No. of Quantization levels $L = 2^n$

Step size $\Delta = \dfrac{x_{max} - x_{min}}{L}$

i = round off $\left(\dfrac{x - x_{min}}{\Delta}\right)$

Index corresponding to binary code.

Quantization Level $x_q = x_{min} + i\Delta$

Length (duration) $N = N_2 - N_1 + 1$

→ A right sided sequence $x[n] = 0$ for $n < N_1$

If $N_1 \geq 0$, a RSS is called causal sequence

→ A left sided sequence $x[n] = 0$ for $n > N_2$

If $N_2 < 0$, a LSS is called an anti-causal sequence.

Norm :- Strength of discrete time signal

$$\boxed{L_p : \|x\|_p = \left(\sum_{n=-\infty}^{\infty} |x[n]|^p\right)^{1/p}}$$

→ $p = 1 \Rightarrow \left(\sum_{n=-\infty}^{\infty} |x[n]|\right)$ Mean absolute value.

Mean absolute value of length-N sequence $\|x\|_1 / N$

→ $p = 2 \Rightarrow \left(\sum_{n=-\infty}^{\infty} |x[n]|^2\right)^{1/2}$ Root mean squared value.

$\|x\|_2 / \sqrt{N}$:- RMS value of length-N sequence

→ $p = \infty \Rightarrow \left(\sum_{n=-\infty}^{\infty} |x[n]|^{\infty}\right)^{1/\infty} = 1$ Peak absolute value.

$\|x\|_\infty = |x|_{max}$

$$\boxed{R = \frac{F_s'}{F_s}}$$

→ Up Sampling = Adding more samples = Interpolation $(R > 1)$

→ Down Sampling = Discarding samples = Decimation $(R < 1)$

$x[n]$ is a periodic sequence & its period is $N = \dfrac{2\pi K}{\omega_0}$, $N_{min} = \dfrac{2\pi}{\omega_0}$

→ If $x_a[n]$ and $x_b[n]$ are two periodic sequences with fundamental period N_a & N_b, then the sequence $y[n] = x_a + x_b$ will have fundamental period.

$$N = \frac{N_a N_b}{GCD(N_a, N_b)}$$

→ If $x[n] = x_{re}[n] + x_{im}[n]$ is periodic, then $x_{re}[n]$ & $x_{im}[n]$ are periodic.

$N = LCM(N_{re}, N_{im})$

N_{re} & N_{im} are fundamental period.

$y[n] = a\,x[n]$

$a \to +ve$, $a > 1$, amplification

$a \to -ve$, $a < 1$, attenuation

$y[n] = x[n - n_0]$

$n_0 \to +ve$, $n_0 > 1$, delay

$n_0 \to -ve$, $n_0 < 1$, advance

Relative Error $= \left[\dfrac{\sum_{n=0}^{N-1} |y[n] - x[n]|^2}{\sum_{n=0}^{N-1} |x[n]|^2}\right]^{1/2}$

Convolution

$$y[n] = \sum_{K=-\infty}^{\infty} x[K]\,h[n-K]$$
$$= \sum_{K=-\infty}^{\infty} x[n-K]\,h[K]$$
$$= x[n] * h[n], \quad -\infty \le n \le \infty$$

Based on Symmetry

$$X[n] = X_{CS}[n] + X_{CA}[n]$$
$$X_{CS}[n] = \frac{x[n] + x^*[-n]}{2} = X^*_{CS}[-n]$$
$$X_{CA}[n] = \frac{x[n] - x^*[-n]}{2} = -X^*_{CA}[-n]$$

$\boxed{x_{CS}[n] = x^*_{CS}[-n]}$ $\boxed{x_{CA}[n] = -x^*_{CA}[-n]}$

★ Similarly with odd & even symmetry

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

★ If amplitude of $x[n]$ does not $\to 0$ when $n \to \infty$, need to measure time average of energy i.e. power P_x because $E_x = \infty$.

$$P_x = \lim_{N \to \infty} \frac{1}{2N_0+1} \sum_{n=-N_0}^{N_0} |x[n]|^2$$
$$= \lim_{N \to \infty} \frac{1}{N_0} \sum_{n=0}^{N_0-1} |x[n]|^2$$

★ P_x is the time average of signal amplitude squared i.e. RMS value of $x(t)$ is $\sqrt{P_x}$

➜ A DT signal can be either energy or power signal but can't be both at the same time.

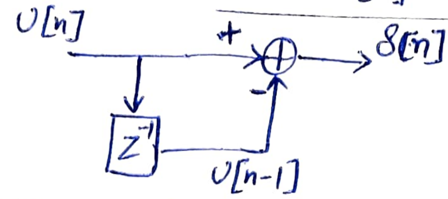
$$x[n] = x_e[n] + x_o[n] \qquad x_e[n] = \frac{x[n] + x[-n]}{2}$$
$$x_o[n] = \frac{x[n] - x[-n]}{2}$$

Discrete Time Impulse

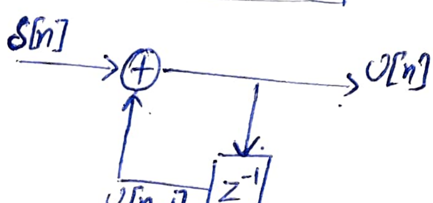
$$\Rightarrow x[n] = \sum_{K=0}^{\infty} x[K]\,\delta[n-K]$$

→ Unit Stepsequence $\Rightarrow u[n] = \sum_{K=-\infty}^{\infty} u[K]\,\delta[n-K]$

$$\boxed{\delta[n] = u[n] - u[n-1]}$$



Unit step $\boxed{u[n] = u[n-1] + \delta[n]}$



M-point moving average system

$$y[n] = \frac{1}{M_1 + M_2 + 1} \sum_{K=-M_1}^{M_2} x[n-K]$$

Average ~~values~~ of the 3 most recent values of input signal is called moving average system

→ Time-invariance

$$x[n] \Rightarrow y[n]$$
If $x[n] = x[n-n_0]$
$$\Rightarrow y[n] = y[n-n_0]$$

→ Causal :- If output depends on past & present inputs only.

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \ne 0 \end{cases}$$

$$u[n] = \begin{cases} 1, & n \ge 0 \\ 0, & n < 0 \end{cases} \qquad \delta[n] = u[n] - u[n-1]$$

$$u[n] = \delta[n] + \delta[n-1] + \dots$$

$e^{ST} = Z \qquad Z^n = e^{STn}$

$S = \sigma + j\omega$

$Z = e^{ST} = e^{(\sigma + j\omega)T} = e^{\sigma T} e^{j\omega T}$

$|Z| = |e^{\sigma T}||e^{j\omega T}| = |e^{\sigma T}| \qquad \because |e^{j\omega T}| = 1$

→ $Re\{S\} < 0$ (LHP), $|Z^n| < 1$.
exponential decreases

→ $Re\{S\} > 0$ (RHP), $|Z^n| > 1$.
exponential grows.

→ $Im\{S\}$, $|Z^n| = 1$
const. - amplitude & oscillates.

⇒ Z^{-n}	s plane	z plane
$\sigma < 0$	LHP	inside unitcircle
$\sigma > 0$	RHP	outside unitcircle.
$\sigma = 0$	Imaginary axis	On unit circle.

→ $x[n] = A\alpha^n \qquad -\infty < n < \infty$.

$A = |A| e^{j\phi} \qquad \alpha = e^{Z}$.

$Z = \sigma_0 + j\omega_0$

$x[n] = |A| e^{\sigma_0 n}[\cos(\omega_0 n + \phi) + j\sin(\omega_0 n + \phi)]$

→ $\sigma_0 > 0$, two sequence increase.

→ $\sigma_0 < 0$, two sequence decrease

→ If A & α are real, $\omega_0 n + \phi = 0$

$$\boxed{x[n] = |A| e^{\sigma_0 n}}$$

$y[n] = x[n] * h[n]$

$= \sum_{K=-\infty}^{\infty} x[K]h[n-K]$

$$\sum_{K=0}^{M} P_K x[n-K] \longrightarrow \boxed{T} \longrightarrow \sum_{K=0}^{N} d_K y[n-K]$$

order of system $= Max\{N, M\}$

Classification

- → Acc. to length of impulse response sequences
- → Acc. to Method of calculation

Impulse Respone

- → Finite Impulse Response (FIR)
- → Infinite Impulse Response (IIR)

Output Calculation

- → Recursive (with feedback) IIR
- → Non-recursive (without feedback) FIR.

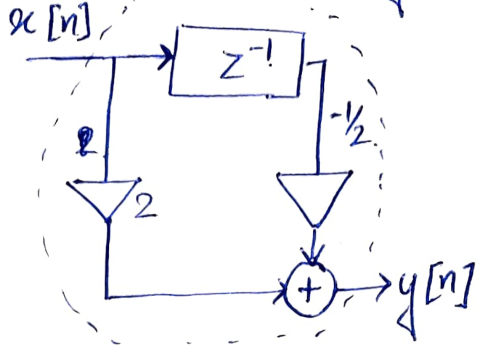
FIR System

$h[n] \rightarrow$ finite length

$$h[n] = 0, \quad n < N_1$$
$$n > N_2$$
$$\therefore N_1 < N_2$$

$$y[n] = x[n] * h[n]$$
$$= \sum_{K=N_1}^{N_2} x[K]h[n-K]$$

Non-recursive system



FIR Filter

NOTE:-

$$r_{yx}[l] = \sum_{n=-\infty}^{\infty} y[n]x[n-l]$$
$$m = n-l$$
$$= \sum_{m=-\infty}^{\infty} y[m+l]x[m]$$
$$\boxed{r_{yx}[l] = r_{xy}[-l]}$$

Auto Correlation

$$r_{xx}[l] = \sum_{n=-\infty}^{\infty} x[n]x[n-l] = r_{xx}[-l]$$

$$r_{xx}[0] = \sum_{n=-\infty}^{\infty} x^2[n]$$
$$= E_x$$

energy of sequence $x[n]$

IIR System

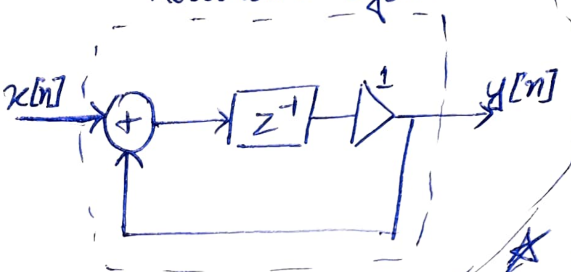
$h[n] \rightarrow$ infinite length

Causal IIR

$$y[n] = \sum_{K=0}^{n} x[K]h[n-K]$$
$$= \sum_{K=0}^{n} h[K]x[n-K]$$

☆ limits are due to Causality of LTI system.

Recursive system



☆ $r_{xy}[n] = x[n] * y[-n]$

$\rightarrow |r_{xy}|^2 \leq r_{xx} r_{yy}$

$\rightarrow |r_{xy}|^2 = r_{xx} r_{yy}$

Correlation Coefficient

$$\rho_{xy} = \frac{|r_{xy}|}{\sqrt{r_{xx} r_{yy}}}$$

$$0 \leq \rho_{xy} \leq 1$$

uncorrelated — strongly correlated.

⇒ Auto-Regressive is IIR system.

$$y[n] = x[n] - \sum_{K=0}^{N} d_K y[n-K]$$

Pole system

⇒ Auto-Regressive Moving Average

$$y[n] = \sum_{K=0}^{M} p_K x[n-K] - \sum_{K=0}^{N} d_K y[n-K]$$

Pole & Zero System

⇒ Cross Correlation

$$r_{xy}[l] = \sum_{n=-\infty}^{\infty} x[n]y[n-l]$$

lag :- time shift b/w pair of signals

Correlation $\Rightarrow r_{xy}[n] = \sum_{K=-\infty}^{\infty} x[K]y[K-n]$

Convolution $\Rightarrow x[n] * y[n] = \sum_{K=-\infty}^{\infty} x[K]y[n-K]$

⇒ Normalised Autocorrelation of $x[n]$

$$r_{xx}[l] \le r_{xx}[0]$$

$$\Rightarrow \left| \frac{r_{xx}[l]}{r_{xx}[0]} \right| \le 1$$

⇒ For cross-correlation

$$r_{xy}[l] \le \sqrt{E_x E_y}$$

$$\Rightarrow \frac{r_{xy}[l]}{\sqrt{r_{xx}[0]\, r_{yy}[0]}} \le 1$$

☆ $\langle xy \rangle = \sum x_i y_i = |x||y|\cos\theta$

when $x \parallel y$, $\cos\theta = 1$

else, $\cos\theta < 1$

Auto Correlation of Periodic Sequence

$$x[n] = x[n+N]$$

$$r_{xx}[l] = \frac{1}{N}\sum_{N=0}^{N-1} x[n]\, x[n-l]$$

$$r_{xx}[0] = \frac{1}{N}\sum_{n=0} x^2[n] = P_x$$

Average energy per sample or Power of x

$$r_{xx}[l+N] = r_{xx}[l]$$

⇒ Autocorrelation of periodic sequence is periodic.

Difference Equation

$$y[n+N] + a_1 y[n+N-1] + \dots + a_{N-1} y[n+1] + a_N y[n] = b_{N-M} x[n+M] + b_{N-M+1} x[n+M-1] + \dots + b_{N-1} x[n+1] + b_N x[n]$$

order = max{N, M}

→ Causal Condition : N=M.

E-Operator

↳ operation for advancing the sequence by one unit.

$$E\, x[n] = x[n+1]$$

$$E^2 x[n] = x[n+2]$$

$$E^N x[n] = x[n+N]$$

→ General N^{th} order difference eq. with E-operator.

$$(E^N + a_1 E^{N-1} + \dots + a_{N-1} E + a_N)\, y[n] = (b_0 E^N + b_1 E^{N-1} + \dots + b_{N-1} E + b_N)\, x[n]$$

$$Q(E)\, y[n] = P[E]\, x[n]$$

Response of Linear DT Systems

Total response = Zero Input Response + Zero-State Response

ZIR $y_0[n]$ is solution when $x[n] = 0$

$$Q[E]\, y_0[n] = 0$$

- This is true only for an exponential function

If $y_0[n] = c\gamma^n$, $E^K y_0[n] = y_0[n+K] = c\gamma^{n+K}$

$$\Rightarrow Q(E)\, h[n] = P(E)\, \delta[n] \quad ; \quad IC = 0$$

$$h[-1] = h[-2] = \dots = h[-n] = 0$$

ZSR

If $\delta[n] \Rightarrow h[n]$

then $\delta[n-m] \Rightarrow h[n-m]$

$$x[k]\,\delta[n-K] \Rightarrow x[K]\, h[n-K]$$

$$\sum_{K=-\infty}^{\infty} x[K]\,\delta[n-K] \Rightarrow \sum_{K=-\infty}^{\infty} x[K]\, h[n-K]$$

$$y[n] = \sum_{K=-\infty}^{\infty} x[K]\, h[n-K]$$

Two-sided Bilateral Z Transform

$$X(z) = Z\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

* For non-causal signals, the summation should start from minus infinity and the Z transform so defined is called two sided Z transform.

$$z = e^{j\Omega}$$

$z^{-1} \rightarrow$ delay of one unit.

One-Sided Unilateral Z-Transform

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

Inverse Z-Transform

$$x[n] = Z^{-1}\{X(n)\}$$

$$= \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

↖ closed contour integral.

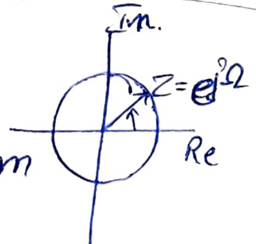
$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$z = e^{j\Omega}$$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$

$$X(\Omega) = X(z)\big|_{z=e^{-j\Omega}}$$

→ DTFT is used to evaluate z-transform on a unit circle.



ROC for DT signal $x[n]$ is defined as a continuous region in z plane where ZT converges.

ROC $\quad X(z) = \frac{P(z)}{Q(z)} \rightarrow$ polynomial.

⇒ Roots of $P(z) = 0 \rightarrow$ zeros

⇒ Roots of $Q(z) = 0 \rightarrow$ poles.

⇒ ZT of causal & anticausal signals are identical except their ROC.

→ $\delta[n] \Longleftrightarrow 1 z^0$ $\quad$ Z-plane including 0 & ∞

→ $u[n] \Longleftrightarrow \frac{z}{z-1}$ $\quad |z| > 1$

→ $e^{\pm j\beta n} u[n] \Longleftrightarrow \frac{z}{z - e^{\pm j\beta}}$ $\quad |z| > |e^{\pm j\beta}| = 1$

$$\cos\beta n = (e^{j\beta n} + e^{-j\beta n})/2$$

⇒ $Z\{a x_1[n] + b x_2[n]\} = a X_1(z) + b X_2(z)$ $\quad z \in R_1 \cap R_2$

⇒ $x[n-n_0]u[n-n_0] \Longleftrightarrow z^{-n_0} X(z)$ $\quad z \in R_x$

⇒ $x[n+n_0]u[n+n_0] \Longleftrightarrow z^{n_0} X(z)$ $\quad z \in R_x$

⇒ $\gamma^{n} x[n] u[n] \Longleftrightarrow X(\gamma^{-1} z)$ $\quad z \in |\gamma| R_x$

⇒ $n\, x[n] u[n] \Longleftrightarrow -z \frac{dX(z)}{dz}$ $\quad z \in R_x$

⇒ $x[-n] \Longleftrightarrow X(z^{-1})$ $\quad z \in \frac{1}{R_x}$

⇒ $Z\{a^n x(n)\} = X\left(\frac{z}{a}\right)$

⇒ $-\frac{(-a)^n}{n} u(n-1) \Longleftrightarrow \log(1 + a z^{-1})$

⇒ $n a^n u[n] \Longleftrightarrow \frac{a z^{-1}}{(1 - a z^{-1})^2}$

Initial Value Theorem

If $x(n)$ is causal

i.e. $x(n) = 0$, for $n<0$

then $x[0] = \lim_{z\to\infty} X(z)$

$$\Rightarrow X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$= x(0) + \underbrace{x(1)z^{-1} + x(2)z^{-2} + \dots}_{\Rightarrow 0 \text{ as } z\to\infty}$$

Final Value Theorem

If $x[n] \Longleftrightarrow X(z)$

then $\lim_{n\to\infty} x[n] = x(\infty) = \lim_{z\to 1} (1-z^{-1})X(z)$

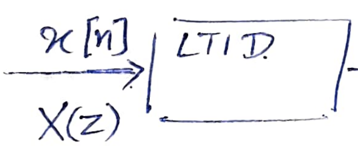
i.e. limit exist if all the poles of $(1-z^{-1})X(z)$ lie inside the unit circle

Magnitude of Poles < 1.

Covolution

$$x_1[n] * x_2[n] \Longleftrightarrow X_1(z)\, X_2(z)$$

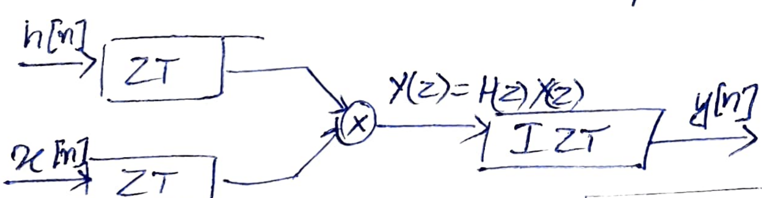
$$z \in R_1 \cap R_2$$

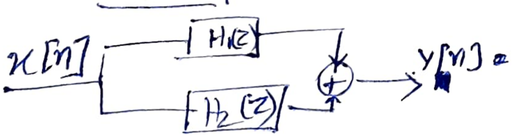


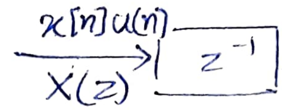
$y[n] = x[n] * h[n]$

$Y(z) = H(z)X(z)$

& $H(z) = \dfrac{Y(z)}{X(z)} = \dfrac{Z(\text{Zero State Response})}{Z(\text{Input})}$







$y[n] = x(n-1)u(n-1)$

$Y(z) = z^{-1}X(z)$

For filter

$y[n] = x[n] - x[n-1]$

z-plane.

$$Y(z) = X(z) - z^{-1}X(z) = (1-z^{-1})X(z)$$

FIR $\Rightarrow$ Finite Impulse Response

$\Rightarrow$ Output is non-recursive

$\Rightarrow$ T.F. is polynomial in z^{-1}

IIR $\Rightarrow$ Infinite Impulse Response.

$\Rightarrow$ Output is recursive

$\Rightarrow$ T.F. is rational function in z^{-1}

$\boxed{\text{FIR}}$

$h[n] = 0 \qquad N_1 \le n \le N_2$

$$Y(z) = \left(\sum_{n=N_1}^{N_2} h[n]z^{-n}\right) X(z)$$

$$H(z) = \sum_{n=N_1}^{N_2} h[n] z^{-n}$$

$$\Rightarrow y(n) = -\sum_{K=1}^{N} a_K \underbrace{y(n-K)}_{Y(z)z^{-K}} + \sum_{K=0}^{M} b_K \underbrace{x(n-K)}_{X(z)z^{-K}}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{K=0}^{M} b_K z^{-K}}{1 + \sum_{K=1}^{N} a_K z^{-K}}$$

$\Rightarrow$ LTI system is BIBO stable if $S = \sum_{n=-\infty}^{\infty} |h(n)| < \infty$

$\rightarrow$ If ROC contains a unit circle LTI system is stable with bounded impulse response.

$\rightarrow$ Causal FIR filter is always stable. since all poles are at origin

$\rightarrow$ Causal IIR is stable if all poles are inside unit circle.

Auto correlation $\quad R_{xx}(z) = X(z)\,X(z^{-1})$

$\rightarrow$ ROC of BIBO stable contain unit circle.

DTFS

$$x[n] = \sum_{r=0}^{N_0-1} D_r e^{j\Omega r n}, \quad -\infty < r < \infty$$

SYNTHESIS EQUATION

$$D_r = \frac{1}{N_0} \sum_{n=0}^{N_0-1} x[n] e^{-j\frac{2\pi r n}{N_0}}, \quad -\infty < r < \infty$$

$\downarrow$

ANALYSIS EQUATION

DTFS or Spectral Coefficients.

Harmonic values of $\Omega_0 = \frac{2\pi}{N_0}$

Series $\rightarrow D_0, D_1 e^{j\Omega_0 n}, D_2 e^{j2\Omega_0 n}, \ldots$

$\ldots, D_{N_0-1} e^{j(N_0-1)\Omega_0 n}$

Frequency Components $\rightarrow 0, \Omega_0, 2\Omega_0, \ldots, (N_0-1)\Omega_0$

$$\Omega_0 = \frac{2\pi}{N_0}$$

$$D_r = |D_r| e^{j\angle D_r}$$

$$D_{r+N_0} = \frac{1}{N_0} \sum_{n=0}^{N_0-1} x[n] e^{-j\frac{2\pi(r+N_0)n}{N_0}}$$

$$= \frac{1}{N_0} \sum_{n=0}^{N_0-1} x[n] e^{-j\frac{2\pi r n}{N_0}} e^{-j2\pi n}$$

$$e^{-j2\pi n} = \cos(2\pi n) - j\sin(2\pi n)$$
$$= 1.$$

$$D_{r+N_0} = D_r.$$

For r^{th} harmonic, frequency $\Omega_r = r\Omega$

$\rightarrow$ The frequency spacing b/w the consecutive spectral lines, called the frequency resolution, is Ω_0Hz.

Parseval's Theorem

$$\frac{1}{N_0} \sum_{n=0}^{N_0-1} |x[n]|^2 = \sum_{r=0}^{N_0-1} |D_r|^2.$$

$\rightarrow$ If $x[n] \Leftrightarrow X_r$, $y[n] \Leftrightarrow Y_r$

$$a x[n] + b y[n] \Leftrightarrow a X_r + b Y_r$$

$\rightarrow$ $x[n] \Leftrightarrow D_r$

$$x[n-n_0] \Leftrightarrow D_r e^{-jr\Omega n_0}.$$

$\rightarrow$ $x[n] \Leftrightarrow D_r$

$$x[-n] \Leftrightarrow D_{-r} = -D_r.$$

$\rightarrow$ $x[n] \Leftrightarrow D_r$

$$x\left[\frac{n}{m}\right] \Leftrightarrow \frac{1}{m} D_r$$