MTH-222, MTH-6031: Probability and Statistics

Tutorial # 5 (Expectation)

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1. Let X be a continuous random variable with pdf f. Suppose X is symmetric (i.e.,X and $-X$ have the same distribution) and $X^2 \sim \exp(\lambda)$. Find f.

2. Let X be a random variable such that $P\{|X-1| = 2\} = 0$. Is X a continuous random variable? Also write down $P\{|X-1| \geq 2\}$ in terms of distribution function of X.

3. Let X be Poisson with parameter λ. Find mean of $\dfrac{1}{1+X}$.

4. Find the variance of Binomial(n,p), Poisson(λ), exponential(λ) and $N(\mu, \sigma^2)$ random variables.

5. Let X be exponentially distributed with parameter λ. Find the 4th moment.

6. Let X be standard normal random variable. Find $E|X|$.

7. Let X be a Binomial random variable with parameters $n = 4, p = \dfrac{1}{2}$. Find $E\left[\sin\left(\dfrac{\pi x}{2}\right)\right]$.

8. Let X be a geometrically distributed random variable with parameter p and M be a positive integer. Find $E[\min\{X, M\}]$.

9. Let X be a discrete random variable with pmf
$$f_X(k) = P(X = k) = \frac{1}{9}, \quad \text{for } k = -4, -3, -2, \cdots, 3, 4$$
Find $var(X)$.

10. Let
$$P_X(x) = \begin{cases} \dfrac{1}{2^x}, & x = 1, 2, 3, \cdots, \\ 0, & otherwise. \end{cases}$$
be the pmf of the random variable X, then find all the possible values of t such that $E[e^{tX}]$ exist.

Q1.

1. Let X be a continuous random variable with pdf f. Suppose X is symmetric (i.e.,X and $-X$ have the same distribution) and $X^2 \sim \exp(\lambda)$. Find f.

Solution $\rightarrow$ $F_X(x) = P(X \leq x)$

$= F_{-X}(x) = P(-X \leq x) = P(X \geq -x)$

$= 1 - P(X < -x)$

$= 1 - F_X(-x)$

$\because F_X$ is continuous

$$\boxed{F_X(x) = 1 - F_X(-x) \quad \forall\, x \in \mathbb{R}}$$

$$F_{X^2}(x) = P(X^2 \leq x)$$

$$= \begin{cases} 0 & ; \ x < 0 \\ P(-\sqrt{x} \leq X \leq \sqrt{x}), & x \geq 0 \end{cases}$$

$$F_{X^2}(x) = \begin{cases} 0 & ; \ x < 0 \\ F_X(\sqrt{x}) - F_X(-\sqrt{x}), & x \geq 0 \end{cases}$$

CDF of $\exp(\lambda)$ distribution is

$$= \begin{cases} 0, & x < 0 \\ 1 - e^{-\lambda x}; & x \geq 0 \end{cases}$$

So for $x \geq 0$, we have

$$F_X(\sqrt{x}) - F_X(-\sqrt{x}) = 1 - e^{-\lambda x}$$

$$F_X(\sqrt{x}) - (1 - F_X(\sqrt{x}))$$

$$2F_X(\sqrt{x}) - 1 = 1 - e^{-\lambda x}$$

$$F_X(\sqrt{x}) = \frac{2 - e^{-\lambda x}}{2}, \quad x \geq 0$$

$$\Rightarrow \quad F_X(x) = \frac{2 - e^{-\lambda x^2}}{2}, \quad x \geq 0$$

$$\Rightarrow \quad f_X(x) = \frac{+\lambda(2x)e^{-\lambda x^2}}{2}; \quad x \geq 0$$

$$\boxed{f_X(x) = \lambda x e^{-\lambda x^2}}, \quad x \geq 0$$

For $x < 0$,

$$F_X(x) = 1 - F_X(-x)$$

$$= 1 - \left(\frac{2 - e^{-\lambda x^2}}{2}\right)$$

$$= e^{-\lambda x^2}/2$$

$$\boxed{f_X(x) = -\lambda x e^{-\lambda x^2} \quad ; \quad x < 0}$$

$$\boxed{f_X(x) = \lambda |x| e^{-\lambda x^2}} \quad , x \in \mathbb{R}$$

2. Let X be a random variable such that $P\{|X-1| = 2\} = 0$. Is X a continuous random variable? Also write down $P\{|X-1| \geq 2\}$ in terms of distribution function of X.

Sol^n $\rightarrow$

$$P(X \in \{3, -1\}) = 0$$

$$\Omega = \{3, -3, 1, -1\}$$

$$X(\omega) = \omega$$

$$P(1) = 1/2 \qquad P(-3) = 1/2$$

$$P(3) = 0 = P(-1)$$

$\hookrightarrow$ Discrete r.v.

$$P(|X-1| \geq 2)$$

$$= P(X-1 \geq 2) + P(-(X-1) \geq 2)$$

$$= P(X \geq 3) + P(X \leq -1)$$

$$= 1 - P(X < 3) + F_X(-1)$$

$$\boxed{= 1 - F_X(3-) + F_X(-1)}$$

3. Let X be Poisson with parameter λ. Find mean of $\dfrac{1}{1+X}$.

$$\frac{1}{1+X} = \sum_{K=0}^{\infty} \frac{1}{1+K} P(X=k)$$

$$= \sum_{k=0}^{\infty} \frac{1}{1+k} e^{-\lambda} \frac{\lambda^k}{k!}$$

$$= \frac{e^{-\lambda}}{\lambda} \sum_{k=0}^{\infty} \frac{\lambda^{k+1}}{(K+1)!}$$

$$= \frac{e^{-\lambda}}{\lambda} (e^{\lambda} - 1)$$

4. Find the variance of Binomial(n, p), Poisson(λ), exponential(λ) and $N(\mu, \sigma^2)$ random variables.

• Variance $B(n, p)$

1)

$$EX = np$$

$$E(X^2) = \sum_{k=0}^{\infty} k^2 P(X=k)$$

$$= \sum_{k=1}^{n} k^2 \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} \frac{k^2 \, n!}{k!\,(n-k)!} p^k (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} \frac{k \quad n!}{(k-1)!\,(n-k)!} p^k (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} \frac{(k-1+1) \; n!}{(k-1)!\,(n-k)!} p^k (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} \frac{(k-1) \; n!}{(k-1)!\,(n-k)!} p^k (1-p)^{n-k} + \underbrace{\sum_{k=1}^{n} \frac{n!\, p^k (1-p)^{n-k}}{(k-1)!\,(n-k)!}}_{= EX = np}$$

$$= \sum_{k=2}^{n} \frac{n!}{(k-2)!\,(n-k)!} p^k (1-p)^{n-k} + np$$

$$= n(n-1)p^2 \sum_{k=2}^{n} \frac{(n-2)!}{(k-2)!\,(n-k)!} p^{k-2} (1-p)^{n-k}$$

$$= n(n-1)p^2 \underbrace{\sum_{k=0}^{n-2} \frac{(n-2)!}{k!\,(n-k)!} p^k (1-p)^{n-k}}_{= 1}$$

$$EX^2 = n^2p^2 - np^2 + np$$

$$Var(X) = E(X^2) - (EX)^2$$

$$= n^2p^2 - np^2 + np - n^2p^2$$

$$= np(1-p)$$

ii) Var Poisson (λ)

$$EX = \lambda$$

$$E(X^2) = \sum_{k=0}^{\infty} k^2 \frac{e^{-\lambda}\lambda^k}{k!}$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{k(\lambda^k)}{(k-1)!}$$

$$= e^{-\lambda} \left(\sum_{k=1}^{\infty} (k-1+1) \frac{\lambda^k}{(k-1)!} \right)$$

$$= e^{-\lambda} \left(\sum_{k=2}^{\infty} \frac{(k-1)\lambda^k}{(k-1)!} + \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!} \right.$$

$$= e^{-\lambda} \left(\sum_{k=2}^{\infty} \frac{\lambda^k}{(k-2)!} + \lambda \underbrace{\sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!}}_{= e^{\lambda}} \right.$$

$$= e^{-\lambda} \left(\lambda^2 \sum_{k=2}^{\infty} \frac{\lambda^{k-2}}{(k-2)!} + \lambda e^{\lambda} \right)$$

$$= e^{-\lambda} \left(\lambda^2 e^{\lambda} + \lambda e^{\lambda} \right)$$

$$E(X^2) = \lambda^2 + \lambda$$

$$Var(X) = E(X^2) - (EX)^2$$

$$= \lambda^2 + \lambda - \lambda^2 = \lambda$$

5. Let X be exponentially distributed with parameter λ. Find the 4th moment.

$$X \sim \exp(\lambda)$$

$$E(X^4) = \int_0^\infty x^4 f_X(x)\, dx$$

$$= \int_0^\infty \underbrace{x^4}_{\mathrm{I}} \underbrace{\lambda e^{-\lambda x}}_{\mathrm{II}}\, dx$$

$$= \frac{4!}{\lambda^4}$$

6. Let X be standard normal random variable. Find $E|X|$.

$$X \sim N(0,1)$$

$$E|X| = \int_{-\infty}^{\infty} |x| \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$$

$$= \lim_{a \to \infty} \int_{-a}^{a} \frac{|x|}{\sqrt{2\pi}} e^{-x^2/2}\, dx$$

$$= 2\int_0^{\infty} \frac{x\, e^{-x^2/2}}{\sqrt{2\pi}}\, dx$$

$$x^2/2 = u$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^{\infty} \left[e^{-u}\right]_0^{\infty}$$

$$= \sqrt{\frac{2}{\pi}}$$

8. Let X be a geometrically distributed random variable with parameter p and M be a positive integer. Find $E[\min\{X, M\}]$.

$$X \sim \text{geometric}\,(p)$$

$$E\,[\min\{X, m\}]$$

$$= \sum_{k=1}^{\infty} \min\{k, m\}\, P(X = k)$$

$$= \sum_{k=1}^{m} \min\{k, m\}\, p(1-p)^{k-1} + \sum_{k=m+1}^{\infty} \min\{k, m\}\, p(1-p)^{k-1}$$

$$= \sum_{k=1}^{m} k p (1-p)^{k-1} + m \sum_{k=m+1}^{\infty} p(1-p)^{k-1}$$

$$= p \sum_{k=1}^{m} k (1-p)^{k-1} + mp \sum_{k=m+1}^{\infty} (1-p)^{k-1}$$

$$= p \sum_{k=1}^{m} \frac{d}{dp} \left((1-p)^k \right) + mp$$

10. Let

$$P_X(x) = \begin{cases} \frac{1}{2^x}, & x = 1, 2, 3, \cdots, \\ 0, & otherwise. \end{cases}$$

be the pmf of the random variable X, then find all the possible values of t such that $E[e^{tX}]$ exist.

$\downarrow$

moment generating function of r.v X.

$$E[e^{tX}] = \sum_{k=1}^{\infty} e^{tk} p(X=k)$$

$$= \sum_{k=1}^{\infty} e^{tk} \frac{1}{2^k}$$

Ratio Test M-I

$$\lim_{k\to\infty} \frac{e^{t(k+1)}}{2^{k+1}} \times \frac{2^k}{2^{tk}} = e^t/2$$

The series converges if

$$e^t/2 < 1$$

diverges if $e^t/2 > 1$

$e^t < 2$

$\boxed{t < \ln 2}$ $\rightarrow$ for these values

expectation of $E[e^{tx}] < \infty$

$\forall\ t > \ln 2 \quad E[e^{tx}] = \infty$