The Discrete Time Fourier Transform (DTFT)

- The discrete-time Fourier transform (DTFT) **X(e$^{j\omega}$)** of a discrete-time signal x[n]:

$$X\left(e^{j\omega}\right)=\sum_{n=-\infty}^{\infty}x[n]e^{-j\omega n}$$

DTFT X(e$^{j\omega}$) is a complex function of ω

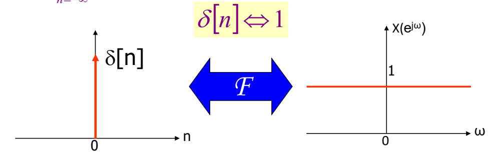
DTFT of the Unit Impulse Sequence

$$\delta[n]=\begin{cases}1, & n=0\\ 0, & else\end{cases}$$

$$X\left(e^{j\omega}\right)=\sum_{n=-\infty}^{\infty}x[n]e^{-jn\omega}$$

$$X\left(e^{j\omega}\right)=\sum_{n=-\infty}^{\infty}\delta[n]e^{-jn\omega}=....+\delta[-1]+\delta[0]+\delta[+1]+....=1$$

$$\delta[n]\Leftrightarrow 1$$



DTFT of the Causal Sequence

- Find the DTFT of $x[n]=\alpha^n u[n]; \quad |\alpha|<1$

$$u[n]=\begin{cases}1, & n\geq 0\\ 0, & else\end{cases}$$

$$X\left(e^{j\omega}\right)=\sum_{n=-\infty}^{\infty}x[n]e^{-jn\omega}$$

$$X\left(e^{j\omega}\right)=\sum_{n=0}^{\infty}\alpha^n e^{-jn\omega}=\sum_{n=0}^{\infty}\left(\alpha e^{-j\omega}\right)^n=\frac{1}{1-\alpha e^{j\omega}}\qquad \because\left|\alpha e^{-j\omega}\right|=|\alpha|<1$$

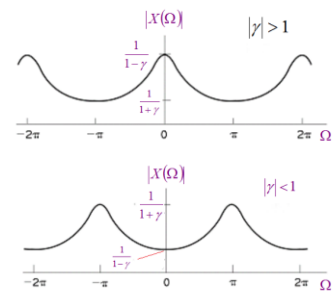
$$X\left(e^{j\omega}\right)=\frac{1}{1-\alpha e^{j\omega}}=\frac{1}{1-\alpha[\cos\omega+j\sin\omega]}=\frac{1}{[1-\alpha\cos\omega]+j[\sin\omega]}$$

Magnitude and phase response for

$$\left|X\left(e^{j\omega}\right)\right|=\frac{1}{\sqrt{1-2\alpha\cos\omega+\alpha^2}}$$

$$\omega=0:\left|X\left(e^{j\omega}\right)\right|=\frac{1}{\sqrt{1-2\alpha+\alpha^2}}=\frac{1}{1-\alpha}=\frac{1}{1-0.5}=\frac{1}{0.5}=2$$

$$\omega=\pi:\left|X\left(e^{j\omega}\right)\right|=\frac{1}{\sqrt{1+2\alpha+\alpha^2}}=\frac{1}{1+\alpha}=\frac{1}{1+0.5}=\frac{1}{1.5}=0.6$$



- The DTFT X(e$^{j\omega}$) of x[n] is a continuous function of ω
- **It is also a periodic function of ω with a period 2π:**

$$X\left(e^{j(\omega+2\pi k)}\right)=\sum_{n=-\infty}^{\infty}x[n]e^{-jn(\omega+2\pi k)}\qquad e^{-j2\pi kn}=1\qquad \forall k,n\varepsilon Z$$

$$=\sum_{n=-\infty}^{\infty}x[n]e^{-j\omega n}e^{-j2\pi kn}=\sum_{n=-\infty}^{\infty}x[n]e^{-j\omega n}$$

$$=X\left(e^{j\omega}\right)\qquad \forall k\varepsilon Z$$

- Thus $X\left(e^{j\omega}\right)=\sum_{n=-\infty}^{\infty}x[n]e^{-jn\omega}$

represents the Fourier series representation of the periodic function

- IDTFT: $x[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi}X\left(e^{j\omega}\right)e^{j\omega n}d\omega$

IDTFT

- The IDFT represents the time-domain signal x[n] in terms of complex exponential functions

$$x[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi}X\left(e^{j\omega}\right)e^{j\omega n}d\omega$$

Integration can be done over any 2π interval

$$x[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left\{\sum_{l=-\infty}^{\infty}x[l]e^{-j\omega l}\right\}e^{j\omega n}d\omega$$

$$x[n]=\sum_{l=-\infty}^{\infty}x[l]\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-j\omega l}e^{j\omega n}d\omega\right]$$

$$x[n]=\sum_{l=-\infty}^{\infty}x[l]\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{j\omega(n-l)}d\omega\right]$$

$$x[n]=\sum_{l=-\infty}^{\infty}x[l]\frac{\sin\pi(n-l)}{\pi(n-l)}$$

$$\frac{\sin\pi(n-l)}{\pi(n-l)}=\begin{cases}1, & n=l\\ 0, & n\neq l\end{cases}$$

$$=\delta[n-l]$$

- Hence

$$x[n]=\sum_{l=-\infty}^{\infty}x[l]\delta[n-l]$$

$$=x[n]$$

Properties of DTFT (I)

❑ If $g[n] = G(e^{j\omega})$ and $h[n] = H(e^{j\omega})$

then $\alpha g[n] + \beta h[n] \Leftrightarrow \alpha G(e^{j\omega}) + \beta H(e^{j\omega})$

Linearity

$$\sum_{n=-\infty}^{\infty} \{\alpha g[n] + \beta h[n]\} e^{-j\Omega n} = \alpha \sum_{n=1}^{\infty} g[n] e^{-j\omega n} + \beta \sum_{n=1}^{\infty} g[n] e^{-j\omega n}$$

$$= \alpha G(e^{j\omega}) + \beta H(e^{j\omega})$$

Properties of DTFT (II)

❑ If $x[n]$ is real

then $X(e^{j\omega}) = X^*(e^{-j\omega})$

Conjugate Symmetry

❑ If $x[n]$ is real

then $X(e^{j\omega}) = -X^*(e^{-j\omega})$

Conjugate-Antisymmetric Function

Properties of DTFT (III)

❑ Time reversal: leads to frequency reversal in the DTFT

If $g[n] \Leftrightarrow G(e^{j\omega})$

Then $g[-n] \Leftrightarrow G(e^{-j\omega})$

$$\mathcal{F}\{g[-n]\} = \sum_{n=-\infty}^{\infty} g[-n] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} g[-n] e^{-j(-n)\omega}$$

$$= G(e^{-j\omega})$$

Properties of DTFT (IV)

❑ Multiplication by n: Frequency Differentiation

If $g[n] \Leftrightarrow G(e^{j\omega})$

Then $ng[n] \Leftrightarrow j\dfrac{dG(e^{j\omega})}{d\omega}$

$$\mathcal{F}\{ng[n]\} = \sum_{n=-\infty}^{\infty} ng[n] e^{-j\omega n}$$

$$= \frac{1}{-j} \sum_{n=-\infty}^{\infty} g[n] \frac{de^{-j\omega n}}{d\omega}$$

$$= j\frac{d}{d\omega} \sum_{n=-\infty}^{\infty} g[n] e^{-j\omega n} = j\frac{d}{d\omega} G(e^{j\omega})$$

❑ Can't take derivative w.r.t. n in DT, but can take derivative w.r.t. ω in the frequency domain.

Properties of DTFT (V)

❑ Time-Shifting Property → Phase Change (Similar to CTFT)

If $g[n] \Leftrightarrow G(e^{j\omega})$

Then $g[n - n_d] \Leftrightarrow G(e^{j\omega}) e^{-jn_d\omega}, n_d \in I$

Delaying a signal by n_d units does not change its amplitude spectrum, but the phase spectrum is changed by $-n_d\omega$. i.e. phase is a linear function of ω with slope $-n_d$.

$$\mathcal{F}[g(n - n_d)] = \sum_{n=-\infty}^{\infty} g[n - n_d] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} g[n] e^{-j\omega(n+n_d)}$$

$$= e^{-j\omega n_d} \sum_{n=-\infty}^{\infty} g[n] e^{-j\omega n} = e^{-j\omega n_d} G(e^{j\omega})$$

Properties of DTFT (VI)

❑ Frequency-Shifting (Modulation) Property → Signal Modulation (Similar to CTFT)

Multiplying a sequence by a complex exponential correspond to shifting its DTFT in the frequency domain.

If $g[n] \Leftrightarrow G(e^{j\omega})$

Then $g[n] e^{j\omega_s n} \Leftrightarrow G(e^{j(\omega - \omega_s)})$ 'delay' in frequency

This property is the dual of the time-shifting property.

$$\mathcal{F}\{e^{j\omega_s n} g[n]\} = \sum_{n=-\infty}^{\infty} e^{j\omega_s n} g[n] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} g[n] e^{-j(\omega - \omega_s) n}$$

$$= G(e^{j(\omega - \omega_s)})$$

Properties of DTFT (VII)

- Time and Frequency Convolution Property (Similar to CTFT)

If $g[n] \Leftrightarrow G(e^{j\omega})$ and $h[n] \Leftrightarrow H(e^{j\omega})$

Then

$$g[n] * h[n] \Leftrightarrow G(e^{j\omega})H(e^{j\omega})$$

and

$$g[n]h[n] \Leftrightarrow \frac{1}{2\pi} G(e^{j\omega}) * H(e^{j\omega})$$

where

$$g[n] * h[n] = \sum_{m=-\infty}^{\infty} g[m]h[n-m]$$

and

$$G(e^{j\omega}) * H(e^{j\omega}) = \int_{2\pi} G(u) * H(e^{j(\omega-u)})du$$

- Since n is discrete while ω is continuous, there is no dulity property for DTFT

Parseval's Theorem

- Parseval's theorem: relates total energy in a sequence to its DTFT.

- If $x[n] \Leftrightarrow X(e^{j\omega})$

Then $E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi} \int_{2\pi} |X(e^{j\omega})|^2 d\omega$

- Proof:

$$E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \sum_{n=-\infty}^{\infty} x^*[n]\left[\frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n} d\omega\right]$$

$$= \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})\left[\sum_{n=-\infty}^{\infty} x^*[n]e^{j\omega n}\right]d\omega$$

$$= \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})X^*(e^{j\omega})d\Omega = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

Example

- Determine the DTFT of y[n]

$$y[n] = (n+1)\alpha^n u[n], \quad |\alpha| < 1$$

$$y[n] = n\alpha^n u[n] + \alpha^n u[n], \quad |\alpha| < 1$$

x[n]

$$y[n] = nx[n] + x[n]$$

Apply time-differentiation property of DTFT

$$X(e^{j\omega}) = \frac{1}{1-\alpha e^{-j\omega}}$$

$$nx[n] = j\frac{dX(e^{j\omega})}{d\omega} = j\frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right) = \frac{\alpha e^{-j\omega}}{(1-\alpha e^{-j\omega})^2}$$

$$Y(e^{j\omega}) = \left(\frac{1}{1-\alpha e^{-j\omega}}\right) + \left\{\frac{\alpha e^{-j\omega}}{(1-\alpha e^{-j\omega})^2}\right\} = \frac{1}{(1-\alpha e^{-j\omega})^2}$$

Energy Density Spectrum

- The total energy of a finite-energy sequence g[n] is given by

$$E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

- From Parseval's relation we observe that

$$E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

Energy density spectrum $S_{xx}(\omega)$

$$S_{xx}(\omega) = |X(e^{j\omega})|^2$$

- The area under this curve in the range − π ≤ ω ≤ π divided by 2π is the energy of the sequence

Frequency Response of LTI Systems

- Consider the LTI discrete-time system with an impulse response {h[n]}

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

- Its input-output relationship in the time-domain is given by the convolution sum

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

- If the input is of the form $x[n] = e^{j\omega n}, \quad -\infty < n < \infty$

then output

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]e^{j\omega(n-k)} = \left(\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}\right)e^{j\omega n}$$

$$y[n]=\sum_{k=-\infty}^{\infty}h[k]e^{j\omega(n-k)}=\underbrace{\left(\sum_{k=-\infty}^{\infty}h[k]e^{-j\omega k}\right)}_{H(e^{j\omega})}e^{j\omega n}$$

$$y[n]=H\underbrace{(e^{j\omega})}e^{j\omega n}$$

Frequency response of the LTI discrete-time system

Eigen function

- Thus, for a complex exponential input signal $e^{j\omega n}$, the output of an LTI discrete-time system is also a complex exponential signal of the same frequency multiplied by a complex constant $H(e^{j\omega})$.

$$\theta(\omega)=\arg H(e^{j\omega})$$

Phase response of LTI DT system

- If the impulse response h[n] is real then the magnitude function is an even function of ω

$$\left|H(e^{j\omega})\right|=\left|H(e^{-j\omega})\right|$$

and the phase function is an odd function of ω:

$$\theta(\omega)=\theta(-\omega)$$

- Likewise, for a real impulse response h[n], $H_{re}(e^{j\omega})$ is even and $H_{im}(e^{j\omega})$ is odd

- $H(e^{j\omega})$ in general, is a complex function of ω with a period 2π, with its real and imaginary parts as follows:

$$H(e^{j\omega})=H_{re}(e^{j\omega})+jH_{im}(e^{j\omega})$$

Or, in terms of its magnitude and phase,

$$H(e^{j\omega})=\left|H(e^{j\omega})\right|e^{j\theta(\omega)}$$

where,

Magnitude response of LTI DT system

$$G(\omega)=20\log_{10}H(e^{j\omega})\quad dB$$

- The negative of the gain function $A(\omega)=-G(\omega)$ is called the attenuation or loss function.