Finding Limits

In Exercises 9–28, find the limit or explain why it does not exist.

9. $\lim \dfrac{x^2 - 4x + 4}{x^3 + 5x^2 - 14x}$

a. as $x \to 0$ **b.** as $x \to 2$

10. $\lim \dfrac{x^2 + x}{x^5 + 2x^4 + x^3}$

a. as $x \to 0$ **b.** as $x \to -1$

11. $\lim\limits_{x \to 1} \dfrac{1 - \sqrt{x}}{1 - x}$

12. $\lim\limits_{x \to a} \dfrac{x^2 - a^2}{x^4 - a^4}$

13. $\lim\limits_{h \to 0} \dfrac{(x + h)^2 - x^2}{h}$

14. $\lim\limits_{x \to 0} \dfrac{(x + h)^2 - x^2}{h}$

15. $\lim\limits_{x \to 0} \dfrac{\dfrac{1}{2 + x} - \dfrac{1}{2}}{x}$

16. $\lim\limits_{x \to 0} \dfrac{(2 + x)^3 - 8}{x}$

17. $\lim\limits_{x \to 1} \dfrac{x^{1/3} - 1}{\sqrt{x} - 1}$

18. $\lim\limits_{x \to 64} \dfrac{x^{2/3} - 16}{\sqrt{x} - 8}$

19. $\lim\limits_{x \to 0} \dfrac{\tan(2x)}{\tan(\pi x)}$

20. $\lim\limits_{x \to \pi^-} \csc x$

21. $\lim\limits_{x \to \pi} \sin\left(\dfrac{x}{2} + \sin x\right)$

22. $\lim\limits_{x \to \pi} \cos^2(x - \tan x)$

23. $\lim\limits_{x \to 0} \dfrac{8x}{3 \sin x - x}$

24. $\lim\limits_{x \to 0} \dfrac{\cos 2x - 1}{\sin x}$

Limits at Infinity

Find the limits in Exercises 37–46.

37. $\lim_{x \to \infty} \frac{2x + 3}{5x + 7}$

38. $\lim_{x \to -\infty} \frac{2x^2 + 3}{5x^2 + 7}$

39. $\lim_{x \to -\infty} \frac{x^2 - 4x + 8}{3x^3}$

40. $\lim_{x \to \infty} \frac{1}{x^2 - 7x + 1}$

41. $\lim_{x \to -\infty} \frac{x^2 - 7x}{x + 1}$

42. $\lim_{x \to \infty} \frac{x^4 + x^3}{12x^3 + 128}$

43. $\lim_{x \to \infty} \frac{\sin x}{\lfloor x \rfloor}$ (If you have a grapher, try graphing the function for $-5 \le x \le 5$.)

44. $\lim_{\theta \to \infty} \frac{\cos \theta - 1}{\theta}$ (If you have a grapher, try graphing $f(x) = x(\cos(1/x) - 1)$ near the origin to "see" the limit at infinity.)

45. $\lim_{x \to \infty} \frac{x + \sin x + 2\sqrt{x}}{x + \sin x}$

46. $\lim_{x \to \infty} \frac{x^{2/3} + x^{-1}}{x^{2/3} + \cos^2 x}$

9. (a) $\lim\limits_{x\to 0} \frac{x^2-4x+4}{x^3+5x^2-14x} = \lim\limits_{x\to 0} \frac{(x-2)(x-2)}{x(x+7)(x-2)} = \lim\limits_{x\to 0} \frac{x-2}{x(x+7)}$, $x \neq 2$; the limit does not exist because $\lim\limits_{x\to 0^-} \frac{x-2}{x(x+7)} = \infty$ and $\lim\limits_{x\to 0^+} \frac{x-2}{x(x+7)} = -\infty$

(b) $\lim\limits_{x\to 2} \frac{x^2-4x+4}{x^3+5x^2-14x} = \lim\limits_{x\to 2} \frac{(x-2)(x-2)}{x(x+7)(x-2)} = \lim\limits_{x\to 2} \frac{x-2}{x(x+7)}$, $x \neq 2$, and $\lim\limits_{x\to 2} \frac{x-2}{x(x+7)} = \frac{0}{2(9)} = 0$

10. (a) $\lim\limits_{x\to 0} \frac{x^2+x}{x^5+2x^4+x^3} = \lim\limits_{x\to 0} \frac{x(x+1)}{x^3(x^2+2x+1)} = \lim\limits_{x\to 0} \frac{x+1}{x^2(x+1)(x+1)} = \lim\limits_{x\to 0} \frac{1}{x^2(x+1)}$, $x \neq 0$ and $x \neq -1$.
Now $\lim\limits_{x\to 0^-} \frac{1}{x^2(x+1)} = \infty$ and $\lim\limits_{x\to 0^+} \frac{1}{x^2(x+1)} = \infty \Rightarrow \lim\limits_{x\to 0} \frac{x^2+x}{x^5+2x^4+x^3} = \infty$.

(b) $\lim\limits_{x\to -1} \frac{x^2+x}{x^5+2x^4+x^3} = \lim\limits_{x\to -1} \frac{x(x+1)}{x^3(x^2+2x+1)} = \lim\limits_{x\to -1} \frac{1}{x^2(x+1)}$, $x \neq 0$ and $x \neq -1$. The limit does not exist because $\lim\limits_{x\to -1^-} \frac{1}{x^2(x+1)} = -\infty$ and $\lim\limits_{x\to -1^+} \frac{1}{x^2(x+1)} = \infty$.

11. $\lim\limits_{x\to 1} \frac{1-\sqrt{x}}{1-x} = \lim\limits_{x\to 1} \frac{1-\sqrt{x}}{(1-\sqrt{x})(1+\sqrt{x})} = \lim\limits_{x\to 1} \frac{1}{1+\sqrt{x}} = \frac{1}{2}$

12. $\lim\limits_{x\to a} \frac{x^2-a^2}{x^4-a^4} = \lim\limits_{x\to a} \frac{(x^2-a^2)}{(x^2+a^2)(x^2-a^2)} = \lim\limits_{x\to a} \frac{1}{x^2+a^2} = \frac{1}{2a^2}$

13. $\lim\limits_{h\to 0} \frac{(x+h)^2-x^2}{h} = \lim\limits_{h\to 0} \frac{(x^2+2hx+h^2)-x^2}{h} = \lim\limits_{h\to 0} (2x+h) = 2x$

14. $\lim\limits_{x\to 0} \frac{(x+h)^2-x^2}{h} = \lim\limits_{x\to 0} \frac{(x^2+2hx+h^2)-x^2}{h} = \lim\limits_{x\to 0} (2x+h) = h$

15. $\lim\limits_{x\to 0} \frac{\frac{1}{2+x}-\frac{1}{2}}{x} = \lim\limits_{x\to 0} \frac{2-(2+x)}{2x(2+x)} = \lim\limits_{x\to 0} \frac{-1}{4+2x} = -\frac{1}{4}$

16. $\lim\limits_{x\to 0} \frac{(2+x)^3-8}{x} = \lim\limits_{x\to 0} \frac{(x^3+6x^2+12x+8)-8}{x} = \lim\limits_{x\to 0} (x^2+6x+12) = 12$

17. $\lim\limits_{x\to 1} \frac{x^{1/3}-1}{\sqrt{x}-1} = \lim\limits_{x\to 1} \frac{(x^{1/3}-1)}{(\sqrt{x}-1)} \cdot \frac{(x^{2/3}+x^{1/3}+1)(\sqrt{x}+1)}{(\sqrt{x}+1)(x^{2/3}+x^{1/3}+1)} = \lim\limits_{x\to 1} \frac{(x-1)(\sqrt{x}+1)}{(x-1)(x^{2/3}+x^{1/3}+1)} = \lim\limits_{x\to 1} \frac{\sqrt{x}+1}{x^{2/3}+x^{1/3}+1} = \frac{1+1}{1+1+1} = \frac{2}{3}$

18. $\lim\limits_{x\to 64} \frac{x^{2/3}-16}{\sqrt{x}-8} = \lim\limits_{x\to 64} \frac{(x^{1/3}-4)(x^{1/3}+4)}{\sqrt{x}-8} = \lim\limits_{x\to 64} \frac{(x^{1/3}-4)(x^{1/3}+4)}{\sqrt{x}-8} \cdot \frac{(x^{2/3}+4x^{1/3}+16)(\sqrt{x}+8)}{(\sqrt{x}+8)(x^{2/3}+4x^{1/3}+16)}$

$= \lim\limits_{x\to 64} \frac{(x-64)\,(x^{1/3}+4)\,(\sqrt{x}+8)}{(x-64)\,(x^{2/3}+4x^{1/3}+16)} = \lim\limits_{x\to 64} \frac{(x^{1/3}+4)\,(\sqrt{x}+8)}{x^{2/3}+4x^{1/3}+16} = \frac{(4+4)\,(8+8)}{16+16+16} = \frac{8}{3}$

19. $\lim\limits_{x\to 0} \frac{\tan 2x}{\tan \pi x} = \lim\limits_{x\to 0} \frac{\sin\ 2x}{\cos\ 2x} \cdot \frac{\cos \pi x}{\sin \pi x} = \lim\limits_{x\to 0} \left(\frac{\sin\ 2x}{2x}\right)\left(\frac{\cos \pi x}{\cos\ 2x}\right)\left(\frac{\pi x}{\sin \pi x}\right)\left(\frac{2x}{\pi x}\right) = 1 \cdot 1 \cdot 1 \cdot \frac{2}{\pi} = \frac{2}{\pi}$

20. $\lim\limits_{x\to \pi^-} \csc\ x = \lim\limits_{x\to \pi^-} \frac{1}{\sin x} = \infty$

21. $\lim\limits_{x\to \pi} \sin\left(\frac{x}{2} + \sin\ x\right) = \sin\left(\frac{\pi}{2} + \sin\ \pi\right) = \sin\left(\frac{\pi}{2}\right) = 1$

22. $\lim\limits_{x\to \pi} \cos^2(x - \tan x) = \cos^2\ (\pi - \tan \pi)\ =\ \cos^2(\pi) = (-1)^2 = 1$

23. $\lim\limits_{x\to 0} \frac{8x}{3\sin\ x - x} = \lim\limits_{x\to 0} \frac{8}{3\frac{\sin x}{x} - 1} = \frac{8}{3(1)-1} = 4$

24. $\lim\limits_{x\to 0} = \frac{\cos\ 2x - 1}{\sin x} = \lim\limits_{x\to 0} \left(\frac{\cos\ 2x - 1}{\sin x} \cdot \frac{\cos\ 2x + 1}{\cos\ 2x + 1}\right) = \lim\limits_{x\to 0} \frac{\cos^2\ 2x - 1}{\sin x(\cos\ 2x + 1)} = \lim\limits_{x\to 0} \frac{-\sin^2 2x}{\sin\ x(\cos\ 2x + 1)}$

$= \lim\limits_{x\to 0} \frac{-4\sin x \cos^2 x}{\cos\ 2x + 1} = \frac{-4(0)(1)^2}{1+1} = 0$

37. $\lim\limits_{x\to \infty} \frac{2x+3}{5x+7} = \lim\limits_{x\to \infty} \frac{2+\frac{3}{x}}{5+\frac{7}{x}} = \frac{2+0}{5+0} = \frac{2}{5}$

38. $\lim\limits_{x\to -\infty} \frac{2x^2+3}{5x^2+7} = \lim\limits_{x\to -\infty} \frac{2+\frac{3}{x^2}}{5+\frac{7}{x^2}} = \frac{2+0}{5+0} = \frac{2}{5}$

39. $\lim\limits_{x\to -\infty} \frac{x^2-4x+8}{3x^3} = \lim\limits_{x\to -\infty} \left(\frac{1}{3x} - \frac{4}{3x^2} + \frac{8}{3x^3}\right) = 0 - 0 + 0 = 0$

40. $\lim\limits_{x\to \infty} \frac{1}{x^2-7x+1} = \lim\limits_{x\to \infty} \frac{\frac{1}{x^2}}{1-\frac{7}{x}+\frac{1}{x^2}} = \frac{0}{1-0+0} = 0$

41. $\lim\limits_{x\to -\infty} \frac{x^2-7x}{x+1} = \lim\limits_{x\to -\infty} \frac{x-7}{1+\frac{1}{x}} = -\infty$

42. $\lim\limits_{x\to \infty} \frac{x^4+x^3}{12x^3+128} = \lim\limits_{x\to -\infty} \frac{x+1}{12+\frac{128}{x^3}} = \infty$

43. $\lim\limits_{x\to \infty} \frac{\sin x}{\lfloor x \rfloor} \le \lim\limits_{x\to \infty} \frac{1}{\lfloor x \rfloor} = 0$ since $\lfloor x \rfloor \to \infty$ as $x \to \infty \Rightarrow \lim\limits_{x\to \infty} \frac{\sin x}{\lfloor x \rfloor} = 0.$

44. $\lim\limits_{\theta\to \infty} \frac{\cos\ \theta - 1}{\theta} \le \lim\limits_{\theta\to \infty} \frac{2}{\theta} = 0 \Rightarrow \lim\limits_{\theta\to \infty} \frac{\cos \theta - 1}{\theta} = 0.$

45. $\lim\limits_{x\to \infty} \frac{x+\sin\ x+2\sqrt{x}}{x+\sin\ x} = \lim\limits_{x\to \infty} \frac{1+\frac{\sin x}{x}+\frac{2}{\sqrt{x}}}{1+\frac{\sin x}{x}} = \frac{1+0+0}{1+0} = 1$

46. $\lim\limits_{x\to \infty} \frac{x^{2/3}+x^{-1}}{x^{2/3}+\cos^2 x} = \lim\limits_{x\to \infty} \left(\frac{1+x^{-5/3}}{1+\frac{\cos^2 x}{x^{2/3}}}\right) = \frac{1+0}{1+0} = 1$