MTH-222, MTH-6031: Probability and Statistics

Tutorial # 8 (Conditional distributions, Conditional Expectation)

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1. The speed of a typical vehicle that drives past a police radar is modeled as an exponentially distributed random variable X with mean 50 miles per hour. The police radar's measurement Y of the vehicle's speed has an error which is modeled as a normal random variable with zero mean and standard deviation equal to one tenth of the vehicle's speed. What is the joint PDF of X and Y?

2. Let X be a uniformly distributed random variable over $(0, 1)$, and let Y be a uniformly distributed random variable over $(0, X)$. Find the joint density of X and Y and the marginal density of Y.

3. Let X and Y have the joint density f given by $f(x, y) = \frac{\sqrt{3}}{4\pi} e^{-\frac{x^2 - xy + y^2}{2}}$, $x, y \in \mathbb{R}$. Find the conditional density of Y given X.

4. Let Y be a discrete random variable having a binomial distribution with parameters n and p. Suppose now that p varies as random variable X having a uniform density over $(0, 1)$. Find the conditional density of X given $Y = y$.

5. Let X and Y be continuous random variables having a joint density f. Suppose that Y and $\phi(X)Y$ have finite expectation. Show that
$$E[\phi(X)Y] = \int_{-\infty}^{\infty} \phi(x)E[Y|X = x]f_X(x)dx$$

6. Let X and Y be continuous random variables having a joint density, and let $\mathrm{Var}[Y|X = x]$ denote the variance of the conditional density of Y given $X = x$. Show that if $E[Y|X = x] = \mu$ independently of X, then $EY = \mu$ and
$$\mathrm{Var}(Y) = \int_{-\infty}^{\infty} \mathrm{Var}[Y|X = x]f_X(x)dx$$

7. Let X and Y be two independent Poisson distributed random variables having parameters λ_1 and λ_2 respectively. Compute $E[Y|X + Y = z]$ where z is a nonnegative integer.

8. Let X and Y have a joint density f that is uniform over the interior of the triangle with vertices at $(0, 0)$, $(2, 0)$, and $(1, 2)$. Find the conditional expectation of Y given X.

9. Let X and Y be iid exponential random variables with parameter λ, and set $Z = X + Y$. Find the conditional expectation of X given Z.

1. The speed of a typical vehicle that drives past a police radar is modeled as an exponentially distributed random variable X with mean 50 miles per hour. The police radar's measurement Y of the vehicle's speed has an error which is modeled as a normal random variable with zero mean and standard deviation equal to one tenth of the vehicle's speed. What is the joint PDF of X and Y?

When $X \sim exp(\lambda)$ with pdf $f_X(x) = \begin{cases} \lambda e^{-\lambda x} & ; x \geq 0 \\ 0 & ; x < 0 \end{cases}$

then $E[X] = 1/\lambda \Rightarrow 1/\lambda = 50$

$$\Rightarrow \lambda = 1/50$$

$$f_X(x) = \begin{cases} \frac{1}{50} e^{-x/50} & ; x \geq 0 \\ 0 & ; x < 0 \end{cases}$$

$Y \sim N(0,$

$X \rightarrow ve$

given $X = x$

$Y \sim (0, (x^2/10))$

$$f_{\cancel{Y}|X}(y|x) = \frac{1}{\left(\frac{x}{10}\right)\sqrt{2\pi}} e^{-\frac{(y-0)^2}{2(x/10)^2}}$$

$$= \frac{10}{x\sqrt{2\pi}} e^{-50y^2/x^2} \; ; \; y \in \mathbb{R}$$

$$x > 0$$

$$f(x,y) = f_{Y|X}(y|x)\, f_X(x)$$

$$= \begin{cases} \frac{10}{x\sqrt{2\pi}} \, \frac{1}{50} e^{-x/50} e^{-50y^2/x^2} & ; x > 0 \\ 0 & ; x \leq 0 \end{cases}$$

2. Let X be a uniformly distributed random variable over $(0, 1)$, and let Y be a uniformly distributed random variable over $(0, X)$. Find the joint density of X and Y and the marginal density of Y.

$X \sim U(0,1)$

$Y \sim U(0,X)$

Given $X = x$ then $Y \sim \mu(0,x)$

$$f_{Y|X}(y|x) = \begin{cases} 1/x & ; 0<y<x, 0<x<1 \\ 0 & ; \text{otherwise} \end{cases}$$

$$f_X(x) = \begin{cases} 1 & ; 0<x<1 \\ 0 & ; \text{otherwise} \end{cases}$$

$$f(x,y) = \begin{cases} 1/x & , 0<y<x<1 \\ 0 & , \text{otherwise.} \end{cases}$$

$$f_Y(y) = \begin{cases} -\ln y & \text{if } 0<y<1 \\ 0 & ; \text{otherwise.} \end{cases}$$

3. Let X and Y have the joint density f given by $f(x,y) = \frac{\sqrt{3}}{4\pi} e^{-\frac{x^2-xy+y^2}{2}}$, $x, y \in \mathbb{R}$. Find the conditional density of Y given X.

$$f(x,y) = \frac{\sqrt{3}}{4\pi} e^{-\frac{x^2-xy+y^2}{2}} \quad , (x,y) \in \mathbb{R}^2$$

$X, Y \sim N(0, 4/3)$ [From Last tut]

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(y-\frac{x}{2}\right)^2}$$

$\downarrow$

Normal distribution

with mean $= x/2$

and variance 1.

4. Let Y be a discrete random variable having a binomial distribution with parameters n and p. Suppose now that p varies as random variable X having a uniform density over $(0,1)$. Find the conditional density of X given $Y=y$.

$Y \sim B(n, p)$

$p \sim U(0,1)$

$$f_{P|Y}(p|y) = \begin{cases} \frac{f(p,y)}{f_Y(y)} & ; \ f_Y(y) > 0 \\ 0 & ; \ f_Y(y) = 0 \end{cases}$$

Given

$$f_{Y|P}(y|p) = \begin{cases} \binom{n}{y} y^p (1-y)^{n-p} & ; \quad y = 0, 1, \dots n \\ \end{cases}$$

$$f_P(p) = \begin{cases} 1 & ; \quad 0 < p < 1 \\ 0 & ; \quad \text{otherwise.} \end{cases}$$

$$f(p, y) = f_{Y|P}(y|p) f_P(p)$$

$$= \begin{cases} \binom{n}{y} y^p (1-y)^{n-p} & ; \quad 0 < p < 1, \; y = 0, 1, 2 \dots \\ 0 & ; \quad \text{otherwise.} \end{cases}$$

$$f_Y(y) = \int_0^1 f(p, y)\, dp$$

$$= \binom{n}{y} \int_0^1 y^p (1-y)^{n-p} \, dp$$

exam me value dege

$$= \binom{n}{y} \frac{y! \, (n-y)!}{(n+1)!}$$

$$= 1/_{n+1}$$

5. Let X and Y be continuous random variables having a joint density f. Suppose that Y and $\phi(X)Y$ have finite expectation. Show that

$$E[\phi(X)Y] = \int_{-\infty}^{\infty} \phi(x) E[Y|X = x] f_X(x) dx$$

$$E[\phi(X)Y] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(x) y f(x, y) \, dx \, dy$$

$$= \int_{-\infty}^{\infty} \phi(x) \left[\int_{-\infty}^{\infty} y f(x, y) \, dy \right] dx$$

$$=$$

$$\therefore \quad E[Y|X=x] = \int_{-\infty}^{\infty} y \underbrace{f_{Y|X}(y|x)}_{\parallel \; \frac{f(x,y)}{f_X(x)}} dy$$

$$\int_{-\infty}^{\infty} \phi(x) \left(\int_{-\infty}^{\infty} y \, f_{Y|X}(y|x) f_X(x) \, dy \right) dx$$

$$= \int_{-\infty}^{\infty} \phi(x) \left(\int_{-\infty}^{\infty} y \, f_{Y|X}(y|x) \, dy \right) f_X(x) dx.$$

6. Let X and Y be continuous random variables having a joint density, and let $\text{Var}[Y|X = x]$ denote the variance of the conditional density of Y given $X = x$. Show that if $E[Y|X = x] = \mu$ independently of X, then $EY = \mu$ and

$$\text{Var}(Y) = \int_{-\infty}^{\infty} \text{Var}[Y|X = x] f_X(x) dx$$

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$$EY = \int_{-\infty}^{\infty} E[Y|X=x] \, f_X(x) dx$$

$$= \mu \int_{-\infty}^{\infty} f_X(x) \, dx = \mu.$$

② To show

$$Var(Y) = \int_{-\infty}^{\infty} Var[Y \mid X = x] \, f_X(x) \, dx$$

$$Var[Y \mid X = x] = \int_{-\infty}^{\infty} (y - E[Y \mid X = x])^2 f_{Y|X}(y \mid x) \, dy$$

$$= \int_{-\infty}^{\infty} (y - \mu)^2 f_{Y|\cancel{X}}(y \mid x) \, dy$$

$$Var(Y) = \int_{-\infty}^{\infty} y^2 f_Y(y) \, dy - \left(\int_{-\infty}^{\infty} y f_Y(y) \, dy \right)^2$$

$$= \int_{-\infty}^{\infty} (y - E_Y)^2 f_Y(y) \, dy$$

Substituting in above eq^n

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (y - \mu)^2 f_{Y|X}(y \mid x) f_X(x) \, dx \, dy$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(y-\mu)^2 f(x,y)\,dx\,dy$$

$$\int_{-\infty}^{\infty}(y-\mu)^2\left[\int_{-\infty}^{\infty} f(x,y)\,dx\right]dy$$

$$= \int_{-\infty}^{\infty}(y-EY)^2 f_Y(y)\,dy$$

$$= Var(Y)$$

7. Let X and Y be two independent Poisson distributed random variables having parameters λ_1 and λ_2 respectively. Compute $E[Y|X+Y=z]$ where z is a nonnegative integer.

$$E[Y|X+Y=z] = \sum_y y\, f_{Y|X+Y}(y|z)$$

$$z = X+Y \sim \text{Poisson}(\lambda_1+\lambda_2)$$

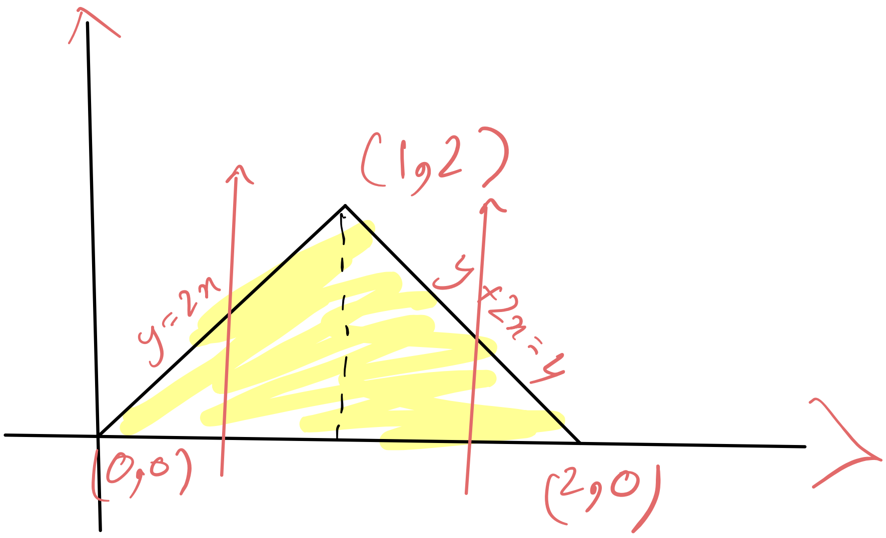
$f_{Y|Z} \rightarrow$ Already in ~~tu~~t I.

$$f_{Y|Z}(m|z) = \binom{z}{m}\lambda_1^{z-m}\lambda_2^{m} \quad , m=0,1,2..$$

DIY!

$\overline{\quad\quad} \quad (\lambda_1+\lambda_2)^2$

8. Let X and Y have a joint density f that is uniform over the interior of the triangle with vertices at $(0,0)$, $(2,0)$, and $(1,2)$. Find the conditional expectation of Y given X.



$$f(x,y) = \begin{cases} \frac{1}{2}, & \triangle \\ 0 & ; \text{ otherwise} \end{cases}$$

$$E[Y \mid X=x] = f_{Y|X}(y|x)$$

$$f_X(x) = \begin{cases} x \; ; & 0 < x \le 1 \\ 2 - x \; ; & 1 < x \le 2 \end{cases}$$

Ans

$$= \begin{cases} x \; ; & 0 < x \le 1 \\ 2 - x \; ; & 1 < x \le 2 \end{cases}$$

9. Let X and Y be iid exponential random variables with parameter λ, and set $Z = X + Y$. Find the conditional expectation of X given Z.

$X, Y \sim \text{iid } \exp(\lambda)$

$Z = X + Y$

$E[X \mid Z = z] =$