Hari Probability Playing Cards

Bored Boy

June 2024

I pick cards randomly and solve them.

A ♣
Solution.

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2 ♣
Solution.

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3 ♣
Solution.

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4 ♣
Solution.

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5 ♣
Solution.

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6 ♣
Solution.

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7 ♣
Solution.

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8 ♣
Solution.

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9 ♣
Solution.

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10 ♣. Draw cards from a shuffled deck until you draw the Jc, Qc, and Kc. How many draws will it take on average?
Solution. Let X be the smallest u such that all 3 of the relevant cards are in the top u cards of the deck. We seek

$$\sum_{i=1}^{52} P(X \geq i)$$

$$= \sum_{i=1}^{5} 21 - (1 - P(X \geq i)) = 52 - \sum_{i=1}^{5} 2P(X < i),$$

where $P(X < i)$ is the probability that the first $i-1$ cards contain the three relevant cards. Let Y_i be the event that the first i cards contain the relevant cards, and this is now

$$= 52 - \sum_{i=1}^{5} 2P(Y_{i-1}$$

$$= 52 - \sum_{i=0}^{5} 1P(Y_i).$$

Now, we compute $P(Y_i)$. The probability the first i cards contain the three relevant cards is equal to the number of ways to choose i cards containing these three cards, divided by the number of ways to choose i cards. This is

$$\frac{\binom{52-3}{i-3}}{\binom{52}{i}},$$

or

$$\frac{\frac{49!}{(i-3)!(52-i)!}}{\frac{52!}{i!(52-i)!}}$$

$$= \frac{1}{52 \cdot 51 \cdot 50}(i)(i-1)(i-2).$$

Finally, we seek

$$= 52 - \sum_{i=0}^{51} \frac{1}{52 \cdot 51 \cdot 50}(i)(i-1)(i-2).$$

$$= 52 - \frac{1}{52 \cdot 51 \cdot 50} \sum_{i=-1}^{50} (i-1)(i)(i+1)$$

$$= 52 - \frac{1}{52 \cdot 51 \cdot 50} \sum_{i=0}^{50} i^3 - i$$

$$= 52 - \frac{1}{52 \cdot 51 \cdot 50}(\frac{(50)(51)}{2}^2 - \frac{(50)(51)}{2})$$

$$= 52 - \frac{1}{52 \cdot 51 \cdot 50}((\frac{(50)(51)}{2})(\frac{(50)(51)}{2} - 1))$$

$$= 52 - \frac{1}{104}(2549)$$

$$= \boxed{27\frac{51}{104}}$$

J ♣
Solution.

Q ♣
Solution.

K ♣ You and I are playing with a strange deck of cards that consists of the 1 through N ♣ s. I remove r cards from the deck. Without knowing which r I removed, how large of a sum-free subset can you promise me you can find in your cards?
Solution. All I have is conjecture.

$$\boxed{f(N, R) = \lceil \frac{N-R}{2} \rceil + 1},$$

with $f(N, R) : R \geq N = 0$. Proceed via a solution for $R = 0$, and then the above is trivially an upper bound via induction. Hopefully it is also a lower bound.

May come back to this one later.

A $\diamondsuit$
Solution.

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2 $\diamondsuit$
Solution.

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3 $\diamondsuit$
Solution.

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4 $\diamondsuit$
Solution.

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5 $\diamondsuit$
Solution.

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6 $\diamondsuit$
Solution.

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7 $\diamondsuit$
Solution.

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8 $\diamondsuit$
Solution.

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9 $\diamondsuit$
Solution.

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10 $\diamondsuit$
Solution.

J $\diamondsuit$
Solution.

Q $\diamondsuit$
Solution.

K $\diamondsuit$
Solution.

A $\heartsuit$
Solution.

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2 $\heartsuit$ Two planes are travelling at 60mph and 180mph respectively. They both stop independently after 0-60 seconds (uniformly at random.) What is the probability that the planes travel less than a mile total in sum?
Solution. This is the same as adding one Unif(0,1) and a Unif(0,3) and asking the likelihood the sum is less than 1. The area of the triangle bounded by y=0, x=0, and y+x=1 is $\frac{1}{2}$ and the total area of the space of possibilities is 3, for a likelihood of $\frac{\frac{1}{2}}{3} = \boxed{\frac{1}{6}}$.

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3 $\heartsuit$
Solution.

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4 $\heartsuit$
Solution.

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5 $\heartsuit$
Solution.

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6 $\heartsuit$
Solution.

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7 $\heartsuit$
Solution.

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8 $\heartsuit$
Solution.

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9 $\heartsuit$ Cars of length 2 are parking on a street of length 100, with parking meters evenly spaced every length 1. Each minute, a car arrives and parks at a uniformly randomly open spot, aligning its front bumper with a parking meter. When no more cars can possibly fit, roughly what fraction of parking meters are unoccupied by a car?
Solution. Let F_n be the answer to the above question, with "100" replaced with n and "fraction" replaced with "number". So the answer is $\frac{F_100}{100}$. Let's case on the position on the first car that parks, just looking to find F_n. If it parks at the very left (with probability $\frac{1}{n-1}$, we are left with an expected F_{n-2} spaces unoccupied. If it parks one space over, we are left with $1+F_{n-3}$. In particular, if the first car's left endpoint is placed i spaces in (with i in $[0, n-2]$), then we are left with $F_i + F_{n-2-i}$. This occurs for each i with probability $\frac{1}{n-1}$, so

$$F_n = \frac{1}{n-1} \sum_{i=0}^{n-2} F_i + F_{n-2-i}$$

$$= \frac{2}{n-1} \sum_{i=0}^{n-2} F_i.$$

Now, note

$$F_{n-1} = \frac{2}{n-2} \sum_{i=0}^{n-3} F_i,$$

so

$$F_n = \frac{2\sum_{i=0}^{n-3} F_i}{n-1} + \frac{2F_{n-2}}{n-1} = \frac{(n-2)F_{n-1} + 2F_{n-2}}{n-1}.$$

At this point we may note that if we were more clever it may have been possible to derive this recurrence directly, but alas I could not.

Now, we look at the generating function for this recurrence, denoted $F(x)$. Here, we assume familiarity with generating fucntions and differentiating them in the interest of space.

$$F(x) = F_0 x^0 + F_1 x^1 + \sum_{i=2}^{\infty} F_i x^i$$

$$= x + \sum_{i=2}^{\infty} \frac{i-2}{i-1} F_{i-1} x^i + 2\sum_{i=2}^{\infty} \frac{F_{i-2}}{i-1} x^i$$

$$= x + \sum_{i=2}^{\infty} F_{i-1} x^i - \sum_{i=2}^{\infty} \frac{1}{i-1} F_{i-1} x^i + 2\sum_{i=2}^{\infty} \frac{F_{i-2}}{i-1} x^i$$

$$= x + xF(x) - \sum_{i=2}^{\infty} \frac{1}{i-1} F_{i-1} x^i + 2\sum_{i=2}^{\infty} \frac{F_{i-2}}{i-1} x^i$$

We can look at the third term as follows: let (we henceforth denote functions just by the letter, with F denoting $F(x)$ and similar)

$$G = \sum_{i=2}^{\infty} \frac{1}{i-1} F_{i-1} x^i$$

$$= x(\frac{G}{x})',$$

$$G = x \int \frac{F}{x} dx.$$

Similarly, let

$$H = \sum_{i=2}^{\infty} \frac{F_{i-2}}{i-1} x^i,$$

$$(\frac{H}{x})' = \sum_{i=2}^{\infty} F_{i-2} x^{i-2} = F,$$

so

$$H = x \int F dx.$$

Then, going back to our initial expression,

$$F = x + xF - x\int \frac{F}{x} dx + 2x \int F dx,$$

$$\frac{F}{x} = 1 + F - \int \frac{F}{x} dx + 2\int F dx,$$

and differentiating yields

$$\frac{xF' - F}{x^2} = F' - \frac{F}{x} + 2F,$$

$$xF' - F = x^2 F' - xF + 2x^2 F,$$

$$\implies \frac{F'}{F} = \frac{2x^2 - x + 1}{x - x^2},$$

$$\implies -\frac{dF}{F} = \frac{2x^2 - x + 1}{x^2 - x} dx.$$

Integrating and long dividing yields

$$-\log F + C = \int (2 + \frac{x+1}{x^2 - x}) dx,$$

and partial fractions yields

$$-\log F + C = \int (2 + \frac{2}{x-1} - \frac{1}{x}) dx.$$

$$\implies \log F + C = \log x - 2\log(x-1) - 2x,$$

$$\implies CF = \frac{e^{-2x} x}{(x-1)^2},$$

or (multiplying together the Taylor series of the components and convolving,)

$$F = C \sum_{i=0}^{\infty} x^i \sum_{j=0}^{i-1} \frac{(-2)^j}{j!} (i - j).$$

Then $F_i = C \sum_{j=0}^{i-1} \frac{(-2)^j}{j!} (i-j)$, and we can quickly see that plugging in $i = 1$ that $C = 1$. So

$$F_i = \sum_{j=0}^{i-1} \frac{(-2)^j}{j!} (i - j).$$

In particular, our solution is $F_{100}/100 =$

$$\frac{1}{100} \sum_{j=0}^{99} \frac{(-2)^j}{j!} (100 - j).$$

Let's instead consider the solution for the proportion $P_i = F_i/i$:

$$P_i = \frac{1}{i} \sum_{j=0}^{i-1} \frac{(-2)^j}{j!} (i - j),$$

$$= \sum_{j=0}^{i-1} \frac{(-2)^j}{j!} - \frac{1}{i} \sum_{j=0}^{i-1} \frac{(-2)^j}{(j-1)!},$$

and we may note that the terms of both summations start getting quite small in magnitude quickly. So we can extend the summations to infinity without adding too much,

$$\approx \sum_{j=0}^{\infty} \frac{(-2)^j}{j!} - \frac{1}{i} \sum_{j=0}^{\infty} \frac{(-2)^j}{(j-1)!},$$

$$\approx e^{-2} - \frac{1}{i}(1 + -2e^{-2})$$

$$\approx \boxed{e^{-2} - \frac{1}{100}(1 - 2e^{-2})}.$$

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10 ♡
Solution.

J $\heartsuit$
Solution.

Q $\heartsuit$
Solution.

K $\heartsuit$
Solution.

A $\spadesuit$ Two crabs are facing each other on the beach. Each second they randomly walk one step to the right or one step to the left. After 6 seconds, what is the probability they are still across from each other?
Solution. This is

$$\frac{\sum_{i=0}^{6}\binom{6}{i}^2}{2^{12}},$$

and here we will include a non-combinatorial way to derive

$$\sum_{i=0}^{n}\binom{n}{i}^2$$

for the reader.

Note that this is the coefficient of x^n in $(x+1)^{2n} = \sum_{i=0}^{n}\binom{n}{i}x^i \sum_{i=0}^{n}\binom{n}{i}x^{n-i}$. But

$$(x+1)^{2n} = \sum_{i=0}^{2n}\binom{2n}{i}x^i,$$

and the coefficient of x^n is $\binom{2n}{n}$.

With that aside out of the way, we end up with

$$= \binom{12}{6}/2^{12} = \boxed{\frac{231}{1028}}$$

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2 $\spadesuit$
Solution.

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3 $\spadesuit$
Solution.

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4 $\spadesuit$
Solution.

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5 $\spadesuit$
Solution.

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6 $\spadesuit$

Solution.

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7 ♠
Solution.

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8 ♠
Solution.

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9 ♠
Solution.

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10 ♠
Solution.

J ♠
Solution.

Q ♠
Solution.

K ♠
Solution.