Technische Universität Berlin Winter term 2025/26
Institute of Mathematics
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Numerical Mathematics II for Engineers
Tutorial 9

Topics: Admissible meshes, Matrix assembly in 2D, linear FEM in 2D, FEM analysis

Discussion in the tutorials of the week 05 – 09 January 2026

Disclaimers:

- To test your code, we provide tests. This should help you to locate precisely your errors, and code more efficiently. To run the test, you need to install pytest.
- Exercises should be solved in **fixed groups of 3 students**.
- Hand in the solution in **one folder** labeled **hw[hw_number]_group[group_number]** and containing:
  - **One pdf** for the theoretical questions and comments on the numerical results,
  - **Three python files: fem2D.py, femlib2D.py, mesh.py**.
  - Write the group number and all names of your members **in each file**.

Mandatory: Exercise evaluation. The mandatory course evaluation takes place from 05.01 until 16.01. Please answer the survey for the exercises.

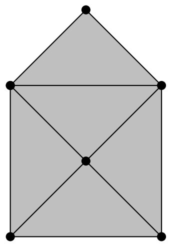
Attention! This is a different survey than for the lecture!

The survey is available here: `https://befragung.tu-berlin.de/evasys/online.php?pswd=WKCPQ`

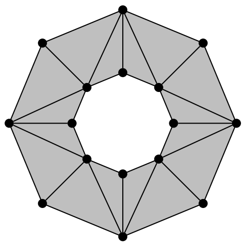
You can also access the survey with this QR code:

Exercise 9.1: Admissible meshes

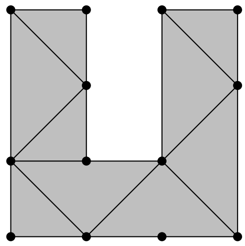
Consider the following two-dimensional decompositions:



House (i)



Annulus (ii)

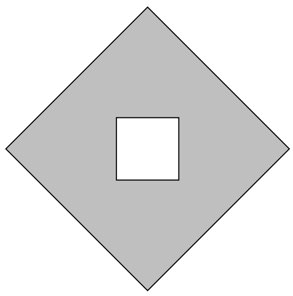


U Shape (iii)

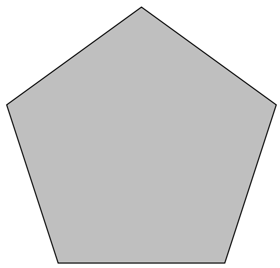
(a) Which of the meshes (i), (ii), and (iii) are admissible? Count for every admissible mesh the number of points n_p, the number of faces/edges n_f, and the number of elements n_e. Then, for every admissible mesh, count the number of points and faces on the boundary $\partial\Omega$ of the domain.

(b) What is the dimension of the finite element discretization $V^h \subset V = H^1(\Omega)$ for piecewise linear and for piecewise quadratic basis functions on the admissible meshes provided above? Similarly, what is the dimension if we instead consider $V_0^h \subset V_0 = H_0^1(\Omega)$.

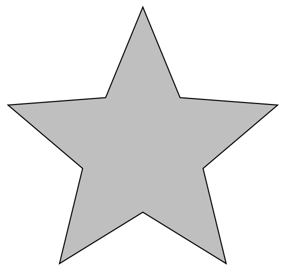
(c) Decompose the following domains into an admissible mesh using triangles. Do it so that the number of elements is $n_e = 8, 9, 10$ for the meshes (iv), (v), (vi), respectively.



Mesh (iv)



Mesh (v)



Mesh (vi)

Exercise 9.2: Matrix assembly for 2D FEM

This exercise is a preliminary exercise for the FEM matrix assembly in 2D. We assume that we have a polygonal domain $\overline{\Omega}$ which has been decomposed into triangles, and consider one of those triangle $\overline{\Omega}_\ell$ with vertices p_1, p_2, $p_3 \in \mathbb{R}^2$.

We introduce the reference triangle $\overline{\Omega}_{\mathrm{ref}}$ with nodes $\widehat{p}_1 = (0,0)$, $\widehat{p}_2 = (1,0)$, and $\widehat{p}_3 = (0,1)$ and assume that we are given an affine linear transformation $T_\ell(x,y) = F_\ell(x,y) + p_1$, mapping $\overline{\Omega}_{\mathrm{ref}}$ onto $\overline{\Omega}_\ell$.

Finally, we introduce the linear basis functions $\psi_1, \psi_2, \psi_3 \colon \overline{\Omega} \to \mathbb{R}$ defined on $\overline{\Omega}$ and characterized by $\psi_i(p_j) = \delta_{i,j}$ for i, $j \in \{1, 2, 3\}$, and the basis functions on the reference triangle $\varphi_1, \widehat{\varphi}_2, \widehat{\varphi}_3 \colon \overline{\Omega}_{\mathrm{ref}} \to \mathbb{R}$ defined by

$$\widehat{\varphi}_1(x,y) := 1 - x - y, \quad \widehat{\varphi}_2(x,y) := x, \quad \widehat{\varphi}_3(x,y) := y.$$

(a) Determine the matrix $S_{xx} \in \mathbb{R}^{3\times 3}$ defined by

$$[S_{xx}]_{ij} = \int_{\overline{\Omega}_{\mathrm{ref}}} \frac{\partial \widehat{\varphi}_i(x,y)}{\partial x} \frac{\partial \widehat{\varphi}_j(x,y)}{\partial x} \mathrm{d}x\mathrm{d}y.$$

Hint: Try to make use of symmetries to avoid unnecessary computations.

(b) Assume that we are given the matrices $S_{xy}, S_{yy} \in \mathbb{R}^{3\times 3}$ defined by

$$[S_{xy}]_{ij} = \int_{\overline{\Omega}_{\mathrm{ref}}} \frac{\partial \widehat{\varphi}_i(x,y)}{\partial x} \frac{\partial \widehat{\varphi}_j(x,y)}{\partial y} \mathrm{d}x\mathrm{d}y. \quad [S_{yy}]_{ij} = \int_{\overline{\Omega}_{\mathrm{ref}}} \frac{\partial \widehat{\varphi}_i(x,y)}{\partial y} \frac{\partial \widehat{\varphi}_j(x,y)}{\partial y} \mathrm{d}x\mathrm{d}y,$$

respectively. Explain how the integrals

$$a_{ij}^{\ell} = \int_{\overline{\Omega}_\ell} \nabla \psi_i \cdot \nabla \psi_j \mathrm{d}x\mathrm{d}y, \qquad \text{for } i,j \in \{1,2,3\}$$

may be computed using the matrices S_{xx}, S_{xy}, S_{yy} and the linear transformation T_ℓ.

Hint: Since T_ℓ is an affine function, what is the general form of $F_\ell(x,y)$? Moreover, you can use that $\psi_i(T_\ell(x,y)) = \widehat{\varphi}_i(x,y)$ holds for all $(x,y) \in \overline{\Omega}_{\mathrm{ref}}$ and for all $i \in \{1,\ 2,\ 3\}$, without proving it.

Exercise 9.3: 2D finite element method

This is a long and complex task, so please read through the entire exercise before starting to code and do not hesitate to ask us questions.

Please note that we provide you the file `getTriangulationMatrices.py`.

PART 1: PROBLEM DEFINITION. Consider the two-dimensional boundary value problem

$$\begin{cases} -\Delta u = f & \text{in} \quad \Omega = (0,1)^2 \subset \mathbb{R}^2, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

(a) Write the associated weak formulation using as trial and test space $H_0^1(\Omega)$.

(b) Write the finite element problem using piecewise linear basis functions.

PART 2: MESH GENERATION. In this section, we will generate a 2D grid for the domain Ω. The grid, in this case a *triangulation*, consists of three things: the list of vertices (or nodes), the list of triangles, and a correspondence table specifying which vertices are in which triangle. The triangulation should also specify which vertices are located on the boundary.

(c) We provide you the function `B, C, D, E = getTriangulationMatrices(N)`. Try to understand what the function does. Then, in the `if __name__ == "__main__"` part of the `mesh.py` file, call the function for $N = 5$ and plot the resulting mesh.

Hint: You may use the function `plt.triplot` from the `matplotlib` library to plot the triangulation.

PART 3: FEM - MATRIX ASSEMBLY. In this section, we will assemble the Galerkin matrix A_n and the right-hand side f_n.

We first introduce some notations. The triangulation obtained in the previous part has n_t triangles and n_v vertices. The list of vertices is $[p_1, \ldots p_{n_v}]$. Each vertex has the coordinates $p_k = (x_k, y_k) \in \overline{\Omega}$. With this triangulation, you also have a correspondence table $[k_1, \ldots, k_{n_t}]$.

For $\ell \in \{1, \dots, n_\mathrm{t}\}$, the entry $k_\ell \colon \{1,2,3\} \to \{1, \dots, n_\mathrm{v}\}$ specifies the three vertices of the triangle ℓ.

As in exercise 9.2, we introduce the reference triangle $\overline{\Omega}_\mathrm{ref}$, the basis functions $\widehat{\varphi}_1, \widehat{\varphi}_2, \widehat{\varphi}_3$, and the transformation T_ℓ from the reference triangle to the ℓth triangle of the triangulation

$$T_\ell(x,y) = F_\ell \begin{pmatrix} x \\ y \end{pmatrix} + p_{k_\ell(1)}, \text{ with } F_\ell = \begin{pmatrix} x_{k_\ell(2)} - x_{k_\ell(1)} & x_{k_\ell(3)} - x_{k_\ell(1)} \\ y_{k_\ell(2)} - y_{k_\ell(1)} & y_{k_\ell(3)} - y_{k_\ell(1)} \end{pmatrix}.$$

Finally, we introduce the matrices

$$S_{xx} = \left[\int_{\overline{\Omega}_\mathrm{ref}} \frac{\partial \widehat{\varphi}_i(x,y)}{\partial x} \frac{\partial \widehat{\varphi}_j(x,y)}{\partial x} \mathrm{d}x\mathrm{d}y \right]_{i,j=1,2,3} = \frac{1}{2} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$S_{xy} = \left[\int_{\overline{\Omega}_\mathrm{ref}} \frac{\partial \widehat{\varphi}_i(x,y)}{\partial x} \frac{\partial \widehat{\varphi}_j(x,y)}{\partial y} \mathrm{d}x\mathrm{d}y \right]_{i,j=1,2,3} = \frac{1}{2} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix},$$

$$S_{yy} = \left[\int_{\overline{\Omega}_\mathrm{ref}} \frac{\partial \widehat{\varphi}_i(x,y)}{\partial y} \frac{\partial \widehat{\varphi}_j(x,y)}{\partial y} \mathrm{d}x\mathrm{d}y \right]_{i,j=1,2,3} = \frac{1}{2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

With all those definitions, we can now describe the matrix assembly algorithm, which has a similar structure to the matrix assembly for 1D FEM in the Homework 8.

Inputs:

- Integer N such that the triangulation has $N \times N$ inner nodes
- right-hand side $f \colon \Omega \to \mathbb{R}$ of the PDE

Outputs:

- Galerkin matrix $A_n \in \mathbb{R}^{n \times n}$ (n - number of inner vertices)
- right-hand side $f_n \in \mathbb{R}^n$ of the Galerkin linear system

$[p_1, \dots p_{n_v}], [k_1, \dots, k_{n_t}] \leftarrow$ triangulation
Initialize $\widetilde{A} \in \mathbb{R}^{n_\mathrm{v} \times n_\mathrm{v}}$ and $\widetilde{f} \in \mathbb{R}^{n_\mathrm{v}}$ with all entries being zero
for $\ell = 1, \dots, n_\mathrm{t}$ **do**
  $F_\ell \leftarrow \begin{bmatrix} x_{k_\ell(2)} - x_{k_\ell(1)} & x_{k_\ell(3)} - x_{k_\ell(1)} \\ y_{k_\ell(2)} - y_{k_\ell(1)} & y_{k_\ell(3)} - y_{k_\ell(1)} \end{bmatrix}$
  $d_\ell \leftarrow |\det(F_\ell)|$
  $B_\ell \leftarrow F_\ell^{-1} F_\ell^{-\top}$
  $S_\ell \leftarrow d_\ell([B_\ell]_{11} S_{xx} + [B_\ell]_{22} S_{yy} + [B_\ell]_{12} S_{xy})$
  for $i = 1,2,3$ **do**
    $[f_\ell]_i \leftarrow$ quadrature approximation of $d_\ell \int\limits_{\Omega_\mathrm{ref}} f(T_\ell(x,y)) \widehat{\varphi}_i(x,y) \, \mathrm{d}x\mathrm{d}y$
    $[\widetilde{f}]_{k_\ell(i)} \leftarrow [\widetilde{f}]_{k_\ell(i)} + [f_\ell]_i$
    for $j = 1,2,3$ **do**
      $[\widetilde{A}]_{k_\ell(i), k_\ell(j)} \leftarrow [\widetilde{A}]_{k_\ell(i), k_\ell(j)} + [S_\ell]_{ij}$
    end for
  end for
end for
Remove those entries from $\widetilde{f}$ and those rows and columns from $\widetilde{A}$ which correspond to vertices at the boundary of Ω and obtain f_n and A_n, respectively

(d) Write a function `An, fn = get_matrix_rhs` that sets up the full Galerkin matrix A_n and the right hand side f_n of the linear system $A_n\alpha = f_n$. You may follow the algorithm outlined above and use the matrices returned by `getTriangulationMatrices`.

Hints:

- For the approximation of an integral over the reference triangle Ω_{ref}, you may use the quadrature rule

$$\int_{\Omega_{\text{ref}}} g(x,y)\mathrm{d}x\mathrm{d}y \approx \tfrac{1}{2}g(\tfrac{1}{3},\tfrac{1}{3}).$$

- You can build the Galerkin matrix as a full matrix, even if the resulting algorithm is relatively slow.

PART 3: SOLVE THE LINEAR SYSTEM AND PLOT THE SOLUTION

(e) Complete the function `my_driver` to solve the linear system and apply the homogeneous Dirichlet boundary conditions.

(f) Solve the problem for the right-hand side $f(x,\, y) = 10\pi^2\sin(3\pi x)\sin(\pi y)$, which corresponds to the exact solution $u(x,y) = \sin(3\pi x)\sin(\pi y)$. Plot your solution for $N = 30$.

Exercise 9.4: FEM Analysis

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and denote by Γ its boundary. Consider the following boundary value problem

$$\begin{cases} -\nabla\cdot(b\nabla u) + cu = f & \text{in } \Omega \subset \mathbb{R}^d \\ \nabla u \cdot \mathbf{n} = g & \text{on } \Gamma \end{cases} \tag{1}$$

with given functions $b, c, f : \Omega \to \mathbb{R}$ and $g : \Gamma \to \mathbb{R}$. Suppose in addition that there are constants $b_m, b_M, c_m, c_M, f_M, g_M > 0$ such that

$$b_m \le b(x) \le b_M, \quad c_m \le c(x) \le c_M, \quad |f(x)| \le f_M \quad \text{and} \quad |g(x)| \le g_M$$

hold for all $x \in \Omega$.

(a) Write the weak formulation of (1) using as trial space and test space $V = H^1(\Omega)$.

(b) Show that the assumptions of the Lax-Milgram theorem hold.

Hint: You can apply the Lax-Milgram theorem even if $V \neq H_0^1(\Omega)$, but you *cannot* use the Poincaré inequality. You can use the following *trace theorem* without proving it: there is a constant $C > 0$ such that

$$\|v\|_{L^2(\Gamma)} \le C\,\|v\|_{H^1(\Omega)}, \qquad \text{for all } v \in H^1(\Omega)$$