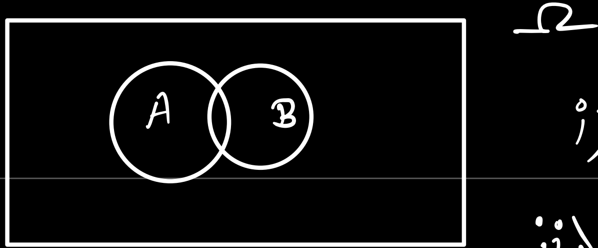
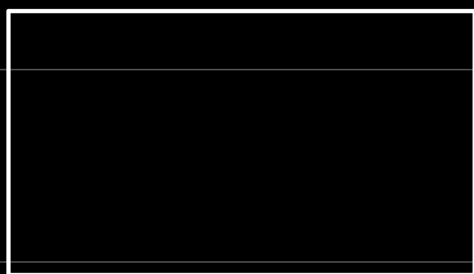
Q1. (a) event which occurs if exactly one of the events A, B occurs




i) $(A \cup B) / (A \cap B)$

ii) $(A \cap B^c) \cup (B \cap A^c)$

b) The event which occurs if none of the events A, B, C occurs.



$(A \cup B \cup C)^c$

c) Event which occurs if exactly two of events A, B, C occurs

$(A \cap B), (A \cap C), (B \cap C)$

$\cup \{((A \cap B) \cap C^c), ((A \cap C) \cap B^c), (B \cap C) \cap A^c\}$

Q2. $P\{e\} = 2p$ , $P\{odd\} = p$

$e = 2, 4, 6$ $\quad$ $odd = 1, 3, 5$

$3P\{e\} + 3P\{odd\} = 1$

$9p = 1$

$p = 1/9.$

$P$ (outcome is less than 4) $= P(\{1,2,3\})$

$= \frac{2}{9} + 2/9 = 4/9.$

Q3. $\Omega \neq \phi$

$P: \mathcal{F} \to [0,1]$

(a) $P(\Omega) = 1$

b) $P(A \quad) = P(A) + P(B)$ if $A \cap B = \phi$

c) If $(A_n) \downarrow$ s.t $\bigcap_{n=1}^{\infty} A_n = \phi$ then

$\lim_{n\to\infty} P(A_n)$

Then to show that $P$ is a probability measure.

Let $B_1, B_2, ----$ be a seq. of pairwise disjoint events.

Define $A_1 = \bigcup_{n=1}^{\infty} B_n$ $\qquad A_2 \subset A_1$

$A_2 = \bigcup_{n=2}^{\infty} B_n$

$\vdots$

$A_K = \bigcup_{n=K}^{\infty} B_n$

$$P\left(\bigcup_{n=1}^{\infty} B_n\right) = P\left(\left\{\bigcup_{n=1}^{k-1} B_n\right\} \cup \left\{\bigcup_{n=k}^{\infty} B_n\right\}\right)$$

$$= P\left(\bigcup_{n=1}^{k-1} B_n\right) + P\left(\bigcup_{n=k}^{\infty} B_n\right)$$

$$P\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{k-1} P(B_n) + P(A_k)$$

$$= \lim_{k\to\infty} \sum_{n=1}^{k-1} P(B_n) + \lim_{k\to\infty} P(A_k)$$

$$\lim_{k\to\infty} \sum_{n=1}^{k-1} P(B_n) = \sum_{n=1}^{\infty} P(B_n), \qquad \lim_{k\to\infty} P(A_k) = 0$$

Q4. $A_1, A_2 \; ------$ s.t $P(A_i \cap A_j) = 0$ if $i \neq j$

To show that

$$P\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} P(A_k)$$

Recall $\rightarrow$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$

Inclusion - Exclusion formula

If $A_1, A_2, \cdots A_n$ are events then

$$P(A_1 \cup A_2 \cdots \cup A_n) = \sum_{i=1}^{n} P(A_i) - \sum_{1 \leq i < j \leq n} P(A_i \cap A_j)$$

$$+ \sum_{1 \leq i < j \leq n} P(A_i \cap A_j \cap A_k)$$

$$+ (-1)^{n+1} P(A_1 \cap \cdots \cap A_n)$$

Define $B_1 = A_1$

$B_2 = A_1 \cup A_2$

$\vdots$

$B_n = \bigcup_{k=1}^{n} A_k$ $\qquad \bigcup_{n=1}^{\infty} B_k = \bigcup_{n=1}^{\infty} A_n$

$(B_n)_{n \geq 1} \uparrow \quad \bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$

$$P(B_n) = P(A_1 \cup \cdots \cup A_n)$$

$$= \sum_{k=1}^{n} P(A_k) - \sum_{i<j} P(A_i \cap A_j)$$

Given $P(A_i \cap A_j) = 0$ if $i \neq j$

$$\sum_{i<j<k} P(A_i \cap A_j \cap A_k) = 0$$

$$(A_i \cap A_j \cap A_k) \subset (A_i \cap A_j)$$

$$P(B_n) = \sum_{k=1}^{n} P(A_k) \quad \forall\ n \geq 1$$

$$\lim_{n\to\infty} P(B_n) = \sum_{k=1}^{\infty} P(A_k)$$

By continuity of probability measure

$$\lim_{n\to\infty} P(B_n) = P\left(\bigcup_{n=1}^{\infty} B_n\right)$$

$$= P\left(\bigcup_{k=1}^{\infty} A_k\right)$$

Q5. $\Omega = \{ (i, j) : i, j \in \{1, 2, \dots 6\} \}$

$A = \{ \max . \{ X, Y\} = 3 \}$

$= \{ (i, 3), (3, j), \ i, j \in \{1, 2\}, (3, 3) \}$

$B = \{ \min \{ X, Y \} = 2 \}$

$= \{ (2, i), (i, 2) : i \in \{2, 3, 4, 5, 6\} \}$

$i \in \{3, 4, 5, 6\}$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{2/36}{9/36} = 2/9$$

Q6. Let $A$ be the event that aircraft is present & $B$ be the event that radar generates an alarm

$$P(B|A) = 0.99$$

$$P(B|A^c) = 0.10$$

$$P(A) = 0.05$$

$$P(A^c \cap B) = P(B|A^c)\,P(A^c)$$

$$= (0.10)(1 - P(A))$$

$$P(A \cap B^c) = P(B^c|A)\,P(A)$$

$$P(B^c|A) = 1 - P(B|A)$$

Q8. Conditional version of total probability

Thm → Let $\{A_1, -- A_N\}$ be a partition of $\Omega$. Let $B \in \mathcal{F}$ be the event s.t $P(A_i \cap B) > 0 \quad \forall i = 1, 2 .. N$. Then for any event $A \in \mathcal{F}$

Show that

$$P(A|B) = \sum_{i=1}^{N} P(A_i | B) \, P(A | A_i \cap B)$$

$P(\cdot | B)$

$$A \cap B = (A \cap B) \cap \Omega = (A \cap B) \cap \left( \bigcup_{i=1}^{N} A_i \right)$$

$$= \bigcup_{i=1}^{N} (A \cap B \cap A_i)$$

$$= \bigcup_{i=1}^{N} (A \cap (B \cap A_i))$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{1}{P(B)} \left( P\left( \bigcup_{i=1}^{N} A \cap (B \cap A_i) \right) \right)$$

$$= \sum_{i=1}^{N} P(A \cap (B \cap A_i)) \Big/ P(B)$$

$$= \sum_{i=1}^{N} \frac{P(A \mid B \cap A_i)\, P(B \cap A_i)}{P(B)}$$

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**Q7.**

| ① | ② | ③ |
|---|---|---|
| G \| G | G \| S | S \| S |

A chest is chosen at random and a drawer is opened. If the drawer contains a gold coin What is the probability that the other drawer also contain a gold coin.

Solⁿ ⟹ $A_i \to$ choosing $i$th chest

$$P(A_i) = 1/3 \quad \forall\, i = 1, 2, 3.$$

$B \rightarrow$ drawer contains a gold coin

$$P(A_1 \mid B) = \frac{P(B|A_1) \times P(A_1)}{\sum_{i=1}^{3} P(B|A_i) \times P(A_i)}$$

$P(B|A_1) = 1$

$P(B|A_2) = 1/2$

$P(B|A_3) = 0$

$$= \frac{1}{1 + 1/2}$$

$$= 2/3$$