Hari Probability Playing Cards

Bored Boy

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I pick cards randomly and solve them.

A ♣ What is the expected value of a randomly drawn card from a standard deck? If you draw two cards randomly, what is the expected value of their sum?
Solution. The EV of a random card is $\boxed{7}$. Then, the EV of the sum of two random cards is (by LOE) $\boxed{14}$.

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2 ♣
Solution.

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3 ♣ 20 people are each holding a playing card. These cards are all shuffled and redistributed randomly. What is the expected number of people who get their original card back?
Solution. Let λ_i be an indicator variable for person i getting their card back. We seek

$$E(\sum_{i=1}^{20} \lambda_i$$

$$= 20\lambda_1$$

$$= 20\frac{1}{20}$$

$$= \boxed{1}.$$

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4 ♣
Solution.

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5 ♣
Solution.

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6 ♣
Solution.

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7 ♣ While showing off a fancy shuffling trick, you accidentally drop almost half the deck down a storm drain. You want to figure out which cards you still have, so you try to match up cards with the same rank and color (As with Ac, for example.) What is the expected number of matches found?
Solution. Every card has a partner. We want to find the expected number of pairs present in the top 34 cards of a deck.

Note the number of pairs present in the top 34 cards of a deck is

$$\frac{1}{2}\sum_{i=1}^{34} \lambda_i,$$

where λ_i is 1 if card i's partner is in the first 34 cards and 0 otherwise.

But then the expectation we seek is

$$E(\frac{1}{2}\sum_{i=1}^{34} \lambda_i) = \frac{1}{2}(34)E(\lambda_1)$$

$$= 17 * \frac{33}{51}$$
$$= \boxed{11}$$

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8 ♣
Solution.

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9 ♣
Solution.

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10 ♣. Draw cards from a shuffled deck until you draw the Jc, Qc, and Kc. How many draws will it take on average?
Solution. The three cards partition the deck into 4 piles of cards, each of which have the same expected size. The expected size is thus $\frac{49}{3}$, and then the expected number of draws among the first three piles is $\frac{147}{4}$. Then, the expected number of draws to get all three of these cards is $\boxed{\frac{159}{4}}$

J ♣ We have $2N$ cards: 1 through N of both clubs and spades. Remove the 1♣ and a random ♠. Remove a pair of cards: the 2♣ and the associated ♠ (the 2♠ if you can find it, or else a random ♠). Continue similarly (with the 3♣, 4♣ , etc.) until all cards are paired and removed. What's the chance N♣ and N♠ form a pair? On average, how many pairs have matching rank?
Solution. For the first problem, note that if player 1 goes to the u of spades, then everything up to $u-1$ of spades gets filled in. Then, if we thereafter have the u of clubs map to the v of spades for $v \neq u$, everything from u to $v-1$ is filled in.

Let's call x_1 the rank of the card that the 1♣ is paired with, and x_2 the rank of the card that the x_1♣ is paired with, and so on. Note that when x_τ equals 1 or n, the cards from $x_\tau+1$ to $n-1$ ♣ get paired with their corresponding ♠ card. Then, n♣ will get paired with n♠ if $x_\tau = 1$ and with 1♠ if $x_\tau = n$. These events occur with equal likelihood, so the solution to the first question is

$$\boxed{\frac{1}{2}.}$$

For the second part, let us have for each x♣ an indicator variable λ_x equal to 1 if it is paired with itself and 0 otherwise. We seek $E(\sum\limits_{i=1}^{N} \lambda_i) = \sum\limits_{i=1}^{N} E(\lambda_i)$.

The probability $\lambda_i = 1$ is the probability that the sequence x_j equals 1 or $i+1, \ldots, n$ before it equals i (the sequence must displace i♣ before it fixes i♣ by either overshooting it or equalling 1.) The first value among

$$\{1, i, i+1, \ldots, n\}$$

to be taken on by the sequence is equally likely to be any of these values by symmetry, so the probability it is exactly i is

$$\frac{1}{n-i+2}$$

and so the expecation of λ_i is

$$1 - \frac{1}{n-i+2}$$

except for $i = 1$, where the sequence cannot equal "1" before "1" and the expectation of λ_1 is

$$\frac{1}{n}.$$

Then,

$$\sum_{i=1}^{n} E(\lambda_i)$$

$$= \frac{1}{n} + \sum_{i=2}^{n} 1 - \frac{1}{n-i+2}$$

$$= \frac{1}{n} + \sum_{i=0}^{n-2} 1 - \frac{1}{n-i}$$

$$= \frac{1}{n} + n - 1 - \sum_{i=0}^{n-2} \frac{1}{n-i} = \frac{1}{n} + n - 1 - \frac{1}{n} - \sum_{i=2}^{n-1} \frac{1}{i}$$

$$= \boxed{n - H_{n-1}}$$

Q ♣
Solution.

K ♣ You and I are playing with a strange deck of cards that consists of the 1 through N ♣ s. I remove r cards from the deck. Without knowing which r I removed, how large of a sum-free subset can you promise me you can find in your cards?
Solution. Our answer is

$$\boxed{f(N,R) = \lfloor \frac{N-R}{2} \rfloor + 1},$$

with $f(N,R) : R \geq N = 0$.

Firstly, note the solution holds for $r = 0$; just take all the odd numbers (n odd) or the last $\frac{n+2}{2}$ numbers (n even). We can't do better as whatever the largest number is (say U,) we can only pick one of $1, U-1$, $2, U-2$, etc. which leads to at most our expression.

We now attempt to proceed inductively. Note that $f(n,r) \geq f(n-1,r)$: just ignore the nth number and draw from the mapping from $|s| = r$ to $|u| \subseteq s, |u| = f(n-1,r)$ with u sumfree guaranteed by induction. Also, $f(n,r) \leq f(n-1,r-1)$: if the last number is removed, we can do only as well as in the $f(n-1,r-1)$ case.

The latter rule directly gives $f(N,R) \leq f(N,R) = \lfloor \frac{N-R}{2} \rfloor + 1$

for nontrivial pairs (N,R.)

Finally, proceed by (R, then N) induction on the problem. Suppose our formula first does not hold for some (N', R'). Then note that $N' - R'$ must be even, as in the odd case we directly get the result from our lower and upper bounds via induction.

I do not know how to show this inductive step.

A ♢ A standard deck of cards is shuffled. What is the chance that the third card from the top is either a diamond or an ace?
Solution.

$$\boxed{4}13$$

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2 ♢
Solution.

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3 $\diamondsuit$
Solution.

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4 $\diamondsuit$
Solution.

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5 $\diamondsuit$ I have all the diamonds. You randomly choose one of the cards I have without looking at it. I look at my cards and then place 11 of them face up. The 5 of diamonds is not among them. You and I both want the 5 of diamonds. Should you trade the card you initially chose for my remaining card?
Solution. Let us first make a short note; we assume (as in monty hall: see https://en.wikipedia.org/wiki/Monty_Hall_problem) the game show host is restricted to randomly choosing 11 of the cards from the cards both not chosen by me and not containing the 5$\diamondsuit$ to show me. The second constaint can be shown to be optimal; if the game show host is allowed to show you your own card when it doesn't contain the 5$\diamondsuit$, then their odds vastly improve. I will not solve for the nash equilibrium of this updated game and simply assume this is a monty hall clone.

In the monty hall case, (by conditional probability and letting A be our picked card, B be the other unpicked card,) the probability B is the 5 is

$$\frac{\frac{1}{13}}{\frac{1}{13} + \frac{1}{13}\frac{1}{12}} = \boxed{\frac{12}{13}},$$

where we note that conditioning on B being wrong (and the fact that A is right,) the probability B is the unflipped card is $\frac{1}{13}$. One important thing to note is that we cannot simply say "since A could not be picked, we gain no new information about A and so B "absorbs" the probabilities of all flipped cards" which is a common misconception; see my CMIMC TCS problem at haridesikan.com to see why this is the case. The uniform prior distribution is a special case.

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6 $\diamondsuit$
Solution.

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7 $\diamondsuit$
Solution.

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8 $\diamondsuit$ We are playing with a random deck. I remove one $\diamondsuit$ and some non $\diamondsuit$ but I don't tell you how many! I secretly flip a fair coin; on heads I return the missing $\diamondsuit$, on tails I return one of the other missing cards. You shufle the deck and draw one card; it's a $\diamondsuit$. What's the probability my coin came up heads?
Solution. We assume that the number of nondiamonds removed is uniformly chosen from 0 to 39 (1 to 39 gives the same result.) Then $P(H|d) =$

$$\frac{\frac{1}{2}\frac{1}{40}\sum_{i=0}^{39}\frac{13}{52-i}}{\frac{1}{2}\frac{1}{40}\sum_{i=0}^{39}\frac{13}{52-i} + +\frac{1}{2}\frac{1}{40}\sum_{i=0}^{39}\frac{12}{52-i}}$$
$$= \boxed{\frac{13}{25}}.$$

Note whether we begin at $i = 0$ or 1 doesn't matter, so the 2 interpretations of the problem give the same result.

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9 $\diamondsuit$

Solution.

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10 $\diamondsuit$
Solution.

J $\diamondsuit$
Solution.

Q $\diamondsuit$ Take a random permutation of [13]. Given the longest increasing subsequence has length 10, find the probability the $K\diamondsuit$ is the last card.
Solution. After much work, I searched OEIS to see if my pattern (number of permutations of $[n]$ with longest increasing subsequence of length $n-3$) was recognized, which is basically giving up. It was recognized, but not with any closed form – there is no closed form. Further investigation shows that only asymptotics are available.

Searching $OEIS$ yields
https://oeis.org/A245666 https://oeis.org/A245665
which together yield

$$\boxed{\frac{153341}{275705}}$$

I do not plan on doing bashy counting on this problem. I may look at asymptotics, but the problem definitely should have been phrased that way to begin.

K $\diamondsuit$
Solution.

A $\heartsuit$
Solution.

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2 $\heartsuit$ Two planes are travelling at 60mph and 180mph respectively. They both stop independently after 0-60 seconds (uniformly at random.) What is the probability that the planes travel less than a mile total in sum?
Solution. This is the same as adding one Unif(0,1) and a Unif(0,3) and asking the likelihood the sum is less than 1. The area of the triangle bounded by y=0, x=0, and y+x=1 is $\frac{1}{2}$ and the total area of the space of possibilities is 3, for a likelihood of $\frac{\frac{1}{2}}{3} = \boxed{\frac{1}{6}}$.

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3 $\heartsuit$
Solution.

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4 $\heartsuit$
Solution.

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5 $\heartsuit$
Solution.

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6 ♡
Solution.

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7 ♡ N cyclists are biking around a circular track at a rate of 1 rpm. They start in uniformly random locations and random orientations. Just before a pair would collide, they instantly reverse direction. What is the expected number of times all the cyclists reverse direction each minute?
Solution. Cyclists reverse direction on a collision, so we seek the expected number of collisions. The primary observation is to note that each cyclist can be viewed as simply passing through opposing cyclists − when a cyclist A and their bumper B reverse directions, if we label A 'B' and B 'A' then the situation looks like the cyclists passed through each other. Then it is clear that in this pass through version, each cyclist passes through $\frac{N-1}{2}$ of the other cyclists in expectation (the ones facing the opposite direction.) They do this twice per revolution, so each cyclist does their little phasing trick $N-1$ times per revolution in expectation. Finally, note the question asks "number of times cyclists reverse direction each minute" so depending on interpretation we can answer $\boxed{N(N-1)}$ (every instance of a turn around, counting a phase through twice for the perspective of each cycler being called a "time") or $\boxed{\frac{(N)(N-1)}{2}}$ if we take the meaning of time in the literal sense; as in periods of time of length 0, counting each phase-through once.

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8 ♡
Solution.

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9 ♡ Cars of length 2 are parking on a street of length 100, with parking meters evenly spaced every length 1. Each minute, a car arrives and parks at a uniformly randomly open spot, aligning its front bumper with a parking meter. When no more cars can possibly fit, roughly what fraction of parking meters are unoccupied by a car?
Solution. Let F_n be the answer to the above question, with "100" replaced with n and "fraction" replaced with "number". So the answer is $\frac{F_100}{100}$. Let's case on the position on the first car that parks, just looking to find F_n. If it parks at the very left (with probability $\frac{1}{n-1}$, we are left with an expected F_{n-2} spaces unoccupied. If it parks one space over, we are left with $1+F_{n-3}$. In particular, if the first car's left endpoint is placed i spaces in (with i in $[0, n-2]$), then we are left with $F_i + F_{n-2-i}$. This occurs for each i with probability $\frac{1}{n-1}$, so

$$F_n = \frac{1}{n-1} \sum_{i=0}^{n-2} F_i + F_{n-2-i}$$
$$= \frac{2}{n-1} \sum_{i=0}^{n-2} F_i.$$

Now, note

$$F_{n-1} = \frac{2}{n-2} \sum_{i=0}^{n-3} F_i,$$

so

$$F_n = \frac{2 \sum_{i=0}^{n-3} F_i}{n-1} + \frac{2F_{n-2}}{n-1} = \frac{(n-2)F_{n-1} + 2F_{n-2}}{n-1}.$$

At this point we may note that if we were more clever it may have been possible to derive this recurrence directly, but alas I could not.

Now, we look at the generating function for this recurrence, denoted $F(x)$. Here, we assume familiarity with generating fucntions and differentiating them in the interest of space.

$$F(x) = F_0 x^0 + F_1 x^1 + \sum_{i=2}^{\infty} F_i x^i$$

$$= x + \sum_{i=2}^{\infty} \frac{i-2}{i-1} F_{i-1} x^i + 2 \sum_{i=2}^{\infty} \frac{F_{i-2}}{i-1} x^i$$

$$= x + \sum_{i=2}^{\infty} F_{i-1} x^i - \sum_{i=2}^{\infty} \frac{1}{i-1} F_{i-1} x^i + 2 \sum_{i=2}^{\infty} \frac{F_{i-2}}{i-1} x^i$$

$$= x + xF(x) - \sum_{i=2}^{\infty} \frac{1}{i-1} F_{i-1} x^i + 2 \sum_{i=2}^{\infty} \frac{F_{i-2}}{i-1} x^i$$

We can look at the third term as follows: let (we henceforth denote functions just by the letter, with F denoting $F(x)$ and similar)

$$G = \sum_{i=2}^{\infty} \frac{1}{i-1} F_{i-1} x^i$$

$$= x(\frac{G}{x})',$$

$$G = x \int \frac{F}{x} dx.$$

Similarly, let

$$H = \sum_{i=2}^{\infty} \frac{F_{i-2}}{i-1} x^i,$$

$$(\frac{H}{x})' = \sum_{i=2}^{\infty} F_{i-2} x^{i-2} = F,$$

so

$$H = x \int F dx.$$

Then, going back to our initial expression,

$$F = x + xF - x \int \frac{F}{x} dx + 2x \int F dx,$$

$$\frac{F}{x} = 1 + F - \int \frac{F}{x} dx + 2 \int F dx,$$

and differentiating yields

$$\frac{xF' - F}{x^2} = F' - \frac{F}{x} + 2F,$$

$$xF' - F = x^2 F' - xF + 2x^2 F,$$

$$\implies \frac{F'}{F} = \frac{2x^2 - x + 1}{x - x^2},$$

$$\implies -\frac{dF}{F} = \frac{2x^2 - x + 1}{x^2 - x} dx.$$

Integrating and long dividing yields

$$-\log F + C = \int (2 + \frac{x+1}{x^2 - x}) dx,$$

and partial fractions yields

$$-\log F + C = \int (2 + \frac{2}{x-1} - \frac{1}{x}) dx.$$

$$\implies \log F + C = \log x - 2\log(x-1) - 2x,$$

$$\implies CF = \frac{e^{-2x} x}{(x-1)^2},$$

or (multiplying together the Taylor series of the components and convolving,)

$$F = C\sum_{i=0}^{\infty} x^i \sum_{j=0}^{i-1} \frac{(-2)^j}{j!}(i-j).$$

Then $F_i = C\sum_{j=0}^{i-1} \frac{(-2)^j}{j!}(i-j)$, and we can quickly see that plugging in $i = 1$ that $C = 1$. So

$$F_i = \sum_{j=0}^{i-1} \frac{(-2)^j}{j!}(i-j).$$

In particular, our solution is $F_{100}/100 =$

$$\frac{1}{100}\sum_{j=0}^{99} \frac{(-2)^j}{j!}(100-j).$$

Let's instead consider the solution for the proportion $P_i = F_i/i$:

$$P_i = \frac{1}{i}\sum_{j=0}^{i-1} \frac{(-2)^j}{j!}(i-j),$$

$$= \sum_{j=0}^{i-1} \frac{(-2)^j}{j!} - \frac{1}{i}\sum_{j=0}^{i-1} \frac{(-2)^j}{(j-1)!},$$

and we may note that the terms of both summations start getting quite small in magnitude quickly. So we can extend the summations to infinity without adding too much,

$$\approx \sum_{j=0}^{\infty} \frac{(-2)^j}{j!} - \frac{1}{i}\sum_{j=0}^{\infty} \frac{(-2)^j}{(j-1)!},$$

$$\approx e^{-2} - \frac{1}{i}(1 + -2e^{-2})$$

$$\approx \boxed{e^{-2} - \frac{1}{100}(1 - 2e^{-2})}.$$

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10 ♡
Solution.

J ♡
Solution.

Q ♡
Solution.

K ♡
Solution.

A ♠ Two crabs are facing each other on the beach. Each second they randomly walk one step to the right or one step to the left. After 6 seconds, what is the probability they are still across from each other?

Solution. This is

$$\frac{\sum\limits_{i=0}^{6}\binom{6}{i}^2}{2^{12}},$$

and here we will include a non-combinatorial way to derive

$$\sum_{i=0}^{n}\binom{n}{i}^2$$

for the reader.

Note that this is the coefficient of x^n in $(x+1)^{2n} = \sum\limits_{i=0}^{n}\binom{n}{i}x^i \sum\limits_{i=0}^{n}\binom{n}{i}x^{n-i}$. But

$$(x+1)^{2n} = \sum_{i=0}^{2n}\binom{2n}{i}x^i,$$

and the coefficient of x^n is $\binom{2n}{n}$.

With that aside out of the way, we end up with

$$= \binom{12}{6}/2^{12} = \boxed{\frac{231}{1028}}$$

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2 ♠ In a random binary sequence, which is more likely to come first? 00 or 01? What is the expected time to see 00? 01? What about 001?

Solution. Unless we instantly roll BB, we are guaranteed to roll RB first. So BB comes first 1/4 of the time, and $\boxed{RB}$ is most likely to arrive first.

To see BB we have a simple states problem. We solve:

$$E = \frac{1}{4}(2) + \frac{1}{2}(E+1) + \frac{1}{4}(E+2).$$

(These are the cases: BB, (R), BR.) This yields $\boxed{E(t_{BB}) = 6}$ as the average time to see BB.

To see RB we have

$$E = \frac{1}{2}(E+1) + \frac{1}{2}(3),$$

which yields $\boxed{E(t_{RB} = 4}$ as the average time to see BB. (We can also see this as the average time to see a red, which is 2, summed with the average time to see a blue after, which is also 2, by LOE.)

For BBR, we have

$$E = \frac{1}{2}(E+1) + \frac{1}{4}(E+2) + \frac{1}{4}(4),$$

and so

$$\boxed{E(t_{BBR}) = 8}$$

Putting this here: https://martingalemeasure.wordpress.com/2014/02/02/monkey-typing-abracadabra-14/

I read this 2 years ago, didn't fully understand it, and have been looking for it ever since. Pretty important moment for me.

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3 ♠

Solution.

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4 ♠ A bunny cannot decide where to take a nap. There is a rose bush 4 steps to the left and sunflowers 8 steps to the right. The bunny is 25% to step left and 75% to step right. How many steps will it take in expectation until the bunny can get some rest?

Solution. Approaching this problem with states sounds annoying. Instead, we solve the more general problem via function equation / martingale techniques. First, we phrase the more general problem:

A random walk with probability p of moving right begins at point a on the number line, and terminates upon reaching 0 or L. Find the expected number of steps taken.

Our FE is

$$f(x) = 1 + pf(x+1) + (1-p)f(x-1)$$

subject to $f(0) = f(L) = 0$.

I spent a good amount of time working on this FE, and a quick write-up cannot do it justice. (To be fair, time spent on an FE is infinitely more enjoyable than time spent on a states problem.)

First, we note that this is almost the gamblers ruin FE, with martingale-solution

$$(\frac{1-p}{p})^x.$$

A quick look reveals that this is not a solution (duh). The boundary conditions, for one, are doomed. We need to subtract 1, and a natural way of doing this is to add ax such that $pa(x+1) + (1-p)a(x-1) = -1 + ax$ (we isolate the '1' in our FE and try to add a linear term to get rid of it) or

$$(2p-1)(a) = -1 \implies a = \frac{-1}{2p-1}.$$

Now, we have a better attempt at a solution:

$$(\frac{1-p}{p})^x - \frac{1}{2p-1}x.$$

Importantly, it is not clear how to use the boundary conditions. Our "solution" doesn't even reference L! I was stumped for a while until my friend mentioned another functional equation – the same FE, but with only the boundary condition at x=0 – and that it was "underconstrained". This led to me realize that subtracting 1 from our existing solution was a "valid solution" for some L – that is,

$$(\frac{1-p}{p})^x - \frac{1}{2p-1}x - 1$$

is a martingale and satisfies $f(0) = 0$, and probably satisfies $f(x) = 0$ for some $x \neq 0$. Now, this x may not be an integer, so in the strictest sense this might be a solution with no physical interpretation for an invalid L. But this naturally leads to the question of how to incorporate our "free variable". A bit of thinking yields

$$f(x) = c(L)((\frac{1-p}{p})^x - 1) - \frac{1}{2p-1}x.$$

This is basically unfindable without first underconstraining the problem, as $c(L)$ is a complicated function in L. Indeed, we have that

$$c(L) = \frac{1 + \frac{L}{2p-1}}{(\frac{1-p}{p})^L - 1},$$

and our final solution is

$$f(x) = \frac{1 + \frac{L}{2p-1}}{(\frac{1-p}{p})^L - 1}((\frac{1-p}{p})^x - 1) - \frac{1}{2p-1}x.$$

Finally, lets look at our particular case. We have $L = 12$ and $p = \frac{3}{4}$ for a solution of

$$\boxed{16\frac{4593}{6643}}$$

TODO: seriously? check this, no chance a p4 has this bs as an answer

5 ♠
Solution.

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6 ♠
Solution.

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7 ♠ A crow and a raven are each picking up coins off the street at the same pace. They are clever and keep track of the number of heads they picked up. They each pick up N coins total. What is the chance they never collected the same number of heads after starting? **Solution.**

Let the crow and raven pick up coins in order, so the Crow's first coin precedes the Raven's first coin precedes the Crow's second coin .... There are $2N$ coins picked up in this way.

The key observation is that we can view the sequence of 2N coins picked up as a random up-right walk of length $2N$, where we move up if either the current coin is heads and being picked up by raven or tails and being picked up by crow, and move right if the current coin is tails and picked by raven or heads and picked by crow. Note that this new random walk meets $y = x$ precisely when the number of heads by both birds is equal.

Now, we simply seek the likelihood that a random walk of this sort doesn't meet $y = x$. Such a walk must end at $(2N - A, A)$ for $A \in [0, 2N]$. We compute the probability that we have such a walk for $A \in [0, N - 1]$ and then multiply by 2. (Note paths ending at [N,N] can't fulfill the criteria.)

First, we count the number of paths beginning with a right move that end at $(2N - A, A)$.

For some A, the number of paths of this form is

$$\binom{2N-1}{A}.$$

Now, we make the following observation: for all A in our desired range, there exists a bijection between paths to $(2N - A, A)$ passing through $y = x$ and paths to $(2N - A + 1, A - 1)$ (the term "paths" here refers to paths starting with a right move.) Just flip every step made in a path of the former variety before and including the first step onto $y = x$ and we end up with a path to $(2N - A + 1, A - 1)$ (up/right gets swapped.)

Then, we can view the number of paths to $(2N - A, A)$ for A in the desired range as

$$\sum_{A=0}^{N-1} A_A - B_A,$$

where A_A denotes the number of right-beginning paths to $(2N - A, A)$ and B_A denotes the number of paths of the above variety passing through $y = x$. But our bijection above says

$$B_A = A_{A-1},$$

and so the sum simply evaluates to

$$A_{N-1} - A_{-1} = \binom{2N-1}{N} - 0.$$

Doubling this yields

$$2\binom{2N-1}{N} = \frac{(2N-1)!(2N)}{(N!)(N-1)!(N)} = \binom{2N}{N},$$

and so the probability of such a path being valid is the number of such valid paths over the total number of possible paths,

$$= \boxed{\frac{\binom{2n}{n}}{4^n}}$$

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8 ♠ B blue jays and C cardinals are sitting together in a tree. The birds take turns flying away. What is the likelihood that there is a time when the same number of blue jays and cardinals have flown away?

Solution. $t = 0$ doesn't count.

We look for the probability that there is no such time. We call the event that there is no such time 'N' (for NOT happened.). WLOG $B > C$. Note if a C goes first then there is no chance of N occurring. So it suffices to find the number of paths from $0, 0$ to (B, C) that stay strictly underneath the $B = C$ diagonal except for the beginning and then divide by $\binom{B+C}{B}$, and then finally take the complement.

If you look at the Wikipedia page for catalan numbers, you will see the now-standard argument to be used in these situations (search for 'bad paths'.) The idea is that the first step must be in the positive B direction. Then, we must stay below or on the $C = B - 1$ diagonal. There is a bijection between paths (starting at (1,0) that cross the $C = B - 1$ diagonal and touch the $C = B$ diagonal before arriving at (B,C) (therefore not being part of our event N) and paths from $(1, 0)$ to (C-1,B+1), by taking all up moves and right moves after the first incidence with the line $B = C$ and swapping them. So there are

$$\binom{B+C-1}{C-2}$$

bad paths that cross the diagonal,

$$\binom{B+C-1}{B-1}$$

paths in total that start at (1,0) and end at (B,C), and so there are

$$\binom{B+C-1}{B-1} - \binom{B+C-1}{C-2}$$

good paths. Finally, there are

$$\binom{B+C}{C} - (\binom{B+C-1}{B-1} - \binom{B+C-1}{C-2})$$

paths that cross the B=C diagonal at some point, and the desired probability is

$$\boxed{1 - \frac{\binom{B+C-1}{B-1} - \binom{B+C-1}{C-2}}{\binom{B+C}{C}}}$$

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9 ♠
Solution.

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10 ♠
Solution.

What is the probability a 2d random walk returns to the origin?

Solution. First, we find the expected number of returns to the origin.

Let R_n denote the event that the walk returns to the origin after $2n$ steps (possibly not for the first time.) Also, note that the walk can only return to the origin after an even number of steps.

Then of these 2n steps, we need 'a' lefts-and-rights and 'b' ups-and-downs such that $a + b = n$. In other words, we have the number of walks of length $2n$ that return to the origin after 2n steps is

$$\sum_{a+b=n} \binom{2n}{a, a, b, b},$$

where we use a multinomial coefficient to make clear that we pick a lefts, a rights, b ups, and b downs in our path.

$$= \sum_{a+b=n} \frac{(2n)!}{a!a!b!b!}.$$

$$= \sum_{a+b=n} \binom{2n}{n} \frac{n!^2}{a!a!b!b!},$$

motivated by the fact that this substitution matches each $a!b!$ term in the denominator with an $n!$ term in the numerator,

$$= \binom{2n}{n} \sum_{a+b=n} \binom{n}{a,b}^2,$$

and now the multinomial coefficient is a little silly.

$$= \binom{2n}{n} \sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}^2,$$

and the probability of return is

$$= \frac{\binom{2n}{n}^2}{4^{2n}}.$$

The expected number of returns is just (by LoE) the sum from n=1 to infinity of this term, which is

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2}{4^{2n}}$$

Stirling's approximation yields

$$\sqrt{2\pi n}(\frac{n}{e})^n \leq n! \leq \sqrt{2\pi n}(\frac{n}{e})^n e^{\frac{1}{12n}},$$

so

$$\sum_{n=1}^{\infty} \frac{((2n!)/(n!)^2)^2}{4^{2n}} \geq \sum_{n=1}^{\infty} \frac{(\sqrt{2\pi(2n)}(\frac{2n}{e})^{2n}/(\sqrt{2\pi n}(\frac{n}{e})^n e^{\frac{1}{12n}})^2)^2}{4^{2n}}$$

$$= \sum_{n=1}^{\infty} \frac{4\pi n 2^{4n}(\frac{n}{e})^{4n}}{4\pi^2 n^2 (\frac{n}{e})^{4n} e^{\frac{1}{3n}} 4^{2n}} = \sum_{i=1}^{n} \frac{1}{\pi n e^{\frac{1}{3n}}} \geq \sum_{i=1}^{n} \frac{1}{\pi n e^{\frac{1}{3}}},$$

which diverges, so the expected number of returns is infinite.

Then, let the probability of return be p. Then the expected number of returns E is

$$E = p(1 + E)$$

(with probability p, you return and then get E more returns in expectation. If you don't return, you get no returns,) or

$$p = \frac{E}{1+E}, E = \frac{p}{1-p}.$$

It's clear now that $\boxed{p = 1}$. The argument is direct when E is noninfinite; in any case, we can see that $p \in [0, 1]$, and $p \in [0, 1)$ results in a contradiction, justifying our intuition for the division by 0 or limit shenanigans a little more rigorously.

J ♠
Solution.

Q ♠
Solution.

K ♠
Solution.