

Numerical Mathematics II for Engineers Tutorial 5

Topics : Norms, consistency, general boundary conditions

Discussion in the tutorials of the week 24–28 November 2025

Disclaimers:

- To test your code, we provide tests. This should help you to locate precisely your errors, and code more efficiently. To run the test, you need to install pytest.
- Exercises should be solved in **fixed groups of 3 students**.
- Hand in the solution in **one folder** labeled **hw[hw_number]_group[group_number]** and containing:
 - **One pdf** for the theoretical questions and comments on the numerical results,
 - **One python file per programming exercise**.
 - Write the group number and all names of your members **in each file**.

Exercise 5.1: Norms

In the lecture we have seen that, given a norm $\|\cdot\|$ on \mathbb{R}^N , we may define a corresponding induced norm $\|A\|$ on $\mathbb{R}^{N \times N}$ via

$$\|A\| := \sup_{v \in \mathbb{R}^N \setminus \{0\}} \frac{\|Av\|}{\|v\|} = \sup_{v \in \mathbb{R}^N, \|v\|=1} \|Av\|.$$

- a) Show that the norm induced by $\|\cdot\|_\infty$ satisfies

$$\|A\|_\infty = \max_{i \in \{1, \dots, N\}} \sum_{j=1}^N |a_{ij}| \quad \text{for all } A \in \mathbb{R}^{N \times N}.$$

- b) You may know from linear algebra that all norms on \mathbb{R}^N are equivalent: for any two norms $\|\cdot\|_a, \|\cdot\|_b$, there exist two positive constants m, M such that

$$m\|w\|_a \leq \|w\|_b \leq M\|w\|_a, \quad \text{for all } w \in \mathbb{R}^n.$$

This is also true for operator norms. For the particular case of $\|A\|_\infty$ and $\|A\|_2$, there holds

$$\|A\|_2 \leq \sqrt{N} \|A\|_\infty$$

for all $A \in \mathbb{R}^{N \times N}$ (we do **not** ask you to prove this!).

Does this imply that a finite difference scheme $L_h u_h = f_h$ (with $L_h \in \mathbb{R}^{N \times N}$ and $h = \frac{1}{N+1}$) stable with respect to the system of norms $\|\cdot\|_{\infty, h}$ is also stable with respect to the system of norms $\|\cdot\|_{2, h}$?

Exercise 5.2: Consistency of 2D schemes

We study the consistency of two FD schemes approximating the 2D Poisson equation with homogeneous Dirichlet boundary conditions. The first scheme is the five-point stencil that was introduced in the lecture

$$-\Delta u \approx L_h^{(5)} u = -\frac{1}{h^2} (u(x_1 \pm h, x_2) + u(x_1, x_2 \pm h) - 4u(x_1, x_2))$$

The second scheme is the so-called "compact nine-point stencil"

$$\begin{aligned} -\Delta u \approx L_h^{(9)} u = & -\frac{1}{6h^2} (4u(x_1 + h, x_2) + 4u(x_1, x_2 + h) + 4u(x_1 - h, x_2) + 4u(x_1, x_2 - h) \\ & + u(x_1 + h, x_2 + h) + u(x_1 - h, x_2 - h) + u(x_1 + h, x_2 - h) + u(x_1 - h, x_2 + h) \\ & - 20u(x_1, x_2)). \end{aligned}$$

The compact nine-point stencil can be written in the compact way

$$-\Delta u \approx L_h^{(9)} u = -\frac{1}{6h^2} (4u(x_1 \pm h, x_2) + 4u(x_1, x_2 \pm h) + u(x_1 \pm h, x_2 \pm h) - 20u(x_1, x_2)).$$

- (a) Show that the five-point stencil is consistent of order 2 with respect to the maximum norm.

Hint: Use the auxiliary functions $x \mapsto u(x, y)$ and $y \mapsto u(x, y)$, for fixed y and x respectively.

- (b) Show that the compact nine-point stencil is also consistent of order at least 2.

Hint: Show that the nine-point stencil is an approximation of order at least 2 of the Laplacian. Consider each derivative separately and see which terms cancel out.

Remark: one can show that the compact nine-point stencil is consistent of order four when using the modified right-hand side (we do **not** ask you to prove this!)

$$f_h(x_1, x_2) = f(x_1, x_2) + \frac{1}{12} h^2 L_h^{(5)} f(x_1, x_2).$$

Exercise 5.3: General boundary conditions

We consider the linear second-order partial differential equation on (a, b) with general boundary conditions

$$-u''(x) + g(x)u(x) = f(x), \tag{1}$$

$$\alpha u(a) + \beta u'(a) = 1, \quad \delta u(b) + \gamma u'(b) = 1, \tag{2}$$

where f, g are real functions, and $\alpha, \beta, \delta, \gamma$ are real numbers. The aim of this exercise is to implement a FD solver for this generic equation. The template `second_order_template.py` is provided.

Note: For consistency with the previous programming exercises, the intervals (a, b) is discretized into $N + 1$ sub-intervals, so $x_0 = a$ and $x_{N+1} = b$.

- (a) Four test-cases are implemented in the `get_testproblem` function. Specify for each test case the type of boundary conditions (Dirichlet, Neumann or Robin). Do you expect the linear problem to be solvable for all test cases?

- (b) Write the discrete system corresponding to the PDE (1)-(2). Use the centered difference quotient for the inner points, and the one sided difference quotient for the approximation of the derivative of u at the boundary points (see Lecture 07).

Hint: As for Neumann boundary conditions, the boundary points u_0, u_{N+1} are now unknowns and should be obtained by solving the linear system. The matrix of the discretized linear system has shape $(N + 2) \times (N + 2)$.

- (c) Implement the matrix and right-hand side in the function `get_rhs_diag`.
- (d) Solve the system for each test case for $N = 80$. Plot the exact solution and the solution of the linear system. Are your results consistent with the question (a)?