Frequency Response of LTI Systems.

Lets find the resulting response if system is excited with a single I/p frequency

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = h(n) * x(n)$$

$$\underset{\text{Amplitude.}}{A} \cdot e^{j\omega n} \longrightarrow \boxed{h(n)} \longrightarrow y(n)$$

$$y(n) = h(n) * x(n)$$

$$= \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$$

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, A\, e^{j\omega(n-k)}$$

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, A\, e^{j\omega n}\, e^{-j\omega k}$$

$$= A\left[\sum_{k=-\infty}^{\infty} h(k)\, e^{-j\omega k}\right] e^{j\omega n}$$

$$= A\, H(\omega)\, e^{j\omega n}$$

$$y(n) = H(\omega)\, A\, e^{j\omega n}$$

$$\boxed{y(n) = H(\omega) \cdot x(n)}$$

$$y(n) = [\underset{\text{mag.}}{|H(\omega)|}\ \underset{\text{angle}}{\angle H(\omega)}]\, x(n)$$

→ The o/p response $y(n)$ is scaled version of $x(n)$ & its scaled by $|H(\omega)|$.

→ O/p is relying on the phase content added by LTI system $\angle H(\omega)$

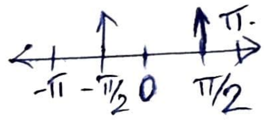
→ $e^{j\omega n}$ is called eigen function of the corresponding $|H(\omega)|$ is called eigen value

Ex

$$h(n) = \left(\frac{1}{2}\right)^n u(n)$$

$$x(n) = e^{j\frac{\pi n}{2}}$$

$$y(n) = H(\omega) \cdot x(n)$$



$$h(n) = \left(\frac{1}{2}\right)^n u(n) \Rightarrow H(z) = \frac{1}{1 - \frac{z^{-1}}{2}}$$

$$z = e^{j\omega}$$

$$y(n) = \underbrace{\left[\frac{1}{1 - \frac{1}{2}e^{-j\omega}}\right]}_{H(\omega)} e^{j\frac{\pi n}{2}}$$

At $\omega = \frac{\pi}{2}$.

$$y(n) = \left[\frac{1}{1 - \frac{1}{2}e^{-j\pi/2}}\right] \cdot x(n)$$

$$= \frac{1}{1 - \frac{1}{2}\left(\cos\frac{\pi}{2} - j\sin\frac{\pi}{2}\right)} \cdot x(n)$$

$$= \frac{1}{1 - \frac{1}{2}(-j)}\, x(n)$$

$$y(n) = \frac{x(n)}{1 + j/2}. \qquad H(\omega) = \frac{1}{1 + \frac{1}{2}j}$$

$$|H(\omega)| = \left|\frac{1}{\frac{2+j}{2}}\right| = \frac{\sqrt{4}}{\sqrt{4+1}} = \frac{2}{\sqrt{5}}$$

$$H(\omega) = \frac{2}{2+j} \cdot \frac{2-j}{2-j} = \frac{2(2-j)}{5} = \frac{4}{5} - \frac{2}{5}j$$

$$\angle H(\omega) = \tan^{-1}\left(\frac{-2/5}{4/5}\right) = \tan^{-1}\left(-\frac{1}{2}\right)$$

$$y(n) = \left[\frac{2}{\sqrt{5}}\ \angle \tan^{-1}\left(-\frac{1}{2}\right)\right] e^{j\frac{\pi n}{2}}$$

If $x(n) = e^{j\pi n}$

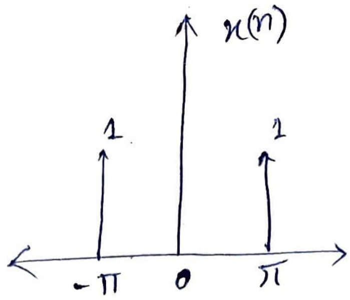
$$H(\omega) = \frac{1}{1 - \frac{1}{2}e^{-j\omega}}$$

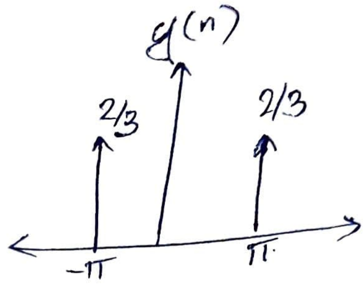
$$H(\omega)\Big|_{\omega=\pi} = \frac{1}{1-\frac{1}{2}e^{-j\pi}}$$

$$= \frac{1}{1-\frac{1}{2}(-1)}$$

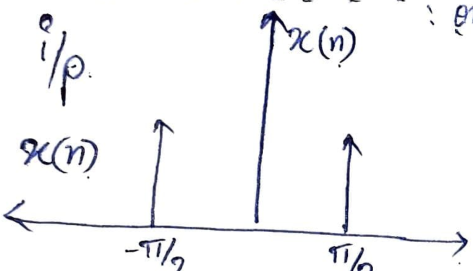
$$= \frac{2}{3}$$

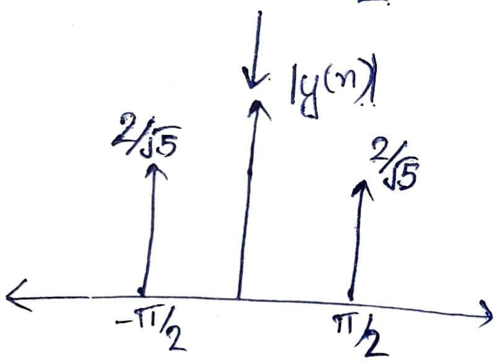
$$\boxed{y(n) = \frac{2}{3} e^{j\pi n}}$$ This is scaled in magnitude only

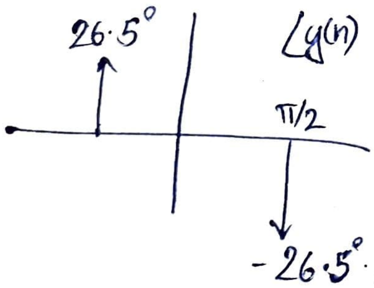




i/p.







Frequency Response of LTI system will depend on pole zero location.

$$H(z) = \frac{\sum_{K=0}^{q} b_K z^{-K}}{1 + \sum_{K=1}^{p} a_K z^{-K}}$$ Generalised system

q :- zeros
p :- poles

$$H(z) = \frac{b_0 + b_1 z^{-1} + \dots + b_q z^{-q}}{1 + a_1 z^{-1} + \dots + a_p z^{-p}}$$

$$= \frac{b_0\left[1 + \frac{b_1}{b_0} z^{-1} + \dots + \frac{b_q}{b_0} z^{-q}\right]}{\left[1 + a_1 z^{-1} + \dots + a_p z^{-p}\right]} \begin{matrix} \leftarrow m \text{ roots} \\ \leftarrow n \text{ roots.} \end{matrix}$$

$$H(z) = b_0 \frac{\prod_{K=1}^{m} (1 - q_K z^{-K})}{\prod_{K=1}^{n} (1 - P_K z^{-K})}$$

$$H(z)\Big|_{z=e^{j\omega}} = b_0 \frac{\prod_{K=1}^{m} (1 - q_K e^{-j\omega K})}{\prod_{K=1}^{n} (1 - P_K e^{-j\omega K})}$$

Taking $e^{-j\omega}$ common

$$\boxed{H(e^{j\omega}) = b_0 \frac{e^{j\omega m}}{e^{j\omega n}} \frac{\prod_{K=1}^{m} (e^{j\omega} - q_K)}{\prod_{K=1}^{n} (e^{j\omega} - P_K)}}$$

Frequency Response Function.

Since each pole contributes in magnitude scaling & phase in the response to an eigen function "$e^{j\omega n}$"

$$H(e^{j\omega}) = b_0 \, e^{j\omega(n-m)} \frac{\overbrace{V_0(\omega)}^{(e^{j\omega}-q_1)} \; \overbrace{V_1(\omega)}^{(e^{j\omega}-q_2)} \cdots V_m(\omega)}{\underbrace{U_0(\omega)}_{(e^{j\omega}-p_1)} \; \underbrace{U_1(\omega)}_{(e^{j\omega}-p_2)} \cdots U_n(\omega)}$$

∴ Response depend on overall magnitude response of LTI system.

Magnitude Response →

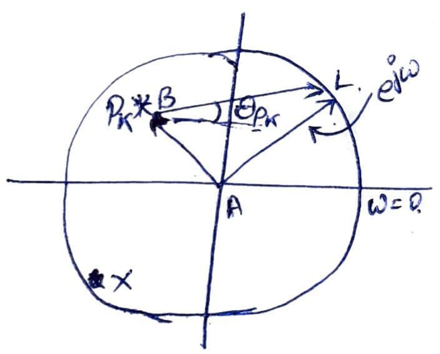
$$\boxed{|H(e^{j\omega})| = |b_0| \, |e^{j\omega(n-m)}| \frac{|V_0(\omega)| \, |V_1(\omega)| \cdots |V_m(\omega)|}{|U_0(\omega)| \, |U_1(\omega)| \cdots |U_n(\omega)|}}$$

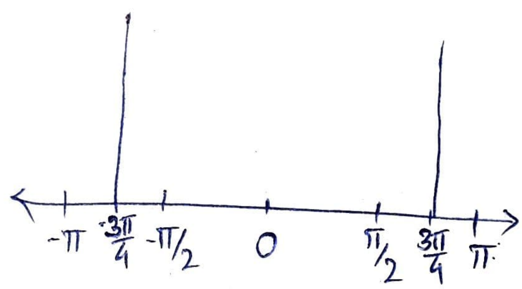
Phase Response

$$\boxed{\angle H(\omega) = \angle b_0 + \angle e^{j\omega(n-m)} + \angle V_0(\omega) + \angle V_1(\omega) + \cdots + \angle V_m(\omega) - U_0(\omega) - U_1(\omega) - \cdots - U_n(m)}$$

Let us consider a pole located at $\frac{3\pi}{4}$

$$z = 0.9 \, e^{\pm j\frac{3\pi}{4}}$$





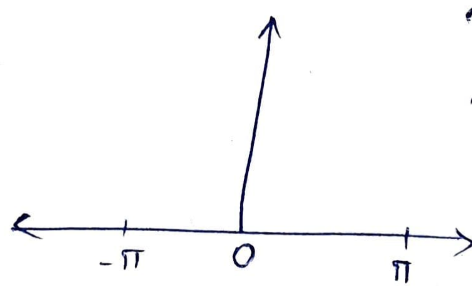
$AB + BL = AL$

$P_K + BL = e^{j\omega}$

$BL = e^{j\omega} - P_K$

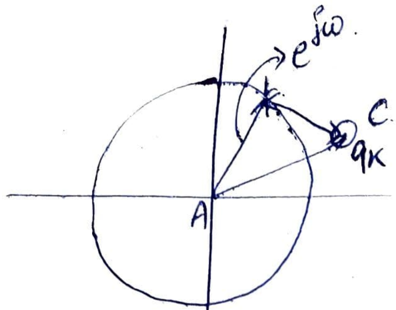
$\Rightarrow |BL| = |e^{j\omega} - P_K|$

θ_{PK} : angle contribution for that particular ω.

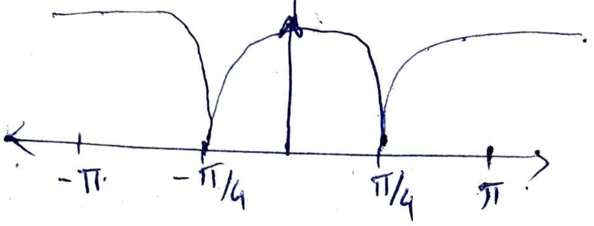


Phase response.

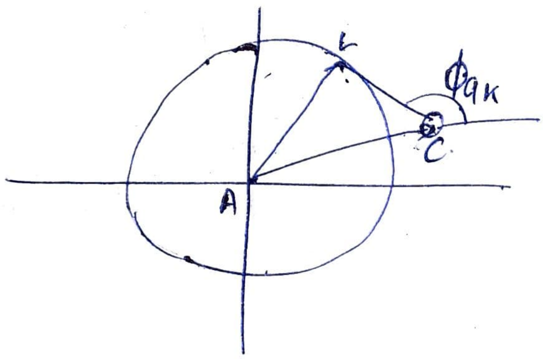
If zero is outside the unit circle

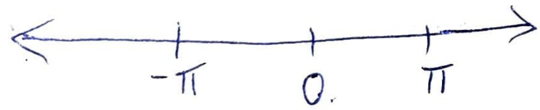


$$AC + CL = AL$$
$$q_K + CL = e^{j\omega}$$
$$CL = e^{j\omega} - q_K$$
$$|CL| = |e^{j\omega} - q_K|$$



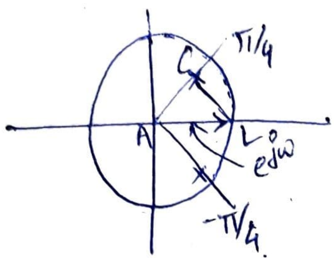
Magnitude Response.



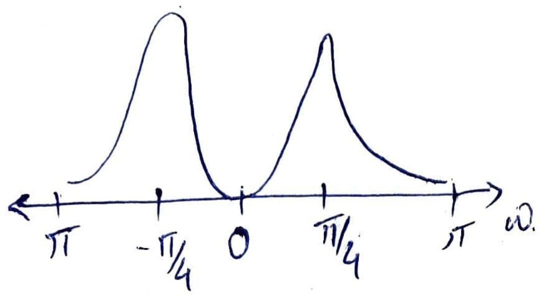


Basic LTI Systems

Digital Resonators



Magnitude $= \frac{1}{|CL|} = \frac{1}{\text{Decreasing}} \Rightarrow$ increasing



At $\omega = \frac{\pi}{4}$, $|CL| \to 0$, Mag $\to \infty$

Using a pole pair / complex conjugate poles

$$\boxed{z = r e^{\pm j\omega_0}}$$

ω_0: corresponding resonant frequency.

The overall system $H(z)$ is

$$H(z) = \frac{b_0}{(1 - r e^{j\omega_0} z^{-1})(1 - r e^{-j\omega_0} z^{-1})}$$

r is kept less than 1 ($r<1$) to ensure poles are inside the unit circle & guarantees stability.

At $\omega = \omega_0$

$$H(z) = \frac{b_0}{(1 - r e^{j\omega_0} z^{-1})(1 - r e^{-j\omega_0} z^{-1})}$$

$$H(z)\Big|_{z = e^{j\omega}} = \frac{b_0}{(1 - r e^{j\omega_0} e^{-j\omega})(1 - r e^{-j\omega_0} e^{-j\omega})}$$

$$H(\omega_0) = \frac{b_0}{(1 - r e^{+j\omega_0} e^{-j\omega_0})(1 - r e^{-j\omega_0} e^{-j\omega_0})}$$

$$\boxed{H(\omega_0) / H(z)\Big|_{z = e^{j\omega_0}} = \frac{b_0}{(1-r)(1 - r e^{-2j\omega_0})}}$$

Digital resonator at resonant frequency.

$$\boxed{H(z)\Big|_{z=e^{j\omega_0}} = \frac{b_0}{U_0(\omega_0)\;U_1(\omega_0)}}$$

Magnitude Response $\quad |H(z)| = \dfrac{|b_0|}{|U_0(\omega_0)|\;|U_1(\omega_0)|}$

$$= \frac{|b_0|}{|1-re^{j\omega_0}z^{-1}|\;|1-re^{-j\omega_0}z^{-1}|}$$

$$|U_0(\omega)| = |1-re^{j\omega_0}z^{-1}| = |1-re^{j\omega_0}e^{-j\omega}|$$

$$|U_0(\omega)| = |1-re^{j(\omega_0-\omega)}| = |1-r(\cos(\omega_0-\omega)+j\sin(\omega_0-\omega))|$$

$$|U_0(\omega)| = \sqrt{1-2r\cos(\omega_0-\omega)+r^2\cos^2(\omega_0-\omega)+r^2\sin^2(\omega_0-\omega)}$$

$$= \sqrt{1-2r\cos(\omega_0-\omega)+r^2}$$

$$|U_1(\omega)| = |1-re^{j\omega_0}e^{-j\omega}|$$

$$|U_1(\omega)| = \sqrt{1-2r\cos(\omega_0+\omega)+r^2}$$

b_0 term → normalization term
The magnitude at resonant frequency has a max. value $=1$

$$H(z)\Big|_{z=e^{j\omega_0}} = \frac{b_0}{(1-r)(1-re^{-j2\omega_0})}$$

$$|H(z)|_{\omega=\omega_0} = 1 = \frac{|b_0|}{|(1-r)(1-re^{-j2\omega_0})|}$$

$$b_0 = |(1-r)\;(1-re^{-j2\omega_0})|$$

$$\boxed{b_0 = (1-r)\sqrt{1-2r\cos 2\omega_0+r^2}}$$

↑ to normalize the magnitude $=1$ at resonant frequency

Phase Response

$$H(z)\Big|_{z=e^{j\omega}} = \frac{b_0}{(1-re^{j\omega_0}e^{-j\omega})(1-re^{-j\omega_0}e^{-j\omega})}$$

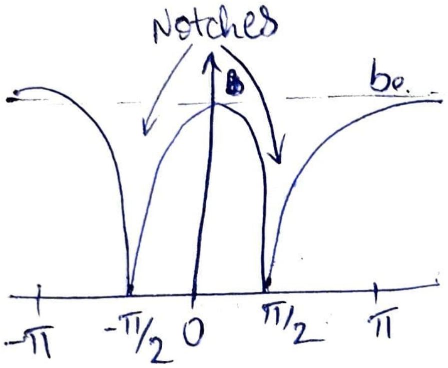
To calculate phase, put $e^{j\omega}$ in Pr form.

$$H(z)\Big|_{z=e^{j\omega}} = \frac{b_0}{e^{-j\omega}(e^{j\omega}-re^{j\omega_0})\;e^{-j\omega}(e^{j\omega}-re^{-j\omega_0})}$$

$$H(z)\Big|_{z=e^{j\omega}} = e^{2j\omega} \frac{b_0}{(e^{j\omega}-P_K)(e^{j\omega}-P_K^*)}$$

$$\boxed{\angle H(z) = 2\omega - \angle V_0(\omega) - \angle V_1(\omega)}$$ Phase response of Digital resonator.

Notch Filter



We want to create null's in spectrum or the magnitude response will be zero at $\pm\frac{\pi}{2}$.

$$\boxed{H(z) = b_0(1-e^{j\omega_0}e^{-j\omega})(1-e^{-j\omega_0}e^{-j\omega})}$$

Complex pair of zeros should be placed at the frequency where magnitude zero is required.

If we want notches at $\pm\frac{\pi}{3}$ & $\pm\frac{\pi}{4}$ then

$$H(z) = b_0(1-e^{j\pi/3}e^{-j\omega})(1-e^{-j\pi/3}e^{-j\omega})(1-e^{j\pi/4}e^{-j\omega})(1-e^{-j\pi/4}e^{-j\omega})$$

All Pass System

It passes all the frequencies with same magnitude but it adds up phase and delay in the system

$$H_{All}(z) = \frac{z^{-1}-\alpha_K}{1-\alpha_K z^{-1}}$$

For example $\alpha_K = 0.5$.

$$H_{All}(z) = \frac{z^{-1}-0.5}{1-0.5z^{-1}}$$

$$|H_{All}(z)| = \left|\frac{e^{-j\omega}-0.5}{1-0.5e^{-j\omega}}\right| = \left|\frac{\cos\omega - j\sin\omega - 0.5}{1-0.5\cos\omega + 0.5j\sin\omega}\right|$$

$$|H_{All}(z)| = \frac{0.25+\cos\omega+1}{1+\cos\omega+0.25} = 1.$$

Generalised: $$H_{All}(z) = \frac{\prod_{k=1}^{a} (z^{-1} - \alpha_k)}{\prod_{k=1}^{a} (1 - \alpha_k z^{-1})} * \frac{\prod_{k=1}^{M} (z^{-1} - \beta_k)(z^{-1} - \beta_k^*)}{\prod_{l=1}^{M} (1 - \beta_l z^{-1})(1 - \beta_l^* z^{-1})}$$

Minimum Phase System

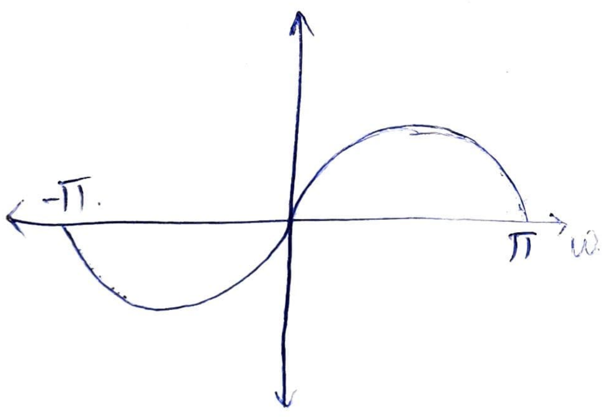
Let's take a system having a single zero.

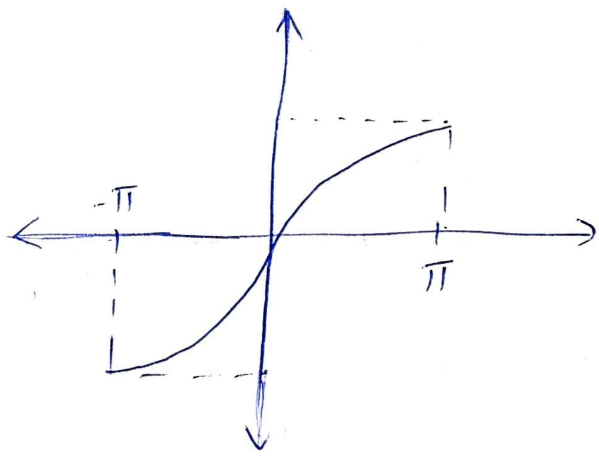
$H_1(z) = 1 + \frac{1}{2} z^{-1}$ — zero at $z = -\frac{1}{2}$ — Inside the unit circle

$H_2(z) = \frac{1}{2} + z^{-1}$ — zero at $z = -2$ — Outside the unit circle

$$H_1(z) = 1 + \frac{1}{2} e^{-j\omega}$$
$$= e^{-j\omega}\left(e^{j\omega} + \frac{1}{2}\right)$$
$$\angle H_1(z) = -\omega + \tan^{-1}\left(\frac{\sin\omega}{\cos\omega + \frac{1}{2}}\right)$$

$$H_2(z) = \frac{1}{2} + e^{-j\omega}$$
$$= e^{-j\omega}\left(1 + \frac{1}{2} e^{j\omega}\right)$$
$$\angle H_2(z) = -\omega + \tan^{-1}\left(\frac{\frac{1}{2}\sin\omega}{\frac{1}{2}\cos\omega + 1}\right)$$





Phase response is antisymmetric.

	When zero is inside unit circle	When zero is outside unit circle
Overall Phase Change	$\angle H(\pi) - \angle H(0) = 0$	$\angle H(\pi) - \angle H(0) = \pi$

When all the zeros are lie inside the unit circle, then these systems are <u>minimum phase systems</u>.

When all the zeros ~~are~~ lie outside the unit circle, then these systems are <u>maximum phase systems</u>.

When zeros lie outside as well as inside the unit circle, then these systems are <u>mixed phase systems</u>.

→ Poles should always lie inside the unit circle to maintain stability.

<u>Inverse Systems</u>

$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = h(n) * x(n)$ ← convolution in time domain

$Y(\omega) = H(\omega) \cdot X(\omega)$ ← multiplication in frequency domain.

<u>Ex</u>.

$X(\omega) \longrightarrow \boxed{H(z) = \dfrac{1+2z^{-1}}{1+0.3z^{-1}}} \longrightarrow Y(\omega)$

Pole is inside but zero is outside the unit circle.

$$\boxed{Y(z) = \left[\frac{1+2z^{-1}}{1+0.3z^{-1}}\right] X(z)}$$

Response of LTI system which is maximum phase.

$$Y(\omega) = \left[\frac{1+2e^{-j\omega}}{1+0.3e^{-j\omega}}\right] X(\omega)$$

$X(z) \longrightarrow \boxed{H(z)} \longrightarrow Y(z) \longrightarrow \boxed{\text{Inv. of } H(z)} \longrightarrow X(z)$

↑

$$H(z) = \frac{1+0.3z^{-1}}{1+2z^{-1}}.$$

system is not stable since pole is lying outside the unit circle.

Now we multiply inverse of H(z) with all pass system.

$$X(Z) \longrightarrow \boxed{\frac{1+2z^{-1}}{1+0.3z^{-1}}} \xrightarrow{Y(Z)} \boxed{\left(\frac{1+0.3z^{-1}}{\cancel{1+2z^{-1}}}\right)\cdot\left(\frac{\cancel{1+2z^{-1}}}{z^{-1}+2}\right) = \frac{1+0.3z^{-1}}{z^{-1}+2}} \longrightarrow X(z)$$

Inverse of $H(z)$ — All pass system.

Now the system is stable ~~and~~ as pole is inside ~~unit~~ ~~ca~~ circle, and it is also a minimum phase system as zero is inside unit circle.

Filter Design

Finite Impulse Response (FIR) Filter

$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$ $h(n)$ is impulse response.

$$h(n) = h_0\,\delta(n) + h_1\,\delta(n-1) + \cdots + h_{N-1}\,\delta(n-(N-1))$$

Z-domain $H(z) = h_0 + h_1 z^{-1} + h_2 z^{-2} + \cdots + h_{N-1} z^{-(N-1)}$

$$\boxed{H(z) = \sum_{i=0}^{N-1} h_i z^{-i}} \longrightarrow \text{Polynomial form}$$

Pole zero domain $\boxed{H(z) = \prod_{i=0}^{q-1} (1 - \tilde{h}_i z^{-i})}$ $\boxed{\tilde{h}_i = \frac{h_i}{h_0}}$

FIR filters have finite length of N and it is made up of only zeros of system.

$$H(z) = z^{-q} \prod_{i=0}^{q-1} (z - \tilde{h}_i)$$

$$H(z) = \frac{1}{z^q} \prod_{i=0}^{q-1} (z - \tilde{h}_i)$$

Infinite Impulse Response (IIR) Filters

$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$ $Y(z) = H(z)X(z)$ Z-domain

$y(n) = h(n) * x(n)$ $Y(e^{j\omega}) = H(e^{j\omega}) X(e^{j\omega})$ Frequency Domain.

$$H(z) = \frac{Y(z)}{X(z)}$$

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_{q-1} z^{-(q-1)}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_{p-1} z^{-(p-1)}}$$

Pole-Zero Form $\boxed{H(z) = \frac{\prod_{i=0}^{q-1} \left(1 - \frac{b_i}{b_0} z^{-1}\right)}{\prod_{i=0}^{p-1} (1 - a_i z^{-1})}}$

For a single Pole system.

$$\frac{Y(z)}{X(z)} = H(z) = \frac{1}{1+az^{-1}}$$

$$Y(z)\,(1+az^{-1}) = X(z)$$

$$Y(z) + a z^{-1} Y(z) = X(z)$$

Taking inverse Z transform

$$y(n) + a\,y(n-1) = x(n)$$

$$y(n) = x(n) - a\,y(n-1)$$

If $n = 0$, let $y(0) = 0$

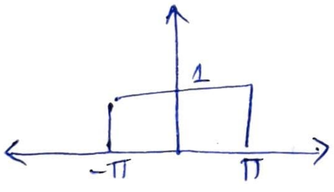
$n = 1$, $y(1) = x(1) - a\,y(0) \Rightarrow y(1) = x(1)$

$n = 2$, $y(2) = x(2) - a\,y(1)$

$\vdots$

$\therefore$ This system is recursive and have infinite impulse response.

Linear Phase Filter



$$H(\omega) = \underbrace{C\,e^{-j\omega n_d}}_{}$$ $C = 1$

Scaling factor.

Time domain shifting by n_d samples.

$|H(\omega)| = C.$ $\theta(\omega) = \angle H(\omega) = -\omega n_d$

Group delay $T_g = -\frac{d}{d\omega}\theta(\omega) = -\frac{d}{d\omega}(-\omega n_d)$

$\boxed{T_g = n_d} \rightarrow$ constant.

Since $\theta(\omega)$ is a linear function of ω, T_g is constant.

→ If the phase is linear function of ω than, the group delay is constant which indirectly says that any frequency at the i/p of this filter will pass through this system with constant delay.

Audio → $h(n)$ → o/p is without phase distortion.

Low frequency + high frequency — Linear phase

Both frequency component will pass with equal delay.
If this is not the situation then it will result in phase distortion

Statement :- "FIR filters are always linear phase filter."

Proof :- Start with symmetry and later generalise.

$$h(n) = h(m-1-n) \qquad 0 \le n \le m-1, \text{ symmetric sequence. } m \text{ is odd.}$$

$$H(z) = h_0 + h_1 z^{-1} + h_2 z^{-2} + \dots + h_{m-1} z^{-(m-1)}$$

$$H(z) = h_0 + h_1 z^{-1} + \dots + h\left(\tfrac{m-1}{2}\right) z^{-\left(\frac{m-1}{2}\right)} + \dots + h_{m-1} z^{-(m-1)}$$

$h(n) = h(m-1-n)$, $0 \le n \le m-1$

$n=0$ $\quad h(0) = h(m-1)$

$h(1) = h(m-2)$

$\vdots$

$$H(z) = h_0 + h_1 z^{-1} + \dots + h_{\frac{m-1}{2}} z^{-\left(\frac{m-1}{2}\right)} + \dots + h(1) z^{-(m-2)} + h(0) z^{-(m-1)}$$

Now taking $z^{-\left(\frac{m-1}{2}\right)}$ common.

$$H(z) = z^{-\frac{(m-1)}{2}} \left[h_0 z^{\frac{m-1}{2}} + h_1 z^{\left(\frac{m-1}{2}-1\right)} + \dots + h_{\frac{m-1}{2}} + \dots + h_0 z^{-\left(\frac{m-1}{2}\right)} \right]$$

$$= z^{-\left(\frac{m-1}{2}\right)} \left[h_0 \left(z^{\frac{m-1}{2}} + z^{-\left(\frac{m-1}{2}\right)}\right) + h_1 \left(z^{\left(\frac{m-1}{2}-1\right)} + z^{-\left(\frac{m-1}{2}-1\right)}\right) + \dots + h_{\frac{m-1}{2}} \right]$$

Converting to fourier domain.

$$H(z)\Big|_{z=e^{j\omega}} = e^{-j\omega\left(\frac{m-1}{2}\right)} \left[h_0 \left(e^{j\omega\left(\frac{m-1}{2}\right)} + e^{-j\omega\left(\frac{m-1}{2}\right)}\right) + h_1 \left(e^{j\omega\left(\frac{m-1}{2}-1\right)} + e^{-j\omega\left(\frac{m-1}{2}-1\right)}\right) + \dots + h_{\frac{m-1}{2}} \right]$$

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{m-1}{2}\right)} \left[h\left(\frac{m-1}{2}\right) + \sum_{n=0}^{\frac{m-1}{2}-1} 2h_n \cos\left(\omega\left(\frac{m-1}{2}-n\right)\right) \right]$$

$$\boxed{H(e^{j\omega}) = e^{-j\omega\left(\frac{m-1}{2}\right)} H_r(\omega)}$$

$$\angle H(e^{j\omega}) = -\omega\left(\frac{m-1}{2}\right), \quad H_r(\omega) > 0$$

$$= -\omega\left(\frac{m-1}{2}\right) + \pi, \quad H_r(\omega) < 0$$

$$\angle H(e^{j\omega}) = -\omega\left(\frac{m-1}{2}\right)$$

$$T_g(H(e^{j\omega})) = \frac{m-1}{2}$$

for $h(n) = h(m-1-n)$, $m = \text{even}$

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{m-1}{2}\right)}\left[\sum_{n=0}^{\frac{m}{2}-1} h(n)\cos\left(\omega\left(\frac{m-1}{2}-n\right)\right)\right]$$

Asymmetric case.

→ $h(n) = -h(m-1-n)$, $m = \text{odd}$. ← $h\left(\frac{m-1}{2}\right) = 0$

→ $h(n) = -h(m-1-n)$, $m = \text{even}$

$m = \text{odd} \Rightarrow H(e^{j\omega}) = e^{-j\omega\left(\frac{m-1}{2}\right)} e^{j\pi/2}\left[2\sum_{n=0}^{\frac{m-1}{2}-1} h(n)\cdot\sin\left(\omega\left(\frac{m-1}{2}-n\right)\right)\right]$

$m = \text{even} \Rightarrow H(e^{j\omega}) = e^{-j\omega\left(\frac{m-1}{2}\right)} e^{j\pi/2}\left[2\sum_{n=0}^{\frac{m}{2}-1} h(n)\sin\left(\omega\left(\frac{m-1}{2}-n\right)\right)\right]$

$h_0\left(e^{j\omega\left(\frac{m-1}{2}\right)} - e^{-j\omega\left(\frac{m-1}{2}\right)}\right)$

$\Rightarrow h_0\, 2j\sin\left(\omega\left(\frac{m-1}{2}\right)\right)$

$j = e^{j\pi/2}$.

$$\boxed{\begin{aligned}\angle H(e^{j\omega}) &= -\omega\left(\frac{m-1}{2}\right) + \frac{\pi}{2}, \quad H_r(\omega) > 0 \\ &= -\omega\left(\frac{m-1}{2}\right) + \frac{3\pi}{2}, \quad H_r(\omega) < 0\end{aligned}}$$

for asymmetric case.

Hence, proved for all symmetricities.

⇒ Whether any FIR filter is linear phase or not?

For example, → $h(n) = [\underset{\uparrow}{2}, 1, 4, 3]$

Any sequence given can be represented into decomposition of odd & even.

$$h(n) = h_o(n) + h_e(n)$$

$h_o(n) = \frac{1}{2}[h(n) - h(-n)]$ $h_e(n) = \frac{1}{2}[h(n) + h(-n)]$

$h(-n) = [3, 4, 1, \underset{\uparrow}{2}]$ $h_e(n) = \frac{1}{2}\left[(\underset{\uparrow}{2}, 1, 4, 3) + (3, 4, 1, \underset{\uparrow}{2})\right]$

$h_e(n) = 1.5, 2, 0.5, \underset{\uparrow}{2}, 0.5, 2, 1.5$

$h_o(n) = \frac{1}{2}\left[(\underset{\uparrow}{2}, 1, 4, 3) - (3, 4, 1, \underset{\uparrow}{2})\right]$

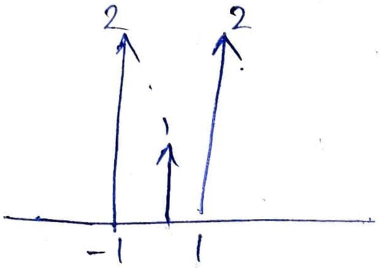
$h_o(n) = (-1.5, -2, -0.5, \underset{\uparrow}{0}, 0.5, 2, 1.5)$

h_e & h_o are symmetric sequences.

∴ Both of them will have linear phase.

Since IIR is not having finite length, the middle point can never be found & therefore linear phase is not gauaoranteed.

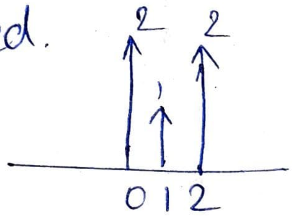
Ex



$$H(z) = 2z^{1} + 1 + 2z^{-1}$$

$$H(e^{j\omega}) = 2e^{j\omega} + 1 + 2e^{-j\omega}$$

When shifted.



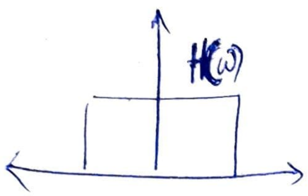
$$H(z) = 2 + z^{-1} + 2z^{-2}$$

$$H(z) = z^{-1}(2z^{+1} + 1 + 2z^{-1})$$

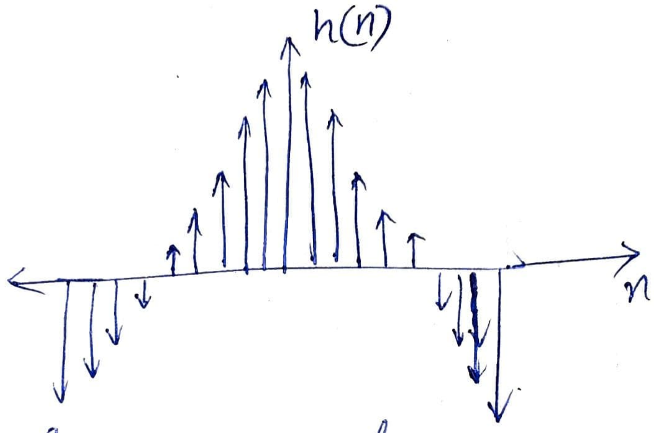
Causality

$h(n) = 0, \quad n < 0,$ O/p depends on present & past values but not future values.

Ideal Filter



$\longrightarrow$



For ideal filter system is non-causal.

$\boxed{H(\omega) = 1 e^{-j\omega n_d}}$ Shift h(n) to right by n_d samples.

$\Theta(\omega) = -\omega n_d$ $\boxed{T_g = -\frac{d}{d\omega}\Theta(\omega) = n_d}$ Consistent.

Paley - Wiener Criteria For System Causality

→ If h(n) has finite energy & $h(n) = 0$ for $n < 0$. or.
If this integral is finite with $H(\omega)$ as square integrable then also h(n) is causal.

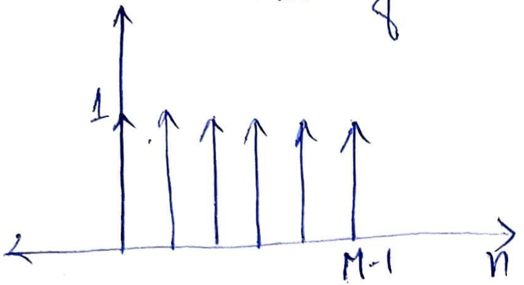
$$\int_{-\pi}^{\pi} \ln(|H(\omega)|)\, d\omega < \infty$$

$\log(0) = \infty$ This means that $H(\omega)$ can be zero at some frequencies but it can be zero over a band of frequencies.

It can be zero at zero crossing.

Window Based Filter Design

Rectangular window



Multiplication in Time domain $\xleftrightarrow{F.T.}$ Convolution in Frequency Domain.

$\boxed{h_d(n) \cdot \omega(n)}$ $\xleftrightarrow{F.T.}$ $\int H_d(v)\, \omega(\omega - v)\, dv.$

Sample by sample multiplication

$$\boxed{H_d(\omega) = 1 \cdot e^{-j\omega\left(\frac{M-1}{2}\right)}}$$

$$\boxed{\omega(n) = 1 \,, \quad \text{for } 0 \le n \le M-1}$$

M :- window length

$$W(\omega) = \sum_{n=0}^{M-1} \omega(n) \cdot e^{-j\omega n}.$$

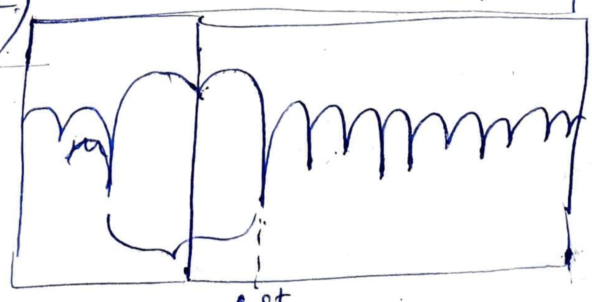
For rectangle window.

$$W(\omega) = \sum_{n=0}^{M-1} e^{-j\omega n} \qquad \left(\sum_{n=0}^{M-1} a^n = \frac{1-a^n}{1-a} \right)$$

$$W(\omega) = \frac{1 - e^{-j\omega M}}{1 - e^{-j\omega}}$$

$$\boxed{W(\omega) = \frac{e^{-j\omega\frac{M}{2}}\left(e^{j\omega\frac{M}{2}} - e^{-j\omega\frac{M}{2}}\right)}{e^{-j\omega/2}\left(e^{j\omega/2} - e^{-j\omega/2}\right)} = e^{-j\omega\left(\frac{M-1}{2}\right)} \frac{\sin\left(\frac{\omega M}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}}$$

$$|W(\omega)| = \left| \frac{\sin\left(\frac{\omega M}{2}\right)}{\sin\left(\frac{\omega}{2}\right)} \right|$$



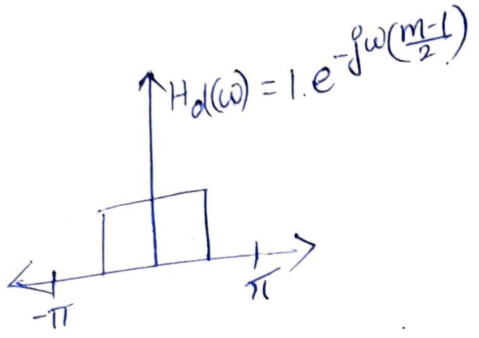
$$\sin\left(\frac{\omega M}{2}\right) = 0$$

$$\frac{\omega M}{2} = \pm m\pi \,, \quad m = 1$$

$$\omega = \frac{2\pi}{M} \,, \text{ first null occurs}$$

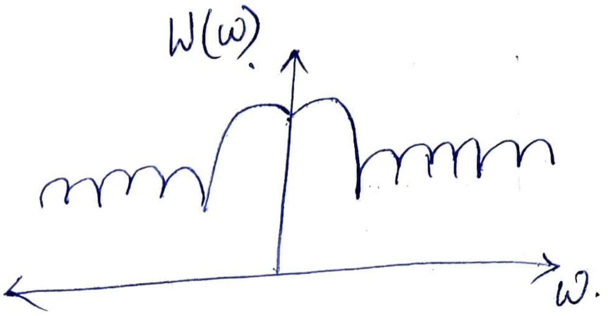
Main lobe width $= \frac{4\pi}{m}$

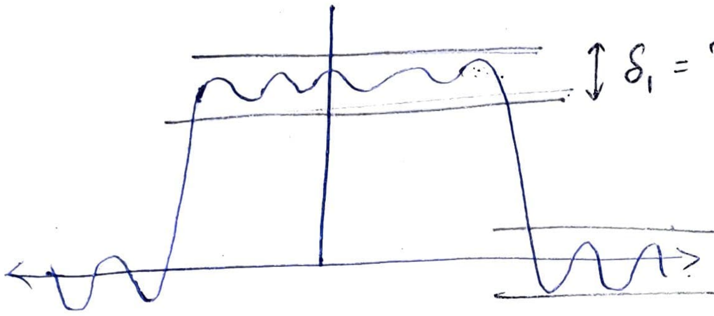
Main lobe width is inversely proportional to window length. m.



$\circledast$

Convolution.





δ_1 = Pass band ripple

δ_2 = Stopband ripple.

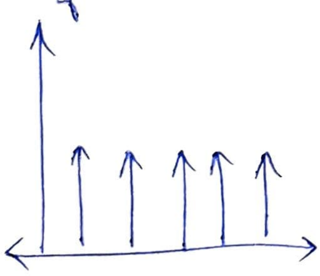
Since main lobe width is inversely proportional to m

If m ↑ , width ↓

m ↓ , width ↑.

For 2 different values of m, the width will be different as well as the height because the effective area over the lobes is same i.e. if width changes, then height will also changes to adjust the area to be same.

Why not choose smooth function for windowing.



Sharp. discontinuities

Less continuity & hence side lobes are of minimum size with less power.

Hamming window $H(n) = 0.54 - 0.46 \cos\left(\frac{2\pi n}{m-1}\right)$

more smooth. Rectangle window $R(n) = 1$, for $0 \le n \le m-1$

Kaiser window $K(n) = \dfrac{I_0\left(\alpha \sqrt{\left(\frac{m-1}{2}\right) + \left(n - \left(\frac{m-1}{2}\right)\right)^2}\right)}{I_0\left(\alpha \sqrt{\frac{m-1}{2}}\right)}$

All window functions like Hamming, Kaiser window, other than rectangle window, provides smooth transition, & hence limits the side lobes.