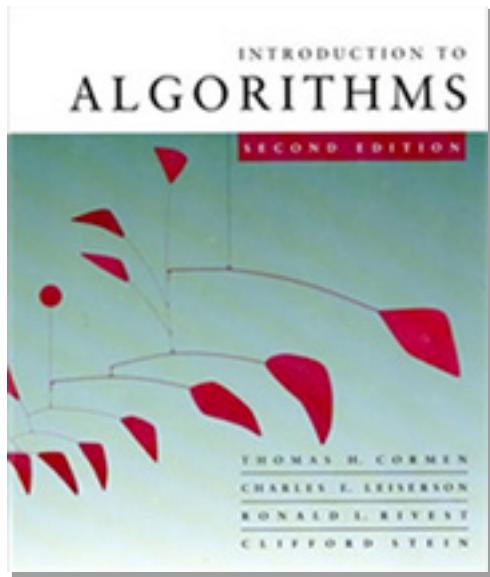


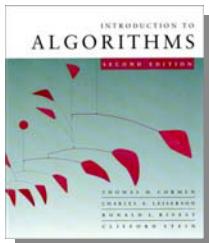
# *Introduction to Algorithms*



## **Divide and Conquer**

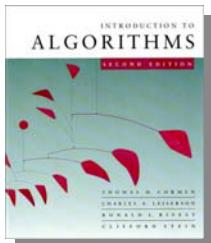
- Binary search
- Powering a number
- Fibonacci numbers
- Matrix multiplication
- Strassen's algorithm
- VLSI tree layout

**Prof. Erik D. Demaine**



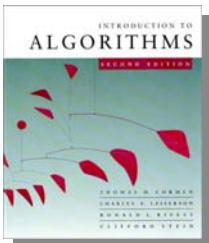
# The divide-and-conquer design paradigm

1. *Divide* the problem (instance) into subproblems.
2. *Conquer* the subproblems by solving them recursively.
3. *Combine* subproblem solutions.



# Merge sort

1. ***Divide:*** Trivial.
2. ***Conquer:*** Recursively sort 2 subarrays.
3. ***Combine:*** Linear-time merge.



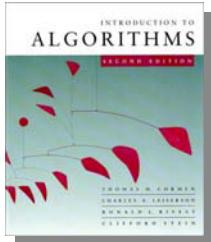
# Merge sort

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$$T(n) = 2 T(n/2) + \Theta(n)$$

# subproblems      ↗  
subproblem size      ↗  
work dividing  
and combining

The diagram shows the recurrence relation  $T(n) = 2 T(n/2) + \Theta(n)$  with three yellow circles containing the terms 2,  $T(n/2)$ , and  $\Theta(n)$ . Three arrows point from the text labels '# subproblems', 'subproblem size', and 'work dividing and combining' to these respective terms in the equation.



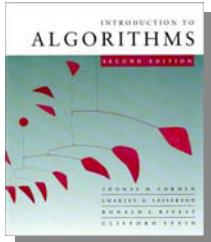
# Master theorem (reprise)

$$T(n) = a T(n/b) + f(n)$$

**CASE 1:**  $f(n) = O(n^{\log_b a - \varepsilon})$ , constant  $\varepsilon > 0$   
 $\Rightarrow T(n) = \Theta(n^{\log_b a})$ .

**CASE 2:**  $f(n) = \Theta(n^{\log_b a} \lg^k n)$ , constant  $k \geq 0$   
 $\Rightarrow T(n) = \Theta(n^{\log_b a} \lg^{k+1} n)$ .

**CASE 3:**  $f(n) = \Omega(n^{\log_b a + \varepsilon})$ , constant  $\varepsilon > 0$ ,  
and regularity condition  
 $\Rightarrow T(n) = \Theta(f(n))$ .



# Master theorem (reprise)

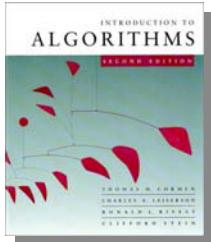
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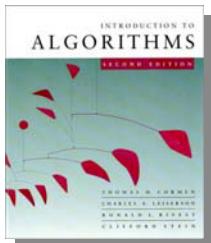
**Merge sort:**  $a = 2$ ,  $b = 2 \Rightarrow n^{\log_b a} = n^{\log_2 2} = n$   
 $\Rightarrow$  CASE 2 ( $k = 0$ )  $\Rightarrow T(n) = \Theta(n \lg n)$ .



# Binary search

Find an element in a sorted array:

- 1. *Divide*:** Check middle element.
- 2. *Conquer*:** Recursively search **1** subarray.
- 3. *Combine*:** Trivial.



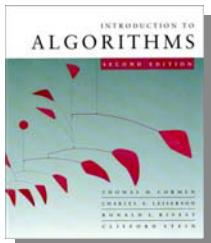
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*Example:* Find 9

3    5    7    8    9    12    15



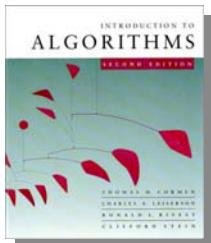
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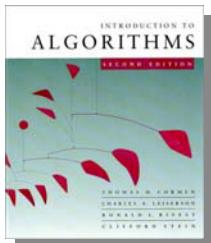
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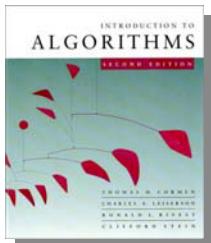
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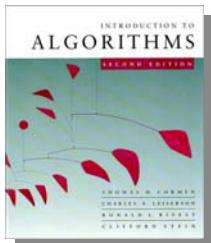
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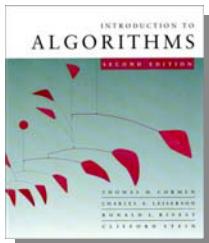
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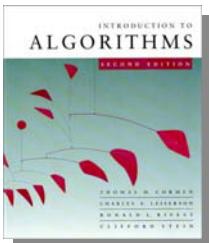


# Recurrence for binary search

$$T(n) = 1 T(n/2) + \Theta(1)$$

# subproblems      ↗  
subproblem size      ↙  
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and combining

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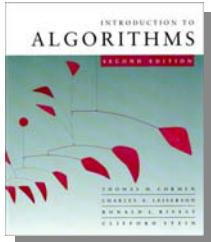
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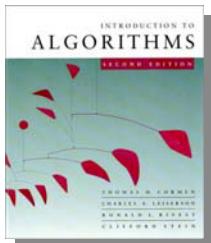
$$\begin{aligned} n^{\log_b a} &= n^{\log_2 1} = n^0 = 1 \Rightarrow \text{CASE 2 } (k=0) \\ \Rightarrow T(n) &= \Theta(\lg n). \end{aligned}$$



# Powering a number

**Problem:** Compute  $a^n$ , where  $n \in \mathbb{N}$ .

**Naive algorithm:**  $\Theta(n)$ .



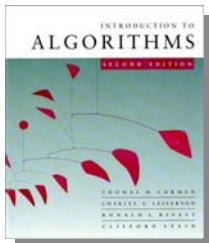
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**Divide-and-conquer algorithm:**

$$a^n = \begin{cases} a^{n/2} \cdot a^{n/2} & \text{if } n \text{ is even;} \\ a^{(n-1)/2} \cdot a^{(n-1)/2} \cdot a & \text{if } n \text{ is odd.} \end{cases}$$



# Powering a number

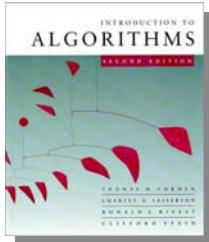
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$$T(n) = T(n/2) + \Theta(1) \Rightarrow T(n) = \Theta(\lg n) .$$

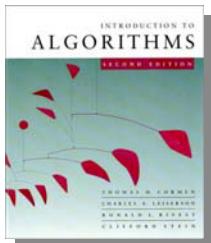


# Fibonacci numbers

Recursive definition:

$$F_n = \begin{cases} 0 & \text{if } n = 0; \\ 1 & \text{if } n = 1; \\ F_{n-1} + F_{n-2} & \text{if } n \geq 2. \end{cases}$$

0    1    1    2    3    5    8    13    21    34    ...



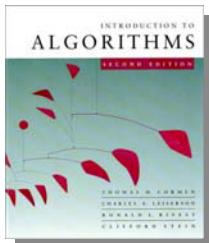
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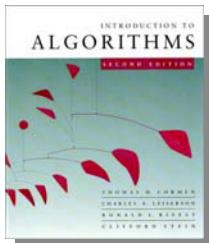
Naive recursive algorithm:  $\Omega(\phi^n)$   
(exponential time), where  $\phi=(1+\sqrt{5})/2$   
is the *golden ratio*.



# Computing Fibonacci numbers

## Bottom-up:

- Compute  $F_0, F_1, F_2, \dots, F_n$  in order, forming each number by summing the two previous.
- Running time:  $\Theta(n)$ .



# Computing Fibonacci numbers

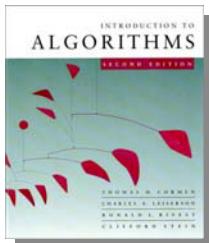
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## Naive recursive squaring:

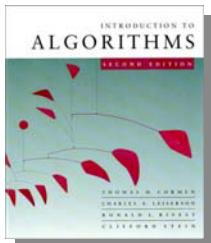
$$F_n = \phi^n / \sqrt{5} \text{ rounded to the nearest integer.}$$

- Recursive squaring:  $\Theta(\lg n)$  time.
- This method is unreliable, since floating-point arithmetic is prone to round-off errors.



# Recursive squaring

**Theorem:**  $\begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n.$

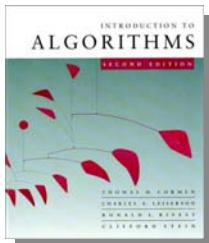


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**Algorithm:** Recursive squaring.

Time =  $\Theta(\lg n)$ .



# Recursive squaring

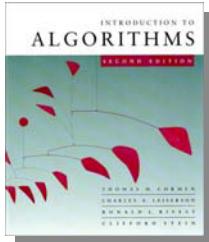
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**Algorithm:** Recursive squaring.

Time =  $\Theta(\lg n)$ .

*Proof of theorem.* (Induction on  $n$ .)

Base ( $n = 1$ ): 
$$\begin{bmatrix} F_2 & F_1 \\ F_1 & F_0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^1.$$

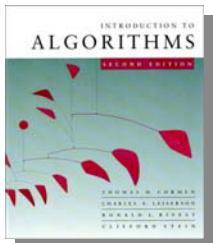


# Recursive squaring

Inductive step ( $n \geq 2$ ):

$$\begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} = \begin{bmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{n-1} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n$$

■

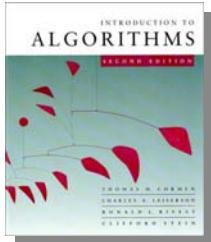


# Matrix multiplication

**Input:**  $A = [a_{ij}], B = [b_{ij}] \cdot \left. \begin{array}{l} \\ \end{array} \right\} i, j = 1, 2, \dots, n.$   
**Output:**  $C = [c_{ij}] = A \cdot B.$

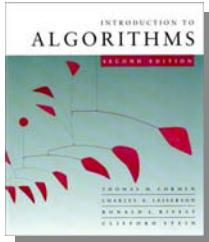
$$\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

$$c_{ij} = \sum_{k=1}^n a_{ik} \cdot b_{kj}$$



# Standard algorithm

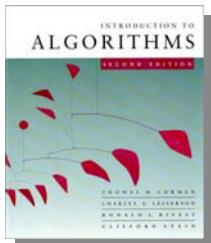
```
for  $i \leftarrow 1$  to  $n$ 
    do for  $j \leftarrow 1$  to  $n$ 
        do  $c_{ij} \leftarrow 0$ 
        for  $k \leftarrow 1$  to  $n$ 
            do  $c_{ij} \leftarrow c_{ij} + a_{ik} \cdot b_{kj}$ 
```



# Standard algorithm

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for  $i \leftarrow 1$  to  $n$ 
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```

Running time =  $\Theta(n^3)$



# Divide-and-conquer algorithm

**IDEA:**

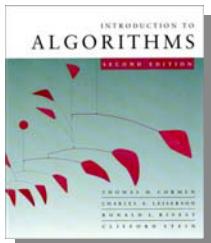
$n \times n$  matrix =  $2 \times 2$  matrix of  $(n/2) \times (n/2)$  submatrices:

$$\begin{bmatrix} r & s \\ t & u \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

$$C = A \cdot B$$

$$\left. \begin{array}{l} r = ae + bg \\ s = af + bh \\ t = ce + dg \\ u = cf + dh \end{array} \right\}$$

8 mults of  $(n/2) \times (n/2)$  submatrices  
4 adds of  $(n/2) \times (n/2)$  submatrices



# Divide-and-conquer algorithm

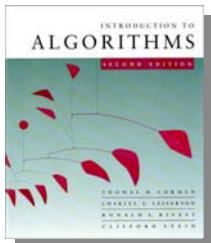
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$$C = A \cdot B$$

$$\left. \begin{array}{l} r = ae + bg \\ s = af + bh \\ t = ce + dh \\ u = cf + dg \end{array} \right\} \begin{array}{l} \text{recursive} \\ \hline 8 \text{ mults of } (n/2) \times (n/2) \text{ submatrices} \\ 4 \text{ adds of } (n/2) \times (n/2) \text{ submatrices} \end{array}$$



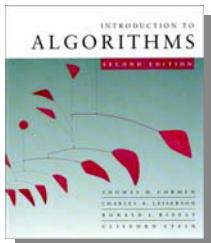
# Analysis of D&C algorithm

$$T(n) = 8T(n/2) + \Theta(n^2)$$

# submatrices      ↗  
submatrix size

work adding  
submatrices

The diagram illustrates the recurrence relation for the D&C algorithm. The equation  $T(n) = 8T(n/2) + \Theta(n^2)$  is shown in black text. Three yellow circles highlight the term  $8T(n/2)$ , the term  $\Theta(n^2)$ , and the entire equation. Arrows point from the text labels to these highlighted terms: one arrow points from "# submatrices" to the  $8T(n/2)$  term, another from "submatrix size" to the  $\Theta(n^2)$  term, and a third from "work adding submatrices" to the entire equation.



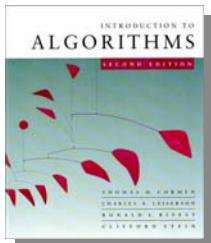
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$$n^{\log_b a} = n^{\log_2 8} = n^3 \Rightarrow \text{CASE 1} \Rightarrow T(n) = \Theta(n^3).$$



# Analysis of D&C algorithm

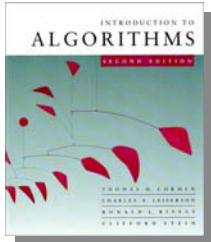
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# submatrices      ↗  
submatrix size      ↙  
work adding  
submatrices

The equation  $T(n) = 8T(n/2) + \Theta(n^2)$  is displayed in the center. Three arrows point from the text below to specific parts of the equation: one arrow from "# submatrices" points to the coefficient 8, another from "submatrix size" points to the term  $n^2$ , and a third from "work adding submatrices" points to the term  $\Theta(n^2)$ .

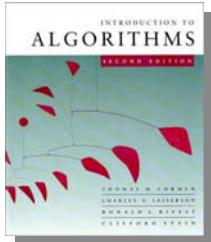
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**No better than the ordinary algorithm.**



# Strassen's idea

- Multiply  $2 \times 2$  matrices with only 7 recursive mults.



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$$P_1 = a \cdot (f - h)$$

$$P_2 = (a + b) \cdot h$$

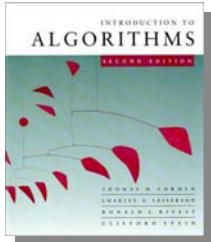
$$P_3 = (c + d) \cdot e$$

$$P_4 = d \cdot (g - e)$$

$$P_5 = (a + d) \cdot (e + h)$$

$$P_6 = (b - d) \cdot (g + h)$$

$$P_7 = (a - c) \cdot (e + f)$$



# Strassen's idea

- Multiply  $2 \times 2$  matrices with only 7 recursive mults.

$$P_1 = a \cdot (f - h)$$

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$$P_3 = (c + d) \cdot e$$

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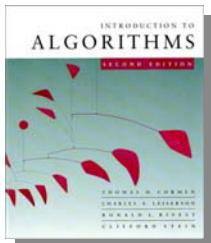
$$P_7 = (a - c) \cdot (e + f)$$

$$r = P_5 + P_4 - P_2 + P_6$$

$$s = P_1 + P_2$$

$$t = P_3 + P_4$$

$$u = P_5 + P_1 - P_3 - P_7$$



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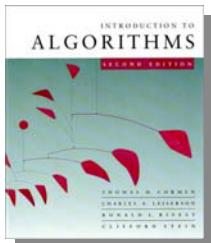
$$s = P_1 + P_2$$

$$t = P_3 + P_4$$

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7 mults, 18 adds/subs.

**Note:** No reliance on commutativity of mult!



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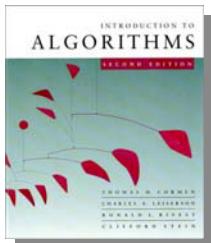
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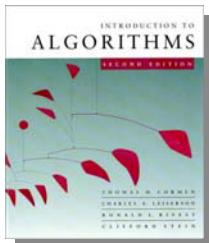
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$$\begin{aligned} r &= P_5 + P_4 - P_2 + P_6 \\ &= (a + d)(e + h) \\ &\quad + d(g - e) - (a + b)h \\ &\quad + (b - d)(g + h) \\ &= ae + ah + de + dh \\ &\quad + dg - de - ah - bh \\ &\quad + bg + bh - dg - dh \\ &= ae + bg \end{aligned}$$



# Strassen's algorithm

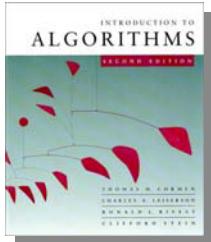
1. ***Divide:*** Partition  $A$  and  $B$  into  $(n/2) \times (n/2)$  submatrices. Form terms to be multiplied using  $+$  and  $-$ .
2. ***Conquer:*** Perform 7 multiplications of  $(n/2) \times (n/2)$  submatrices recursively.
3. ***Combine:*** Form  $C$  using  $+$  and  $-$  on  $(n/2) \times (n/2)$  submatrices.



# Strassen's algorithm

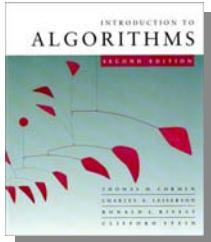
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$$T(n) = 7 T(n/2) + \Theta(n^2)$$



# Analysis of Strassen

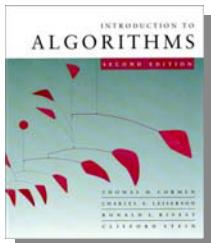
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# Analysis of Strassen

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$$n^{\log_b a} = n^{\log_2 7} \approx n^{2.81} \Rightarrow \text{CASE 1} \Rightarrow T(n) = \Theta(n^{\lg 7}).$$

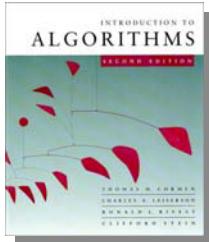


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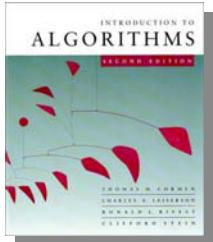
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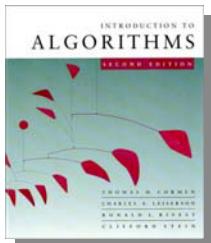
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**Best to date** (of theoretical interest only):  $\Theta(n^{2.3728\dots})$ .



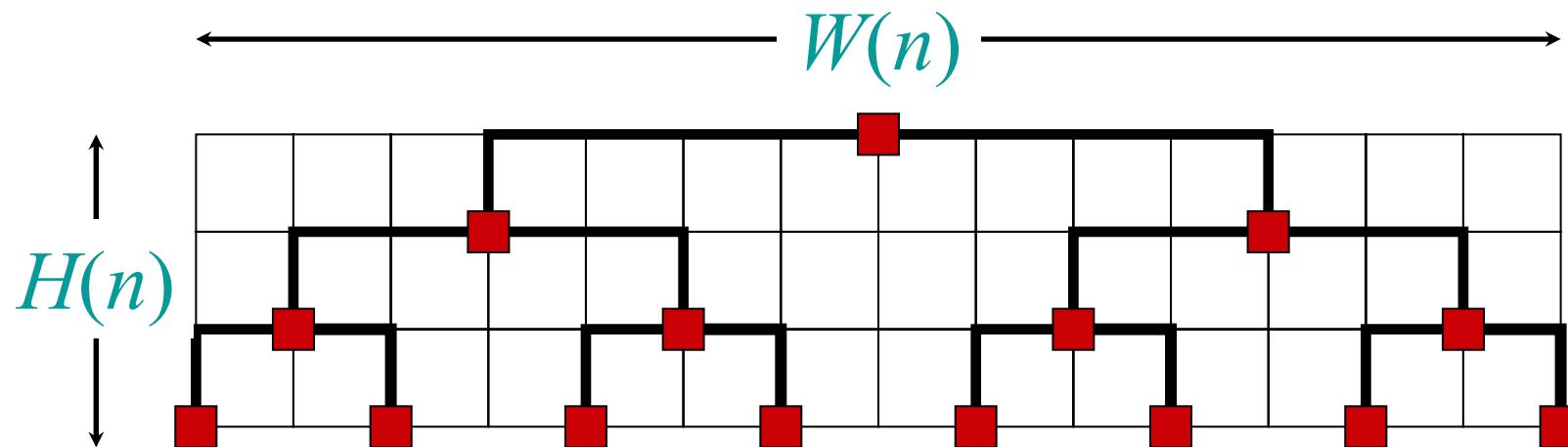
# VLSI layout

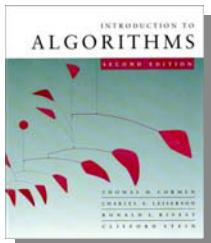
**Problem:** Embed a complete binary tree with  $n$  leaves in a grid using minimal area.



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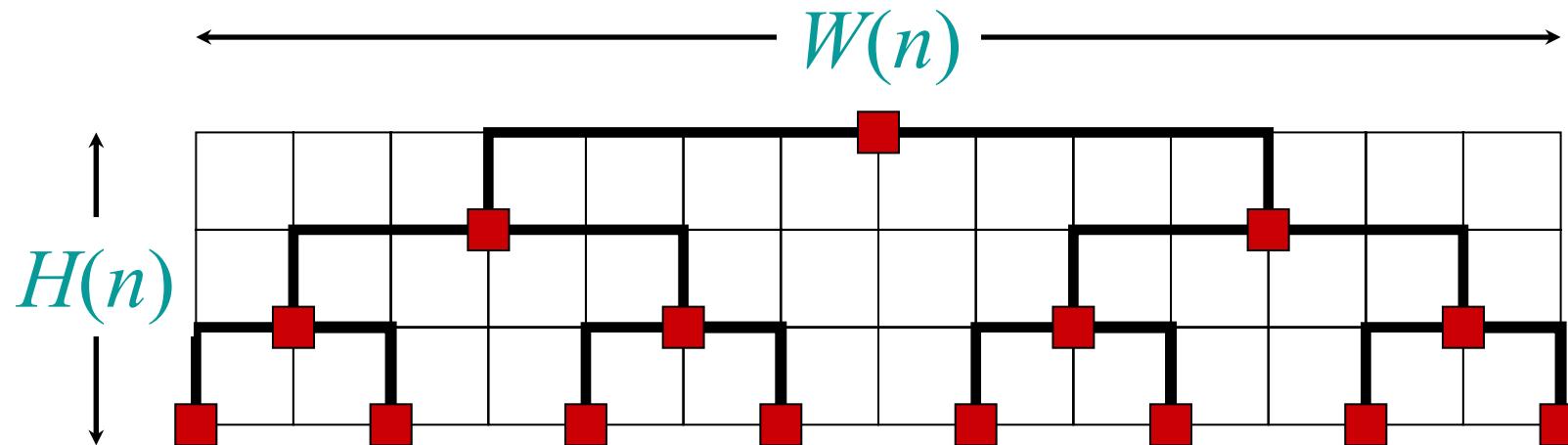
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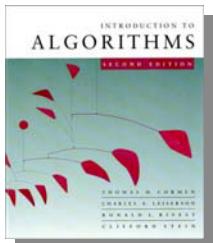


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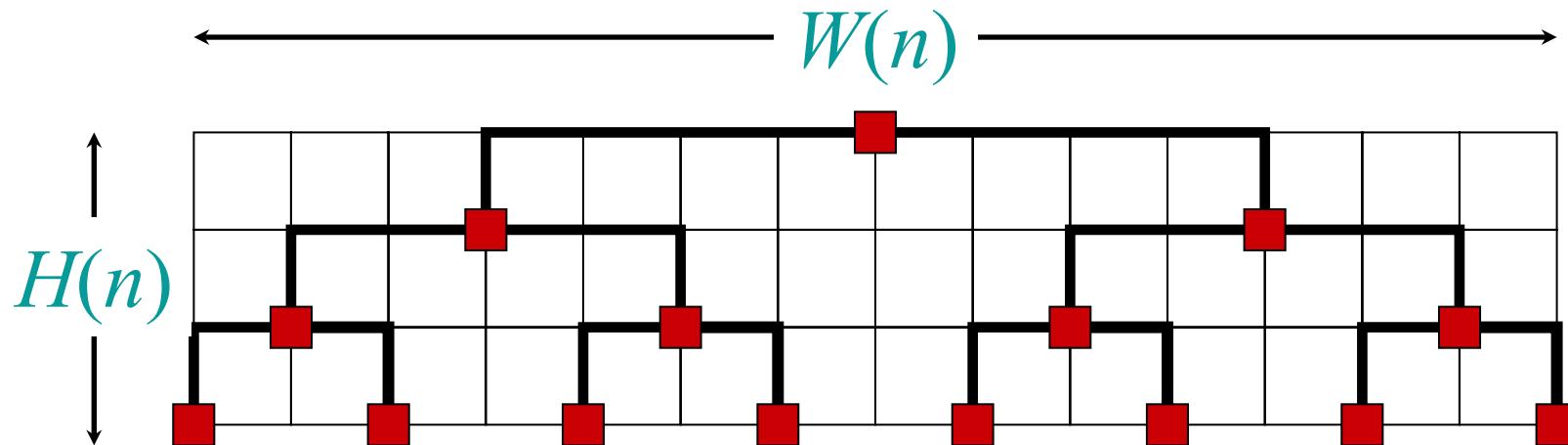


$$\begin{aligned} H(n) &= H(n/2) + \Theta(1) \\ &= \Theta(\lg n) \end{aligned}$$

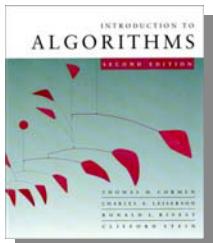


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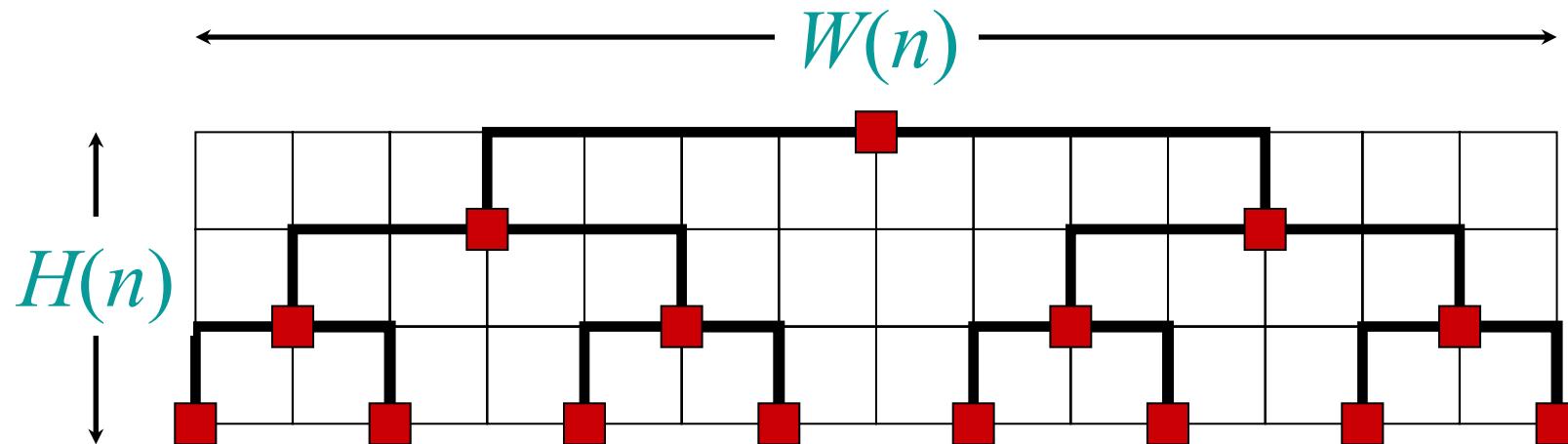


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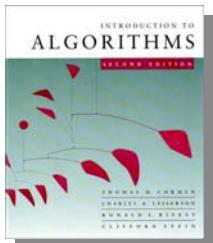
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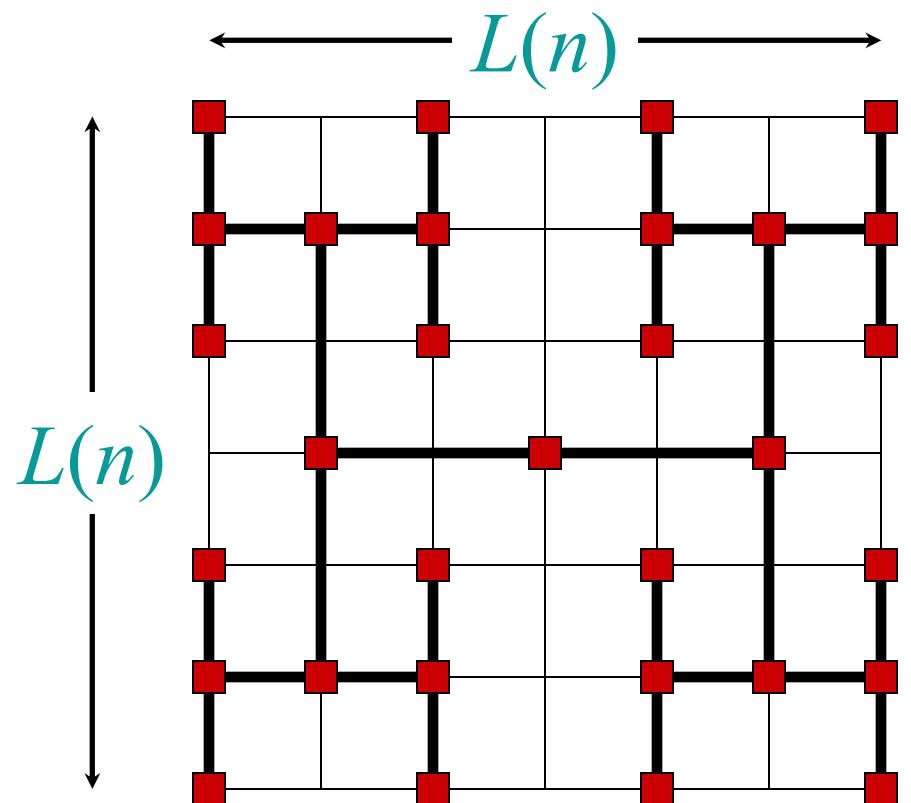
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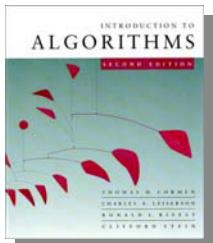
$$\begin{aligned} W(n) &= 2 W(n/2) + \Theta(1) \\ &= \Theta(n) \end{aligned}$$

$$\text{Area} = \Theta(n \lg n)$$

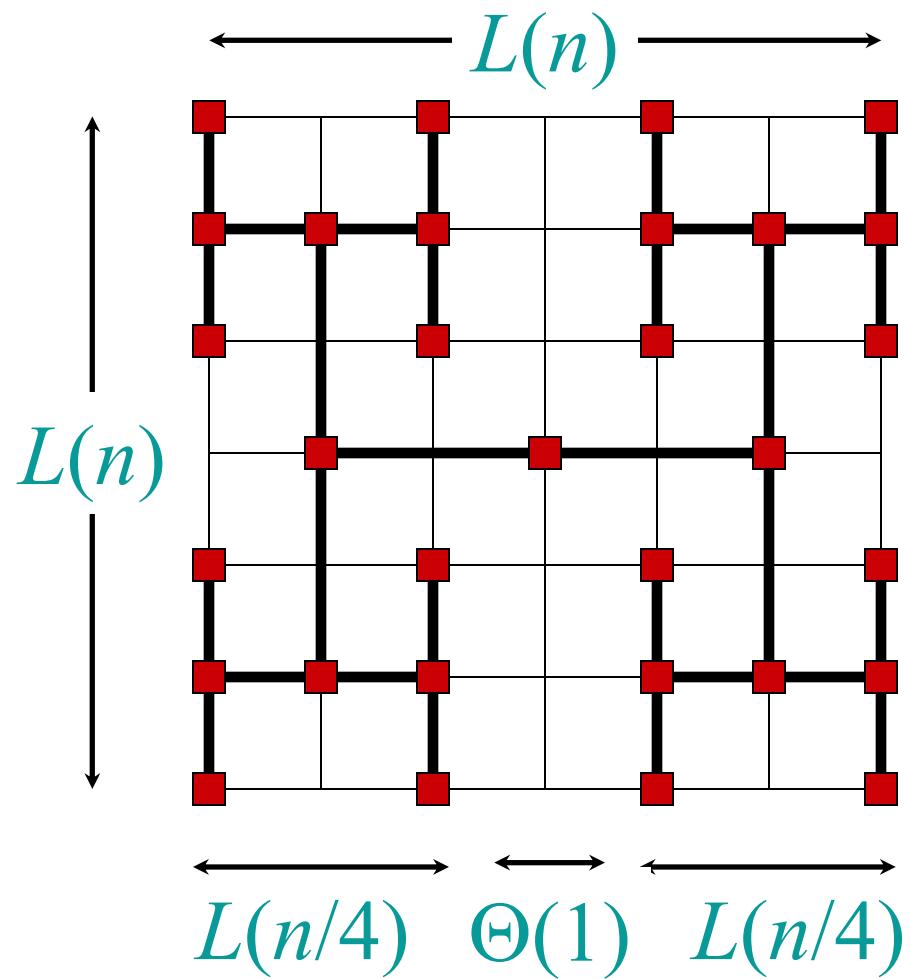


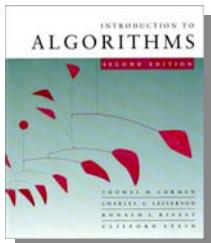
# H-tree embedding



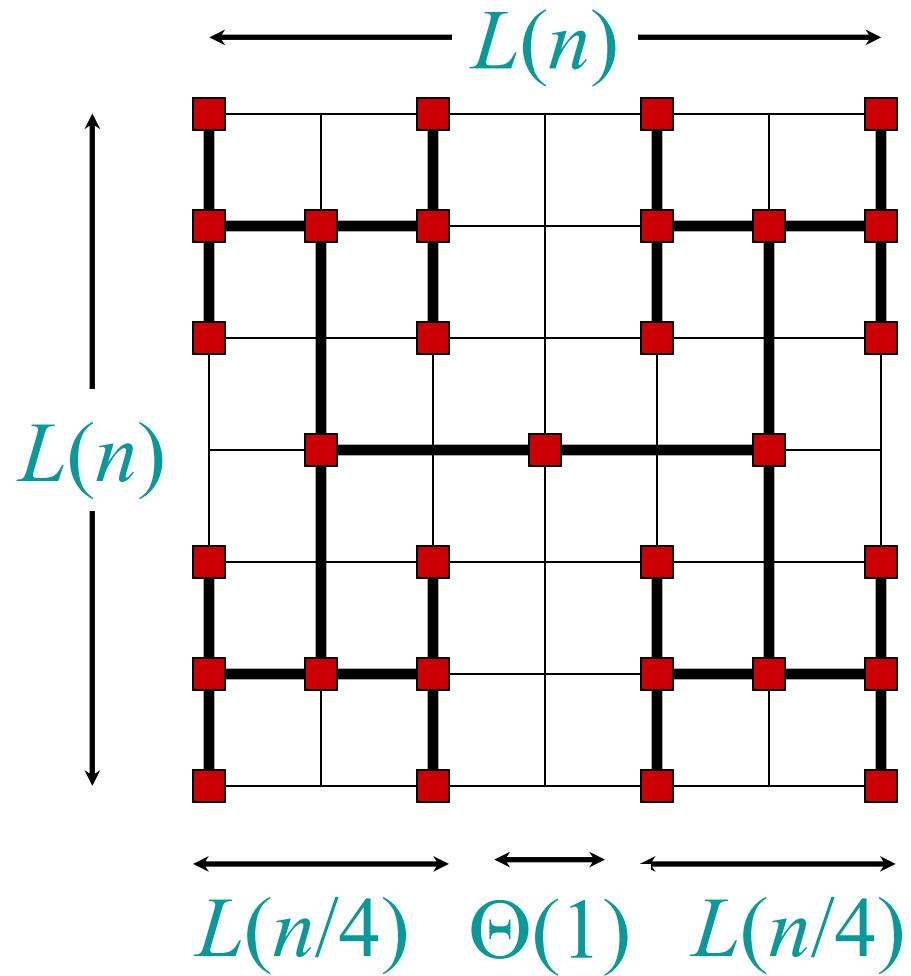


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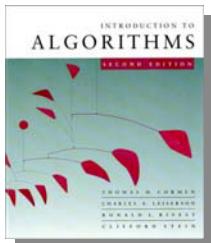


# H-tree embedding



$$\begin{aligned}L(n) &= 2L(n/4) + \Theta(1) \\&= \Theta(\sqrt{n})\end{aligned}$$

Area =  $\Theta(n)$



# Conclusion

- Divide and conquer is just one of several powerful techniques for algorithm design.
- Divide-and-conquer algorithms can be analyzed using recurrences and the master method (so practice this math).
- The divide-and-conquer strategy often leads to efficient algorithms.