PSO

* suppose $f : \mathbb{R}^n \longrightarrow \mathbb{R}$

vector

$$\Rightarrow \nabla f(x) = \begin{bmatrix} \frac{\partial f(x)}{\partial x_1} \\ \frac{\partial f(x)}{\partial x_2} \\ \vdots \\ \frac{\partial f(x)}{\partial x_n} \end{bmatrix}_{n\times 1} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$$

gradient!

$$\nabla^2 f(x) = \begin{bmatrix} \frac{\partial^2 f(x)}{\partial x_1^2} & \frac{\partial^2 f(x)}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f(x)}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f(x)}{\partial x_2 \partial x_1} & \frac{\partial^2 f(x)}{\partial x_2^2} & & \vdots \\ \vdots & & \ddots & \\ \frac{\partial^2 f(x)}{\partial x_n \partial x_1} & \cdots & & \frac{\partial^2 f(x)}{\partial x_n^2} \end{bmatrix}_{n\times n}$$

Hessian!

Symmetric by definition for contious f^ns.

(a) $f(x) = \frac{1}{2} x^T \underbrace{A}_{\text{symmetric matrix}} x + b^T x$, $b \in \mathbb{R}^n$, vector

$$\nabla f(x) = \nabla \left(\frac{1}{2} x^T A x + b^T x \right)$$

* $\boxed{\nabla_x b^T x = b}$ $\quad$ $\boxed{\text{*} \ \nabla_x x^T A x = Ax}$

$= Ax + b$

(b) $f(x) = g(h(x))$

$g : \mathbb{R} \longrightarrow \mathbb{R}$

$h : \mathbb{R}^n \longrightarrow \mathbb{R}$

$\nabla f(x) = g'(h(x)) \times \nabla h(x)$

$\rightarrow$ linear $\Rightarrow \nabla^2 (b^T x) = 0$

(c) $\nabla^2 f(x) = \frac{1}{2} \nabla^2 (x^T A x) + \nabla^2 (b^T x)$

$= \underline{\underline{A}}$

(d) $f(x) = g(a^T x) \quad g : \mathbb{R} \longrightarrow \mathbb{R} \quad \& \ a \in \mathbb{R}^n$

$\nabla f(x) = g'(a^T x) \times a$

$\nabla^2 f(x) = \underline{\underline{g''(a^T x)\, a a^T}}$

Q2 (a) $\cancel{x^T A x = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} x_i x_j}$

$\cancel{= \sum_{i=1}^{n} \sum_{j=1}^{n} z_i z_j x_i x_j}$

(a) $x^T A x = x^T z z^T x$

$= x^T z (x^T z)^T$

$= \underline{\underline{\sum_{i=1}^{n} (x^T z)_i^2}} \geq 0$

(b) $A = z z^T$

Null space : $\{x : Ax = 0\}$

$\{x : z z^T x = 0\}$

$\Rightarrow \underline{\{x : z^T x = 0\}} \rightarrow R(\text{nullspace}) = \underline{\underline{n-1}}$

$\hookrightarrow$ hyperplane in n dimensions.

~~$R(A) = R(ZZ^T) = 1$?~~

$$R(A) = \underbrace{n}_{\text{combined dimensionality of nullspace \& rowspace}} - \overbrace{(n-1)}^{\text{dimensionality of the nullspace of } A} = 1$$

(c) $x^T B A B^T x = (x^T B) A (x^T B)^T$

let $x' = (x^T B)^T$

$\Rightarrow x^T B A B^T x = (x')^T A x' \geq 0$

$\Rightarrow$ PSD

Q3 (a) ~~$At^{(i)} = T \Lambda T^{-1} t^{(i)}$~~

$AT = T\Lambda$ $\Rightarrow At^{(i)} = \lambda_i t^{(i)}$

$[At^{(1)} \ At^{(2)} \ \ldots \ At^{(n)}] = [\lambda_1 t^{(1)} \ \lambda_2 t^{(2)} \ \ldots \ \lambda_n t^{(n)}]$

(b) $A = U\Lambda U^T$

$AU = U\Lambda(U^T U)$

$AU = U\Lambda$

$\Rightarrow Au^{(i)} = \lambda_i u^{(i)}$

(c) $x^T A x \geq 0 \quad \forall x$

$x^T U \Lambda U^T x \geq 0 \quad \forall x$

$(U^T x)^T \Lambda U^T x \geq 0 \quad \forall x$

$(x')^T \Lambda x' \geq 0 \quad \forall x$

$\Rightarrow \Lambda$ is PSD

$At^{(i)} = \lambda_i t^{(i)}$

$t^{T(i)} A t^{(i)} = \lambda_i \|t^{(i)}\|_2 \geq 0$

$\Rightarrow \lambda_i \geq 0$.