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CHAPTER 7

Curve Fitting

Chapter Outline

7.1 INTRODUCTION

Curve fitting is the process of finding the 'best-fit' curve for a given set of data. It is the representation of the relationship between two variables by means of an algebraic equation. On the basis of this mathematical equation, predictions can be made in many statistical problems.

Suppose a set of n points of values $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ of the two variables x and y are given. These values are plotted on a rectangular coordinate system, i.e., the xy-plane. The resulting set of points is known as a *scatter diagram* (Fig. 7.1). The scatter diagram exhibits the trend and it is possible to visualize a smooth curve approximating the data. Such a curve is known as an *approximating curve*.

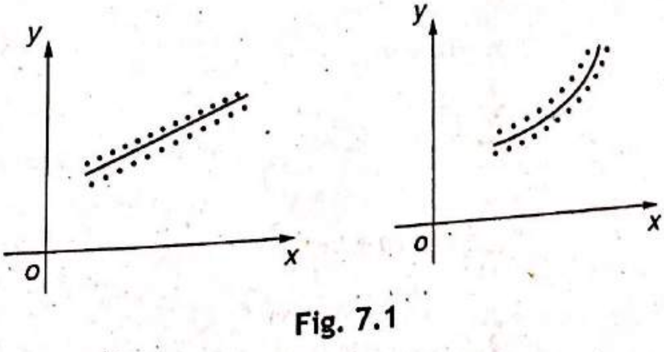

Fig. 7.1

7.2 LEAST SQUARE METHOD

From a scatter diagram, generally, more than one curve may be seen to be appropriate to the given set of data. The method of least squares is used to find a curve which passes through the maximum number of points.

Let $P(x_i, y_i)$ be a point on the scatter diagram (Fig. 7.2). Let the ordinate at P meet the curve $y = f(x)$ at Q and the x-axis at M.

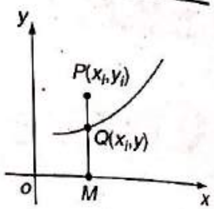

Fig. 7.2

Distance
$$QP = MP - MQ$$
$$= y_i - y$$
$$= y_i - f(x_i)$$

The distance QP is known as *deviation*, *error*, or *residual* and is denoted by d_i. It may be positive, negative, or zero depending upon whether P lies above, below, or on the curve. Similar residuals or errors corresponding to the remaining $(n-1)$ points may be obtained. The sum of squares of residuals, denoted by E, is given as

$$E = \sum_{i=1}^{n} d_i^2 = \sum_{i=1}^{n} [y_i - f(x_i)]^2$$

If $E = 0$ then all the n points will lie on $y = f(x)$. If $E \neq 0$, $f(x)$ is chosen such that E is minimum, i.e., the best fitting curve to the set of points is that for which E is minimum. This method is known as the least square method. This method does not attempt to determine the form of the curve $y = f(x)$ but it determines the values of the parameters of the equation of the curve.

7.3 FITTING OF LINEAR CURVES

Let (x_i, y_i), $i = 1, 2, \ldots, n$ be the set of n values and let the relation between x and y be $y = a + bx$. The constants a and b are selected such that the straight line is the best fit to the data.

The residual at $x = x_i$ is

$$d_i = y_i - f(x_i)$$
$$= y_i - (a + bx_i) \qquad i = 1, 2, \ldots, n$$
$$E = \sum_{i=1}^{n} d_i^2$$
$$= \sum_{i=1}^{n} \left[y_i - (a + bx_i)\right]^2$$
$$= \sum_{i=1}^{n} (y_i - a - bx_i)^2$$

For E to be minimum,

(i) $\dfrac{\partial E}{\partial a} = 0$

$$\sum_{i=1}^{n} 2(y_i - a - bx_i)(-1) = 0$$
$$\sum_{i=1}^{n} (y_i - a - bx_i) = 0$$
$$\sum_{i=1}^{n} y_i = a\sum_{i=1}^{n} 1 + b\sum_{i=1}^{n} x_i$$
$$\sum y = na + b\sum x$$

(ii) $\dfrac{\partial E}{\partial b} = 0$

$$\sum_{i=1}^{n} 2(y_i - a - bx_i)(-x_i) = 0$$
$$\sum_{i=1}^{n} (x_i y_i - ax_i - bx_i^2) = 0$$
$$\sum_{i=1}^{n} x_i y_i = a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} x_i^2$$
$$\sum xy = a\sum x + b\sum x^2$$

These two equations are known as *normal equations*. These equations can be solved simultaneously to give the best values of a and b. The best fitting straight line is obtained by substituting the values of a and b in the equation $y = a + bx$.

Example 1

Fit a straight line to the following data:

x	1	2	3	4	6	8
y	2.4	3	3.6	4	5	6

Solution

Let the straight line to be fitted to the data be

$$y = a + bx$$

The normal equations are

$$\sum y = na + b\sum x \qquad \ldots(1)$$
$$\sum xy = a\sum x + b\sum x^2 \qquad \ldots(2)$$

Here, $n = 6$

x	y	x^2	xy
1	2.4	1	2.4
2	3	4	6
3	3.6	9	10.8
4	4	16	16
6	5	36	30
8	6	64	48
$\sum x = 24$	$\sum y = 24$	$\sum x^2 = 130$	$\sum xy = 113.2$

Substituting these values in Eqs (1) and (2),

$$24 = 6a + 24b \quad \text{...(3)}$$
$$113.2 = 24a + 130b \quad \text{...(4)}$$

Solving Eqs (3) and (4),

$$a = 1.9764$$
$$b = 0.5059$$

Hence, the required equation of the straight line is

$$y = 1.9764 + 0.5059x$$

Note $\sum x, \sum y, \sum x^2, \sum xy$ can be directly obtained with the help of scientific calculator.

Example 2

Fit a straight line to the following data. Also, estimate the value of y at $x = 2.5$.

x	0	1	2	3	4
y	1	1.8	3.3	4.5	6.3

Solution

Let the straight line to be fitted to the data be

$$y = a + bx$$

The normal equations are

$$\sum y = na + b\sum x \quad \text{...(1)}$$
$$\sum xy = a\sum x + b\sum x^2 \quad \text{...(2)}$$

Here, $n = 5$

x	y	x^2	xy
0	1	0	0
1	1.8	1	1.8
2	3.3	4	6.6
3	4.5	9	13.5
4	6.3	16	25.2
$\sum x = 10$	$\sum y = 16.9$	$\sum x^2 = 30$	$\sum xy = 47.1$

Substituting these values in Eqs (1) and (2),

$$16.9 = 5a + 10b \quad \text{...(3)}$$
$$47.1 = 10a + 30b \quad \text{...(4)}$$

Solving Eqs (3) and (4),

$$a = 0.72$$
$$b = 1.33$$

Hence, the required equation of the straight line is

$$y = 0.72 + 1.33x$$

At $x = 2.5$, $\quad y(2.5) = 0.72 + 1.33(2.5) = 4.045$

Example 3

A simply supported beam carries a concentrated load P(lb) at its midpoint. Corresponding to various values of P, the maximum deflection Y(in) is measured. The data is given below:

P	100	120	140	160	180	200
Y	0.45	0.55	0.60	0.70	0.80	0.85

Find a law of the form $Y = a + bP$ using the least square method.

[Summer 2015]

Solution

Let the straight line to be fitted to the data be

$$Y = a + bP$$

The normal equations are

$$\sum Y = na + b\sum P \quad \text{...(1)}$$

$$\sum PY = a\sum P + b\sum P^2 \quad ...(2)$$

Here, $n = 6$

P	Y	P^2	PY
100	0.45	10000	45
120	0.55	14400	66
140	0.60	19600	84
160	0.70	25600	112
180	0.80	32400	144
200	0.85	40000	170
$\sum P = 900$	$\sum Y = 3.95$	$\sum P^2 = 142000$	$\sum PY = 621$

Substituting these values in Eqs (1) and (2),

$$3.95 = 6a + 900\,b \quad ...(3)$$

$$621 = 900\,a + 142000\,b \quad ...(4)$$

Solving Eqs (3) and (4),

$$a = 0.0476$$

$$b = 0.0041$$

Hence, the required equation of the straight line is

$$Y = 0.0476 + 0.0041\,P$$

Example 4

Fit a straight line to the following data. Also, estimate the value of y at $x = 70$.

x	71	68	73	69	67	65	66	67
y	69	72	70	70	68	67	68	64

Solution

Since the values of x and y are larger, we choose the origin for x and y at 69 and 67 respectively,

Let $X = x - 69$ and $Y = y - 67$

Let the straight line to be fitted to the data be

$$Y = a + bX$$

The normal equations are

$$\sum Y = na + b\sum X \quad ...(1)$$

$$\sum XY = a\sum X + b\sum X^2 \quad ...(2)$$

Here, $n = 8$

x	y	X	Y	X^2	XY
71	69	2	2	4	4
68	72	−1	5	1	−5
73	70	4	3	16	12
69	70	0	3	0	0
67	68	−2	1	4	−2
65	67	−4	0	16	0
66	68	−3	1	9	−3
67	64	−2	−3	4	6
		$\sum X = -6$	$\sum Y = 12$	$\sum X^2 = 54$	$\sum XY = 12$

Substituting these values in Eqs (1) and (2),

$$12 = 8a - 6b \quad ...(3)$$

$$12 = -6a + 54b \quad ...(4)$$

Solving Eqs (3) and (4),

$$a = 1.8182$$

$$b = 0.4242$$

Hence, the required equation of the straight line is

$$Y = 1.8182 + 0.4242X$$

$$y - 67 = 1.8182 + 0.4242(x - 69)$$

$$y = 0.4242x + 39.5484$$

$$y(x = 70) = 0.4242(70) + 39.5484 = 69.2424$$

Note Since $\sum x$, $\sum y$, $\sum x^2$, $\sum xy$ can be directly obtained with the help of scientific calculator, the problem can be solved without shifting the origin.

Example 5

Fit a straight line to the following data taking x as the dependent variable.

x	1	3	4	6	8	9	11	14
y	1	2	4	4	5	7	8	9

Solution

If x is considered the dependent variable and y the independent variable, the equation of the straight line to be fitted to the data is

$$x = a + by$$

The normal equations are

$$\sum x = na + b\sum y \quad \ldots(1)$$

$$\sum xy = a\sum y + b\sum y^2 \quad \ldots(2)$$

Here, $n = 8$

x	y	y^2	xy
1	1	1	1
3	2	4	6
4	4	16	16
6	4	16	24
8	5	25	40
9	7	49	63
11	8	64	88
14	9	81	126
$\sum x = 56$	$\sum y = 40$	$\sum y^2 = 256$	$\sum xy = 364$

Substituting these values in Eqs (1) and (2),

$$56 = 8a + 40b \quad \ldots(3)$$

$$364 = 40a + 256b \quad \ldots(4)$$

Solving Eqs (3) and (4),

$$a = -0.5$$

$$b = 1.5$$

Hence, the required equation of the straight line is

$$x = -0.5 + 1.5y$$

Example 6

If P is the pull required to lift a load W by means of a pulley block, find a linear law of the form $P = mW + c$ connecting P and W using the following data:

P	12	15	21	25
W	50	70	100	120

where P and W are taken in kg-wt. Compute P when $W = 150$ kg.

Solution

Let the linear curve (straight line) fitted to the data be

$$P = mW + c = c + mW$$

The normal equations are

$$\sum P = nc + m\sum W \quad \ldots(1)$$

$$\sum PW = c\sum W + m\sum W^2 \quad \ldots(2)$$

Here, $n = 4$

P	W	W^2	PW
12	50	2500	600
15	70	4900	1050
21	100	10000	2100
25	120	14400	3000
$\sum P = 73$	$\sum W = 340$	$\sum W^2 = 31800$	$\sum PW = 6750$

Substituting these values in Eqs (1) and (2),

$$73 = 4c + 340\,m \quad \ldots(3)$$

$$6750 = 340\,c + 31800\,m \quad \ldots(4)$$

Solving Eqs (3) and (4),

$$c = 2.2759$$

$$m = 0.1879$$

Hence, the required equation of the straight line is

$$P = 0.1879\,W + 2.2759$$

When $W = 150$ kg,

$$P = 0.1879(150) + 2.2759 = 30.4609$$

EXERCISE 7.1

1. Fit the line of best fit to the following data:

x	0	5	10	15	20	25
y	12	15	17	22	24	30

[Ans.: $y = 0.7x + 11.28$]

2. The results of a measurement of electric resistance R of a copper bar at various temperatures t°C are listed below:

t°C	19	25	30	36	40	45	50
R	76	77	79	80	82	83	85

Find a relation $R = a + bt$ where a and b are constants to be determined.

[Ans.: $R = 70.0534 + 0.2924t$]

3. Fit a straight line to the following data:

x	1.53	1.78	2.60	2.95	3.42
y	33.50	36.30	40.00	45.85	53.40

$\left[\textbf{Ans.: } y = 19 + 9.7x\right]$

4. Fit a straight line to the following data:

x	100	120	140	160	180	200
y	0.45	0.55	0.60	0.70	0.80	0.85

$\left[\textbf{Ans.: } y = 0.0475 + 0.00407x\right]$

5. Find the relation of the type $R = aV + b$, when some values of R and V obtained from an experiment are

V	60	65	70	75	80	85	90
R	109	114	118	123	127	130	133

$\left[\textbf{Ans.: } R = 0.8071V + 61.4675\right]$

7.4 FITTING OF QUADRATIC CURVES

Let (x_i, y_i), $i = 1, 2, \ldots, n$ be the set of n values and let the relation between x and y be $y = a + bx + cx^2$. The constants a, b, and c are selected such that the parabola is the best fit to the data. The residual at $x = x_i$ is

$$d_i = y_i - f(x_i)$$
$$= y_i - \left(a + bx_i + cx_i^2\right)$$

$$E = \sum_{i=1}^{n} d_i^2$$
$$= \sum_{i=1}^{n} \left[y_i - \left(a + bx_i + cx_i^2\right)\right]^2$$
$$= \sum_{i=1}^{n} \left(y_i - a - bx_i - cx_i^2\right)^2$$

For E to be minimum,

(i) $\dfrac{\partial E}{\partial a} = 0$

$$\sum_{i=1}^{n} 2\left(y_i - a - bx_i - cx_i\right)(-1) = 0$$

$$\sum_{i=1}^{n} \left(y_i - a - bx_i - cx_i\right) = 0$$
$$\sum_{i=1}^{n} y_i = a\sum_{i=1}^{n} 1 + b\sum_{i=1}^{n} x_i + c\sum_{i=1}^{n} x_i^2$$
$$\sum y = na + b\sum x + c\sum x^2$$

(ii) $\dfrac{\partial E}{\partial b} = 0$

$$\sum_{i=1}^{n} 2(y_i - a - bx_i - cx_i)(-x_i) = 0$$
$$\sum_{i=1}^{n} \left(x_i y_i - ax_i - bx_i^2 - cx_i^3\right) = 0$$
$$\sum_{i=1}^{n} x_i y_i = a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} x_i^2 + c\sum_{i=1}^{n} x_i^3$$
$$\sum xy = na + b\sum x^2 + c\sum x^3$$

(iii) $\dfrac{\partial E}{\partial c} = 0$

$$\sum_{i=1}^{n} 2(y_i - a - bx_i - cx_i^2)(x_i^2) = 0$$
$$\sum_{i=1}^{n} x_i^2 y_i - ax_i^2 - bx_i^3 - cx_i^4 = 0$$
$$\sum_{i=1}^{n} x_i^2 y_i = a\sum_{i=1}^{n} x_i^2 + b\sum_{i=1}^{n} x_i^3 + c\sum_{i=1}^{n} x_i^4$$
$$\sum x^2 y = a\sum x^2 + b\sum x^3 + c\sum x^4$$

These equations are known as *normal equations*. These equations can be solved simultaneously to give the best values of a, b, and c. The best fitting parabola is obtained by substituting the values of a, b, and c in the equation $y = a + bx + cx^2$.

Example 1

Fit a least squares quadratic curve to the following data:

x	1	2	3	4
y	1.7	1.8	2.3	3.2

Estimate $y(2.4)$.

Solution

Let the equation of the least squares quadratic curve (parabola) be $y = a + bx + cx^2$.
The normal equations are

$$\sum y = na + b\sum x + c\sum x^2 \qquad \ldots(1)$$

$$\sum xy = a\sum x + b\sum x^2 + c\sum x^3 \qquad \ldots(2)$$

$$\sum x^2 y = a\sum x^2 + b\sum x^3 + c\sum x^4 \qquad \ldots(3)$$

Here, $n = 4$

x	y	x^2	x^3	x^4	xy	x^2y
1	1.7	1	1	1	1.7	1.7
2	1.8	4	8	16	3.6	7.2
3	2.3	9	27	81	6.9	20.7
4	3.2	16	64	256	12.8	51.2
$\Sigma x = 10$	$\Sigma y = 9$	$\Sigma x^2 = 30$	$\Sigma x^3 = 100$	$\Sigma x^4 = 354$	$\Sigma xy = 25$	$\Sigma x^2 y = 80.8$

Substituting these values in Eqs (1), (2), and (3),

$$9 = 4a + 10b + 30c \qquad \ldots(4)$$

$$25 = 10a + 30b + 100c \qquad \ldots(5)$$

$$80.8 = 30a + 100b + 354c \qquad \ldots(6)$$

Solving Eqs (4), (5), and (6),

$$a = 2$$

$$b = -0.5$$

$$c = 0.2$$

Hence, the required equation of least squares quadratic curve is

$$y = 2 - 0\cdot 5x + 0\cdot 2x^2$$

$$y(2\cdot 4) = 2 - 0\cdot 5(2\cdot 4) + 0\cdot 2(2.4)^2 = 1\cdot 952$$

Note Σx, Σy, Σx^2, Σx^3, Σx^4, Σxy, $\Sigma x^2 y$ can be directly obtained with the help of scientific calculator.

Example 2

Fit a second-degree polynomial using least square method to the following data:

x	0	1	2	3	4
y	1	1.8	1.3	2.5	6.3

[Summer 2015]

Solution

Let the equation of the least squares quadratic curve be $y = a + bx + cx^2$. The normal equations are

$$\sum y = na + b\sum x + c\sum x^2 \qquad \ldots(1)$$

$$\sum xy = a\sum x + b\sum x^2 + c\sum x^3 \qquad \ldots(2)$$

$$\sum x^2 y = a\sum x^2 + b\sum x^3 + c\sum x^4 \qquad \ldots(3)$$

Here, $n = 5$

x	y	x^2	x^3	x^4	xy	x^2y
0	1	0	0	0	0	0
1	1.8	1	1	1	1.8	1.8
2	1.3	4	8	16	2.6	5.2
3	2.5	9	27	81	7.5	22.5
4	6.3	16	64	256	25.2	100.8
$\Sigma x = 10$	$\Sigma y = 12.9$	$\Sigma x^2 = 30$	$\Sigma x^3 = 100$	$\Sigma x^4 = 354$	$\Sigma xy = 37.1$	$\Sigma x^2 y = 130.3$

Substituting these values in Eqs (1), (2), and (3),

$$12.9 = 5a + 10b + 30c \qquad \ldots(4)$$

$$37.1 = 10a + 30b + 100c \qquad \ldots(5)$$

$$130.3 = 30a + 100b + 354c \qquad \ldots(6)$$

Solving Eqs (4), (5), and (6),

$$a = 1.42$$

$$b = -1.07$$

$$c = 0.55$$

Hence, the required equation of the least squares quadratic curve is

$$y = 1.42 - 1.07x + 0.55x^2$$

Example 3

By the method of least squares, fit a parabola to the following data:

x	1	2	3	4	5
y	5	12	26	60	97

Also, estimate y at x = 6.

Solution

Let the equation of the parabola be $y = a + bx + cx^2$. The normal equations are

$$\sum y = na + b\sum x + c\sum x^2 \qquad \ldots(1)$$

$$\sum xy = a\sum x + b\sum x^2 + c\sum x^3 \qquad \ldots(2)$$

$$\sum x^2y = a\sum x^2 + b\sum x^3 + c\sum x^4 \qquad \ldots(3)$$

Here, $n = 5$

x	y	x^2	x^3	x^4	xy	x^2y
1	5	1	1	1	5	5
2	12	4	8	16	24	48
3	26	9	27	81	78	234
4	60	16	64	256	240	960
5	97	25	125	625	485	2425
$\sum x = 15$	$\sum y = 200$	$\sum x^2 = 55$	$\sum x^3 = 225$	$\sum x^4 = 979$	$\sum xy = 832$	$\sum x^2y = 3672$

Substituting these values in Eqs (1), (2), and (3),

$$200 = 5a + 15b + 55c \qquad \ldots(4)$$

$$832 = 15a + 55b + 225c \qquad \ldots(5)$$

$$3672 = 55a + 225b + 979c \qquad \ldots(6)$$

Solving Eqs (4), (5), and (6),

$$a = 10.4$$

$$b = -11.0857$$

$$c = 5.7143$$

Hence, the required equation of the parabola is

$$y = 10.4 - 11.0857\,x + 5.7143\,x^2$$

$$y(6) = 10.4 - 11.0857(6) + 5.7143(6)^2 = 149.6006$$

Example 4

Fit a second-degree parabolic curve to the following data.

x	1	2	3	4	5	6	7	8	9
y	2	6	7	8	10	11	11	10	9

Solution

Let

$$X = x - 5$$

$$Y = y - 10$$

Let the equation of the parabola be $Y = a + bX + cX^2$. ...(1)

The normal equations are

$$\sum Y = na + b\sum X + c\sum X^2$$

$$\sum XY = a\sum X + b\sum X^2 + c\sum X^3 \qquad \ldots(2)$$

$$\sum X^2Y = a\sum X^2 + b\sum X^3 + c\sum X^4 \qquad \ldots(3)$$

Here, $n = 9$

x	y	X	Y	X^2	X^3	X^4	XY	X^2Y
1	2	−4	−8	16	−64	256	32	−128
2	6	−3	−4	9	−27	81	12	−36
3	7	−2	−3	4	−8	16	6	−12
4	8	−1	−2	1	−1	1	2	−2
5	10	0	0	0	0	0	0	0
6	11	1	1	1	1	1	1	1
7	11	2	1	4	8	16	2	4
8	10	3	0	9	27	81	0	0
9	9	4	−1	16	64	256	−4	−16
		$\Sigma X = 0$	$\Sigma Y = -16$	$\Sigma X^2 = 60$	$\Sigma X^3 = 0$	$\Sigma X^4 = 708$	$\Sigma XY = 51$	$\Sigma X^2Y = -189$

Substituting these values in Eqs (1), (2), and (3),

$$-16 = 9a + 60c \qquad \ldots(4)$$

$$51 = 60b \qquad \ldots(5)$$

$$-189 = 60a + 708c \qquad \ldots(6)$$

Solving Eqs (4), (5), and (6),

$$a = 0.0043$$

$$b = 0.85$$

$$c = -0.2673$$

Hence, the required equation of the parabola is

$$Y = 0.0043 + 0.85X - 0.2673X^2$$

$$y - 10 = 0.0043 + 0.85(x-5) - 0.2673(x-5)^2$$

$$y = 10 + 0.0043 + 0.85(x-5) - 0.2673(x^2 - 10x + 25)$$

$$= 10 + 0.0043 + 0.85x - 4.25 - 0.2673x^2 + 2.673x - 6.6825$$

$$= -0.9282 + 3.523x - 0.2673x^2$$

Note Since $\Sigma x, \Sigma y, \Sigma x^2, \Sigma x^3, \Sigma x^4, \Sigma xy, \Sigma x^2y$ can be directly obtained with the help of scientific calculator, the problem can be solved without shifting the origin.

Example 5

Fit a second-degree parabola $y = a + bx^2$ to the following data:

x	1	2	3	4	5
y	1.8	5.1	8.9	14.1	19.8

Solution

Let the curve to be fitted to the data be

$$y = a + bx^2$$

The normal equations are

$$\sum y = na + b\sum x^2 \qquad \text{...(1)}$$

$$\sum x^2 y = a\sum x^2 + b\sum x^4 \qquad \text{...(2)}$$

Here, $n = 5$

x	y	x^2	x^4	x^2y
1	1.8	1	1	1.8
2	5.1	4	16	20.4
3	8.9	9	81	80.1
4	14.1	16	256	225.6
5	19.8	25	625	495
	$\Sigma y = 49.7$	$\Sigma x^2 = 55$	$\Sigma x^4 = 979$	$\Sigma x^2 y = 822.9$

Substituting these values in Eqs (1) and (2),

$$49.7 = 5a + 55b \qquad \text{...(3)}$$

$$822.9 = 55a + 979\,b \qquad \text{...(4)}$$

Solving Eqs (3) and (4),

$$a = 1.8165$$

$$b = 0.7385$$

Hence, the required equation of the curve is

$$y = 1.8165 + 0.7385\,x^2$$

Example 6

Fit a curve $y = ax + bx^2$ for the following data:

x	1	2	3	4	5	6
y	2.51	5.82	9.93	14.84	20.55	27.06

Solution

Let the curve to be fitted to the data be

$$y = ax + bx^2$$

The normal equations are

$$\sum xy = a\sum x^2 + b\sum x^3 \qquad \text{...(1)}$$

$$\sum x^2 y = a\sum x^3 + b\sum x^4 \qquad \text{...(2)}$$

x	y	x^2	x^3	x^4	xy	x^2y
1	2.51	1	1	1	2.51	2.51
2	5.82	4	8	16	11.64	23.28
3	9.93	9	27	81	29.79	89.37
4	14.84	16	64	256	59.36	237.44
5	20.55	25	125	625	102.75	513.75
6	27.06	36	216	1296	162.36	974.16
		$\Sigma x^2 = 91$	$\Sigma x^3 = 441$	$\Sigma x^4 = 2275$	$\Sigma xy = 368.41$	$\Sigma x^2 y = 1840.51$

Substituting these values in Eqs (1) and (2),

$$368 \cdot 41 = 91a + 441b \qquad \text{...(3)}$$

$$1840 \cdot 51 = 441\,a + 2275\,b \qquad \text{...(4)}$$

Solving Eqs (3) and (4),

$$a = 2.11$$

$$b = 0.4$$

Hence, the required equation of the curve is

$$y = 2 \cdot 11x + 0 \cdot 4x^2$$

EXERCISE 7.2

1. Fit a parabola to the following data:

x	–2	–1	0	1	2
y	1.0	1.8	1.3	2.5	6.3

[Ans.: $y = 1.48 + 1.13x + 0.55x^2$]

2. Fit a curve $y = ax + bx^2$ to the following data:

x	–2	–1	0	1	2
y	–72	–46	–12	35	93

[Ans.: $y = 41.1x + 2.147x^2$]

3. Fit a parabola $y = a + bx + cx^2$ to the following data:

x	0	2	5	10
y	4	7	6.4	−6

[Ans.: $y = 4.1 + 1.979x - 0.299x^2$]

4. Fit a curve $y = a_0 + a_1x + a_2x^2$ for the given data:

x	3	5	7	9	11	13
y	2	3	4	6	5	8

[Ans.: $y = 0.7897 + 0.4004x + 0.0089x^2$]

7.5 FITTING OF EXPONENTIAL AND LOGARITHMIC CURVES

Let (x_i, y_i), $i = 1, 2, \ldots, n$ be the set of n values and let the relation between x and y be $y = ab^x$.

Taking logarithm on both the sides of the equation $y = ab^x$,

$$\log_e y = \log_e a + x \log_e b$$

Putting $\log_e y = Y$, $\log_e a = A, x = X$, and $\log_e b = B$,

$$Y = A + BX$$

This is a linear equation in X and Y. The normal equations are

$$\sum Y = nA + B\sum X$$

$$\sum XY = A\sum X + B\sum X^2$$

Solving these equations, A and B, and, hence, a and b can be found. The best fitting exponential curve is obtained by substituting the values of a and b in the equation $y = ab^x$.

Similarly, the best fitting exponential curves for the relation $y = ax^b$ and $y = ae^{bx}$ can be obtained.

Example 1

Find the law of the form $y = ab^x$ to the following data:

x	1	2	3	4	5	6	7	8
y	1	1.2	1.8	2.5	3.6	4.7	6.6	9.1

Solution

$$y = ab^x$$

Taking logarithm on both the sides,

$$\log_e y = \log_e a + x\log_e b$$

Putting $\log_e y = Y$, $\log_e a = A$, $x = X$ and $\log_e b = B$,

$$Y = A + BX$$

The normal equations are

$$\sum Y = nA + B\sum X \quad \ldots(1)$$

$$\sum XY = A\sum X + B\sum X^2 \quad \ldots(2)$$

Here, $n = 8$

x	y	X	Y	X^2	XY
1	1	1	0.0000	1	0.0000
2	1.2	2	0.1823	4	0.3646
3	1.8	3	0.5878	9	1.7634
4	2.5	4	0.9163	16	3.6652
5	3.6	5	1.2809	25	6.4045
6	4.7	6	1.5476	36	9.2856
7	6.6	7	1.8871	49	13.2097
8	9.1	8	2.2083	64	17.6664
		$\sum X = 36$	$\sum Y = 8.6103$	$\sum X^2 = 204$	$\sum XY = 52.3594$

Substituting these values in Eqs (1) and (2),

$$8.6103 = 8A + 36B \quad \ldots(3)$$

$$52.3594 = 36A + 204B \quad \ldots(4)$$

Solving Eqs (3) and (4),

$$A = -0.3823$$

$$B = 0.3241$$

$$\log_e a = A$$

$$\log_e a = -0.3823$$

$$a = 0.6823$$

$$\log_e b = B$$

$$\log_e b = 0.3241$$

$$b = 1.3828$$

Hence, the required law is

$$y = 0.6823\,(1.3828)^x$$

Example 2

Fit a curve of the form $y = ab^x$ to the following data by the method of least squares:

x	1	2	3	4	5	6	7
y	87	97	113	129	202	195	193

Solution

$$y = ab^x$$

Taking logarithm on both the sides,

$$\log_e y = \log_e a + x \log_e b$$

Putting $\log_e y = Y$, $\log_e a = A$, $x = X$ and $\log_e b = B$,

$$Y = A + BX$$

The normal equations are

$$\sum Y = nA + B\sum X \qquad \text{...(1)}$$

$$\sum XY = A\sum X + B\sum X^2 \qquad \text{...(2)}$$

Here, $n = 7$

x	y	X	Y	X^2	XY
1	87	1	4.4659	1	4.4659
2	97	2	4.5747	4	9.1494
3	113	3	4.7274	9	14.1822
4	129	4	4.8598	16	19.4392
5	202	5	5.3083	25	26.5415
6	195	6	5.2730	36	31.6380
7	193	7	5.2627	49	36.8389
		$\sum X = 28$	$\sum Y = 34.4718$	$\sum X^2 = 140$	$\sum XY = 142.2551$

Substituting these values in Eqs (1) and (2),

$$34.4718 = 7A + 28B \qquad \text{...(3)}$$

$$142.2551 = 28A + 140B \qquad \text{...(4)}$$

Solving Eqs (3) and (4),

$$A = 4.3006$$

$$B = 0.156$$

$$\log_e a = A$$

$$\log_e a = 4.3006$$

$$a = 73.744$$

$$\log_e b = B$$

$$\log_e b = 0.156$$

$$b = 1.1688$$

Hence, the required curve is

$$y = 73.744\,(1.1688)^x$$

Example 3

Fit a curve of the form $y = ax^b$ to the following data:

x	20	16	10	11	14
y	22	41	120	89	56

Solution

$$y = ax^b$$

Taking logarithm on both the sides,

$$\log_e y = \log_e a + b \log_e x$$

Putting $\log_e y = Y$, $\log_e a = A$, $b = B$ and $\log_e x = X$,

$$Y = A + BX$$

The normal equations are

$$\sum Y = nA + B\sum X \qquad \text{...(1)}$$

$$\sum XY = A\sum X + B\sum X^2 \qquad \text{...(2)}$$

Here, $n = 5$

x	y	X	Y	X^2	XY
20	22	2.9957	3.0910	8.9742	9.2597
16	41	2.7726	3.7136	7.6873	10.2963
10	120	2.3026	4.7875	5.3019	11.0237
11	89	2.3979	4.4886	5.7499	10.7632
14	56	2.6391	4.0254	6.9648	10.6234
		$\sum X = 13.1079$	$\sum Y = 20.1061$	$\sum X^2 = 34.6781$	$\sum XY = 51.9663$

Substituting these values in Eqs (1) and (2),

$$20.1061 = 5A + 13.1079B \qquad \text{...(3)}$$

$$51.9663 = 13.1079A + 34.6781B \qquad \text{...(4)}$$

Solving Eqs (3) and (4),

$$A = 10.2146$$
$$B = -2.3624$$
$$\log_e a = A$$
$$\log_e a = 10.2146$$
$$a = 27298.8539$$
$$\text{and } b = B = -2.3624$$

Hence, the required equation of the curve is

$$y = 27298.8539\, x^{-2.3624}$$

Example 4

Fit a curve of the form $y = ae^{bx}$ *to the following data:*

x	1	3	5	7	9
y	115	105	95	85	80

Solution

$$y = ae^{bx}$$

Taking logarithm on both the sides,

$$\log_e y = \log_e a + bx \log_e e$$
$$= \log_e a + bx$$

Putting $\log_e y = Y$, $\log_e a = A$, $b = B$ and $x = X$,

$$Y = A + BX$$

The normal equations are

$$\sum Y = nA + B\sum X \quad \text{...(1)}$$
$$\sum XY = A\sum X + B\sum X^2 \quad \text{...(2)}$$

Here, $n = 5$

x	y	X	Y	X^2	XY
1	115	1	4.7449	1	4.7449
3	105	3	4.6539	9	13.9617
5	95	5	4.5539	25	22.7695
7	85	7	4.4427	49	31.0989
9	80	9	4.3820	81	39.438
		$\sum X = 25$	$\sum Y = 22.7774$	$\sum X^2 = 165$	$\sum XY = 112.013$

Substituting these values in Eqs (1) and (2),

$$22.7774 = 5A + 25B \quad \text{...(3)}$$
$$112.013 = 25A + 165B \quad \text{...(4)}$$

Solving Eqs (3) and (4),

$$A = 4.7897$$
$$B = -0.0469$$
$$\log_e a = A$$
$$\log_e a = 4.7897$$
$$a = 120.2653$$

and

$$b = B = -0.0469$$

Hence, the required equation of the curve is

$$y = 120.2653\, e^{-0.0469x}$$

Example 5

Fit the exponential curve $y = ae^{bx}$ *to the following data:*

x	0	2	4	6	8
y	150	63	28	12	5.6

[Summer 2015]

Solution

$$y = ae^{bx}$$

Taking logarithm on both the sides,

$$\log_e y = \log_e a + bx \log_e e$$
$$= \log_e a + bx$$

Putting $\log_e y = Y$, $\log_e a = A$, $b = B$ and $x = X$,

$$Y = A + BX$$

The normal equations are

$$\sum Y = nA + b\sum X \quad \text{...(1)}$$
$$\sum XY = A\sum X + B\sum X^2 \quad \text{...(2)}$$

Here, $n = 5$

x	y	X	Y	X^2	XY
0	150	0	5.0106	0	0
2	63	2	4.1431	4	8.2862
4	28	4	3.3322	16	13.3288
6	12	6	2.4849	36	14.9094
8	5.6	8	1.7228	64	13.7824
		$\sum X = 20$	$\sum Y = 16.6936$	$\sum X^2 = 120$	$\sum XY = 50.3068$

Substituting these values in Eqs (1) and (2),

$$16.6936 = 5A + 20B \quad \text{...(3)}$$

$$50.3068 = 20A + 120B \quad \text{...(4)}$$

Solving Eqs (3) and (4),

$$A = 4.9855$$

$$B = -0.4117$$

$$\log_e a = A$$

$$\log_e a = 4.9855$$

$$a = 146.28$$

and $\quad b = B = -0.4117$

Hence, the required equation of the curve is

$$y = 146.28\, e^{-0.4117x}$$

Example 6

The pressure and volume of a gas are related by the equation $PV^\gamma = c$. Fit this curve to the following data:

P	0.5	1.0	1.5	2.0	2.5	3.0
V	1.62	1.00	0.75	0.62	0.52	0.46

Solution

$$PV^\gamma = c$$

Taking logarithm on both the sides,

$$\log_e P + \gamma \log_e V = \log_e c$$

$$\log_e V = \frac{1}{\gamma}\log_e c - \frac{1}{\gamma}\log_e P$$

Putting $\log_e V = y, \frac{1}{\gamma}\log_e c = a, \log_e P = x, -\frac{1}{\gamma} = b,$

$$y = a + bx$$

The normal equations are

$$\sum y = na + b\sum x$$

$$\sum xy = a\sum x + b\sum x^2$$

Here, $n = 6$

P	V	x	y	x^2	xy
0.5	1.62	−0.6931	0.4824	0.4804	−0.3343
1.0	1.00	0	0	0	0
1.5	0.75	0.4055	−0.2877	0.1644	−0.1166
2.0	0.62	0.6931	−0.4780	0.4804	−0.3313
2.5	0.52	0.9163	−0.6539	0.8396	−0.5992
3.0	0.46	1.0986	−0.7765	1.2069	−0.8531
		$\sum x = 2.4204$	$\sum y = -1.7137$	$\sum x^2 = 3.1717$	$\sum xy = -2.2345$

Substituting these values in Eqs (1) and (2),

$$-1.7137 = 6a + 2.4204\,b \quad \text{...(3)}$$

$$-2.2345 = 2.4204a + 3.1717\,b \quad \text{...(4)}$$

Solving Eqs (3) and (4),

$$a = -0.002$$

$$b = -0.7029$$

$$-\frac{1}{\gamma} = b$$

$$\gamma = 1.4227$$

$$\frac{1}{\gamma}\log_e c = a$$

$$\frac{1}{1.4227}\log_e c = -0.002$$

$$c = 0.9972$$

Hence, the required equation of the curve is

$$PV^{(1.4227)} = 0.9972$$

EXERCISE 7.3

1. Fit the curve $y = ab^x$ to the following data:

x	2	3	4	5	6
y	144	172.3	207.4	248.8	298.5

[Ans.: $y = 100\,(1.2)$

2. Fit the curve $y = ae^{bx}$ to the following data:

x	0	2	4
y	5.012	10	31.62

[Ans.: $y = 4.642e^{0.46x}$]

3. Fit the curve $y = ax^b$ to the following data:

x	1	2	3	4
y	2.50	8.00	19.00	50.00

[Ans.: $y = 2.227x^{2.09}$]

4. Estimate γ by fitting the ideal gas law $PV^{\gamma} = c$ to the following data:

P	16.6	39.7	78.5	115.5	195.3	546.1
V	50	30	20	15	10	5

[Ans.: $\gamma = 1.504$]