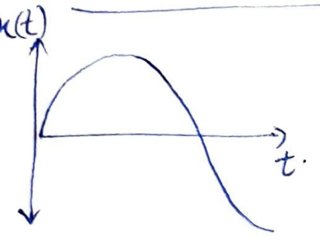
→ Continuous Time Periodic Signal
→ Continuous " Aperiodic "
→ Discrete " Periodic "
→ " " Aperiodic Signal

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

$$\mathrm{rect}(t) \longleftrightarrow \frac{\sin(\pi f)}{\pi f} = \mathrm{sinc}(\pi f)$$

Continuous Time Periodic Signal



$$x(t) = A \sin(2\pi f_0 t)$$

f_0 := Fundamental frequency

$f_0 = \frac{1}{T_p} \rightarrow$ Time Period.

Any periodic signal, can represented as follows

$$\boxed{x(t) = \sum_{K=-\infty}^{\infty} C_K e^{j2\pi K f_0 t}}$$

↑ Fourier Coefficient.

Multiply both sides by $e^{-j2\pi l f_0 t}$ & integrate over one period.

$$\int_{t_0}^{t_0+T_p} x(t) \cdot e^{-j2\pi l f_0 t} \, dt = \int_{t_0}^{t_0+T_p} \sum_{K=-\infty}^{\infty} C_K e^{j2\pi K f_0 t} \cdot e^{-j2\pi l f_0 t} \, dt$$

$$\text{RHS} = \int_{t_0}^{t_0+T_p} \sum_{K=-\infty}^{\infty} C_K \cdot e^{j2\pi (K-l) f_0 t} \, dt$$

Case I :- $K = l$.

$$\text{RHS} = \int_{t_0}^{t_0+T_p} \sum_{K=-\infty}^{\infty} C_K \, e^{j2\pi (K-l) f_0 t} \, dt$$

$$= \int_{t_0}^{t_0+T_p} \cancel{\Sigma} \, C_l \, e^{j0} \, dt = C_l \int_{t_0}^{t_0+T_p} dt = C_l T_p$$

Now $$\int_{t_0}^{t_0+T_p} x(t) \, e^{-j2\pi l f_0 t} \, dt = C_l T_p$$

$$\boxed{C_l = \frac{1}{T_p} \int_{t_0}^{t_0+T_p} x(t) \cdot e^{-j2\pi l f_0 t} \, dt}$$

Case II : $K \neq l$

$$\int_{t_0}^{t_0+T_p} x(t)\, e^{-j2\pi l f_0 t} dt = \int_{t_0}^{t_0+T_p} \sum_{K=-\infty}^{\infty} C_K e^{j2\pi(K-l) f_0 t} dt$$

$$RHS = \sum_{K=-\infty}^{\infty} C_K \int_{t_0}^{t_0+T_p} e^{j2\pi(K-l) f_0 t} dt$$

$$RHS = \sum_{K=-\infty}^{\infty} C_K \left[\frac{e^{j2\pi(K-l) f_0 t}}{j2\pi(K-l) f_0} \right]_{t_0}^{t_0+T_p}$$

$$RHS = \sum_{K=-\infty}^{\infty} C_K \left[\frac{e^{j2\pi(K-l) f_0 (t_0+T_p)} - e^{j2\pi(K-l) f_0 t_0}}{j2\pi(K-l) f_0} \right]$$

$f_0 = \frac{1}{T_p}$

$f_0 T_p = 1$

$$RHS = \sum_{K=-\infty}^{\infty} C_K \left[\frac{e^{j2\pi(K-l) f_0 t_0} e^{j2\pi(K-l) f_0 T_p} - e^{j2\pi(K-l) f_0 t_0}}{j2\pi(K-l) f_0} \right]$$

$\therefore$ RHS = 0

Sum of weighted exponentials

$$x(t) = \sum_{K=-\infty}^{\infty} C_K e^{j2\pi K f_0 t} \rightarrow \text{Synthesis Equation}$$

weight is fourier series coefficient.

$$\text{Analysis Equation} \leftarrow \boxed{C_K = \frac{1}{T_p} \int_{T_p} x(t) \cdot e^{-j2\pi K f_0 t} dt}$$

Discrete in nature

$C_K \rightarrow |C_K| e^{j\theta_K}$ (complex coefficient) $\quad \theta_K = \angle C_K$

$C_{-K} = |C_K| e^{-j\theta_K}$

Magnitude is +ve, even symmetric & Phase is odd symmetric.

Alternate Representation

$$x(t) = \sum_{K=-\infty}^{\infty} C_K e^{j2\pi K f_0 t} = \sum_{K=-\infty}^{-1} C_K e^{j2\pi K f_0 t} + C_0 e^{j0} + \sum_{K=1}^{\infty} C_K e^{j2\pi K f_0 t}$$

$$= C_0 + \sum_{K=-\infty}^{-1} C_K e^{j2\pi K f_0 t} + \sum_{K=1}^{\infty} C_K e^{j2\pi K f_0 t}$$

$\Downarrow$

$K \rightarrow -K$

$$x(t) = C_0 + \sum_{K=1}^{\infty} C_{-K} e^{-j2\pi K f_0 t} + \sum_{K=1}^{\infty} C_K e^{j2\pi K f_0 t}$$

$$x(t) = C_0 + \sum_{K=1}^{\infty} |C_K| e^{-j\theta_K} e^{-j2\pi K f_0 t} + \sum_{K=1}^{\infty} |C_K| e^{j\theta_K} e^{j2\pi K f_0 t}$$

$$x(t) = a_0 + \sum_{K=1}^{\infty} |C_K| \left[e^{-j(2\pi K f_0 t + \theta_K)} + e^{j(2\pi K f_0 t + \theta_K)} \right]$$

$$= a_0 + 2 \sum_{K=1}^{\infty} |C_K| \left[\cos(2\pi K f_0 t + \theta_K) \right]$$

$$= a_0 + 2 \sum_{K=1}^{\infty} |C_K| \left[\cos(2\pi K f_0 t) \cos\theta_K - \sin(2\pi K f_0 t) \sin\theta_K \right]$$

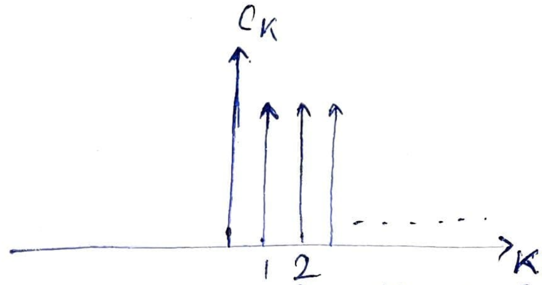
$$\boxed{x(t) = a_0 + \sum_{K=1}^{\infty} a_K \cos(2\pi K f_0 t) - \sum_{K=1}^{\infty} b_K \sin(2\pi K f_0 t)}$$

Alternate representation of Fourier series.

$a_K = 2|C_K| \cos\theta_K \quad b_K = 2|C_K| \sin\theta_K$

Spectrum

$$x(t) = \sum_{K=-\infty}^{\infty} C_K e^{j2\pi K f_0 t}$$



$K = 0 \rightarrow x(t) = C_0$

$K = 1 \rightarrow x(t) = C_1 e^{j2\pi f_0 t}$

$K = 2 \rightarrow x(t) = C_2 e^{j2\pi (2) f_0 t}$

Line spectrum is discrete in nature.

Dirichlet Conditions (Sufficient conditions)

→ The given signal should have finite no. of discontinuity

→ The given signal ~~has~~ " " " no. of maxima & minima.

→ The given signal should be integrable over one period

$$\int_{T_p} |x(t)| dt < \infty \quad [\text{finite}]$$

If the fourier series doesn't fulfil dirichlet conditions, then it is not necessary that F.S. not exist

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \infty \quad \left[\text{Infinite energy}\right]$$

Power of signal $= \frac{1}{T_p} \int_{T_p} |x(t)|^2 dt < \infty$

Finite power signal.

$$P_x = \frac{1}{T_p} \int_{T_p} |x(t)|^2 dt$$

$$= \frac{1}{T_p} \int_{T_p} x(t)\, x^*(t) dt = \frac{1}{T_p} \int_{T_p} x(t) \left[\sum_{K=-\infty}^{\infty} C_K^* e^{-j2\pi K f_0 t} \right] dt$$

$$= \cancel{\frac{1}{T_p} \int_{T_p}} \sum_{K=-\infty}^{\infty} C_K^* \left[\frac{1}{T_p} \int_{T_p} x(t) e^{-j2\pi K f_0 t} dt \right]$$

Changing order of summation & integration

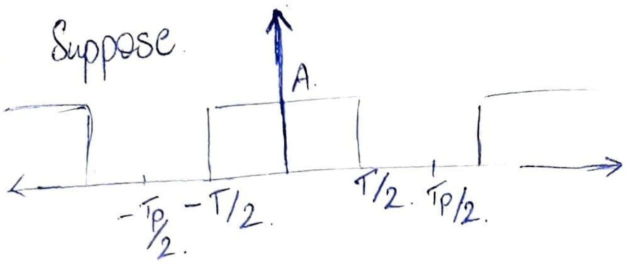
$$= \sum_{K=-\infty}^{\infty} C_K^* C_K = \sum_{K=-\infty}^{\infty} |C_K|^2.$$

$$P_x = \frac{1}{T_P}\int_{T_P} |x(t)|^2 dt = \sum_{K=-\infty}^{\infty} |C_K|^2$$

Parseval's Theorem

Whether time or spectrum domain, power is going to be the same.

Ex. Suppose



$$x(t) = \begin{cases} A & ; -T/2 < t < T/2 \\ 0 & ; \text{otherwise} \end{cases}$$

Analysis Expression $\quad C_K = \frac{1}{T_P}\int_{T_P} x(t)\, e^{-j2\pi K f_0 t}\, dt$

$$C_K = \frac{1}{T_P}\int_{-T_P/2}^{T_P/2} x(t)\, e^{-j2\pi K f_0 t}\, dt = \frac{1}{T_P}\int_{-T/2}^{T/2} A\, e^{-j2\pi K f_0 t}\, dt$$

$$= \frac{A}{T_P}\left[\frac{e^{-j2\pi K f_0 t}}{-j2\pi K f_0}\right]_{-T/2}^{T/2} = \frac{A}{T_P}\left[\frac{e^{-j2\pi K f_0 T/2} - e^{+j2\pi K f_0 T/2}}{-j2\pi K f_0}\right]$$

$$= \frac{A}{T_P}\left[\frac{\sin(2\pi K f_0 T/2)}{\pi K f_0}\right] = \frac{AT}{T_P}\,\frac{\sin(\pi K f_0 T)}{\pi K f_0 T}$$

$$\boxed{C_K = \frac{AT}{T_P}\,\mathrm{sinc}(\pi K f_0 T)} \Rightarrow \text{for } K = 0$$

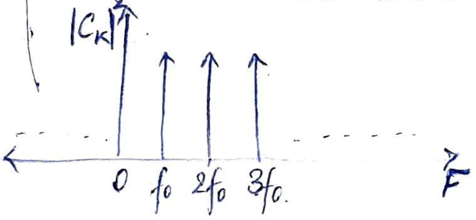
$$C_K = \frac{AT}{T_P}\,\frac{\sin(\pi K f_0 T)}{\pi K f_0 T} = \frac{0}{0} \quad \text{L'Hospital rule.}$$

$$\frac{\frac{d}{dK}\{\sin(\pi K f_0 T)\}}{\frac{d}{dK}\,\pi K f_0 T} = \frac{\cos(\pi K f_0 T)\cdot \cancel{\pi f_0 T}}{\cancel{\pi f_0 T}}$$

$$|C_K|_{K=0} = \frac{AT}{T_P}\cos(\pi K f_0 T) = \frac{AT}{T_P} = C_0$$

T: lobe width.

Line Spectrum



$$C_K = \frac{AT}{T_P}\,\frac{\sin(\pi K f_0 T)}{\pi K f_0 T}$$

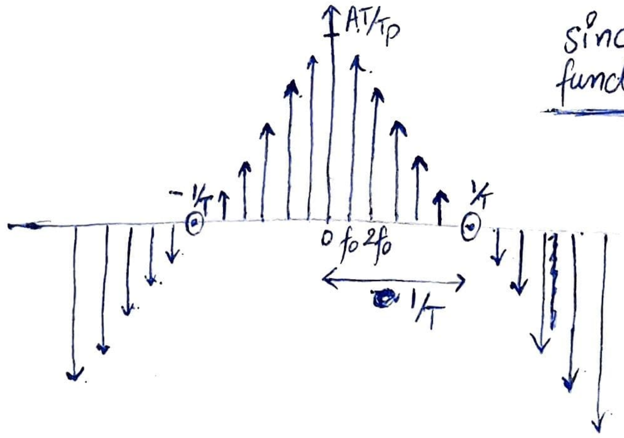
$\sin\theta = 0 \qquad \theta = \pm m\pi$

$\pi K f_0 T = \pm m\pi$

$K f_0 T = \pm m$

$f = K f_0 \qquad fT = \pm m$

At $f = \pm \frac{m}{T}$, spectrum is having zero value



sinc function

Effects of T & T_p on Spectrum

→ When T_p is fixed.

If we increase T, then $\frac{1}{T}$ decreases and hence the spectrum contracts.

If we decrease T, then $\frac{1}{T}$ increases and hence the spectrum widens.

No. of frequency components also increases when T decreases.

→ When T is fixed.

If we increase T_p, then spectral lines come closer enough.

If we decrease T_p, then spectral lines go far from each other.

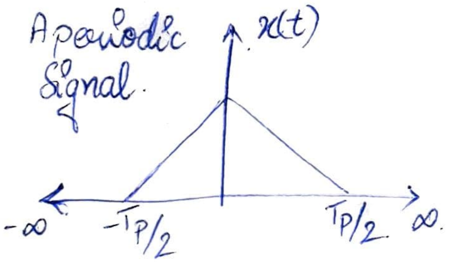
Worst Scenario.

$T_p \to \infty$, $\frac{1}{T_p} \to 0$

Spectrum looks continuous.

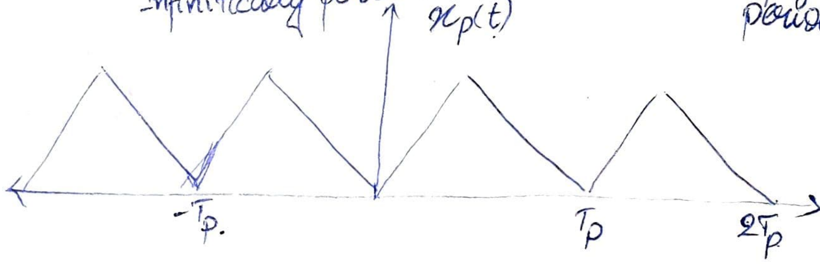
Continuous Time Aperiodic Signal

Aperiodic Signal.



⇒

Infinitically periodic :- periodic with infinite period.



$$\lim_{T_p \to \infty} x_p(t) = x(t) \qquad x(t) = \sum_{K=-\infty}^{\infty} C_K e^{j2\pi K f_0 t}$$

$$x_p(t) = \sum_{K=-\infty}^{\infty} C_K e^{j2\pi K f_0 t}$$

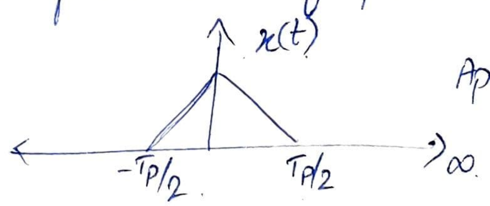
$$C_K = \frac{1}{T_p} \int_{-T_p/2}^{T_p/2} x_p(t) e^{-j2\pi K f_0 t} dt$$

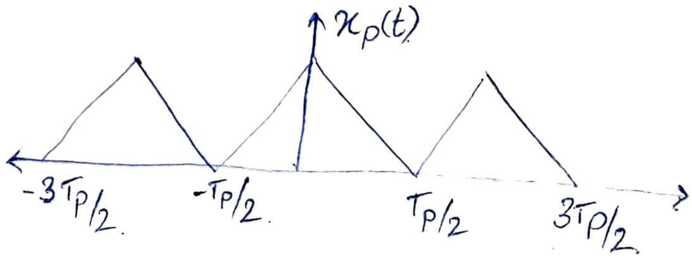
$$C_K = \frac{1}{T_p} \int_{-T_p/2}^{T_p/2} x(t) e^{-j2\pi K f_0 t} dt$$

$$\text{I} \quad \boxed{C_K = \frac{1}{T_p} \int_{-\infty}^{\infty} x(t) e^{-j2\pi K f_0 t} dt}$$

Fourier Representation of Aperiodic Signal.

Aperiodic.





$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt \quad \text{— II}$$

$$C_K = \frac{1}{T_p}\int_{-\infty}^{\infty} x(t)\, e^{-j2\pi K f_0 t}\, dt$$

Analysis Expression

$$\boxed{C_K = \frac{1}{T_p} X(K f_0)}$$

Now substituting this in synthesis eq.

$$\cancel{C_K = \frac{1}{T_p} X(} \quad x_p(t) = \sum_{K=-\infty}^{\infty} C_K\, e^{j2\pi K f_0 t} \Rightarrow x_p(t) = \sum_{K=-\infty}^{\infty} \frac{1}{T_p} X(K f_0)\, e^{j2\pi K f_0 t}$$

$$\Rightarrow \lim_{T_p \to \infty} x_p(t) = x(t)$$

$$x_p(t) = \frac{1}{T_p}\sum_{K=-\infty}^{\infty} (\Delta f)\, X(K\Delta f)\, e^{j2\pi K \Delta f t}$$

$\therefore \left(\frac{1}{T_p}\right) \to 0$ Infinitesimal points/elements

replace $\to \Delta f$

$$\lim_{T_p \to \infty} x_p(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi f t}\, df$$

$$\boxed{x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi f t}\, dt}$$

Synthesis Expression for Aperiodic signal.

→ Fourier Transform exist for dirichlet conditions

~~cond~~

1. Given signal $x(t)$ has finite no. of discontinuities
2. Signal $x(t)$ has finite no. of maxima & minima.
3. Signal should be integrable $\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$

Energy of the Signal.

Signal is aperiodic → Energy signal

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

Parseval's Theorem

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} x(t)\, x^*(t)\, dt$$

$$= \int_{-\infty}^{\infty} x(t) \left[\int_{-\infty}^{\infty} X(f)\, e^{-j2\pi f t}\, df\right]^* dt$$

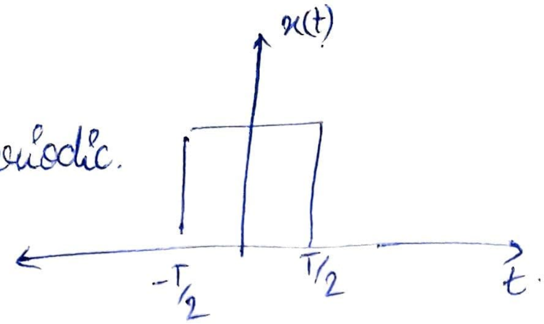
$$= \int_{-\infty}^{\infty} x(t) \int_{-\infty}^{\infty} X^*(f)\, e^{-j2\pi f t}\, df\, dt$$

$$= \int_{-\infty}^{\infty} X^*(f) \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt.\, df$$

$$= \int_{-\infty}^{\infty} X^*(f) \cdot X(f)\, df = \int_{-\infty}^{\infty} |X(f)|^2 df$$

Ex

Aperiodic.



Spectrum ~~of~~ → F.T. of signal

$$X(f) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j2\pi ft} dt$$

$$= \int_{-T/2}^{T/2} A e^{-j2\pi ft} dt$$

$$= A \left[\frac{e^{-j2\pi ft}}{-j2\pi f} \right]_{-T/2}^{T/2}$$

$$= A \left[\frac{e^{-j2\pi fT/2} - e^{j2\pi fT/2}}{-j2\pi f} \right]$$

$$X(f) = A \frac{\sin(\pi f T)}{\pi f T}$$

⇒ T ↑, lobe width ↓.
T ↓, lobe width ↑.

for $f = 0$

$$X(f) = 0/0$$

By applying L-Hospitals rule.

$$X(f) = \frac{A\cancel{T} \cos(\pi f T)\, \cancel{\pi} T}{\cancel{\pi}\cancel{T}}$$

$$= AT \cos(\pi f T)$$

$$X(0) = AT$$

$$X(f) = \begin{cases} AT & , f = 0 \\ AT \frac{\sin \pi f T}{\pi f T} & , f \neq 0 \end{cases}$$

$$\pi f T = \pm m\pi$$

$$f = \pm \frac{m}{T}$$