Infinite series

Textbook sections: 10.2

In the next part of the course we will discuss how one can find sum of all elements of given infinite sequence.

Definition 1. Suppose we have a sequence a_n. The formal infinite sum $\sum\limits_{i=1}^{\infty} a_i$ is called a series.

Denote $S_n = \sum\limits_{i=1}^{n} a_i$. If the sequence S_i converges to S then

$\sum\limits_{i=1}^{\infty} a_i = \lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} a_i = \lim\limits_{n\to\infty} S_n = S$

We call S the sum of the series.

In case the sequence S_n diverges we say that the series diverges.

Example 1. Analyze the series $\sum\limits_{i=1}^{\infty} \frac{1}{2^i}$ for convergence. Compute the sum in case it converges.

Let us look at the partial sums of this series.

$$S_1 = a_1 = \frac{1}{2} = 1 - \frac{1}{2}$$

$$S_2 = a_1 + a_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} = 1 - \frac{1}{4}$$

$$S_3 = a_1 + a_2 + a_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8} = 1 - \frac{1}{8}$$

$$S_4 = a_1 + a_2 + a_3 + a_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16} = 1 - \frac{1}{16}$$

We can see the pattern. Generally, $S_n = 1 - \frac{1}{2^n}$. We can prove it by induction. The base of induction was shown above. Induction step:

$$S_{n+1} = S_n + a_{n+1} = (1 - \frac{1}{2^n}) + \frac{1}{2^{n+1}} = 1 - \frac{2}{2^{n+1}} + \frac{1}{2^{n+1}} = 1 - \frac{1}{2^{n+1}}$$

Since we know the general term of the partial sums sequence, it is sufficient to compute the limit of this sequence.

$$\sum_{i=1}^{\infty} \frac{1}{2^i} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} (1 - \frac{1}{2^n}) = 1 - \lim_{n\to\infty} \frac{1}{2^n} = 1 - 0 = 1$$

Answer: converges to 1.

Example 2. Analyze the series $\sum\limits_{i=1}^{\infty} \frac{1}{i(i+1)}$ for convergence. Compute the sum in case it converges.

As in the previous example we try to find the general term for the sequence of partial sums.

First of all we transform the term of the series:

$$\frac{1}{i(i+1)} = \frac{(i+1)-i}{i(i+1)} = \frac{i+1}{i(i+1)} - \frac{i}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1}$$

Now let us find the partial sum S_n

$$S_n = a_1 + a_2 + a_3 + \ldots + a_n =$$

$$= (\frac{1}{1} - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + \ldots + (\frac{1}{n} - \frac{1}{n+1}) = 1 - \frac{1}{n+1}$$

Then we can find the sum of the series.

$$\sum_{i=1}^{\infty} \frac{1}{i(i+1)} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} (1 - \frac{1}{n+1}) = 1 - \lim_{n\to\infty} \frac{1}{n+1} = 1 - 0 = 1$$

Answer: converges to 1.

Theorem 1. *The geometric series* $\sum\limits_{i=0}^{\infty} cr^i$ *converges to* $\frac{c}{1-r}$ *if* $|r| < 1$ *and diverges if* $|r| \geq 1$ *and* $c \neq 0$.

Proof. 1)Assume $|r| < 1$.

Then $S_n = c + cr + \ldots + cr^{n-1}$. On the one hand, $S_{n+1} = S_n + cr^n$. On the other hand, $S_{n+1} = c + cr + \ldots + cr^n = c + r(c + cr + \ldots + cr^{n-1}) = c + rS_n$. Thus we have an equation $S_n + cr^n = c + rS_n$. Solving it for S_n we get $S_n = c\frac{1-r^n}{1-r}$. So

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} c\frac{1-r^n}{1-r} = \frac{c}{1-r}$$

2)Assume $|r| = 1$.

For the case $r = 1$ we have partial sums $S_n = nc$, so the series diverges to ∞.

For the case $r = -1$ we have partial sums alternating with two values c and 0, so the series diverges.

3)Assume $|r| > 1$.

By the same method as in 1) we get $S_n = c\frac{1-r^n}{1-r}$, though in this case the sequence of partial sums diverges. So the series itself diverges. □

Example 3. Does the series $\sum_{i=0}^{\infty} 7(-\frac{3}{4})^i$ converge? Find the sum of the series in case it converges.

This is a geometric series with the base $-\frac{3}{4}$. Since the absolute value of the base is less than 1 this series converges. We can find the sum of the series.

$$\sum_{i=0}^{\infty} 7(-\frac{3}{4})^i = \frac{7}{1-(-\frac{3}{4})} = \frac{7}{\frac{7}{4}} = 4$$

Answer: converges to 4.

Sometimes the series satisfies the definition of geometric series after certain transformations.

Example 4. Does the series $\sum_{i=1}^{\infty} 7(\frac{1}{2})^{3i}$ converge? Find the sum of the series in case it converges.

First of all we use the fact that $(\frac{1}{2})^{3i} = (\frac{1}{8})^i$. Thus our series is $\sum_{i=1}^{\infty} 7(\frac{1}{8})^i$. Now it is almost geometric in the sense of the definition but the summation starts from 1, not from 0. However it may be fixed easily. Let us take out the factor $\frac{1}{8}$ from each term of the series. We get

$$\sum_{i=1}^{\infty} 7(\frac{1}{8})^i = \sum_{i=1}^{\infty} 7\frac{1}{8}(\frac{1}{8})^{i-1} = \sum_{i=0}^{\infty} \frac{7}{8}(\frac{1}{8})^i = \frac{\frac{7}{8}}{1-\frac{1}{8}} = 1$$

Answer: converges to 1.

Theorem 2. *If a series* $\sum_{i=1}^{\infty} a_i$ *converges, then* $\lim_{n\to\infty} a_n = 0$

Proof. Consider the equation $a_n = S_n - S_{n-1}$, where S_i is an i-th partial sum of this series for any positive integer i. Take the limit for both parts of this equation. We assume that the sum of the series equals S. Then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0$$

$\square$

Using this theorem we can obtain the sufficient condition on divergence of the series.

Proposition 3 (n-th term test for divergence)**.** *If a sequence a_n diverges or converges to L, $L \neq 0$, then* $\sum_{i=1}^{\infty} a_i$ *diverges.*

Example 5. Does the series $\sum_{i=1}^{\infty} \frac{i+2}{3i}$ converge?

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+2}{3n} = \lim_{n\to\infty} \frac{n}{3n} + \frac{2}{3n} = \lim_{n\to\infty} \frac{1}{3} + \lim_{n\to\infty} \frac{2}{3n} = \frac{1}{3}$$

The sequence converges but its limit is not equal to 0. Hence by the n-th term test we obtain that the series diverges.

Answer: diverges.

Remark (**IMPORTANT!**). The sequence and the series based on this sequence are different concepts. The situation when the sequence converges but the corresponding series diverges is shown in the previous example.

The following facts directly follow from the corresponding theorems for sequences.

Theorem 4. *If $\sum_{i=1}^{\infty} a_i$ and $\sum_{i=1}^{\infty} b_i$ both converge, then $\sum_{i=1}^{\infty}(a_i + b_i)$ converges to $\sum_{i=1}^{\infty} a_i + \sum_{i=1}^{\infty} b_i$ and $\sum_{i=1}^{\infty}(a_i - b_i)$ converges to $\sum_{i=1}^{\infty} a_i - \sum_{i=1}^{\infty} b_i$; for any constant c, $\sum_{i=1}^{\infty} ca_i$ converges to $c\sum_{i=1}^{\infty} a_i$.*

If $\sum_{i=1}^{\infty} a_i$ converges and $\sum_{i=1}^{\infty} b_i$ diverges, then $\sum_{i=1}^{\infty}(a_i+b_i)$ diverges and $\sum_{i=1}^{\infty}(a_i-b_i)$ diverges; for any constant c, $c \neq 0$, $\sum_{i=1}^{\infty} cb_i$ diverges.

Remark. In case both series are divergent, no conclusion about their sum or difference can be made.

Example 6. Analyze for convergence $\sum_{i=1}^{\infty}(\frac{3}{2^i} + 5i)$. Find the sum in case the series converges.

Consider this series as the sum of $\sum_{i=1}^{\infty} \frac{3}{2^i}$ and $\sum_{i=1}^{\infty} 5i$. The first of these two is just geometric because $\sum_{i=1}^{\infty} \frac{3}{2^i} = \frac{1}{2}\sum_{i=0}^{\infty} \frac{3}{2^i}$ and it converges to $\frac{1}{2} \cdot \frac{3}{1-\frac{1}{2}} = 3$. The sequence $a_n = 5n$ diverges to infinity. Thus the series $\sum_{i=1}^{\infty} 5i$ diverges by the n-th term test. So we have the sum of the convergent and divergent series and it must diverge.

Answer: diverges.

Example 7. Analyze for convergence $\sum\limits_{i=1}^{\infty}(\frac{2}{i(i+1)} - \frac{7}{(-6)^i})$. Find the sum in case the series converges.

$$\sum_{i=1}^{\infty}(\frac{2}{i(i+1)} - \frac{7}{(-6)^i}) = \sum_{i=1}^{\infty}\frac{2}{i(i+1)} - \sum_{i=1}^{\infty}\frac{7}{(-6)^i} =$$

$$= 2\sum_{i=1}^{\infty}\frac{1}{i(i+1)} - \frac{1}{-6}\sum_{i=0}^{\infty}\frac{7}{(-6)^i}$$

The first series in this difference was in one of the examples. The second series is geometric. Since both of them converge, the resulting series converges. Now we can calculate its sum.

$$2\sum_{i=1}^{\infty}(\frac{1}{i(i+1)} - \frac{1}{-6}\sum_{i=0}^{\infty}\frac{7}{(-6)^i}) = 2\cdot 1 - \frac{1}{-6}\cdot\frac{7}{1-(-\frac{1}{6})} =$$

$$= 2 + \frac{1}{6}\cdot\frac{7}{\frac{7}{6}} = 2 + 1 = 3$$

Answer: converges to 3.

Homework problems

1. Determine whether the sequence $a_n = \frac{2^n+n^2}{3^n}$ is bounded or not.

2. Analyze the sequence $a_n = \frac{3^n}{2^n+n^4}$ for convergence. Compute the limit if a_n is convergent.

3. Is the sequence $a_n = \frac{n!}{n^n}$ convergent or divergent? Find the limit if a_n is convergent.

4. Analyze for convergence $\sum\limits_{i=1}^{\infty}\frac{6}{i(i+3)}$. Find the sum in case the series converges.