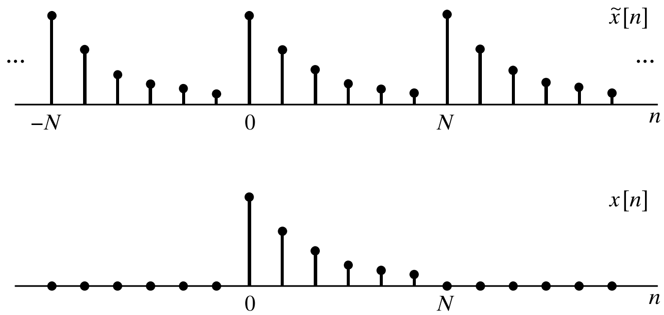


Figure 4 Periodic sequence $\tilde{x}[n]$ formed by repeating a finite-length sequence, $x[n]$, periodically. Alternatively, $x[n] = \tilde{x}[n]$ over one period and is zero otherwise.

of $x[n]$ with the periodic impulse train $\tilde{p}[n]$ of Example 5:

$$\tilde{x}[n] = x[n] * \tilde{p}[n] = x[n] * \sum_{r=-\infty}^{\infty} \delta[n - rN] = \sum_{r=-\infty}^{\infty} x[n - rN]. \tag{41}$$

Equation (41) states that $\tilde{x}[n]$ consists of a set of periodically repeated copies of the finite-length sequence $x[n]$. Figure 4 illustrates how a periodic sequence $\tilde{x}[n]$ can be formed from a finite-length sequence $x[n]$ through Eq. (41). The Fourier transform of $x[n]$ is $X(e^{j\omega})$, and the Fourier transform of $\tilde{x}[n]$ is

$$\begin{aligned}\tilde{X}(e^{j\omega}) &= X(e^{j\omega})\tilde{P}(e^{j\omega}) \\ &= X(e^{j\omega}) \sum_{k=-\infty}^{\infty} \frac{2\pi}{N}\delta\left(\omega - \frac{2\pi k}{N}\right) \\ &= \sum_{k=-\infty}^{\infty} \frac{2\pi}{N} X(e^{j(2\pi/N)k})\delta\left(\omega - \frac{2\pi k}{N}\right).\end{aligned} \tag{42}$$

Comparing Eq. (42) with Eq. (35), we conclude that

$$\tilde{X}[k] = X(e^{j(2\pi/N)k}) = X(e^{j\omega})\big|_{\omega=(2\pi/N)k}. \tag{43}$$

In other words, the periodic sequence $\tilde{X}[k]$ of DFS coefficients in Eq. (11) has an discrete-time interpretation as equally spaced samples of the DTFT of the finite-length sequence obtained by extracting one period of $\tilde{x}[n]$; i.e.,

$$x[n] = \begin{cases} \tilde{x}[n], & 0 \le n \le N-1, \\ 0, & \text{otherwise.} \end{cases} \tag{44}$$

This is also consistent with Figure 4, where it is clear that $x[n]$ can be obtained from $\tilde{x}[n]$ using Eq. (44). We can verify Eq. (43) in yet another way. Since $x[n] = \tilde{x}[n]$ for $0 \le n \le N-1$ and $x[n] = 0$ otherwise,

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n]e^{-j\omega n} = \sum_{n=0}^{N-1} \tilde{x}[n]e^{-j\omega n}. \tag{45}$$

Comparing Eq. (45) and Eq. (11), we see again that

$$\tilde{X}[k] = X(e^{j\omega})|_{\omega=2\pi k/N}. \tag{46}$$

This corresponds to sampling the Fourier transform at N equally spaced frequencies between $\omega = 0$ and $\omega = 2\pi$ with a frequency spacing of $2\pi/N$.

Example 6 Relationship Between the Fourier Series Coefficients and the Fourier Transform of One Period

We again consider the sequence $\tilde{x}[n]$ of Example 3, which is shown in Figure 1. One period of $\tilde{x}[n]$ for the sequence in Figure 1 is

$$x[n] = \begin{cases} 1, & 0 \le n \le 4, \\ 0, & \text{otherwise.} \end{cases} \tag{47}$$

The Fourier transform of one period of $\tilde{x}[n]$ is given by

$$X(e^{j\omega}) = \sum_{n=0}^{4} e^{-j\omega n} = e^{-j2\omega} \frac{\sin(5\omega/2)}{\sin(\omega/2)}. \tag{48}$$

Equation (46) can be shown to be satisfied for this example by substituting $\omega = 2\pi k/10$ into Eq. (48), giving

$$\tilde{X}[k] = e^{-j(4\pi k/10)} \frac{\sin(\pi k/2)}{\sin(\pi k/10)},$$

which is identical to the result in Eq. (18). The magnitude and phase of $X(e^{j\omega})$ are sketched in Figure 5. Note that the phase is discontinuous at the frequencies where $X(e^{j\omega}) = 0$. That the sequences in Figures 2(a) and (b) correspond to samples of Figures 5(a) and (b), respectively, is demonstrated in Figure 6, where Figures 2 and 5 have been superimposed.

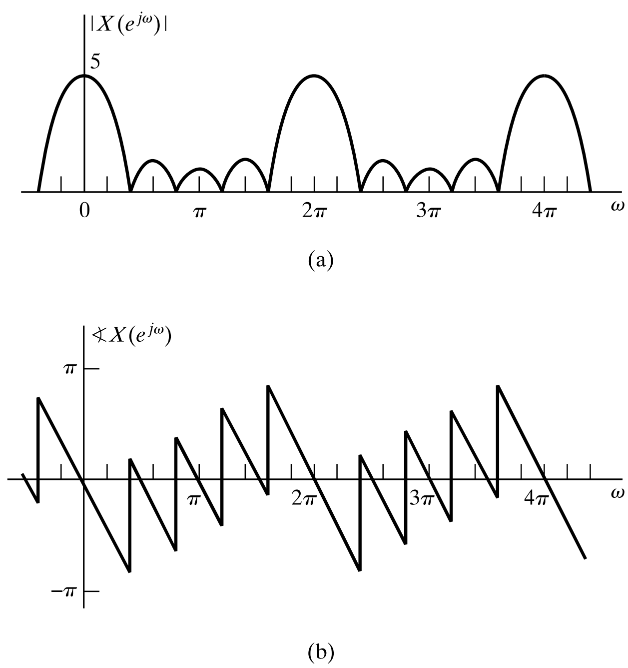


Figure 5 Magnitude and phase of the Fourier transform of one period of the sequence in Figure 1.

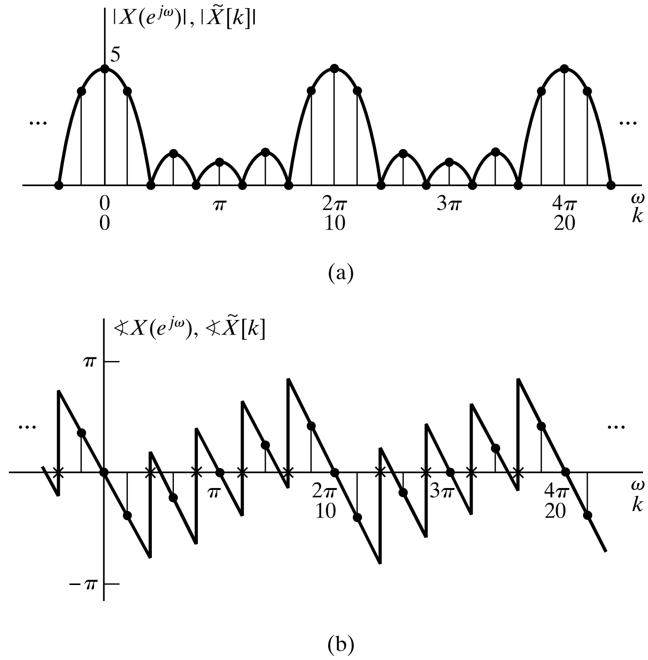


Figure 6 Overlay of Figures 2 and 5 illustrating the DFS coefficients of a periodic sequence as samples of the Fourier transform of one period.

4 SAMPLING THE FOURIER TRANSFORM

In this section, we discuss with more generality the relationship between an aperiodic sequence with Fourier transform $X(e^{j\omega})$ and the periodic sequence for which the DFS coefficients correspond to samples of $X(e^{j\omega})$ equally spaced in frequency. We will find this relationship to be particularly important when we discuss the discrete Fourier transform and its properties later in the chapter.

Consider an aperiodic sequence $x[n]$ with Fourier transform $X(e^{j\omega})$, and assume that a sequence $\tilde{X}[k]$ is obtained by sampling $X(e^{j\omega})$ at frequencies $\omega_k = 2\pi k/N$; i.e.,

$$\tilde{X}[k] = X(e^{j\omega})|_{\omega=(2\pi/N)k} = X(e^{j(2\pi/N)k}). \tag{49}$$

Since the Fourier transform is periodic in ω with period 2π, the resulting sequence is periodic in k with period N. Also, since the Fourier transform is equal to the z-transform evaluated on the unit circle, it follows that $\tilde{X}[k]$ can also be obtained by sampling $X(z)$ at N equally spaced points on the unit circle. Thus,

$$\tilde{X}[k] = X(z)|_{z=e^{j(2\pi/N)k}} = X(e^{j(2\pi/N)k}). \tag{50}$$

These sampling points are depicted in Figure 7 for $N = 8$. The figure makes it clear that the sequence of samples is periodic, since the N points are equally spaced starting with zero angle. Therefore, the same sequence repeats as k varies outside the range $0 \le k \le N-1$ since we simply continue around the unit circle visiting the same set of N points.

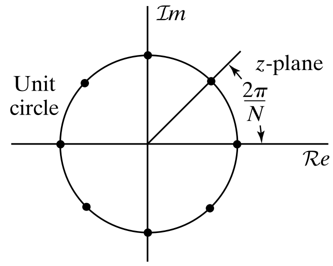


Figure 7 Points on the unit circle at which $X(z)$ is sampled to obtain the periodic sequence $\tilde{X}[k]$ ($N = 8$).

Note that the sequence of samples $\tilde{X}[k]$, being periodic with period N, could be the sequence of DFS coefficients of a sequence $\tilde{x}[n]$. To obtain that sequence, we can simply substitute $\tilde{X}[k]$ obtained by sampling into Eq. (12):

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]W_N^{-kn}. \tag{51}$$

Since we have made no assumption about $x[n]$ other than that the Fourier transform exists, we can use infinite limits to indicate that the sum is

$$X(e^{j\omega}) = \sum_{m=-\infty}^{\infty} x[m]e^{-j\omega m} \tag{52}$$

is over all nonzero values of $x[m]$.

Substituting Eq. (52) into Eq. (49) and then substituting the resulting expression for $\tilde{X}[k]$ into Eq. (51) gives

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_{m=-\infty}^{\infty} x[m]e^{-j(2\pi/N)km}\right]W_N^{-kn}, \tag{53}$$

which, after we interchange the order of summation, becomes

$$\tilde{x}[n] = \sum_{m=-\infty}^{\infty} x[m]\left[\frac{1}{N}\sum_{k=0}^{N-1} W_N^{-k(n-m)}\right] = \sum_{m=-\infty}^{\infty} x[m]\tilde{p}[n-m]. \tag{54}$$

The term in brackets in Eq. (54) can be seen from either Eq. (7) or Eq. (16) to be the Fourier series representation of the periodic impulse train of Examples 1 and 2. Specifically,

$$\tilde{p}[n-m] = \frac{1}{N}\sum_{k=0}^{N-1} W_N^{-k(n-m)} = \sum_{r=-\infty}^{\infty}\delta[n-m-rN] \tag{55}$$

and therefore,

$$\tilde{x}[n] = x[n] * \sum_{r=-\infty}^{\infty}\delta[n-rN] = \sum_{r=-\infty}^{\infty} x[n-rN], \tag{56}$$

where $*$ denotes aperiodic convolution. That is, $\tilde{x}[n]$ is the periodic sequence that results from the aperiodic convolution of $x[n]$ with a periodic unit-impulse train. Thus, the

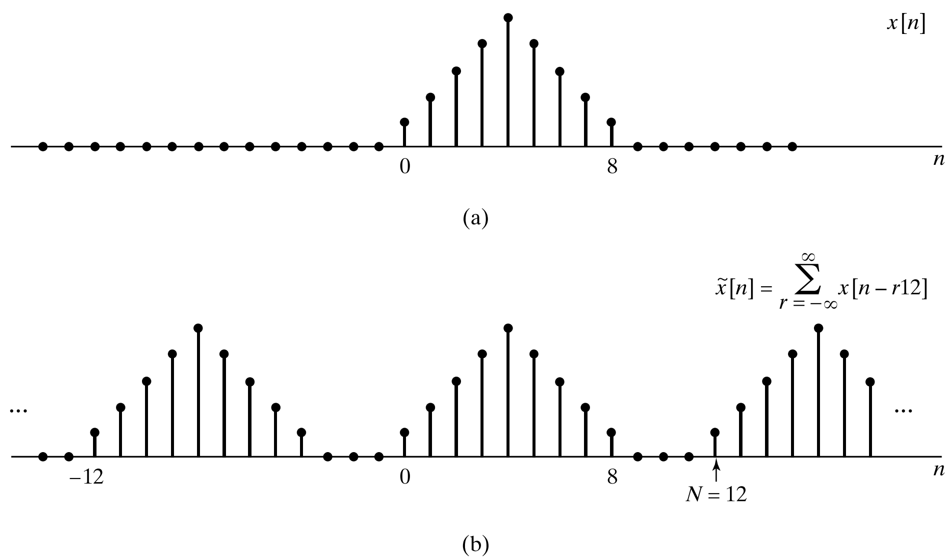


Figure 8 (a) Finite-length sequence $x[n]$. (b) Periodic sequence $\tilde{x}[n]$ corresponding to sampling the Fourier transform of $x[n]$ with $N = 12$.

periodic sequence $\tilde{x}[n]$, corresponding to $\tilde{X}[k]$ obtained by sampling $X(e^{j\omega})$, is formed from $x[n]$ by adding together an infinite number of shifted replicas of $x[n]$. The shifts are all the positive and negative integer multiples of N, the period of the sequence $\tilde{X}[k]$. This is illustrated in Figure 8, where the sequence $x[n]$ is of length 9 and the value of N in Eq. (56) is $N = 12$. Consequently, the delayed replications of $x[n]$ do not overlap, and one period of the periodic sequence $\tilde{x}[n]$ is recognizable as $x[n]$. This is consistent with the discussion in Section 3 and Example 6, wherein we showed that the Fourier series coefficients for a periodic sequence are samples of the Fourier transform of one period. In Figure 9 the same sequence $x[n]$ is used, but the value of N is now $N = 7$. In this case, the replicas of $x[n]$ overlap and one period of $\tilde{x}[n]$ is no longer identical to $x[n]$. In both cases, however, Eq. (49) still holds; i.e., in both cases, the DFS coefficients of $\tilde{x}[n]$ are samples of the Fourier transform of $x[n]$ spaced

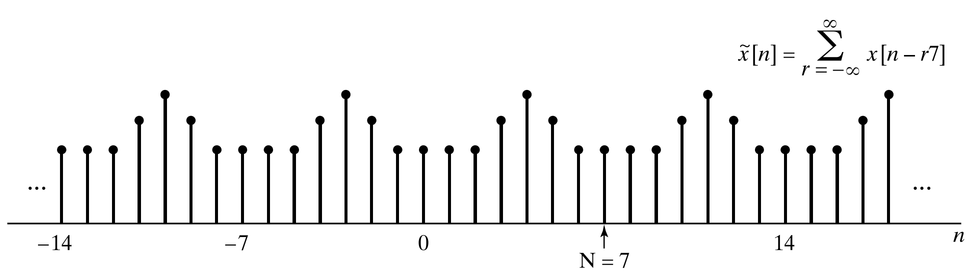


Figure 9 Periodic sequence $\tilde{x}[n]$ corresponding to sampling the Fourier transform of $x[n]$ in Figure 8(a) with $N = 7$.

in frequency at integer multiples of $2\pi/N$. This discussion should be reminiscent of that of sampling. The difference is that here we are sampling in the frequency domain rather than in the time domain. However, the general outlines of the mathematical representations are very similar.

For the example in Figure 8, the original sequence $x[n]$ can be recovered from $\tilde{x}[n]$ by extracting one period. Equivalently, the Fourier transform $X(e^{j\omega})$ can be recovered from the samples spaced in frequency by $2\pi/12$. In contrast, in Figure 9, $x[n]$ cannot be recovered by extracting one period of $\tilde{x}[n]$, and, equivalently, $X(e^{j\omega})$ cannot be recovered from its samples if the sample spacing is only $2\pi/7$. In effect, for the case illustrated in Figure 8, the Fourier transform of $x[n]$ has been sampled at a sufficiently small spacing (in frequency) to be able to recover it from these samples, whereas Figure 9 represents a case for which the Fourier transform has been undersampled. The relationship between $x[n]$ and one period of $\tilde{x}[n]$ in the undersampled case can be thought of as a form of aliasing in the time domain, essentially identical to the frequency-domain aliasing that results from undersampling in the time domain. Obviously, time-domain aliasing can be avoided only if $x[n]$ has finite length, just as frequency-domain aliasing can be avoided only for signals that have bandlimited Fourier transforms.

This discussion highlights several important concepts that will play a central role in the remainder of the chapter. We have seen that samples of the Fourier transform of an aperiodic sequence $x[n]$ can be thought of as DFS coefficients of a periodic sequence $\tilde{x}[n]$ obtained through summing periodic replicas of $x[n]$. If $x[n]$ is finite length and we take a sufficient number of equally spaced samples of its Fourier transform (specifically, a number greater than or equal to the length of $x[n]$), then the Fourier transform is recoverable from these samples, and, equivalently, $x[n]$ is recoverable from the corresponding periodic sequence $\tilde{x}[n]$. Specifically, if $x[n] = 0$ outside the interval $n = 0$, $n = N - 1$, then

$$x[n] = \begin{cases} \tilde{x}[n], & 0 \le n \le N-1, \\ 0, & \text{otherwise}. \end{cases} \tag{57}$$

If the interval of support of $x[n]$ is different than $0, N - 1$ then Eq. (57) would be appropriately modified.

A direct relationship between $X(e^{j\omega})$ and its samples $\tilde{X}[k]$, i.e., an interpolation formula for $X(e^{j\omega})$, can be derived (see Problem 57). However, the essence of our previous discussion is that to represent or to recover $x[n]$, it is not necessary to know $X(e^{j\omega})$ at all frequencies if $x[n]$ has finite length. Given a finite-length sequence $x[n]$, we can form a periodic sequence using Eq. (56), which in turn can be represented by a DFS. Alternatively, given the sequence of Fourier coefficients $\tilde{X}[k]$, we can find $\tilde{x}[n]$ and then use Eq. (57) to obtain $x[n]$. When the Fourier series is used in this way to represent finite-length sequences, it is called the discrete Fourier transform or DFT. In developing, discussing, and applying the DFT, it is always important to remember that the representation through samples of the Fourier transform is in effect a representation of the finite-duration sequence by a periodic sequence, one period of which is the finite-duration sequence that we wish to represent.

5 FOURIER REPRESENTATION OF FINITE-DURATION SEQUENCES: THE DFT

In this section, we formalize the point of view suggested at the end of the previous section. We begin by considering a finite-length sequence $x[n]$ of length N samples such that $x[n] = 0$ outside the range $0 \le n \le N-1$. In many instances, we will want to assume that a sequence has length N, even if its length is $M \le N$. In such cases, we simply recognize that the last $(N-M)$ samples are zero. To each finite-length sequence of length N, we can always associate a periodic sequence

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n - rN]. \tag{58a}$$

The finite-length sequence $x[n]$ can be recovered from $\tilde{x}[n]$ through Eq. (57), i.e.,

$$x[n] = \begin{cases} \tilde{x}[n], & 0 \le n \le N-1, \\ 0, & \text{otherwise.} \end{cases} \tag{58b}$$

Recall from Section 4 that the DFS coefficients of $\tilde{x}[n]$ are samples (spaced in frequency by $2\pi/N$) of the Fourier transform of $x[n]$. Since $x[n]$ is assumed to have finite length N, there is no overlap between the terms $x[n - rN]$ for different values of r. Thus, Eq. (58a) can alternatively be written as

$$\tilde{x}[n] = x[(n \text{ modulo } N)]. \tag{59}$$

For convenience, we will use the notation $((n))_N$ to denote $(n \text{ modulo } N)$; with this notation, Eq. (59) is expressed as

$$\tilde{x}[n] = x[((n))_N]. \tag{60}$$

Note that Eq. (60) is equivalent to Eq. (58a) only when $x[n]$ has length less than or equal to N. The finite-duration sequence $x[n]$ is obtained from $\tilde{x}[n]$ by extracting one period, as in Eq. (58b).

One informal and useful way of visualizing Eq. (59) is to think of wrapping a plot of the finite-duration sequence $x[n]$ around a cylinder with a circumference equal to the length of the sequence. As we repeatedly traverse the circumference of the cylinder, we see the finite-length sequence periodically repeated. With this interpretation, representation of the finite-length sequence by a periodic sequence corresponds to wrapping the sequence around the cylinder; recovering the finite-length sequence from the periodic sequence using Eq. (58b) can be visualized as unwrapping the cylinder and laying it flat so that the sequence is displayed on a linear time axis rather than a circular (modulo N) time axis.

As defined in Section 1, the sequence of DFS coefficients $\tilde{X}[k]$ of the periodic sequence $\tilde{x}[n]$ is itself a periodic sequence with period N. To maintain a duality between the time and frequency domains, we will choose the Fourier coefficients that we associate with a finite-duration sequence to be a finite-duration sequence corresponding to one period of $\tilde{X}[k]$. This finite-duration sequence, $X[k]$, will be referred to as the DFT. Thus, the DFT, $X[k]$, is related to the DFS coefficients, $\tilde{X}[k]$, by

$$X[k] = \begin{cases} \tilde{X}[k], & 0 \le k \le N-1, \\ 0, & \text{otherwise}, \end{cases} \tag{61}$$

and

$$\tilde{X}[k] = X[(k \text{ modulo } N)] = X[((k))_N]. \tag{62}$$

From Section 1, $\tilde{X}[k]$ and $\tilde{x}[n]$ are related by

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn}, \tag{63}$$

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-kn}. \tag{64}$$

where $W_N = e^{-j(2\pi/N)}$.

Since the summations in Eqs. (63) and (64) involve only the interval between zero and $(N-1)$, it follows from Eqs. (58b) to (64) that

$$X[k] = \begin{cases} \displaystyle\sum_{n=0}^{N-1} x[n] W_N^{kn}, & 0 \le k \le N-1, \\ 0, & \text{otherwise}, \end{cases} \tag{65}$$

$$x[n] = \begin{cases} \displaystyle\frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}, & 0 \le n \le N-1, \\ 0, & \text{otherwise}. \end{cases} \tag{66}$$

Generally, the DFT analysis and synthesis equations are written as follows:

$$\text{Analysis equation:} \quad X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \qquad 0 \le k \le N-1, \tag{67}$$

$$\text{Synthesis equation:} \quad x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}, \qquad 0 \le n \le N-1. \tag{68}$$

That is, the fact that $X[k] = 0$ for k outside the interval $0 \le k \le N-1$ and that $x[n] = 0$ for n outside the interval $0 \le n \le N-1$ is implied, but not always stated explicitly. The relationship between $x[n]$ and $X[k]$ implied by Eqs. (67) and (68) will sometimes be denoted as

$$x[n] \overset{\mathcal{DFT}}{\longleftrightarrow} X[k]. \tag{69}$$

In recasting Eqs. (11) and (12) in the form of Eqs. (67) and (68) for finite-duration sequences, we have not eliminated the inherent periodicity. As with the DFS, the DFT $X[k]$ is equal to samples of the periodic Fourier transform $X(e^{j\omega})$, and if Eq. (68) is evaluated for values of n outside the interval $0 \le n \le N-1$, the result will not be zero, but rather a periodic extension of $x[n]$. The inherent periodicity is always present. Sometimes, it causes us difficulty, and sometimes we can exploit it, but to totally ignore it is to invite trouble. In defining the DFT representation, we are simply recognizing that we are interested in values of $x[n]$ only in the interval $0 \le n \le N-1$, because $x[n]$ is really zero outside that interval, and we are interested in values of $X[k]$ only in the interval $0 \le k \le N-1$ because these are the only values needed in Eq. (68) to reconstruct $X[n]$.

Example 7 The DFT of a Rectangular Pulse

To illustrate the DFT of a finite-duration sequence, consider $x[n]$ shown in Figure 10(a). In determining the DFT, we can consider $x[n]$ as a finite-duration sequence with any length greater than or equal to $N = 5$. Considered as a sequence of length $N = 5$, the periodic sequence $\tilde{x}[n]$ whose DFS corresponds to the DFT of $x[n]$ is shown in Figure 10(b). Since the sequence in Figure 10(b) is constant over the interval $0 \le n \le 4$, it follows that

$$\begin{aligned} \tilde{X}[k] &= \sum_{n=0}^{4} e^{-j(2\pi k/5)n} = \frac{1 - e^{-j2\pi k}}{1 - e^{-j(2\pi k/5)}} \\ &= \begin{cases} 5, & k = 0, \pm 5, \pm 10, \dots, \\ 0, & \text{otherwise}; \end{cases} \end{aligned} \tag{70}$$

i.e., the only nonzero DFS coefficients $\tilde{X}[k]$ are at $k = 0$ and integer multiples of $k = 5$ (all of which represent the same complex exponential frequency). The DFS coefficients are shown in Figure 10(c). Also shown is the magnitude of the DTFT, $|X(e^{j\omega})|$. Clearly, $\tilde{X}[k]$ is a sequence of samples of $X(e^{j\omega})$ at frequencies $\omega_k = 2\pi k/5$. According to Eq. (61), the five-point DFT of $x[n]$ corresponds to the finite-length sequence obtained by extracting one period of $\tilde{X}[k]$. Consequently, the five-point DFT of $x[n]$ is shown in Figure 10(d).

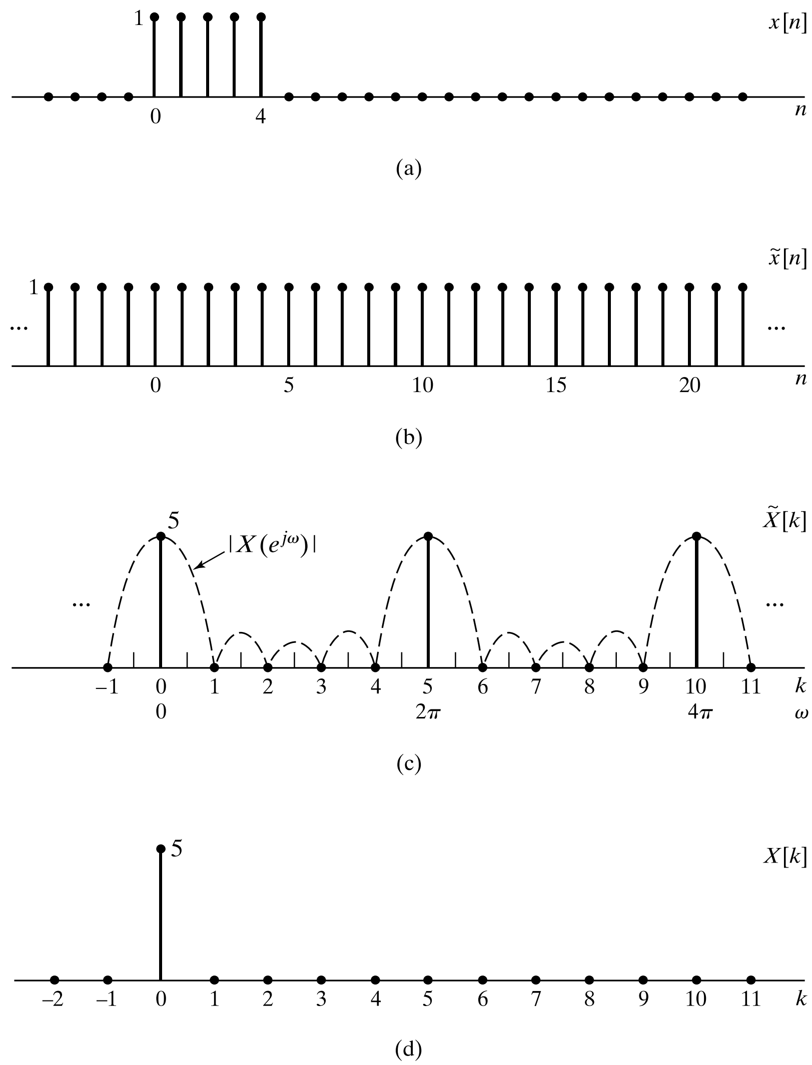


Figure 10 Illustration of the DFT. (a) Finite-length sequence $x[n]$. (b) Periodic sequence $\tilde{x}[n]$ formed from $x[n]$ with period $N = 5$. (c) Fourier series coefficients $\tilde{X}[k]$ for $\tilde{x}[n]$. To emphasize that the Fourier series coefficients are samples of the Fourier transform, $|X(e^{j\omega})|$ is also shown. (d) DFT of $x[n]$.

If, instead, we consider $x[n]$ to be of length $N = 10$, then the underlying periodic sequence is that shown in Figure 11(b), which is the periodic sequence considered in Example 3. Therefore, $\tilde{X}[k]$ is as shown in Figures 2 and 6, and the 10-point DFT $X[k]$ shown in Figures 11(c) and 11(d) is one period of $\tilde{X}[k]$.

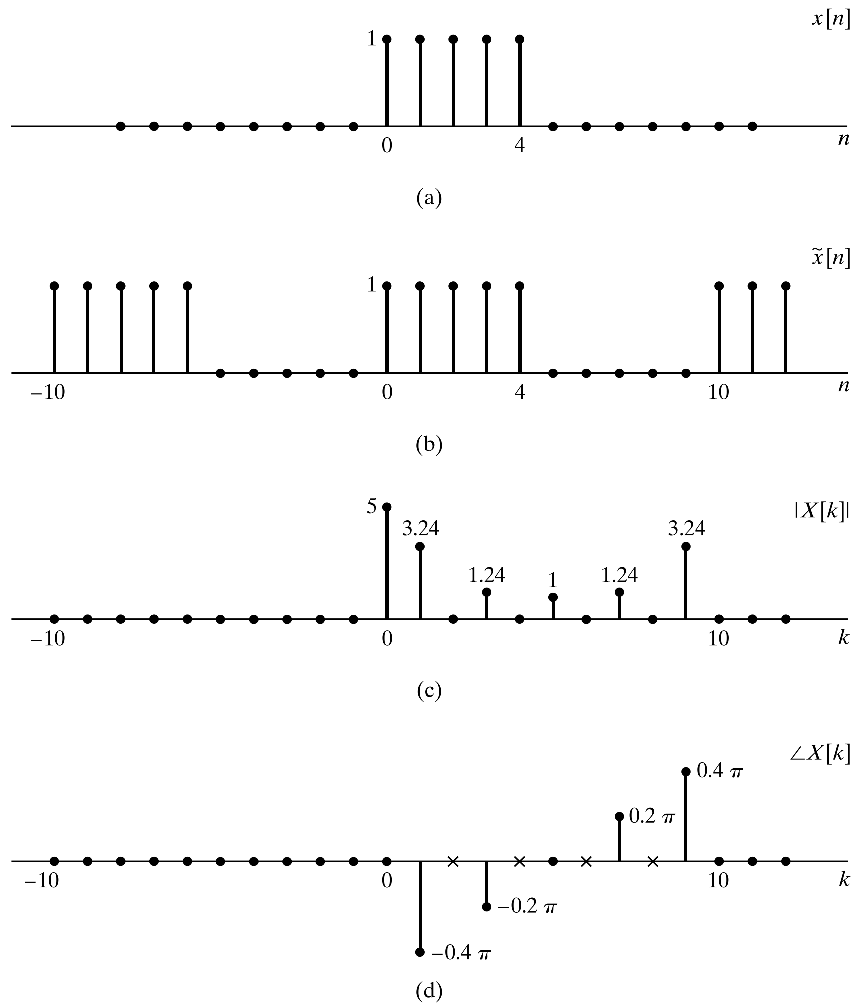


Figure 11 Illustration of the DFT. (a) Finite-length sequence $x[n]$. (b) Periodic sequence $\tilde{x}[n]$ formed from $x[n]$ with period $N = 10$. (c) DFT magnitude. (d) DFT phase. (x's indicate indeterminate values.)

The distinction between the finite-duration sequence $x[n]$ and the periodic sequence $\tilde{x}[n]$ related through Eqs. (57) and (60) may seem minor, since, by using these equations, it is straightforward to construct one from the other. However, the distinction becomes important in considering properties of the DFT and in considering the effect on $x[n]$ of modifications to $X[k]$. This will become evident in the next section, where we discuss the properties of the DFT representation.

6 PROPERTIES OF THE DFT

In this section, we consider a number of properties of the DFT for finite-duration sequences. Our discussion parallels the discussion of Section 2 for periodic sequences. However, particular attention is paid to the interaction of the finite-length assumption and the implicit periodicity of the DFT representation of finite-length sequences.

6.1 Linearity

If two finite-duration sequences $x_1[n]$ and $x_2[n]$ are linearly combined, i.e., if

$$x_3[n] = ax_1[n] + bx_2[n], \tag{71}$$

then the DFT of $x_3[n]$ is

$$X_3[k] = aX_1[k] + bX_2[k]. \tag{72}$$

Clearly, if $x_1[n]$ has length N_1 and $x_2[n]$ has length N_2, then the maximum length of $x_3[n]$ will be $N_3 = \max(N_1, N_2)$. Thus, in order for Eq. (72) to be meaningful, both DFTs must be computed with the same length $N \geq N_3$. If, for example, $N_1 < N_2$, then $X_1[k]$ is the DFT of the sequence $x_1[n]$ augmented by $(N_2 - N_1)$ zeros. That is, the N_2-point DFT of $x_1[n]$ is

$$X_1[k] = \sum_{n=0}^{N_1-1} x_1[n]W_{N_2}^{kn}, \qquad 0 \leq k \leq N_2 - 1, \tag{73}$$

and the N_2-point DFT of $x_2[n]$ is

$$X_2[k] = \sum_{n=0}^{N_2-1} x_2[n]W_{N_2}^{kn}, \qquad 0 \leq k \leq N_2 - 1. \tag{74}$$

In summary, if

$$x_1[n] \overset{\mathcal{DFT}}{\longleftrightarrow} X_1[k] \tag{75a}$$

and

$$x_2[n] \overset{\mathcal{DFT}}{\longleftrightarrow} X_2[k], \tag{75b}$$

then

$$ax_1[n] + bx_2[n] \overset{\mathcal{DFT}}{\longleftrightarrow} aX_1[k] + bX_2[k], \tag{76}$$

where the lengths of the sequences and their DFTs are all equal to at least the maximum of the lengths of $x_1[n]$ and $x_2[n]$. Of course, DFTs of greater length can be computed by augmenting both sequences with zero-valued samples.

6.2 Circular Shift of a Sequence

If $X(e^{j\omega})$ is the discrete-time Fourier transform of $x[n]$, then $e^{-j\omega m}X(e^{j\omega})$ is the Fourier transform of the time-shifted sequence $x[n-m]$. In other words, a shift in the time domain by m points (with positive m corresponding to a time delay and negative m to a time advance) corresponds in the frequency domain to multiplication of the Fourier transform by the linear-phase factor $e^{-j\omega m}$. In Section 2.2, we discussed the corresponding property for the DFS coefficients of a periodic sequence; specifically, if a periodic sequence $\tilde{x}[n]$ has Fourier series coefficients $\tilde{X}[k]$, then the shifted sequence $\tilde{x}[n-m]$ has Fourier series coefficients $e^{-j(2\pi k/N)m}\tilde{X}[k]$. Now we will consider the operation in the time domain that corresponds to multiplying the DFT coefficients of a finite-length sequence $x[n]$ by the linear-phase factor $e^{-j(2\pi k/N)m}$. Specifically, let $x_1[n]$ denote the finite-length sequence for which the DFT is $e^{-j(2\pi k/N)m}X[k]$; i.e., if

$$x[n] \overset{\mathcal{DFT}}{\longleftrightarrow} X[k], \tag{77}$$

then we are interested in $x_1[n]$ such that

$$x_1[n] \overset{\mathcal{DFT}}{\longleftrightarrow} X_1[k] = e^{-j(2\pi k/N)m}X[k] = W_N^m X[k]. \tag{78}$$

Since the N-point DFT represents a finite-duration sequence of length N, both $x[n]$ and $x_1[n]$ must be zero outside the interval $0 \le n \le N-1$, and consequently, $x_1[n]$ cannot result from a simple time shift of $x[n]$. The correct result follows directly from the result of Section 2.2 and the interpretation of the DFT as the Fourier series coefficients of the periodic sequence $x_1[((n))_N]$. In particular, from Eqs. (59) and (62) it follows that

$$\tilde{x}[n] = x[((n))_N] \overset{\mathcal{DFS}}{\longleftrightarrow} \tilde{X}[k] = X[((k))_N], \tag{79}$$

and similarly, we can define a periodic sequence $\tilde{x}_1[n]$ such that

$$\tilde{x}_1[n] = x_1[((n))_N] \overset{\mathcal{DFS}}{\longleftrightarrow} \tilde{X}_1[k] = X_1[((k))_N], \tag{80}$$

where, by assumption,

$$X_1[k] = e^{-j(2\pi k/N)m}X[k]. \tag{81}$$

Therefore, the DFS coefficients of $\tilde{x}_1[n]$ are

$$\tilde{X}_1[k] = e^{-j[2\pi((k))_N/N]m} X[((k))_N]. \tag{82}$$

Note that

$$e^{-j[2\pi((k))_N/N]m} = e^{-j(2\pi k/N)m}. \tag{83}$$

That is, since $e^{-j(2\pi k/N)m}$ is periodic with period N in both k and m, we can drop the notation $((k))_N$. Hence, Eq. (82) becomes

$$\tilde{X}_1[k] = e^{-j(2\pi k/N)m} \tilde{X}[k], \tag{84}$$

so that it follows from Section 2.2 that

$$\tilde{x}_1[n] = \tilde{x}[n-m] = x[((n-m))_N]. \tag{85}$$

Thus, the finite-length sequence $x_1[n]$ whose DFT is given by Eq. (81) is

$$x_1[n] = \begin{cases} \tilde{x}_1[n] = x[((n-m))_N], & 0 \le n \le N-1, \\ 0, & \text{otherwise.} \end{cases} \tag{86}$$

Equation (86) tells us how to construct $x_1[n]$ from $x[n]$.

Example 8 Circular Shift of a Sequence

The circular shift procedure is illustrated in Figure 12 for $m = -2$; i.e., we want to determine $x_1[n] = x[((n+2))_N]$ for $N = 6$, which we have shown will have DFT $X_1[k] = W_6^{-2k} X[k]$. Specifically, from $x[n]$, we construct the periodic sequence $\tilde{x}[n] = x[((n))_6]$, as indicated in Figure 12(b). According to Eq. (85), we then shift $\tilde{x}[n]$ by 2 to the left, obtaining $\tilde{x}_1[n] = \tilde{x}[n+2]$ as in Figure 12(c). Finally, using Eq. (86), we extract one period of $\tilde{x}_1[n]$ to obtain $x_1[n]$, as indicated in Figure 12(d).

A comparison of Figures 12(a) and (d) indicates clearly that $x_1[n]$ does not correspond to a linear shift of $x[n]$, and in fact, both sequences are confined to the interval between 0 and $(N-1)$. By reference to Figure 12, we see that $x_1[n]$ can be formed by shifting $x[n]$, so that as a sequence value leaves the interval 0 to $(N-1)$ at one end, it enters at the other end. Another interesting point is that, for the example shown in Figure 12(a), if we form $x_2[n] = x[((n-4))_6]$ by shifting the sequence by 4 to the right modulo 6, we obtain the same sequence as $x_1[n]$. In terms of the DFT, this results because $W_6^{4k} = W_6^{-2k}$ or, more generally, $W_N^{mk} = W_N^{-(N-m)k}$, which implies that an N-point circular shift in one direction by m is the same as a circular shift in the opposite direction by $N-m$.

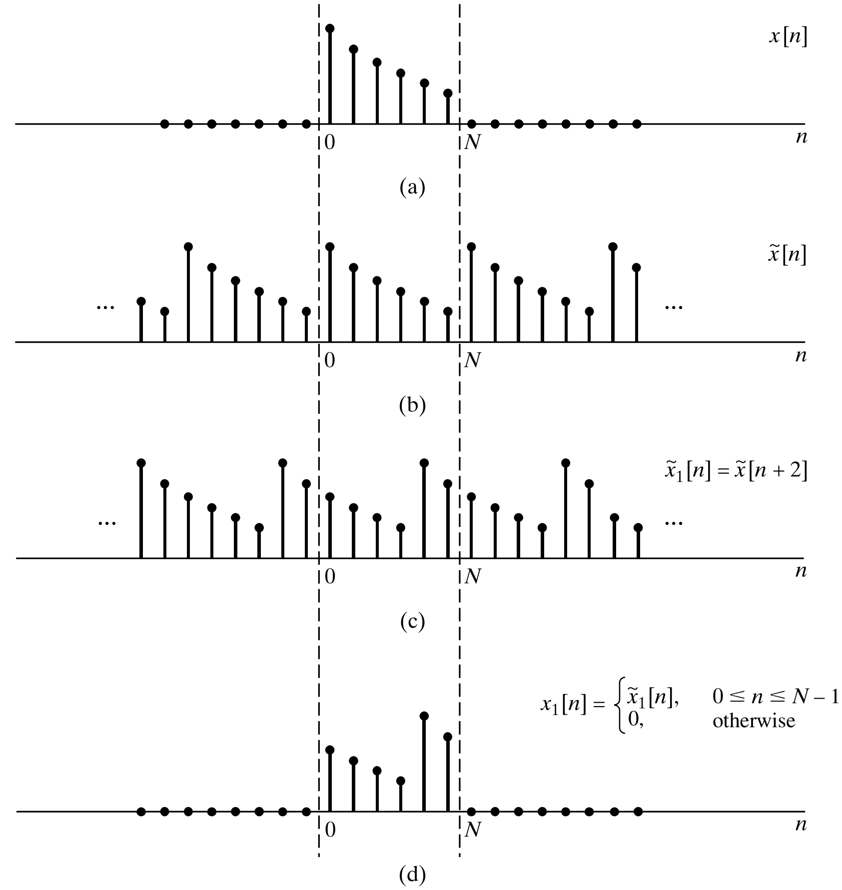


Figure 12 Circular shift of a finite-length sequence; i.e., the effect in the time domain of multiplying the DFT of the sequence by a linear-phase factor.

In Section 5, we suggested the interpretation of forming the periodic sequence $\tilde{x}[n]$ from the finite-length sequence $x[n]$ by displaying $x[n]$ around the circumference of a cylinder with a circumference of exactly N points. As we repeatedly traverse the circumference of the cylinder, the sequence that we see is the periodic sequence $\tilde{x}[n]$. A linear shift of this sequence corresponds, then, to a rotation of the cylinder. In the context of finite-length sequences and the DFT, such a shift is called a *circular* shift or a *rotation* of the sequence within the interval $0 \leq n \leq N - 1$.

In summary, the circular shift property of the DFT is

$$x[((n-m))_N], \quad 0 \leq n \leq N-1 \overset{\mathcal{DFT}}{\longleftrightarrow} e^{-j(2\pi k/N)m} X[k] = W_N^m X[k]. \tag{87}$$

6.3 Duality

Since the DFT is so closely associated with the DFS, we would expect the DFT to exhibit a duality property similar to that of the DFS discussed in Section 2.3. In fact, from an examination of Eqs. (67) and (68), we see that the analysis and synthesis equations differ only in the factor $1/N$ and the sign of the exponent of the powers of W_N.

The DFT duality property can be derived by exploiting the relationship between the DFT and the DFS as in our derivation of the circular shift property. Toward this end, consider $x[n]$ and its DFT $X[k]$, and construct the periodic sequences

$$\tilde{x}[n] = x[((n))_N], \tag{88a}$$

$$\tilde{X}[k] = X[((k))_N], \tag{88b}$$

so that

$$\tilde{x}[n] \stackrel{\mathcal{DFS}}{\longleftrightarrow} \tilde{X}[k]. \tag{89}$$

From the duality property given in Eqs. (25),

$$\tilde{X}[n] \stackrel{\mathcal{DFS}}{\longleftrightarrow} N\tilde{x}[-k]. \tag{90}$$

If we define the periodic sequence $\tilde{x}_1[n] = \tilde{X}[n]$, one period of which is the finite-length sequence $x_1[n] = X[n]$, then the DFS coefficients of $\tilde{x}_1[n]$ are $\tilde{X}_1[k] = N\tilde{x}[-k]$. Therefore, the DFT of $x_1[n]$ is

$$X_1[k] = \begin{cases} N\tilde{x}[-k], & 0 \le k \le N-1, \\ 0, & \text{otherwise}, \end{cases} \tag{91}$$

or, equivalently,

$$X_1[k] = \begin{cases} Nx[((-k))_N], & 0 \le k \le N-1, \\ 0, & \text{otherwise}. \end{cases} \tag{92}$$

Consequently, the duality property for the DFT can be expressed as follows: If

$$x[n] \stackrel{\mathcal{DFT}}{\longleftrightarrow} X[k], \tag{93a}$$

then

$$X[n] \stackrel{\mathcal{DFT}}{\longleftrightarrow} Nx[((-k))_N], \qquad 0 \le k \le N-1. \tag{93b}$$

The sequence $Nx[((-k))_N]$ is $Nx[k]$ index reversed, modulo N. Index-reversing modulo N corresponds specifically to $((-k))_N = N-k$ for $1 \le k \le N-1$ and $((-k))_N = ((k))_N$ for $k = 0$. As in the case of shifting modulo N, the process of index-reversing modulo N is usually best visualized in terms of the underlying periodic sequences.

Example 9 The Duality Relationship for the DFT

To illustrate the duality relationship in Eqs. (93), let us consider the sequence $x[n]$ of Example 7. Figure 13(a) shows the finite-length sequence $x[n]$, and Figures 13(b) and 13(c) are the real and imaginary parts, respectively, of the corresponding 10-point DFT $X[k]$. By simply relabeling the horizontal axis, we obtain the complex sequence $x_1[n] = X[n]$, as shown in Figures 13(d) and 13(e). According to the duality relation in Eqs. (93), the 10-point DFT of the (complex-valued) sequence $X[n]$ is the sequence shown in Figure 13(f).

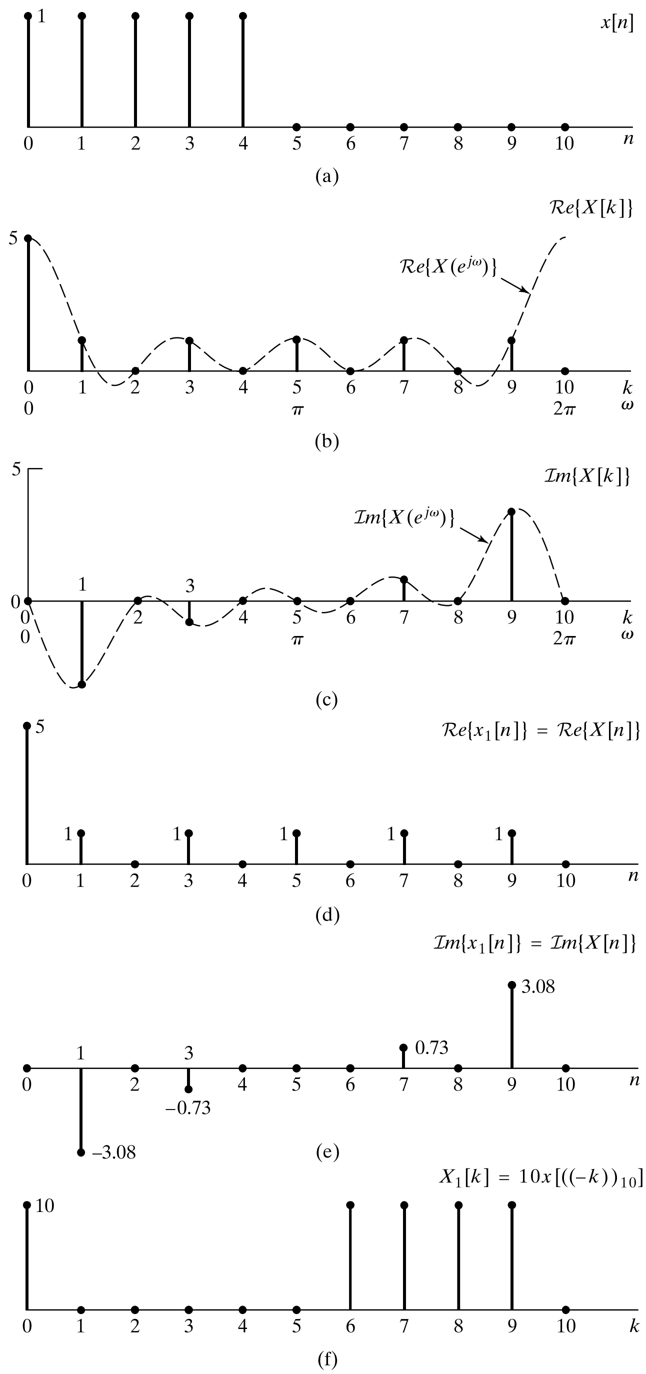


Figure 13 Illustration of duality. (a) Real finite-length sequence $x[n]$. (b) and (c) Real and imaginary parts of corresponding DFT $X[k]$. (d) and (e) The real and imaginary parts of the dual sequence $x_1[n] = X[n]$. (f) The DFT of $x_1[n]$.

6.4 Symmetry Properties

Since the DFT of $x[n]$ is identical to the DFS coefficients of the periodic sequence $\tilde{x}[n] = x[((n))_N]$, symmetry properties associated with the DFT can be inferred from the symmetry properties of the DFS summarized in Table 1 in Section 2.6. Specifically, using Eqs. (88) together with Properties 9 and 10 in Table 1, we have

$$x^*[n] \overset{\mathcal{DFT}}{\longleftrightarrow} X^*[((-k))_N], \qquad 0 \le n \le N-1, \tag{94}$$

and

$$x^*[((-n))_N] \overset{\mathcal{DFT}}{\longleftrightarrow} X^*[k], \qquad 0 \le n \le N-1. \tag{95}$$

Properties 11–14 in Table 1 refer to the decomposition of a periodic sequence into the sum of a conjugate-symmetric and a conjugate-antisymmetric sequence. This suggests the decomposition of the finite-duration sequence $x[n]$ into the two finite-duration sequences of duration N corresponding to one period of the conjugate-symmetric and one period of the conjugate-antisymmetric components of $\tilde{x}[n]$. We will denote these components of $x[n]$ as $x_{\text{ep}}[n]$ and $x_{\text{op}}[n]$. Thus, with

$$\tilde{x}[n] = x[((n))_N] \tag{96}$$

and the conjugate-symmetric part being

$$\tilde{x}_e[n] = \tfrac{1}{2}\{\tilde{x}[n] + \tilde{x}^*[-n]\}, \tag{97}$$

and the conjugate-antisymmetric part being

$$\tilde{x}_o[n] = \tfrac{1}{2}\{\tilde{x}[n] - \tilde{x}^*[-n]\}, \tag{98}$$

we define $x_{\text{ep}}[n]$ and $x_{\text{op}}[n]$ as

$$x_{\text{ep}}[n] = \tilde{x}_e[n], \qquad 0 \le n \le N-1, \tag{99}$$

$$x_{\text{op}}[n] = \tilde{x}_o[n], \qquad 0 \le n \le N-1, \tag{100}$$

or, equivalently,

$$x_{\text{ep}}[n] = \tfrac{1}{2}\{x[((n))_N] + x^*[((-n))_N]\}, \qquad 0 \le n \le N-1, \tag{101a}$$

$$x_{\text{op}}[n] = \tfrac{1}{2}\{x[((n))_N] - x^*[((-n))_N]\}, \qquad 0 \le n \le N-1, \tag{101b}$$

with both $x_{\text{ep}}[n]$ and $x_{\text{op}}[n]$ being finite-length sequences, i.e., both zero outside the interval $0 \le n \le N-1$. Since $((-n))_N = (N-n)$ and $((n))_N = n$ for $0 \le n \le N-1$, we can also express Eqs. (101) as

$$x_{\text{ep}}[n] = \tfrac{1}{2}\{x[n] + x^*[N-n]\}, \qquad 1 \le n \le N-1, \tag{102a}$$

$$x_{\text{ep}}[0] = \mathcal{R}e\{x[0]\}, \tag{102b}$$

$$x_{\text{op}}[n] = \tfrac{1}{2}\{x[n] - x^*[N-n]\}, \qquad 1 \le n \le N-1, \tag{102c}$$

$$x_{\text{op}}[0] = j\mathcal{I}m\{x[0]\}. \tag{102d}$$

This form of the equations is convenient, since it avoids the modulo N computation of indices.

Clearly, $x_{\text{ep}}[n]$ and $x_{\text{op}}[n]$ are not equivalent to $x_e[n]$ and $x_o[n]$. However, it can be shown (see Problem 59) that

$$x_{\text{ep}}[n] = \{x_e[n] + x_e[n-N]\}, \qquad 0 \le n \le N-1, \tag{103}$$

and

$$x_{\text{op}}[n] = \{x_o[n] + x_o[n-N]\}, \qquad 0 \le n \le N-1. \tag{104}$$

In other words, $x_{\text{ep}}[n]$ and $x_{\text{op}}[n]$ can be generated by time-aliasing $x_e[n]$ and $x_o[n]$ into the interval $0 \le n \le N-1$. The sequences $x_{\text{ep}}[n]$ and $x_{\text{op}}[n]$ will be referred to as *the periodic conjugate-symmetric* and *periodic conjugate-antisymmetric* components, respectively, of $x[n]$. When $x_{\text{ep}}[n]$ and $x_{\text{op}}[n]$ are real, they will be referred to as the *periodic even* and *periodic odd* components, respectively. Note that the sequences $x_{\text{ep}}[n]$ and $x_{\text{op}}[n]$ are not periodic sequences; they are, however, finite-length sequences that are equal to one period of the periodic sequences $\tilde{x}_e[n]$ and $\tilde{x}_o[n]$, respectively.

Equations (101) and (102) define $x_{\text{ep}}[n]$ and $x_{\text{op}}[n]$ in terms of $x[n]$. The inverse relation, expressing $x[n]$ in terms of $x_{\text{ep}}[n]$ and $x_{\text{op}}[n]$, can be obtained by using Eqs. (97) and (98) to express $\tilde{x}[n]$ as

$$\tilde{x}[n] = \tilde{x}_e[n] + \tilde{x}_o[n]. \tag{105}$$

Thus,

$$x[n] = \tilde{x}[n] = \tilde{x}_e[n] + \tilde{x}_o[n], \qquad 0 \le n \le N-1. \tag{106}$$

Combining Eqs. (106) with Eqs. (99) and (100), we obtain

$$x[n] = x_{\text{ep}}[n] + x_{\text{op}}[n]. \tag{107}$$

Alternatively, Eqs. (102), when added, also lead to Eq. (107). The symmetry properties of the DFT associated with properties 11–14 in Table 1 now follow in a straightforward way:

$$\mathcal{R}e\{x[n]\} \stackrel{\mathcal{DFT}}{\longleftrightarrow} X_{\text{ep}}[k], \tag{108}$$

$$j\mathcal{I}m\{x[n]\} \stackrel{\mathcal{DFT}}{\longleftrightarrow} X_{\text{op}}[k], \tag{109}$$

$$x_{\text{ep}}[n] \stackrel{\mathcal{DFT}}{\longleftrightarrow} \mathcal{R}e\{X[k]\}, \tag{110}$$

$$x_{\text{op}}[n] \stackrel{\mathcal{DFT}}{\longleftrightarrow} j\mathcal{I}m\{X[k]\}. \tag{111}$$

6.5 Circular Convolution

In Section 2.5, we showed that multiplication of the DFS coefficients of two periodic sequences corresponds to a periodic convolution of the sequences. Here, we consider two *finite-duration* sequences $x_1[n]$ and $x_2[n]$, both of length N, with DFTs $X_1[k]$ and $X_2[k]$, respectively, and we wish to determine the sequence $x_3[n]$, for which the DFT is $X_3[k] = X_1[k]X_2[k]$. To determine $x_3[n]$, we can apply the results of Section 2.5. Specifically, $x_3[n]$ corresponds to one period of $\tilde{x}_3[n]$, which is given by Eq. (27). Thus,

$$x_3[n] = \sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m], \qquad 0 \le n \le N-1, \tag{112}$$

or, equivalently,

$$x_3[n] = \sum_{m=0}^{N-1} x_1[((m))_N]x_2[((n-m))_N], \qquad 0 \le n \le N-1. \tag{113}$$

Since $((m))_N = m$ for $0 \le m \le N-1$, Eq. (113) can be written

$$x_3[n] = \sum_{m=0}^{N-1} x_1[m]x_2[((n-m))_N], \qquad 0 \le n \le N-1. \tag{114}$$

Equation (114) differs from a linear convolution of $x_1[n]$ and $x_2[n]$ in some important respects. In linear convolution, the computation of the sequence value $x_3[n]$ involves multiplying one sequence by a time-reversed and linearly shifted version of the other and then summing the values of the product $x_1[m]x_2[n-m]$ over all m. To obtain successive values of the sequence formed by the convolution operation, the two sequences are successively shifted relative to each other along a linear axis. In contrast, for the convolution defined by Eq. (114), the second sequence is circularly time reversed and circularly shifted with respect to the first. For this reason, the operation of combining two finite-length sequences according to Eq. (114) is called *circular convolution.* More specifically, we refer to Eq. (114) as an N-point circular convolution, explicitly identifying the fact that both sequences have length N (or less) and that the sequences are shifted modulo N. Sometimes, the operation of forming a sequence $x_3[n]$ for $0 \le n \le N-1$ using Eq. (114) will be denoted

$$x_3[n] = x_1[n] \circledN x_2[n], \tag{115}$$

i.e., the symbol $\circledN$ denotes N-point circular convolution.

Since the DFT of $x_3[n]$ is $X_3[k] = X_1[k]X_2[k]$ and since $X_1[k]X_2[k] = X_2[k]X_1[k]$, it follows with no further analysis that

$$x_3[n] = x_2[n] \circledN x_1[n], \tag{116}$$

or, more specifically,

$$x_3[n] = \sum_{m=0}^{N-1} x_2[m]x_1[((n-m))_N]. \tag{117}$$

That is, circular convolution, like linear convolution, is a commutative operation.

Since circular convolution is really just periodic convolution, Example 4 and Figure 3 are also illustrative of circular convolution. However, if we use the notion of circular shifting, it is not necessary to construct the underlying periodic sequences as in Figure 3. This is illustrated in the following examples.

Example 10 Circular Convolution with a Delayed Impulse Sequence

An example of circular convolution is provided by the result of Section 6.2. Let $x_2[n]$ be a finite-duration sequence of length N and

$$x_1[n] = \delta[n-n_0], \tag{118}$$

where $0 < n_0 < N$. Clearly, $x_1[n]$ can be considered as the finite-duration sequence

$$x_1[n] = \begin{cases} 0, & 0 \le n < n_0, \\ 1, & n = n_0, \\ 0, & n_0 < n \le N-1. \end{cases} \tag{119}$$

as depicted in Figure 14 for $n_0 = 1$.

The DFT of $x_1[n]$ is

$$X_1[k] = W_N^{kn_0}. \tag{120}$$

If we form the product

$$X_3[k] = W_N^{kn_0} X_2[k], \tag{121}$$

we see from Section 6.2 that the finite-duration sequence corresponding to $X_3[k]$ is the sequence $x_2[n]$ rotated to the right by n_0 samples in the interval $0 \le n \le N-1$. That is, the circular convolution of a sequence $x_2[n]$ with a single delayed unit impulse results in a rotation of $x_2[n]$ in the interval $0 \le n \le N-1$. This example is illustrated in Figure 14 for $N = 5$ and $n_0 = 1$. Here, we show the sequences $x_2[m]$

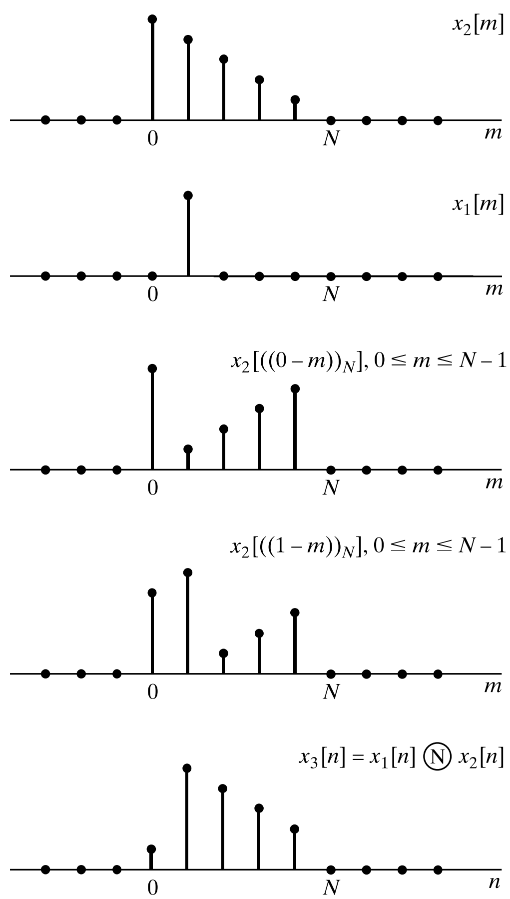


Figure 14 Circular convolution of a finite-length sequence $x_2[n]$ with a single delayed impulse, $x_1[n] = \delta[n-1]$.

and $x_1[m]$ and then $x_2[((0-m))_N]$ and $x_2[((1-m))_N]$. It is clear from these two cases that the result of circular convolution of $x_2[n]$ with a single shifted unit impulse will be to circularly shift $x_2[n]$. The last sequence shown is $x_3[n]$, the result of the circular convolution of $x_1[n]$ and $x_2[n]$.

Example 11 Circular Convolution of Two Rectangular Pulses

As another example of circular convolution, let

$$x_1[n] = x_2[n] = \begin{cases} 1, & 0 \le n \le L-1, \\ 0, & \text{otherwise,} \end{cases} \tag{122}$$

where, in Figure 15, $L = 6$. If we let N denote the DFT length, then, for $N = L$, the N-point DFTs are

$$X_1[k] = X_2[k] = \sum_{n=0}^{N-1} W_N^{kn} = \begin{cases} N, & k = 0, \\ 0, & \text{otherwise.} \end{cases} \tag{123}$$

If we explicitly multiply $X_1[k]$ and $X_2[k]$, we obtain

$$X_3[k] = X_1[k]X_2[k] = \begin{cases} N^2, & k = 0, \\ 0, & \text{otherwise,} \end{cases} \tag{124}$$

from which it follows that

$$x_3[n] = N, \qquad 0 \le n \le N-1. \tag{125}$$

This result is depicted in Figure 15. Clearly, as the sequence $x_2[((n-m))_N]$ is rotated with respect to $x_1[m]$, the sum of products $x_1[m]x_2[((n-m))_N]$ will always be equal to N.

Of course, it is possible to consider $x_1[n]$ and $x_2[n]$ as $2L$-point sequences by augmenting them with L zeros. If we then perform a $2L$-point circular convolution of the augmented sequences, we obtain the sequence in Figure 16, which can be seen to be identical to the linear convolution of the finite-duration sequences $x_1[n]$ and $x_2[n]$. This important observation will be discussed in much more detail in Section 7.

Note that for $N = 2L$, as in Figure 16,

$$X_1[k] = X_2[k] = \frac{1 - W_N^{Lk}}{1 - W_N^k},$$

so the DFT of the triangular-shaped sequence $x_3[n]$ in Figure 16(e) is

$$X_3[k] = \left(\frac{1 - W_N^{Lk}}{1 - W_N^k}\right)^2,$$

with $N = 2L$.

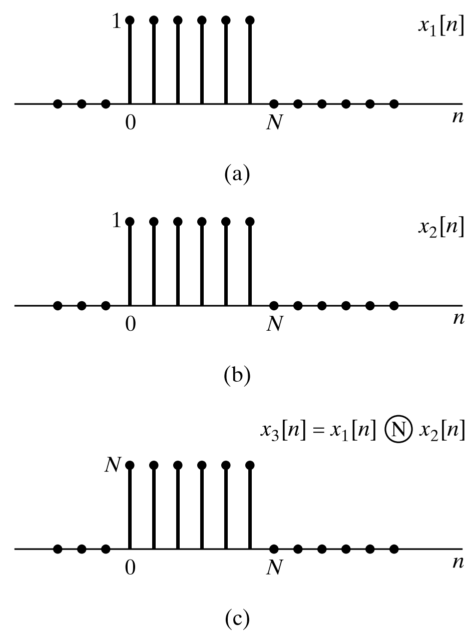


Figure 15 N-point circular convolution of two constant sequences of length N.

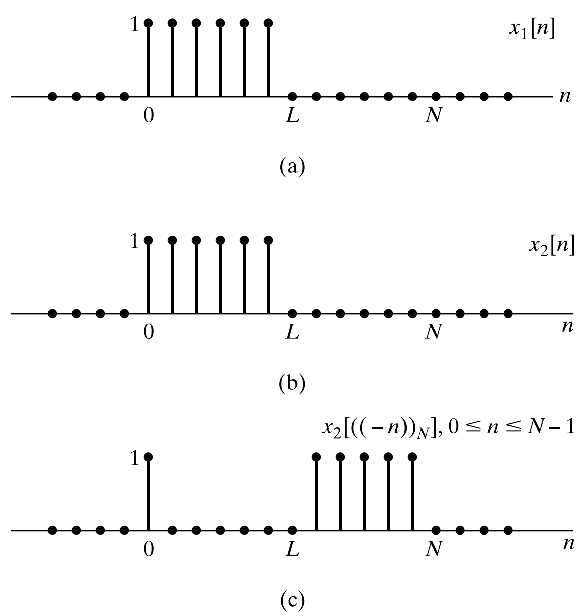
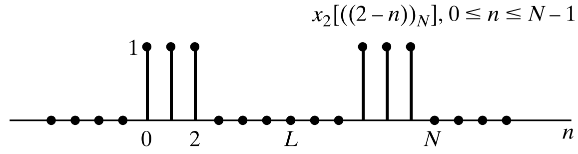
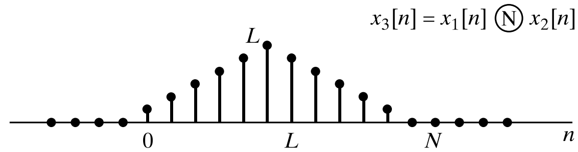


Figure 16 $2L$-point circular convolution of two constant sequences of length L.



(d)



(e)

Figure 16 (*continued*)

The circular convolution property is represented as

$$x_1[n] \; \text{Ⓝ} \; x_2[n] \overset{\mathcal{DFT}}{\longleftrightarrow} X_1[k]X_2[k]. \tag{126}$$

In view of the duality of the DFT relations, it is not surprising that the DFT of a product of two N-point sequences is the circular convolution of their respective DFTs. Specifically, if $x_3[n] = x_1[n]x_2[n]$, then

$$X_3[k] = \frac{1}{N}\sum_{\ell=0}^{N-1} X_1[\ell]X_2[((k-\ell))_N] \tag{127}$$

or

$$x_1[n]x_2[n] \overset{\mathcal{DFT}}{\longleftrightarrow} \frac{1}{N} X_1[k] \; \text{Ⓝ} \; X_2[k]. \tag{128}$$

6.6 Summary of Properties of the DFT

The properties of the DFT that we discussed in Section 6 are summarized in Table 2. Note that for all of the properties, the expressions given specify $x[n]$ for $0 \le n \le N-1$ and $X[k]$ for $0 \le k \le N-1$. Both $x[n]$ and $X[k]$ are equal to zero outside those ranges.

TABLE 2 SUMMARY OF PROPERTIES OF THE DFT

Finite-Length Sequence (Length N)	N-point DFT (Length N)
1. $x[n]$	$X[k]$
2. $x_1[n], x_2[n]$	$X_1[k], X_2[k]$
3. $ax_1[n] + bx_2[n]$	$aX_1[k] + bX_2[k]$
4. $X[n]$	$Nx[((-k))_N]$
5. $x[((n-m))_N]$	$W_N^{km} X[k]$
6. $W_N^{-\ell n} x[n]$	$X[((k-\ell))_N]$
7. $\sum_{m=0}^{N-1} x_1[m]x_2[((n-m))_N]$	$X_1[k]X_2[k]$
8. $x_1[n]x_2[n]$	$\frac{1}{N}\sum_{\ell=0}^{N-1} X_1[\ell]X_2[((k-\ell))_N]$
9. $x^*[n]$	$X^*[((-k))_N]$
10. $x^*[((-n))_N]$	$X^*[k]$
11. $\mathcal{R}e\{x[n]\}$	$X_{\text{ep}}[k] = \frac{1}{2}\{X[((k))_N] + X^*[((-k))_N]\}$
12. $j\mathcal{I}m\{x[n]\}$	$X_{\text{op}}[k] = \frac{1}{2}\{X[((k))_N] - X^*[((-k))_N]\}$
13. $x_{\text{ep}}[n] = \frac{1}{2}\{x[n] + x^*[((-n))_N]\}$	$\mathcal{R}e\{X[k]\}$
14. $x_{\text{op}}[n] = \frac{1}{2}\{x[n] - x^*[((-n))_N]\}$	$j\mathcal{I}m\{X[k]\}$
Properties 15–17 apply only when $x[n]$ is real.	
15. Symmetry properties	$\begin{cases} X[k] = X^*[((-k))_N] \\ \mathcal{R}e\{X[k]\} = \mathcal{R}e\{X[((-k))_N]\} \\ \mathcal{I}m\{X[k]\} = -\mathcal{I}m\{X[((-k))_N]\} \\ \lvert X[k]\rvert = \lvert X[((-k))_N]\rvert \\ \angle\{X[k]\} = -\angle\{X[((-k))_N]\} \end{cases}$
16. $x_{\text{ep}}[n] = \frac{1}{2}\{x[n] + x[((-n))_N]\}$	$\mathcal{R}e\{X[k]\}$
17. $x_{\text{op}}[n] = \frac{1}{2}\{x[n] - x[((-n))_N]\}$	$j\mathcal{I}m\{X[k]\}$

7 COMPUTING LINEAR CONVOLUTION USING THE DFT

Efficient algorithms are available for computing the DFT of a finite-duration sequence. These are known collectively as FFT algorithms. Because these algorithms are available, it is computationally efficient to implement a convolution of two sequences by the following procedure:

(a) Compute the N-point DFTs $X_1[k]$ and $X_2[k]$ of the two sequences $x_1[n]$ and $x_2[n]$, respectively.

(b) Compute the product $X_3[k] = X_1[k]X_2[k]$ for $0 \le k \le N-1$.

(c) Compute the sequence $x_3[n] = x_1[n] \circledN x_2[n]$ as the inverse DFT of $X_3[k]$.