Q1. A is finite

B is countably infinite

To show $A \cup B$ is countably infinite

Solⁿ → Let A has 'n' elements

$$A = \{x_1, x_2, x_3 \cdots x_n\}$$

B is countably infinite so it can be written as sequence of distinct terms

say $B = \{b_1, b_2, \cdots\}$

Then $A \cup B = \{x_1, x_2, x_3 \cdots x_n, b_1, b_2 \cdots \cdots\}$

Therefore $A \cup B$ is countably infinite

Q2. Given each A_n, $n \in \mathbb{N}$ is countably infinite. To show $\bigcup_{n=1}^{\infty} A_n$ countably infinite

Solution → Since each A_n is countably infinite we may write its elements as a sequence of distinct term

$$\begin{array}{llll}
A_1 & x_{11} & x_{12} & x_{13} \cdots\cdots \\
A_2 & x_{21} & x_{22} & x_{23} \cdots\cdots \\
A_3 & x_{31} & x_{32} & x_{33}
\end{array}$$

we will move along the following way and we will not take repeating elements.

Q3. 1 2 10

'at random'

$$\Omega = \{ (i,j) : i,j \in \{1,2,3,\dots 10\}, \; i \neq j \}$$

$$\# \Omega = 10 \times 9 = 90$$

$$P(\omega) = \frac{1}{10} \times \frac{1}{9} = 1/90$$

Q4. Given $P(A) = 0.7$, $P(B) = 0.6$, $P(C) = 0.5$

$P(A\cap B) = 0.4$, $P(A\cap B) = 0.3$, $P(B\cap C) = 0.2$

$P(A\cap B\cap C) = 0.1$

To find $P(A\cup B\cup C) =$

Recall

$$P(A\cup B) \leq P(A) + P(B)$$

$$A\cup B = A \cup (B \setminus A)$$

$$A\cup B = A \cup (B \setminus (A\cap B))$$

$$P(A\cup B) = P(A) + P(B \setminus (A\cap B))$$

$$P(A\cup B) = P(A) + P(B) - P(A\cap B)$$

$$= P(A\cup B) + P(C) - P((A\cup B)\cap C)$$

$$= P(A) + P(B) - P(A\cap B) + P(C) - P((A\cap C)\cup(B\cap C))$$

$$P((A\cap C)\cup(B\cap C)) = P(A\cap C) + P(B\cap C) - P((A\cap C)\cap(B\cap C))$$

$$= P(A\cap C) + P(B\cap C) - P(A\cap B\cap C)$$

$$P(A\cup B\cup C) = P(A) + P(B) + P(C) - P(A\cap B) - P(A\cap C) - P(B\cap C) + P(A\cap B\cap C)$$

(ii) To find $P(A^c \cap C)$

$$= P(C \setminus (A\cap C))$$

$$= P(C) - P(A\cap C)$$

Q5. Claim → The statement is false

Let $\Omega = [0,1]$

P = length of subsets of Ω.

$A = [0, 1/2]$ $\quad B = [1/2, 1]$

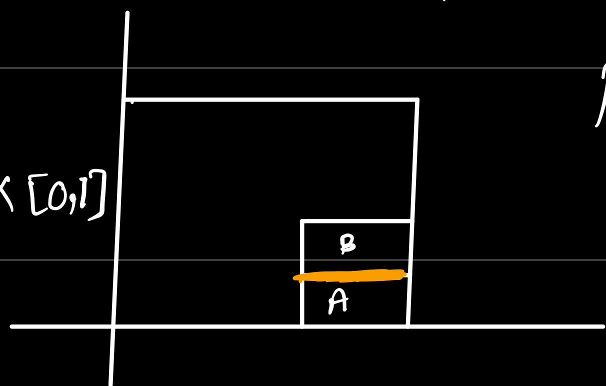
$P(A) = \frac{1}{2} = P(B)$

$P(A \cap B) = P\left(\left\{\frac{1}{2}\right\}\right) = 0$

$A \cap B \neq \emptyset$

Second Example

$\Omega = [0,1] \times [0,1]$



P = Area of subset of Ω

Q6. Does there exist a probability measure P s.t.

$P(A) = 0.6$, $P(B) = 0.8$, $P(C) = 0.7$

$P(A \cap B) = 0.5$, $P(A \cap C) = 0.4$, $P(B \cap C) = 0.5$

$P(A \cap B \cap C) = 0.1$.

$P(A \cup B \cup C) = 1$ it doesn't settle the Question

$$P((A \cap B) \cup C) = P(A \cap B) + P(C) - P(A \cap B \cap C)$$

$$= 0.5 + 0.7 - 0.1$$

$$= 1.1$$

Q7. For events $A_1, A_2, A_3 \ldots\ldots A_n$

$$P(A_1 \cap A_2 \ldots \cap A_n) \geq \sum_{i=1}^{n} P(A_i) - n + 1$$

Solution $\longrightarrow$ claim $\rightarrow$ $P(A_1 \cap A_2) \geq P(A_1) + P(A_2) - 2 + 1$

proof $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$

$$P(A_1 \cap A_2) = P(A_1) + P(A_2) - P(A_1 \cup A_2)$$

$$\geq P(A_1) + P(A_2) - 1$$

$$(\because P(A_1 \cup A_2) \leq 1)$$

Now suppose inequality is true for $n = m$

Now for event $A_1, A_2 \ldots\ldots A_m, A_{m+1}$

$$P(\underbrace{A_1 \cap A_2 \cap \ldots \cap A_m}_{E} \cap A_{m+1}) \geq P(A_1 \cap \ldots \cap A_m)$$

$$+ P(A_{m+1}) - 1$$

$$\geq \sum_{i=1}^{m} P(A_i) - m + 1 + P(A_{m+1}) - 1$$

$$= \sum_{i=1}^{m+1} P(A_i) - (m+1) + 1$$

So by induction principle we have truthness of (1) for all $n \in \mathbb{N}$

$\emptyset$ $\Omega = \mathbb{N}$ $\quad P : \mathcal{P}(\Omega) \to [0,1]$

$$P(A) = \begin{cases} 0 & \text{if } A \text{ is finite} \\ 1 & \text{if } A \text{ is infinite} \end{cases}$$

- P isn't countably additive

Take $A_n = \{n\} \ \forall n = 1, 2, \ldots$

$$P(A_n) = 0 \quad \forall n \in \mathbb{N}.$$

$$A_i \cap A_j = \phi \text{ if } i \neq j$$

$$\bigcup_{n=1}^{\infty} A_n = \Omega$$

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = 1 \neq \sum_{n=1}^{\infty} P(A_n) = 0$$

• Alter → Take $A = \{2, 4, 6, \ldots\}$

$B = \{1, 3, 5, \ldots\}$

$$P(A) = 1 = P(B)$$

$$A \cap B = \phi$$

$$P(A \cup B) = P(\Omega) = 1$$

$$\neq P(A) + P(B)$$

Q9. Let $(A_n)_{n \geq 1}$ be a sequence of events.

Then show that

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} P(A_n)$$

Solution →

If $A_1, A_2, \ldots A_n$ are events

then by finite subadditivity

$$P\left(\bigcup_{n=1}^{\infty} A_k\right) \leq \sum_{K=1}^{n} P(A_k)$$

$$\lim_{n \to \infty} P\left(\bigcup_{K=1}^{n} A_K\right) \leq \lim_{n \to \infty} \sum_{K=1}^{n} P(A_k)$$

Define $B_1 = A_1$

$$B_2 = A_1 \cup A_2$$

$$B_3 = A_1 \cup A_2 \cup A_3$$

$$B_1 \subset B_2 \subset B_3 \ldots\ldots$$

$$B_n = \bigcup_{n=1}^{n} A_k$$

$$\bigcup_{k=1}^{n} B_k = \bigcup_{k=1}^{n} \Rightarrow \bigcup_{k=1}^{\infty} B_k = \bigcup_{k=1}^{\infty} A_k$$

$$P\left(\bigcup_{k=1}^{n} B_k\right) = P\left(\bigcup_{k=1}^{n} A_k\right)$$

$$\leq \sum_{k=1}^{n} P(A_k)$$

$$\operatorname*{lt}_{n\to\infty} P\left(\bigcup_{k=1}^{n} B_k\right) = \operatorname*{lt}_{n\to\infty} P(B_n)$$

$$= P\left(\bigcup_{n=1}^{\infty} B_n\right)$$ "By continuity of probability"

$$= P\left(\bigcup_{n=1}^{\infty} A_n\right)$$