Derivation of Root-finding Formulas for Some Special Fourth-order Equations of One Variable

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For general quartic equations of one variable, there is the complex Ferrari formula. Here I will discuss some methods of solving equations that satisfy certain coefficient relationships.

For a certain quartic equation (The coefficient of the highest order term is changed to "1")

$$x^4 + ax^3 + bx^2 + cx + d = 0$$

It may not be able to be transformed into the form of

$$(x^2 + px + q)^2 + m(x^2 + px + q) + n = 0 \ldots\ldots(3)$$

Now suppose that something can be converted into this form, then we have

$$\begin{cases} {x_1}^2 + px_1 + q = -\dfrac{m}{2} + \dfrac{1}{2}\sqrt{m^2 - 4n} \ldots\ldots(1) \\ {x_2}^2 + px_2 + q = -\dfrac{m}{2} - \dfrac{1}{2}\sqrt{m^2 - 4n} \ldots\ldots(2) \end{cases}$$

As for (1), let $s = q + \frac{m}{2} - \frac{1}{2}\sqrt{m^2 - 4n}$, then

$$\begin{cases} x_{11} = -\dfrac{p}{2} + \dfrac{1}{2}\sqrt{p^2 - 4s} \\ x_{12} = -\dfrac{p}{2} - \dfrac{1}{2}\sqrt{p^2 - 4s} \end{cases}$$

As for (2), let $t = q + \frac{m}{2} + \frac{1}{2}\sqrt{m^2 - 4n}$, then

$$\begin{cases} x_{21} = -\dfrac{p}{2} + \dfrac{1}{2}\sqrt{p^2 - 4t} \\ x_{22} = -\dfrac{p}{2} - \dfrac{1}{2}\sqrt{p^2 - 4t} \end{cases}$$

Equation (3) can be written as

$$x^4 + 2px^3 + (p^2 + 2q + m)x^2 + (2pq + mp)x + q^2 + mp + n = 0$$

It is the same equation as the original equation, so the coefficients should satisfy

$$\begin{cases} 2p = a \ldots\ldots(4) \\ p^2 + 2q + m = b \ldots\ldots(5) \\ 2pq + mp = c \ldots\ldots(6) \\ q^2 + mq + n = d \ldots\ldots(7) \end{cases}$$

By eliminating p and q by substitution, we get

$$\begin{cases} \dfrac{a^2}{4} + \dfrac{2c}{a} = b \ldots\ldots(8) \\ \left(\dfrac{c}{a} - \dfrac{m}{2}\right)^2 + m\left(\dfrac{c}{a} - \dfrac{m}{2}\right) + n = d \ldots\ldots(9) \end{cases}$$

Where

$$\begin{cases} p = \dfrac{a}{2} \\ q = \dfrac{c}{a} - \dfrac{m}{2} \end{cases}$$

From (8) we get:

$$a^3 + 8c = 4ab \dots\dots (10)$$

Which is a relationship about the original coefficient, which shows that only by satisfying this relationship can the hypothesis be established.

From (9), we get:

$$m = 2\sqrt{n - d + \frac{c^2}{a^2}}$$

Substituting q, we get

$$q = \frac{c}{a} - \sqrt{n - d + \frac{c^2}{a^2}}$$

Where n is still unknown, but it doesn't matter, it can be eliminated later in the substitution process.

Substituting m and q into s and t, we get

$$s = \frac{c}{a} - \sqrt{\frac{c^2}{a^2} - d},\ t = \frac{c}{a} + \sqrt{\frac{c^2}{a^2} - d}$$

Substituting s, t and p into x, we get

$$\begin{cases} x_{11} = -\dfrac{a}{4} + \sqrt{\dfrac{a^2}{16} - \dfrac{c}{a} + \sqrt{\dfrac{c^2}{a^2} - d}} \\ x_{12} = -\dfrac{a}{4} - \sqrt{\dfrac{a^2}{16} - \dfrac{c}{a} + \sqrt{\dfrac{c^2}{a^2} - d}} \\ x_{21} = -\dfrac{a}{4} + \sqrt{\dfrac{a^2}{16} - \dfrac{c}{a} - \sqrt{\dfrac{c^2}{a^2} - d}} \\ x_{22} = -\dfrac{a}{4} - \sqrt{\dfrac{a^2}{16} - \dfrac{c}{a} - \sqrt{\dfrac{c^2}{a^2} - d}} \end{cases} \dots\dots (11)$$

To sum up, if the fourth equation of one variable $x^4 + ax^3 + bx^2 + cx + d = 0$ satisfies $a^3 + 8c = 4ab \dots\dots (10)$

Then its four solutions are shown as (11)

Example: Solving Equations

$$x^4 - \frac{3}{2}x^3 - \frac{133}{48}x^2 + \frac{5}{2}x + \frac{3}{2} = 0$$

Test the coefficients, they satisfy

$$\left(-\frac{3}{2}\right)^3 + 8 \times \frac{5}{2} = 4 \times \left(-\frac{3}{2}\right) \times \left(-\frac{133}{48}\right)$$

Substituting into the formula we get:

$$\begin{cases} x_{11} = \frac{3}{8} + \sqrt{\frac{1.5^2}{16} + \frac{2.5}{1.5} + \sqrt{\frac{2.5^2}{1.5^2} - 1.5}} \approx 2.089 \\ x_{12} = \frac{3}{8} - \sqrt{\frac{1.5^2}{16} + \frac{2.5}{1.5} + \sqrt{\frac{2.5^2}{1.5^2} - 1.5}} \approx -1.339 \\ x_{21} = \frac{3}{8} + \sqrt{\frac{1.5^2}{16} + \frac{2.5}{1.5} - \sqrt{\frac{2.5^2}{1.5^2} - 1.5}} \approx 1.198 \\ x_{22} = \frac{3}{8} - \sqrt{\frac{1.5^2}{16} + \frac{2.5}{1.5} - \sqrt{\frac{2.5^2}{1.5^2} - 1.5}} \approx -0.448 \end{cases}$$

All are found to be the roots of the original equation.

This method of solving a quartic equation with specific coefficients is very simple. Of course, the four solutions in this example are all real numbers, but it doesn't matter if they are complex numbers. Now let's observe $a^3 + 8c = 4ab \ldots\ldots (10)$ and the formula itself, found that there is no coefficient b in the formula, but in fact, $b = \frac{a^3+8c}{4a}$.f