Problem $\rightarrow$ Let $f : \mathbb{R} \to \mathbb{R}$ be a right continuous function s.t.

$$f(x+y) = f(x) + f(y)$$

$$\forall\, x, y \in \mathbb{R}$$

Then show that $f(x) = cx$ where $c = f(1)$

Solution $f(x+0) = f(x) + f(0)$

$$\boxed{f(0) = 0}$$

$$f(x + (-x)) = f(x) + f(-x)$$

$$f(-x) = -f(x)$$

$\hookrightarrow$ odd function.

Now we show that $f(x) = cx$ for some constant c. It is sufficient to prove this for $x > 0$.

If $x < 0$ then $-x > 0$

Then $f(-x) = c(-x) = f(x) = cx.$

Let $m \in \mathbb{N}$. Then $f(mx) = f(\underbrace{x + x + \cdots x}_{m \text{ times}})$

$$f(mx) = \underbrace{f(x) + f(x) + f(x)}_{m \text{ times}}$$

$$f(mx) = m f(x) \text{ for } x > 0, \; m \in \mathbb{N}$$

Take $x = n/m$ where $n \in \mathbb{N}$

$$f(n) = f\left(m \cdot \frac{n}{m}\right) = m f\left(\frac{n}{m}\right)$$

$$f(n) = f(1 + 1 + 1 \cdots)$$

$$= n f(1)$$

$$f\left(\frac{n}{m}\right) = \frac{n}{m} f(1)$$

Set $f(1) = c$. Then

$$f(x) = cx \quad \text{for } x > 0, \; x \in \mathbb{Q}$$

Suppose $x > 0$ & $x \in \mathbb{Q}^c$. Then

Choose a seq. of rational no.s $x_n \downarrow x$.

Then $f(x) = \underset{x_n \downarrow x}{lt} f(x_n)$

$= \underset{x_n \downarrow x}{lt} c x_n = cx.$

Q7) tut #4

Let x be a r.v taking values in $[0,1]$

$P(x < X \leq y) = f(y-x)$

$\forall \, 0 \leq x \leq y \leq 1$

where f is some function.

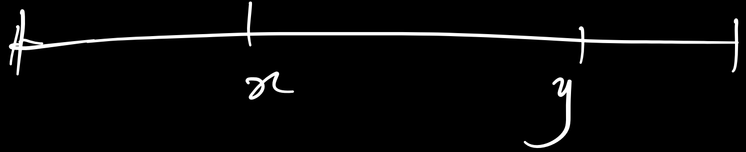
To show $\rightarrow$ $X \sim U[0,1]$

We will show that the cdf of X is equal to is

$$F(x) = \begin{cases} 0 & ; \; x < 0 \\ x & ; \; 0 \leq x \leq 1 \\ 1 & ; \; x > 1 \end{cases}$$

$$0 \le x \le y \le 1$$

$$0 \le y - x \le 1$$



$$f : \mathbb{R} - \mathbb{R}$$

$$f(x+y) = P(0 < X \le x+y)$$

$$\downarrow ?$$

$$= P(0 < X \le x) + P(x < X \le x+y)$$

$$= f(x) + f((x+y) - x)$$

$$f(x+y) = f(x) + f(y)$$

Claim f is right continuous.

We need to show the right-continuity of f $\forall\ x \in [0,1]$. Fix $c \in [0,1]$

Let $(x_n)_{n \ge 1}$ be a seq. s.t. $x_n \downarrow c$

$$\{0 \le X \le x_n\} \downarrow \{0 < X \le c\}$$

$$\lim_{n\to\infty} P(0 < X \le x_n) = P(0 < X \le c) = f(c)$$

$$\underset{f(x_n)}{\parallel}$$

So by previous then only right continuous soln to

$$f(x) = c\,x \text{ for } c = f(1)$$

For $0 \le x \le 1$,

$$F_X(x) = P(X \le 0) + P(0 < X \le x)$$

$$= P(X = 0) + f(x)$$

$$F_X(x) = cx, \quad 0 \le x \le 1$$

$$F_X(1) = P(X \le 1) = 1$$

$$\boxed{c = 1}$$

Q8. X taking values in $[0, \infty)$

$$P(X > r+s \mid X > s) = P(X > 0) \quad \text{— ①} \quad \forall\, r, s \geq 0$$

Let F be CDF of X. Then

$$P(X > s) = 1 - P(X \leq s)$$

$$= 1 - F(s)$$

① is equivalent to

$$= \frac{P(X > r+s,\ X > r)}{P(X > s)}$$

$$= \frac{P(X > r+s)}{P(X > s)} = \frac{1 - F(r+s)}{1 - F(s)}$$

$$= 1 - F(r)$$

$$1 - F(r+s) = (1 - F(r))(1 - F(s)) \quad \text{— ②} \quad \forall\, r, s \geq 0$$

Define $g(x) = \ln(1 - F(x))$

$$0 \leq F(x) \leq 1$$

$$g(x) \in (-\infty, 0]$$

Take log both sides on ②.

$$\ln(1 - F(r+s)) = \ln(1 - F(r)) + \ln(1 - F(s))$$

$$g(r+s) = g(r) + g(s)$$

$$r, s \geq 0$$

$$F(x) = 0 \quad \text{if } x < 0$$

$$g(x) = 0 \quad \text{if } x < 0$$

$$g : [0, \infty) \longrightarrow \mathbb{R}.$$

$$g(x+y) = g(x) + g(y)$$

g is right continuous

$\ln(1-F(x)) = g(x) = cx$

$F(x) = 1 - e^{cx}$ if $c < 0$

$\because \lim_{x \to} F(x) = 1$

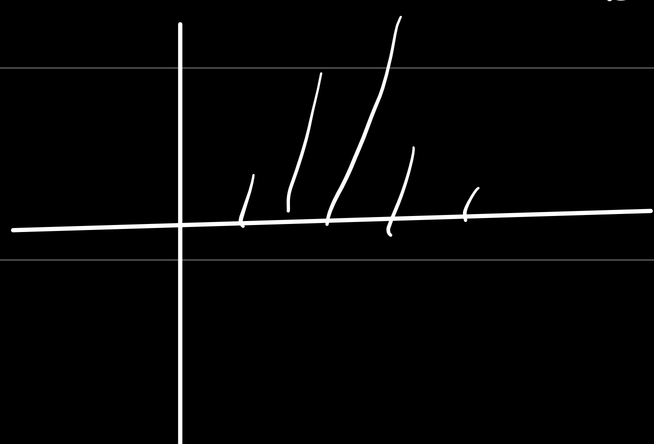
$\lambda = -c > 0$

Q1° $X \sim$ binomial (n, p)

$k^* = [(n+1)p]$

$f_X(k) \uparrow [0, k^*]$ $\quad k = 0, 1, 2, \ldots$

$\downarrow [k^*, n]$



$$\frac{f_X(k)}{f_X(k-1)} = \frac{\binom{n}{k} p^k (1-p)^{n-k}}{\binom{n}{k-1} p^{k-1} (1-p)^{n-(k-1)}}$$

$$= \frac{(n-k+1)p}{k(1-p)} = \frac{(n+1)p-kp}{k-kp} \geq 1$$

If $k \leq k^* = [(n+1)p] \Rightarrow k \leq (n+1)p - kp$

$f_X(k) \geq f_X(k-1)$ if $k \leq k^*$

$f_X(k^*) \dots \geq f_X(2) \geq f_X(1) \geq f_X(0)$

If $k > k^* \Rightarrow k > (n+1)p$

$[x] \leq x < [x]+1$

Q2.

$$P(n \leq X \leq N) = \sum_{k=n}^{N} P(X=k)$$

Q3.

$$f_\theta(x) = \begin{cases} \theta^2 x e^{-\theta x} & \text{if } x > 0 \\ 0 & ; x \leq 0 \end{cases}$$

where $\theta > 0$

$f(x) \geq 0 \quad \forall \; x \in \mathbb{R}$

$$\int_{-\infty}^{\infty} \theta^2 x e^{-\theta x} dx = 1$$

ii) $f_\theta(x)$ is pdf of X

$$P(X \geq 1) = \int_1^{\infty} f_\theta(x) dx$$

Q40 X is continuous r.v

s.t $P(X \geq x) = \begin{cases} 1 & \text{if } x < 0 \\ \left(1 + \frac{x}{\lambda}\right)^{-\lambda} & \text{if } x \geq 0 \end{cases}$

$\lambda > 0$ const.

Suppose $f_X(t)$ is density of X

$$P(X \geq x) = \int_x^{\infty} f_X(t) dt$$

$$\left(1 + \frac{x}{\lambda}\right)^{-\lambda} = \int_x^{\infty} f_X(t) dt$$

$$+\cancel{\lambda}\left(1+\frac{x}{\lambda}\right)^{-\lambda-1}\cdot\frac{1}{\cancel{\lambda}} = -f_X(\cancel{x})$$

$$f_X(x) = \begin{cases} \left(1+\frac{x}{\lambda}\right)^{-\lambda-1} & ; x \geq 0 \\ 0 & ; x < 0 \end{cases}$$

Q6. X is taking values in $\mathbb{N}$

$$P(X > n+m \mid X > m) = P(X > n) \quad \text{— ①}$$

$m, n \geq 0$

m, n integers.

Let $P(X = k) =: p_k \quad k = 1, 2, \ldots$

Then $$P(X > n) = \sum_{k=n+1}^{\infty} p_k =: q_n$$

Take $n=1$ in ①

$$P(X > m+1 \mid X > m) = P(X > 1)$$
$$= q_1$$

$$\frac{P(X > m+1, X > m)}{P(X > m)}$$

$$= \frac{P(X > m+1)}{P(X > m)} = \frac{q_{m+1}}{q_m}$$

$$\Rightarrow q_{m+1} = q_1 q_m \quad \text{where } m \geq 0$$

$$q_0 = 1$$

$$q_1 = P(X > 1) = 1 - P(X = 1)$$
$$= 1 - p_1$$

$$q_2 = q_1 q_1 = (1 - p_1)^2$$

$$q_3 = q_1 q_2 = (1 - p_1)^3$$

$$\boxed{q_n = (1-p_1)^n}$$

$$p_k = q_{k-1} - q_{k-2} = P(X > k-1) - P(X > k-2)$$

$$P(X \geq k) - P(X \geq k-1)$$

$$= (1-p_1)^{k-1} - (1-p_1)^{k-2}$$

$$= (1-p_1)^{k-1}$$