DTFS (Synthesis FS)

- The DTFS representation of an N_0–periodic signal x[n] is

$$x[n] = \sum_{r=0}^{N_0-1} D_r e^{j\frac{2\pi rn}{N_0}} = \sum_{r=0}^{N_0-1} D_r e^{j\Omega_n rn}, -\infty < r < \infty$$

SYNTHESIS EQUATION

where $$D_r = \frac{1}{N_0}\sum_{n=0}^{N_0-1} x[n]e^{-j\frac{2\pi rn}{N_0}}, -\infty < r < \infty$$

ANALYSIS EQUATION, DTFS coefficients or Spectral coefficients of x[n]

- A periodic signal x[n] of period N_0 is represented in its DTFS by the SUM of N_0 complex exponentials at harmonics values of $\Omega_0 = 2\pi/N_0$
- i.e. Harmonics $\boldsymbol{\Omega_r = r\Omega}$

Fourier Spectra of a Periodic Signal

- The Fourier series consists of N_0 components

$$D_0, D_1 e^{j\Omega_0 n}, D_2 e^{j2\Omega_0 n}, \ldots, D_{N_0-1} e^{j(N_0-1)\Omega_0 n}$$

- The frequency of these components are

$$0, \Omega_0, 2\Omega_0, \ldots, (N_0-1)\Omega_0; \text{ where } \Omega_0 = \frac{2\pi}{N_0}$$

- The Fourier spectrum of r^{th} harmonic (or Fourier coefficient) D_r as a function of index r or frequency Ω,
- In polar form $$D_r = |D_r| e^{j\angle D_r}$$

Magnitude (Amplitude) Spectrum

$|D_r| = |D_{-r}|$ $|D_r| \; vs \; r\Omega_0$

Phase Spectrum

$\angle D_r = -\angle D_{-r}$ $\angle D_r \; vs \; r\Omega_0$

Periodicity of DTFS coefficients

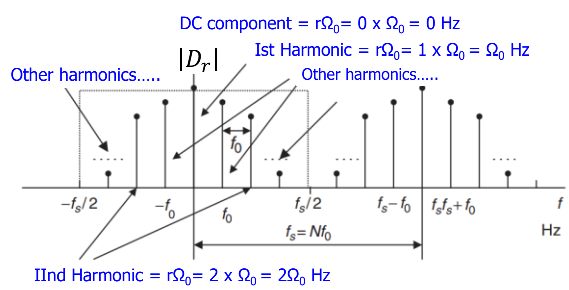
$$D_r = \frac{1}{N_0}\sum_{n=0}^{N_0-1} x[n]e^{-j\frac{2\pi rn}{N_0}}, -\infty < r < \infty$$

$$D_{r+kN_0} = \frac{1}{N_0}\sum_{n=0}^{N_0-1} x[n]e^{-j\frac{2\pi(r+N_0)n}{N_0}} = \frac{1}{N_0}\sum_{n=0}^{N_0-1} x[n]e^{-j\frac{2\pi rn}{N_0}} e^{-j2\pi n}$$

$$e^{-j2\pi n} = \cos(2\pi n) - j\sin(2\pi n) = 1$$

$$D_{r+kN_0} = D_r$$

Fourier Spectra of a Periodic Signal



- For the r^{th} harmonic, the frequency is $\boldsymbol{\Omega_r = r\Omega}$
- The frequency spacing between the consecutive spectral lines, called the frequency resolution, is Ω_0 Hz.

Fourier Spectra of a Periodic Signal

Compute the DTFS of the periodic signal

$$x[n] = \{\ldots, \underline{24}, 8, 12, 16, 24, 8, 12, 16, \ldots\}.$$

$$N_0 = 4$$

$$\Omega_0 = \frac{2\pi}{N_0} = \frac{\pi}{2}$$

$$D_r = \frac{1}{N_0}\sum_{n=0}^{N_0-1} x[n]e^{-jr\Omega_0 n} = \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-jr\Omega_0 n}$$

$$D_0 = \frac{1}{N_0}\sum_{n=0}^{N_0-1} x[n] = \tfrac{1}{4}(24+8+12+16) = 15.$$

$$D_1 = \frac{1}{4}\sum_{n=0}^{3} x[n]\, e^{-j\Omega_0 n}$$

$$= \frac{1}{4}(24e^0 + 8e^{-j\pi/2} + 12e^{-j\pi} + 16e^{-j3\pi/2})$$

$$= \frac{1}{4}(24 - j8 - 12 + j16) = 3 + j2 = 3.6e^{j33.7^\circ}$$

$$D_2 = \tfrac{1}{4}(24 - 8 + 12 - 16) = 3$$

$$D_3 = D_1^* = 3 - j2 = 3.6e^{-j33.7^\circ}$$

$$x[n] = \sum_{r=0}^{N_0-1} D_r e^{jr\Omega_0 n}$$
$$= D_0 + D_1 e^{j\pi n/2} + D_2 e^{j\pi n} + D_3 e^{j3\pi n/2}$$
$$= 15 + 3.6e^{j33.7^\circ} e^{j\pi n/2} + 3e^{j\pi n} + 3.6e^{-j33.7^\circ} e^{j3\pi n/2}.$$
$$x[n] = 15 + 7.2\cos\left(\frac{\pi n}{2} + 33.7^\circ\right) + 3\cos(\pi n)$$

Parseval's Theorem

$$\frac{1}{N_0} \sum_{n=0}^{N_0-1} |x[n]|^2 = \sum_{r=0}^{N_0-1} |D_r|^2$$

In the time domain, the average power is

$$P_{\text{av}} = \frac{1}{4}\left[(24)^2 + (8)^2 + (12)^2 + (16)^2\right] = 260.$$

In the frequency domain, the average power is

$$P_{\text{av}} = (15)^2 + (3.6)^2 + 3^2 + (3.6)^2 = 260.$$

Compute the DTFS of $4\cos(0.15\pi n + 1)$.

$$x[n] = 4\cos(0.15\pi n + 1)$$
$$= 4\cos\left(2\pi\left(\frac{3}{40}\right)n + 1\right)$$
$$= 2\left[e^{j(2\pi(\frac{3}{40})n+1)} + e^{-j(2\pi(\frac{3}{40})n+1)}\right]$$
$$= 2e^{j1}e^{j2\pi(\frac{3}{40})n} + 2e^{-j1}e^{-j2\pi(\frac{3}{40})n}e^{j2\pi}$$
$$= 2e^{j1}e^{j2\pi(\frac{3}{40})n} + 2e^{-j1}e^{j2\pi(1-\frac{3}{40})n}$$
$$= 2e^{j1}e^{j2\pi(\frac{3}{40})n} + 2e^{-j1}e^{j2\pi(\frac{37}{40})n}.$$
$$D_3 = 2e^{j1}, \quad D_{37} = 2e^{-j1},$$

Time domain: The average power of the *periodic* sinusoid is $\boxed{\frac{4^2}{2} = 8.}$

Frequency domain: The average powers of the two *periodic* complex exponentials are $\boxed{|2e^{j1}|^2 + |2e^{-j1}|^2 = 8.}$
Hence, the average powers are identical.

- Find the DTFS for the discrete-time signal $x[n] = 4\cos[2.4\pi n] + 2\sin[3.2\pi n]$. Sketch their spectra $|D_r|$ and $arg\ D_r$ for $0 \le r \le (N_0 - 1)$.

$$\begin{aligned} x[n] &= 4\cos 2.4\pi n + 2\sin 3.2\pi n \\ &= 4\cos 0.4\pi n + 2\sin 1.2\pi n \\ &= 2[e^{j0.4\pi n} + e^{-j0.4\pi n}] + \frac{1}{j}[e^{j1.2\pi n} - e^{-j1.2\pi n}] \\ &= 2e^{j0.4\pi n} + 2e^{-j0.4\pi n} + e^{j(1.2\pi n - \pi/2)} + e^{-j(1.2\pi n - \pi/2)} \end{aligned}$$

The fundamental $\Omega_0 = 0.4\pi$ and $N_0 = \frac{2\pi}{\Omega_0} = 5$. Note also that.

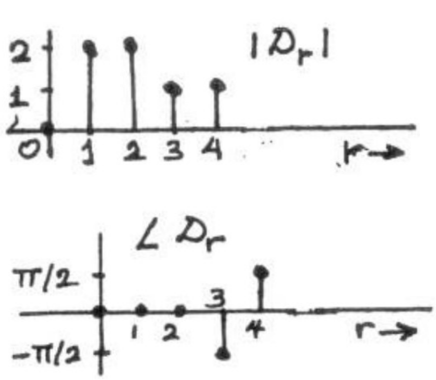
$$e^{-j0.4\pi n} = e^{j1.6\pi n} \quad \text{and} \quad e^{-j1.2\pi n} = e^{j0.8\pi n}$$

Therefore

$$x[n] = 2e^{j0.4\pi n} + 2e^{j1.6\pi n} + e^{j(1.2\pi n - \pi/2)} + e^{j(0.8\pi n + \pi/2)}$$

We have first, second, third and fourth harmonics with coefficients

$$D_1 = D_2 = 2 \quad D_3 = -j \quad D_4 = j$$
$$|D_1| = |D_2| = 2 \quad |D_3| = |D_4| = 1$$
$$\angle D_1 = \angle D_2 = 0 \quad \angle D_3 = -\frac{\pi}{2} \quad \text{and} \quad \angle D_4 = \frac{\pi}{2}$$



Reliaziation of Digital Filter

- DF-1
- DF-2
- Cascade
- Parallel

Reliaziation of Digital Filter: DF 1

$$H(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

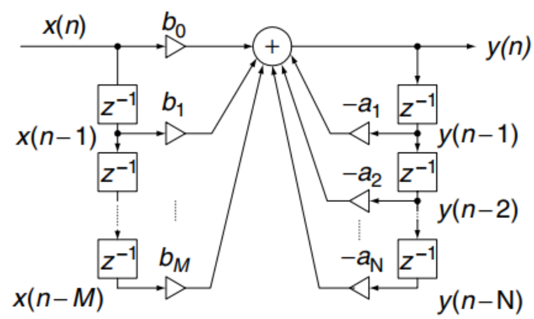
$$Y(z) = H(z)X(z),$$

$$Y(z) = \left(\frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}}\right) X(z)$$

$$\begin{aligned} y(n) =& b_0 x(n) + b_1 x(n-1) + \cdots + b_M x(n-M) \\ & - a_1 y(n-1) - a_2 y(n-2) - \cdots - a_N y(n-N) \end{aligned}$$

Second-order IIR filter (M = N = 2)

Reliaziation of Digital Filter: DF 1



Second-order IIR filter (M = N = 2)

Reliaziation of Digital Filter: DF 2

$$Y(z) = H(z)X(z) = \frac{B(z)}{A(z)}X(z) = B(z)\left(\frac{X(z)}{A(z)}\right)$$

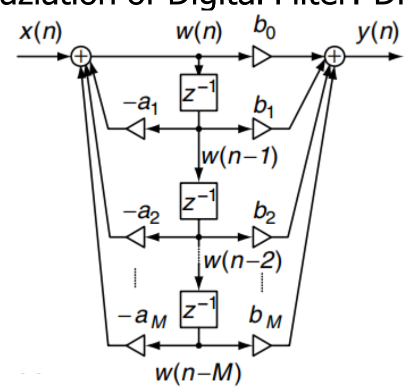
$$= (b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}) \underbrace{\left(\frac{X(z)}{1 + a_1 z^{-1} + \cdots + a_M z^{-M}}\right)}_{W(z)}$$

$$w(n) = x(n) - a_1 w(n-1) - a_2 w(n-2) - \cdots - a_M w(n-M)$$

and

$$y(n) = b_0 w(n) + b_1 w(n-1) + \ldots + b_M w(n-M).$$

Reliaziation of Digital Filter: DF 2

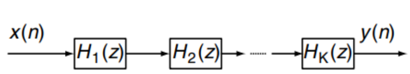


Reliaziation of Digital Filter: Cacade (Series) Realization

$$H(z) = H_1(z) \cdot H_2(z) \cdots H_k(z),$$

$$H_k(z) = \frac{b_{k0} + b_{k1} z^{-1}}{1 + a_{k1} z^{-1}}$$

OR $$H_k(z) = \frac{b_{k0} + b_{k1} z^{-1} + b_{k2} z^{-2}}{1 + a_{k1} z^{-1} + a_{k2} z^{-2}},$$



Reliaziation of Digital Filter: Parallel Realization

$$H(z) = H_1(z) + H_2(z) + \cdots + H_k(z),$$

$$H_k(z) = \frac{b_{k0}}{1 + a_{k1} z^{-1}}$$

OR $$H_k(z) = \frac{b_{k0} + b_{k1} z^{-1}}{1 + a_{k1} z^{-1} + a_{k2} z^{-2}},$$