PS-4

Q1 Code

Q2 (a) Importance Sampling,

$$\mathbb{E}_{\substack{s \sim p(s) \\ a \sim \pi_0(s,a)}} \left[\frac{\pi_1(s,a)}{\hat{\pi}_0(s,a)} R(s,a) \right]$$

$$= \sum_{(s,a)} R(s,a)\, p(s,a) \times \frac{\pi_1(s,a)}{\hat{\pi}_0(s,a)}$$

$$= \sum_{(s,a)} R(s,a) \times \frac{\pi_1(s,a)}{\hat{\pi}_0(s,a)} \times \underbrace{\pi_0(s,a) \times p(s)}_{\hookrightarrow p(s,a)}$$

$$\cancel{= \sum_{(s,a)}}$$

clearly if $\hat{\pi}_0(s,a) = \pi_0(s,a)$,

$$\mathbb{E}_{\substack{s \sim p(s) \\ a \sim \pi_0(s,a)}} \left[\frac{\pi_1(s,a)}{\hat{\pi}_0(s,a)} R(s,a) \right] = \sum_{(s,a)} R(s,a)\, \pi_1(s,a) \times p(s)$$

$$= \mathbb{E}_{\substack{s \sim p(s) \\ a \sim \pi_1(s,a)}} \left[R(s,a) \right]$$

(b) clearly, numerator $= \mathbb{E}_{\substack{s \sim p(s) \\ a \sim \pi_1(s,a)}} (R(s,a))$ from (a)

$\Rightarrow$ we have to show that the denominator $= 1$.

$$\mathbb{E}_{\substack{s\sim p(s)\\ a\sim \pi_0(s,a)}}\left[\frac{\pi_1(s,a)}{\hat{\pi}_0(s,a)}\right] = \sum_{(s,a)} \frac{\pi_1(s,a)}{\hat{\pi}_0(s,a)}\, p(s,a)$$

$$= \sum_{(s,a)} \frac{\pi_1(s,a)}{\hat{\pi}_0(s,a)} \times p(s) \times \pi_0(s,a)$$

clearly, if $\hat{\pi}_0(s,a) = \pi_0(s,a)$

$$\mathbb{E}_{\substack{s\sim p(s)\\ a\sim \pi_0(s,a)}}\left[\frac{\pi_1(s,a)}{\hat{\pi}_0(s,a)}\right] = \sum_{(s,a)} p(s)\,\pi_1(s,a)$$

$$= \mathbb{E}_{\substack{s\sim p(s)\\ a\sim \pi_1(s,a)}}[1]$$

$$= \underline{\underline{1}}$$

(c) if there is only 1 sample in the dataset,

$$\frac{\mathbb{E}_{\substack{s\sim p(s)\\ a\sim \pi_0(s,a)}}\left[\frac{\pi_1(s,a)\,R(s,a)}{\hat{\pi}_0(s,a)}\right]}{\mathbb{E}_{\substack{s\sim p(s)\\ a\sim \pi_0(s,a)}}\left[\frac{\pi_1(s,a)}{\hat{\pi}_0(s,a)}\right]} \approx \frac{\frac{\pi_1(s,a)}{\hat{\pi}_0(s,a)} \times R(s,a) \times \cancel{p(s)\times\pi_0(s,a)}}{\frac{\pi_1(s,a)}{\hat{\pi}_0(s,a)} \times \cancel{p(s)\times\pi_0(s,a)}}$$

$$\approx R(s,a)$$

$$\neq \mathbb{E}_{\substack{s\sim p(s)\\ a\sim \pi_1(s,a)}}[R(s,a)]$$

if $\underline{\underline{\pi_1 \neq \pi_0}}$

(d) (i) $E_{\substack{s\sim p(s)\\ a\sim\pi_0(s,a)}}\left[E_{a\sim\pi_1(s,a)}\left[\hat{R}(s,a)\right]\right]$

$$= E_{\substack{s\sim p(s)\\ a\sim\pi_1(s,a)}}\left[\hat{R}(s,a)\right]$$

$$= \sum_{(s,a)} \hat{R}(s,a)\, p(s)\, \pi_1(s,a)$$

& $E_{\substack{s\sim p(s)\\ a\sim\pi_0(s,a)}}\left[\frac{\pi_1(s,a)}{\hat{\pi}_0(s,a)}\left(R(s,a)-\hat{R}(s,a)\right)\right]$

$$= \sum_{(s,a)} \frac{\pi_1(s,a)\, R(s,a)}{\hat{\pi}_0(s,a)} \times p(s) \times \pi_0(s,a)$$

$$- \sum_{(s,a)} \frac{\pi_1(s,a)}{\hat{\pi}_0(s,a)} R(s,a) \times p(s) \times \pi_0(s,a)$$

$\Rightarrow$ if $\hat{\pi}_0(s,a) = \pi_0(s,a)$, we precisely end up with

$$E_{\substack{s\sim p(s)\\ a\sim\pi_1(s,a)}}\left[R(s,a)\right]$$

(ii) notice that the first term will not change & the 2nd term goes to 0 at $R(s,a) = \hat{R}(s,a)$.

1st term does become $\sum_{(s,a)} R(s,a)\, p(s)\, \pi_1(s,a)$

$$= E_{\substack{s\sim p(s)\\ a\sim\pi_1(s,a)}}\left[R(s,a)\right]$$

~~(d)~~

(e) (i) Importance sampling clearly, $\because$ estimating $\hat{R}(s,a)$ is very tough opposed to estimating $\hat{\pi}_0(s,a)$.

(ii) opposite of (i) $\because$ $\hat{R}(s,a)$ is simple ~~to~~ to estimate compared to $\hat{\pi}_0(S,a)$.

Q3 $f_u(x) = \arg\min_{v \in V} \|x - v\|^2$

$= \arg\min_{\alpha \in \mathbb{R}} \{(x-v)^T (x-v)\}$

$= \arg\min_{\alpha \in \mathbb{R}} (x^T x - 2x^T v + v^T v)$

$= \arg\min_{\alpha \in \mathbb{R}} (x^T x - 2\alpha x^T u + \alpha^2 \cancel{u^T u}^{\,1})$

$$= \arg\min_{\alpha \in \mathbb{R}} \underbrace{(x^T x - 2\alpha x^T u + \alpha^2)}_{g(\alpha)}$$

$\frac{dg(\alpha)}{d\alpha} = -2x^T u + 2\alpha$

setting to 0, we get

$\underline{\alpha = x^T u}$

$\Rightarrow f_u(x) = (x^T u)\, u = (u^T x)\, u$

now objective, find u constraint to

$$\arg\min_{u^T u = 1} \sum_{i=1}^{m} \| x^{(i)} - f_u(x^{(i)}) \|_2^2$$

$$\equiv \arg\min_{u : u^T u = 1} \sum_{i=1}^{m} (x^{(i)} - (u^T x^{(i)}) u)^T (x^{(i)} - (u^T x^{(i)}) u)$$

$$\equiv \arg\min_{u : u^T u = 1} \sum_{i=1}^{m} \Big(\|x^{(i)}\|_2^2 - (u^T x^{(i)})(x^{(i)T} u) - (u^T x^{(i)})(u^T x^{(i)}) + (u^T x^{(i)})^2 \cancel{u^T u}^{\,1} \Big)$$

$$= \arg\min_{u : u^T u = 1} \sum_{i=1}^{m} \left(\|x^{(i)}\|_2^2 - 2u^T (x^{(i)} x^{(i)T}) u + x^{(i)T} \cancel{(u u^T)}^{\,1} x^{(i)} \right)$$

$$\equiv \arg\max_{u : u^T u = 1} u^T \left(\sum_{i=1}^{m} x^{(i)T} x^{(i)} \right) u \equiv \arg\max_{u : u^T u = 1} u^T \Sigma u$$

Q5 $V'(s) = R(s) + \gamma \max_{a \in A} \sum_{s' \in S} P_{sa}(s') V(s')$

(a) $B(V_1) = V_1'$, $B(V_2) = V_2'$

$$\|B(V_1) - B(V_2)\|_\infty \overset{\max_{s \in S}}{=} \| \gamma \max_{a \in A} \sum_{s' \in S} P_{sa}(s') V_1(s') - \gamma \max_{a \in A} \sum_{s' \in S} P_{sa}(s') V_2(s') \|_\infty$$

$$= \gamma \max_{s' \in S} \| \underbrace{\max_{a \in A}}_{\text{becomes useless}} \sum_{s' \in S} P_{sa}(s') (V_1(s') - V_2(s')) \|$$

identity, if $\sum_i p_i = 1$ & $p_i \geq 0 \ \forall i$

$$\Rightarrow \sum_i p_i x_i \leq \max_i x_i$$

$$\leq \gamma \|V_1 - V_2\|_\infty$$

(b) assume $B(V)$ has 2 fixed pts.

$$\Rightarrow B(V_1) = V_1 \ \& \ B(V_2) = V_2 \quad \exists V_1, V_2.$$

$$\Rightarrow \|B(V_1) - B(V_2)\|_\infty \leq \gamma \|V_1 - V_2\|_\infty$$

$$\|V_1 - V_2\|_\infty \leq \gamma \|V_1 - V_2\|_\infty$$

$\because \gamma$ can be taken arbitrarily small, $\Rightarrow \|V_1 - V_2\|_\infty = 0$

$$\Rightarrow V_1 = V_2$$

Q4 (a) $s_j \sim \mathcal{N}(0,1)$

$$\Rightarrow g'(w_j^T x^{(i)}) = \frac{1}{\sqrt{2\pi}} e^{-(w_j^T x^{(i)})^2/2}$$

$$\log g'(w_j^T x^{(i)}) = -\frac{(w_j^T x^{(i)})^2}{2} - \frac{1}{2}\log(2\pi)$$

$$\frac{\partial \log g'(w_j^T x^{(i)})}{\partial w_j} = -\frac{\cancel{2}(w_j^T x^{(i)}) \times x^{(i)}}{\cancel{2}}$$

$$= -(w_j^T x^{(i)})\, x^{(i)}$$

note that W has rows of w_j^T

$$\cancel{\Rightarrow \frac{\partial \sum_{j=1}^{d} \log g'(w_j^T x^{(i)})}{\partial W}}$$

$$\nabla_W \sum_{j=1}^{d} \log g'(w_j^T x^{(i)}) = -\underbrace{(Wx^{(i)}) * x^{(i)T}}_{\text{only way shapes match.}}$$

elementwise product

$$\nabla_W l(W) = \sum_{i=1}^{n} \left(\frac{1}{|W|} \times |W| \times (W^{-1})^T - (Wx^{(i)})\, x^{(i)T} \right)$$

$$= n (W^{-1})^T - W\Sigma$$

$$= 0 \Rightarrow W\Sigma = n(W^{-1})^T$$

$$\cancel{\Rightarrow W \Sigma W^T = n}$$

$$W X^T X = n W^{-T}$$

$$W^T W X^T X = n(W^T W^{-1}) \rightarrow I$$

$$W^T W = n(X^T X)^{-1}$$

now, let R be any ~~(orthog)~~ rotation matrix

~~s.t.~~

$\Rightarrow R^T R = I$

if $W' = RW$

$$(W'^T) W' = W^T R^T R W$$
$$= W^T W$$

$\Rightarrow$ all W's are also solutions

if W is a solution.

(b) $s_j \sim \mathcal{L}(0,1)$

$$\Rightarrow g'(w_j^T x^{(i)}) = \frac{1}{2} e^{-|w_j^T x^{(i)}|}$$

$$\log g'(w_j^T x^{(i)}) = -|w_j^T x^{(i)}| - \log 2$$

$$\frac{\partial \log g'(w_j^T x^{(i)})}{\partial w_j} = \begin{cases} -x^{(i)} & w_j^T x^{(i)} \geq 0 \\ x^{(i)} & w_j^T x^{(i)} < 0 \end{cases}$$

$$= (-1)^{I\{w_j^T x^{(i)} \geq 0\}} x^{(i)}$$

$$\Rightarrow \nabla_W \sum_{j=1}^{d} \log g'(w_j^T x^{(i)}) = \begin{bmatrix} (-1)^{I\{w_1^T x^{(i)} \geq 0\}} x^{(i)T} \\ (-1)^{I\{w_2^T x^{(i)} \geq 0\}} x^{(i)T} \\ \vdots \\ (-1)^{I\{w_n^T x^{(i)} \geq 0\}} x^{(i)T} \end{bmatrix}$$

$$\nabla_W l(W) = \sum_{i=1}^{n} \left(W^{-T} + \phi(\cdots) M^{(i)T} \right)$$

$$= \underbrace{\sum_{i=1}^{n} \left(W^{-T} - \operatorname{sgn}(W M^{(i)}) M^{(i)T} \right)}$$

$$\Rightarrow W := W + \alpha (\cdots\cdots)$$

(c) Code.

Q6 Code