3D Vision 3

Week 9

Two-view Geometry: Epipolar Geometry

Two-view Geometry: Essential and Fundamental Matrices

Announcements

- **Tutorial 6:** available now on Wattle, runs across Weeks 9–10
- **Survey 2 – Week 8|9** is live on Wattle, please provide a response to improve the course for future years
- **Class Representatives:** another mechanism to provide feedback
  - Arjun Raj (u7526852@anu.edu.au) [COMP4528]
  - Ruxuan Li (u7763307@anu.edu.au) [ENGN4528]
  - Jiabao Han (u7765516@anu.edu.au) [COMP6528]
  - Xingchen Zhang (u7670173@anu.edu.au) [COMP6528]
- **Exam:** 9:00am–12:15pm Saturday 1 June 2024
  - 7-11 Barry Drive, First Floor Left Side

Weekly Study Plan: Overview

Wk	Starting	Lecture	Lab	Assessment
1	19 Feb	Introduction	✗	
2	26 Feb	Low-level Vision 1	1	
3	4 Mar	Low-level Vision 2	1	
		Mid-level Vision 1		
4	11 Mar	Mid-level Vision 2	1	CLab1 report due Friday
		High-level Vision 1		
5	18 Mar	High-level Vision 2	2	
6	25 Mar	High-level Vision 3[1]	2	
	1 Apr	Teaching break	✗	
	8 Apr	Teaching break	✗	
7	15 Apr	3D Vision 1	2	CLab2 report due Friday
8	22 Apr	3D Vision 2	3	
9	29 Apr	3D Vision 3	3	
10	6 May	3D Vision 4	3	
		Mid-level Vision 3		
11	13 May	High-level Vision 4	✗	CLab3 report due Friday
12	20 May	Course Review	✗	

Weekly Study Plan: Part B

Wk	Starting	Lecture	By
7	15 Apr	3D vision: introduction, camera model, single-view geometry	Dylan
8	22 Apr	3D vision: camera calibration, two-view geometry (homography)	Dylan
9	29 Apr	3D vision: two-view geometry (epipolar geometry, triangulation, stereo)	Dylan
10	6 May	3D vision: multiple-view geometry	Weijian
		Mid-level vision: optical flow, shape-from-X	Dylan
11	13 May	High-level vision: self-supervised learning, detection, segmentation	Dylan
12	20 May	Course review	Dylan

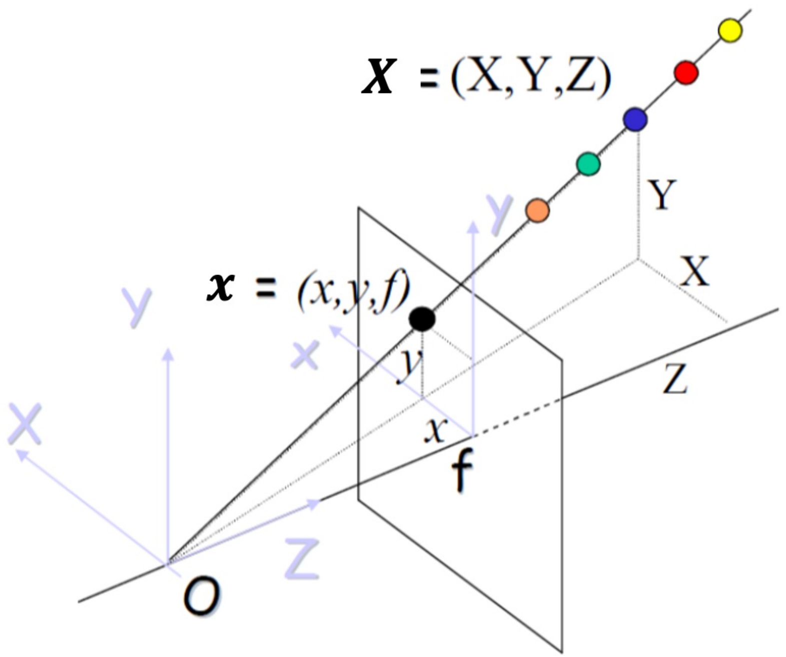
Outline

1. Two-view Geometry: Epipolar Geometry
2. Two-view Geometry: Essential and Fundamental Matrices
3. Two-view Geometry: Triangulation
4. Two-view Geometry: Stereo

Epipolar Geometry

Two-view Geometry

Why Two Views?

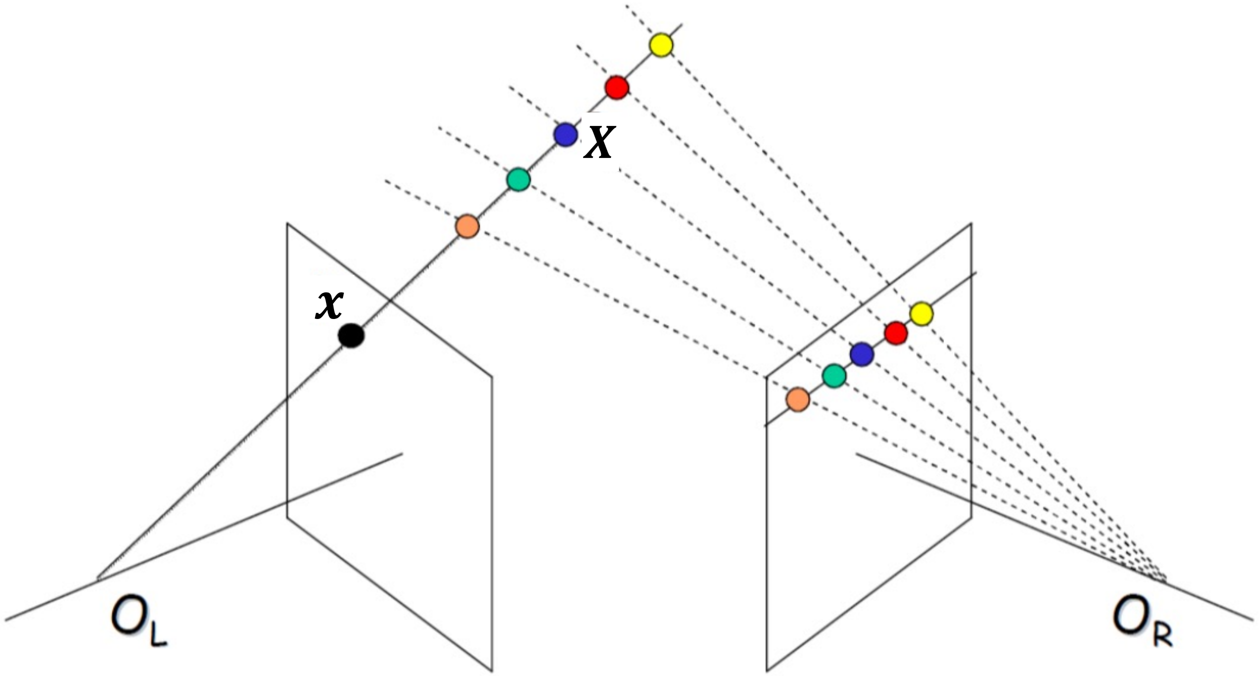


$$x = f\frac{X}{Z} = f\frac{kX}{kZ}$$
$$y = f\frac{Y}{Z} = f\frac{kY}{kZ}$$

- A fundamental *ambiguity*: any point on the ray OX has projection point $\boldsymbol{x}$ on the image plane
  - Note that here the camera center is (0,0,0) in the world coordinate system

Image Credit: Robert Collins

Why Two Views?

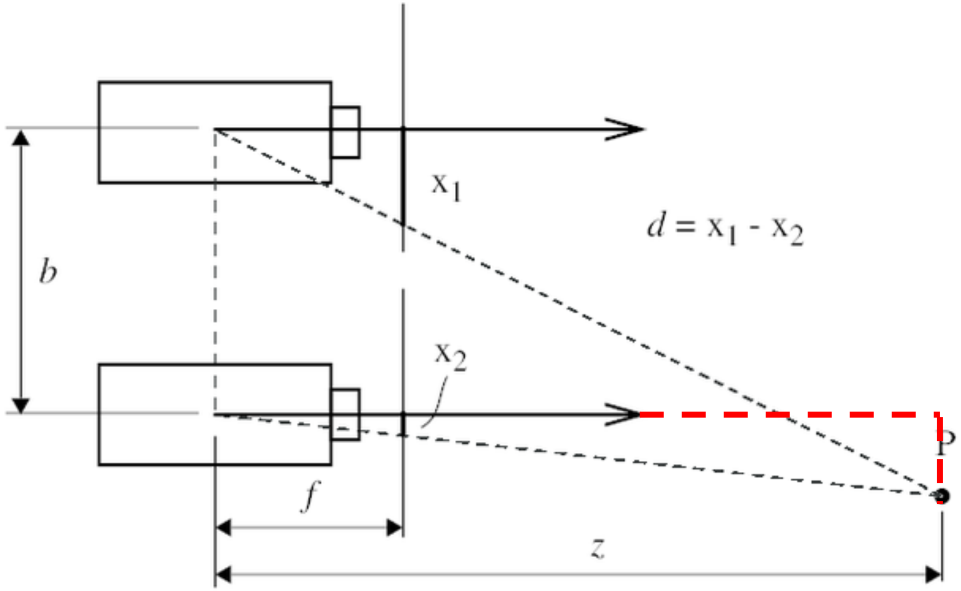


- A second camera can resolve the ambiguity, enabling measurement of depth via triangulation

Image Credit: Robert Collins

General Two-view Geometry

- Human stereopsis
- Stereograms
- Epipolar geometry and the epipolar constraint
  - Parallel optical axes
  - General case with calibrated cameras



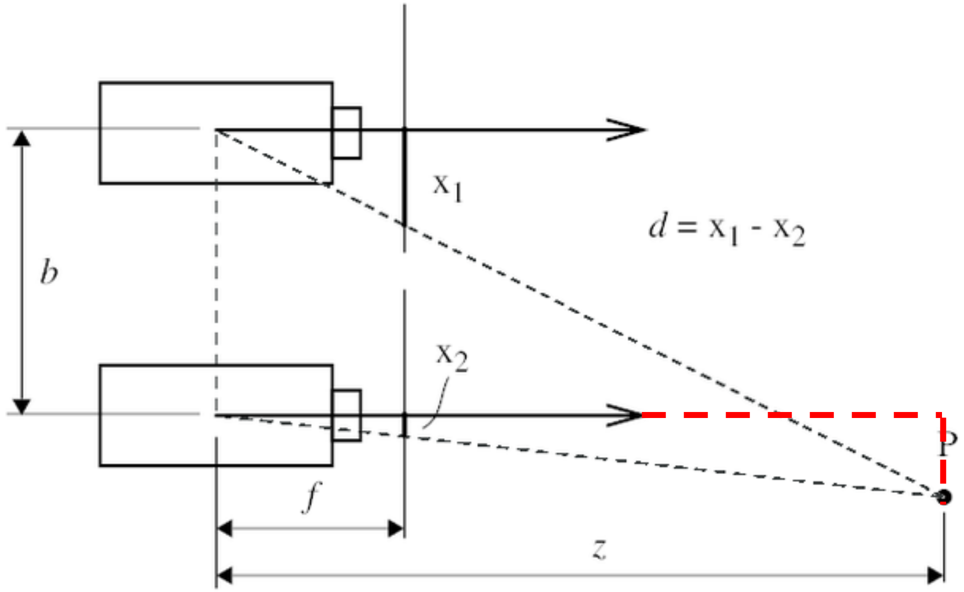
For similar triangles,

$$\frac{d}{b} = \frac{f}{z}$$

Parallel Optical Axes

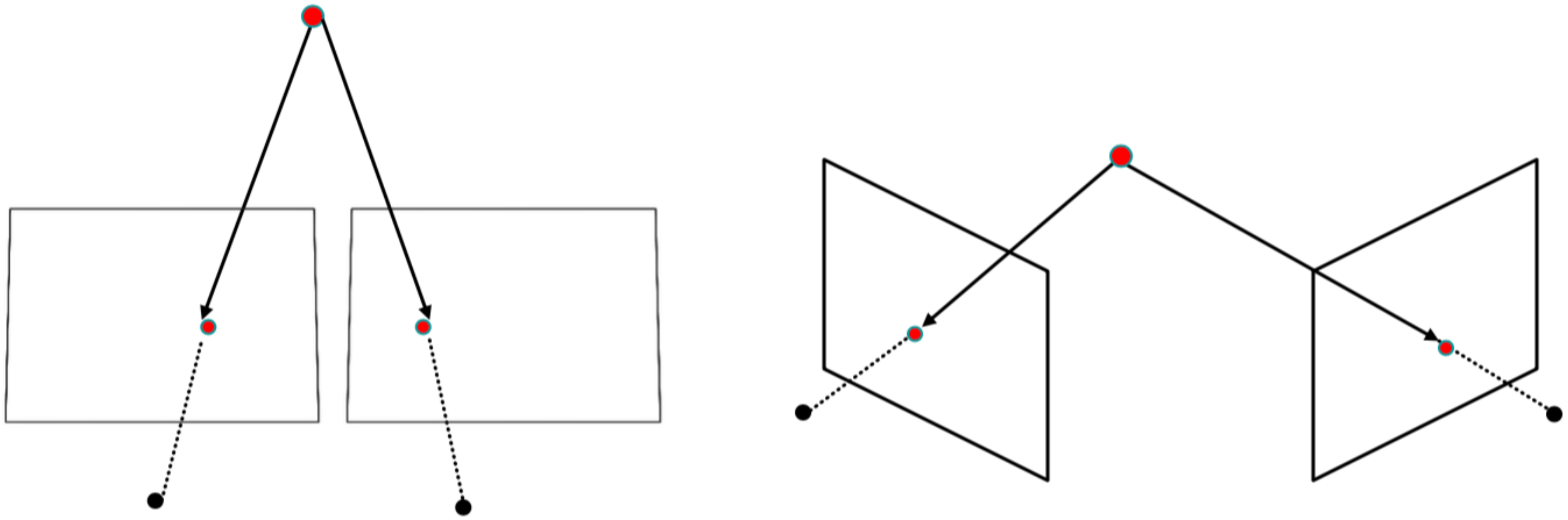
- Depth from disparity d

1. $\frac{z-f}{z} = \frac{b-x_1+x_2}{b}$ [similar △s]
2. $1 - \frac{f}{z} = 1 - \frac{d}{b}$
3. $z = \frac{bf}{d}$ [depth of 3D point]



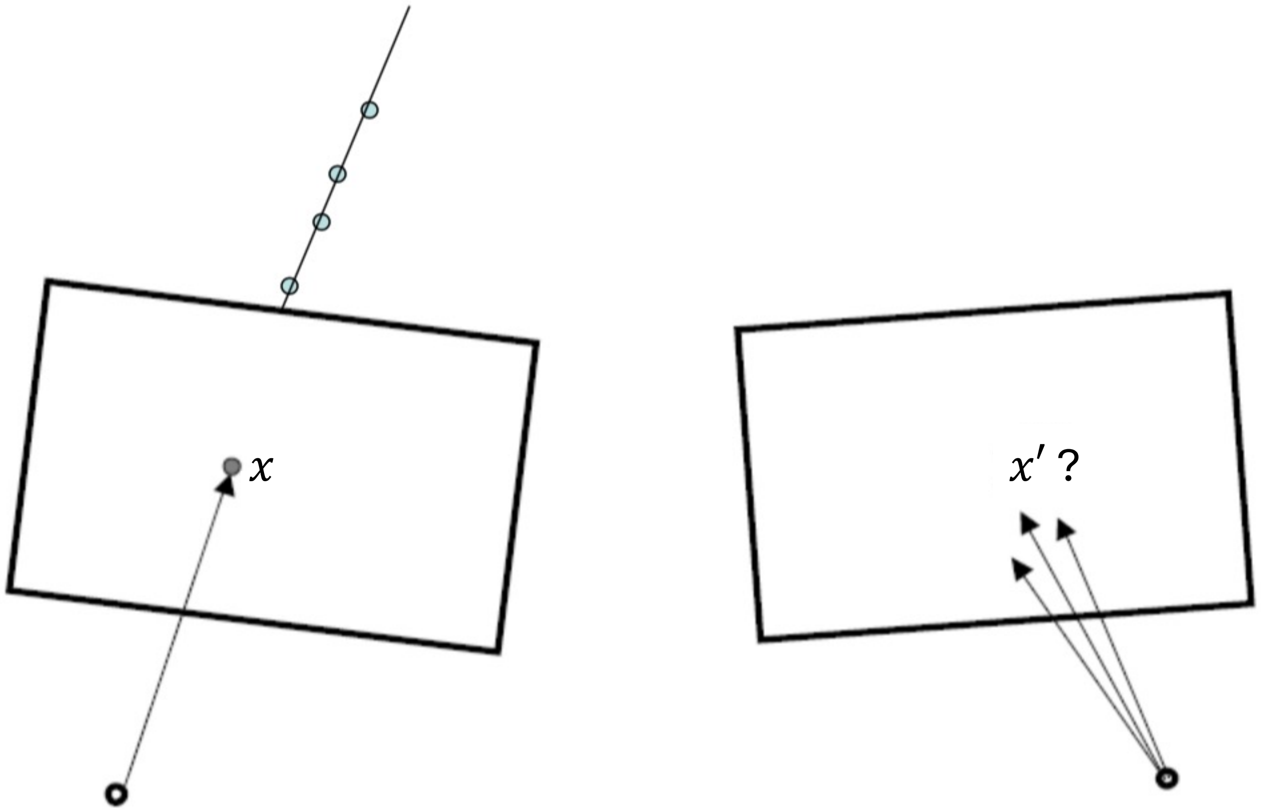
General Case with Calibrated Cameras

- The two cameras need not have parallel optical axes



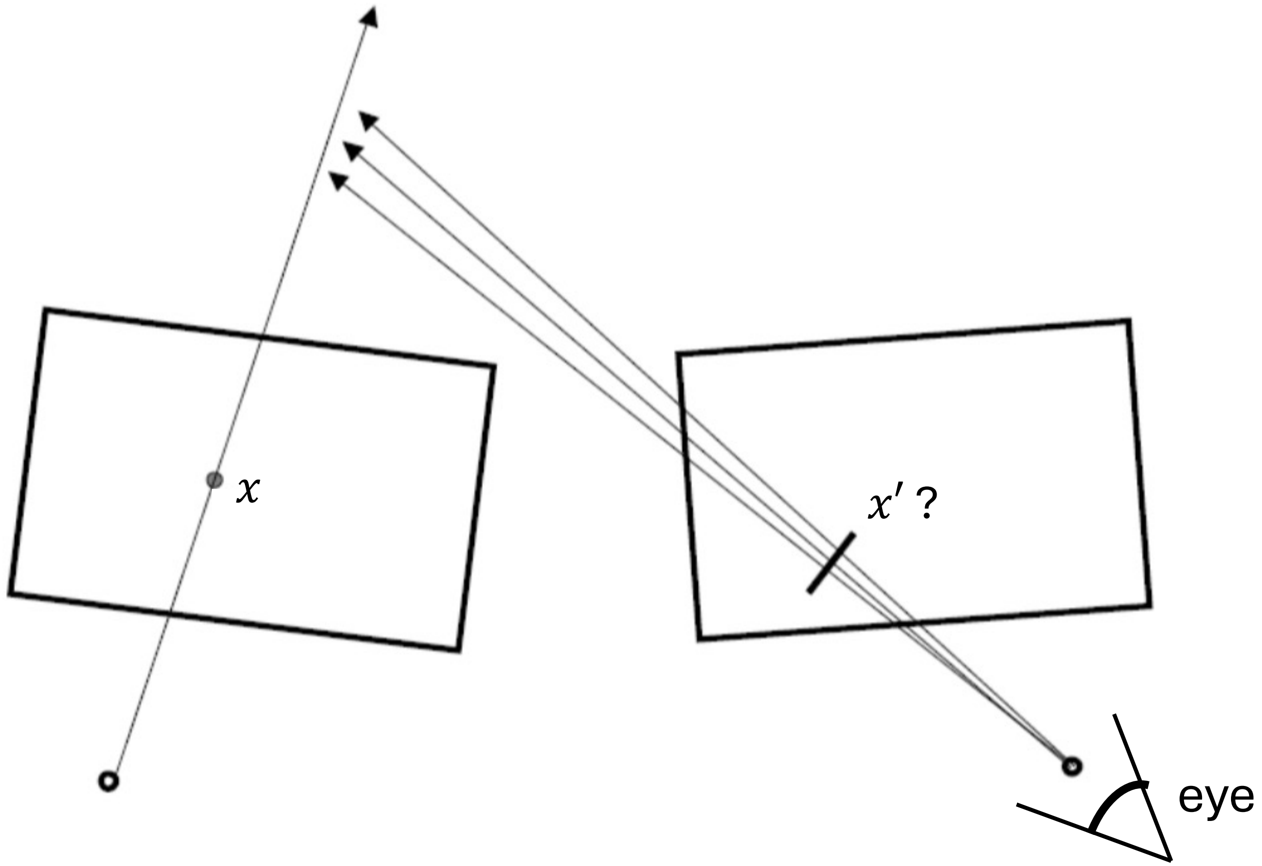
Parallel Vs. Non-parallel

Stereo Correspondence Constraints



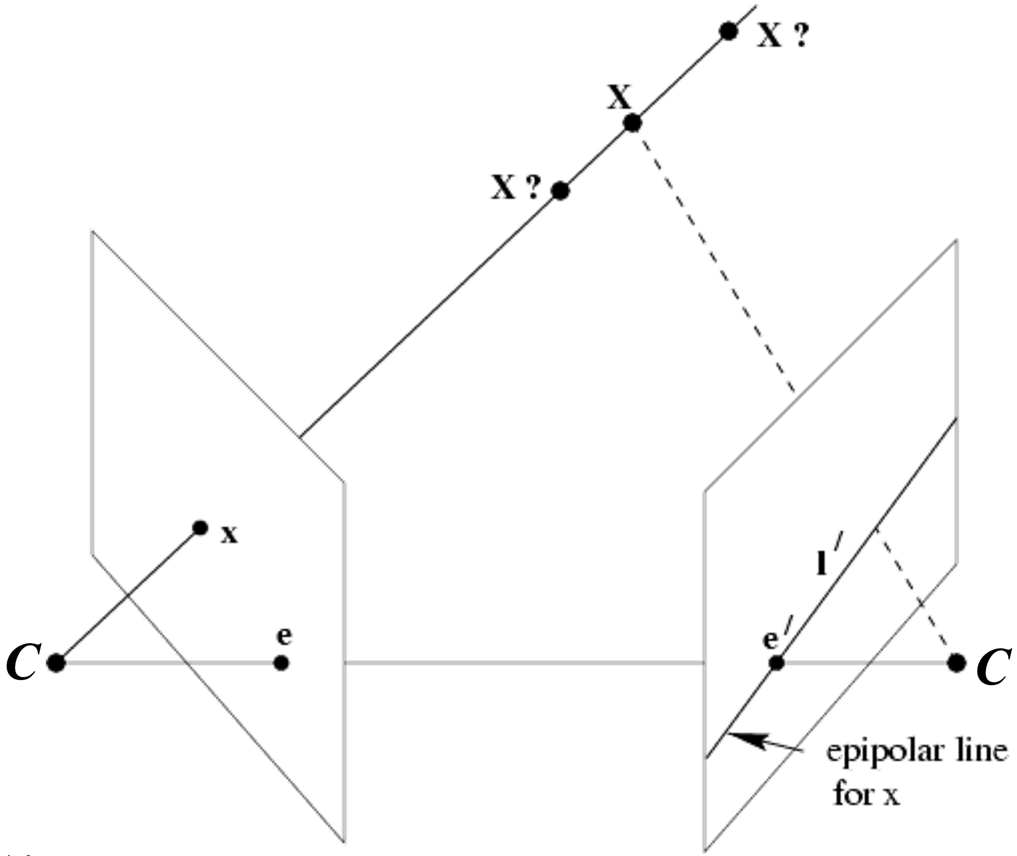
- Given x in the left image, where can the corresponding point x' be?

Stereo Correspondence Constraints



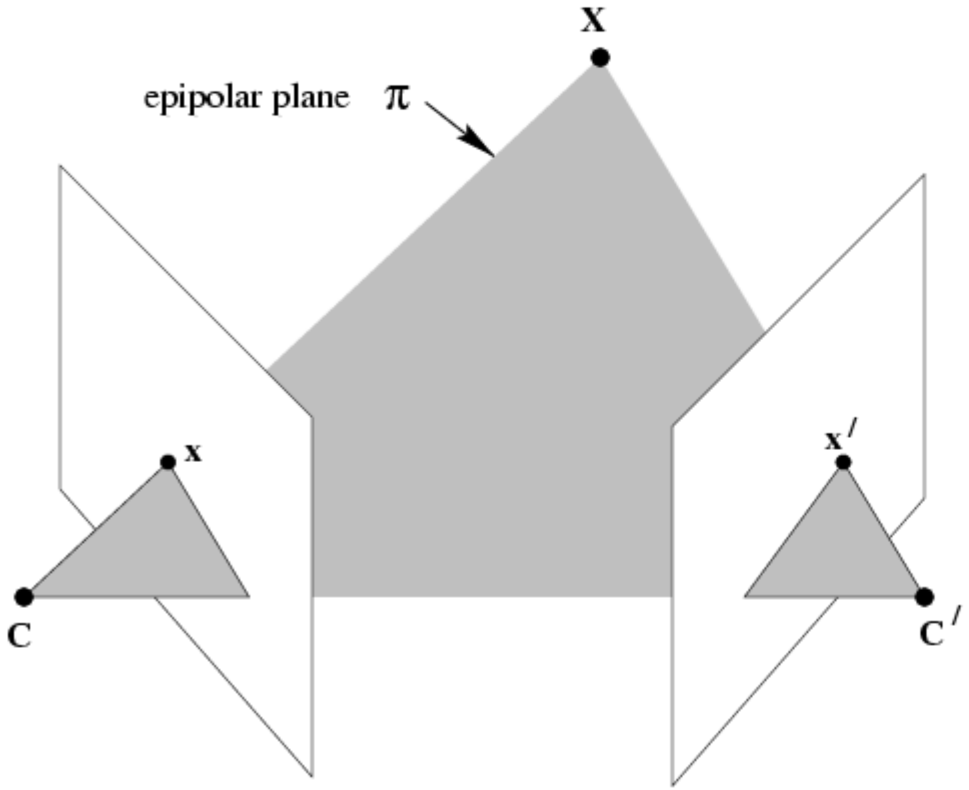
- If we, from the second image, view the ray going through x, this ray will form a **line** on the second image
- Point x' will be on this line in the second image

Epipolar Geometry



- What if only C, C', and x are known?
- We can obtain the epipolar line on the second image

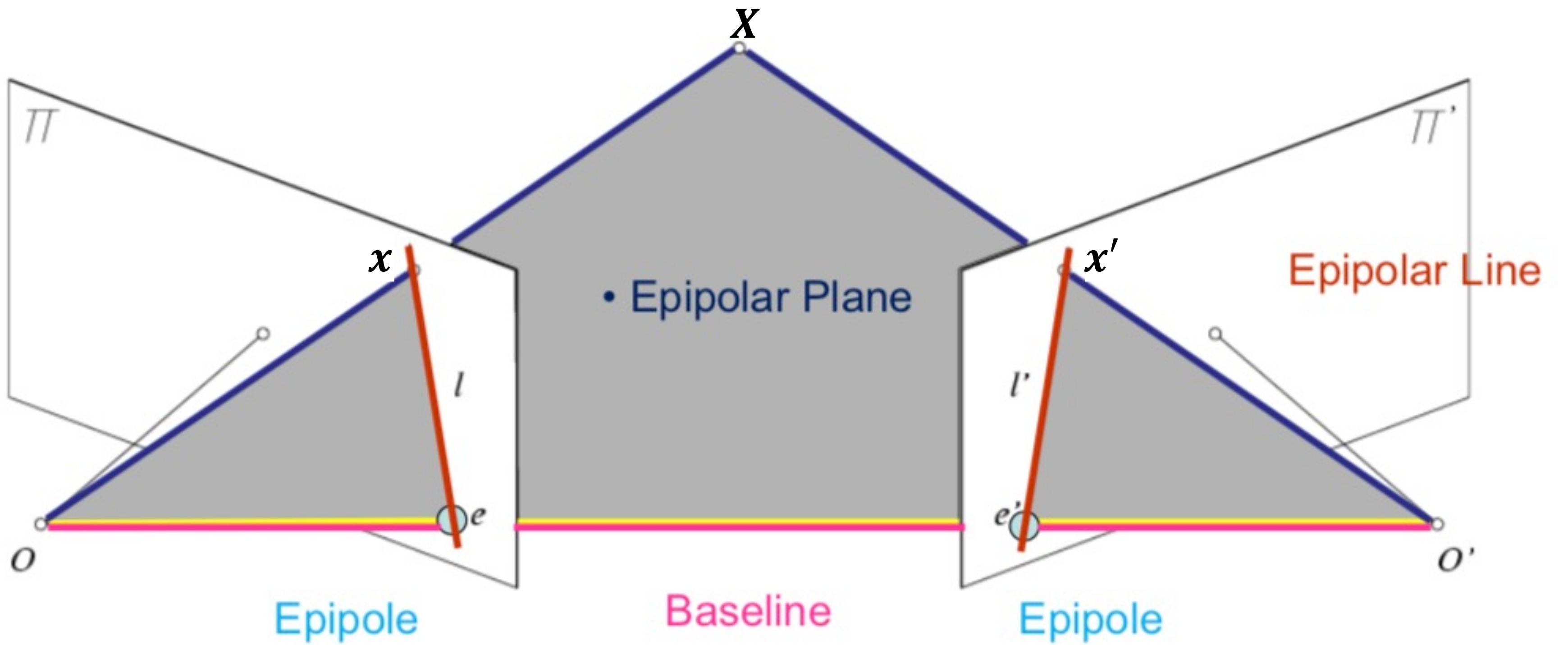
Epipolar Geometry



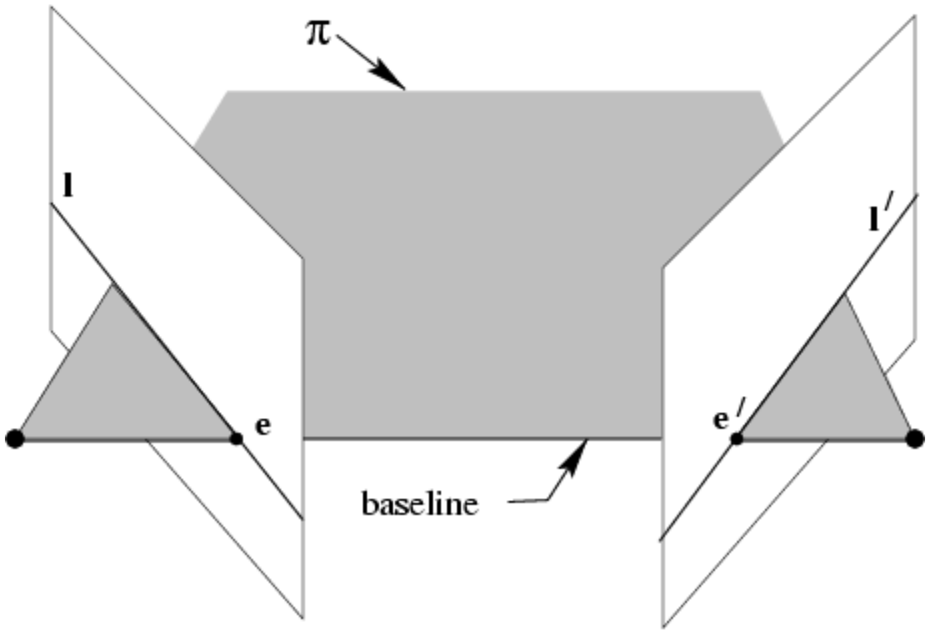
- C, C', x, x' and X are coplanar

Epipolar Geometry Terminology

- **Baseline:** line joining the camera centres
- **Epipole:** point of intersection of baseline with image plane
- **Epipolar plane:** plane containing baseline and world point
- **Epipolar line:** intersection of epipolar plane with the image plane



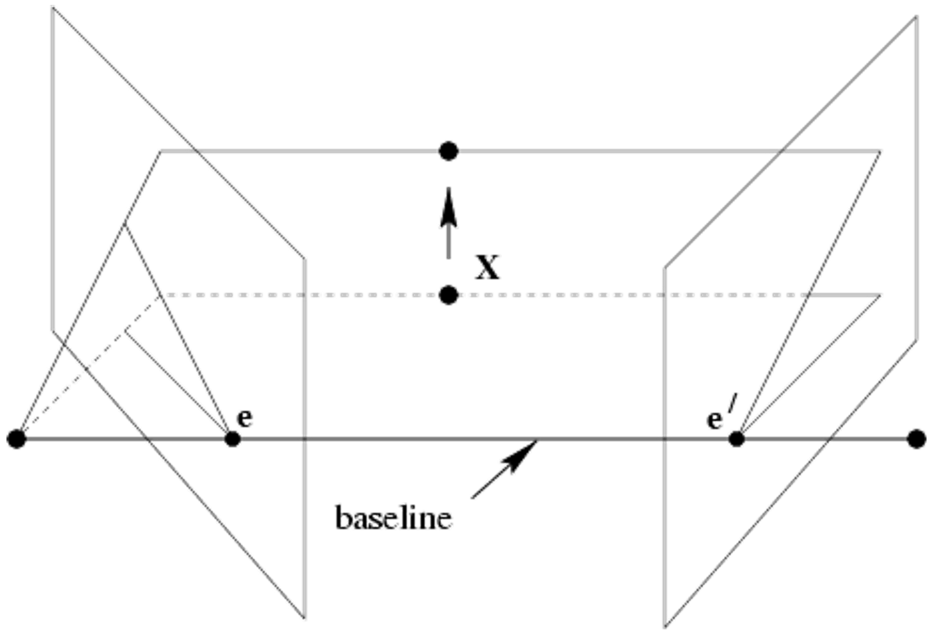
The Epipolar Plane



- All points on π project on *l* and *l’*

Epipolar Geometry

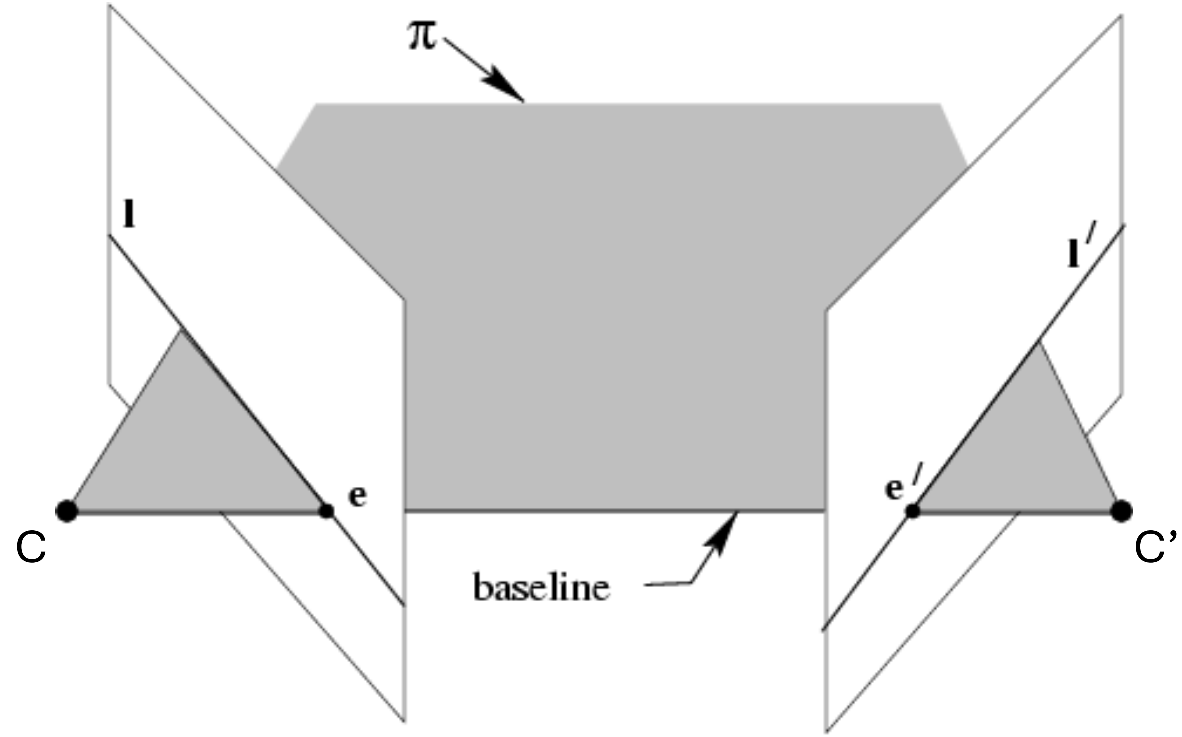
- All epipolar lines intersect at the epipole



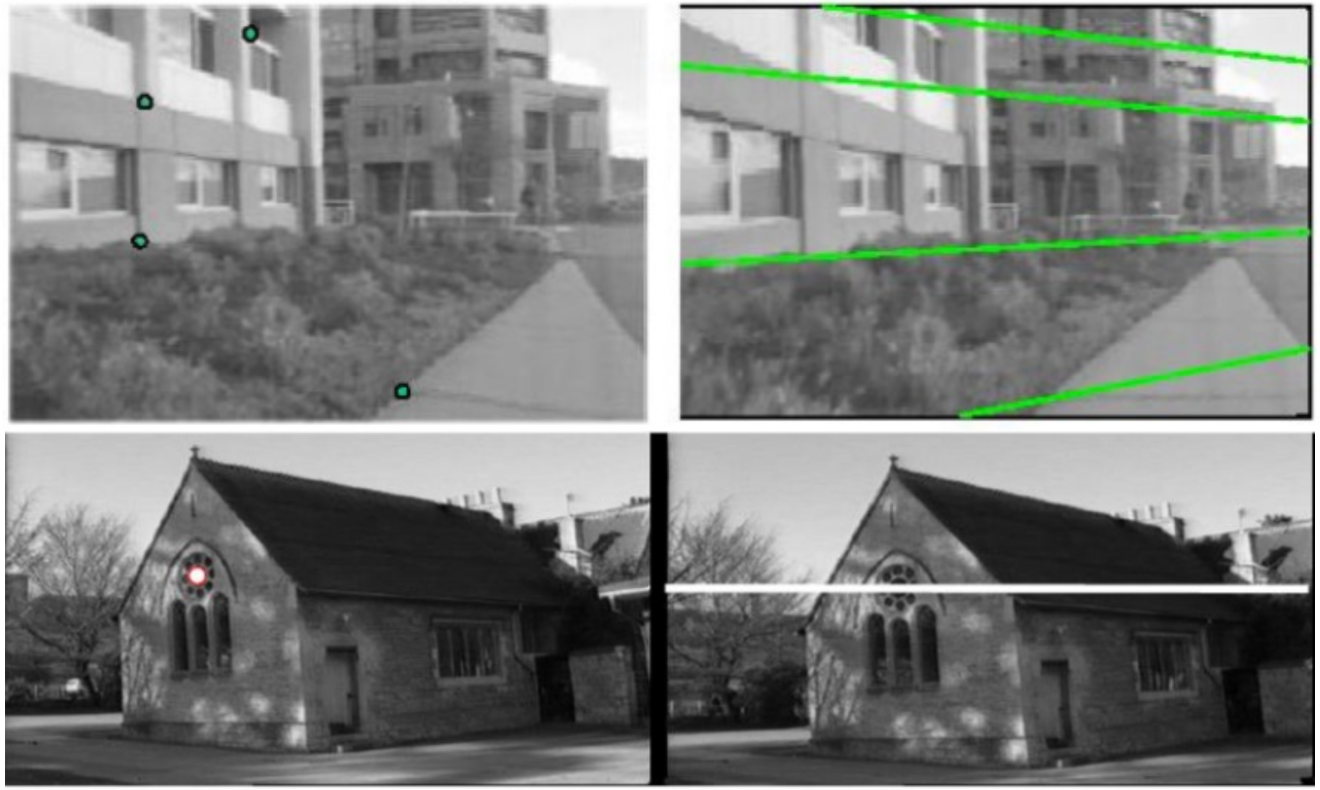
- Family of planes π and lines *l* and *l'* intersect at *e* and *e'*

Epipolar Geometry

- Epipole = intersection of baseline with image plane
  = projection of one optical centre in the other image

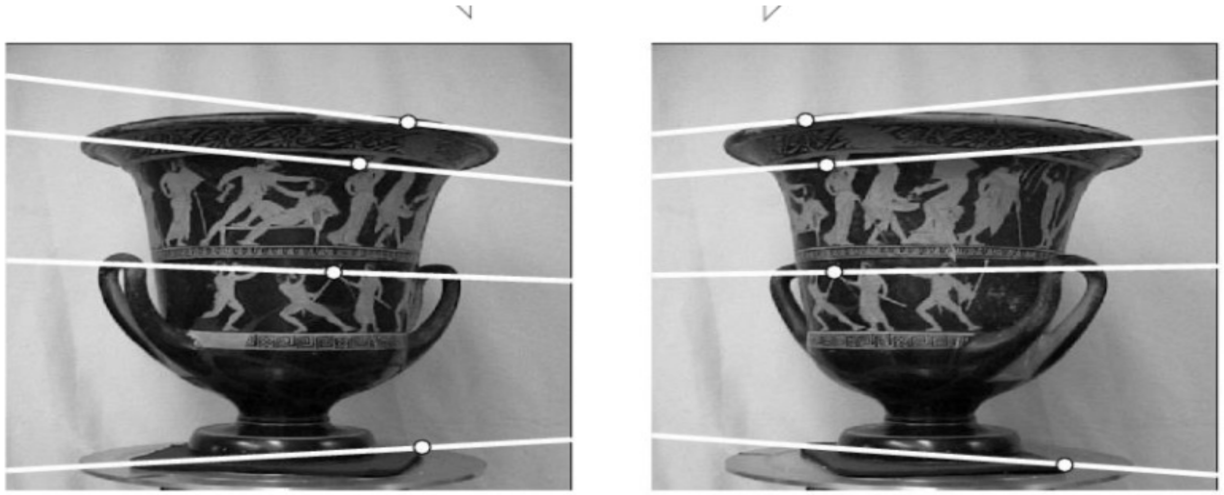


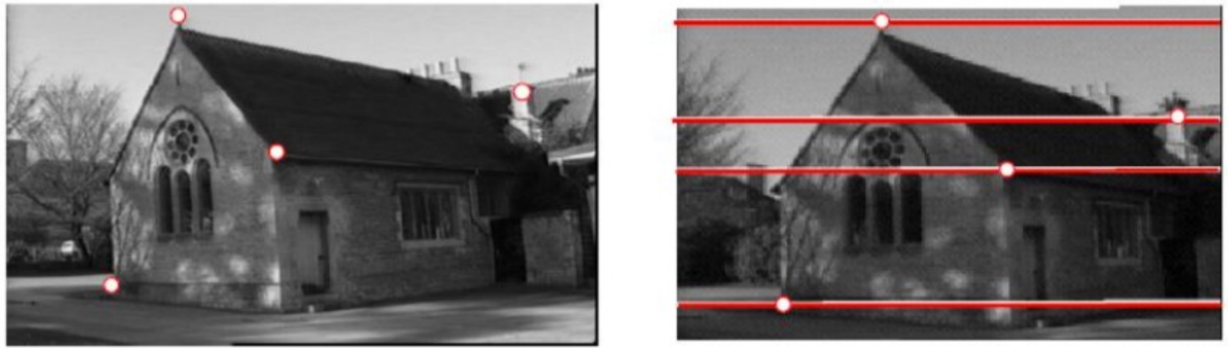
Examples



- Useful because it reduces the correspondence problem to a 1D search along an epipolar line

Examples

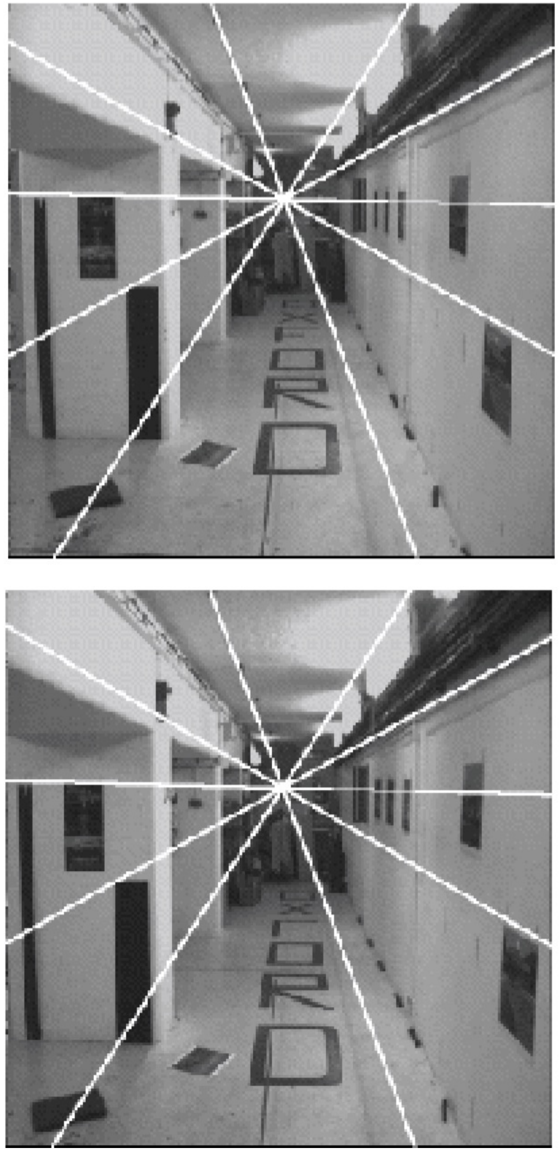




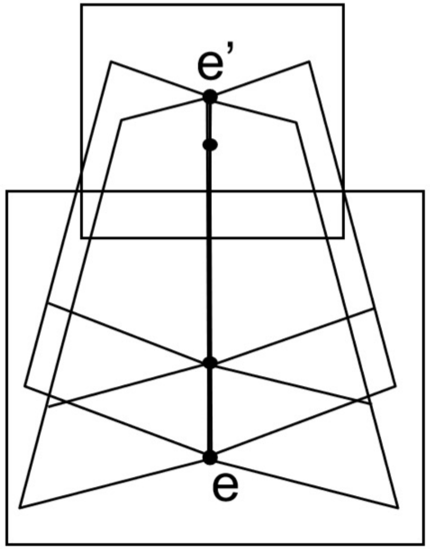
Examples

Image Credit: .gs8

Example: Forward Motion



- All epipolar lines go through the epipole, which is visible in the image plane in this case



- The epipole is the projection of an optical centre onto the other image

Epipolar Geometry: Summary

- The intrinsic geometry of 2 different views of the same 3D scene

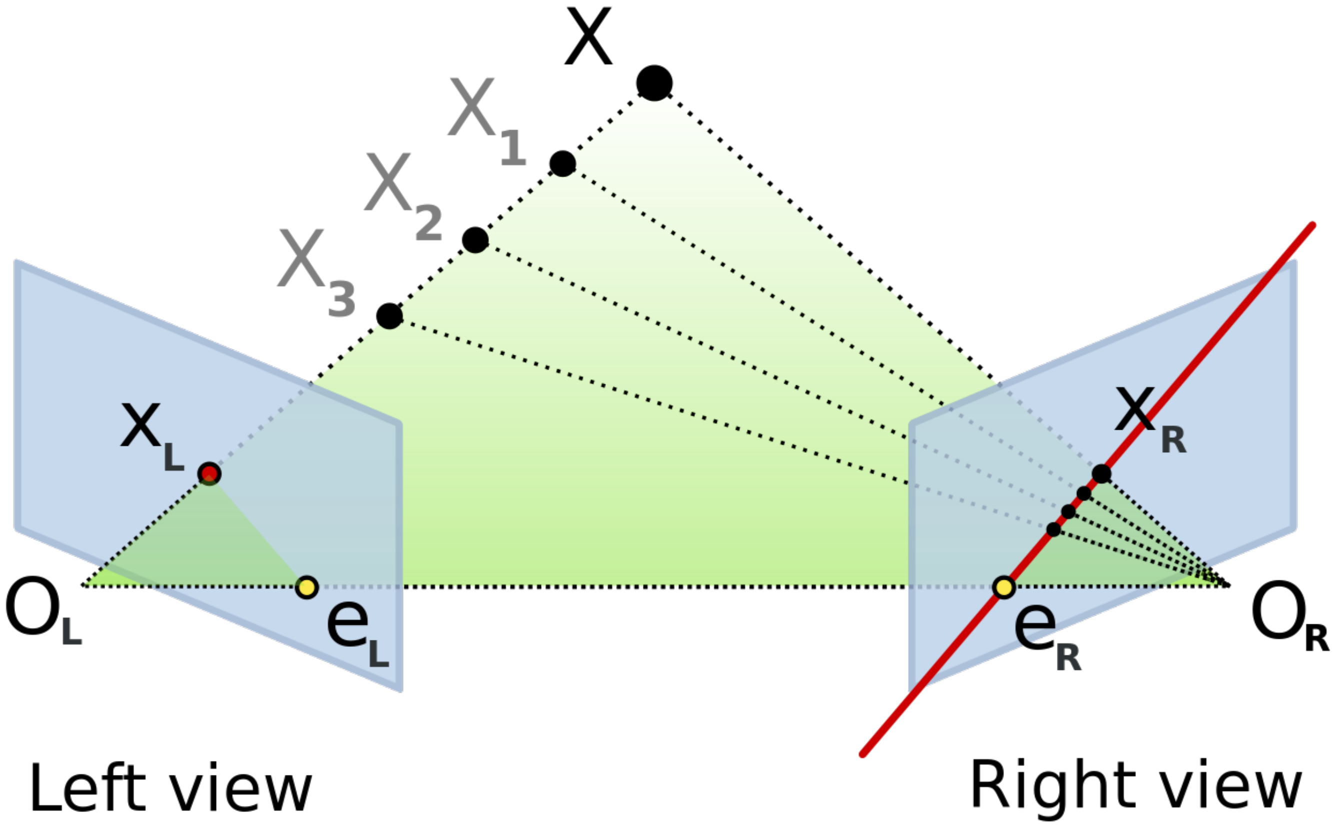


Image Credit: Arne Nordmann

Epipolar Lines: The Game

- https://shenshen.mit.edu/cv_book/figures/3d_scene_understanding/epipolar_game_play.png
- Answers: A-2, B-5, C-4, D-6, E-3, F-1

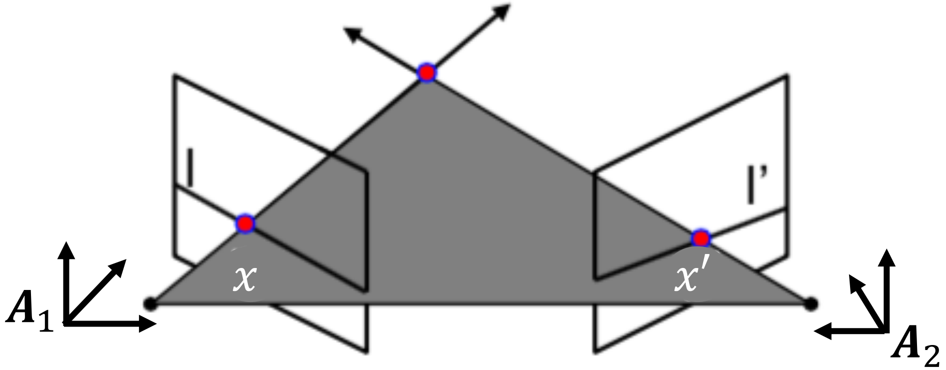
Essential and Fundamental Matrices

Two-view Geometry

The Essential Matrix E

- Encodes the relationship between two images **when the intrinsics of both cameras are known**
  - If it's the same calibrated camera, moved in space, use E
- A point in one image is mapped to a line in the other image
  - The *epipolar line*

The Essential Matrix E



- The coordinates of x and x' are x_c and x'_c in the camera coordinate frames of camera 1 and camera 2, respectively [how? $x = Kx_c$]
- We can transform one coordinate to the other by the relative rotation and translation between the two cameras [why? See next]

$$x'_c = Rx_c + t$$

1. Take the **cross** product of both sides by t

$$t \times x'_c = t \times Rx_c + \cancelto{0}{t \times t}$$

2. Take the **dot** product of both sides by x'_c

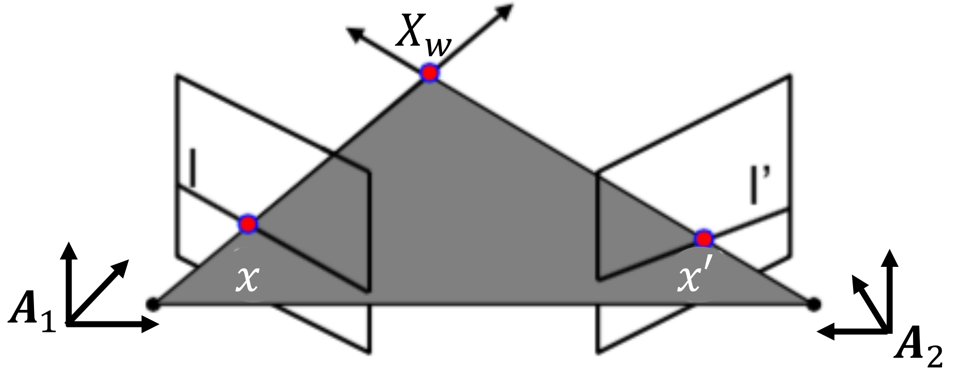
$$\cancelto{0}{x'^{\top}_c (t \times x'_c)} = x'^{\top}_c (t \times Rx_c)$$

($t \times x'_c$ is $\perp$ to both t and x'_c)

3. So we have:

$$0 = x'^{\top}_c (t \times Rx_c) = x'^{\top}_c [t]_\times Rx_c$$

The Essential Matrix E



- We can transform one coordinate to the other by the relative rotation and translation between the two cameras

$$x_c = R_1[I \mid -C_1]\begin{bmatrix} X_w \\ 1 \end{bmatrix} \Rightarrow X_w = R_1^{-1}x_c + C_1$$

$$x_c' = R_2[I \mid -C_2]\begin{bmatrix} X_w \\ 1 \end{bmatrix}$$

$$x_c' = R_2[I \mid -C_2]\begin{bmatrix} R_1^{-1}x_c + C_1 \\ 1 \end{bmatrix}$$

$$x_c' = R_2[R_1^{-1}x_c + C_1 - C_2]$$

$$\therefore x_c' = Rx_c + t, \text{where}$$

$$R = R_2R_1^{-1} \text{ and } t = R_2(C_1 - C_2)$$

$$\text{w.l.o.g. set } R_2 = I \text{ and } C_2 = 0, \text{then}$$

$$R = R_1^{-1} \text{ and } t = C_1$$

(since we are free to choose the world coordinate system)

The Essential Matrix E

$$x_c'^{\top}[t]_\times R x_c = x_c'^{\top} E x_c = 0 \Rightarrow \boxed{E \triangleq [t]_\times R}$$

- $[t]_\times$: translation cross-product matrix in $\mathbb{R}^{3\times3}$ of rank 2

$$[t]_\times = \begin{bmatrix} 0 & -t_z & t_y \\ t_z & 0 & -t_x \\ -t_y & t_x & 0 \end{bmatrix}$$

- R: rotation matrix in $\mathbb{R}^{3\times3}$
- x_c and x_c': homogeneous point vectors in $\mathbb{R}^3$
- $E \triangleq [t]_\times R$ in $\mathbb{R}^{3\times3}$ is defined as the Essential matrix
    - Relates the *camera* coordinate frames of camera 1 to camera 2

The Essential Matrix E: Properties

- $E \triangleq [t]_{\times} R$
- Properties:
  - 5 DoF
  - Rank 2
  - Singular values: $\sigma_1 = \sigma_2$ and $\sigma_3 = 0$
- Constraints:
  - $\det(E) = 0$
  - $2EE^\top E - \mathrm{tr}(EE^\top)E = 0$
- Estimation:
  - “The Five-Point Algorithm” [Nister 2004; Li & Hartley 2006]
  - “The Eight-Point Algorithm” (i.e., DLT) – see later slides

The Essential Matrix E: Recovering R and t

- $E \triangleq [t]_\times R$

1. SVD: $E = U\Sigma V^\top$ where $\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
2. Then,

$$R = UW^{-1}V^\top$$
$$[t]_\times = UW\Sigma U^\top$$

   where

$$W = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3. To get all 4 solutions, solve for all combinations of $W \in \{W, W^{-1}\}$
4. Select solution where all of the 3D points are in front of both cameras

The Fundamental Matrix F

- Encodes the relationship between two images **when the intrinsics of both cameras are *not* known**
- A point in one image is mapped to a line in the other image
  - The *epipolar line*

The Fundamental Matrix F

- We have:

$$x_c'^{\top} E x_c = 0$$
$$x = K x_c$$

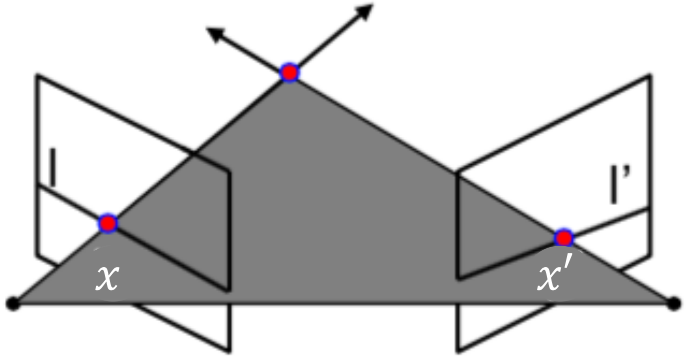
- So we have:

$$(K'^{-1}x')^{\top} E (K^{-1}x) = 0$$
$$x'^{\top} \underbrace{K'^{-\top} E K^{-1}} x = 0$$

$$x'^{\top} F x = 0$$
$$F \triangleq K'^{-\top} E K^{-1} = K'^{-\top} [t]_{\times} R K^{-1}$$

- F relates *pixel* coordinates of image 1 to those in image 2

The Fundamental Matrix F



- We have: $x'^{\top} F x = 0$
- And $x'^{\top} l' = 0$
  - Since x' is on the epipolar line l'
- Therefore: $l' = Fx$
- Similarly: $l = F^{\top} x'$
- That is, for a given point x in the first image, we can find the **epipolar line** $l' = Fx$ in the second image
  - On which we can search for the corresponding point

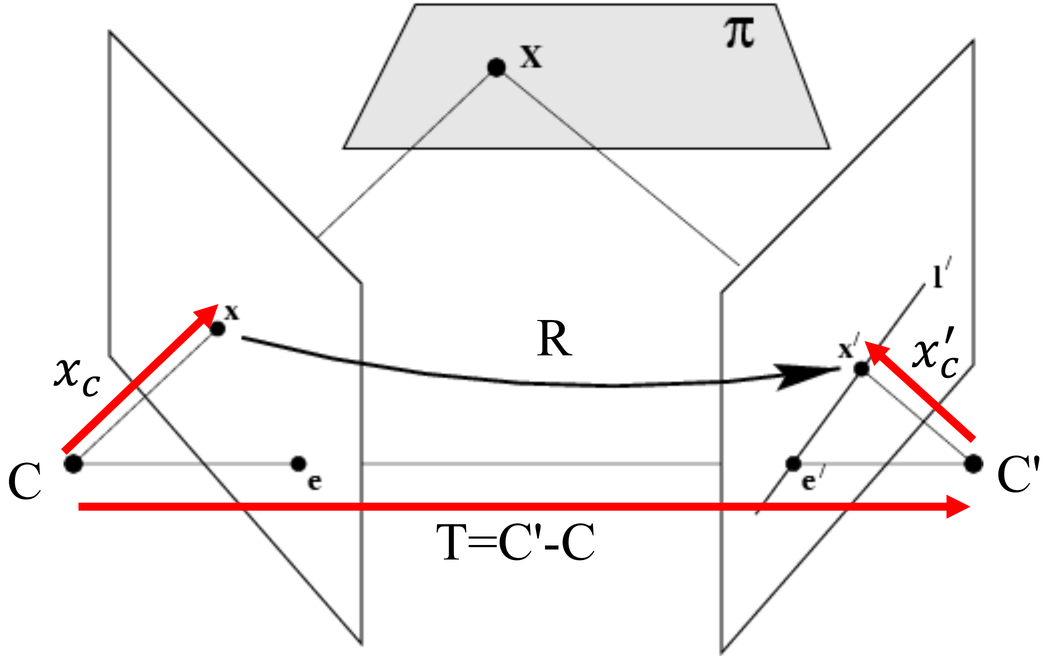
2D line representation:

- In 2D, a line is represented as
  $$ax + by + c = 0$$
  or equivalently,
  $$\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$$
- We use $l = [a, b, c]^{\top}$
  - Orthogonal to vector (a, b)
  - Dot product of the line vector and any point on the line is 0

The Fundamental Matrix F: Epipole

- $x'^{\top} F x = 0$
- We know that all epipolar lines (regardless of the point x') in the first image go through the epipole
- So the epipole e must satisfy
$$Fe = 0$$
- How to solve?
  - SVD (again!) or eigendecomposition

F vs E



$$x'^{\top} F x = 0$$
$$x_c'^{\top} E x_c = 0$$

The Fundamental Matrix F: Properties

- F is the unique 3x3 rank 2 matrix that satisfies $x'^{\top}Fx = 0$ for all $x' \leftrightarrow x$
  1. **Transpose:** if F is fundamental matrix for (x, x'), then $F^{\top}$ is the fundamental matrix for (x', x)
  2. **Epipolar lines:** F maps from a point x to a line $l' = Fx$
     - Similarly, $l = F^{\top}x'$
  3. **Epipoles:** lie on all epipolar lines, thus $x'^{\top}Fx = 0 \; \forall x \Rightarrow Fe = 0$
     - Similarly, $F^{\top}e' = 0$

Direct Linear Transformation (DLT) Algorithm for Fundamental Matrix Estimation

- AKA **"The 8-Point Algorithm"**
- The Fundamental matrix F is defined by $x'^{\top} F x = 0$ for any pair of matches x and x' in two images
- Let $x = (u, v, 1)^{\top}$ and $x' = (u', v', 1)^{\top}$,
  then each match gives a linear equation

  $$uu' f_{11} + vu' f_{12} + u' f_{13} + uv' f_{21} + vv' f_{22} + v' f_{23} + u f_{31} + v f_{32} + f_{33} = 0$$

  where

$$F = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}$$

The 8-Point Algorithm

$$\begin{bmatrix} u_1u_1' & v_1u_1' & u_1' & u_1v_1' & v_1v_1' & v_1' & u_1 & v_1 & 1 \\ u_2u_2' & v_2u_2' & u_2' & u_2v_2' & v_2v_2' & v_2' & u_2 & v_2 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ u_nu_n' & v_nu_n' & u_n' & u_nv_n' & v_nv_n' & v_n' & u_n & v_n & 1 \end{bmatrix} \begin{bmatrix} f_{11} \\ f_{12} \\ f_{13} \\ f_{21} \\ f_{22} \\ f_{23} \\ f_{31} \\ f_{32} \\ f_{33} \end{bmatrix} = 0$$

- Instead of solving $Af = 0$, we seek f to minimise $\|Af\|$: the least right-singular vector of A (the least eigenvector of $A^\top A$)

The 8-Point Algorithm: Rank 2 Constraint

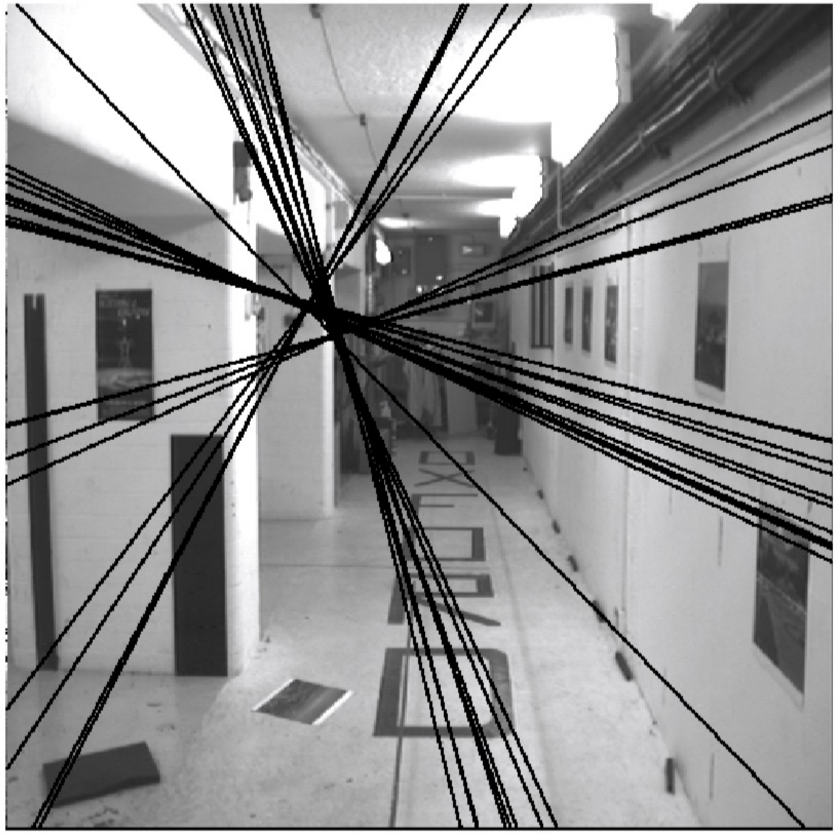
- But $\mathrm{rank}(F) = 2$
- Therefore, replace current estimate F' with

$$F = \mathrm{minimise}_F \ \|F - F'\|$$
$$\text{subject to } \det(F) = 0$$

- How? SVD, again! Decompose $F' = U\Sigma V^\top$ and set $F = U\Sigma' V^\top$ where

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}, \text{ and } \Sigma' = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The 8-Point Algorithm: Rank 2 Constraint



Impose
$\mathrm{rank}(F) = 2$

The 8-Point Algorithm

- Advantages:
  - Linear
  - Easy to implement
  - Fast
  - Can extract R and *t*
- Disadvantages:
  - Susceptible to noise
  - Optimises a non-physical algebraic quantity

The 8-Point Algorithm: Normalisation

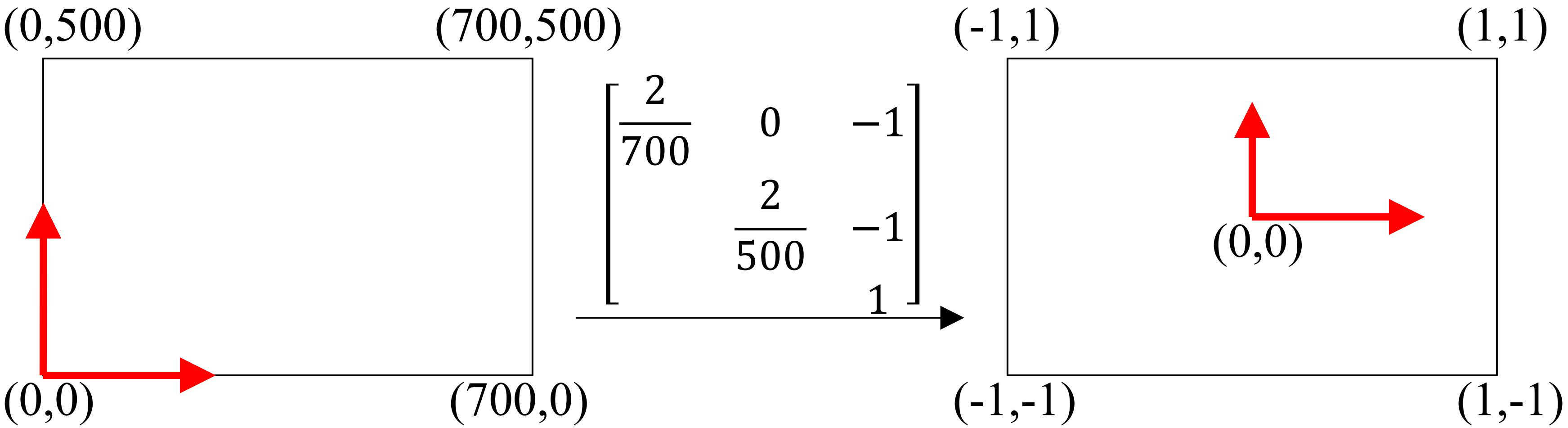
$$\begin{array}{c} \begin{bmatrix} u_1u_1' & v_1u_1' & u_1' & u_1v_1' & v_1v_1' & v_1' & u_1 & v_1 & 1 \\ u_2u_2' & v_2u_2' & u_2' & u_2v_2' & v_2v_2' & v_2' & u_2 & v_2 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ u_nu_n' & v_nu_n' & u_n' & u_nv_n' & v_nv_n' & v_n' & u_n & v_n & 1 \end{bmatrix} \\ \begin{matrix} \sim 10000 & \sim 10000 & \sim 100 & \sim 10000 & \sim 10000 & \sim 100 & \sim 100 & \sim 100 & 1 \end{matrix} \end{array} \begin{bmatrix} f_{11} \\ f_{12} \\ f_{13} \\ f_{21} \\ f_{22} \\ f_{23} \\ f_{31} \\ f_{32} \\ f_{33} \end{bmatrix} = 0$$

!

Orders of magnitude difference
between columns of data matrix
$\rightarrow$ least-squares yields poor results

The Normalised 8-Point Algorithm

- Normalised least squares yields good results
- Transform image to ~[-1,1]x[-1,1]



$$\begin{bmatrix} \frac{2}{700} & 0 & -1 \\ & \frac{2}{500} & -1 \\ & & 1 \end{bmatrix}$$

The Normalised 8-Point Algorithm

1. Transform input by $\hat{x}_i = Tx_i, \hat{x}'_i = Tx'_i$
2. Call the 8-point algorithm on $\hat{x}_i, \hat{x}'_i$ to obtain $\hat{F}$
3. Then, $F = T'^{\top}\hat{F}T$
    - Since $x'^{\top}Fx = 0 \Rightarrow \hat{x}'^{\top}T'^{\top}FT\hat{x}' = 0 \Rightarrow \hat{F} = T'^{\top}FT$

- Example normalisation matrices:

$$T = \begin{bmatrix} w/2 & 0 & w/2 \\ 0 & h/2 & h/2 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 2/w & 0 & -1 \\ 0 & 2/h & -1 \\ 0 & 0 & 1 \end{bmatrix}; \quad T' = \begin{bmatrix} 2/w' & 0 & -1 \\ 0 & 2/h' & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

The Normalised 8-Point Algorithm

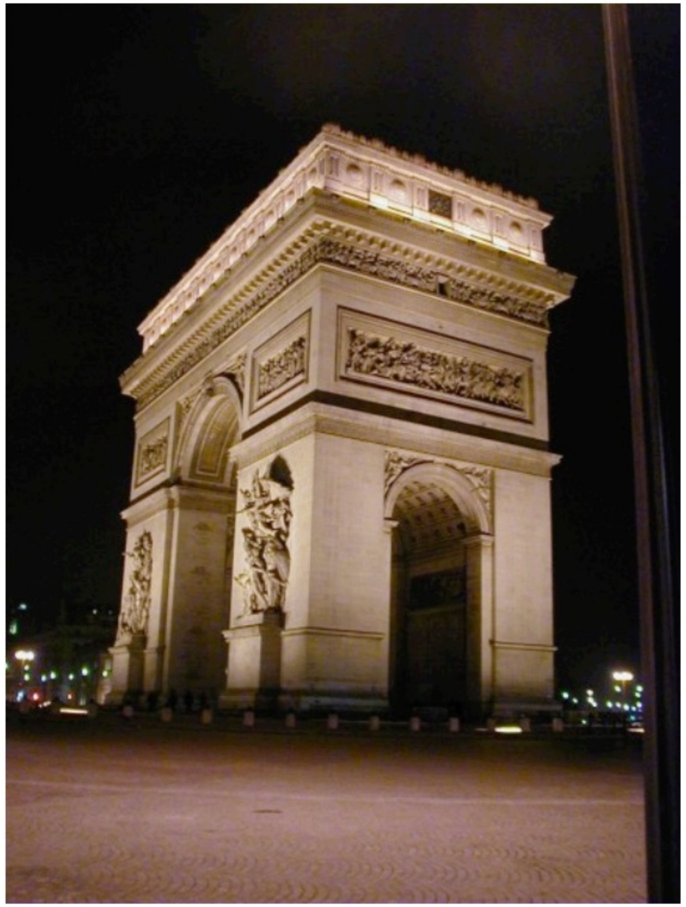
- Example normalisation matrices:

$$x_i' \leftarrow x_i - \bar{x}$$

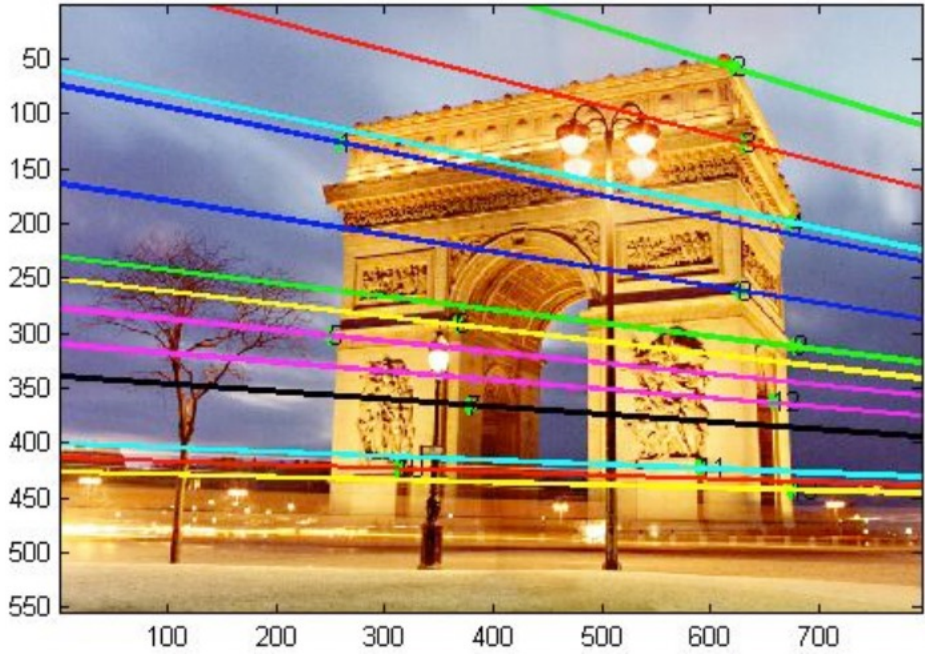
$$s \leftarrow \frac{\sqrt{2}}{\frac{1}{n}\sum_{i=1}^{n} ||x_i'||}$$

$$T = \begin{bmatrix} s & 0 & -s\bar{x}_1 \\ 0 & s & -s\bar{x}_2 \\ 0 & 0 & 1 \end{bmatrix}$$

Example



Example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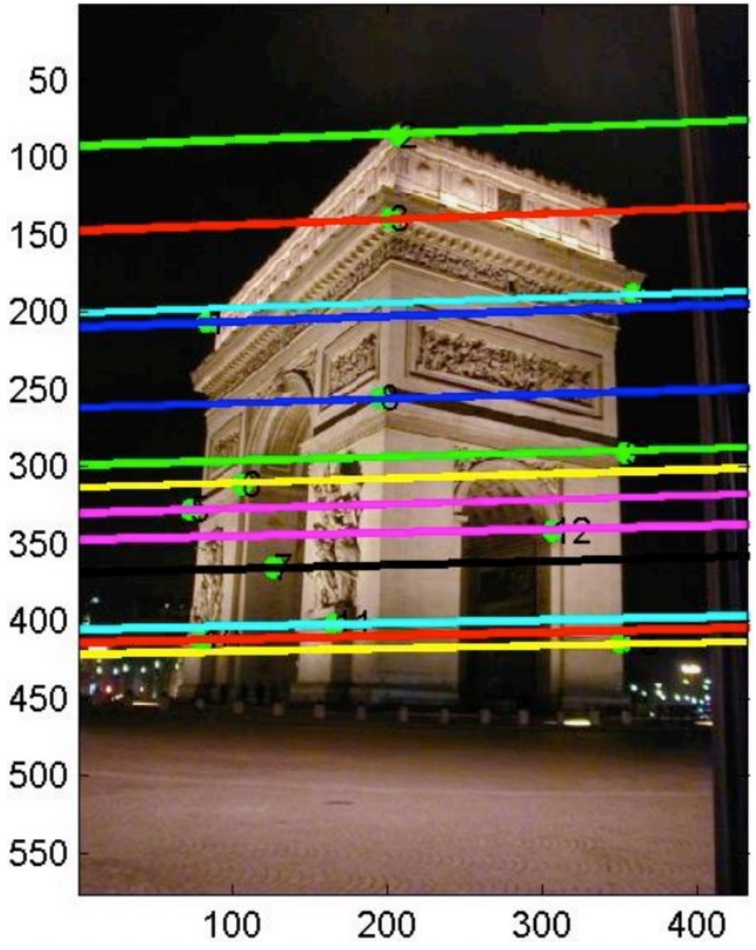
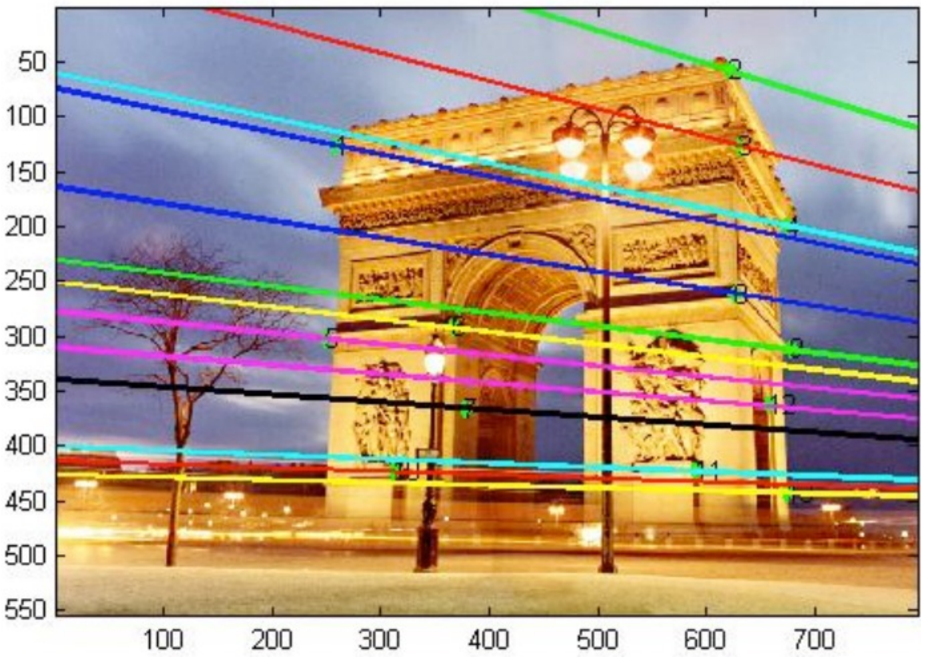
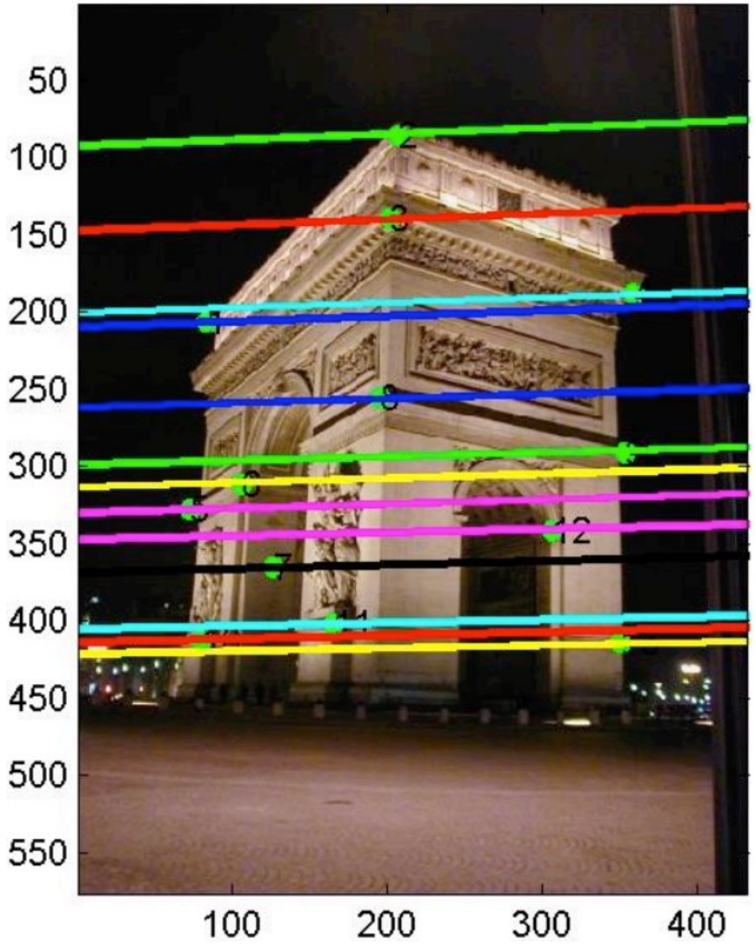
600
700


50
100
150
200
250
300
350
400
450
500
550
100
200
300
400

Example





$$F = \begin{pmatrix} -0.00310695 & -0.0025646 & 2.96584 \\ -0.028094 & -0.00771621 & 56.3813 \\ 13.1905 & -29.2007 & -9999.79 \end{pmatrix}$$

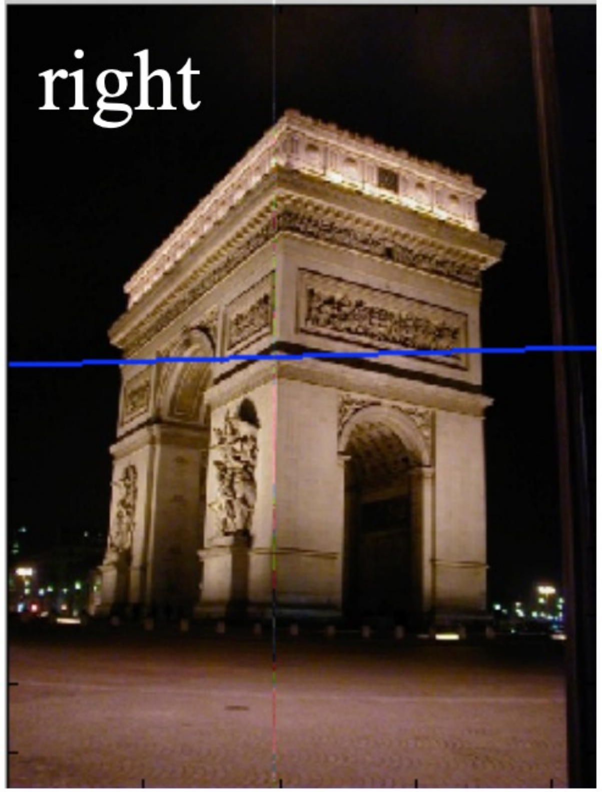
Example

$$\boxed{l' = Fx}$$

$$\begin{pmatrix} -0.00310695 & -0.0025646 & 2.96584 \\ -0.028094 & -0.00771621 & 56.3813 \\ 13.1905 & -29.2007 & -9999.79 \end{pmatrix} \begin{pmatrix} 343.53 \\ 221.70 \\ 1.0 \end{pmatrix}$$

x = 343.5300 y = 221.7005

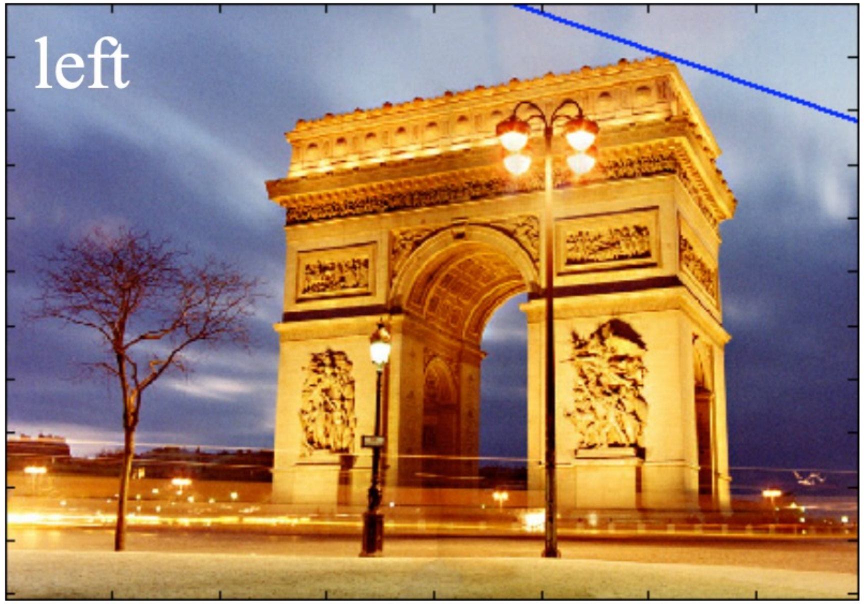
$$\begin{bmatrix} 1.3299 \\ 45.0195 \\ -1.1942\times10^4 \end{bmatrix}$$



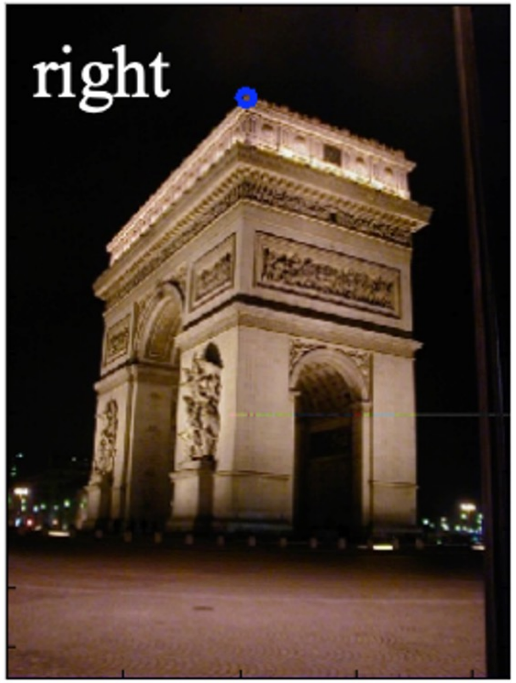
Example

$$\boxed{l = F^{\top} x'}$$

$$\begin{pmatrix}205.5526 & 80.5 & 1.0\end{pmatrix}\begin{pmatrix}-0.00310695 & -0.0025646 & 2.96584 \\ -0.028094 & -0.00771621 & 56.3813 \\ 13.1905 & -29.2007 & -9999.79\end{pmatrix}$$

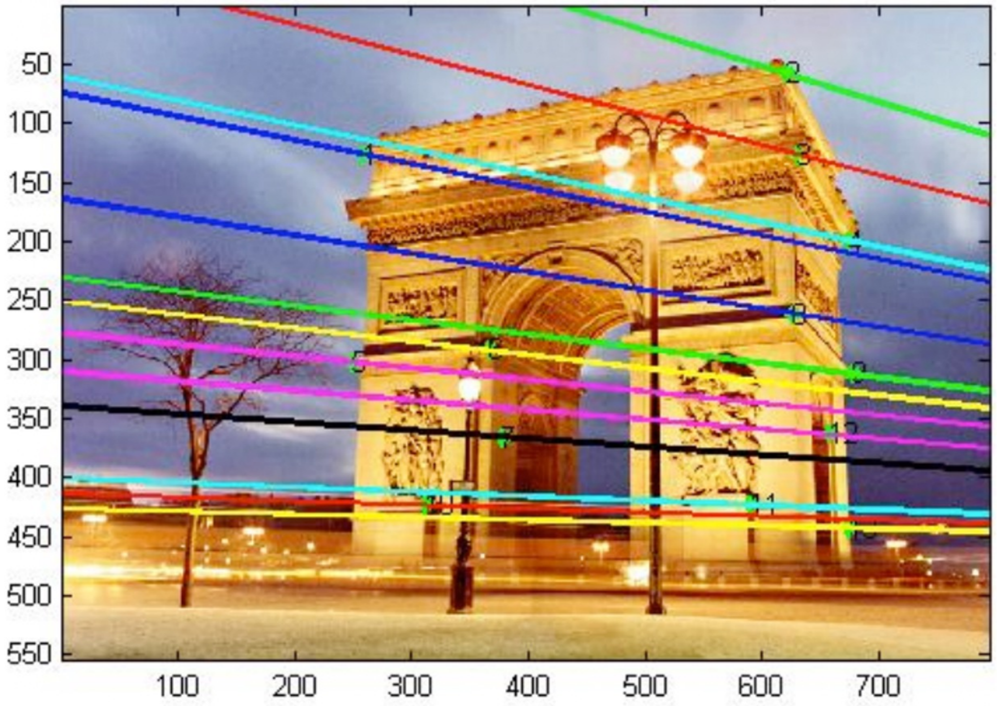


$$\begin{bmatrix}10.29 \\ -30.34 \\ -4851.5\end{bmatrix}$$



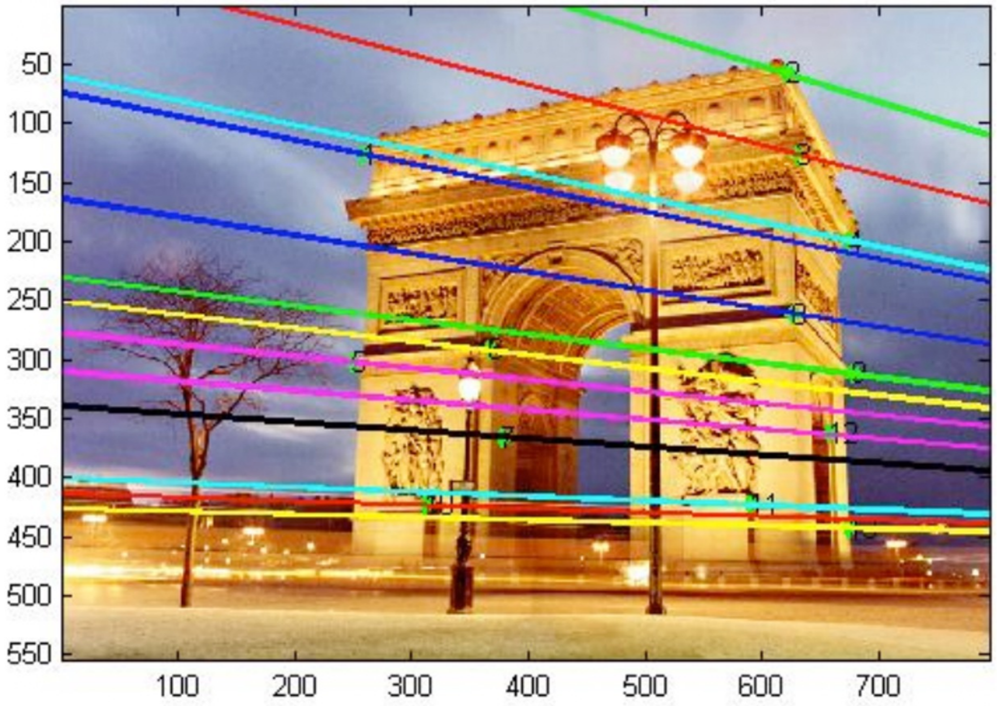
x = 205.5526 y = 80.5000

Example



- Where is the epipole?
- $Fe = 0$
- Vector in the right null space of matrix F
- However, due to noise, F may not be singular. So instead, next best thing is eigenvector associated with smallest eigenvalue of $F^\top F$

Example



- Eigendecomposition:
  - $[\mathrm{u}, \mathrm{d}] = \mathrm{eigs}(\mathrm{F}' * \mathrm{F})$

$$\mathrm{u} = \begin{matrix} -0.0013 & 0.2586 & -0.9660 \\ 0.0029 & -0.9660 & -0.2586 \\ 1.0000 & 0.0032 & -0.0005 \end{matrix} \qquad \mathrm{d} = 1.0\mathrm{e}8^{*} \begin{matrix} -1.0000 & 0 & 0 \\ 0 & -0.0000 & 0 \\ 0 & 0 & -0.0000 \end{matrix}$$

- Eigenvector associated with smallest eigenvalue:
  - $e = u(:,3)$
  - $e = (-0.9660 \quad -0.2586 \quad -0.0005)$
  - $e = \mathrm{e}/\mathrm{e}(3)$ to get pixel cords
  - $e = (1861.02 \quad 498.21 \quad 1)$

Use RANSAC to Handle Outliers

1. Repeat:
   1. select minimal sample (8 pixel matches)
   2. compute solution(s) for *F*
   3. count the number of inliers

   until Γ(#inliers, #samples) > 95% or bored
2. Recompute *F* using all inliers

The Fundamental Matrix F: Recovering P & P'

- Since we can set the world coordinate system arbitrarily, set
$$P = [I \mid 0]$$
$$P' = \left[[e']_\times F \mid e' \right]$$
where epipole can be found by solving
$$F^\top e' = 0$$
via SVD/eigendecomposition
[HZ p. 256]

Essential & Fundamental Matrices: Summary

- Algebraic representations of epipolar geometry:
  - Projection matrices (given intrinsics + extrinsics)
  - **Essential matrix** (given intrinsics): $\mathbf{E} = [\boldsymbol{t}]_{\times}\mathbf{R}$
  - **Fundamental matrix**: $\mathbf{F} = \mathbf{K}'^{-\top}[\boldsymbol{t}]_{\times}\mathbf{R}\mathbf{K}^{-1} = [\boldsymbol{e}']_{\times}\mathbf{K}'\mathbf{R}\mathbf{K}^{-1}$ [HZ p.244]
    - Using identity $[\boldsymbol{v}]_{\times}\mathbf{M} = \mathbf{M}^{-\top}[\mathbf{M}^{-1}\boldsymbol{v}]_{\times}$ and $\boldsymbol{e}' = \mathbf{K}'\boldsymbol{t}$ (image of camera centre 1)
- Epipolar constraint for corresponding points $\{\boldsymbol{x}, \boldsymbol{x}'\}$:

$$\boldsymbol{x}'^{\top}\mathbf{F}\boldsymbol{x} = \mathbf{0}$$

  - $\mathbf{F}$ is rank 2 and is known only up to scale → 7 DoF
  - What is $\mathbf{F}\boldsymbol{x}$?
- **Estimation:** DLT (again!); assemble a matrix $\mathbf{A}$, compute the SVD, enforce rank 2 (with another SVD); or use a nonlinear solver

Next Lecture

- Two-view geometry: Triangulation
- Two-view geometry: Stereo