Technische Universität Berlin Winter term 2025/26
Institute of Mathematics
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Numerical Mathematics II for Engineers
Tutorial 7

Topics : Weak derivatives, Variational problems, Galerkin method.

Discussion in the tutorials of the week 08–12 December 2025

Disclaimers:

- To test your code, we provide tests. This should help you to locate precisely your errors, and code more efficiently. To run the test, you need to install pytest.
- Exercises should be solved in **fixed groups of 3 students**.
- Hand in the solution in **one folder** labeled **hw[hw_number]_group[group_number]** and containing:
  - **One pdf** for the theoretical questions and comments on the numerical results,
  - **One python file per programming exercise**.
  - Write the group number and all names of your members **in each file**.

Exercise 7.1: Weak derivatives

Let $\Omega = (0,1)$. Determine whether the following functions defined in $L^2(\Omega)$ are weakly differentiable, and if so, calculate the weak derivative.

(a) $f(x) = \begin{cases} 4\left(x - \frac{1}{4}\right), & \text{if } \frac{1}{4} < x \leq \frac{1}{2}, \\ 4\left(\frac{3}{4} - x\right), & \text{if } \frac{1}{2} < x < \frac{3}{4}, \\ 0, & \text{otherwise .} \end{cases}$

(b) $g(x) = \begin{cases} 0, & \text{if } 0 < x \leq \frac{1}{2}, \\ 1, & \text{if } \frac{1}{2} < x < 1 \end{cases}$

Bonus question: Draw the graph of f and g.

Hint: You might need the Dirac "function" $\delta(\cdot)$, defined by

$$\delta(x) = \begin{cases} \infty, & \text{if } \quad x = 0, \\ 0, & \text{else.} \end{cases}$$

More precisely, we call it the δ *distribution*, as it is not a function in the classical sense. The Dirac distribution is not in $L^2(\Omega)$ (you can prove this!), and has the following property:

$$\int\limits_{\Omega} \delta(x)\phi(x)\,\mathrm{d}x = \phi(0), \qquad \forall \phi \in C_c^\infty(\Omega).$$

Exercise 7.2: Variational problems

Let $\Omega \subset \mathbb{R}^n$ with $\partial\Omega$ piecewise smooth, $\Gamma \subset \partial\Omega$, $\Gamma_n = \partial\Omega \backslash \Gamma$. We introduce the functional space

$$V_0 = H_0^1(\Omega) := \{v \in H^1(\Omega) | v = 0 \text{ on } \Gamma\},$$

the linear form

$$F(v) = \int_\Omega f v \mathrm{d}x + \int_{\Gamma_n} g v \mathrm{d}s \quad \text{for some } f \in L^2(\Omega),$$

and the bilinear forms

$$a(u, v) = \int_\Omega h \nabla u \cdot \nabla v + c u v \, \mathrm{d}x, \tag{1}$$

$$a(u, v) = \int_\Omega h \nabla u \cdot \nabla v \mathrm{d}x + \int_{\Gamma_n} \alpha u v \, \mathrm{d}s, \tag{2}$$

$$a(u, v) = \int_\Omega h \nabla u \cdot \nabla v + [\mathbf{b} \cdot \nabla u] v \, \mathrm{d}x, \tag{3}$$

with the functions $h, c, f \colon \Omega \to \mathbb{R}$, $\mathbf{b} \colon \Omega \to \mathbb{R}^n$ and $\alpha, g \colon \Gamma_n \to \mathbb{R}$.

For each bilinear form, we consider the variational problem: "Seek $u \in V_0$ such that $a(u, v) = F(v)$ for all $v \in V_0$". Assuming some extra regularity, derive their corresponding elliptic PDE and boundary conditions for u.

Hint: Use Green's identity and/or the divergence theorem. We also assume that we can use the fundamental lemma of calculus of variation for $\Omega \in \mathbb{R}^n$, and that the considered functions have enought continuity to apply this lemma.

Exercise 7.3: Galerkin method

In this exercise we introduce a showcase for Galerkin methods in 1D, using polynomials as trial and test functions.

Let $\Omega = (-1, 1)$. We want to solve the Poisson equation with homogeneous Dirichlet boundary conditions

$$\begin{cases} -u''(x) = f(x) & \text{for } x \in \Omega := (-1, 1), \\ u(-1) = u(1) = 0. \end{cases} \tag{4}$$

As you have seen in the lecture, the weak formulation of the problem (4) is

$$\text{Find } u \in V \text{ such that } a(u, v) = F(v) \text{ for all } v \in V, \tag{5}$$

with

$$V = H_0^1(\Omega), \quad a(u, v) = \int_\Omega u'(x) v'(x) dx, \quad F(v) = \int_\Omega f(x) v(x) dx.$$

For this Galerkin method, we consider the space of polynomials

$$\begin{aligned} V_n &:= \operatorname{span}\{\phi_1, \ldots, \phi_n\}, \quad \text{where} \\ \phi_k(x) &:= (1 - x^2) x^k \quad \text{for } k = 1, \ldots, n \end{aligned}$$

(a) What is the difference between this Galerkin method and the linear finite elements in 1D seen in the lecture?

(b) Does each space V_n satisfy the homogeneous Dirichlet boundary condition? Write down the Galerkin equation of (5). Using the specified trial and test space V_n, write the corresponding linear system of equation.

(c) The Galerkin matrix $A_n = [a(\phi_i, \phi_j)]_{ij} \in \mathbb{R}^{n,n}$ and the vector $f_n = [F(v_i)]_i \in \mathbb{R}^n$ are given by

$$(A_n)_{i,k} = \int_\Omega \phi_i'(x)\phi_k'(x)dx, \quad (f_n)_k = \int f(x)\phi_k(x)dx,$$

Write a function `get_galerkin_matrix` that returns the **dense** matrix $A_n \in \mathbb{R}^{n,n}$, and a function `get_rhs` that returns the vector $f_n \in \mathbb{R}^n$.

Hint: Since the number n of basis functions stays small in this example, you can use `for` loops for constructing A_n and f_n. You can use the function `scipy.integrate.fixed_quad` with an order of $n+2$ in Python to compute the integrals.

(d) Complete the function `my_driver` to create the solution $u_n \in V_n$.

Hint: Remember that the solution can be written as $u_n(x) = \sum_{i=1}^n \alpha_i \phi_i(x)$.

(e) Use your code to solve the problem with the source $f = \pi^2 \sin(\pi x)$ for $n = 1, 2, 5, 10$. Create a plot comparing your approximate solution and the exact solution on the equidistant grid with step size $h = \frac{1}{N}$, with $N = 100$.

Remark: The grid is now an input of the function `my_driver`, you can create it in the `__main__` part of the file. Think about why we do not define it in `my_driver` anymore!

(f) Compute the L^2 error

$$e_n := \left(\int_{-1}^{1} |u_n(x) - u(x)|^2 dx \right)^{\frac{1}{2}}$$

for $n = 1, \ldots, 10$, and plot e_n against n in a `semilogy plot`. To compute the integral, you can use the build in function `scipy.integrate.fixed_quad` in Python (with an order of $n+2$ again).