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Chap=1, chap.2 → 14 Marks
Chap 3, chap 4 → 14 Marks
Chap=5 → 18 Marks
Chap=6 → 17 Marks
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70 Marks.

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# CHAPTER 2

# Random Variables

**Chapter Outline**

2.1 Introduction
2.2 Random Variables
2.3 Probability Mass Function
2.4 Discrete Distribution Function
2.5 Probability Density Function
2.6 Continuous Distribution Function
2.7 Two-Dimensional Discrete Random Variables
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## 2.1 INTRODUCTION

The outcomes of random experiments are, in general, abstract quantities or, in other words, most of the time they are not in any numerical form. However, the outcomes of a random experiment can be expressed in quantitative terms, in particular, by means of real numbers. Hence, a function can be defined that takes a definite real value corresponding to each outcome of an experiment. This gives a rationale for the concept of random variables about which probability statements can be made. In probability and statistics, a probability distribution assigns a probability to each measurable subset of the possible outcomes of a random experiment. Important and commonly encountered probability distributions include binomial distribution, Poisson distribution, and normal distribution.

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## 2.2 RANDOM VARIABLES

A random variable $X$ is a real-valued function of the elements of the sample space of a random experiment. In other words, a variable which takes the real values, depending on the outcome of a random experiment is called a *random variable*, e.g.,

(i) When a fair coin is tossed, $S = \{H, T\}$. If $X$ is the random variable denoting the number of heads,

$X(H) = 1$ and $X(T) = 0$

Hence, the random variable $X$ can take values 0 and 1.

(ii) When two fair coins are tossed, $S = \{HH, HT, TH, TT\}$. If $X$ is the random variable denoting the number of heads,

$X(HH) = 2, X(HT) = 1, X(TH) = 1, X(TT) = 0.$

Hence, the random variable $X$ can take values 0, 1, and 2.

(iii) When a fair die is tossed, $S = \{1, 2, 3, 4, 5, 6\}$.

If $X$ is the random variable denoting the square of the number obtained,

$X(1) = 1, X(2) = 4, X(3) = 9, X(4) = 16, X(5) = 25, X(6) = 36$

Hence, the random variable $X$ can take values 1, 4, 9, 16, 25, and 36.

### Types of Random Variables

There are two types of random variables:

(i) Discrete random variables
(ii) Continuous random variables

**Discrete Random Variables** A random variable $X$ is said to be discrete if it takes either finite or countably infinite values. Thus, a discrete random variable takes only isolated values, e.g.,

(i) Number of children in a family
(ii) Number of cars sold by different companies in a year
(iii) Number of days of rainfall in a city
(iv) Number of stars in the sky
(v) Profit made by an investor in a day

**Continuous Random Variables** A random variable $X$ is said to be continuous if it takes any values in a given interval. Thus, a continuous random variable takes uncountably infinite values, e.g.,

(i) Height of a person in cm
(ii) Weight of a bag in kg
(iii) Temperature of a city in degree Celsius
(iv) Life of an electric bulb in hours
(v) Volume of a gas in cc.



### Example 1

*Identify the random variables as either discrete or continuous in each of the following cases:*

*(i) A page in a book can have at most 300 words*
*$X$ = Number of misprints on a page*

*(ii) Number of students present in a class of 50 students*

*(iii) A player goes to the gymnasium regularly*
*$X$ = Reduction in his weight in a month*

*(iv) Number of attempts required by a candidate to clear the IAS examination*

*(v) Height of a skyscraper*

### Solution

(i) $X$ = Number of misprints on a page
The page may have no misprint or 1 misprint or 2 misprint … or 300 misprints. Thus, $X$ takes values 0, 1, 2, …, 300. Hence, $X$ is a discrete random variable.

(ii) Let $X$ be the random variable denoting the number of students present in a class. $X$ takes values 0, 1, 2, …, 50. Hence, $X$ is a discrete random variable.

(iii) Reduction in weight cannot take isolated values 0, 1, 2, etc., but it takes any continuous value.
Hence, $X$ is a continuous random variable.

(iv) Let $X$ be a random variable denoting the number of attempts required by a candidate. Thus, $X$ takes values 1, 2, 3, …. Hence, $X$ is a discrete random variable.

(v) Since height can have any fractional value, it is a continuous random variable.

## 2.3 PROBABILITY MASS FUNCTION

Probability distribution of a random variable is the set of its possible values together with their respective probabilities. Let $X$ be a discrete random variable which takes the values $x_1, x_2, \ldots x_n$. The probability of each possible outcome $x_i$ is $p_i = p(x_i) = P(X = x_i)$ for $i = 1, 2, \ldots, n$. The number $p(x_i)$, $i = 1, 2, \ldots$ must satisfy the following conditions:

(i) $p(x_i) \geq 0$ for all values of $i$

(ii) $\sum_{i=1}^{\infty} p(x_i) = 1$

The function $p(x_i)$ is called the probability function or probability mass function of the random variable $X$. The set of pairs $\{x, p(x_i)\}$, $i = 1, 2, \ldots, n$ is called the probability

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distribution of the random variable which can be displayed in the form of a table as shown below:

| $X = x_i$ | $x_1$ | $x_2$ | $x_3$ | $\ldots x_i$ | $\ldots x_n$ |
|---|---|---|---|---|---|
| $p(x_i) = P(X = x_i)$ | $p(x_1)$ | $p(x_2)$ | $p(x_3)$ | $\ldots p(x_i)$ | $\ldots p(x_n)$ |

## 2.4 DISCRETE DISTRIBUTION FUNCTION

Let $X$ be a discrete random variable which takes the values $x_1, x_2, \ldots$ such that $x_1 < x_2 < \ldots$ with probabilities $p(x_1), p(x_2) \ldots$ such that $p(x_i) \geq 0$ for all values of $i$ and $\sum_{i=1}^{x} p(x_i) = 1$.

The distribution function $F(x)$ of the discrete random variable $X$ is defined by

$$F(x) = P(X \leq x) = \sum_{i=1}^{x} p(x_i)$$

where $x$ is any integer. The function $F(x)$ is also called the cumulative distribution function. The set of pairs $\{x_i, F(x)\}$, $i = 1, 2, \ldots$ is called the cumulative probability distribution.

| $X$ | $x_1$ | $x_2$ | ... |
|---|---|---|---|
| $F(x)$ | $p(x_1)$ | $p(x_1) + p(x_2)$ | ... |

### Example 1

*A fair die is tossed once. If the random variable is getting an even number, find the probability distribution of X.*

#### Solution

When a fair die is tossed,

$$S = \{1, 2, 3, 4, 5, 6\}$$

Let $X$ be the random variable of getting an even number. Hence, $X$ can take the values 0 and 1.

$$P(X = 0) = P(1, 3, 5) = \frac{3}{6} = \frac{1}{2}$$

$$P(X = 1) = P(2, 4, 6) = \frac{3}{6} = \frac{1}{2}$$



Hence, the probability distribution of $X$ is

| $X = x$ | 0 | 1 |
|---|---|---|
| $P(X = 1)$ | $\frac{1}{2}$ | $\frac{1}{2}$ |

Also, $\sum P(X = x) = \frac{1}{2} + \frac{1}{2} = 1$

### Example 2

*Find the probability distribution of the number of heads when three coins are tossed.*

#### Solution

When three coins are tossed,

$$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

Let $X$ be the random variable of getting heads in tossing of three coins. Hence $X$ can take the values 0, 1, 2, 3.

$$P(X = 0) = P(\text{no head}) = P(TTT) = \frac{1}{8}$$

$$P(X = 1) = P(\text{one head}) = P(HTT, THT, TTH) = \frac{3}{8}$$

$$P(X = 2) = P(\text{two heads}) = P(HHT, THH, HTH) = \frac{3}{8}$$

$$P(X = 3) = P(\text{three heads}) = P(HHH) = \frac{1}{8}$$

Hence, the probability distribution of $X$ is

| $X = x$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $P(X = x)$ | $\frac{1}{8}$ | $\frac{3}{8}$ | $\frac{3}{8}$ | $\frac{1}{8}$ |

Also, $\sum P(X = x) = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = 1$

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## Example 3

*State with reasons whether the following represent the probability mass function of a random variable:*

*(i)*

| $X = x$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $P(X = x)$ | 0.4 | 0.3 | 0.2 | 0.1 |

*(ii)*

| $X = x$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $P(X = x)$ | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{6}$ | $\frac{1}{4}$ |

*(iii)*

| $X = x$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $P(X = x)$ | $-\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{4}$ | $\frac{3}{4}$ |

### Solution

(i) Here, $0 \le P(X = x) \le 1$ is satisfied for all values of $X$.

$$\sum P(X = x) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)$$
$$= 0.4 + 0.3 + 0.2 + 0.1$$
$$= 1$$

Since $\sum P(X = x) = 1$, it represents probability mass function.

(ii) Here, $0 \le P(X = x) \le 1$ is satisfied for all values of $X$.

$$\sum P(X = x) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)$$
$$= \frac{1}{2} + \frac{1}{3} + \frac{1}{6} + \frac{1}{4}$$
$$= \frac{5}{4} > 1$$

Since $\sum (P(X = x) > 1$, it does not represent a probability mass function.

(iii) Here, $0 \le P(X = x) \le 1$ is not satisfied for all the values of $X$ as $P(X = 0) = -\frac{1}{2}$.

Hence, $P(X = x)$ does not represent a probability mass function.



## Example 4

*Verify whether the following functions can be regarded as the probability mass function for the given values of X:*

*(i)* $P(X = x) = \frac{1}{5}$ *for* $x = 0, 1, 2, 3, 4$
$= 0$ *for otherwise*

*(ii)* $P(X = x) = \frac{x-2}{5}$ *for* $x = 1, 2, 3, 4, 5$
$= 0$ *for otherwise*

*(iii)* $P(X = x) = \frac{x^2}{30}$ *for* $x = 0, 1, 2, 3, 4$
$= 0$ *for otherwise*

### Solution

(i) $P(X = 0) = P(X = 1) = P(X = 2) = P(X = 3) = P(X = 4) = \frac{1}{5}$

$P(X = x) \ge 0$ for all values of $x$

$$\sum P(X = x) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)$$
$$= \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5}$$
$$= 1$$

Hence, $P(X = x)$ is a probability mass function.

(ii) $P(X = 1) = \frac{1-2}{5} = -\frac{1}{5} < 0$

Hence, $P(X = x)$ is not a probability mass function.

(iii) $P(X = 0) = 0$

$P(X = 1) = \frac{1}{30}$

$P(X = 2) = \frac{4}{30}$

$P(X = 3) = \frac{9}{30}$

$P(X = 4) = \frac{16}{30}$

$P(X = x) \ge 0$ for all values of $x$

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$$\sum P(X=x) = P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4)$$
$$= 0 + \frac{1}{30} + \frac{4}{30} + \frac{9}{30} + \frac{16}{30}$$
$$= 1$$

Hence, $P(X=x)$ is a probability mass function.

## Example 5

*A random variable X has the probability mass function given by*

| $X$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $P(X=x)$ | 0.1 | 0.2 | 0.5 | 0.2 |

*Find (i) $P(2 \le x < 4)$, (ii) $P(X > 2)$, (iii) $P(X$ is odd), and (iv) $P(X$ is even).*

### Solution

(i) $P(2 \le X < 4) = P(X=2) + P(X=3)$
$= 0.2 + 0.5$
$= 0.7$

(ii) $P(X > 2) = P(X=3) + P(X=4)$
$= 0.5 + 0.2$
$= 0.7$

(iii) $P(X \text{ is odd}) = P(X=1) + P(X=3)$
$= 0.1 + 0.5$
$= 0.6$

(iv) $P(X \text{ is even}) = P(X=2) + P(X=4)$
$= 0.2 + 0.2$
$= 0.4$

## Example 6

*If the random variable X takes the value 1, 2, 3, and 4 such that $2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4)$. Find the probability distribution.*

### Solution

Let $2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4) = k$

$$P(X=1) = \frac{k}{2}$$
$$P(X=2) = \frac{k}{3}$$



$$P(X=3) = k$$
$$P(X=4) = \frac{k}{5}$$

Since $\sum(P(X=x) = 1$,

$$\frac{k}{2} + \frac{k}{3} + k + \frac{k}{5} = 1$$
$$k = \frac{30}{61}$$

Hence, the probability distribution is

| $X$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $P(X=x)$ | $\frac{15}{61}$ | $\frac{10}{61}$ | $\frac{30}{61}$ | $\frac{6}{61}$ |

## Example 7

*A random variable X has the following probability distribution:*

| $X$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $P(X=x)$ | $a$ | $4a$ | $3a$ | $7a$ | $8a$ | $10a$ | $6a$ | $9a$ |

*(i) Find the value of a.*

*(ii) Find $P(X < 3)$.*

*(iii) Find the smallest value of m for which $P(X \le m) \ge 0.6$.*

### Solution

(i) Since $P(X=x)$ is a probability distribution function,

$$\sum(P(X=x) = 1$$
$$P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4) + P(X=5) + P(X=6) + P(X=7) = 1$$
$$a + 4a + 3a + 7a + 8a + 10a + 6a + 9a = 1$$
$$a = \frac{1}{48}$$

(ii) $P(X < 3) = P(X=0) + P(X=1) + P(X=2)$
$= a + 4a + 3a$
$= 8a$
$= 8\left(\frac{1}{48}\right)$
$= \frac{1}{6}$

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(iii) $P(X \le 4) = P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4)$

$$= a + 4a + 3a + 7a + 8a$$
$$= 23a$$
$$= 23\left(\frac{1}{48}\right)$$
$$= 0.575$$

$$P(X \le 5) = P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4) + P(X=5)$$
$$= a + 4a + 3a + 7a + 8a + 10a$$
$$= 33a$$
$$= 33\left(\frac{1}{48}\right)$$
$$= 0.69$$

Hence, the smallest value of $m$ for which $P(X \le m) \ge 0.6$ is 5.

## Example 8

*The probability mass function of a random variable X is zero except at the points X = 0, 1, 2. At these points, it has the values $P(X = 0) = 3c^3$, $P(X = 1) = 4c - 10c^2$, $P(X = 2) = 5c - 1$. Find (i) c, (ii) $P(X < 1)$, (iii) $P(1 < X \le 2)$, and (iv) $P(0 < X \le 2)$.*

### Solution

(i) Since $P(X = x)$ is a probability mass function,

$$\sum(P(X=x) = 1$$
$$P(X=0) + P(X=1) + P(X=2) = 1$$
$$3c^3 + 4c - 10c^2 + 5c - 1 = 1$$
$$3c^3 - 10c^2 + 9c - 2 = 0$$
$$(3c-1)(c-2)(c-1) = 0$$
$$c = \frac{1}{3},\ 2,\ 1$$

But $c < 1$, otherwise given probabilities will be greater than one or less than zero.

$$\therefore \quad c = \frac{1}{3}$$

Hence, the probability distribution is

| $X$ | 0 | 1 | 2 |
|---|---|---|---|
| $P(X=x)$ | $\frac{1}{9}$ | $\frac{2}{9}$ | $\frac{2}{3}$ |



(ii) $P(X < 1) = P(X = 0) = \frac{1}{9}$

(iii) $P(1 < X \le 2) = P(X = 2) = \frac{2}{3}$

(iv) $P(0 < X \le 2) = P(X = 1) + P(X = 2)$

$$= \frac{2}{9} + \frac{2}{3}$$
$$= \frac{8}{9}$$

## Example 9

*From a lot of 10 items containing 3 defectives, a sample of 4 items is drawn at random. Let the random variable X denote the number of defective items in the sample. Find the probability distribution of X.*

### Solution

The random variable $X$ can take the value 0, 1, 2, or 3.

Total number of items = 10

Number of good items = 7

Number of defective items = 3

$$P(X=0) = P(\text{no defective}) = \frac{{}^7C_4}{{}^{10}C_4} = \frac{1}{6}$$

$$P(X=1) = P(\text{one defective and three good items}) = \frac{{}^3C_1\,{}^7C_3}{{}^{10}C_4} = \frac{1}{2}$$

$$P(X=2) = P(\text{two defectives and two good items}) = \frac{{}^3C_2\,{}^7C_2}{{}^{10}C_4} = \frac{3}{10}$$

$$P(X=3) = P(\text{three defectives and one good item}) = \frac{{}^3C_3\,{}^7C_1}{{}^{10}C_4} = \frac{1}{30}$$

Hence, the probability distribution of the random variable is

| $X$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $P(X=x)$ | $\frac{1}{6}$ | $\frac{1}{2}$ | $\frac{3}{10}$ | $\frac{1}{30}$ |

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## Example 10

*Construct the distribution function of the discrete random variable X whose probability distribution is as given below:*

| $X$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $P(X=x)$ | 0.1 | 0.15 | 0.25 | 0.2 | 0.15 | 0.1 | 0.05 |

### Solution

Distribution function of $X$

| $X$ | $P(X=x)$ | $F(x)$ |
|---|---|---|
| 1 | 0.1 | 0.1 |
| 2 | 0.15 | 0.25 |
| 3 | 0.25 | 0.5 |
| 4 | 0.2 | 0.7 |
| 5 | 0.15 | 0.85 |
| 6 | 0.1 | 0.95 |
| 7 | 0.05 | 1 |

## Example 11

*A random variable X has the probability function given below:*

| $X$ | 0 | 1 | 2 |
|---|---|---|---|
| $P(X=x)$ | $k$ | $2k$ | $3k$ |

*Find (i) k, (ii) $P(X<2)$, $P(X\le 2)$, $P(0<X<2)$, and (iii) the distribution function.*

### Solution:

(i) Since $P(X=x)$ is a probability mass function,

$$\sum(P(X=x)=1$$
$$k+2k+3k=1$$
$$6k=1$$
$$k=\frac{1}{6}$$

Hence, the probability distribution is

| $X$ | 0 | 1 | 2 |
|---|---|---|---|
| $P(X=x)$ | $\frac{1}{6}$ | $\frac{2}{6}$ | $\frac{3}{6}$ |



(ii) $P(X<2)=P(X=0)+P(X=1)=\frac{1}{6}+\frac{2}{6}=\frac{1}{2}$

$P(X\le 2)=P(X=0)+P(X=1)+P(X=2)=\frac{1}{6}+\frac{2}{6}+\frac{3}{6}=1$

$P(0<X<2)=P(X=1)=\frac{1}{3}$

(iii) Distribution function

| $X$ | $P(X=x)$ | $F(x)$ |
|---|---|---|
| 0 | $\frac{1}{6}$ | $\frac{1}{6}$ |
| 1 | $\frac{2}{6}$ | $\frac{1}{2}$ |
| 2 | $\frac{3}{6}$ | 1 |

## Example 12

*A random variable X takes the values –3, –2, –1, 0, 1, 2, 3, such that*
$P(X=0)=P(X>0)=P(X<0)$,
$P(X=-3)=P(X=-2)=P(X=-1)=P(X=1)=P(X=2)=P(X=3)$.
*Obtain the probability distribution and the distribution function of X.*

### Solution

Let $P(X=0)=P(X>0)=P(X<0)=k_1$

Since $\sum P(X=x)=1$

$$k_1+k_1+k_1=1$$
$$\therefore \quad k_1=\frac{1}{3}$$
$$P(X=0)=P(X>0)=P(X<0)=\frac{1}{3}$$

Let $\quad P(X=1)=P(X=2)=P(X=3)=k_2$

$$P(X>0)=P(X=1)+P(X=2)+P(X=3)$$
$$\frac{1}{3}=k_2+k_2+k_2$$
$$\therefore \quad k_2=\frac{1}{9}$$
$$P(X=1)=P(X=2)=P(X=3)=\frac{1}{9}$$

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Similarly, $P(X=-3)=P(X=-2)=P(X=-1)=\frac{1}{9}$

Probability distribution and distribution function

| $X$ | $P(X=x)$ | $F(x)$ |
|---|---|---|
| $-3$ | $\frac{1}{9}$ | $\frac{1}{9}$ |
| $-2$ | $\frac{1}{9}$ | $\frac{2}{9}$ |
| $-1$ | $\frac{1}{9}$ | $\frac{3}{9}$ |
| $0$ | $\frac{1}{3}$ | $\frac{6}{9}$ |
| $1$ | $\frac{1}{9}$ | $\frac{7}{9}$ |
| $2$ | $\frac{1}{9}$ | $\frac{8}{9}$ |
| $3$ | $\frac{1}{9}$ | $1$ |

## Example 13

*A discrete random variable X has the following distribution function:*

$$F(x)=\begin{cases}0 & x<1\\ \frac{1}{3} & 1\le x<4\\ \frac{1}{2} & 4\le x<6\\ \frac{5}{6} & 6\le x<10\\ 1 & x\ge 10\end{cases}$$

*Find (i) $P(2<X\le 6)$, (ii) $P(X=5)$, (iii) $P(X=4)$, (iv) $P(X\le 6)$, and (v) $P(X=6)$.*



### Solution

(i) $P(2<X\le 6)=F(6)-F(2)=\frac{5}{6}-\frac{1}{3}=\frac{3}{6}=\frac{1}{2}$

(ii) $P(X=5)=P(X\le 5)-P(X<5)=F(5)-P(X<5)=\frac{1}{2}-\frac{1}{2}=0$

(iii) $P(X=4)=P(X\le 4)-P(X<4)=F(4)-P(X<4)=\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$

(iv) $P(X\le 6)=F(6)=\frac{5}{6}$

(v) $P(X=6)=P(X\le 6)-P(X<6)=F(6)-P(X<6)=\frac{5}{6}-\frac{1}{2}=\frac{1}{3}$

## EXERCISE 2.1

1. Verify whether the following functions can be considered as probability mass functions:

   (i) $P(X=x)=\frac{x^2+1}{18}$, $x=0,1,2,3$ [Ans.: Yes]

   (ii) $P(X=x)=\frac{x^2-2}{8}$, $x=1,2,3$ [Ans.: No]

   (iii) $P(X=x)=\frac{2x+1}{18}$, $x=0,1,2,3$ [Ans.: No]

2. The probability mass function of a random variable $X$ is

| $X$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $P(X=x)$ | $k$ | $3k$ | $5k$ | $7k$ | $9k$ | $11k$ | $13k$ |

   Find $P(X<4)$ and $P(3<X\le 6)$.

   $\left[\text{Ans.: } \frac{16}{49}, \frac{33}{49}\right]$

3. A random variable $X$ has the following probability distribution:

| $X$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $P(X=x)$ | $k$ | $2k$ | $3k$ | $k^2$ | $k^2+k$ | $2k^2$ | $4k^2$ |

   Find (i) $k$, (ii) $P(X<5)$, (iii) $P(X>5)$, and (iv) $P(0\le X\le 5)$

   $\left[\text{Ans.: } \frac{1}{8}\text{ (ii) } \frac{49}{64}\text{ (iii) } \frac{3}{32}\text{ (iv) } \frac{29}{32}\right]$

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4. A discrete random variable $X$ has the following probability distribution:

| $X$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|
| $P(X = x)$ | 0.1 | $k$ | 0.2 | $2k$ | 0.3 | $3k$ |

Find (i) $k$, (ii) $P(X \geq 2)$, and (iii) $P(-2 < X < 2)$.

[Ans.: $\frac{1}{15}$ (ii) $\frac{1}{2}$ (iii) $\frac{2}{5}$]

5. Given the following probability function of a discrete random variable $X$:

| $X$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $P(X = x)$ | 0 | $c$ | $2c$ | $2c$ | $3c$ | $c^2$ | $2c^2$ | $7c^2 + c$ |

Find (i) $c$, (ii) $P(X \geq 6)$, (iii) $P(X < 6)$, and (iv) Find $k$ if $P(X \leq k) > \frac{1}{2}$, where $k$ is a positive integer.

[Ans.: (i) 0.1 (ii) 0.19 (iii) 0.81 (iv) 4]

6. A random variable $X$ assumes four values with probabilities $\frac{1+3x}{4}, \frac{1-x}{4}, \frac{1+2x}{4}$ and $\frac{1-4x}{4}$. For what value of $x$ do these values represent the probability distribution of $X$?

[Ans.: $-\frac{1}{3} \leq X \leq \frac{1}{4}$]

7. Let $X$ denote the number of heads in a single toss of 4 fair coins. Determine (i) $P(X < 2)$, and (ii) $P(1 < X \leq 3)$.

[Ans.: (i) $\frac{5}{16}$ (ii) $\frac{5}{8}$]

8. If 3 cars are selected from a lot of 6 cars containing 2 defective cars, find the probability distribution of the number of defective cars.

[Ans.:

| $X$ | 0 | 1 | 2 |
|---|---|---|---|
| $P(X = x)$ | $\frac{1}{5}$ | $\frac{3}{5}$ | $\frac{2}{5}$ |
]

9. Five defective bolts are accidentally mixed with 20 good ones. Find the probability distribution of the number of defective bolts, if four bolts are drawn at random from this lot.



[Ans.:

| $X$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $P(X = x)$ | $\frac{969}{2530}$ | $\frac{1140}{2530}$ | $\frac{380}{2530}$ | $\frac{40}{2530}$ | $\frac{1}{2530}$ |
]

10. Two dice are rolled at once. Find the probability distribution of the sum of the numbers on them.

[Ans.:

| $X$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $P(X = x)$ | $\frac{1}{36}$ | $\frac{2}{36}$ | $\frac{3}{36}$ | $\frac{4}{36}$ | $\frac{5}{36}$ | $\frac{6}{36}$ | $\frac{5}{36}$ | $\frac{4}{36}$ | $\frac{3}{36}$ | $\frac{2}{36}$ | $\frac{1}{36}$ |
]

11. A random variable X takes three values 0, 1, and 2 with probabilities $\frac{1}{3}, \frac{1}{6}$, and $\frac{1}{2}$ respectively. Obtain the distribution function of $X$.

[Ans.: $F(0) = \frac{1}{3}, F(1) = \frac{1}{2}, F(2) = 1$]

12. A random variable $X$ has the following probability function:

| $x$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $P(X = x)$ | $k$ | $3k$ | $5k$ | $7k$ | $9k$ |

Find (i) the value of $k$, (ii) $P(X < 3)$, $P(X \geq 3)$, $P(0 < X < 4)$, and (iii) distribution function of $X$.

[Ans.: (i) $\frac{1}{25}$, (ii) $\frac{9}{25}, \frac{16}{25}, \frac{3}{5}$
(iii) $F(0) = \frac{1}{25}, F(1) = \frac{4}{25}, F(2) = \frac{9}{25}, F(3) = \frac{16}{25}, F(4) = 1$]

13. A random variable $X$ has the probability function

| $X$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|
| $P(X = x)$ | 0.1 | $k$ | 0.2 | $2k$ | 0.3 | $k$ |

Find (i) $k$, (ii) $P(X \leq 1)$, (iii) $P(-2 < X < 1)$, and (iv) obtain the distribution function of $X$.

[Ans.: (i) 0.1 (ii) 0.6 (iii) 0.3]

14. The following is the distribution function $F(x)$ of a discrete random variable $X$:

| $X$ | $-3$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $P(X = x)$ | 0.08 | 0.2 | 0.4 | 0.65 | 0.8 | 0.9 | 1 |

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Find (i) the probability distribution of $X$, (ii) $P(-2 \le X \le 1)$, and (iii) $P(X \ge 1)$.

Ans.: (i)

| $X$ | −3 | −2 | −1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $P(X = x)$ | 0.08 | 0.12 | 0.2 | 0.25 | 0.15 | 0.1 | 0.1 |

(ii) 0.72 (ii) 0.35

## 2.5 PROBABILITY DENSITY FUNCTION

Let $X$ be a continuous random variable such that the probability of the variable $X$ falling in the small interval $x-\frac{1}{2}dx$ to $x+\frac{1}{2}dx$ is $f(x)\,dx$, i.e.,

$$P\left(x-\frac{1}{2}dx \le X \le x+\frac{1}{2}dx\right) = f(x)dx$$

The function $f(x)$ is called the probability density function of the random variable $X$ and the continuous curve $y = f(x)$ is called the probability curve.

### Properties of Probability Density Function

(i) $f(x) \ge 0, \quad -\infty < x < \infty$

(ii) $\int_{-\infty}^{\infty} f(x)dx = 1$

(iii) $P(a < x < b) = \int_a^b f(x)\,dx$

## 2.6 CONTINUOUS DISTRIBUTION FUNCTION

If $X$ is a continuous random variable having the probability density function $f(x)$ then the function

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)dx, \quad -\infty < x < \infty$$

is called the distribution function or cumulative distribution function of the random variable $X$.

### Properties of Cumulative Distribution Function

(i) $F(-\infty) = 0$
(ii) $F(\infty) = 1$
(iii) $0 \le F(x) \le 1, \quad -\infty < x < \infty$
(iv) $P(a < X < b) = F(b) - F(a)$



(v) $F'(x) = \frac{d}{dx}F(x) = f(x), \quad f(x) \ge 0$

## Example 1

*Show that the function $f(x)$ defined by*

$$f(x) = \frac{1}{7} \quad 1 < x < 8$$
$$= 0 \quad \textit{otherwise}$$

*is a probability density function for a random variable. Hence, find $P(3 < X < 10)$.*

### Solution

$f(x) \ge 0$ in $1 < x < 8$

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{1} f(x)dx + \int_{1}^{8} f(x)dx + \int_{8}^{\infty} f(x)dx$$
$$= 0 + \int_{1}^{8} \frac{1}{7}dx + 0$$
$$= \frac{1}{7}|x|_1^8$$
$$= \frac{1}{7}(8-1)$$
$$= 1$$

Hence, $f(x)$ is a probability density function.

$$P(3 < X < 10) = \int_{3}^{10} f(x)dx$$
$$= \int_{3}^{8} f(x)dx + \int_{8}^{10} f(x)dx$$
$$= \int_{3}^{8} \frac{1}{7}dx + 0$$
$$= \frac{1}{7}|x|_3^8$$
$$= \frac{1}{7}(8-3)$$
$$= \frac{5}{7}$$

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## Example 2

*Is the function f(x) defined by*

$$f(x) = e^{-x} \quad x \geq 0$$
$$= 0 \quad x < 0$$

*is a probability density function. If so, find the probability that the variate having this density falls in the interval* (1, 2).

### Solution

$f(x) \geq 0$ in $(0, \infty)$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx$$
$$= 0 + \int_{0}^{\infty} e^{-x}\,dx$$
$$= \left|-e^{-x}\right|_{0}^{\infty}$$
$$= -e^{-\infty} + 1$$
$$= 1$$

Hence, $f(x)$ is a probability density function.

$$P(1 \leq X \leq 2) = \int_{1}^{2} f(x)\,dx$$
$$= \int_{1}^{2} e^{-x}\,dx$$
$$= \left|-e^{-x}\right|_{1}^{2}$$
$$= -e^{-2} + e^{-1}$$
$$= 0.233$$

## Example 3

*If a random variable has the probability density function f(x) as*

$$f(x) = 2e^{-2x} \quad x > 0$$
$$= 0 \quad x \leq 0$$

*Find the probabilities that it will take on a value (i) between* 1 *and* 3, *and (ii) greater than 0.5.*



### Solution

(i) Probability that the variable will take a value between 1 and 3

$$P(1 < X < 3) = \int_{1}^{3} f(x)\,dx$$
$$= \int_{1}^{3} 2e^{-2x}\,dx$$
$$= 2\left|\frac{e^{-2x}}{-2}\right|_{1}^{3}$$
$$= -(e^{-6} - e^{-2})$$
$$= e^{-2} - e^{-6}$$

(ii) Probability that the variable will take a value greater than 0.5

$$P(X > 0.5) = \int_{0.5}^{\infty} f(x)\,dx$$
$$= \int_{0.5}^{\infty} 2e^{-2x}\,dx$$
$$= 2\left|\frac{e^{-2x}}{-2}\right|_{0.5}^{\infty}$$
$$= -(e^{-\infty} - e^{-1})$$
$$= e^{-1}$$

## Example 4

*Find the constant k such that the function*

$$f(x) = kx^2 \quad 0 < x < 3$$
$$= 0 \quad \textit{otherwise}$$

*is a probability density function and compute (i)* $P(1 < x < 2)$, *(ii)* $P(X < 2)$, *and (iii)* $P(X \geq 2)$.

### Solution

Since $f(x)$ is a probability density function,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{-\infty}^{0} f(x)\,dx + \int_{0}^{3} f(x)\,dx + \int_{3}^{\infty} f(x)\,dx = 1$$

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$$0+\int_0^3 kx^2\,dx+0=1$$

$$k\left|\frac{x^3}{3}\right|_0^3=1$$

$$\frac{k}{3}(27-0)=1$$

$$9k=1$$

$$k=\frac{1}{9}$$

Hence, $f(x)=\frac{1}{9}x^2 \quad 0<x<3$

$\quad =0 \quad$ otherwise

(i) $P(1<X<2)=\int_1^2 f(x)\,dx$

$$=\int_1^2 \frac{1}{9}x^2\,dx$$

$$=\frac{1}{9}\left|\frac{x^3}{3}\right|_1^2$$

$$=\frac{1}{27}(8-1)$$

$$=\frac{7}{27}$$

(ii) $P(X<2)=\int_{-\infty}^2 f(x)\,dx$

$$=\int_{-\infty}^0 f(x)\,dx+\int_0^2 f(x)\,dx$$

$$=0+\int_0^2 \frac{1}{9}x^2\,dx$$

$$=\frac{1}{9}\int_0^2 x^2\,dx$$

$$=\frac{1}{9}\left|\frac{x^3}{3}\right|_0^2$$

$$=\frac{1}{27}(8-0)$$

$$=\frac{8}{27}$$



(iii) $P(X\geq 2)=1-P(X<2)$

$$=1-\frac{8}{27}$$

$$=\frac{19}{27}$$

## Example 5

*If the probability density function of a random variable is given by*

$$f(x)=k(1-x^2) \quad 0<x<1$$
$$=0 \qquad \textit{otherwise}$$

*Find the value of k and the probabilities that a random variable having this probability density will take on a value (i) between* 0.1 *and* 0.2, *and (ii) greater than* 0.5.

### Solution

Since $f(x)$ is a probability density function,

$$\int_{-\infty}^{\infty} f(x)\,dx=1$$

$$\int_{-\infty}^{0} f(x)\,dx+\int_0^1 f(x)\,dx+\int_1^{\infty} f(x)\,dx=1$$

$$0+\int_0^1 k(1-x^2)\,dx+0=1$$

$$k\left|x-\frac{x^3}{3}\right|_0^1=1$$

$$k\left(1-\frac{1}{3}\right)=1$$

$$k=\frac{3}{2}$$

Hence, $f(x)=\frac{3}{2}(1-x^2) \quad 0<x<1$

$\quad =0 \quad$ otherwise

(i) Probability that the variable will take on a value between 0.1 and 0.2

$$P(0.1<X<0.2)=\int_{0.1}^{0.2} f(x)\,dx$$

$$=\int_{0.1}^{0.2} \frac{3}{2}(1-x^2)\,dx$$

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$$= \frac{3}{2}\left|x - \frac{x^3}{3}\right|_{0.1}^{0.2}$$

$$= \frac{3}{2}\left[\left(0.2 - \frac{0.008}{3}\right) - \left(0.1 - \frac{0.001}{3}\right)\right]$$

$$= 0.1465$$

(ii) Probability that the variable will take on a value greater than 0.5

$$P(X > 0.5) = \int_{0.5}^{\infty} f(x)\,dx$$

$$= \int_{0.5}^{1} f(x)\,dx + \int_{1}^{\infty} f(x)\,dx$$

$$= \int_{0.5}^{1} \frac{3}{2}(1 - x^2)\,dx + 0$$

$$= \frac{3}{2}\left|x - \frac{x^3}{3}\right|_{0.5}^{1}$$

$$= \frac{3}{2}\left[\left(1 - \frac{1}{3}\right) - \left(0.5 - \frac{0.125}{3}\right)\right]$$

$$= 0.3125$$

## Example 6

*If X is a continuous random variable with pdf*

$$f(x) = x^2 \quad 0 \le x \le 1$$
$$= 0 \quad \textit{otherwise}$$

*If* $P(a \le X \le 1) = \frac{19}{81}$, *find the value of a.*

### Solution

$$P(a \le X \le 1) = \frac{19}{81}$$

$$\int_a^1 f(x)\,dx = \frac{19}{81}$$

$$\int_a^1 x^2\,dx = \frac{19}{81}$$



$$\left|\frac{x^3}{3}\right|_a^1 = \frac{19}{81}$$

$$\frac{1}{3}(1 - a) = \frac{19}{81}$$

$$1 - a = \frac{19}{27}$$

$$a = \frac{46}{27}$$

## Example 7

*Let X be a continuous random variable with pdf*

$$f(x) = kx\,(1 - x),\ 0 \le x \le 1$$

*Find k and determine a number b such that* $P(X \le b) = P(X \ge b)$.

### Solution

Since $f(x)$ is a probability density function,

$$\int_{-\infty}^{\infty} f(x) = 1$$

$$\int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{\infty} f(x)\,dx = 1$$

$$0 + \int_0^1 kx(1 - x)\,dx + 0 = 1$$

$$k\int_0^1 (x - x^2)\,dx = 1$$

$$k\left|\frac{x^2}{2} - \frac{x^3}{3}\right|_0^1 = 1$$

$$k\left[\left(\frac{1}{2} - \frac{1}{3}\right) - (0 - 0)\right] = 1$$

$$k\left(\frac{1}{6}\right) = 1$$

$$k = 6$$

Hence, $f(x) = 6(x - x^2) \quad 0 \le x \le 1$

Since total probability is 1 and $P(X \le b) = P(X \ge b)$,

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$$P(X \le b) = \frac{1}{2}$$

$$\int_0^b f(x)\,dx = \frac{1}{2}$$

$$6\int_0^b (x - x^2)\,dx = \frac{1}{2}$$

$$6\left|\frac{x^2}{2} - \frac{x^3}{3}\right|_0^b = \frac{1}{2}$$

$$\frac{b^2}{2} - \frac{b^3}{3} = \frac{1}{12}$$

$$6b^2 - 4b^3 = 1$$

$$4b^3 - 6b^2 + 1 = 0$$

$$(2b-1)(2b^2 - 2b - 1) = 0$$

$$b = \frac{1}{2} \text{ or } b = \frac{1 \pm \sqrt{3}}{2}$$

$b$ lies in (0, 1).

$\therefore \quad b = \frac{1}{2}$

## Example 8

*The length of time (in minutes) that a certain lady speaks on the telephone is found to be a random phenomenon, with a probability function specified by the function*

$$f(x) = A\,e^{-\frac{x}{5}} \quad x \ge 0$$
$$= 0 \quad \textit{otherwise}$$

*(i) Find the value of A that makes f(x) a probability density function.*
*(ii) What is the probability that the number of minutes that she will take over the phone is more than 10 minutes?*

### Solution

(i) For $f(x)$ to be a probability density function,

$$\int_{-\infty}^{\infty} f(x)\,x = 1$$

$$\int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx = 1$$



$$0 + \int_0^{\infty} A e^{-\frac{x}{5}}\,dx = 1$$

$$A\left|\frac{e^{-\frac{x}{5}}}{-\frac{1}{5}}\right|_0^{\infty} = 1$$

$$-5A(e^{-\infty} - e^{-0}) = 1$$

$$-5A(0-1) = 1$$

$$5A = 1$$

$$A = \frac{1}{5}$$

Hence, $f(x) = \frac{1}{5}e^{-\frac{x}{5}} \quad x \ge 0$.

$= 0 \quad$ otherwise

(ii) $P(X > 10) = \int_{10}^{\infty} f(x)\,dx$

$$= \int_{10}^{\infty} \frac{1}{5} e^{-\frac{x}{5}}\,dx$$

$$= \frac{1}{5}\left|\frac{e^{-\frac{x}{5}}}{-\frac{1}{5}}\right|_{10}^{\infty}$$

$$= -(e^{-\infty} - e^{-2})$$

$$= -(0 - e^{-2})$$

$$= e^{\frac{1}{2}}$$

## Example 9

*A continuous random variable X has a pdf $f(x)^2 = 3x^2$, $0 \le x \le 1$. Find a and b such that*

*(i)* $P(X \le a) = P(X > a)$ and

*(ii)* $P(X > b) = 0.05$

### Solution

Since total probability is 1 and $P(X \le a) = P(X > a)$,

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$$P(X \le a) = \frac{1}{2}$$

$$\int_0^a f(x)\,dx = \frac{1}{2}$$

$$\int_0^a 3x^2\,dx = \frac{1}{2}$$

$$3\left|\frac{x^3}{3}\right|_0^a = \frac{1}{2}$$

$$a^3 = \frac{1}{2}$$

$$a = \left(\frac{1}{2}\right)^{\frac{1}{3}}$$

$$P(X > b) = 0.05$$

$$\int_b^1 f(x)\,dx = 0.05$$

$$\int_b^1 3x^2\,dx = 0.05$$

$$3\left|\frac{x^3}{3}\right|_b^1 = 0.05$$

$$1 - b^3 = 0.05$$

$$b^3 = \frac{19}{20}$$

$$b = \left(\frac{19}{20}\right)^{\frac{1}{3}}$$

## Example 10

*Let the continuous random variable X have the probability density function*

$$f(x) = \frac{2}{x^3} \qquad 1 < x < \infty$$

$$= 0 \qquad \textit{otherwise}$$

*Find F(x).*



### Solution

$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{1} f(x)\,dx + \int_{1}^{x} f(x)\,dx$$

$$= 0 + \int_1^x \frac{2}{x^3}\,dx$$

$$= 2\left|\frac{x^{-2}}{-2}\right|_1^x$$

$$= -\left|\frac{1}{x^2}\right|_1^x$$

$$= -\left(\frac{1}{x^2} - 1\right)$$

$$= 1 - \frac{1}{x^2}$$

Hence, $F(x) = 1 - \frac{1}{x^2} \qquad 1 < x < \infty$

$= 0 \qquad$ otherwise

## Example 11

*Verify that the function F(x) is a distribution function.*

$$F(x) = 0 \qquad x < 0$$

$$= 1 - e^{-\frac{x}{4}} \qquad x \ge 0$$

*Also, find the probabilities* $P(X \le 4), P(X \ge 8), P(4 \le X \le 8)$.

### Solution

For the function $F(x)$,

(i) $F(-\infty) = 0$

(ii) $F(\infty) = 1 - e^{-\infty} = 1 - 0 = 1$

(iii) $0 \le F(x) \le 1 \qquad -\infty < x < \infty$

If $f(x)$ is the corresponding probability density function,

$$f(x) = F'(x) = 0 \qquad x < 0$$

$$= \frac{1}{4} e^{-\frac{x}{4}} \qquad x \ge 0$$

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$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx$$

$$= 0 + \int_{0}^{\infty} \frac{1}{4} e^{-\frac{x}{4}}\,dx$$

$$= \frac{1}{4}\left| \frac{e^{-\frac{x}{4}}}{-\frac{1}{4}} \right|_{0}^{\infty}$$

$$= -\left| e^{-\frac{x}{4}} \right|_{0}^{\infty}$$

$$= -(0-1)$$

$$= 1$$

Hence, $F(x)$ is a distribution function.

$$P(X \le 4) = F(4)$$

$$= 1 - e^{-1}$$

$$= 1 - \frac{1}{e}$$

$$= \frac{e-1}{e}$$

$$P(X \ge 8) = 1 - P(X \le 8)$$

$$= 1 - F(8)$$

$$= 1 - (1 - e^{-2})$$

$$= e^{-2}$$

$$= \frac{1}{e^2}$$

$$P(4 \le X \le 8) = F(8) - F(4)$$

$$= (1 - e^{-2}) - (1 - e^{-1})$$

$$= e^{-1} - e^{-2}$$

$$= \frac{1}{e} - \frac{1}{e^2}$$

$$= \frac{e-1}{e^2}$$

## Example 12

*The troubleshooting capacity of an IC chip in a circuit is a random variable X whose distribution function is given by*



$$F(x) = 0 \qquad x \le 3$$

$$= 1 - \frac{9}{x^2} \qquad x > 3$$

*where x denotes the number of years. Find the probability that the IC chip will work properly (i) less than 8 years, (i) beyond 8 years, (iii) between 5 to 7 years, and (iv) anywhere from 2 to 5 years.*

### Solution

(i) $P(X \le 8) = F(8)$

$$= 1 - \frac{9}{8^2}$$

$$= 0.8594$$

(ii) $P(X > 8) = 1 - P(X \le 8)$

$$= 1 - F(8)$$

$$= 1 - 0.8594$$

$$= 0.1406$$

(iii) $P(5 \le X \le 7) = F(7) - F(5)$

$$= \left(1 - \frac{9}{7^2}\right) - \left(1 - \frac{9}{5^2}\right)$$

$$= 0.1763$$

(iv) $P(2 \le X \le 5) = F(5) - F(2)$

$$= \left(1 - \frac{9}{5^2}\right) - 0$$

$$= 0.64$$

## Example 13

*The probability density function of a continuous random variable X is given by*

$$f(x) = \begin{cases} ax & 0 \le x \le 1 \\ a & 1 \le x \le 2 \\ 3a - ax & 2 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

*(i) Find the value of a, and (ii) find the cdf of X.*

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## Solution

(i) Since $f(x)$ is a probability density function,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{-\infty}^{\infty} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx + \int_{2}^{3} f(x)\,dx = 1$$

$$0 + \int_{0}^{1} ax\,dx + \int_{1}^{2} a\,dx + \int_{2}^{3} (3a - ax)\,dx = 1$$

$$a\left|\frac{x^2}{2}\right|_0^1 + a\left|x\right|_1^2 + \left|3ax - \frac{ax^2}{2}\right|_2^3 = 1$$

$$a\left(\frac{1}{2} - 0\right) + a(2-1) + \left[\left(9a - \frac{9a}{2}\right) - (6a - 2a)\right] = 1$$

$$\frac{1}{2}a + a + \frac{9a}{2} - 4a = 1$$

$$2a = 1$$

$$a = \frac{1}{2}$$

(ii) $F(x) = \int_{-\infty}^{x} f(x)\,dx$

For $0 \le x \le 1$,

$$F(x) = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{x} f(x)\,dx$$

$$= 0 + \int_{0}^{x} ax\,dx$$

$$= a\left|\frac{x^2}{2}\right|_0^x$$

$$= \frac{ax^2}{2}$$

For $1 \le x \le 2$,

$$F(x) = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{x} f(x)\,dx$$

$$= 0 + \int_{0}^{1} ax\,dx + \int_{1}^{x} a\,dx$$



$$= a\left|\frac{x^2}{2}\right|_0^1 + a\left|x\right|_1^x$$

$$= a\left(\frac{1}{2} - 0\right) + a(x-1)$$

$$= \frac{a}{2} + ax - a$$

$$= ax - \frac{a}{2}$$

For $2 \le x \le 3$,

$$F(x) = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx + \int_{2}^{x} f(x)\,dx$$

$$= 0 + \int_{0}^{1} ax\,dx + \int_{1}^{2} a\,dx + \int_{2}^{x} (3a - ax)\,dx$$

$$= a\left|\frac{x^2}{2}\right|_0^1 + a\left|x\right|_1^2 + \left|3ax - \frac{ax^2}{2}\right|_2^x$$

$$= a\left(\frac{1}{2} - 0\right) + a(2-1) + \left[\left(3ax - \frac{ax^2}{2}\right) - (6a - 2a)\right]$$

$$= \frac{a}{2} + a + 3ax - \frac{ax^2}{2} - 4a$$

$$= 3ax - \frac{ax^2}{2} - \frac{5a}{2}$$

Hence, $F(x) = \dfrac{ax^2}{2}$ $\quad 0 \le x \le 1$

$= ax - \dfrac{a}{2}$ $\quad 1 \le x \le 2$

$= 3ax - \dfrac{ax^2}{2} - \dfrac{5a}{2}$ $\quad 2 \le x \le 3$

## Example 14

*The pdf of a continuous random variable X is*

$$f(x) = \frac{1}{2}e^{-|x|}$$

*Find cdf F(x).*

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### Solution

$$f(x)=\frac{1}{2}e^{x} \quad -\infty<x<0$$
$$=\frac{1}{2}e^{-x} \quad 0<x<\infty$$

$$F(x)=\int_{-\infty}^{x} f(x)\,dx$$

For $x \le 0$,

$$F(x)=\int_{-\infty}^{x}\frac{1}{2}e^{x}dx$$
$$=\frac{1}{2}\left|e^{x}\right|_{-\infty}^{x}$$
$$=\frac{1}{2}(e^{x}-e^{-\infty})$$
$$=\frac{1}{2}e^{x}$$

For $x > 0$,

$$F(x)=\int_{-\infty}^{0} f(x)\,dx+\int_{0}^{x} f(x)\,dx$$
$$=\int_{-\infty}^{0}\frac{1}{2}e^{x}dx+\int_{0}^{x}\frac{1}{2}e^{-x}\,dx$$
$$=\frac{1}{2}\left|e^{x}\right|_{-\infty}^{0}+\frac{1}{2}\left|-e^{-x}\right|_{0}^{x}$$
$$=\frac{1}{2}(1-e^{-\infty})+\frac{1}{2}(-e^{-x}+e^{0})$$
$$=\frac{1}{2}-\frac{1}{2}e^{-x}+\frac{1}{2}$$
$$=1-\frac{1}{2}e^{-x}$$

Hence, $F(x)=\frac{1}{2}e^{x} \quad x\le 0$

$$=1-\frac{1}{2}e^{-x} \quad x>0$$



## Example 15

*Find the value of k and the distribution function F(x) given the probability density function of a random variable X as*

$$f(x)=\frac{k}{x^{2}+1} \quad -\infty<x<\infty$$

### Solution

Since $f(x)$ is the probability density function,

$$\int_{-\infty}^{\infty} f(x)\,dx=1$$

$$\int_{-\infty}^{\infty}\frac{k}{x^{2}+1}dx=1$$
$$k\int_{-\infty}^{\infty}\frac{1}{x^{2}+1}dx=1$$
$$k\left|\tan^{-1}x\right|_{-\infty}^{\infty}=1$$
$$k\left[\tan^{-1}\infty-\tan^{-1}(-\infty)\right]=1$$
$$k\left[\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right]=1$$
$$k\pi=1$$
$$k=\frac{1}{\pi}$$

Hence, $f(x)=\frac{1}{\pi}\frac{1}{x^{2}+1} \quad -\infty<x<\infty$

$$F(x)=\int_{-\infty}^{x} f(x)\,dx$$
$$=\frac{1}{\pi}\int_{-\infty}^{x}\frac{1}{x^{2}+1}dx$$
$$=\frac{1}{\pi}\left|\tan^{-1}x\right|_{-\infty}^{x}$$
$$=\frac{1}{\pi}\left[\tan^{-1}x-\tan^{-1}(-\infty)\right]$$
$$=\frac{1}{\pi}\left(\tan^{-1}x+\frac{\pi}{2}\right)$$

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## Example 16

*Find the constant k such that*

$$f(x) = kx^2 \quad 0 < x < 3$$
$$= 0 \quad \textit{otherwise}$$

*is a probability function. Also, find the distribution function $F(x)$ and $P(1 < X \le 2)$.*

### Solution

Since $f(x)$ is probability density function,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{-\infty}^{\infty} f(x)\,dx + \int_{0}^{3} f(x)\,dx + \int_{3}^{\infty} f(x)\,dx = 1$$

$$0 + \int_{0}^{3} kx^2\,dx + 0 = 1$$

$$k\left|\frac{x^3}{3}\right|_0^3 = 1$$

$$k(9-0) = 1$$

$$k = \frac{1}{9}$$

Hence, $f(x) = \frac{1}{9}x^2 \quad 0 < x < 3$

$= 0 \quad$ otherwise

$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{x} f(x)\,dx$$

$$= 0 + \int_0^x \frac{1}{9}x^2\,dx$$

$$= \frac{1}{9}\left|\frac{x^3}{3}\right|_0^x$$

$$= \frac{1}{27}x^3$$



Hence, $F(x) = \frac{1}{27}x^3 \quad 0 < x < 3$

$= 0 \quad$ otherwise

$$P(1 < x \le 2) = \int_1^2 f(x)\,dx$$

$$= \int_1^2 \frac{1}{9}x^2\,dx$$

$$= \frac{1}{9}\left|\frac{x^3}{3}\right|_1^2$$

$$= \frac{1}{27}(8-1)$$

$$= \frac{7}{27}$$

## EXERCISE 2.2

1. Verify whether the following functions are probability density functions:

   (i) $f(x) = k\,e^{-kx} \qquad x \ge 0, k > 0$

   (ii) $f(x) = \frac{1}{2}e^{-|x|} \qquad -\infty < x < \infty$

   (iii) $f(x) = \frac{2}{9}x\left(2 - \frac{x}{2}\right) \qquad 0 \le x \le 3$

   [**Ans.:** (i) Yes (ii) Yes (iii) Yes]

2. Find the value of $k$ if the following are probability density functions:

   (i) $f(x) = k(1+x) \qquad 2 \le x \le 5$

   (ii) $f(x) = k(x - x^2) \qquad 0 \le x \le 1$

   (iii) $f(x) = kx\,e^{-4x^2} \qquad 0 \le x \le \infty$

   (iv) $f(x) = kx\,e^{-\frac{x^2}{4}} \qquad 0 \le x \le \infty$

   [**Ans.:** (i) $\frac{2}{27}$ (ii) 6 (iii) 8 (iv) $\frac{1}{2}$]

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3. A function is defined as

$$f(x)=\begin{cases}0 & x<2\\ \dfrac{2x+3}{18} & 2\le x\le 4\\ 0 & x>4\end{cases}$$

Show that $f(x)$ is a probability density function and find $P(2<X<3)$.

$\left[\text{Ans.: } \frac{4}{9}\right]$

4. Let $X$ be a continuous random variable with probability distribution

$$f(x)=\begin{cases}\dfrac{x}{6}+k & 0\le x\le 3\\ 0 & \text{otherwise}\end{cases}$$

Find $k$, and $P(1\le X\le 2)$.

$\left[\text{Ans.: } 1, \frac{1}{3}\right]$

5. Find the value of $k$ such that $f(x)$ is a probability density function. Find also, $P(X\le 1.5)$.

$$f(x)=\begin{cases}kx & 0\le x\le 1\\ k & 1\le x\le 2\\ k(3-x) & 2\le x\le 3\end{cases}$$

$\left[\text{Ans.: } \frac{1}{2}, \frac{1}{2}\right]$

6. If $X$ is a continuous random variable whose probability density function is given by

$$f(x)=k(4x-2x^2) \qquad 0<x<2$$
$$=0 \qquad \text{otherwise}$$

Find (i) the value of $k$, and (ii) $P(X>1)$.

$\left[\text{Ans.: (i) } \frac{3}{8} \text{ (ii) } \frac{1}{2}\right]$

7. If a random variable has the probability density function

$$f(x)=k(x^2-1) \qquad -1\le x\le 3$$
$$=0 \qquad \text{otherwise}$$



Find (i) the value of $k$, and (ii) $P\left(\frac{1}{2}\le X\le \frac{5}{2}\right)$.

$\left[\text{Ans.: (i) } \frac{3}{28} \text{ (ii) } \frac{19}{56}\right]$

8. The probability density function is

$$f(x)=k(3x^2-1) \qquad -1\le x\le 2$$
$$=0 \qquad \text{otherwise}$$

Find (i) the value of $k$, and (ii) $P(-1\le X\le 0)$.

$\left[\text{Ans.: (i) } \frac{1}{6} \text{ (ii) } 0\right]$

9. Is the function defined by

$$f(x)=0 \qquad x<2$$
$$=\frac{1}{18}(2x+3) \qquad 2\le x\le 4$$
$$=0 \qquad x>4$$

a probability density function? Find the probability that a variate having $f(x)$ as density function will fall in the interval $2\le X\le 3$.

$\left[\text{Ans.: Yes, } \frac{4}{9}\right]$

10. A random variable $X$ gives measurements $x$ between 0 and 1 with a probability function

$$f(x)=12x^3-21x^2+10x \qquad 0\le x\le 1$$
$$=0 \qquad \text{otherwise}$$

(i) Find $P\left(X\le \frac{1}{2}\right)$ and $P\left(X>\frac{1}{2}\right)$.

(ii) Find a number $k$ such that $P(X\le k)=\frac{1}{2}$.

$\left[\text{Ans.: (i) } \frac{7}{16} \text{ (ii) } 0.452\right]$

11. The distribution function of a random variable $X$ is given by

$$F(x)=\begin{cases}1-e^{-x^2} & x>0\\ 0 & \text{otherwise}\end{cases}$$

Find the probability density function.

$\left[\text{Ans.: } f(x)=2xe^{-x^2} \quad x>0;\ =0 \quad \text{otherwise}\right]$

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**12.** The cdf of a continuous random variable $X$ is given by

$$F(x)=\begin{cases}0 & x<0\\ x^2 & 0\le x\le 1\\ 1 & x>1\end{cases}$$

Find the pdf and $P\left(\frac{1}{2}\le X\le\frac{4}{5}\right)$. [Ans.: 0.195]

**13.** Find the distribution function corresponding to the following probability density functions:

(i) $f(x)=\begin{cases}\frac{1}{2}x^2e^{-x} & 0\le x<\infty\\ 0 & \text{otherwise}\end{cases}$

(ii) $f(x)=x \quad 0\le x\le 1$
$\quad =2-x \quad 1\le x\le 2$
$\quad =0 \quad \text{otherwise}$

(iii) $f(x)=\lambda(x-1)^4 \quad 1\le x\le 3,\ \lambda>0$
$\quad =0 \quad \text{otherwise}$

Ans.: (i) $F(x)=\begin{cases}1-e^{-x}\left(1+x+\frac{x^2}{2}\right) & x\ge 0\\ 0 & \text{otherwise}\end{cases}$

(ii) $F(x)=\begin{cases}0 & x<0\\ \frac{x^2}{2} & 0\le x\le 1\\ 2x-0.5x^2-1 & 1\le x\le 2\\ 1 & x>2\end{cases}$

(iii) $\lambda=\frac{5}{32},\quad F(x)=\begin{cases}0 & x\le 1\\ \frac{5}{32}(x-1)^4 & 1\le x\le 3\\ 1 & x\ge 3\end{cases}$

**14.** A continuous random variable $X$ has the following probability density function

$$f(x)=\frac{a}{x^5}\quad 2\le x\le 10$$



Determine the constant $a$, distribution function of $X$, and find the probability of the event $4\le x\le 7$.

$\left[\text{Ans.: } \frac{2500}{39},\ F(x)=\frac{625}{39}\left(\frac{1}{16}-\frac{1}{x^4}\right),\ 0.056\right]$

## 2.7 TWO-DIMENSIONAL DISCRETE RANDOM VARIABLES

In one-dimensional random variable, the outcome of any experiment had only one characteristic. In many situations, the outcome of a random experiment depends on two or more characteristics e.g., both voltage and current are measured in certain experiment.

Let $X$ and $Y$ be two random variables defined on the same sample space $S$, then the function $(X, Y)$ that assigns a point in $R^2$ is called a two-dimensional random variable.

A two-dimensional random variable is said to be discrete if it takes at most a countable number of points in $R^2$. When $(X, Y)$ is a two-dimensional discrete random variable, the possible values of $(X, Y)$ may be represented as $(x_i, y_j)$, $i = 1, 2, \ldots, m, \ldots$; $j = 1, 2, \ldots, n, \ldots$

### 2.7.1 Joint Probability Mass Function

If $(X, Y)$ is a two-dimensional discrete random variable, then the joint discrete function of $X$, $Y$, also called the joint probability mass function of $X$, $Y$, denoted by $p_{XY}$ is defined by

$$p_{XY}(x_i, y_j)=P(X=x_i, Y=y_j) \text{ for a value of } (x_i, y_j) \text{ of } (X, Y)$$

and $\quad p_{XY}(x_i, y_j)=0, \quad \text{otherwise}$

Following conditions should be satisfied for a function to be a probability mass function:

(i) $p_{XY}(x_i, y_j)\ge 0$, for all $i$ and $j$

(ii) $\sum_{j=1}^{n}\sum_{i=1}^{m} p_{XY}(x_i, y_j)=1$

### 2.7.2 Cumulative Distribution Function

If $(X, Y)$ is a two-dimensional discrete random variable, then $F_{XY}(x, y) = P(X \le x, Y \le y)$ is called the cumulative distribution function (cdf) of $(X, Y)$ and is defined by

$$F_{XY}(x,y)=\sum_{j=1}^{n}\sum_{i=1}^{m} p_{XY}(x_i, y_j)=\Sigma\Sigma\, p_{ij}$$

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**Properties of cdf**

(i) $F(-\infty, y) = 0 = F(x, \infty)$ and $F(\infty, \infty) = 1$

(ii) $P(a < X < b,\ Y \le y) = F(b, y) - F(a, y)$

(iii) $P(X \le x,\ c < Y < d) = F(x, d) - F(x, c)$

(iv) $P(a < X < b,\ c < Y < d) = F(b, d) - F(a, d) - F(b, c) + F(a, c)$

(v) For a discrete random variable, $F_{XY}(x,y)$ will have step discontinuities. Derivatives at such discontinuities are not defined. At points of continuity,

$$\frac{\partial^2 F}{\partial x\, \partial y} = f(x, y)$$

### 2.7.3 Marginal Probability function

Let $(X, Y)$ be a two-dimensional discrete random variable which takes up countable number of values $(x_i, y_j)$. Then the probability distribution of $X$ is given by

$$\begin{aligned} p_X(x_i) &= P(X = x_i) \\ &= P(X = x_i, Y = y_1) + P(X = x_i, Y = y_2) + \ldots + P(X = x_i, Y = y_m) \\ &= p_{i1} + p_{i2} + \cdots + p_{ij} + \cdots + p_{im} \\ &= \sum_{j=1}^{m} p_{ij} \\ &= \sum_{j=1}^{m} p(x_i, y_j) \\ &= p_{i*} \end{aligned}$$

and is known as marginal probability mass function or discrete marginal density function of $X$.

Similarly,

$$p_Y(y_j) = P(Y = y_j) = \sum_{i=1}^{n} p_{ij} = \sum_{i=1}^{n} p(x_i, y_j) = p_{*j}$$

is the marginal probability mass function of $Y$.

### 2.7.4 Conditional Probability Function

Let $(X, Y)$ be a two-dimensional discrete random variable. Then the conditional discrete density function or conditional probability mass function of $X$, given $Y = y$, denoted by $p_{X/Y}(x/y)$ is defined as

$$p_{X/Y} = P(X = x / Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)}, \text{ provided } P(Y = y) \neq 0.$$



The conditional probability mass function of $Y$, given $X = x$, denoted by $p_{Y/X}(y/x)$ is defined as:

$$p_{Y/X} = P(Y = y / X = x) = \frac{P(X = x, Y = y)}{P(X = x)}, \text{ provided } P(X = x) \neq 0.$$

A necessary and sufficient condition for the discrete random variables $X$ and $Y$ to be independent is

$$P(X = x_i, Y = y_j) = P(X = x_i)\, P(Y = y_j) \text{ for all values } (x_i, y_j) \text{ of } (X, Y)$$

## Example 1

*From the following table for bivariate distribution of (X, Y), find (i) $P(X \le 1)$ (ii) $P(Y \le 3)$ (iii) $P(X \le 1,\ Y \le 3)$, (iv) $P(X \le 1/Y \le 3)$ (v) $P(Y \le 3/X \le 1)$ (vi) $P(X + Y \le 4)$*

| X \ Y | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | $\frac{1}{32}$ | $\frac{2}{32}$ | $\frac{2}{32}$ | $\frac{3}{32}$ |
| 1 | $\frac{1}{16}$ | $\frac{1}{16}$ | $\frac{1}{8}$ | $\frac{1}{8}$ | $\frac{1}{8}$ | $\frac{1}{8}$ |
| 2 | $\frac{1}{32}$ | $\frac{1}{32}$ | $\frac{1}{64}$ | $\frac{1}{64}$ | 0 | $\frac{2}{64}$ |

### *Solution*

Marginal distributions

| X \ Y | 1 | 2 | 3 | 4 | 5 | 6 | $p_X(x)$ |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | $\frac{1}{32}$ | $\frac{2}{32}$ | $\frac{2}{32}$ | $\frac{3}{32}$ | $\frac{8}{32}$ |
| 1 | $\frac{1}{16}$ | $\frac{1}{16}$ | $\frac{1}{8}$ | $\frac{1}{8}$ | $\frac{1}{8}$ | $\frac{1}{8}$ | $\frac{10}{16}$ |
| 2 | $\frac{1}{32}$ | $\frac{1}{32}$ | $\frac{1}{64}$ | $\frac{1}{64}$ | 0 | $\frac{2}{64}$ | $\frac{8}{64}$ |
| $p_Y(y)$ | $\frac{3}{32}$ | $\frac{3}{32}$ | $\frac{11}{64}$ | $\frac{13}{64}$ | $\frac{6}{32}$ | $\frac{16}{64}$ | $\Sigma p(x) = 1$ $\Sigma p(y) = 1$ |

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(i) $P(X \le 1) = P(X=0) + P(X=1)$

$$= \frac{8}{32} + \frac{10}{16}$$

$$= \frac{7}{8}$$

(ii) $P(Y \le 3) = P(Y=1) + P(Y=2) + P(Y=3)$

$$= \frac{3}{32} + \frac{3}{32} + \frac{11}{64}$$

$$= \frac{23}{64}$$

(iii) $P(X \le 1, Y \le 3) = P(X=0, Y=1) + P(X=0, Y=2) + P(X=0, Y=3)$
$+ P(X=1, Y=1) + P(X=1, Y=2) + P(X=1, Y=3)$

$$= 0 + 0 + \frac{1}{32} + \frac{1}{16} + \frac{1}{16} + \frac{1}{8}$$

$$= \frac{9}{32}$$

(iv) $P(X \le 1 / Y \le 3) = \dfrac{P(X \le 1, Y \le 3)}{P(Y \le 3)}$

$$= \frac{\frac{9}{32}}{\frac{23}{64}}$$

$$= \frac{18}{23}$$

(v) $P(Y \le 3 / X \le 1) = \dfrac{P(X \le 1, Y \le 3)}{P(X \le 1)}$

$$= \frac{\frac{9}{32}}{\frac{7}{8}}$$

$$= \frac{9}{28}$$

(vi) $P(X + Y \le 4) = P(X=0, Y=1) + P(X=0, Y=2) + P(X=0, Y=3)$
$+ P(X=0, Y=4) + P(X=1, Y=1) + P(X=1, Y=2)$
$+ P(X=1, Y=3) + P(X=2, Y=1) + P(X=2, Y=2)$



$$= 0 + 0 + \frac{1}{32} + \frac{2}{32} + \frac{1}{16} + \frac{1}{16} + \frac{1}{8} + \frac{1}{32} + \frac{1}{32}$$

$$= \frac{13}{32}$$

## Example 2

*For the following joint distribution of X and Y, find the marginal distributions:*

| Y \ X | 0 | 1 | 2 |
|---|---|---|---|
| 0 | $\frac{3}{28}$ | $\frac{9}{28}$ | $\frac{3}{28}$ |
| 1 | $\frac{3}{14}$ | $\frac{3}{14}$ | 0 |
| 2 | $\frac{1}{28}$ | 0 | 0 |

### Solution

Marginal distributions

| Y \ X | 0 | 1 | 2 | $p_Y(y)$ |
|---|---|---|---|---|
| 0 | $\frac{3}{28}$ | $\frac{9}{28}$ | $\frac{3}{28}$ | $\frac{15}{28}$ |
| 1 | $\frac{3}{14}$ | $\frac{3}{14}$ | 0 | $\frac{6}{14}$ |
| 2 | $\frac{1}{28}$ | 0 | 0 | $\frac{1}{28}$ |
| $p_X(x)$ | $\frac{10}{28}$ | $\frac{15}{28}$ | $\frac{3}{28}$ | $\Sigma p(x) = 1$ <br> $\Sigma p(y) = 1$ |

Marginal distributions of $X$

$$P(X=0) = P(X=0, Y=0) + P(X=0, Y=1) + P(X=0, Y=2)$$

$$= \frac{3}{28} + \frac{3}{14} + \frac{1}{28}$$

$$= \frac{10}{28}$$

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$$P(X=1)=P(X=1,Y=0)+P(X=1,Y=1)+P(X=1,Y=2)$$
$$=\frac{9}{28}+\frac{3}{14}+0$$
$$=\frac{15}{28}$$

$$P(X=2)=P(X=2,Y=0)+P(X=2,Y=1)+P(X=2,Y=2)$$
$$=\frac{3}{28}+0+0$$
$$=\frac{3}{28}$$

Marginal distributions of $Y$

$$P(Y=0)=P(X=0,Y=0)+P(X=1,Y=0)+P(X=2,Y=0)$$
$$=\frac{3}{28}+\frac{9}{28}+\frac{3}{28}$$
$$=\frac{15}{28}$$

$$P(Y=1)=P(X=0,Y=1)+P(X=1,Y=1)+P(X=2,Y=1)$$
$$=\frac{3}{14}+\frac{3}{14}+0$$
$$=\frac{6}{14}$$

$$P(Y=2)=P(X=0,Y=2)+P(X=1,Y=2)+P(X=2,Y=2)$$
$$=\frac{1}{28}+0+0$$
$$=\frac{1}{28}$$

## Example 3

*The joint distribution of X and Y is given by*

$$f(x,y)=\frac{x+y}{21},\ x=1,2,3;\quad y=1,2$$

*Find the marginal distributions.*



### Solution

Marginal distributions

| $Y$ \ $X$ | 1 | 2 | 3 | $p_Y(y)$ |
|---|---|---|---|---|
| 1 | $\frac{2}{21}$ | $\frac{3}{21}$ | $\frac{4}{21}$ | $\frac{9}{21}$ |
| 2 | $\frac{3}{21}$ | $\frac{4}{21}$ | $\frac{5}{21}$ | $\frac{12}{21}$ |
| $p_X(x)$ | $\frac{5}{21}$ | $\frac{7}{21}$ | $\frac{9}{21}$ | $\Sigma p(x)=1$ <br> $\Sigma p(y)=1$ |

Marginal distributions of $X$

$$P(X=1)=P(X=1,Y=1)+P(X=1,Y=2)$$
$$=\frac{2}{21}+\frac{3}{21}$$
$$=\frac{5}{21}$$

$$P(X=2)=P(X=2,Y=1)+P(X=2,Y=2)$$
$$=\frac{3}{21}+\frac{4}{21}$$
$$=\frac{7}{21}$$

$$P(X=3)=P(X=3,Y=1)+P(X=3,Y=2)$$
$$=\frac{4}{21}+\frac{5}{21}$$
$$=\frac{9}{21}$$

Marginal distributions of $Y$

$$P(Y=1)=P(X=1,Y=1)+P(X=2,Y=1)+P(X=3,Y=1)$$
$$=\frac{2}{21}+\frac{3}{21}+\frac{4}{21}$$
$$=\frac{9}{21}$$

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$$P(Y=2)=P(X=1,Y=2)+P(X=2,Y=2)+P(X=3,Y=2)$$
$$=\frac{3}{21}+\frac{4}{21}+\frac{5}{21}$$
$$=\frac{12}{21}$$

## Example 4

*Given is the joint distribution of X and Y*

| Y \ X | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0.02 | 0.08 | 0.1 |
| 1 | 0.05 | 0.2 | 0.25 |
| 2 | 0.03 | 0.12 | 0.15 |

*Find (i) marginal distributions (ii) the conditional distributions of X given Y = 0.*

### Solution

Marginal distributions

| Y \ X | 0 | 1 | 2 | $p_Y(y)$ |
|---|---|---|---|---|
| 0 | 0.02 | 0.08 | 0.1 | 0.2 |
| 1 | 0.05 | 0.2 | 0.25 | 0.5 |
| 2 | 0.03 | 0.12 | 0.15 | 0.3 |
| $p_X(x)$ | 0.1 | 0.4 | 0.5 | $\Sigma p(x)=1$ <br> $\Sigma p(y)=1$ |

Marginal distributions of X

$$P(X=0)=P(X=0,Y=0)+P(X=0,Y=1)+P(X=0,Y=2)$$
$$=0.02+0.05+0.03$$
$$=0.1$$

$$P(X=1)=P(X=1,Y=0)+P(X=1,Y=1)+P(X=1,Y=2)$$
$$=0.08+0.2+0.12$$
$$=0.4$$



$$P(X=2)=P(X=2,Y=0)+P(X=2,Y=1)+P(X=2,Y=2)$$
$$=0.1+0.25+0.15$$
$$=0.5$$

Marginal distributions of Y

$$P(Y=0)=P(X=0,Y=0)+P(X=1,Y=0)+P(X=2,Y=0)$$
$$=0.02+0.08+0.1$$
$$=0.2$$

$$P(Y=1)=P(X=0,Y=1)+P(X=1,Y=1)+P(X=2,Y=1)$$
$$=0.05+0.2+0.25$$
$$=0.5$$

$$P(Y=2)=P(X=0,Y=2)+P(X=1,Y=2)+P(X=2,Y=2)$$
$$=0.03+0.12+0.15$$
$$=0.3$$

Conditional distributions of X for Y = 0

$$P(X=0/Y=0)=\frac{P(X=0,Y=0)}{P(Y=0)}=\frac{0.02}{0.2}=0.1$$

$$P(X=1/Y=0)=\frac{P(X=1,Y=0)}{P(Y=0)}=\frac{0.08}{0.2}=0.4$$

$$P(X=2/Y=0)=\frac{P(X=2,Y=0)}{P(Y=0)}=\frac{0.1}{0.2}=0.5$$

| $X=x$ | 0 | 1 | 2 |
|---|---|---|---|
| $P(X=x/y=0)$ | 0.1 | 0.4 | 0.5 |

## Example 5

*The joint probability distribution of two random variables X and Y is given by*

$$P(X=0,Y=1)=\frac{1}{3},\ P(X=1,Y=-1)=\frac{1}{3}\text{ and } P(X=1,Y=1)=\frac{1}{3}.$$

*Find (i) marginal distributions of X and Y and (ii) the conditional probability distributions of X given Y = 1.*

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## Solution

Marginal distributions

| Y \ X | −1 | 0 | 1 | Marginal Y $p_Y(y)$ |
|---|---|---|---|---|
| −1 | 0 | 0 | $\frac{1}{3}$ | $\frac{1}{3}$ |
| 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{2}{3}$ |
| Marginal X $p_X(x)$ | 0 | $\frac{1}{3}$ | $\frac{2}{3}$ | $\Sigma p(y) = 1$ $\Sigma p(x) = 1$ |

Marginal distributions of $X$

$$P(X=-1) = P(X=-1, Y=-1) + P(X=-1, Y=0) + P(X=-1, Y=1)$$
$$= 0$$

$$P(X=0) = P(X=0, Y=-1) + P(X=0, Y=0) + P(X=0, Y=1)$$
$$= 0 + 0 + \frac{1}{3}$$
$$= \frac{1}{3}$$

$$P(X=1) = P(X=1, Y=-1) + P(X=1, Y=0) + P(X=1, Y=1)$$
$$= \frac{1}{3} + 0 + \frac{1}{3}$$
$$= \frac{2}{3}$$

Marginal distributions of $Y$

$$P(Y=-1) = P(X=-1, Y=-1) + P(X=0, Y=-1) + P(X=1, Y=-1)$$
$$= 0 + 0 + \frac{1}{3}$$
$$= \frac{1}{3}$$

$$P(Y=0) = P(X=-1, Y=0) + P(X=0, Y=0) + P(X=1, Y=0)$$
$$= 0$$



$$P(Y=1) = P(X=-1, Y=1) + P(X=0, Y=1) + P(X=1, Y=1)$$
$$= 0 + \frac{1}{3} + \frac{1}{3}$$
$$= \frac{2}{3}$$

Conditional Probability distributions of $X$ given $Y = 1$ is

$$P(X=x/Y=y) = \frac{P(X=x, Y=y)}{P(Y=y)}$$

$$P(X=-1/Y=1) = \frac{P(X=-1, Y=1)}{P(Y=1)} = 0$$

$$P(X=0/Y=1) = \frac{P(X=0, Y=1)}{P(Y=1)} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$$

$$P(X=1/Y=1) = \frac{P(X=1, Y=1)}{P(Y=1)} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$$

## Example 6

*If the joint probability mass function of (X, Y) is given by*

$$P(x, y) = k(2x + 3y), x = 0, 1, 2; y = 1, 2, 3$$

*Find all the marginal probability distribution. Also, find the probability distribution of (X + Y).*

## Solution

$$P(x, y) = k(2x + 3y)$$

Marginal distributions

| Y \ X | 0 | 1 | 2 | $p_Y(y)$ |
|---|---|---|---|---|
| 1 | 3 k | 5 k | 7 k | 15 k |
| 2 | 6 k | 8 k | 10 k | 24 k |
| 3 | 9 k | 11 k | 13 k | 33 k |
| $p_X(x)$ | 18 k | 24 k | 30 k | 72 k |

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## Solution

Marginal distributions

| Y \ X | −1 | 0 | 1 | Marginal Y $p_Y(y)$ |
|---|---|---|---|---|
| −1 | 0 | 0 | $\frac{1}{3}$ | $\frac{1}{3}$ |
| 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{2}{3}$ |
| Marginal X $p_X(x)$ | 0 | $\frac{1}{3}$ | $\frac{2}{3}$ | $\Sigma p(y) = 1$ $\Sigma p(x) = 1$ |

Marginal distributions of $X$

$$P(X=-1) = P(X=-1, Y=-1) + P(X=-1, Y=0) + P(X=-1, Y=1)$$
$$= 0$$

$$P(X=0) = P(X=0, Y=-1) + P(X=0, Y=0) + P(X=0, Y=1)$$
$$= 0 + 0 + \frac{1}{3}$$
$$= \frac{1}{3}$$

$$P(X=1) = P(X=1, Y=-1) + P(X=1, Y=0) + P(X=1, Y=1)$$
$$= \frac{1}{3} + 0 + \frac{1}{3}$$
$$= \frac{2}{3}$$

Marginal distributions of $Y$

$$P(Y=-1) = P(X=-1, Y=-1) + P(X=0, Y=-1) + P(X=1, Y=-1)$$
$$= 0 + 0 + \frac{1}{3}$$
$$= \frac{1}{3}$$

$$P(Y=0) = P(X=-1, Y=0) + P(X=0, Y=0) + P(X=1, Y=0)$$
$$= 0$$



$$P(Y=1) = P(X=-1, Y=1) + P(X=0, Y=1) + P(X=1, Y=1)$$
$$= 0 + \frac{1}{3} + \frac{1}{3}$$
$$= \frac{2}{3}$$

Conditional Probability distributions of $X$ given $Y = 1$ is

$$P(X=x/Y=y) = \frac{P(X=x, Y=y)}{P(Y=y)}$$

$$P(X=-1/Y=1) = \frac{P(X=-1, Y=1)}{P(Y=1)} = 0$$

$$P(X=0/Y=1) = \frac{P(X=0, Y=1)}{P(Y=1)} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$$

$$P(X=1/Y=1) = \frac{P(X=1, Y=1)}{P(Y=1)} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$$

## Example 6

*If the joint probability mass function of (X, Y) is given by*

$$P(x, y) = k(2x + 3y),\ x = 0, 1, 2;\ y = 1, 2, 3$$

*Find all the marginal probability distribution. Also, find the probability distribution of (X + Y).*

## Solution

$$P(x, y) = k(2x + 3y)$$

Marginal distributions

| Y \ X | 0 | 1 | 2 | $p_Y(y)$ |
|---|---|---|---|---|
| 1 | 3 k | 5 k | 7 k | 15 k |
| 2 | 6 k | 8 k | 10 k | 24 k |
| 3 | 9 k | 11 k | 13 k | 33 k |
| $p_X(x)$ | 18 k | 24 k | 30 k | 72 k |

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$$\because \quad \Sigma p(x) = \Sigma p(y) = 1$$
$$72k = 1$$
$$k = \frac{1}{72}$$

Marginal distributions of X and Y

| Y \ X | 0 | 1 | 2 | $p_Y(y)$ |
|---|---|---|---|---|
| 1 | $\frac{3}{72}$ | $\frac{5}{72}$ | $\frac{7}{72}$ | $\frac{15}{72}$ |
| 2 | $\frac{6}{72}$ | $\frac{8}{72}$ | $\frac{10}{72}$ | $\frac{24}{72}$ |
| 3 | $\frac{9}{72}$ | $\frac{11}{72}$ | $\frac{13}{72}$ | $\frac{33}{72}$ |
| $p_X(x)$ | $\frac{18}{72}$ | $\frac{24}{72}$ | $\frac{30}{72}$ | 1 |

Probability distribution of $(X + Y)$

| $X + Y$ | $P$ |
|---|---|
| 1 | $p_{01} = \frac{3}{72}$ |
| 2 | $p_{02} + p_{11} = \frac{11}{72}$ |
| 3 | $p_{03} + p_{12} + p_{21} = \frac{24}{72}$ |
| 4 | $p_{13} + p_{22} = \frac{21}{72}$ |
| 5 | $p_{23} = \frac{13}{72}$ |
| | Total = 1 |



## Example 7

*Let X and Y have the following marginal probability distributions:*

| X \ Y | 0 | 1 | 2 | $p_X(x)$ |
|---|---|---|---|---|
| 0 | 0.1 | 0.04 | 0.06 | 0.2 |
| 1 | 0.2 | 0.08 | 0.12 | 0.4 |
| 2 | 0.2 | 0.08 | 0.12 | 0.4 |
| $p_Y(y)$ | 0.5 | 0.2 | 0.3 | $\Sigma p(x) = 1$<br>$\Sigma p(y) = 1$ |

### Solution

X and Y are independent, if $p_{ij} = p_{i*}\ p_{*j}$ for all $i$ and $j$.

$$p_{0*} = 0.1 + 0.04 + 0.06 = 0.2$$
$$p_{1*} = 0.2 + 0.08 + 0.12 = 0.4$$
$$p_{2*} = 0.2 + 0.08 + 0.12 = 0.4$$
$$p_{*0} = 0.1 + 0.2 + 0.2 = 0.5$$
$$p_{*1} = 0.04 + 0.08 + 0.08 = 0.2$$
$$p_{*2} = 0.06 + 0.12 + 0.12 = 0.3$$

Now,
$$p_{0*}\ p_{*0} = (0.2)\ (0.5) = 0.1 = p_{00}$$
$$p_{0*}\ p_{*1} = (0.2)\ (0.2) = 0.04 = p_{01}$$
$$p_{0*}\ p_{*2} = (0.2)\ (0.3) = 0.06 = p_{02}$$

Similarly, it can be verified that
$$p_{1*}\ p_{*0} = p_{10};\ p_{1*}\ p_{*1} = p_{11};\ p_{1*} p_{*2} = p_{12}$$
$$p_{2*}\ p_{*0} = p_{20};\ p_{2*}\ p_{*1} = p_{21};\ p_{2*} p_{*2} = p_{22}$$

Hence, the random variables X and Y are independent.

## EXERCISE 2.3

1. Find the marginal distributions of X and Y from the bivariate distribution of (X, Y) given below:

| X \ Y | 1 | 2 |
|---|---|---|
| 1 | 0.1 | 0.2 |
| 2 | 0.3 | 0.4 |

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Ans.:

| $X = x$ | 1 | 2 |
|---|---|---|
| $P(X = x)$ | 0.3 | 0.7 |

| $Y = y$ | 1 | 2 |
|---|---|---|
| $P(Y = y)$ | 0.4 | 0.6 |

2. For the joint probability distribution of two random variables $X$ and $Y$ given below:

| X \ Y | 1 | 2 | 3 | 4 | Total |
|---|---|---|---|---|---|
| 1 | $\frac{4}{36}$ | $\frac{3}{36}$ | $\frac{2}{36}$ | $\frac{1}{36}$ | $\frac{10}{36}$ |
| 2 | $\frac{1}{36}$ | $\frac{3}{36}$ | $\frac{3}{36}$ | $\frac{2}{36}$ | $\frac{9}{36}$ |
| 3 | $\frac{5}{36}$ | $\frac{1}{36}$ | $\frac{1}{36}$ | $\frac{1}{36}$ | $\frac{8}{36}$ |
| 4 | $\frac{1}{36}$ | $\frac{2}{36}$ | $\frac{1}{36}$ | $\frac{5}{36}$ | $\frac{9}{36}$ |
| Total | $\frac{11}{36}$ | $\frac{9}{36}$ | $\frac{7}{36}$ | $\frac{9}{36}$ | 1 |

Find (i) marginal distributions of $X$ and $Y$.

(ii) Conditional distributions of $X$ given the value of $Y = 1$ and that of $Y$ given the value of $X = 2$.

Ans.: (i)

| Value of $X = x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $P(X = x)$ | $\frac{10}{36}$ | $\frac{9}{36}$ | $\frac{8}{36}$ | $\frac{9}{36}$ |

| Value of $Y = y$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $P(Y = y)$ | $\frac{11}{36}$ | $\frac{9}{36}$ | $\frac{7}{36}$ | $\frac{9}{36}$ |

(ii)

| $Y = y$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $P(X = x/Y = 1)$ | $\frac{4}{11}$ | $\frac{1}{11}$ | $\frac{5}{11}$ | $\frac{1}{11}$ |

| $Y = y$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $P(Y = y/X = 2)$ | $\frac{1}{9}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{2}{9}$ |

3. A two-dimensional random variable $(X, Y)$ has the joint probability mass function $p(x, y) = \frac{2x + y}{27}$, where $x$ and $y$ can assume only the integer values 0, 1 and 2. Find the conditional distributions of $Y$ for $X = x$.



| Y \ X | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | $\frac{1}{3}$ | $\frac{2}{3}$ |
| 1 | $\frac{2}{9}$ | $\frac{3}{9}$ | $\frac{4}{9}$ |
| 2 | $\frac{4}{15}$ | $\frac{5}{15}$ | $\frac{6}{15}$ |

4. Let $X$ and $Y$ have the following joint probability distribution:

| Y \ X | 0 | 1 |
|---|---|---|
| 0 | 0.1 | 0.15 |
| 1 | 0.2 | 0.3 |
| 2 | 0.1 | 0.15 |

Show that $X$ and $Y$ are independent.

5. The joint probability distribution of $X$ and $Y$ is given below:

| Y \ X | 1 | 2 | 3 |
|---|---|---|---|
| 2 | $\frac{1}{8}$ | $\frac{1}{24}$ | $\frac{1}{12}$ |
| 4 | $\frac{1}{4}$ | $\frac{1}{4}$ | 0 |
| 6 | $\frac{1}{8}$ | $\frac{1}{24}$ | $\frac{1}{12}$ |

Find $P(X < 4)$, $P(Y > 1)$, $P(X < 4/Y > 1)$, $P(2 \le X \le 5, Y > 1)$, $P(Y = 3/X = 2)$, $P(X + Y \le 7)$.

[Ans.: $\left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}, \frac{3}{8}, \frac{1}{6}, \frac{19}{24}\right)$]

6. For the following joint probability distribution of $X$ and $Y$, find (i) marginal distributions of $X$ and $Y$, (ii) conditional distributions of $X$ given $Y = 2$, (iii) Are $X$ and $Y$ independent?

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The conditional probability density function of $Y$, given $X = x$, denoted by $f(y/x)$ is defined as:

$$f(y/x) = \frac{f(x,y)}{f_X(x)}$$

A necessary and sufficient condition for the continuous random variables $X$ and $Y$ to be independent is

$$f(x,y) = f_X(x)\, f_Y(y)$$

## Example 1

*The joint probability density function of a two dimensional random variable is*

$$f(x,y) = \frac{1}{2}xe^{-y}, \quad 0 < x < 2, y > 0$$
$$= 0, \quad \text{, otherwise}$$

*Find the cumulative distribution function.*

### Solution

The cumulative distribution function is given by

$$F(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f(x,y)\,dx\,dy$$
$$= \int_0^y\int_0^x \frac{1}{2}xe^{-y}dx\,dy$$
$$= \frac{1}{2}\int_0^y e^{-y}\left|\frac{x^2}{2}\right|_0^x dy$$
$$= \frac{1}{4}x^2\left|-e^{-y}\right|_0^y$$
$$= \frac{1}{4}x^2(-e^{-y}+e^0)$$
$$= \frac{1}{4}x^2(-e^{-y}+1)$$

$$F(x,y) = \frac{1}{4}x^2(1-e^{-y}), \quad 0 < x < 2, y > 0$$
$$= 0 \quad \text{, otherwise}$$



## Example 2

*The joint probability density function of a two dimensional random variable (X, Y) is* $f(x,y) = xe^{-x(y+1)}, x > 0, y > 0$. *Examine whether the variables X and Y are independent.*

### Solution

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$
$$= \int_0^{\infty} xe^{-x(y+1)}dy$$
$$= x\left|\frac{e^{-x(y+1)}}{-x}\right|_0^{\infty}$$
$$= -(e^{-\infty} - e^{-x})$$
$$= e^{-x}, x > 0$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$
$$= \int_0^{\infty} xe^{-x(y+1)}dx$$
$$= \left|x\frac{e^{-x(y+1)}}{-(y+1)} - 1\frac{e^{-x(y+1)}}{(y+1)^2}\right|_0^{\infty}$$
$$= \frac{1}{(y+1)^2}, y > 0$$

$$f_X(x)\cdot f_Y(y) = e^{-x}\frac{1}{(y+1)^2}$$
$$f(x,y) = xe^{-x(y+1)}$$
$$f(x,y) \neq f_X(x)\cdot f_Y(y)$$

Hence, $X$ and $Y$ are not independent.

## Example 3

*Two random variables X and Y have the joint pdf*

$$f(x,y) = Ae^{-(2x+y)}, \quad x, y \geq 0$$
$$= 0 \quad \text{, otherwise}$$

*Find* (i) *A* (ii) *marginal pdf of X and Y* (iii) *f(y/x)*

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### Solution

(i) Since $f(x, y)$ is a pdf,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\mathrm{d}x\,\mathrm{d}y = 1$$

$$\int_{0}^{\infty}\int_{0}^{\infty} Ae^{-(2x+y)}\mathrm{d}x\,\mathrm{d}y = 1$$

$$A\int_{0}^{\infty}\left[\int_{0}^{\infty} e^{-2x}\mathrm{d}x\right]e^{-y}\mathrm{d}y = 1$$

$$A\int_{0}^{\infty}\left|\frac{e^{-2x}}{-2}\right|_{0}^{\infty} e^{-y}\mathrm{d}y = 1$$

$$\frac{A}{-2}\int_{0}^{\infty}(e^{-\infty}-e^{0})e^{-y}\mathrm{d}y = 1$$

$$\frac{A}{-2}\int_{0}^{\infty}(-1)e^{-y}\mathrm{d}y = 1$$

$$\frac{A}{2}\left|-e^{-y}\right|_{0}^{\infty} = 1$$

$$\frac{A}{2}(-e^{-\infty}+e^{0}) = 1$$

$$A = 2$$

(ii) $$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\mathrm{d}y$$

$$= \int_{0}^{\infty} Ae^{-(2x+y)}\mathrm{d}y$$

$$= 2\left|-e^{-(2x+y)}\right|_{0}^{\infty}$$

$$= 2(-e^{-\infty}+e^{-2x})$$

$$= 2e^{-2x}, x \geq 0$$

$$\therefore \quad f_X(x) = 2e^{-2x}, \quad x \geq 0$$

$$= 0 \quad , \quad x < 0$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,\mathrm{d}x$$



$$= \int_{0}^{\infty} Ae^{-(2x+y)}\mathrm{d}x$$

$$= 2\left|\frac{e^{-(2x+y)}}{-2}\right|_{0}^{\infty}$$

$$= -\frac{2}{2}(e^{-\infty}-e^{-y})$$

$$= -(0-e^{-y})$$

$$= e^{-y}, y \geq 0$$

$$\therefore f_Y(y) = e^{-y}, \ y \geq 0$$

$$= 0 \ , \ y < 0$$

(iii) $$f(y/x) = \frac{f(x,y)}{f_X(x)}$$

$$= \frac{2e^{-(2x+y)}}{2e^{-2x}}$$

$$= e^{-y}, \ y \geq 0$$

## Example 4

*The joint probability distribution of X and Y is given by*

$$f(x,y) = \frac{6-x-y}{8}, \quad 0 < x < 2, \ 2 < y < 4$$

$$= 0, \qquad \text{otherwise}$$

*Find* $f(y/x = 2)$.

### Solution

$$f(y/x) = \frac{f(x,y)}{f_X(x)}$$

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,\mathrm{d}y$$

$$= \int_{2}^{4}\frac{6-x-y}{8}\,\mathrm{d}y$$

$$= \frac{1}{8}\left|6y - xy - \frac{y^2}{2}\right|_{2}^{4}$$

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$$= \frac{1}{8}[(24-4x-8)-(12-2x-2)]$$

$$= \frac{1}{8}(6-2x)$$

$$f(y/x) = \frac{6-x-y}{6-2x}$$

Putting $x = 2$,

$$f(y/x=2) = \frac{4-y}{2}$$

## Example 5

*The joint pdf of a two dimensional variable (X, Y) is given by*

$$f(x,y) = kxye^{-(x^2+y^2)}, \quad x>0, y>0.$$

*Find the value of k and prove that X and Y are independent.*

### Solution

Since $f(x, y)$ is a pdf,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

$$\int_0^{\infty}\int_0^{\infty} k\,x\,y\,e^{-(x^2+y^2)}dx\,dy = 1$$

$$k\int_0^{\infty} ye^{-y^2}dy \cdot \int_0^{\infty} xe^{-x^2}dx = 1 \qquad \ldots(1)$$

Putting $x^2 = t, x = \sqrt{t}, dx = \frac{1}{2\sqrt{t}}dt$

When $x = 0,\ t = 0$

When $x = \infty,\ t = \infty$

$$\int_0^{\infty} xe^{-x^2}dx = \int_0^{\infty}\sqrt{t}e^{-t}\frac{1}{2\sqrt{t}}dt$$

$$= \frac{1}{2}\left|-e^{-t}\right|_0^{\infty}$$

$$= \frac{1}{2}(-e^{-\infty}+e^0)$$

$$= \frac{1}{2}$$



Similarly, $\int_0^{\infty} ye^{-y^2}dy = \frac{1}{2}$

Putting both integral values in Eq. (1),

$$k\cdot\frac{1}{2}\cdot\frac{1}{2} = 1$$

$$k = 4$$

If $X$ and $Y$ are independent,

$$f_X(x)\cdot f_Y(y) = f(x,y)$$

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_0^{\infty} k\,x\,y\,e^{-(x^2+y^2)}dy$$

$$= k\,x\,e^{-x^2}\int_0^{\infty} y\,e^{-y^2}dy$$

$$= 4x\,e^{-x^2}\cdot\frac{1}{2}$$

$$= 2x\,e^{-x^2}, x>0$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \int_0^{\infty} k\,x\,y\,e^{-(x^2+y^2)}dx$$

$$= k\,y\,e^{-y^2}\int_0^{\infty} xe^{-x^2}dx$$

$$= 4y\,e^{-y^2}\cdot\frac{1}{2}$$

$$= 2ye^{-y^2}, y>0$$

$$f_X(x)\cdot f_Y(y) = 2x\,e^{-x^2}\cdot 2y\,e^{-y^2}, x,y>0$$

$$= 4xy\,e^{-(x^2+y^2)}$$

$$= f(x,y), \quad x>0, y>0$$

Hence, $X$ and $Y$ are independent.

## Example 6

*The joint probability density function of a two dimensional random variable (X, Y) is*

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$$f(x,y) = kx(x-y), \quad 0<x<2, -x<y<x$$
$$= 0 \quad , \quad \text{elsewhere}$$

*Find* (i) $k$ (ii) $f_X(x)$ (iii) $f_Y(y)$ (iv) $f(y/x)$

## Solution

Since $f(x, y)$ is a probability density function,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

$$\int_0^2\int_{-x}^{x} kx(x-y)\,dy\,dx = 1$$

$$k\int_0^2 x\left|xy-\frac{y^2}{2}\right|_{-x}^{x} dx = 1$$

$$k\int_0^2 x\left[\left(x^2-\frac{x^2}{2}\right)-\left(-x^2-\frac{x^2}{2}\right)\right]dx = 1$$

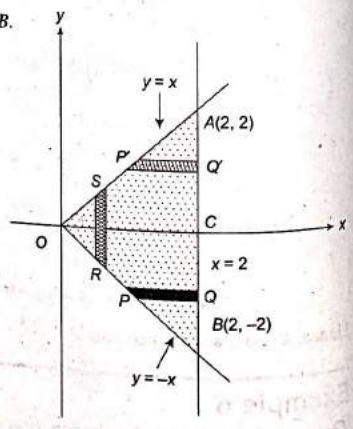
$$k\int_0^2 2x^3 dx = 1$$

$$k\left|\frac{2x^4}{4}\right|_0^2 = 1$$

$$k(8) = 1$$

$$k = \frac{1}{8}$$

(ii) The region of integration is $\Delta OAB$.
In $\Delta OAB$, along vertical strip $RS$,
Limits of $y$ : $y = -x$ to $y = x$
and $x$ varies from $x = 0$ to $x = 2$.




$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_{-x}^{x} kx(x-y)\,dy$$

$$= kx\left|xy-\frac{y^2}{2}\right|_{-x}^{x}$$

$$= kx\left[\left(x^2-\frac{x^2}{2}\right)-\left(-x^2-\frac{x^2}{2}\right)\right]$$

$$= kx\,(2x^2)$$

$$= \frac{1}{8}(2x^3)$$



$$= \frac{x^3}{4}, \quad 0<x<2$$

(iii) For limits of $x$, $\Delta OAB$ is divided into two parts, $\Delta OBC$ and $\Delta OAC$.
In $\Delta OBC$, along horizontal strip $PQ$,
Limits of $x$: $x = -y$ to $x = 2$ and $y$ varies from $y = -2$ to $y = 0$.
In $\Delta OAC$, along horizontal strip $P'Q'$,
Limits of $x$: $x = y$ to $x = 2$ and $y$ varies from $y = 0$ to $y = 2$.

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \int_{-y}^{2} kx(x-y)\,dx, \quad -2 \le y \le 0$$

$$= \int_{y}^{2} kx(x-y)\,dx, \quad 0 \le y \le 2$$

Now,

$$\int_{-y}^{2} kx(x-y)\,dx = k\left|\frac{x^3}{3}-\frac{x^2y}{2}\right|_{-y}^{2}$$

$$= k\left[\left(\frac{8}{3}-2y\right)-\left(-\frac{y^3}{3}-\frac{y^3}{2}\right)\right]$$

$$= \frac{1}{8}\left(\frac{8}{3}-2y+\frac{5y^3}{6}\right)$$

$$= \frac{1}{3}-\frac{y}{4}+\frac{5}{48}y^3$$

Also,

$$\int_{y}^{2} kx(x-y)\,dx = k\left|\frac{x^3}{3}-\frac{x^2y}{2}\right|_{y}^{2}$$

$$= k\left[\left(\frac{8}{3}-2y\right)-\left(\frac{y^3}{3}-\frac{y^3}{2}\right)\right]$$

$$= \frac{1}{8}\left(\frac{8}{3}-2y+\frac{y^3}{6}\right)$$

$$= \frac{1}{3}-\frac{y}{4}+\frac{y^3}{48}$$

Hence, $f_Y(y) = \frac{1}{3}-\frac{y}{4}+\frac{5}{48}y^3, \quad -2 \le y \le 0$

$$= \frac{1}{3}-\frac{y}{4}+\frac{y^3}{48}, \quad 0 \le y \le 2$$

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(iv) $f(y/x) = \dfrac{f(x,y)}{f_X(x)}$

$$= \frac{\frac{1}{8}x(x-y)}{\frac{x^3}{4}}$$

$$= \frac{x-y}{2x^2}, \quad -x < y < x$$

## Example 7

*The joint pdf of a two-dimensional random variable (X, Y) is given by*

$$f_{XY}(x,y) = \frac{8}{9}xy, \qquad 1 \le y \le 2, \quad 1 \le x \le y$$
$$= 0 \quad , \qquad \text{otherwise}$$

*Find the marginal density function of X and Y.*

### Solution

The region of integration is $\Delta ABC$.
In $\Delta ABC$, along vertical strip $PQ$,
limits of $y$: $y = x$ to $y = 2$
and $x$ varies from $x = 1$ to $x = 2$.
Marginal density function of $X$ is

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\mathrm{d}y$$
$$= \int_x^2 \frac{8}{9}xy \, \mathrm{d}y$$
$$= \frac{8}{9}x\left|\frac{y^2}{2}\right|_x^2$$
$$= \frac{4}{9}x(4-x^2), \qquad 1 \le x \le 2$$

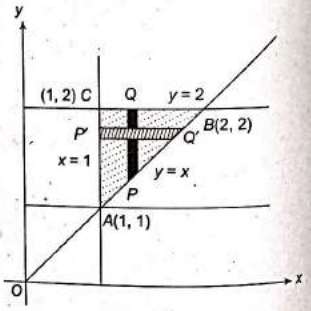


Fig. 2.2

In $\Delta ABC$, along horizontal strip $P'Q'$, limits of $x : x = 1$ to $x = y$ and $y$ varies from $y = 1$ to $y = 2$.
Marginal density function of $Y$ is

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\mathrm{d}x$$
$$= \int_1^y \frac{8}{9}xy \, \mathrm{d}x$$



$$= \frac{8}{9}y\left|\frac{x^2}{2}\right|_1^y$$
$$= \frac{4}{9}y(y^2-1), \qquad 1 \le y \le 2$$

## Example 8

*If the joint distribution function of X and Y is given by*

$$F(x,y) = (1-e^{-x})(1-e^{-y}), \qquad x > 0, y > 0$$
$$= 0 \qquad , \qquad \text{otherwise}$$

*Find* $f_X(x), f_Y(y)$ (ii) *Are X and Y are independent* (iii) *Find* $P(1 < X < 3, 1 < Y < 2)$.

### Solution

$$F(x,y) = (1-e^{-x})(1-e^{-y})$$

The joint pdf is given by

$$f(x,y) = \frac{\partial^2 F}{2x\partial y}$$
$$= \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right)$$
$$= \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}(1-e^{-x})(1-e^{-y})\right]$$
$$= \frac{\partial}{\partial x}(1-e^{-x})(e^{-y})$$
$$= e^{-x}e^{-y}$$
$$= e^{-(x+y)}, x > 0, y > 0$$

$$\therefore \quad f(x,y) = e^{-(x+y)}, \quad x > 0, y > 0$$
$$= 0, \quad , \text{ otherwise}$$

(i) $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,\mathrm{d}y$

$$= \int_0^{\infty} e^{-(x+y)}\mathrm{d}y$$
$$= \left|-e^{-(x+y)}\right|_0^{\infty}$$
$$= (-e^{-\infty} + e^{-x})$$
$$= e^{-x}, x > 0$$

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$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$
$$= \int_0^{\infty} e^{-(x+y)}dx$$
$$= \left|-e^{-(x+y)}\right|_0^{\infty}$$
$$= (-e^{-\infty} + e^{-y})$$
$$= e^{-y}, x > 0$$

(ii) $f_X(x) \cdot f_Y(y) = e^{-x} \cdot e^{-y}$
$$= e^{-(x+y)}, x > 0, y > 0$$
$$= f(x,y)$$

Hence, $X$ and $Y$ are independent.

(iii) Since $X$ and $Y$ are independent,
$$P(1 < X < 3, 1 < Y < 2) = P(1 < X < 3) \cdot P(1 < Y < 2)$$
$$= \int_1^3 f_X(x)\,dx \cdot \int_1^2 f_Y(y)\,dy$$
$$= \int_1^3 e^{-x}\,dx \cdot \int_1^2 e^{-y}\,dy$$
$$= \left|-e^{-x}\right|_1^3 \cdot \left|-e^{-y}\right|_1^2$$
$$= (-e^{-3} + e^{-1}) \cdot (-e^{-2} + e^{-1})$$
$$= e^{-5} - e^{-4} - e^{-3} - e^{-2}$$

## Example 9

*The joint probability density of two random variables is given by*

$$f(x,y) = 15e^{-3x-5y}, \quad x > 0, y > 0$$
$$= 0 \quad , \text{ elsewhere}$$

*Find* (i) $P(1 < X < 2, 0.2 < Y < 0.3)$ (ii) $P(X < 2, Y > 0.2)$ (iii) *marginal probability density functions of X and Y.*

### Solution

(i) $P(1 < X < 2, 0.2 < Y < 0.3) = \int_{0.2}^{0.3}\int_1^2 f(x,y)\,dxdy$
$$= \int_{0.2}^{0.3}\int_1^2 15e^{-3x-5y}dxdy$$
$$= 15\int_{0.2}^{0.3} e^{-5y}\left[\int_1^2 e^{-3x}dx\right]dy$$



$$= 15\int_{0.2}^{0.3} e^{-5y}\left|\frac{e^{-3x}}{-3}\right|_1^2 dy$$
$$= -5\int_{0.2}^{0.3} e^{-5y}(e^{-6} - e^{-3})dy$$
$$= -5(e^{-6} - e^{-3})\left|\frac{e^{-5y}}{-5}\right|_{0.2}^{0.3}$$
$$= (e^{-6} - e^{-3})(e^{-1.5} - e^{-1.0})$$
$$= 6.84 \times 10^{-3}$$

(ii) $P(X < 2, Y > 0.2) = \int_{0.2}^{\infty}\int_0^2 f(x,y)\,dxdy$
$$= \int_{0.2}^{\infty}\int_0^2 15e^{-3x-5y}dxdy$$
$$= 15\int_{0.2}^{\infty}\left[\int_0^2 e^{-3x}dx\right]e^{-5y}dy$$
$$= 15\int_{0.2}^{\infty}\left|\frac{e^{-3x}}{-3}\right|_0^2 e^{-5y}dy$$
$$= -5\int_{0.2}^{\infty}(e^{-6} - 1)e^{-5y}dy$$
$$= -5(e^{-6} - 1)\left|\frac{e^{-5y}}{-5}\right|_{0.2}^{\infty}$$
$$= (e^{-6} - 1)(e^{-\infty} - e^{-1.0})$$
$$= (e^{-6} - 1)(-e^{-1.0})$$
$$= 0.367$$

(iii) The region of integration is the first quadrant. Hence, $x$ and $y$ both varies from 0 to $\infty$.

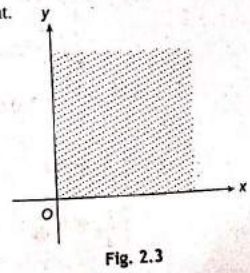


Fig. 2.3

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$
$$= \int_0^{\infty} 15e^{-3x-5y}dy$$
$$= 15e^{-3x}\left|\frac{e^{-5y}}{-5}\right|_0^{\infty}$$

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$$= -3e^{-3x}(e^{-\infty} - e^{0})$$
$$= 3e^{-3x}, \ x > 0$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$
$$= \int_{0}^{\infty} 15e^{-3x-5y}\,dy$$
$$= 15e^{-5y}\left|\frac{e^{-3x}}{-3}\right|_0^{\infty}$$
$$= -5e^{-5y}(e^{-\infty} - e^{0})$$
$$= 5e^{-5y}, \ y > 0$$

## Example 10

*The joint pdf of (X, Y) is given by*

$$f(x,y) = \frac{1}{4}e^{-|x|-|y|}, \ -\infty < x < \infty, -\infty < y < \infty$$

*(i) Are X and Y independent?*
*(ii) Find the probability that $X \le 1$ and $Y < 0$.*

### Solution

$$|x| = -x, \quad -\infty < x \le 0$$
$$= x, \quad 0 \le x < \infty$$

Similarly,
$$|y| = -y, \quad -\infty < y \le 0$$
$$= y, \quad 0 \le y < \infty$$

$$\therefore \quad f(x,y) = \frac{1}{4}e^{-|x|-|y|}$$
$$= \frac{1}{4}e^{x+y}, -\infty < x \le 0, -\infty < y \le 0$$
$$= \frac{1}{4}e^{-x-y}, 0 \le x < 0, 0 \le y < \infty$$

(i) $$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$
$$= \int_{-\infty}^{\infty} \frac{1}{4}e^{-|x|-|y|}\,dy$$



$$= \frac{1}{4}e^{-|x|}\int_{-\infty}^{\infty} e^{-|y|}dy$$
$$= \frac{1}{4}e^{-|x|}\left[\int_{-\infty}^{0} e^{y}dy + \int_{0}^{\infty} e^{-y}dy\right]$$
$$= \frac{1}{4}e^{-|x|}\left[\left|e^{y}\right|_{-\infty}^{0} + \left|-e^{-y}\right|_{0}^{\infty}\right]$$
$$= \frac{1}{4}e^{-|x|}(1+1)$$
$$= \frac{1}{2}e^{-|x|}, \quad -\infty < x < \infty$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$
$$= \int_{-\infty}^{\infty} \frac{1}{4}e^{-|x|-|y|}\,dx$$
$$= \frac{1}{4}e^{-|y|}\int_{-\infty}^{\infty} e^{-|x|}dx$$
$$= \frac{1}{4}e^{-|y|}\left[\int_{-\infty}^{0} e^{x}dx + \int_{0}^{\infty} e^{-x}dx\right]$$
$$= \frac{1}{4}e^{-|y|}\left[\left|e^{x}\right|_{-\infty}^{0} + \left|-e^{-x}\right|_{0}^{\infty}\right]$$
$$= \frac{1}{4}e^{-|y|}(1+1)$$
$$= \frac{1}{2}e^{-|y|}, \quad -\infty < y < \infty$$

$$f_X(x)\cdot f_Y(y) = \frac{1}{2}e^{-|x|}\cdot\frac{1}{2}e^{-|y|}$$
$$= \frac{1}{4}e^{-|x|-|y|}, -\infty < x < \infty, -\infty < y < \infty$$
$$= f(x,y)$$

Hence, $X$ and $Y$ are independent.

(ii) $$P(X \le 1, Y < 0) = \int_{-\infty}^{0}\int_{-\infty}^{1} f(x,y)\,dx\,dy$$
$$= \int_{-\infty}^{0}\int_{-\infty}^{1} \frac{1}{4}e^{-|x|-|y|}\,dx\,dy$$

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$$=\frac{1}{4}\int_{-\infty}^{0} e^{-|y|}\left[\int_{-\infty}^{0} e^{x}dx+\int_{0}^{1} e^{-x}dx\right]dy$$

$$=\frac{1}{4}\int_{-\infty}^{0} e^{-|y|}\left[\left|e^{x}\right|_{-\infty}^{0}+\left|-e^{-x}\right|_{0}^{1}\right]dy$$

$$=\frac{1}{4}\int_{-\infty}^{0} e^{-|y|}(1-e^{-1}+1)\,dy$$

$$=\frac{1}{4}(2-e^{-1})\int_{-\infty}^{0} e^{y}dy$$

$$=\frac{1}{4}(2-e^{-1})\left|e^{y}\right|_{-\infty}^{0}$$

$$=\frac{1}{4}(2-e^{-1})(1)$$

$$=\frac{1}{4}(2-e^{-1})$$

## Example 11

*The joint pdf of (X, Y) is given by*

$$f(x,y)=ke^{-x}\cos y,\quad 0\le x\le 2, 0\le y\le\frac{\pi}{2}$$
$$=0\quad,\quad \text{otherwise}$$

*Find* (i) $k$ (ii) $P\left(X+Y\ge\frac{\pi}{2}\right)$.

### Solution

(i) Since $f(x, y)$ is a pdf,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy=1$$

$$\int_{0}^{\frac{\pi}{2}}\int_{0}^{2} k\,e^{-x}\cos y\,dx\,dy=1$$

$$k\int_{0}^{\frac{\pi}{2}}\cos y\left|-e^{-x}\right|_{0}^{2}dy=1$$

$$k\int_{0}^{\frac{\pi}{2}}\cos y\,(-e^{-2}+1)\,dy=1$$



$$k(1-e^{-2})\left|\sin y\right|_{0}^{\frac{\pi}{2}}=1$$

$$k(1-e^{-2})(1)=1$$

$$k=\frac{1}{1-e^{-2}}$$

(ii) $P\left(X+Y\ge\frac{\pi}{2}\right)=1-P\left(X+Y<\frac{\pi}{2}\right)$

The region of integration $x + y < 1$ is the $\Delta OAB$. In $\Delta OAB$, along horizontal strip $P'Q'$,

Limits of $x: x=0$ to $x=\frac{\pi}{2}-y$

Limits of $y: y=0$ to $y=\frac{\pi}{2}$

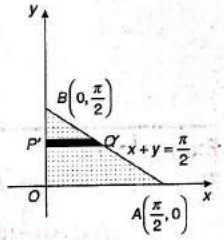


Fig. 2.4

$$P(X+Y<1)=\int_{0}^{\frac{\pi}{2}}\int_{0}^{\frac{\pi}{2}-y} ke^{-x}\cos y\,dx\,dy$$

$$=k\int_{0}^{\frac{\pi}{2}}\cos y\left|-e^{-x}\right|_{0}^{\frac{\pi}{2}-y}dy$$

$$=k\int_{0}^{\frac{\pi}{2}}\cos y\left[-e^{-\left(\frac{\pi}{2}-y\right)}+e^{0}\right]dy$$

$$=k\int_{0}^{\frac{\pi}{2}}\cos y\left[-e^{-\frac{\pi}{2}}e^{y}+1\right]dy$$

$$=k\left[-e^{-\frac{\pi}{2}}\int_{0}^{\frac{\pi}{2}}e^{y}\cos y\,dy+\int_{0}^{\frac{\pi}{2}}\cos y\,dy\right]$$

$$=k\left[-e^{-\frac{\pi}{2}}\left|\frac{e^{y}}{1+1}(\cos y+\sin y)\right|_{0}^{\frac{\pi}{2}}+\left|\sin y\right|_{0}^{\frac{\pi}{2}}\right]$$

$$=k\left[-e^{-\frac{\pi}{2}}\left\{\frac{e^{\frac{\pi}{2}}}{2}\left(\cos\frac{\pi}{2}+\sin\frac{\pi}{2}\right)-\frac{1}{2}\right\}+\sin\frac{\pi}{2}\right]$$

$$=k\left[-e^{-\frac{\pi}{2}}\left\{\frac{e^{\frac{\pi}{2}}}{2}(1)-\frac{1}{2}\right\}+1\right]$$

$$=k\left(-\frac{1}{2}+\frac{e^{-\frac{\pi}{2}}}{2}+1\right)$$

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$$=\frac{k}{2}\left(1+e^{-\frac{\pi}{2}}\right)$$

$$P\left(X+Y\geq\frac{\pi}{2}\right)=1-\frac{k}{2}\left(1+e^{-\frac{\pi}{2}}\right)$$

$$=1-\frac{\left(1+e^{-\frac{\pi}{2}}\right)}{2(1-e^{-2})}$$

## Example 12

*The joint p.d.f of a two-dimensional random variable (X, Y) is given by*

$$f(x,y)=\frac{1}{8}(6-x-y),\qquad 0<x<2,\quad 2<y<4$$
$$=0\qquad\qquad,\quad \text{otherwise}$$

*Find* (i) $P(X<1, Y<3)$ (ii) $P(X<1/Y<3)$.

### Solution

(i) $P(X<1, Y<3)=\int_2^3\int_0^1 f(x,y)\,dx\,dy$

$$=\int_2^3\int_0^1\frac{1}{8}(6-x-y)\,dx\,dy$$

$$=\frac{1}{8}\int_2^3\left|6x-\frac{x^2}{2}-xy\right|_0^1 dy$$

$$=\frac{1}{8}\int_2^3\left(6-\frac{1}{2}-y\right)dy$$

$$=\frac{1}{8}\int_2^3\left(\frac{11}{2}-y\right)dy$$

$$=\frac{1}{8}\left|\frac{11}{2}y-\frac{y^2}{2}\right|_2^3$$

$$=\frac{1}{8}\left[\left(\frac{33}{2}-\frac{9}{2}\right)-(11-2)\right]$$

$$=\frac{3}{8}$$



(ii) $P(X<1/Y<3)=\dfrac{P(X<1,Y<3)}{P(Y<3)}$ ...(1)

$$P(Y<3)=\int_2^3\int_0^2 f(x,y)\,dx\,dy$$

$$=\int_2^3\int_0^2\frac{1}{8}(6-x-y)\,dx\,dy$$

$$=\frac{1}{8}\int_2^3\left|6x-\frac{x^2}{2}-xy\right|_0^2 dy$$

$$=\frac{1}{8}\int_2^3(12-2-2y)\,dy$$

$$=\frac{1}{8}\int_2^3(10-2y)\,dy$$

$$=\frac{1}{8}\left|10y-y^2\right|_2^3$$

$$=\frac{1}{8}[(30-9)-(20-4)]$$

$$=\frac{5}{8}$$

Substituting in Eq (1),

$$P(X<1/Y<3)=\frac{\left(\frac{3}{8}\right)}{\left(\frac{5}{8}\right)}=\frac{3}{5}$$

## Example 13

*The joint pdf of a two-dimensional random variable (X, Y) is given by*

$$f_{XY}(x,y)=\begin{cases}xy^2+\dfrac{x^2}{8}, & 0<x<2,\quad 0<y<1\\ 0\quad, & \text{Otherwise}\end{cases}$$

*Find* (i) $P(X>1)$ (ii) $P\left(Y<\frac{1}{2}\right)$ (iii) $P\left(X>1/Y<\frac{1}{2}\right)$

(iv) $P\left(Y<\frac{1}{2}/X>1\right)$ (v) $P(X<Y)$ (vi) $P(X+Y\leq 1)$

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**Solution**

(i) $P(X>1)=\int_0^1\int_1^2 f(x,y)\,dx\,dy$

$$=\int_0^1\int_1^2\left(xy^2+\frac{x^2}{8}\right)dx\,dy$$

$$=\int_0^1\left|\frac{x^2y^2}{2}+\frac{x^3}{24}\right|_1^2 dy$$

$$=\int_0^1\left(2y^2+\frac{1}{3}\right)-\left(\frac{y^2}{2}+\frac{1}{24}\right)dy$$

$$=\int_0^1\left(\frac{3y^2}{2}+\frac{7}{24}\right)dy$$

$$=\left|\frac{y^3}{2}+\frac{7y}{24}\right|_0^1$$

$$=\frac{1}{2}+\frac{7}{24}$$

$$=\frac{19}{24}$$

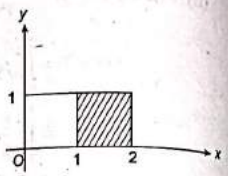

Fig. 2.5

(ii) $P\left(Y<\frac{1}{2}\right)=\int_0^{\frac{1}{2}}\int_0^2 f(x,y)\,dx\,dy$

$$=\int_0^{\frac{1}{2}}\int_0^2\left(xy^2+\frac{x^2}{8}\right)dx\,dy$$

$$=\int_0^{\frac{1}{2}}\left|\frac{x^2y^2}{2}+\frac{x^3}{24}\right|_0^2 dy$$

$$=\int_0^{\frac{1}{2}}\left(2y^2+\frac{1}{3}\right)dy$$

$$=\left|\frac{2y^3}{3}+\frac{1}{3}y\right|_0^{\frac{1}{2}}$$

$$=\frac{1}{12}+\frac{1}{6}$$

$$=\frac{1}{4}$$

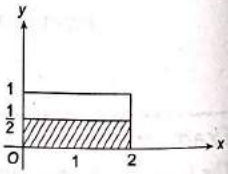

Fig. 2.6



(iii) $P\left(X>1, Y<\frac{1}{2}\right)=\int_0^{\frac{1}{2}}\int_1^2 f(x,y)\,dx\,dy$

$$=\int_0^{\frac{1}{2}}\int_1^2\left(xy^2+\frac{x^2}{8}\right)dx\,dy$$

$$=\int_0^{\frac{1}{2}}\left|\frac{x^2y^2}{2}+\frac{x^3}{24}\right|_1^2 dy$$

$$=\int_0^{\frac{1}{2}}\left(2y^2+\frac{1}{3}\right)-\left(\frac{y^2}{2}+\frac{1}{24}\right)dy$$

$$=\int_0^{\frac{1}{2}}\left(\frac{3y^2}{2}+\frac{7}{24}\right)dy$$

$$=\left|\frac{y^3}{2}+\frac{7y}{24}\right|_0^{\frac{1}{2}}$$

$$=\frac{1}{16}+\frac{7}{48}$$

$$=\frac{5}{24}$$

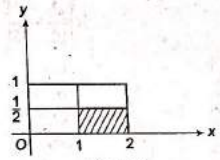

Fig. 2.7

$$P\left(X>1/Y<\frac{1}{2}\right)=\frac{P\left(X>1, Y<\frac{1}{2}\right)}{P\left(Y<\frac{1}{2}\right)}=\frac{\frac{5}{24}}{\frac{1}{4}}=\frac{5}{6}$$

(iv) $P\left(Y<\frac{1}{2}/X>1\right)=\frac{P\left(X>1, Y<\frac{1}{2}\right)}{P(X>1)}=\frac{\frac{5}{24}}{\frac{19}{24}}=\frac{5}{19}$

(v) $P(X<Y)=\int_0^1\int_0^y f(x,y)\,dx\,dy$

$$=\int_0^1\int_0^y\left(xy^2+\frac{x^2}{8}\right)dx\,dy$$

$$=\int_0^1\left|\frac{x^2y^2}{2}+\frac{x^3}{24}\right|_0^y dy$$

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$$= \int_0^1 \left( \frac{y^4}{2} + \frac{y^3}{24} \right) dy$$

$$= \left| \frac{y^5}{10} + \frac{y^4}{96} \right|_0^1$$

$$= \frac{1}{10} + \frac{1}{96}$$

$$= \frac{53}{480}$$

(vi) $P(X+Y \le 1) = \int_0^1 \int_0^{1-y} f(x,y)\, dx\, dy$

$$= \int_0^1 \int_0^{1-y} \left( xy^2 + \frac{x^2}{8} \right) dx\, dy$$

$$= \int_0^1 \left| \frac{x^2y^2}{2} + \frac{x^3}{24} \right|_0^{1-y} dy$$

$$= \int_0^1 \left\{ \frac{(1-y)^2 y^2}{2} + \frac{(1-y)^3}{24} \right\} dy$$

$$= \int_0^1 \left\{ \frac{(1-2y+y^2)y^2}{2} + \frac{(1-y)^3}{24} \right\} dy$$

$$= \int_0^1 \left\{ \frac{1}{2}(y^2 - 2y^3 + y^4) + \frac{1}{24}(1-y)^3 \right\} dy$$

$$= \left| \frac{1}{2}\left( \frac{y^3}{3} - \frac{y^4}{2} + \frac{y^5}{5} \right) + \frac{1}{24} \frac{(1-y)^4}{(-4)} \right|_0^1$$

$$= \frac{1}{2}\left( \frac{1}{3} - \frac{1}{2} + \frac{1}{5} \right) + \frac{1}{24} \cdot \frac{1}{4}$$

$$= \frac{13}{480}$$

## Example 14

*The joint pdf of a two dimensional random variable (X, Y) is given by*

$$f(x,y) = \frac{1}{2\pi a^2} e^{-\frac{x^2+y^2}{2a^2}}, -\infty < x, y < \infty$$



*Find* $P(X^2 + Y^2 \le 4)$.

### Solution

$$P(X^2+Y^2 \le 4) = \iint_{x^2+y^2 \le 4} f(x,y)\, dx\, dy$$

$$= \iint_{x^2+y^2 \le 4} \frac{1}{2\pi a^2} e^{-\frac{x^2+y^2}{2a^2}}\, dx\, dy$$

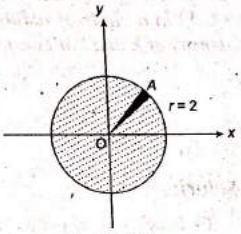


Fig. 2.8

The region of integration is the interior of the circle $x^2 + y^2 = 4$.

Converting to polar coordinates by putting $x = \cos\theta, y = r\sin\theta$, $dxdy = rdrd\theta$, equation of the circle $x^2 + y^2 = 4$ reduces to $r = 2$.

In the region, along elementary radial strip $OA$,

Limits of $r$ : $r = 0$ to $r = 2$

and in the region

Limits of $\theta$: $\theta = 0$ to $\theta = 2\pi$

$$P(X^2+Y^2 \le 4) = \int_0^{2\pi} \int_0^2 \frac{1}{2\pi a^2} e^{-\frac{r^2}{2a^2}} \cdot r dr\, d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \int_0^2 -e^{-\frac{r^2}{2a^2}} \left( -\frac{r}{a^2} \right) dr\, d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left| -e^{-\frac{r^2}{2a^2}} \right|_a^2 d\theta \qquad \left[ \because \int_0 e^{f(x)} f'(x) dx = e^{f(x)} \right]$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left( -e^{-\frac{2}{a^2}} + 1 \right) d\theta$$

$$= \frac{1}{2\pi} \left( 1 - e^{-\frac{2}{a^2}} \right) |\theta|_0^{2\pi}$$

$$= \frac{1}{2\pi} \left( 1 - e^{-\frac{2}{a^2}} \right) (2\pi)$$

$$= 1 - e^{-\frac{2}{a^2}}$$

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## Example 15

*A gun is aimed at a certain point (origin of the co-ordinate system). Because of the random factors, the actual hit point can be any point (X, Y) in a circle of radius 'a' about the origin. Assume that the joint density of X and Y is constant in this circle and is given by*

$$f(x,y)=c, \quad x^2+y^2 \le a^2$$
$$=0, \quad \text{otherwise}$$

*Find* (i) $c$ (ii) $f_X(x)$.

### Solution

(i) Since $f(x, y)$ is a probability density function,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy=1$$

$$\iint_{x^2+y^2\le a^2} c\,dx\,dy=1$$

$$c\iint_{x^2+y^2\le a^2} dx\,dy=1$$

$$c(\text{area of circle } x^2+y^2=a^2)=1$$

$$c(\pi a^2)=1$$

$$c=\frac{1}{\pi a^2}$$

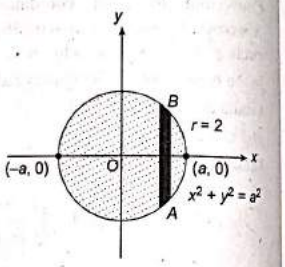


Fig. 2.9

(iii) The region of integration is the interior of the circle $x^2+y^2=a^2$.

In the region along the vertical strip $AB$, Limits of $y=-\sqrt{a^2-x^2}$ to $y=\sqrt{a^2-x^2}$ and $x$ varies from $x=-a$ to $x=a$.

$$f_X(x)=\int_{-\infty}^{\infty} f(x,y)\,dy$$

$$=\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} c\,dy$$

$$=c\left|y\right|_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}$$

$$=c\left(\sqrt{a^2-x^2}+\sqrt{a^2-x^2}\right)$$

$$=\frac{1}{\pi a^2}\left(2\sqrt{a^2-x^2}\right)$$

$$=\frac{2}{\pi a^2}\sqrt{a^2-x^2}, \qquad -a\le x\le a$$



## EXERCISE 2.4

1. The joint pdf of a two dimensional random variable (X, Y) is given by

$$f(x,y)=2,\ 0<x<1,\ 0<y<1$$
$$=0, \quad \text{otherwise}$$

Find (i) marginal density function (ii) conditional density function.

[Ans.:
(i) $f(x)=2x, \quad 0<x<1$
$=0$, otherwise

$f(y)=2(1-y), \quad 0<y<1$
$=0$, otherwise

(ii) $f(y/x)=\frac{1}{x}, \quad 0<x<1$

$f(x/y)=\frac{1}{1-y}, 0<y<1$]

2. The joint pdf of a two dimensional random variable (X, Y) is given by

$$f(x,y)=\frac{1}{ab}, \quad 0<x<a, \quad 0<y<b$$
$$=0, \quad \text{otherwise}$$

Find the joint probability distribution function and marginal density functions. Are X and Y independent?

[Ans.: $F(x,y)=\frac{xy}{ab}, \quad 0<x\le a, \quad 0<y\le b$
$=0$, otherwise

$f_X(x)=\frac{1}{a}, f_Y(y)=\frac{1}{b}$, Yes]

3. The joint pdf of (X, Y) is given by $f(x,y)=k, 0\le x<y\le 2$.

Find (i) $k$ (ii) $f_X(x)$, $f_Y(y)$ (iii) $f(y/x)$, $f(x/y)$

[Ans.: (i) $\frac{1}{2}$, (ii) $f_X(x)=\frac{1}{2}(2-x), 0\le x\le 2; f_Y(y)=\frac{1}{2}y, 0\le y\le 2$

(iii) $f(y/x)=\frac{1}{2-x}, x<y<2; f(x/y)=\frac{1}{y}, 0<x<y$]

4. The joint pdf of (X, Y) is given by

$$f(x,y)=k(x^3y+xy^3), 0\le x\le 2, 0\le y\le 2$$

Find (i) $k$ (ii) $f_X(x)$, $f_Y(y)$ (iii) $f(y/x)$, $f(x/y)$

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$$\left[\text{Ans.: (i) } k=\frac{1}{16}, \text{(ii) } f_X(x)=\frac{1}{8}(x^3+2x), 0\le x\le 2;\ f_Y(y)=\frac{1}{8}(y^3+2y), 0\le y\le 2 \right.$$
$$\left. \text{(iii) } f(y/x)=\frac{y(x^2+y^2)}{2(x^2+2)}, 0\le y\le 2; f(x/y)=\frac{x(x^2+y^2)}{2(y^2+2)}, 0\le x\le 2. \right]$$

5. The joint pdf of $(X, Y)$ is given by

$$f(x,y)=\frac{1}{3}(3x^2+xy),\quad 0<x\le 1,\quad 0<y\le 2$$
$$=0, \qquad \text{, otherwise}$$

Find $P(X+Y\ge 1)$.

$$\left[\text{Ans.: } \frac{65}{72}\right]$$

6. The joint pdf of $(X, Y)$ is given by

$f(x,y) = k(6-x-y),\ 0<x<2,\ 2<y<4$

Find (i) $k$ (ii) $P(X<1, Y<3)$, (iii) $P(X+Y<3)$, (iv) $P(X<1/y<3)$

$$\left[\text{Ans.: (i) } \frac{1}{8}, \text{(ii) } \frac{3}{8}, \text{(iii) } \frac{5}{24}, \text{(iv) } \frac{3}{5}\right]$$

7. The joint pdf of $(X, Y)$ is given by $f(x,y)=e^{-y},\quad x>0,\quad y>x$

$$=0 \quad , \quad \text{otherwise}$$

Find (i) $P(X>1/Y<5)$ (ii) marginal distributions of $X$ and $Y$.

$$\left[\text{Ans.: (i) } \frac{e^4-5}{e^5-6}, \text{(ii) } f_X(x)=e^{-x}, x>0;\ f_Y(y)=ye^{-y}, y>0\right]$$

8. The joint pdf of $(X, Y)$ is given by $f(x,y)=\frac{1}{12}e^{-\frac{x}{4}-\frac{y}{3}}, x\ge 0, y\ge 0$

$$=0, \qquad \text{otherwise}$$

(i) Find conditional density functions of $X$ and $Y$.

(ii) Are $X$, $Y$ independent?

$$\left[\text{Ans.: (i) } f(y/x)=\frac{1}{3}e^{-\frac{y}{3}}, y\ge 0;\ f(x/y)=\frac{1}{4}e^{-\frac{x}{4}}, x\ge 0, \text{ (ii) Yes}\right]$$

9. The joint pdf of $(X, Y)$ is given by $f(x,y)=\frac{2}{(1+x+y)^3},\quad x>0, y>0$

$$=0 \qquad \text{, otherwise}$$

Find (i) $F(x, y)$ (ii) $f_X(x)$ (iii) $f(y/x)$.



$$\left[\text{Ans.: (i) } F(x,y)=1-\frac{1}{1+x}+\frac{1}{1+x+y}-\frac{1}{1+y}\right.$$
$$\text{(ii) } f_X(x)=\frac{1}{(1+x)^2},\quad x>0$$
$$=0 \qquad \text{, otherwise}$$
$$\left.\text{(iii) } f(y/x)=\frac{2(1+x)^2}{(1+x+y)^3}\right]$$

10. The joint pdf of $(X, Y)$ is given by $f(x,y)=x^2+\frac{xy}{3},\quad 0<x<1, 0<y<2$

$$=0 \qquad \text{, otherwise}$$

Find (i) $P\left(X>\frac{1}{2}\right)$ (ii) $P(Y<X)$ (iii) $P\left(Y<\frac{1}{2}/X<\frac{1}{2}\right)$

$$\left[\text{Ans.: (i) } \frac{5}{6}, \text{ (ii) } \frac{7}{24}, \text{ (iii) } \frac{5}{32}\right]$$

11. The joint pdf of $(X, Y)$ is given by $f(x,y)=\frac{1}{4}(1+xy),\ |x|<1,\ |y|<1$

$$=0 \qquad \text{, otherwise}$$

Show that $X$ and $Y$ are independent.

12. The joint pdf of $(X, Y)$ is given by $f(x,y)=Ae^{-|x|-2|y|}$. Show that $X$ and $Y$ are independent.

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