Traditional

25/10/2023

Conventional DTFS for periodic signals. → DTFS Syn. Eq

DTFS Synthesis Eq^n →

$$x[n] = x(n + lN), \quad l \in \mathbb{Z}, \; N \in \mathbb{Z}^+$$

$$x[n] = \sum_{k=0}^{N-1} D_k e^{jk\frac{2\pi}{N}\cdot n} = \sum_{k=0}^{N-1} D(k) e^{jk\frac{2\pi}{N}\cdot n} \quad \forall n \in \mathbb{Z} \qquad (1)$$

Assuming Basis set $\phi_k(n) = e^{j\frac{2\pi}{N}kn}$, $k \in \{0, 1, 2, \ldots, N-1\}$

DTFS Analysis Eq^n → where

$$D_k = D(k) = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk\frac{2\pi}{N}\cdot n}; \forall k \in \mathbb{Z}$$

$$= D(k + rN)$$

$$r \in \mathbb{Z}; \; N \in \mathbb{Z}^+ \qquad (2)$$

→ DTFS Analysis eq^n.

Parseval's Theorem

$$P_x = \frac{1}{N} \sum_{\langle N \rangle} |x(n)|^2$$

$$= \frac{1}{N} \sum_{\langle N \rangle} x[n] x^*(n)$$

$$\frac{1}{N} \sum_{\langle N \rangle} x[n] \left(\sum_{\langle N \rangle} D_k e^{+j\frac{2\pi}{N}kn} \right)^*$$

$$= \sum_{\langle N \rangle} D_k^* \left(\frac{1}{N} \sum_{\langle N \rangle} x(n) e^{-j\frac{2\pi}{N}kn} \right)$$

$$= \sum_{\langle N \rangle} D_k^* \cdot D_k = \sum_{\langle N \rangle} |D_k|^2$$

$$x[n] \xleftrightarrow{\text{DTFS}} D(k) = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk\frac{2\pi}{N}\cdot n}$$

$$= \sum_{k=0}^{N-1} D(k) e^{jk\frac{2\pi}{N}\cdot n}$$

Discrete → Periodic of F.P. 'N'

Periodic of period 'N' (fundamental) → Discrete in F.D.

F.P. → fundamental period.

Proof →

$$\frac{x(-n)}{N} = \frac{1}{N} \sum_{k=0}^{N-1} D(k) e^{-jk\frac{2\pi}{N}\cdot n}$$

$k \to n$; $n \to k$

$$\frac{x(-k)}{N} = \frac{1}{N} \sum_{n=0}^{N-1} D(n) e^{-jn\frac{2\pi}{N}\cdot k} \rightarrow D(n) \xleftrightarrow{\text{DTFS}} \frac{x(-k)}{N}$$

Duality Property exists for DTFS

Duality Property of DTFS →

Assuming Basis set $\phi_k(n) = e^{jk\frac{2\pi}{N}\cdot n}$; $k \in \{0, 1, 2, \ldots, N-1\}$.

$$x[n] \xleftrightarrow{\text{DTFS}} D(k)$$

$$D(n) \xleftrightarrow{\text{DTFS}} \frac{x(-k)}{N}.$$

e.g.

$$\sum_{r=-\infty}^{\infty} \delta[n - rN] \xleftrightarrow{\text{DTFS}} \frac{1}{N} = D(k)$$

$$D(n) = \frac{1}{N} \; \forall n \xleftrightarrow{\text{DTFS}} \frac{\sum_{r=-\infty}^{\infty} \delta[k - rN]}{N}$$

Derived from DTFT by sampling in freq^n domain
$\Downarrow$
periodicity in Time-domain

Discrete-Time Fourier Series :-

→ Discrete-time Periodic signals

Discrete & Periodic with period 'N'.

$$x(n) = x(n+lN) \; ; \; N \in \mathbb{Z}^+ \; ; \; l \in \mathbb{Z}$$

$$\omega_0 = \frac{2\pi}{N} \text{ rps.}$$

$x[n]$ can be represented by a DTFS, made up of complex exponentials of fundamental freq^n $\omega_0 = \frac{2\pi}{N}$ & its harmonics.

$$\boxed{e^{j\frac{2\pi}{N}\cdot k\cdot n} = \phi_k(n).}$$

→ is Periodic with period N.

i.e.

$$\boxed{\phi_k(n) = \phi_k(n+lN)}$$

i.e.

$$\boxed{x[n] = \sum_{k=k_0}^{k_0+N-1} D_k \, e^{j\frac{2\pi}{N}\cdot k\cdot n}}$$

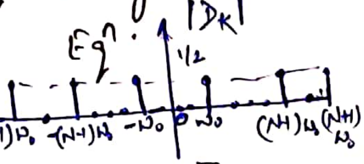
→ any arbitrary integer e.g. '0'.

e.g. $\sin\left[\frac{2\pi}{N}n\right] = \frac{e^{j\frac{2\pi}{N}n} - e^{-j\frac{2\pi}{N}n}}{2j}$

$D_1 = \frac{1}{2j}$; $D_{-1} = \frac{-1}{2j}$

$D_k \in \mathbb{C} \Rightarrow$ DTFS coefficients

$$\boxed{D_k = \frac{1}{N}\sum_{n=n_0}^{n_0+N-1} x[n]\cdot e^{-jk\omega_0 n} \; ; \; \omega_0 = \frac{2\pi}{N}} \Rightarrow \text{Analysis Eq}^n.$$



→ $k \in 0, \pm 1, \pm 2, \ldots$

Discrete & Periodic with period 'N'

i.e. $\boxed{D_k = D_{k+mN}.}$

$|D_k|$ & $\angle D_k$

$\Downarrow$ Mag. spectrum of $x[n]$; $\Downarrow$ Phase spectrum of $x[n]$.

Relationship of DTFS, DTFT & ZT :-

$$D(k) = \frac{X(e^{j\omega})\big|_{\omega=\frac{2\pi}{N}k}}{N}$$

$$= \frac{X(z)}{N}\Bigg|_{z=e^{j\frac{2\pi}{N}k}}$$

5. Discrete Fourier Transform (DFT)

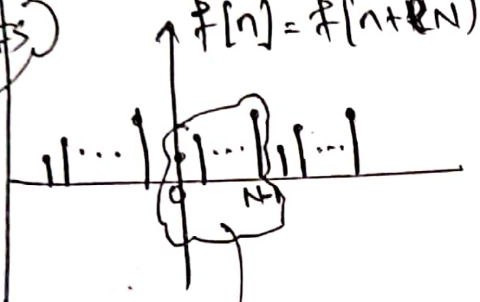
Derived from DTFS by carrying only unique samples in TD & FD

Mainly invented to do spectrum estimation in Digital Machines or Computers.

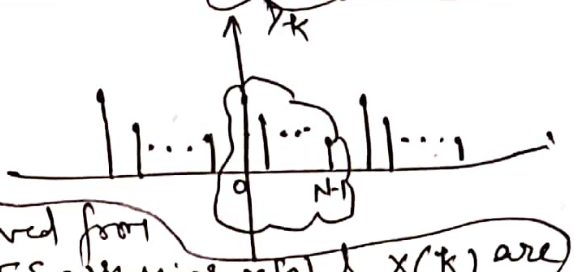
finite length in TD & FD

via DTFS

$f[n] = f[n+kN]$



DTFS ⟷

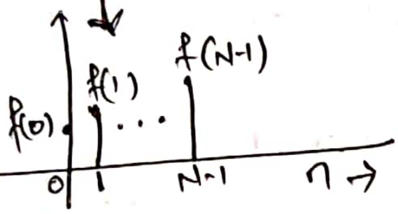


D_k

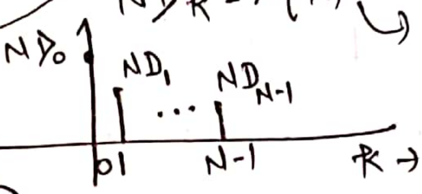
Derived from DTFS assuming $x[n]$ & $X(k)$ are periodic with period 'N'

$N D_k = X(k)$ → DFT Transform sequence of same length 'N'

finite-length sequence of length 'N'



DFT ⟷ considers only unique samples of DTFS



DFT ⇒

$$X(k) = N \cdot D_k = \sum_{n=0}^{N-1} x[n] \cdot e^{-j\frac{2\pi}{N} \cdot k n} \; ; \quad k = 0, 1, 2, \ldots, N-1$$

N-pt. DFT

→ Analysis Eq^n / DFT Eq^n

Relationships →

$$X(k) = N D_k = X\left(e^{j\cdot\frac{2\pi}{N}\cdot k}\right) = X(Z)\Big|_{Z = e^{j\frac{2\pi}{N}\cdot k}}, \; k \in \{0,1,\ldots,N-1\}$$

$k \in \{0,1,\ldots,N-1\}$ $k \in \{0,1,\ldots,N-1\}$

DFT = N·DTFS (Unique) = Sampled DTFT (Unique) = Sampled ZT (Unique)

IDFT ⇒

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{j\frac{2\pi}{N} \cdot k \cdot n} \; ; \quad n = 0, 1, 2, \ldots, N-1.$$

→ Synthesis Eq^n / IDFT

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Why do length$(x[n]) \leq$ No. of Points in DFT ??

N-DFT $\quad X(k) = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}k.n}$; $\quad k = 0,1,2,\ldots,N-1$

N-IDFT $\quad x[n] = \frac{1}{N} \sum_{n=0}^{N-1} X(k) e^{j\frac{2\pi}{N}kn}$; $\quad n = 0,1,2,\ldots,N-1$

'N' unique samples of DTFS, $\tilde{X}(k)$ & IDTFS $\tilde{x}(n)$ respectively. such that :→

Assuming

$$\boxed{X(k) = \begin{cases} \tilde{X}(k), & k = 0,1,2,\ldots,N-1 \\ 0 & \text{o.w.} \end{cases}}$$

$$\boxed{x[n] = \begin{cases} \tilde{x}(n), & n = 0,1,\ldots,N-1 \\ 0 & \text{o.w.} \end{cases}}$$

$$\boxed{\tilde{X}(k) = \sum_{r=-\infty}^{\infty} X(k - rN)}$$

$$\boxed{\tilde{x}(n) = \sum_{r=-\infty}^{\infty} x(n - rN)}$$

⇒ $M > N$ ↳ Aliasing in TD.

DTFS → $\tilde{X}(k) = \sum_{n=0}^{N-1} \tilde{x}(n) e^{-j\frac{2\pi}{N}kn}$

$= \text{DTFT}(x(n))\Big|_{w = \frac{2\pi}{N}.k}$

DTFS, $\tilde{X}(k)$ are samples of $X(e^{jw})$ at $w = \frac{2\pi}{N}k$

$$\boxed{\tilde{X}(k) = X(e^{jw})\Big|_{w = \frac{2\pi}{N}.k}}$$

⇓

$\{\ldots, -\frac{4\pi}{N}, -\frac{2\pi}{N}, 0, \frac{2\pi}{N}, \frac{4\pi}{N}, \ldots\}$.

↓

inter-samples separation $= \frac{2\pi}{N}$.

sampling of DTFT i.e. in FD ⇒ Replication in TD

Otherwise if $M > N$:

$x(n) \neq \begin{cases} \tilde{x}(n) & 0 \leq n \leq M-1 \\ 0 & \text{o.w.} \end{cases}$

$\boxed{\text{To Avoid aliasing in TD ; } x[n] \text{ is assumed to be of finite-length, } M \leq N.}$

so that $x[n]$ can be recovered faithfully from $\tilde{x}[n]$.

~~Unit-1~~ DFT Properties

Finite length Sequence (length N)		N-point DFT (length N)
1. $x[n]$	$\longleftrightarrow$	$X(k)$
2. $x_1[n]\,,\ x_2[n]$	$\longleftrightarrow$	$X_1(k)\,,\ X_2(k)$
3. $a_1 x_1[n] + a_2 x_2[n]$	$\longleftrightarrow$	$aX_1(k) + a_2 X_2(k)$
4. $x[-n]$	$\longleftrightarrow$	$X\left(((-k))_N\right)$
5. $x^*(n)$	$\longleftrightarrow$	$X^*\left[((-k))_N\right]$
6. $x^*\left(((-n))_N\right)$	$\longleftrightarrow$	$X^*(k)$
7. $x\left(((n-m))_N\right)$	$\longleftrightarrow$	$e^{-j\frac{2\pi}{N}km}\, X(k)$
8. $e^{j\frac{2\pi}{N}ln}\, x[n]$	$\longleftrightarrow$	$X\left(((k-l))_N\right)$
9. $\sum_{m=0}^{N-1} x_1(m)\, x_2\left(((n-m))_N\right)$	$\longleftrightarrow$	$X_1(k)\, X_2(k)$
10. $x_1[n]\, x_2[n]$	$\longleftrightarrow$	$\frac{1}{N}\sum_{l=0}^{N-1} X_1[l]\, X_2\left[((k-l))_N\right]$
11. $X[n]$	$\longleftrightarrow$	$N\, x\left[((-k))_N\right]$

12. Parseval's Relation.

$$E_x = \sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X[k]|^2$$

Energy spectral density of $x[n]$.

13. If $x[n]$ is real

$$x[n] = x^*[n] \longleftrightarrow X(k) = X^*\left[((-k))_N\right] = X^*(N-k)$$

$k = \{0, 1, \ldots N-1\}$

for 8-point DFT $\Rightarrow$ e.g. $[X(0)\ \ X(1)\ \ X(2)\ \ X(3)\ \ X(4)\ \ X(5)\ \ X(6)\ \ X(7)]$

Conjugate pairs; i.e. $\boxed{X(7) = X^*(1)}$

Circular Convolution

$$y[n] = \sum_{k=0}^{N-1} f(k)\, h[n-k]_N$$

$$f[n] \circledast_N h[n] \Rightarrow \text{Circular Convolution} \Downarrow \text{Periodic}$$

Considers: $h[n-k]_N \Rightarrow$ circ. shifting (periodic); $h[k]_N \Rightarrow$ circ. reversal (periodic) are same as that of linear shifting & linear reversal of the periodic signal & stick to the unique range.

$$y[n] = f(0)\,h[n] + f(1)\,h[n-1] + f(2)\,h[n-2] + \dots + f(N-1)\,h[n-(N-1)]$$

$$y(0) = f(0)\,h[0] + f(1)\,h[-1] + f(2)\,h[-2] + \dots + f(N-1)\,h[-(N-1)]$$

$\boxed{\therefore h[-n] = \tilde{h}(rN+n);\ 'r' \in \mathbb{Z} \text{ s.t. } (rN-n) \in \{0,1,\dots,N-1\}}$

$$y(0) = f(0)h(0) + f(1)\,h[N-1] + f(2)\,h[N-2] + \dots + f[N-1]\,h[1]$$

Similarly:

$$y(1) = f(0)h(1) + f(1)h(0) + f(2)h(N-1) + \dots + f(N-1)h(2)$$

$$y(N-1) = f(0)h(N-1) + f(0)h(N-2) + f(3)h(N-3) + \dots + f(N-1)h(0) \text{ \& so-on.}$$

$$\underbrace{\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y[N-1] \end{bmatrix}}_{\vec{y}} = \underbrace{\begin{bmatrix} h(0) & h[N-1] & h[N-2] & \cdots & h(3) & h(2) & h(1) \\ h(1) & h(0) & h[N-1] & \cdots & h(3) & h(2) \\ h(2) & h(1) & h(0) & & h(4) & h(3) \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ h(N-3) & h(N-4) & h(N-5) & & h(N-1) & h(N-2) \\ h(N-2) & h(N-3) & h(N-4) & & h(0) & h(N-1) \\ h[N-1] & h[N-2] & h[N-3] & \cdots & h(1) & h(0) \end{bmatrix}}_{\tilde{H}} \underbrace{\begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ \vdots \\ f(N-1) \end{bmatrix}}_{\vec{f}} \Rightarrow \boxed{\vec{y} = \tilde{H}\vec{f}}$$

DFT Property:

$f[n] \xleftrightarrow{DFT_N} F(k)$; $h[n] \xleftrightarrow{DFT_N} H(k)$.

$$f[n] \circledast_N h[n] \xleftrightarrow{DFT} F(k) \cdot H(k).$$

$\Rightarrow$ Convolution in TD $\rightarrow$ Multiplication in FD

$\boxed{f[n] \circledast h[n] = f[n] \circledast_{2N} h[n]}$ (Lin conv. $\circledast$; $\circledast_{2N}$ circ. conv.)

Important Note:

$$\tilde{H} = \begin{bmatrix} h(0) & h(N-1) & \cdots & h(3) & h(2) & h(1) \\ h(1) & h(0) & h(N-1) & \cdots & h(3) & h(2) \\ h(2) & h(1) & h(0) & \cdots & & h(3) \\ \vdots & & & & & \vdots \\ h(N-2) & h(N-3) & \cdots & & h(0) & h(N-1) \\ h(N-1) & h(N-2) & \cdots & & h(1) & h(0) \end{bmatrix}$$

Note:
1. Circ. conv. matrix has principle diagonal with value $h(0)$;
2. LTP of circ. conv. matrix has sub-principle diagonals with values $h(1), h(2) \dots h(N-1)$ as we move lower-side of principle diagonal.
3. UTP of circ. conv. matrix has sub-principle diagonals with values $h(N-1), h(N-2), \dots, h(3), h(2), h(1), h(0)$; moving upward from principal diag.

CT: $x(t)$, $t\in\mathbb{R}$ ↔ Aperiodic FD

Periodic TD ↔ Discrete FD, $k\in\mathbb{Z}$

I-CTFS / TD ↔

$$\tilde{x}(t)=\tilde{x}(t-rT)=\sum_{k=-\infty}^{\infty}c(k)\,e^{j\frac{2\pi}{T}k\cdot t};\quad t\in\mathbb{R},\ r\in\mathbb{Z},\ T\in\mathbb{R}^+$$

$$x(t)=\begin{cases}\tilde{x}(t); & t\in\left(-\frac{T}{2},\frac{T}{2}\right)\\ 0; & o.w.\end{cases}\qquad \tilde{x}(t)=\sum_{r=-\infty}^{\infty}x(t-rT);\ t\in\mathbb{R},\ T\in\mathbb{R}^+$$

CTFS / FD ↔

$$c(k)=\frac{1}{T}\int_T\tilde{x}(t)\,e^{-j\frac{2\pi}{T}k\cdot t}\,dt\quad\forall k\in\mathbb{Z}$$

$$c(k)=\frac{X\left(j\cdot\frac{2\pi}{T}k\right)}{T}$$

TD	FD
1. Continuous ↔	Aperiodic
2. Periodic ↔	Discrete

$\tilde{x}(n)=\tilde{x}(t)\big|_{t=nT_s,\ n\in\mathbb{Z}}$ ↔ $\tilde{X}(k)=\tilde{X}(k-rN)$; $r\in\mathbb{Z}$

Sampling in TD ↔ Periodicity in FD

$\tilde{x}(t)$; $t\in\mathbb{R}$ ↔ $c(k)$; $k\in\mathbb{Z}$

Continuity in TD (Interpolation) ↔ Aperiodic in FD (Truncation or windowing)

DT ↔ $x[n]$; $n\in\mathbb{Z}$ ↔ Periodic FD

I-DTFS / TD :

$$\tilde{x}(n)=\tilde{x}(n-rN)=\sum_{\langle N\rangle}D(k)\,e^{j\frac{2\pi}{N}k\cdot n}\quad\forall n\in\mathbb{Z};\ r\in\mathbb{Z};\ N\in\mathbb{Z}^+$$

$$x(n)=\begin{cases}\tilde{x}(n), & n\in\{0,1,\dots,N-1\}\\ 0 & o.w.\end{cases}\qquad \tilde{x}(n)=\sum_{r=-\infty}^{\infty}x(n-rN)$$

DTFS / FD ↔

$$D(k)=\frac{1}{N}\sum_{\langle N\rangle}\tilde{x}(n)\,e^{-j\frac{2\pi}{N}k\cdot n}\quad\forall k\in\mathbb{Z}$$
$$=D(k-rN);$$

$$D(k)=\frac{X\left(e^{j\frac{2\pi}{N}k}\right)}{N}$$

TD	FD
1. Discrete ↔	Periodic$_N$
2. Periodic$_N$ ↔	Discrete

$$\tilde{X}(k)=N\,D(k)$$

N-point DFT / FD

Unique samples only

$$X(k)=\begin{cases}\tilde{X}(k); & k\in\{0,1,\dots,N-1\}\\ 0 & o.w.\end{cases}$$

N-point IDFT ↔ or TD

Unique samples only

$$x[n]=\begin{cases}\tilde{x}(n); & n\in\{0,1,\dots,N-1\}\\ 0 & o.w.\end{cases}$$

TD	FD
1. N-length	1. N-length
2. Discrete	2. Discrete
3. Aperiodic	3. Aperiodic

Modified DTFS corresponding to N-point DFT / FD:

$$\tilde{X}(k)=N\,D(k)=\sum_{\langle N\rangle}\tilde{x}(n)\,e^{-j\frac{2\pi}{N}k\cdot n};\quad k\in\mathbb{Z}$$
$$=\tilde{X}(k-rN)$$

Modified I-DTFS Corr. to N-pt. DFT / TD:

$$\tilde{x}(n)=\frac{1}{N}\sum_{\langle N\rangle}\tilde{X}(k)\,e^{j\frac{2\pi}{N}k\cdot n}=\tilde{x}(n-rN),\quad n\in\mathbb{Z},\ r\in\mathbb{Z},\ N\in\mathbb{Z}^+$$

TD	FD
1. Discrete ↔	Periodic$_N$
2. Periodic$_N$ ↔	Discrete

$\tilde{x}(t)=x(t)$; i.e. $T\to\infty$ ↔ $X(j\omega)$; $\omega\in\mathbb{R}$

Aperiodicity in TD (Truncation or windowing) ↔ Continuity in FD (Interpolation)

$\tilde{x}(t)=\sum_{r=-\infty}^{\infty}x(t-rT)$ (Periodicity in TD) ↔ $c(k)$; $k\in\mathbb{Z}$ (Discreteness in FD)

Periodicity in TD ↔ Sampling in FD: $D(k)$; $k\in\mathbb{Z}$

Aperiodicity in TD (Truncation / Windowing) ↔ Reconstruction / Interpolation in FD

$$\tilde{X}(k)=X\left(e^{j\frac{2\pi}{N}k}\right)$$

Aperiodic TD ↔ Continuous FD, $\omega\in\mathbb{R}$

CTFT / FD ↔

$$X(j\omega)=\int_{-\infty}^{\infty}x(t)\,e^{-j\omega t}\,dt;\quad\forall\omega\in\mathbb{R}$$

$$X(j\omega)=\lim_{T\to\infty}T\,c\left(k\cdot\frac{2\pi}{T}\right)$$

I-CTFT / TD ↔

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)\,e^{j\omega t}\,d\omega;\quad\forall t\in\mathbb{R}$$
$$=\begin{cases}\tilde{x}(t); & t\in\left(-\frac{T}{2},\frac{T}{2}\right)\\ 0 & o.w.\end{cases}$$

TD	FD
1. Continuous ↔	Aperiodic
2. Aperiodic ↔	Continuous

Sampling in TD ↔ Periodicity in FD

Interpolation in TD ↔ Aperiodicity in FD (Truncation or windowing)

DTFT / FD

$$X(e^{j\omega})=\sum_{n=-\infty}^{\infty}x[n]\,e^{-j\omega n};\quad\omega\in\mathbb{R}$$
$$=X\left(e^{j(\omega-2\pi r)}\right),\ r\in\mathbb{Z}$$

$$X(e^{j\omega})=\lim_{N\to\infty}N\cdot D\left(k\cdot\frac{2\pi}{N}\right)$$

TD	FD
1. Discrete ↔	Periodic$_{2\pi}$
2. Aperiodic ↔	Continuous

IDTFT / TD ↔

$$x[n]=\frac{1}{2\pi}\int_{2\pi}X(e^{j\omega})\,e^{j\omega n}\,d\omega;\quad n\in\mathbb{Z}$$

Note ↔

- DFT does not obey the standard-ppls of transformation in TD|FD.

Hence 'DFT' is not an actual transform that transform TD ↔ FD, naturally.

Sampling in TD [FD] ↔ Periodicity in FD (TD)

Interpolation in TD [FD] ↔ Aperiodicity in FD (TD)

DFT is a Pseudo Transform only. | Just a Mapping Technique → which maps unique 'N' samples of IDTFS & DTFS in TD & FD, respectively.

Invented to compute transforms in Digital Machines

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