Technische Universität Berlin Winter term 2025/26
Institute of Mathematics
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Numerical Mathematics II for Engineers
Tutorial 11

Topics : Numerical fluxes, 1D FVM, method of characteristics

Discussion in the tutorials of the week 19 – 23 January 2026

Disclaimers:

- To test your code, we provide tests. This should help you to locate precisely your errors, and code more efficiently. To run the test, you need to install pytest.
- Exercises should be solved in **fixed groups of 3 students**.
- Hand in the solution in **one folder** labeled **hw[hw_number]_group[group_number]** and containing:
  - **One pdf** for the theoretical questions and comments on the numerical results,
  - **One python file per programming exercise**.
  - Write the group number and all names of your members **in each file**.

Exercise 11.1: Numerical fluxes

(a) Show that for a linear advection equation with constant advection velocity $a \in \mathbb{R}^*$, the LF-DG flux and the LLF flux coincide with the upwind flux.

(b) We consider the LF-FV flux and the LW flux for the linear advection equation with velocity $a \in \mathbb{R}^*$, and assume that the time step Δt and meshsize h satisfy the CFL condition $\Delta t < h/|a|$. Are the LF-FV flux and the LW flux monotone?

Exercise 11.2: Implementation of 1D FVM

We solve the linear advection equation

$$u_t + au_x = 0, \quad a > 0 \text{ constant}, \tag{1}$$

using 3 different schemes:

(I) Upwind,
(II) LF-FV,
(III) Lax-Wendroff.

Consider the interval [0,1] and periodic boundary conditions. Use $a = 1$ and choose the final time $T = 1$. Then the exact solution $u(T, \cdot)$ coincides with the initial data u_0.

We will consider two different initial data:

- Continuous initial data

$$u_0(x) = \sin(2\pi x).$$

- Discontinuous initial data

$$u_0(x) = \begin{cases} 0, & x < 0.25 \text{ or } x > 0.75, \\ 1, & \text{otherwise.} \end{cases}$$

The code is organized as follows: first, generate the grid, then start the time stepping. At each time step, update the boundary conditions, then compute the time step using the CFL condition, then compute the flux, and finally advance in time using the conservative formula. Use a CFL number of 0.8 for all three methods in your tests.

(a) Fill the three functions `upwind`, `LW`, `LF_FV` computing the upwind, Lax-Wendroff and Lax-Friedrichs schemes, respectively. Test functions are provided to test the numerical fluxes.

(b) Fill the functions `my_driver` to set the boundary conditions, compute the time step and advance the solution in time. The periodic boundary conditions are implemented similarly to the finite difference code from Homework 01.

(c) We first consider the continuous initial condition

$$u_0(x) = \sin(2\pi x).$$

Run the three schemes for $N = 40, 80, 160$ and compare the errors and convergence orders. Hand in the plots for $N = 50$ and summarize your observations.

(d) We now consider the discontinuous initial condition

$$u_0(x) = \begin{cases} 0, & x < 0.25 \text{ or } x > 0.75, \\ 1, & \text{otherwise.} \end{cases}$$

Run the three schemes for $N = 100$. Which method show oscillations? For LF-FV and upwind, count how many grid points are needed to resolve the discontinuity as an indicator for how diffusive the method is. Which one of LF-FV and upwind is more diffusive?

(e) For the discontinuous initial condition, run the three schemes for $N = 40, 80, 160$ and compare the convergence orders. What do you observe, compared to the continuous initial condition? Does the behavior make sense?

Exercise 11.3: Method of characteristics for the Burgers' equation

We consider the Burgers' equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0, \quad u(0,x) = u_0(x).$$

For the following initial conditions, compute the solution at time $t = 1$ using the method of characteristics.

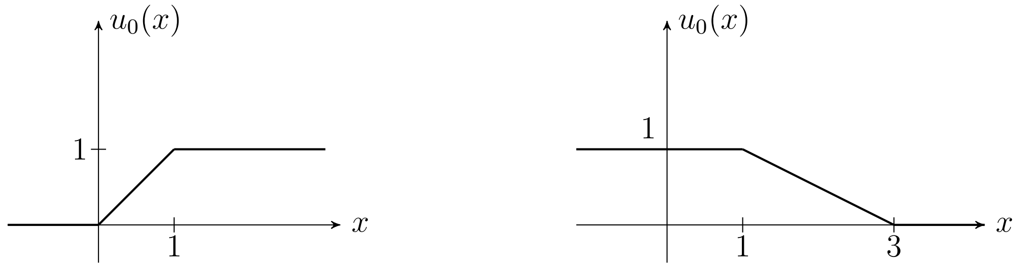

Figure 1: Initial condition $u_0(x)$: left for question (a), right for question (b).

(a) Initial condition

$$u_0(x) = \begin{cases} 0, & x \le 0, \\ x, & 0 < x < 1, \\ 1, & x \ge 1. \end{cases}$$

(b) Initial condition

$$u_0(x) = \begin{cases} 1, & x < 1, \\ (3-x)/2, & x \ge 1, \\ 0, & x \ge 1. \end{cases}$$