ARCHITETTURA DEGLI ELABORATORI

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Università di Napoli Federico II
Corso di Laurea in Informatica

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ALGEBRA DI BOOLE E RETI COMBINATORIE

Boolean Axioms

Number	Axiom	Dual	Name
A1	$B = 0$ if $B \neq 1$	$B = 1$ if $B \neq 0$	Binary Field
A2	$\overline{0} = 1$	$\overline{1} = 0$	NOT
A3	$0 \bullet 0 = 0$	$1 + 1 = 1$	AND/OR
A4	$1 \bullet 1 = 1$	$0 + 0 = 0$	AND/OR
A5	$0 \bullet 1 = 1 \bullet 0 = 0$	$1 + 0 = 0 + 1 = 1$	AND/OR

Dual: Replace: $\bullet$ with $+$
0 with 1

Boolean Theorems of One Variable

Number	Theorem	Dual	Name
T1	$B \bullet 1 = B$	$B + 0 = B$	Identity
T2	$B \bullet 0 = 0$	$B + 1 = 1$	Null Element
T3	$B \bullet B = B$	$B + B = B$	Idempotency
T4	$\overline{\overline{B}} = B$		Involution
T5	$B \bullet \overline{B} = 0$	$B + \overline{B} = 1$	Complements

Dual: Replace: $\bullet$ with $+$
0 with 1

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Boolean Theorems of Several Vars

#	Theorem	Dual	Name
T6	B•C = C•B	B+C = C+B	Commutativity
T7	(B•C) • D = B • (C•D)	(B + C) + D = B + (C + D)	Associativity
T8	B • (C + D) = (B•C) + (B•D)	B + (C•D) = (B+C) (B+D)	Distributivity
T9	B • (B+C) = B	B + (B•C) = B	Covering
T10	(B•C) + (B•$\overline{C}$) = B	(B+C) • (B+$\overline{C}$) = B	Combining
T11	(B•C) + ($\overline{B}$•D) + (C•D) = (B•C) + ($\overline{B}$•D)	(B+C) • ($\overline{B}$+D) • (C+D) = (B+C) • ($\overline{B}$+D)	Consensus

Warning: T8' differs from traditional algebra:
OR (+) distributes over AND (•)

Simplification methods

- **Distributivity (T8, T8')** $B(C+D) = BC + BD$
  $B + CD = (B+C)(B+D)$
- **Covering (T9')** $A + AP = A$
- **Combining (T10)** $P\overline{A} + PA = P$
- **Expansion** $P = P\overline{A} + PA$
  $A = A + AP$
- **Duplication** $A = A + A$
- **"Simplification" theorem** $P\overline{A} + A = P + A$
  $PA + \overline{A} = P + \overline{A}$

Simplifying Boolean Equations

Example 6:

$$Y = (A + BC)(A + DE)$$

Apply T8' first when possible: $W+XZ = (W+X)(W+Z)$

Simplifying Boolean Equations

Example 6:

Y = (A + BC)(A + DE)

Apply T8' first when possible: W+XZ = (W+X)(W+Z)

Make: X = BC, Z = DE and rewrite equation

Y	= (A+X)(A+Z)	substitution (X=BC, Z=DE)
	= A + XZ	T8': Distributivity
	= A + BCDE	substitution

or

Y	= AA + ADE + ABC + BCDE	T8: Distributivity
	= A + ADE + ABC + BCDE	T3: Idempotency
	= **A + ADE** + ABC + BCDE	
	= **A** + ABC + BCDE	T9': Covering
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Simplifying Boolean Equations

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This is called ***multiplying out*** an expression to get sum-of-products (SOP) form.

Multiplying Out: SOP Form

An expression is in **sum-of-products (SOP)** form when all products contain literals only.

- SOP form: Y = AB + BC' + DE
- **NOT** SOP form: Y = DF + E(A'+B)
- SOP form: Z = A + BC + DE'F

Multiplying Out: SOP Form

Example:

$$Y = (A + C + D + E)(A + B)$$

Apply T8' first when possible: $W+XZ = (W+X)(W+Z)$

Make: $X = (C+D+E)$, $Z = B$ and rewrite equation

Y	$= (A+X)(A+Z)$	substitution ($X=(C+D+E)$, $Z=B$)
	$= A + XZ$	T8': Distributivity
	$= A + (C+D+E)B$	substitution
	$= A + BC + BD + BE$	T8: Distributivity

or

Y	$= AA+AB+AC+BC+AD+BD+AE+BE$	T8: Distributivity
$A+AX=A$	$= A+AB+AC+AD+AE+BC+BD+BE$	T3: Idempotency
	$= A + BC + BD + BE$	T9': Covering

Factoring: POS Form

An expression is in **product-of-sums (POS)** form when all sums contain literals only.

- POS form: $Y = (A+B)(C+D)(E'+F)$
- **NOT** POS form: $Y = (D+E)(F'+GH)$
- POS form: $Z = A(B+C)(D+E')$

Factoring: POS Form

Example 1:

$$Y = (A + \overbrace{B'C}^{X}\overbrace{DE}^{Z})$$

Apply T8' first when possible: W+XZ = (W+X)(W+Z)

Make: X = B'C, Z = DE and rewrite equation

Y	= (A+XZ)	substitution (X=B'C, Z=DE)
	= (A+B'C)(A+DE)	T8': Distributivity
	= (A+B')(A+C)(A+D)(A+E)	T8': Distributivity

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Factoring: POS Form

Example 2:

$Y = AB + C'DE + F$

Apply T8' first when possible: $W+XZ = (W+X)(W+Z)$

Make: $W = AB$, $X = C'$, $Z = DE$ and rewrite equation

Y	$= (W+XZ) + F$	substitution $W = AB$, $X = C'$, $Z = DE$
	$= (W+X)(W+Z) + F$	T8': Distributivity
	$= (AB+C')(AB+DE)+F$	substitution
	$= (A+C')(B+C')(AB+D)(AB+E)+F$	T8': Distributivity
	$= (A+C')(B+C')(A+D)(B+D)(A+E)(B+E)+F$	T8': Distributivity
	$= (A+C'+F)(B+C'+F)(A+D+F)(B+D+F)(A+E+F)(B+E+F)$	T8': Distributivity

DeMorgan's Theorem

Number	Theorem	Name
T12	$\overline{B_0 \bullet B_1 \bullet B_2 \ldots} = \overline{B_0} + \overline{B_1} + \overline{B_2} \ldots$	DeMorgan's Theorem

- La negata di un prodotto è uguale alla somma delle negate
- La negata di una somma è uguale al prodotto delle negate

DeMorgan's Theorem: Dual

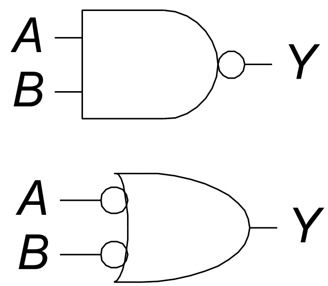
#	Theorem	Dual	Name
T12	$\overline{B_0 \bullet B_1 \bullet B_2 \ldots} = \overline{B_0} + \overline{B_1} + \overline{B_2} \ldots$	$\overline{B_0 + B_1 + B_2 \ldots} = \overline{B_0} \bullet \overline{B_1} \bullet \overline{B_2} \ldots$	DeMorgan's Theorem

The complement of the product
is the
sum of the complements.

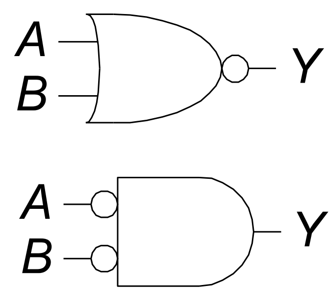
Dual: The **complement** of the **sum**
is the
product of the **complements**.

DeMorgan's Theorem

- $Y = \overline{AB} = \overline{A} + \overline{B}$



- $Y = \overline{A + B} = \overline{A} \bullet \overline{B}$



DeMorgan's Theorem Example 1

$$Y = \overline{(A+\overline{BD})\overline{C}}$$

DeMorgan's Theorem Example 1

$$Y = \overline{(A+\overline{BD})\overline{C}}$$

$$= \overline{(A+\overline{BD})} + C$$

$$= (\overline{A} \bullet (\overline{\overline{BD}})) + C$$

$$= (\overline{A} \bullet (BD)) + C$$

$$= \overline{A}BD + C$$

$\overline{A \cdot B} = \overline{A} + \overline{B}$

SOP

DeMorgan's Theorem Example 2

$$Y = \overline{(\overline{AC}E + \overline{D}) + B}$$

$$(A\,C\overline{E} \cdot D) \cdot \overline{B}$$

DeMorgan's Theorem Example 2

$$Y = \overline{(\overline{ACE}+\overline{D}) + B}$$

$$= \overline{(\overline{ACE}+\overline{D})} \bullet \overline{B}$$

$$= (\overline{\overline{ACE}} \bullet \overline{\overline{D}}) \bullet \overline{B}$$

$$= ((A\overline{\overline{C}}+\overline{E}) \bullet D) \bullet \overline{B}$$

$$= ((AC+\overline{E}) \bullet D) \bullet \overline{B}$$

$$= (ACD + D\overline{E}) \bullet \overline{B}$$

$$= A\overline{B}CD + \overline{B}D\overline{E}$$

SOP

$$\overline{XY} = \overline{X} + \overline{Y}$$

$$X = \overline{AC}$$

$$Y = E$$

$$\overline{AC} =$$

$$\overline{ACE} = E(\overline{AC}) =$$

$$= E(\overline{A} + \overline{C})$$

$$= E\overline{A} + E\overline{C}$$

Teoremi di De Morgan e forme SOP/POS

I teoremi di De Morgan possono essere usati per ridurre una generica espressione E in forma SOP/POS senza «passare» per la tabella di verità

- Applica esaustivamente «De Morgan» per spingere la negazione nella struttura della formula
- Applica esaustivamente la proprietà distributiva dell'AND sull'OR (SOP)
- Applica esaustivamente la proprietà distributiva dell'OR sull'AND (POS)

Esercizi

- Sfruttando i teoremi precedenti verificare le seguenti proprietà della porta XOR

$a \oplus a = 0$ — $A\bar{A} + \bar{A}A = 0 + 0 = 0$
$a \oplus 0 = a$ — $A\bar{0} + \bar{A}0 = A1 + 0 = A$
$a \oplus 1 = \sim a$, where $\sim$ is bit complement.
$a \oplus \sim a = 1$ $A\bar{\bar{A}} + \bar{A}\cdot\bar{A} = A + \bar{A} = 1$
$a \oplus b = b \oplus a$ (commutativity)
$a \oplus (b \oplus c) = (a \oplus b) \oplus c$ (associativity)

Ricordarsi che, date due espressioni booleane E e F,

$$\underbrace{E \oplus F = E\bar{F} + \bar{E}F}$$

E F	$\oplus$
0 0	0
0 1	1
1 0	1
1 1	0

XOR

Esercizi

- Esercizi del libro “Digital Design and Computer Architecture – ARM edition”
  - 2.1 — C SOP $\bar{A}\bar{B}\bar{C} + \bar{A}B\bar{C} + A\bar{B}\bar{C} + A\bar{B}C + ABC$
  - 2.2
  - 2.3 — C POS $(\bar{A}+\bar{B}+C)(A+\bar{B}+\bar{C})(A+B+\bar{C})$
    M_3 $\bar{M}$
  - 2.4
- Verificare in maniera proof-teoretica che
  a ⊕ b $= (\bar{a} + \bar{b})\,(a + b)$
- Dimostrare che la porta NAND è da sola un insieme completo (e minimale) di operatori booleani

$a \oplus b =$

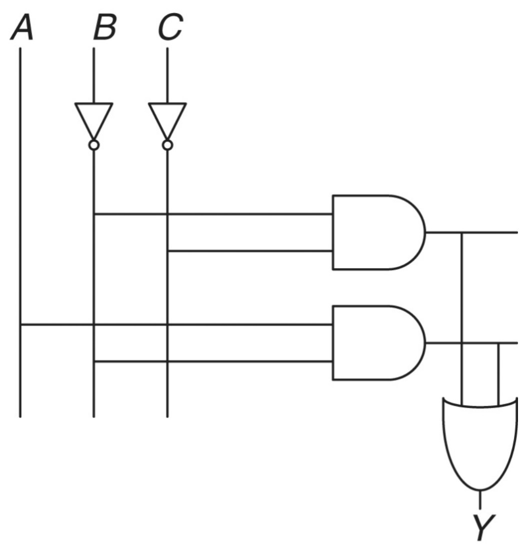
$= \bar{A}B + \bar{B}A = \overline{\overline{\bar{A}B + \bar{B}A}} =$

$\overline{(A+\bar{B})(B+\bar{A})} = \overline{AB + \cancel{\ldots} + \bar{A}\bar{B}} = \overline{AB} \cdot \overline{\bar{A}\bar{B}}$

$(\bar{A}+\bar{B})(A+B)$

Perché semplificare una espressione?

- Importanza della minimizzazione: utilizzare meno porte logiche
- Esercizio: mostrare che $\bar{A}\bar{B}\bar{C} + A\bar{B}\bar{C} + A\bar{B}C = \bar{B}\bar{C}+A\bar{B}$



$$Y = \bar{B}\,\bar{C} + A\bar{B}$$

Esempio 1

$Y = A(AB + ABC)$

$= A(AB(1 + C))$	T8: Distributivity
$= A(AB(1))$	T2': Null Element
$= A(AB)$	T1: Identity
$= (AA)B$	T7: Associativity
$= AB$	T3: Idempotency

Esempio 2

$Y = \overline{A}BC + \overline{A}$ **Recall: $A' = A$**

$= \overline{A}$ T9' Covering: $X + XY = X$

oppure

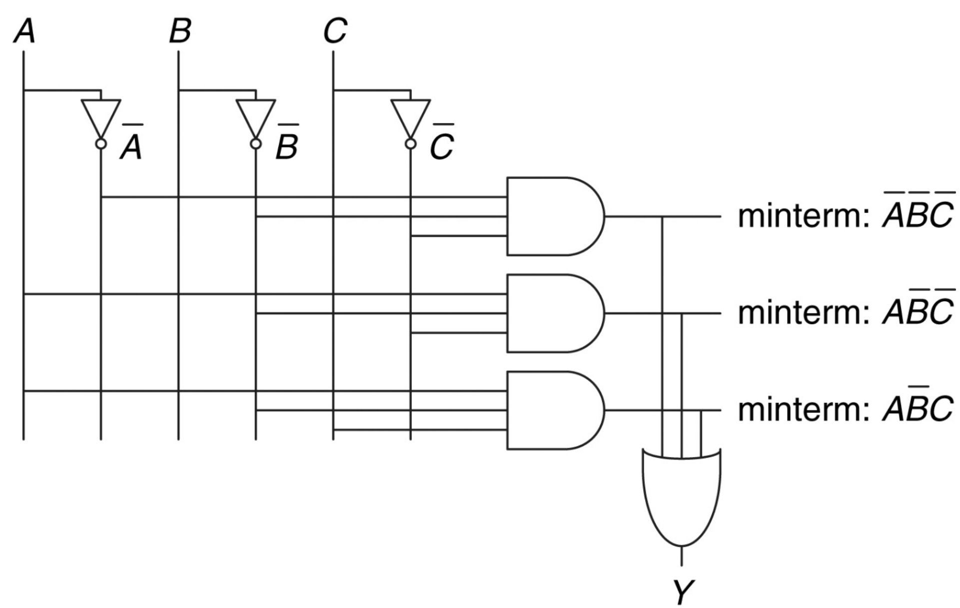
$= \overline{A}(BC + 1)$ T8: Distributivity

$= \overline{A}(1)$ T2': Null Element

$= \overline{A}$ T1: Identity

Schemi circuitali

- Ad ogni espressione booleana corrisponde in circuito combinatorio



$$Y = \overline{A}\,\overline{B}\,\overline{C} + A\overline{B}\,\overline{C} + A\overline{B}C$$

Schemi circuitali SOP

- Le formule in forma SOP hanno degli schemi circuitali molto regolari:
  - Disegna una linea di input per ogni variabile che occorre positiva
  - Aggiungi delle linee con un NOT per ogni variabile che occorre negata
  - Per ogni mintermine disegna una porta AND e aggiungi in ingresso le linee che corrispondono ai relativi litterali
  - Collega tutte le uscite delle porte AND in un unico OR
- Per questo la forma SOP viene detta una logica a due livelli AND-OR

Semplificare formule SOP

- Per T10 $\mathrm{P}\bar{B} + \mathrm{P}B = P$ per ogni implicante *P* .
- Un implicante è detto *implicante primo* se non può essere combinato con altri implicanti della formula per ottenere un nuovo implicante con meno litterali.
- Una espressione SOP è minimale se tutti i suoi implicanti sono primi
- Minimizzazione: *ridurre il numero di implicanti e per ogni implicante ridurre il numero di litterali*

Semplificare formule SOP

- Nel minimizzare una SOP, può essere necessario «sdoppiare» un implicante allorché questo può essere ridotto in modi differenti.

Step	Equation	Justification
	$\overline{A}\,\overline{B}\,\overline{C} + A\overline{B}\,\overline{C} + A\overline{B}C$	
1	$\overline{B}\,\overline{C}(\overline{A} + A) + A\overline{B}C$	T8: Distributivity
2	$\overline{B}\,\overline{C}(1) + A\overline{B}C$	T5: Complements
3	$\overline{B}\,\overline{C} + A\overline{B}C$	T1: Identity

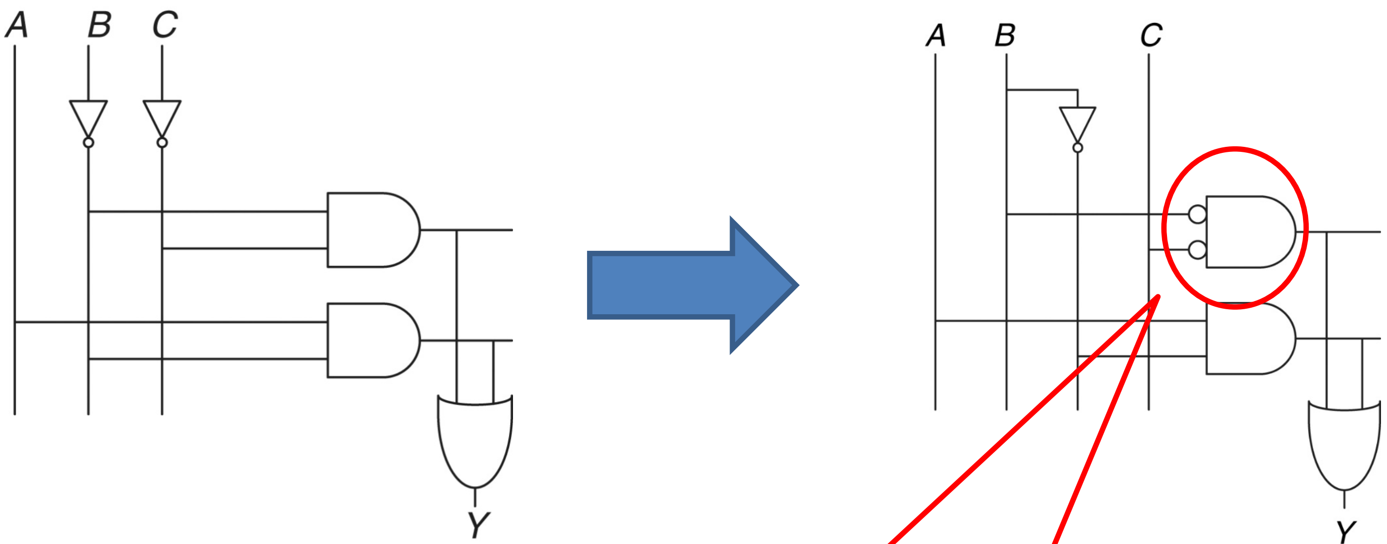
Non è minimale $A\bar{B}C$ e $A\bar{B}\bar{C}$ possono ridursi in $A\bar{B}$

Semplificare formule SOP

Step	Equation	Justification
	$\overline{A}\,\overline{B}\,\overline{C} + A\overline{B}\,\overline{C} + A\overline{B}C$	
1	$\overline{A}\,\overline{B}\,\overline{C} + A\overline{B}\,\overline{C} + A\overline{B}\,\overline{C} + A\overline{B}C$	T3: Idempotency
2	$\overline{B}\,\overline{C}(\overline{A} + A) + A\overline{B}(\overline{C} + C)$	T8: Distributivity
3	$\overline{B}\,\overline{C}(1) + A\overline{B}(1)$	T5: Complements
4	$\overline{B}\,\overline{C} + A\overline{B}$	T1: Identity

Il numero di implicanti è lo stesso ma $A\overline{B}$ ha meno letterali di $A\overline{B}C$

Schemi circuitali e minimizzazione



Per De Morgan: $\bar{B}\bar{C} = \overline{B + C}$ quindi questa porta AND con gli ingressi negati può essere sostituita da un NOR. Nella tecnologia MOSFET il NOR è più veloce di AND

Semplificare le forme POS

- Come si semplifica una forma POS?
- La proprietà principale che si usa è il combining (B+C) • (B+C) = B
- Esempio:

$(A + \bar{B} + \bar{C})(\bar{A} + B + \bar{C})(\bar{A} + \bar{B} + C)(\bar{A} + \bar{B} + \bar{C}) =$

$(A + \bar{B} + \bar{C})(\bar{A} + B + \bar{C})(\bar{A} + \bar{B} + C)(\bar{A} + \bar{B} + \bar{C})(\bar{A} + \bar{B} + \bar{C})(\bar{A} + \bar{B} + \bar{C})=$

$(A + \bar{B} + \bar{C})(\bar{A} + \bar{B} + \bar{C})(\bar{A} + B + \bar{C})(\bar{A} + \bar{B} + \bar{C})(\bar{A} + \bar{B} + C)(\bar{A} + \bar{B} + \bar{C})=$

$(B + \bar{C})(\bar{A} + B + \bar{C})(\bar{A} + \bar{B} + \bar{C})(\bar{A} + \bar{B} + C)(\bar{A} + \bar{B} + \bar{C})=$

$(B + \bar{C})(\bar{A} + \bar{C})\ (\bar{A} + \bar{B} + C)(\bar{A} + \bar{B} + \bar{C})=$

$(B + \bar{C})(\bar{A} + \bar{C})\ (\bar{A} + \bar{B})$

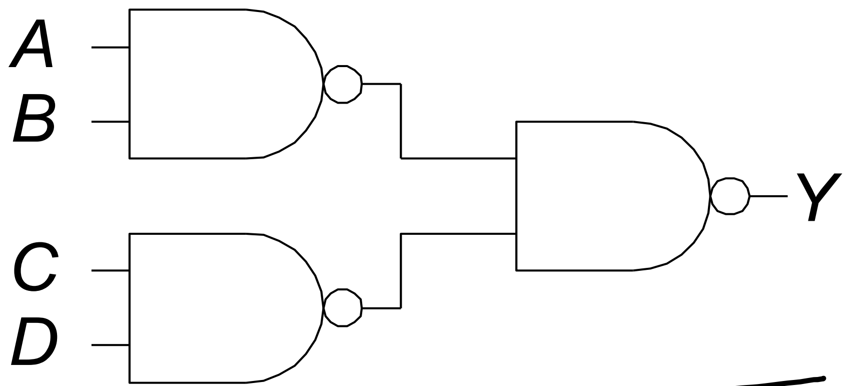
Bubble Pushing

- **Backward:**
  - Body changes
  - Adds bubbles to inputs

- **Forward:**
  - Body changes
  - Adds bubble to output

Bubble Pushing

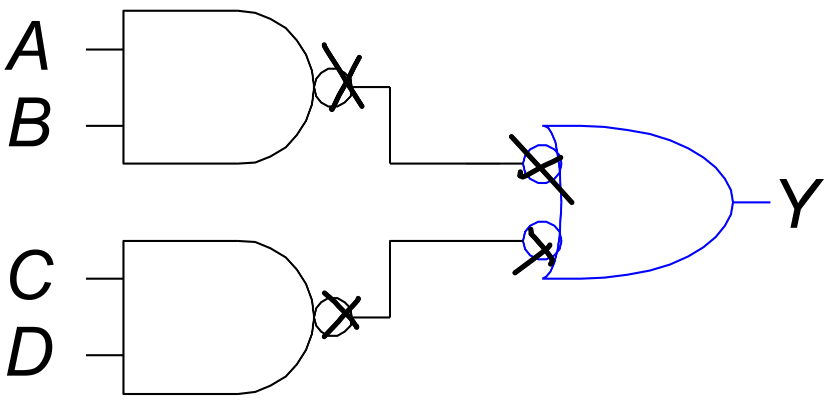
- What is the Boolean expression for this circuit?



$$\overline{\overline{AB} \cdot \overline{CD}} = \underbrace{\overline{\overline{AB}}} + \underbrace{\overline{\overline{CD}}}$$

Bubble Pushing

- What is the Boolean expression for this circuit?

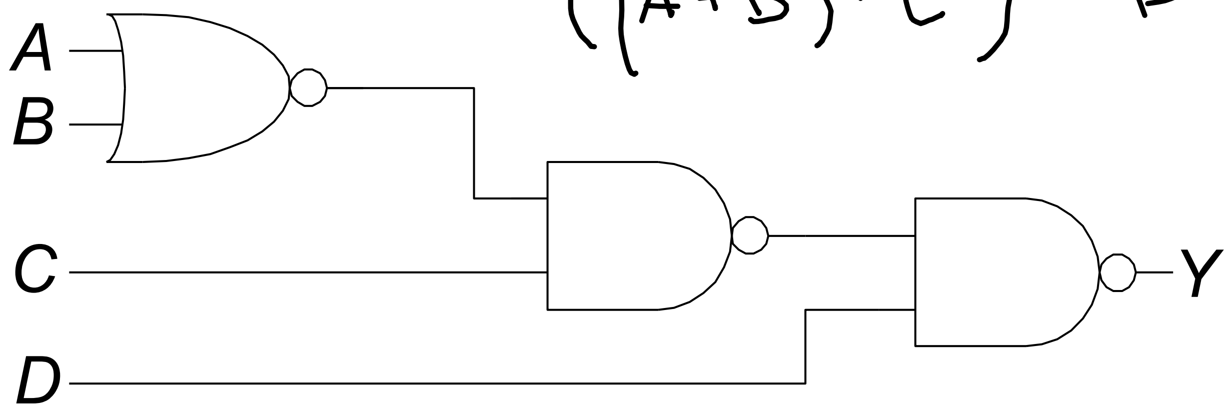


$$Y = AB + CD$$

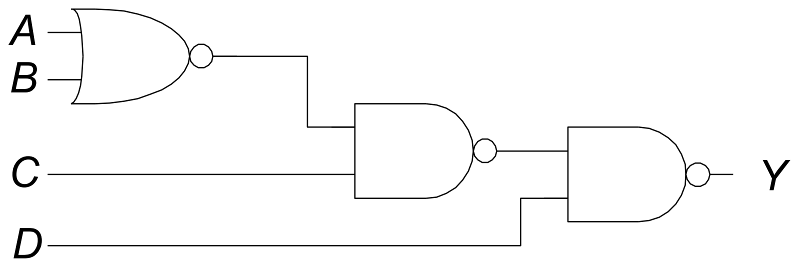
Bubble Pushing Rules

- Begin at output, then work toward inputs
- Push bubbles on final output back
- Draw gates in a form so bubbles cancel

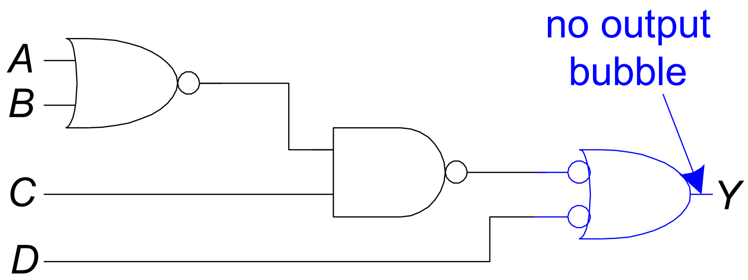
$$\overline{\left(\overline{\overline{A+B}\cdot C}\right)\cdot D}$$



Bubble Pushing Example



Bubble Pushing Example



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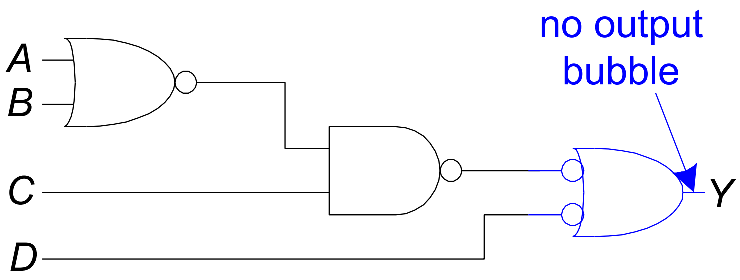
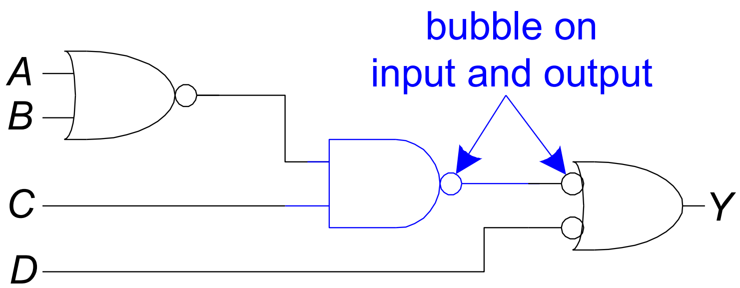
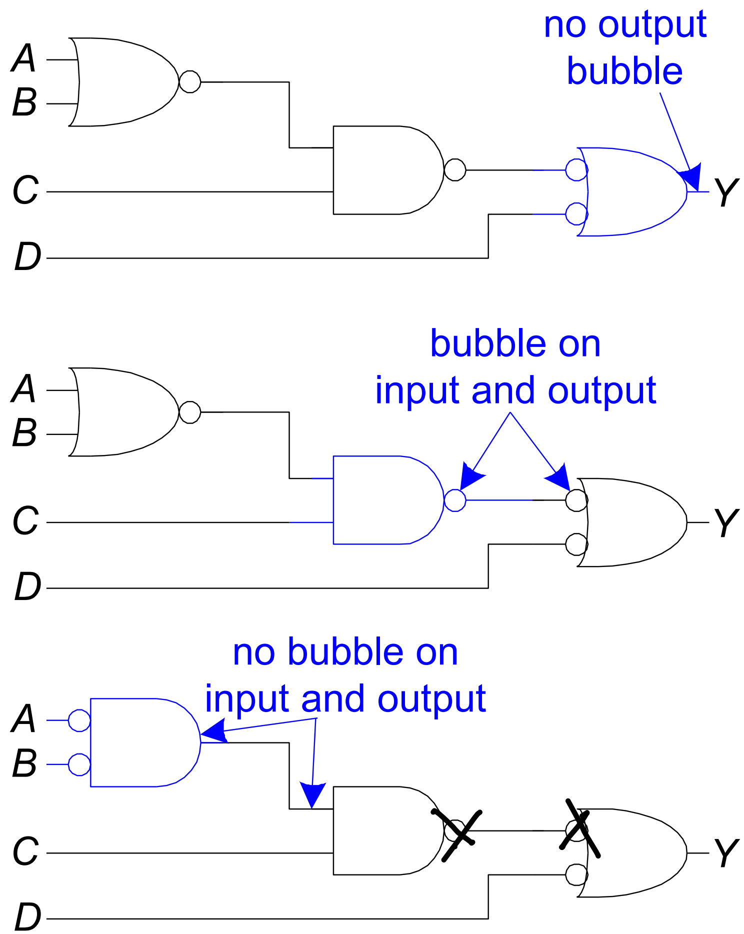




Bubble Pushing Example



$Y = \overline{A}\,\overline{B}C + \overline{D}$