Q1. b) $X, Y$ have joint pdf $f(x, y)$

To compute the joint pdf of $X^2$ & $Y^2$.

Set $Z = X^2$, $W = Y^2$

$Z \geq 0$, $W \geq 0$

For $z, w$ in the Range of $(Z, W)$ $\exists$ four solution

$$x = \pm\sqrt{z}, \quad y = \pm\sqrt{w}$$

$$(\pm\sqrt{z}, \pm\sqrt{w})$$

S.t

$$z = x^2, \quad w = y^2$$

$$J(x, y) = \begin{vmatrix} 2x & 0 \\ 0 & 2y \end{vmatrix} = 4xy$$

So the joint pdf of $(Z, W)$ is

$$h(z, w) = \frac{1}{|J(\sqrt{z}, \sqrt{w})|} f(\sqrt{z}, \sqrt{w}) + \frac{1}{|J(-\sqrt{z}, \sqrt{w})|} f(-\sqrt{z}, \sqrt{w})$$

$$+ \frac{1}{|J(\sqrt{z}, -\sqrt{w})|} f(\sqrt{z}, -\sqrt{w})$$

$$+ \frac{1}{|J(-\sqrt{z}, -\sqrt{w})|} f(-\sqrt{z}, -\sqrt{w})$$

$$g(z,w) = \frac{1}{4\sqrt{zw}} \Big\{ f(\sqrt{z}, \sqrt{w}) + f(-\sqrt{z}, \sqrt{w}) + f(\sqrt{z}, -\sqrt{w}) + f(-\sqrt{z}, -\sqrt{w}) \Big\},$$

$z > 0$

$w > 0$.

$Q_2$. iii) Let $X$ & $Y$ have the joint pdf $f(x,y)$.

To find pdf of $|Y-X|$

Let $Z := |Y-X|$, set $w = X$

$Z \geq 0$

For $(z,w)$ in the range of $(z,w)$

$$Z = |y - \cancel{x}| = \begin{cases} y - x \; ; & y - x \geq 0 \\ -y + x \; ; & y - x < 0 \end{cases}$$

$$x = w$$

$$y = z + w$$

$$y = w - z$$

$$J(x, y) = \begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix} = -1 \; ; \; (w, z + w)$$

$$J(x + y) = \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} = 1 \; ; \; (w, w - z)$$

$$k(z, w) = \frac{1}{|-1|} f(w, z + w) + \frac{1}{|1|} f(w, w - z)$$

$$z \geq 0$$

$$f_z(z) = \int_{-\infty}^{\infty} k(z, w) \, dw$$

$$= \int_{-\infty}^{\infty} \left[ f(w, z + w) + f(w, w - z) \right] dw$$

$; \ z \geq 0$

Q5. Let $X \sim \text{Poisson}(\mu)$

$Y \sim \text{Poisson}(\lambda)$

$X$ & $Y$ are independent.

To show $\rightarrow$ $X+Y \sim \text{Poisson}(\lambda+\mu)$

Range of $X+Y$ is $\{0, 1, 2, \ldots, \infty\}$

For $k \in \{0, 1, 2, \ldots\}$

$$P(X+Y=k) = \sum_{i=0}^{k} P(X=i, X+Y=k)$$

$$= \sum_{i=0}^{k} P(X=i, i+Y=k)$$

$$= \sum_{i=0}^{k} P(X=i, Y=k-i)$$

$$= \sum_{i=0}^{k} P(X=i)\, P(Y=k-i)$$

$$= \frac{e^{-(\mu+\lambda)}}{k!} \sum_{i=0}^{k} \frac{k!\, \mu^{i} \lambda^{k-i}}{i!\,(k-i)!}$$

$$= \sum_{i=0}^{k} \binom{k}{i} \mu^i \lambda^{k-i}$$

$$= \frac{e^{-(\mu+\lambda)} (\mu+\lambda)^k}{k!}$$

Q6. $X \sim P(\lambda_1)$

$Y \sim P(\lambda_2)$

$P = (Y = m \mid X + Y = n)$ for $m = 0, 1, 2 \ldots, n$

Solution →

$$\frac{P(Y = m, X + Y = n)}{P(X + Y = n)}$$

$$= \frac{P(Y = m, X = n - m)}{P(X + Y = n)}$$

$$= \frac{P(Y = m)\, P(X = n - m)}{P(X + Y = n)}$$

Q7 $X, Y$ iid Bernoulli$(\frac{1}{2})$

T.S $\rightarrow$ $X$ & $|X-Y|$ are independent

Range of $|X-Y| = \{0, 1\}$

$$P(X=0, |X-Y|=0) = P(X=0, Y=0)$$

$$= P(X=0) \times \frac{1}{2}$$

$$= P(X=0)\, P(|X-Y|=0)$$

$$P(|X-Y|=0) = P(X=0, Y=0) + P(X=1, Y=1)$$

$$= \frac{1}{2}$$

Check for all case

Q8. $X \sim B(5, \frac{1}{2})$ $\{0, 1, 2, \ldots 5\}$

$Y \sim \mu(0, 1)$ $1 > y > 0$

$X$ & $Y$ are independent.

$$\frac{P(X+Y\le 2)}{P(X+Y\ge 5)}$$

$$P(X+Y\le 2) = P(X+Y\le 2\mid X=0)\,P(X=0)$$

$$+\,P(X+Y\le 2\mid X=1)\,P(X=1)$$

$$= P(0+Y\le 2\mid X=0)\binom{5}{0}\left(\tfrac{1}{2}\right)^{0}\left(\tfrac{1}{2}\right)^{5-0}$$

$$+\,P(1+Y\le 2\mid X=1)\binom{5}{1}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{5-1}$$

$$= P(Y\le 2)\left(\frac{1}{2}\right)^{5} + P(Y\le 1)\,5\times\left(\frac{1}{2}\right)^{5}$$

$$= 6\left(\frac{1}{2}\right)^{5}.$$

$$P(X+Y\ge 5) = P(X+Y\ge 5\mid X=5)\,P(X=5)$$

$$\underbrace{P(Y\ge 0)}_{=1}\binom{5}{5}\left(\tfrac{1}{2}\right)^{5}\left(\tfrac{1}{2}\right)^{0}$$

$$= \left(\frac{1}{2}\right)^5.$$

Q9. $X, Y$ have joint pdf

$$f(x,y) = \frac{\sqrt{3}}{4\pi} e^{-\frac{x^2 - xy + y^2}{2}}, \quad (x,y) \in \mathbb{R}^2$$

To Show $X$ & $Y$ are dependent.

$$f_X(x) = \int_{-\infty}^{\infty} \frac{\sqrt{3}}{4\pi} e^{-\left(\frac{x^2 - xy + y^2}{2}\right)} dy$$

$$= y^2 - 2\frac{x}{2} y + \frac{x^2}{4} - \frac{x^2}{4} + x^2$$

$$= \left(y - \frac{x}{2}\right)^2 + \frac{3}{4} x^2$$

$$= \frac{\sqrt{3}}{4\pi} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(y - \frac{x}{2}\right)^2 - \frac{3}{2\cdot 4} x^2}$$

$$= e^{-\frac{3x^2}{2\cdot 4}} \frac{\sqrt{3}}{4\pi} \times \sqrt{2\pi}$$

$$f_X(x) = \frac{\sqrt{3}}{2\sqrt{2}\sqrt{\pi}} e^{-\frac{x^2}{2(2/\sqrt{3})^2}}$$

$$= \frac{1}{2\pi\left(\frac{2}{\sqrt{3}}\right)} e^{-x^2/2(2/\sqrt{3})^2}$$

$$X \sim N(0, 4/3)$$

$$X \sim N(0, 4/3)$$

$$f(x, y) \neq \frac{\sqrt{3}}{2\sqrt{2\pi}} e^{-3x^2/8} \cdot \frac{\sqrt{3}}{2\sqrt{2\pi}} e^{-3y^2/8}$$

$$f(1, 1) \neq f_X(1) f_Y(1)$$

Q 100 $X, Y$ are independent.

$E(X^4) = 2 \qquad E(X^2) = 1$

$E(Y^2) = 1 \qquad E(Y) = 0$

$Var(X^2Y) = E(X^2Y)^2 - (EX^2Y)^2$

$= E(X^4Y^2) - (EX^2Y)^2$

$= EX^4 \, EY^2 - (EX^2 \, EY)^2$

$= 2$