Unit-II

Z-transform

Inverse Z- transforms

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Inverse Z- transforms:

(1). $Z^{-1}\left[\frac{1}{z-a}\right]=a^{n-1}$

(2). $Z^{-1}\left[\frac{1}{z+a}\right]=(-a)^{n-1}$

(3). $Z^{-1}\left[\frac{1}{(z-a)^2}\right]=(n-1)a^{n-2}$

(4). $Z^{-1}\left[\frac{1}{(z-a)^3}\right]=\frac{(n-1)(n-2)}{2}a^{n-3}$

(5). $Z^{-1}\left[\frac{z}{z-a}\right]=a^{n}$

(6). $Z^{-1}\left[\frac{z}{z+a}\right]=(-a)^{n}$

(7). $Z^{-1}\left[\frac{z^2}{(z-a)^2}\right]=(n+1)a^{n}$

(8). $Z^{-1}\left[\frac{z^3}{(z-a)^3}\right]=\frac{1}{2!}(n+1)(n+2)a^{n}$

Evaluation of Inverse Z- transforms

1. Power series method: This is the simplest method to find the inverse Z-transform. If $U(z)$ is expressed as the ratio of two polynomials which cannot be factorized, we simply divide the numerator by the denominator and take the inverse Z-transform of each term in the quotient.

Example.1: Find the inverse Z-transform of $\log\left(\frac{z}{z+1}\right)$.

Solution: $U(z)=\log\left(\frac{z}{z+1}\right)=\log\left(\frac{z}{z(1+1/z)}\right)=\log\left(\frac{1}{(1+1/z)}\right)$

$$=\log(1)-\log(1+1/z)=0-\left(\frac{1}{z}-\frac{1}{2z^2}+\frac{1}{3z^3}-\frac{1}{4z^4}+\ldots\right)$$

$$U(z) = -\frac{1}{z} + \frac{1}{2z^2} - \frac{1}{3z^3} + \frac{1}{4z^4} - \ldots$$

$$u_n = \begin{cases} 0, & \text{for } n = 0 \\ \frac{(-1)^n}{n}, & \text{for } n \neq 0 \end{cases}.$$

Example.2: Find the inverse Z-transform of $\frac{z}{(z+1)^2}$ by division method.

Solution. $U(z) = \frac{z}{z^2 + 2z + 1} = z^{-1} - \frac{2 + z^{-1}}{z^2 + 2z + 1}$, by actual division

$$= z^{-1} - 2z^{-2} + \frac{3z^{-1} + 2z^{-2}}{z^2 + 2z + 1} = z^{-1} - 2z^{-2} + 3z^{-3} - \frac{4z^{-2} + 3z^{-3}}{z^2 + 2z + 1}$$

Continuing this process of division, we get an infinite series *i.e.*,

$$U(z) = \sum_{n=0}^{\infty} (-1)^{n-1} nz^{-n}$$

Thus $u_n = (-1)^{n-1} n$.

2. Partial Fractions method: This is the similar to that of finding the inverse Laplace transforms using partial fractions. The method consists of decomposing $U(z)/z$ into partial fractions, multiplying the resulting expansion by z and then inverting the same.

Example.3: Find the inverse Z-transform of $\dfrac{2z^2+3z}{(z+2)(z-4)}$.

Solution: We write $U(z)=\dfrac{2z^2+3z}{(z+2)(z-4)}$

$$\frac{U(z)}{z}=\frac{2z+3}{(z+2)(z-4)}=\frac{A}{(z+2)}+\frac{B}{(z-4)} \quad \text{......(1)}$$

For A: $A = \left[\frac{2z+3}{(z-4)}\right]_{z=-2} = \frac{2(-2)+3}{(-2-4)} = \frac{1}{6}$

For B: $B = \left[\frac{2z+3}{(z+2)}\right]_{z=4} = \frac{2(4)+3}{(4+2)} = \frac{11}{6}$

Putting the values of A and B in equation (i), we get

$$\frac{U(z)}{z} = \frac{1}{6}\frac{1}{(z+2)} + \frac{11}{6}\frac{1}{(z-4)}$$

$$U(z) = \frac{1}{6}\frac{z}{(z+2)} + \frac{11}{6}\frac{z}{(z-4)}$$

On inverse Z-transform, we have

$$Z^{-1}\left[U(z)\right] = \frac{1}{6}Z^{-1}\left[\frac{z}{z+2}\right] + \frac{11}{6}Z^{-1}\left[\frac{z}{z-4}\right]$$

$$u_n = \frac{1}{6}(-2)^n + \frac{11}{6}(4)^n.$$

Example.4: Find the inverse Z-transform of $\dfrac{2\left(z^2-5z+6.5\right)}{(z-2)(z-3)^2}$ for $2<|z|<3$.

Solution. Splitting into partial fractions, we obtain

$$U(z) = \frac{2(z^2-5z+6.5)}{(z-2)(z-3)^2} = \frac{A}{z-2} + \frac{B}{z-3} + \frac{C}{(z-3)^2} \qquad \text{where } A = B = C = 1$$

$$\therefore \quad U(z) = \frac{1}{z-2} + \frac{1}{z-3} + \frac{1}{(z-3)^2}$$

$$= \frac{1}{2}\left(1-\frac{2}{z}\right)^{-1} - \frac{1}{3}\left(1-\frac{z}{3}\right)^{-1} + \frac{1}{9}\left(1-\frac{z}{3}\right)^{-2} \qquad \text{so that } 2/z < 1 \text{ and } z/3 < 1$$

$$= \frac{1}{z}\left(1+\frac{2}{z}+\frac{4}{z^2}+\frac{8}{z^3}+\dots\right) - \frac{1}{3}\left(1+\frac{z}{3}+\frac{z^2}{9}+\frac{z^3}{27}+\dots\right) + \frac{1}{9}\left(1+\frac{2z}{3}+\frac{3z^2}{9}+\frac{4z^3}{27}+\dots\right)$$

where $2 < |z| < 3$.

$$= \left(\frac{1}{2} + \frac{2}{z^2} + \frac{2^2}{z^3} + \frac{2^3}{z^4} + \ldots\right) - \left(\frac{1}{3} + \frac{z}{3^2} + \frac{z^2}{3^3} + \frac{z^3}{3^4} + \ldots\right) + \left(\frac{1}{3^2} + \frac{2z}{3^3} + \frac{3z^2}{3^4} + \frac{4z^3}{3^5} + \ldots\right)$$

$$= \sum_{n=1}^{\infty} 2^{n-1} z^{-n} - \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^{n+1} z^n + \sum_{n=0}^{\infty} (n+1)\left(\frac{1}{3}\right)^{n+2} z^n$$

On inversion, we get $u_n = 2^{n-1}, n \geq 1$ and $u_n = -(n+2)3^{n-2}, n \leq 0.$

3. Inversion integral method:

The inverse Z-transform of $U(z)$ is given, by the formula

$$u_n = \frac{1}{2\pi i} \int_c U(z) z^{n-1} dz$$

$=$ sum of residue of $U(z) z^{n-1}$ at the poles of $U(z)$ which are inside the contour C drawn according to the ROC given.

Example.5: Using the inversion integral method, find the inverse Z-transform of $\dfrac{z}{(z-1)(z-2)}$.

Solution: Let $U(z)=\dfrac{z}{(z-1)(z-2)}$. Its poles are at $z=1$ and $z=2$.

Using $U(z)$ in the inversion integral, we have

$$u_n = \frac{1}{2\pi i}\int_C U(z)z^{n-1}\,dz,$$

where C is a circle large enough to enclose both the poles of $U(z)$.

= sum of residues of $U(z)\,z^{n-1}$ at $z=1$ and $z=2$.

Now $$Res\,[U(z)\,z^{n-1}]_{z=1} = \underset{z\to 1}{\mathrm{Lt}}\left\{(z-1)\,.\,\frac{z^n}{(z-1)(z-2)}\right\} = -1$$

and $$Res\,[U(z)\,z^{n-1}]_{z=2} = \underset{z\to 2}{\text{Lt}} \left\{(z-2)\,.\,\frac{z^n}{(z-1)(z-2)}\right\} = 2^n$$

Thus the required inverse Z-transform $u_n = 2^n - 1$, $n = 0, 1, 2, \ldots$

Example.6: Find the inverse Z-transform of $\dfrac{2z}{(z-1)(z^2+1)}$.

Solution: Let $U(z) = \dfrac{2z}{(z-1)(z^2+1)}$. It has three poles at $z = 1, i, -i$.

Using $U(z)$ in the inversion integral, we have

$$u_n = \frac{1}{2\pi i}\int_C U(z)\,.\,z^{n-1}\,dz,$$ where C is a circle large enough to enclose the poles of $U(z)$.

$=$ sum of residues of $U(z)\,.\,z^{n-1}$ at $z = 1$, $z = \pm i$.

Now $$Res\,[U(z)\,z^{n-1}]_{z=1} = \underset{z \to 1}{\text{Lt}} \left\{ (z-1) \frac{2z^n}{(z-1)(z^2+1)} \right\} = 1$$

$$Res\,[U(z)\,z^{n-1}]_{z=i} = \underset{z \to i}{\text{Lt}} \left\{ (z-i) \frac{2z^n}{(z-1)(z+i)(z-i)} \right\} = \frac{-(i)^n}{1+i}$$

$$Res\,[U(z)\,z^{n-1}]_{z=-i} = \underset{z \to -i}{\text{Lt}} \left\{ (z+i) \frac{2z^n}{(z-1)(z+i)(z-i)} \right\} = \frac{(-i)^n}{i-1}$$

Hence $$u_n = 1 - \frac{(i)^n}{1+i} - \frac{(-i)^n}{1-i}.$$

Questions?

Thank You!