# Properties of DFT

DFT : $X(K) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi nK}{N}}$

$$x(n) = \frac{1}{N}\sum_{K=0}^{N-1} X(K)\, e^{j\frac{2\pi nK}{N}}$$

## ① Linearity Property

$$a_1 T[x_1(n)] + a_2 T[x_2(n)] \longleftrightarrow T[a_1 x_1(n) + a_2 x_2(n)]$$

Proof:-

$$LHS = a_1 \sum_{n=0}^{N-1} x_1(n)\, e^{-j\frac{2\pi nK}{N}} + a_2 \sum_{n=0}^{N-1} x_2(n)\, e^{-j\frac{2\pi nK}{N}}$$

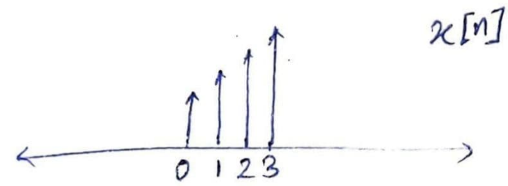
$$RHS = \sum_{n=0}^{N-1} (a_1 x_1(n) + a_2 x_2(n))\, e^{-j\frac{2\pi nK}{N}}$$

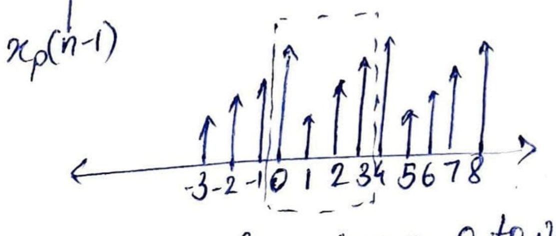
$$= \sum_{n=0}^{N-1} a_1 x_1(n)\, e^{-j\frac{2\pi nK}{N}} + \sum_{n=0}^{N-1} a_2 x_2(n)\, e^{-j\frac{2\pi nK}{N}}$$

$$RHS = LHS.$$

## ② Circular Shift

How a linear shift in a periodic sequence results in circular shift.




Periodic version of $x(n)$.


$x(n) \rightarrow$ values from 0 to $N-1$ $\rightarrow$ aperiodic sequence.

$x((-n))_N = x(N-n)$

Applying right shift operation

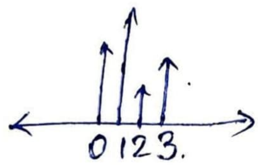
$x(n) \leftrightarrow x(n-K)$ right shifted

$x(n) \leftrightarrow x((n-K))_N$ circular right shift.

$K = 2$

$$n=0 \Rightarrow x((0-2))_4 = x((-2))_4 = x(4-2) = x(2)$$
$$n=1 \Rightarrow x((1-2))_4 = x((-1))_4 = x(4-1) = x(3)$$
$$n=2 \Rightarrow x((2-2))_4 = x((0))_4 = x(0)$$
$$n=3 \Rightarrow x((3-2))_4 = x((1))_4 = x(1)$$


$\xleftrightarrow[K=2]{\text{Circular shift.}}$




If we represent the values in circle form then values will rotate anti-clockwise direction for $x((n-K))_N$

$x(n) = x(N-n)$ :- Even symmetric sequence
$x(n) = -x(N-n)$ :- Odd symmetric sequence.

(3) Circular Convolution in time domain $\xleftrightarrow[\text{pair}]{\text{F.T.}}$ DFT multiplication.

$$x_3(n) = x_1(n) \circledast x_2(n) \xleftrightarrow[\text{pair}]{\text{F.T.}} X_3(K) = X_1(K).X_2(K)$$

symbol of circular convolution.

RHS $\Rightarrow$ $X_3(K) = X_1(K)\, X_2(K)$

$$x_3(m) = \frac{1}{N}\sum_{K=0}^{N-1} X_3(K)\, e^{j\frac{2\pi mK}{N}}$$
$$= \frac{1}{N}\sum_{K=0}^{N-1} [X_1(K)\, X_2(K)]\, e^{j\frac{2\pi mK}{N}}$$

$$x_3(m) = \frac{1}{N}\sum_{K=0}^{N-1}\left(\sum_{n=0}^{N-1} x_1(n)\, e^{-j\frac{2\pi nK}{N}}\right)\left(\sum_{l=0}^{N-1} x_2(l)\, e^{-j\frac{2\pi lK}{N}}\right) e^{j\frac{2\pi mK}{N}}$$
$$= \frac{1}{N}\sum_{K=0}^{N-1}\sum_{n=0}^{N-1}\sum_{l=0}^{N-1} x_1(n)\, x_2(l)\, e^{j\frac{2\pi(m-n-l)K}{N}}$$
$$= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{l=0}^{N-1} x_1(n)\, x_2(l)\left[\sum_{K=0}^{N-1} e^{j\frac{2\pi(m-l-n)K}{N}}\right]$$

$\because \sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a}$, if $a \neq 1$

$= N$, if $a = 1$

$x_3(m) = \frac{1}{N}\sum_{n=0}^{N-1}$

Case I $a = 1$

$m-n-l = \pm pN$, p is integer.

$$\sum_{K=0}^{N-1} e^{j\frac{2\pi(m-l-n)K}{N}} = \sum_{K=0}^{N-1} e^{j\frac{2\pi pNK}{N}} = \sum_{K=0}^{N-1} = N$$

Case II $a \neq 1$, $a = e^{j\frac{2\pi(m-n-l)}{N}}$

$m-n-l \neq \pm pN$.

$$\sum_{K=0}^{N-1} e^{j\frac{2\pi(m-n-l)K}{N}} = \frac{1 - e^{j\frac{2\pi(m-n-l)KN}{N}}}{1 - e^{j\frac{2\pi(m-n-l)}{N}}}$$

$= 0$

Using Case I & II

$$\sum_{K=0}^{N-1} e^{j\frac{2\pi(m-n-l)K}{N}} = \begin{cases} N \\ 0 \end{cases}$$

$\sum_{l=0}^{N-1} \Rightarrow$ Total N values of $l$

$m-n-l = \pm pN.$

$l = m-n\pm pN.$

for remaining $l$ indices answer is zero.

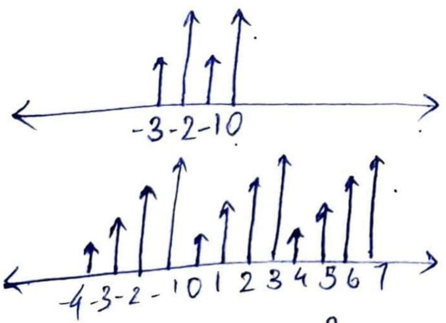
$$x_3(m) = \frac{1}{\cancel{N}}\sum_{n=0}^{N-1} x_1(n)\, x_2(m-n\pm pN) \cdot \cancel{N}$$

$$\boxed{x_3(m) = \sum_{n=0}^{N-1} x_1(n)\, x_2((m-n))_N}$$

Ex

$x_1(n) = [1, 2\ 3\ 4]$

$x_2(n) = [2\ 1\ 2\ 1]$




$x_3(0) = 1(2) + 4(1) + 3(2) + 2(1)$

$= 2 + 4 + 6 + 2$

$= 14$

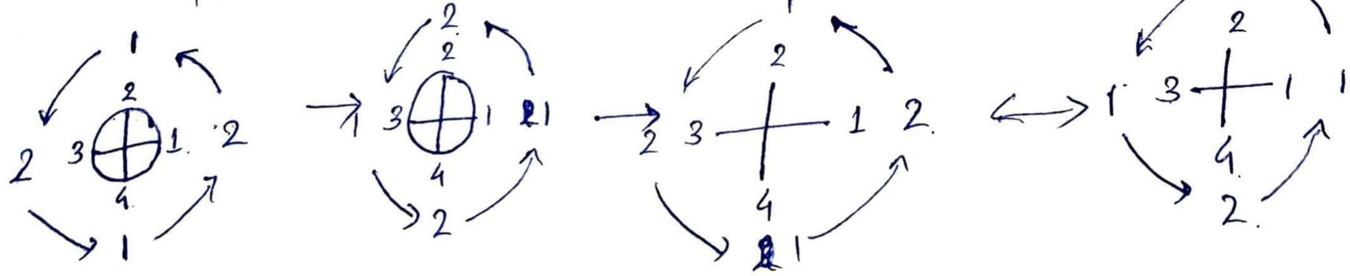
$x_3(1) = (1)(1) + 2(2) + 3(1) + 4(2)$

$= 1 + 4 + 3 + 8$

$= 16$

$x_3(2) = 1(2) + 2(1) + 3(2) + 4(1) = 14$

$x_3(3) = 1(1) + 2(2) + 3(1) + 4(2) = 16$



~~Same example by formula~~

$$\cancel{x_3(m) = \sum_{n=0}^{N-1} x_1(n)\, x_2((m-n))_N}$$

$\cancel{N=4}$ $\quad \cancel{x_1 = [1\ 2\ 3\ 4]}$

$\cancel{x_2 = [2\ 1\ 2\ 1]}$

# Discrete Cosine Transform

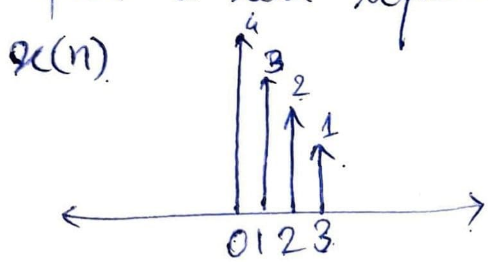
If $x(n)$ is real and even.

DFT. $$X(K) = \sum_{n=0}^{N-1} x_R(n) \cos\left(\frac{2\pi n K}{N}\right)$$

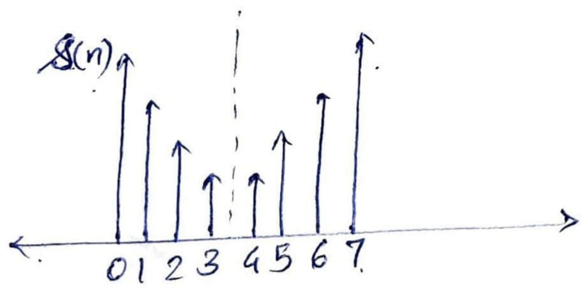
Spectrum is real.

Advantages
- → It's real transform
- → It's more energy compact for real signals.

Given a real sequence




Extended sequence →




$$\Rightarrow \begin{cases} S(n) = x(n), & 0 \le n \le N-1 \\ S(n) = x(2N-n-1), & N \le n \le 2N-1 \end{cases}$$

$$\begin{cases} S(0) = x(0) & S(4) = x(3) \\ S(1) = x(1) & S(5) = x(2) \\ S(2) = x(2) & S(6) = x(1) \\ S(3) = x(3) & S(7) = x(0) \end{cases}$$

Now the resultant sequence consists of 2N points.

∴ DFT should be taken for 2N points.

$$S(K) = \sum_{n=0}^{2N-1} s(n)\, e^{-j\frac{2\pi n K}{2N}}$$

$$= \sum_{n=0}^{2N-1} s(n)\, e^{-j\frac{2\pi n K}{2N}}$$

$$S(K) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi n K}{2N}} + \sum_{n=N}^{2N-1} x(2N-n-1)\, e^{-j\frac{2\pi n K}{2N}}$$

$=$

$m = 2N-n-1, \quad n = 2N-m-1$

$n = N \Rightarrow m = N-1$

$n = 2N-1 \Rightarrow m = 0$

$$S(K) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi n K}{2N}} + \sum_{m=0}^{N-1} x(m)\, e^{-j\frac{2\pi(2N-m-1)K}{2N}}$$

2nd term.

$$2^{nd}\text{ term} = \sum_{m=0}^{N-1} x(m)\, e^{-j2\pi\frac{2NK}{2N}} \cdot e^{j\frac{2\pi mK}{2N}} \cdot e^{j\frac{2\pi K}{2N}}$$

$$= \sum_{m=0}^{N-1} x(m)\ (1)\ e^{j\frac{2\pi mK}{2N}}\ e^{j\frac{2\pi K}{2N}}$$

Change back of variable $m = n$

$$= \sum_{n=0}^{N-1} x(n) \cdot e^{j\frac{2\pi nK}{2N}}\ e^{j\frac{2\pi K}{2N}}$$

$$\boxed{e^{2x} = \frac{e^{x}}{e^{-x}}}$$

$$= \sum_{n=0}^{N-1} x(n)\, e^{j\frac{2\pi nK}{2N}}\ \frac{e^{j\frac{\pi K}{2N}}}{e^{-j\frac{\pi K}{2N}}}$$

$$S(K) = e^{j\frac{\pi K}{2N}} \left[\sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi nK}{2N}}\ e^{-j\frac{\pi K}{2N}} + \sum_{n=0}^{N-1} x(n)\, e^{j\frac{2\pi mK}{2N}}\ e^{j\frac{\pi K}{2N}}\right]$$

$$= e^{j\frac{\pi K}{2N}} \left[\sum_{n=0}^{N-1} x(n)\ e^{-j\frac{\pi K}{N}\left(n+\frac{1}{2}\right)} + \sum_{n=0}^{N-1} x(n)\ e^{j\frac{\pi K}{N}\left(n+\frac{1}{2}\right)}\right]$$

$$S(K) = e^{j\frac{\pi K}{2N}} \left[2\sum_{n=0}^{2N-1} x(n)\ \cos\left(\frac{\pi K}{N}\left(n+\frac{1}{2}\right)\right)\right]$$

Let. ~~S(K)~~ $V(K) = \left[2\sum_{n=0}^{N-1} x(n)\ \cos\left[\frac{\pi K}{N}\left(n+\frac{1}{2}\right)\right]\right]$.

$$S(K) = \left[e^{j\frac{\pi K}{2N}}\ V(K)\right] \quad \text{for } K = 0, 1, 2, \ldots, 2N-1$$

Complex Transform complex factor forward DCT or real transform

## Steps for Forward DCT

1. Take the given real sequence & extend / mirror image to make it real and even sequence.
2. Apply 2N point DFT over $s(n)$
3. Find complex transform $S(K)$

   $S(K) = e^{j\frac{\pi K}{2N}} \cdot V(K)$
4. Multiply both sides with $e^{-j\frac{\pi K}{2N}}$

   $e^{-j\frac{\pi K}{2N}}\ S(K) = V(K)$

   Forward DCT.

If $s(n) = s(2N-n-1)$ then $S(K)$ is complex

If $s(n) = s(2N-n)$ then $S(K)$ is real.

# Inverse DCT

$$S(K) = e^{j\frac{\pi K}{2N}} V(K)$$

$$V(K) = \sum_{n=0}^{N-1} x(n) \cos\left(\frac{\pi K}{N}\left(n+\frac{1}{2}\right)\right)$$

$$S(K) \xrightarrow[DCT]{Inverse} s(n)$$

$$s(n) = \frac{1}{2N} \sum_{K=0}^{2N-1} S(K)\, e^{j\frac{2\pi nK}{2N}}$$

For a real sequence

$$X(K) = \frac{1}{N} \sum_{n=0}^{N-1} x_R(n) . e^{-j\frac{2\pi nK}{2N}}$$

Circular shifted spectrum

$$X((-K))_N = X(N-K) = \frac{1}{N}\sum_{n=0}^{N-1} x_R(n) e^{-j\frac{2\pi(n)(N-K)}{N}}$$

$$= \left[\frac{1}{N}\sum_{n=0}^{N-1} x_R^*(n)\, e^{-j\frac{2\pi nK}{N}}\right]^*$$

$$X(N-K) = [X(K)]^* = X^*(K)$$

After circular shift, spectrum becomes conjugate symmetric.

$$= \frac{1}{2N}\left[\sum_{K=0}^{N-1} S(K)\, e^{j\frac{2\pi nK}{2N}} + \sum_{K=N}^{2N-1} S(K)\, e^{j\frac{2\pi nK}{2N}}\right]$$

$K = 2N - m$

$K = N, m = N$

$K = 2N-1, m = 1$

$$= \frac{1}{2N}\left[\sum_{K=0}^{N-1} S(K)\, e^{j\frac{2\pi nK}{2N}} + \sum_{m=1}^{N} S(2N-m)\, e^{j\frac{2\pi n(2N-m)}{2N}}\right]$$

$$= \frac{1}{2N}\left[\sum_{K=0}^{N-1} S(K)\, e^{j\frac{2\pi nK}{2N}} + \sum_{m=1}^{N} S(2N-m)\, \cancel{e^{j\frac{2\pi n(2N)}{2N}}}^{\,1}\, e^{-j\frac{2\pi nm}{2N}}\right]$$

$$= \frac{1}{2N}\left[\sum_{K=0}^{N-1} S(K)\, e^{j\frac{2\pi nK}{2N}} + \sum_{m=1}^{N} S(2N-m)\, e^{-j\frac{2\pi nm}{2N}}\right]$$

$$s(n) = \frac{1}{2N}\left[\sum_{K=0}^{N-1} S(K)\, e^{j\frac{2\pi nK}{2N}} + \sum_{K=1}^{N} S(2N-K)\, e^{-j\frac{2\pi nK}{2N}}\right]$$ Change of variable $m=K$.

Here $s(n)$ is $2N$ point sequence & $K$ is ranging from $0$ to $2N-1$ then by circular shift spectrum

$$S(2N-K) = S^*(K)$$

$$s(n) = \frac{1}{2N}\left[S(0) + \sum_{K=1}^{N-1} S(K)\, e^{j\frac{2\pi nK}{2N}} + \sum_{K=1}^{N} S^*(K)\, e^{-j\frac{2\pi nK}{2N}}\right]$$

$$= \frac{1}{2N}\left[S(0) + \sum_{K=1}^{N-1} S(K)\, e^{j\frac{2\pi nK}{2N}} + \sum_{K=1}^{N-1} S^*(K)\, e^{-j\frac{2\pi nK}{2N}} + S^*(N)\, e^{-j\frac{2\pi nN}{2N}}\right]$$

Now we substitute $S(K) = V(K)\, e^{j\frac{\pi K}{2N}}$

$$s(n) = \frac{1}{2N}\left[S(0) + \sum_{K=1}^{N-1} V(K)\, e^{j\frac{\pi K}{2N}}\, e^{j\frac{2\pi nK}{2N}} + \sum_{K=1}^{N-1} V^*(K) e^{-j\frac{\pi K}{2N}}\, e^{-j\frac{2\pi nK}{2N}} + 0\right]$$

$$= \frac{1}{2N}\left[S(0) + \sum_{K=1}^{N-1} V(K)\, e^{j\frac{\pi K}{N}\left(n+\frac{1}{2}\right)} + \sum_{K=1}^{N-1} V(K)\, e^{-j\frac{\pi K}{N}\left(n+\frac{1}{2}\right)}\right]$$

$\because S^*(N) = 0$

Since it is real transform $V(K) = V^*(K)$

$$s(n) = \frac{1}{2N}\left[ s(0) + \sum_{K=1}^{N-1} V(K)\left( e^{j\frac{\pi K}{N}(n+\frac{1}{2})} + e^{-j\frac{\pi K}{N}(n+\frac{1}{2})}\right)\right]$$

$$s(n) = \frac{1}{2N}\left[ s(0) + \sum_{K=1}^{N-1} V(K)\cos\left(\frac{\pi K}{N}(n+\tfrac{1}{2})\right)\right]$$

$\because s(0) = V(0)$

$$\boxed{s(n) = \frac{1}{2N}\left[ V(0) + \sum_{K=1}^{N-1} V(K)\cos\left(\frac{\pi K}{N}(n+\tfrac{1}{2})\right)\right]}$$

This is also real transform

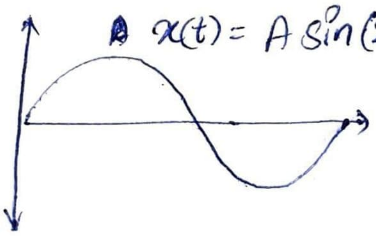
Inverse DCT

## Advantages of DCT

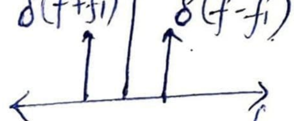
1. For real sequence, DCT is more energy compact as compared to DFT.

Ex

I. $x(t) = A\sin(2\pi f t)$



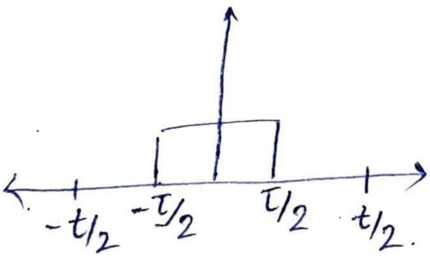
$\xrightarrow{F.T.}$





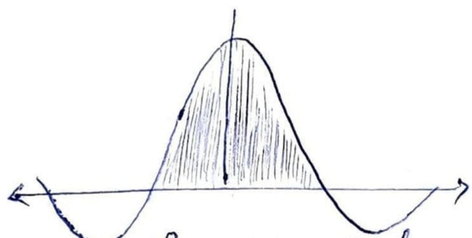
When there is a smooth signal, then it always has less no. of spectrum components.

II



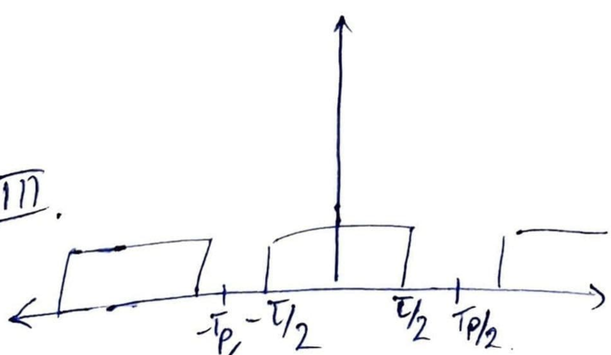


$T_p = \infty$ or $\frac{1}{T_p} = 0$ $\longrightarrow$



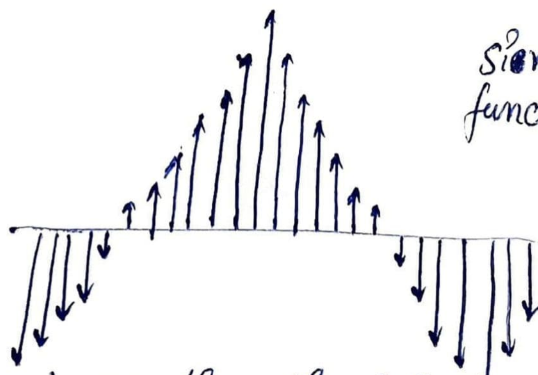
separation b/w spectrum components is zero.

III.





$\xrightarrow{F.T}$

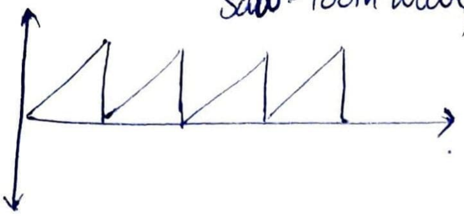


Sinc function.

When signal has sharp discontinuities, spectrum consists of large no. of components.

Ex

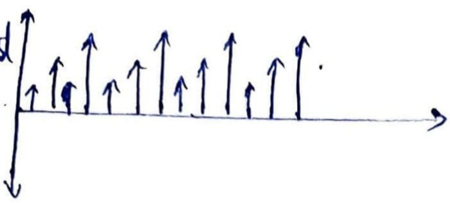
Saw-Tooth Waveform.



Here also, sharp discontinuities lead to many frequency components

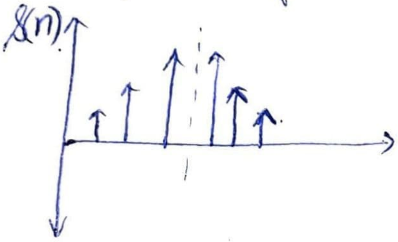
If we convert this signal to smooth one then it will consist of less frequency components.

Sampled signal.



Taking only one period.


By taking mirror image, to make it real and even




s(n) is more smooth as compared to x(n)
Hence, S(K) will surely have less frequency components.
Now if energy is considered ; Energy = $|S(K)|^2$

$$E_{|S(K)|} < E_{|X(K)|}$$

This is DCT. This is DFT.

So, for any real sequence, the extended version of given sequence will always be smooth & therefore, will be more energy compact as compared to DFT.

## Change of Basis.

$D_2$ = DFT matrix 2 $\qquad H_{conv} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

$D_2 H_{conv} D_2^{-1}$ = Diagonal Matrix

## Eigen Values & Eigen Vectors

for matrix A , $|A - \lambda I| = 0 \rightarrow$ characteristic equation

$\rightarrow AV = \lambda V$ , $V$ = eigen vector $\lambda$ = eigen value.

$H = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \qquad |H - \lambda I| = 0$

$$\begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix} = 0 \Rightarrow \begin{array}{l} (1-\lambda)^2 - 4 = 0 \\ \lambda^2 - 2\lambda - 3 = 0 \\ (\lambda-3)(\lambda+1) = 0 \\ \lambda_1 = 3 \quad \lambda_2 = -1 \end{array}$$

$V_1 = \begin{bmatrix} V_{11} \\ V_{12} \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} V_{11} \\ V_{12} \end{bmatrix} = 3 \begin{bmatrix} V_{11} \\ V_{12} \end{bmatrix} \qquad \begin{array}{l} V_{11} + 2V_{12} = 3V_{11} \\ 2V_{11} + V_{12} = 3V_{12} \\ V_{11} = V_{12} = 1 \end{array}$$

$$V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$V_2 = \begin{bmatrix} V_{21} \\ V_{22} \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} V_{21} \\ V_{22} \end{bmatrix} = (-1) \begin{bmatrix} V_{21} \\ V_{22} \end{bmatrix} \qquad \begin{array}{l} V_{21} + 2V_{22} = -V_{21} \\ 2V_{21} + V_{22} = -V_{22} \\ V_{21} = 1 \quad V_{22} = -1 \end{array}$$

$$V_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$$

$$AV = V\lambda$$

$$V^{-1}AV = V^{-1}V\lambda$$

$\boxed{V^{-1}AV = \lambda}$ $\leftarrow$ Diagonalisation.

$\boxed{Q = [V_1 \ V_2]}$ ⟶ Change of basis matrix

$AQ = Q\lambda$ Process → Diagonalisation or similarity transformation

<u>DFT</u> $X(K) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi nK}{N}}$, N has 2 values.

$$\begin{array}{c} K=0 \\ K=1 \end{array} \begin{bmatrix} X(0) \\ X(1) \end{bmatrix} = \begin{array}{cc} n=0 & n=1 \end{array} \begin{bmatrix} e^{-j\frac{2\pi nK}{N}} & e^{-j\frac{2\pi nK}{N}} \\ e^{-j\frac{2\pi nK}{N}} & e^{-j\frac{2\pi nK}{N}} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$D_2$ (Change of basis matrix)

$D_2$ has same eigen ~~value~~ vectors as for $\begin{bmatrix} 2 & 1 & 2 \\ 2 & 1 \end{bmatrix}$

$D_2 H_{conv} D_2^{-1}$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \left(\frac{1}{2}\right) \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} =$$

$$\Rightarrow \left(\frac{1}{2}\right) \begin{bmatrix} 3 & 3 \\ -1 & 1 \end{bmatrix}$$

$D_2^{-1} = \frac{1}{2}[D_2^T]$

$D_2$ is orthogonal matrix