Numerical Mathematics II for Engineers - Homework 6

Group 11

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Exercise 6.1

(a)

Deriving the weak form of the Boundary Value Problem (BVP)
Our original equation with the boundary condition is:

$$\begin{aligned} -u''(x) &= f(x) \text{in } \Omega = (0,1) \\ u(0) &= u(1) = 0 \end{aligned}$$

where the inhomogeneity is

$$f(x) = \begin{cases} 3, & \text{if } 0 < x < \frac{1}{2}, \\ -1, & \text{if } \frac{1}{2} \leq x < 1. \end{cases}$$

We now choose an arbitrary function v and multiply both sides of our equation by $v(x)$ and integrate over Ω.

$$\int_\Omega -u''(x)v(x)dx = \int_\Omega f(x)v(x)dx$$

We can employ the integration by parts method to eliminate the second derivative of u.

$$\begin{aligned} \int_\Omega -u''(x)v(x)dx &= \int_\Omega f(x)v(x)dx \\ [-u'(x)v(x)]_0^1 + \int_\Omega u'(x)v'(x)dx &= \int_\Omega f(x)v(x)dx \\ (-u'(1)v(1) + u'(0)v(0)) + \int_\Omega u'(x)v'(x)dx &= \int_\Omega f(x)v(x)dx \end{aligned}$$

The boundary term $(-u'(1)v(1) + u'(0)v(0))$ must be zero to obtain a simplified weak form. This is achieved by carefully defining the space V for the test function v.

We select the function space $V = H_0^1(\Omega)$. The H^1 space (Sobolev space of order 1) is chosen because it ensures that u and v have square-integrable first (weak) derivatives, making the integral $\int_\Omega u'(x)v'(x)dx$ well-defined and finite1111. The crucial subscript '0' restricts v to satisfy the homogeneous essential (Dirichlet) boundary conditions, meaning $v(0) = 0$ and $v(1) = 0$. This ensures the boundary term vanishes: $(-u'(1)\cdot 0 + u'(0)\cdot 0) = 0$, leading directly to the simplified weak form (these results are from the lecture).

The weak/variational problem can be restated as:

$$\text{Find u} \in H_0^1(\Omega)\text{such that} \int_\Omega u'(x)v'(x)dx = \int_\Omega f(x)v(x)dx\text{for all v} \in H_0^1(\Omega))$$

(b)

Since $f(x)$ is piecewise constant, the solution $u(x)$ is a piecewise quadratic polynomial (because if we wanted to find u(x) we would have to integrate f(x) twice). The solution must satisfy the Dirichlet boundary conditions and the C^1 continuity conditions at the interface $x = 1/2$.

We split the domain as defined in f(x). We solve $u''(x) = -f(x)$ in the two regions $\Omega_1 = (0, 1/2)$ and $\Omega_2 = (1/2, 1)$.

$$\text{If } 0 < x < 1/2 \ (f(x) = 3): \quad u_1''(x) = -3 \implies u_1(x) = \iint -3dx$$

$$= u_1(x) = -\frac{3}{2}x^2 + C_1x + C_2$$

$$\text{If } 1/2 < x < 1 \ (f(x) = -1): \quad u_2''(x) = 1 \implies u_2(x) = \iint 2dx$$

$$= u_2(x) = \frac{1}{2}x^2 + C_3x + C_4$$

We now need to solve for the constants

1. $u(0) = 0 \implies C_2 = 0$ not C_4 because of the domain
2. $u(1) = 0 \implies \frac{1}{2} + C_3 + C_4 = 0 \implies C_3 + C_4 = -1/2 \quad$ (I)
3. While H^1 only strictly requires u to be continuous, the successful elimination of the $[-u'(x)v(x)]_0^1$ in the weak problem relies on the continuity of $u' \implies u_1'(1/2) = u_2'(1/2)$:

$$-3/2 + C_1 = 1/2 + C_3$$

$$C_1 - C_3 = 2 \quad \text{(II)}$$

4. $u_1(1/2) = u_2(1/2)$ (because we need continuity of u since u $\in H_0^1(\Omega)$):

$$-\frac{3}{8} + \frac{C_1}{2} = \frac{1}{8} + \frac{C_3}{2} + C_4$$

$$4C_1 - 4C_3 - 8C_4 = 4 \quad \text{(III)}$$

Substituting $C_1 = C_3 + 2$ from (II) into (III) gives:

$$\begin{aligned} 4(C_3 + 2) - 4C_3 - 8C_4 &= 4 \\ 8 - 8C_4 &= 4 \\ C_4 &= 1/2 \end{aligned}$$

Substituting C_4 into (I):

$$\begin{aligned} C_3 + 1/2 &= -1/2 \\ C_3 &= -1 \end{aligned}$$

Finding C_1 from (II):

$$\begin{aligned} C_1 &= -1 + 2 \\ C_1 &= 1 \end{aligned}$$

The solution to the Weak Problem:

$$u(x) = \begin{cases} u_1(x) = -\frac{3}{2}x^2 + x & \text{if } 0 \le x \le 1/2 \\ u_2(x) = \frac{1}{2}x^2 - x + \frac{1}{2} & \text{if } 1/2 \le x \le 1 \end{cases}$$

(c)

Is the solution of the weak problem also a classical solution of the original BVP?

A function $u(x)$ is a classical solution if it is twice continuously differentiable on the domain, $u \in C^2(\Omega)$, and satisfies the differential equation $-u''(x) = f(x)$ everywhere.

The weak solution $u(x)$ found in (b) is:

$$u(x) = \begin{cases} u_1(x) = -\frac{3}{2}x^2 + x & \text{if } 0 \le x \le 1/2 \\ u_2(x) = \frac{1}{2}x^2 - x + \frac{1}{2} & \text{if } 1/2 \le x \le 1 \end{cases}$$

The second derivatives are:

$$\begin{aligned} u_1''(x) &= -3 \\ u_2''(x) &= 1 \end{aligned}$$

Since $u_1''(1/2) = -3$ and $u_2''(1/2) = 1$, the second derivative $u''(x)$ is discontinuous at $x = 1/2$. This means $u \notin C^2(\Omega)$.

Because the solution $u(x)$ fails the C^2 smoothness requirement, it is not a classical solution.

However, the weak formulation is necessary precisely because it allows for solutions with lower regularity than C^2.

- The trial space $u \in H_0^1(\Omega)$ only requires the first derivative u' to be square-integrable.
- Our solution is $\mathbf{C^1}$-continuous ($u_1'(1/2) = u_2'(1/2) = -1/2$), which is the minimal requirement for the weak formulation to hold, and ensures the flux is continuous across the jump in $f(x)$.

Thus, the weak solution is the mathematical generalisation that holds even when the source term $f(x)$ is not smooth enough to produce a strict classical solution.

Exercise 6.2

(a)

The assumption is

$$|u - u_h|_\infty \leq c\, h^p$$

.

Applying this to mesh size at $h/2$, we get the following:

$$|u - u_{h/2}|_\infty \leq c\, h^p / 2^p$$

.

The maximum difference between u_h and $u_{h/2}$ can also be written as:

$$|(u - u_{h/2}) - (u - u_h)|_\infty$$

Based on the assumption, the right-hand side can also be completed:

$$|u_h - u_{h/2}| \leq c\, h^p + \frac{c\, h^p}{2^p} = c\left(1 + \frac{1}{2^p}\right) h^p$$

We can define $\tilde{c} = c\left(1 + \frac{1}{2^p}\right)$. Then we reach equation (1);

$$\|u_h - u_{h/2}\|_\infty \leq \tilde{c}\, h^p \quad \square$$

(b)

The assumption is

$$|u - u_h|_\infty \leq c\, h^p$$

.

We can copy the logic from part (a) and write this for mesh sizes h, $h/2$ and $h/4$ for experimentation.

Looking at the maximum difference between h and $h/2$, and $h/2$ and $h/4$, we can calculate a ratio of change for each time we halve h. First:

$$u_h - u_{h/2} = (u - u_{h/2}) - (u - u_h) \approx c\,(h/2)^p - c\, h^p = c\, h^p\,((1/2)^p - 1)$$

and:

$$u_{h/2} - u_{h/4} = (u - u_{h/4}) - (u - u_{h/2}) \approx c\,(h/4)^p - c\,(h/2)^p = c\, h^p\,((1/4)^p - (1/2)^p)$$

.

The ratio R is:

$$R = \frac{D_1}{D_2} \approx \frac{|c| h^p \left(1 - \frac{1}{2^p}\right)}{|c| \frac{h^p}{2^p} \left(1 - \frac{1}{2^p}\right)} = 2^p.$$

The ratio of the differences between successive solutions is 2^p, where p is the unknown convergence order.

Here we got the experimental order of convergence as

$$p = \log_2(R)$$

.

Exercise 6.3

(a)

To ensure stability in the maximum norm, we apply the first theorem from Lecture 11. In this theorem, the matrix $M = \tau(1-\theta)L_h$ has to satisfy the condition $M_{ii} \leq 1$. In 2D, the diagonal entry of L_h is $\frac{4}{h^2}$, so the condition becomes

$$\tau(1-\theta)\frac{4}{h^2} \leq 1.$$

Substituting $\tau = \frac{T}{M}$ and $h = \frac{1}{N+1}$, we determine the following condition for M:

$$\frac{T}{M}(1-\theta)\frac{4}{\left(\frac{1}{N+1}\right)^2} \leq 1$$
$$\Rightarrow M \geq T(1-\theta)\frac{4}{\left(\frac{1}{N+1}\right)^2}.$$

If $\theta = 1$, the term $1-\theta$ vanishes and the scheme is unconditionally stable.

(e)

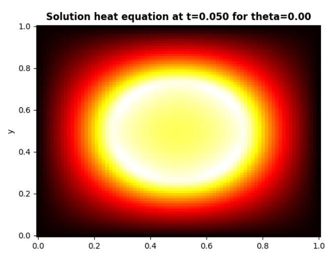

Figure 1: Theta = 0

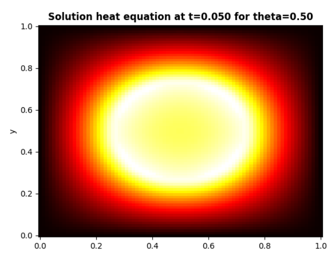

Figure 2: Theta = 0.5

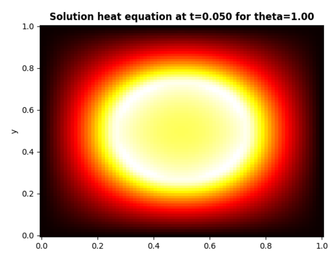

Figure 3: Theta = 1.0

Figure 4: Comparison of the 2D heat equation solution at $T = 0.05$ using different θ-schemes ($N = 80$).