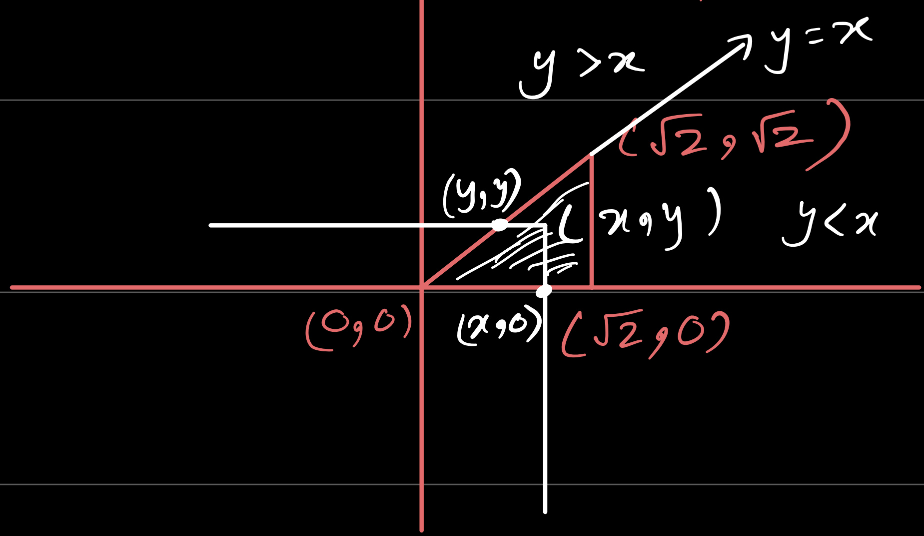
Q1. (Updated problem)



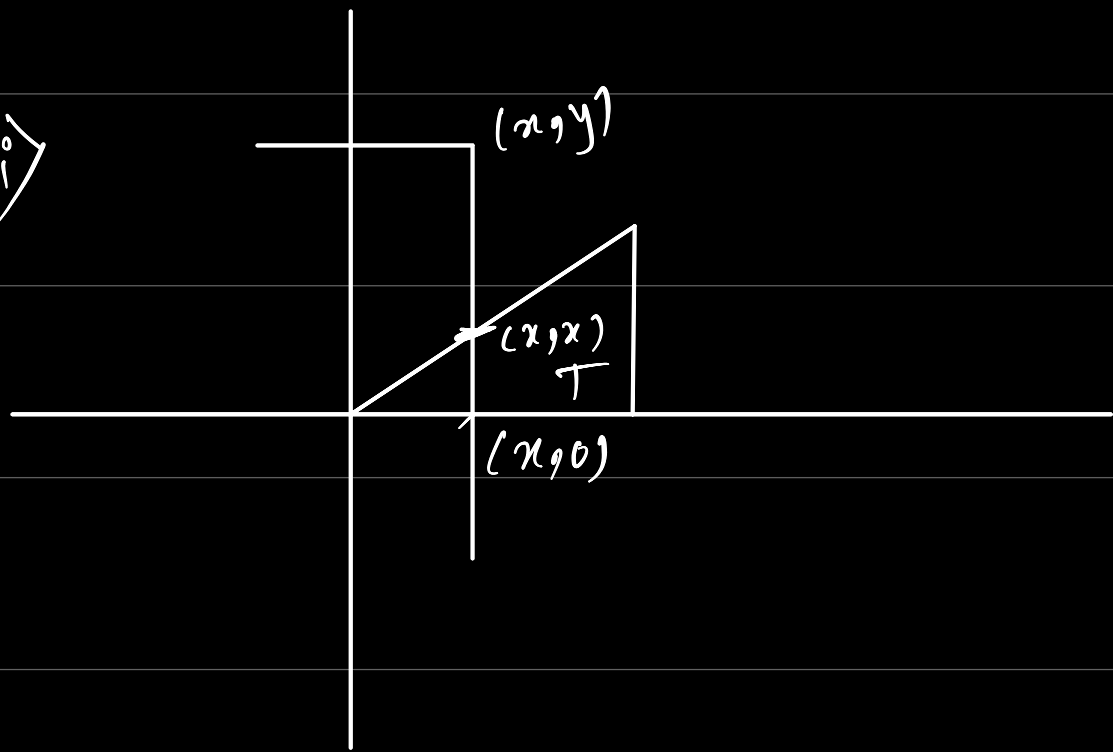
$$F(x,y) = \cancel{(-\infty, x] \times (-\infty, y]}$$
$$= \text{Area of } (A(x,y) \cap T)$$

$$A(x,y) = \{ (x_1, x_2) \quad x_1 \le x ;\ x_2 \le y \}$$
$$= (-\infty, x] \times (-\infty, y]$$

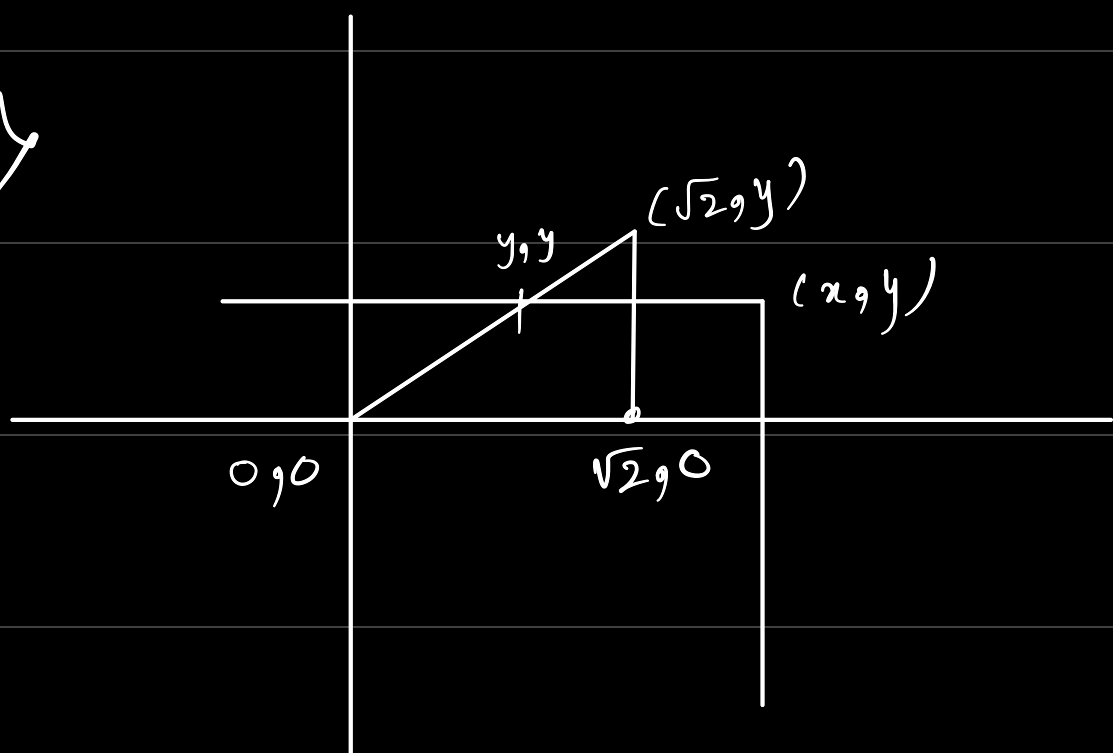
$$F(x,y) = \begin{cases} 0 ; & x \le 0 \text{ or } y \le 0 \\ \dfrac{(2x-y)y}{2} ; & 0 < x < \sqrt{2},\ 0 < y < x \\ x^2/2 ; & 0 < x < \sqrt{2},\ 0 < x \le y \\ \dfrac{(2\sqrt{2}-y)y}{2} ; & x \ge \sqrt{2},\ 0 < y < \sqrt{2} \end{cases}$$

$$\text{Area} = \frac{1}{2} \times \big(\overbrace{x}^{1\ ;\ x \geq \sqrt{2}\ ,\ y \geq \sqrt{2}.} + (x-y)\big) \times y$$

iii)



iv)



Q1. If $c \le 0$ then $F(x,c) \equiv 0 \ \forall x \in \mathbb{R}$

If $0 < c < \sqrt{2}$ then

$$F(x,c) = \begin{cases} 0 , & x \le 0 \\ x^2/2 , & 0 < x < c \\ \dfrac{(2x-c)c}{2} , & c \le x < \sqrt{2} \\ \dfrac{(2\sqrt{2}-c)c}{2} ; & x \geqslant 2 \end{cases}$$

If $c \geqslant \sqrt{2}$ then

$$F(x,c) = \begin{cases} 0 , & x < 0 \\ x^2/2 ; & 0 < x < \sqrt{2} \\ 1 ; & x \geqslant \sqrt{2} \end{cases}$$

If $c \le 0$ then $F(c,y) \equiv 0 \ \forall y \in \mathbb{R}$

If $0 \le c \le \sqrt{2}$ then

$$F(c,y) = \begin{cases} 0 & ; y \le 0 \\ \dfrac{(2c-y)y}{2} & ; 0 < y < c \\ c^2/2 & ; y \geq c \end{cases}$$

If $c \geq \sqrt{2}$ then

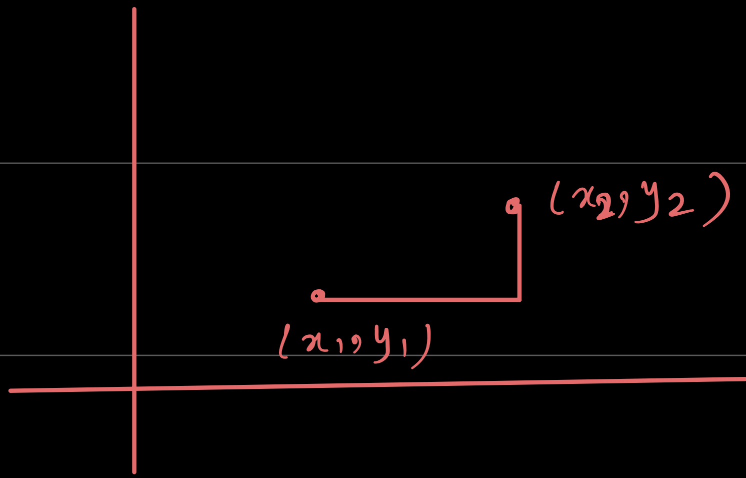
$$F(c,y) = \begin{cases} 0 & , y < 0 \\ \dfrac{(2\sqrt{2} - y)y}{2} & , 0 \le y < \sqrt{2} \\ 1 & ; y \geq \sqrt{2} \end{cases}$$

e) for (x_1, y_1) & (x_2, y_2) with

$x_1 < x_2$, $x_1 < x_2$

$$F(x_2, y_2) + F(x_1, y_1) - F(x_1, x_2) - F(x_2, y_1)$$

$= \text{Area of rectangle } (x_1, x_2) \times (y_1, y_2)$
≥ 0



Theory $\rightarrow$

$$F_X(x) = P(X \le x)$$

$$= P(X \le x, Y < \infty)$$

$$= \lim_{y \to \infty} \underset{\substack{\| \\ P(X \le x,\, Y \le y)}}{F(x, y)}$$

$$F_X(x) = \begin{cases} 0, & x \le 0 \\ x^2/2, & 0 < x < \sqrt{2} \\ 1, & x \geq \sqrt{2} \end{cases}$$

Similarly for $F_Y(y)$

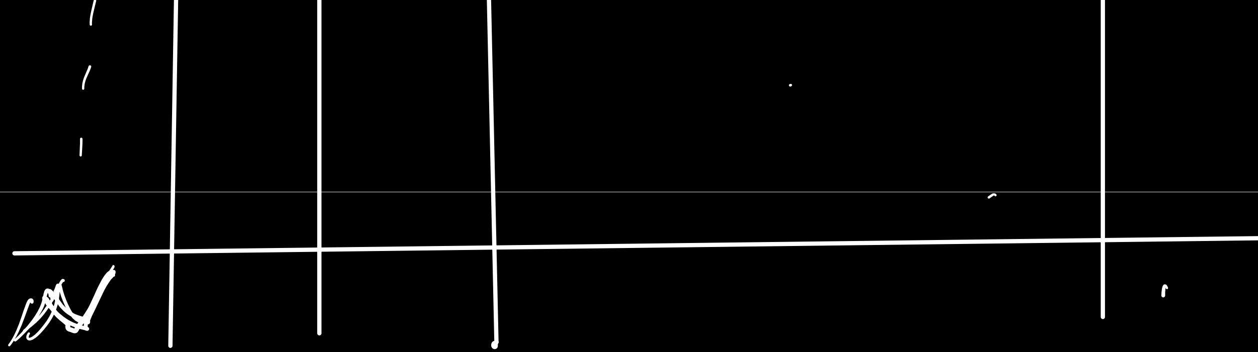
$$F_Y(y) = \begin{cases} 0 & ; y \le 0 \\ \dfrac{(2\sqrt{2} - y)y}{2} & ; 0 < y < \sqrt{2} \\ 1 & ; y \ge \sqrt{2} \end{cases}$$

Q2. $P(X = i, Y = j) = \dfrac{1}{N^2}$ $\quad i, j = 1, 2, \ldots N$

Range of $(X, Y) = \{(i, j),\ i, j = 1, 2, \ldots N\}$

$P(X \ge Y) =$

$X \backslash Y$	1	2	— — — — —	N
1	·			
2		·		
⋮			·	



$$P(X \geq Y) = \sum_{(i,j):\, i \geq j} f(i,j)$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{i} \frac{1}{N^2} = \sum_{j=1}^{N} \sum_{i=j}^{N} \frac{1}{N^2}$$

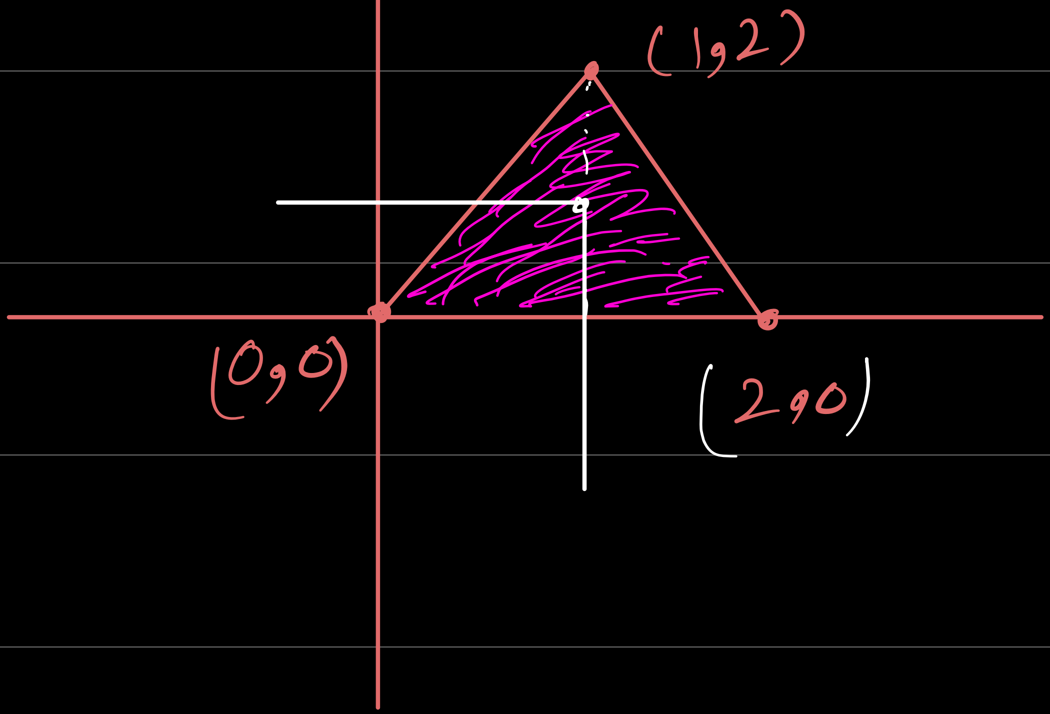
$$= \frac{1}{N^2} \left(\sum_{i=1}^{N} (1+1+1 \cdots i) \right)$$

$$= \frac{1}{N^2} \sum_{i=1}^{N} i$$

$$= \frac{N(N+1)}{2} \frac{1}{N^2}$$

$$= \frac{1}{N} \left(\frac{N+1}{2} \right)$$

Q3.



$$f(x,y) = \begin{cases} \dfrac{1}{\text{Area of Triangle}} & (x,y) \in \Delta \\ 0 & , \text{otherwise} \end{cases}$$

$$P(X \le 1, Y \le 1) = \iint_{\{x \le 1, y \le 1\}} f(x,y)\,dx\,dy$$

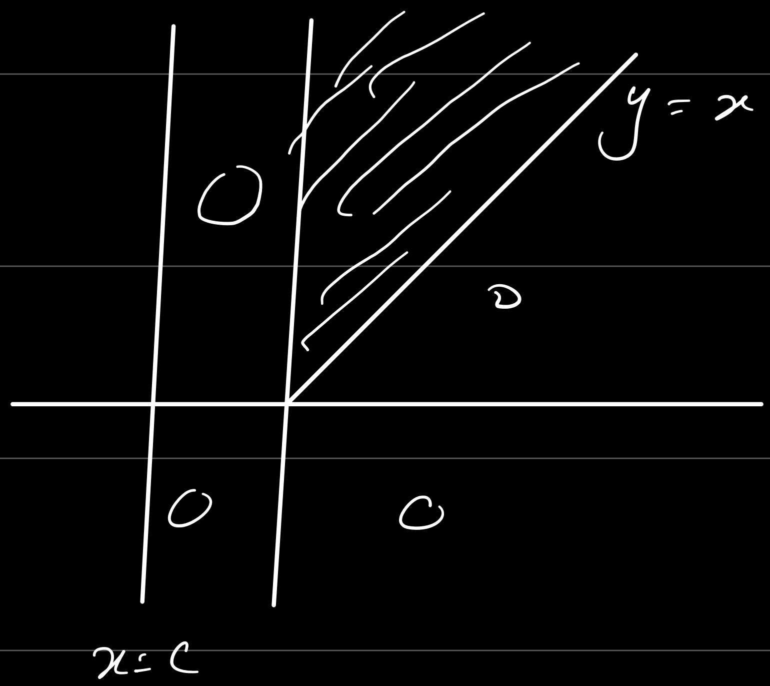
$\frac{1}{2} \times$ Area of $\square$

$$\frac{1}{2} \times \frac{1}{2}\left(\frac{3}{2}\right) = \frac{3}{8}.$$

Q40. $$f(x,y) = \begin{cases} \lambda^2 e^{-\lambda y}, & 0 \le x \le y \\ 0, & \text{otherwise.} \end{cases}$$

To compute marginal pdf of X & Y

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy.$$



if $x < 0$, $f(x,y) = 0 \;\forall\, y \in \mathbb{R}$

So $f_X(x) = 0$.

$$f_X(x) = \int_x^\infty \lambda^2 e^{-\lambda y}\, dy$$

$$= \frac{\lambda^2}{-\lambda}\left[e^{-\lambda y}\right]_x^\infty$$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$X \sim \exp(\lambda)$

$$f_Y(y) = \int_{-\infty}^\infty f(x,y)\, dx$$

$$= \begin{cases} 0 & y < 0 \\ \int_0^y \lambda^2 e^{-\lambda y}\, dx & \text{if } y \geq 0 \end{cases}$$

$$\qquad\qquad = y\lambda^2 e^{-\lambda y}$$

$Y \sim \text{Gamma}(2, \lambda)$

Q 5. $f(x, y) = \begin{cases} e^{-y}(1-e^{-x}), & 0 < x < y < \infty \\ e^{-x}(1-e^{-y}); & 0 < y \leq x < \infty \end{cases}$

Show that X & Y are identically distributed. The CDF of X & Y are same.

$F_X(t) = F_Y(t) \;\; \forall\, t \in \mathbb{R}.$

if $f_X(t) = f_Y(t)$

$\forall\, t \in \mathbb{R}$

$$f_X(x) = \begin{cases} x e^{-x}, & 0 < x < \infty \\ 0 ; & \text{otherwise.} \end{cases}$$

$$f_Y(y) = \begin{cases} y\, e^{-y}, & 0 < y < \infty. \\ 0 ; & \text{otherwise} \end{cases}$$

Q6 $f(x, y, z, u) = \begin{cases} e^{-u}, & 0 < x < y < z < u < \infty \\ 0, & \text{otherwise} \end{cases}$

Is f a joint density function.

① $f \geq 0$

② $\iiiint_{\mathbb{R}^4} f(x, y, z, u)\, dx\, dy\, dz\, du = 1$

Easy ☺

$$= \int_0^\infty \left(\int_x^\infty \left(\int_y^\infty \left(\int_z^\infty e^{-u}\, du \right) dz \right) dy \right) dx.$$

$$= 1.$$

$$P(X \leq 7) = \iiiint_{\{(x,y,z,u)\,:\, x \leq 7} f(x,y,z,u)\,du\,dz\,dy\,dx$$

$$= \int_0^7 e^{-x}\,dx.$$

$$= 1 - e^{-7}$$

Q8. Trick: Drop out.

$$f(x,y,z) = \begin{cases} g(x)g(y)g(z); & x>0, y>0, \\ & z>0 \\ 0, & \text{otherwise.} \end{cases}$$

g is a pdf on $\mathbb{R}$.

if $x \leq 0$ then $f(x,y,z) = 0$

So $f_X(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y,z)\,dy\,dz$

$= 0.$

If $x > 0$ then

$$f_X(x) = \int_0^\infty \int_0^\infty g(x)g(y)g(z)\,dy\,dz$$

$$= g(x)\Big(\int_0^\infty g(y)\,dy\Big)\Big(\int_0^\infty g(z)\,dz\Big)$$

$$\qquad\qquad \underset{1}{\parallel} \qquad\qquad \underset{1.}{\parallel}$$

$$P(X > Y > Z) = \int_0^\infty \Big(\int_0^x \Big(\int_0^y f(x,y,z)\,dz\Big)dy\Big)dx$$

$$= \int_0^\infty g(x)\Big\{\int_0^x g(y)\Big(\underbrace{\int_0^y g(z)\,dz}_{\Downarrow \, F(y)}\Big)dy\Big\}dx$$

$$= \int_0^{\infty} g(x) \left(\int_0^{\infty} g(y) F(y) \, dy \right) dx$$

$$x \quad F'(y) = g(y)$$

$$= \int_0^{x} F(y) \, d(F(y))$$

$$= \left[\frac{(F(y))^2}{2} \right]_0^{x}$$

$$= \frac{1}{2} \int_0^{\infty} [F(x)]^2 g(x) \, dx$$

$$= \frac{1}{2} \int_0^{\infty} [F(x)]^2 \, dF(x)$$

$$= \frac{1}{2} \frac{1}{3} \left\{ [F(x)]^3 \right\}_0^{\infty}$$

$= 1/6 \cdot$ Ans.