Technische Universität Berlin Winter term 2025/26
Institute of Mathematics
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Numerical Mathematics II for Engineers
Tutorial 6

Topics : Weak formulation, experimental convergence, theta-schemes

Discussion in the tutorials of the week 01–05 December 2025

Disclaimers:

- To test your code, we provide tests. This should help you to locate precisely your errors, and code more efficiently. To run the test, you need to install pytest.
- Exercises should be solved in **fixed groups of 3 students**.
- Hand in the solution in **one folder** labeled **hw[hw_number]_group[group_number]** and containing:
  - **One pdf** for the theoretical questions and comments on the numerical results,
  - **One python file per programming exercise**.
  - Write the group number and all names of your members **in each file**.

Exercise 6.1: Weak solutions

Consider the boundary value problem (BVP)

$$-u''(x) = f(x) \quad \text{in } \Omega = (0,1), \qquad u(0) = u(1) = 0$$

with inhomogeneity

$$f(x) = \begin{cases} 3, & \text{if } 0 < x < \frac{1}{2}, \\ -1, & \text{if } \frac{1}{2} \leq x < 1. \end{cases}$$

(a) Derive the weak form of the BVP.

Hint: In general, you may use the integration by parts formula

$$\int_a^b \phi(x)\psi'(x)\,\mathrm{d}x = \phi\psi|_a^b - \int_a^b \phi'(x)\psi(x)\,\mathrm{d}x$$

on intervals where ϕ and ψ are continuously differentiable.

(b) Determine the solution of the weak problem.

Hint: Try a piecewise polynomial function with suitable degree. Which conditions does the function has to fulfill to be a solution to the weak problem?

(c) Is the solution of the weak problem also a classical solution of the original BVP?

Exercise 6.2: Experimental convergence

Assume we are given a PDE on the domain $[x_L, x_R] \subset \mathbb{R}$, and a numerical scheme computing the discrete solution u_h. The domain is discretized into $N+1$ intervals and the the meshsize h is related to N by $h = \frac{x_R - x_L}{N+1}$. We assume that the PDE is well-posed and denote by u its exact solution.

(a) We assume that the scheme is convergent of order p with respect to the system maximum norms: there exists c independent of h such that

$$\|u - u_h\|_\infty \leq ch^p$$

holds for all $N \geq N_0$ for some $N_0 \in \mathbb{N}$. Show that this implies the existence of a constant $\tilde{c} \in \mathbb{R}$ with

$$\|u_h - u_{h/2}\|_\infty \leq \tilde{c}h^p \tag{1}$$

holds for all $N \geq N_0$ for some $N_0 \in \mathbb{N}$.

(b) We assume that the difference $u_h - u$ can be written

$$u_h(x) - u(x) = c(x)h^p + o(h^{p+1})$$

for a function $c(x)$ independent of h. How can you compute the convergence order p of a scheme when you don't know the exact solution?

Exercise 6.3: Implementation of theta-schemes

Consider the 2D heat equation with homogeneous Dirichlet boundary conditions

$$\begin{cases} u_t - \Delta u = f & \text{in } (0,T) \times \Omega, \\ u(t,x) = 0 & \text{for all } (t,x) \in (0,T) \times \partial\Omega, \\ u(0,x) = 0 & \text{for all } x \in \Omega, \end{cases} \tag{2}$$

the spatial domain is $\Omega = (0,1)^2$, the final time $T = 0.05$, and the source term $f \colon \Omega \to \mathbb{R}$ is defined via

$$f(x,y) = \begin{cases} 1000, & \text{if } 0.05 \leq (x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 \leq 0.07, \\ 0, & \text{otherwise.} \end{cases}$$

The goal of this exercise is to discretize this PDE using the theta scheme. With a uniform grid in space with grid size $h = \frac{1}{N+1}$, $N \in \mathbb{N}^*$, and a time discretization with constant step size $\tau = \frac{T}{M}$, $M \in \mathbb{N}^*$, the theta scheme for (2) may be written as

$$\frac{1}{\tau}(u_h^{k+1} - u_h^k) + \theta L_h u_h^{k+1} + (1-\theta) L_h u_h^k = f_h$$

for $k = 0, \ldots, M-1$.

(a) Assuming that N is given, how should M be chosen to ensure the stability of the scheme?

(b) Write a function `get_fh` which returns the source f evaluated on the grid points. The resulting vector should have length N^2.

(c) Write a function `advance_time` which computes one time step of the theta scheme.

(d) Complete the function `my_driver` to update the full solution at each time step.

(e) Plot the solution at final time for the following values:

- $N = 80, M = 1400, \theta = 0$,
- $N = 80, M = 700, \theta = 0.5$,
- $N = 80, M = 80, \theta = 1$.

Are the choices of M consistent with your answer to (a)?