Fusion products of representations of the $\widehat{\mathfrak{sl}_2}$ algebra

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Abstract

In this thesis, we review the representations that appear in the $\widetilde{SL}_2(\mathbb{R})$ WZW model and the degenerate representations that are needed to bootstrap this model. We determine fusion rules of level 1 degenerate representation with affine highest weight representations by solving null vector equations in both spectral flow preserving and violating cases.

Contents

1 Introduction and overview

The $\widetilde{SL}_2(\mathbb{R})$ Wess-Zumino-Witten (WZW) model is a conformal field theory (CFT) with an additional $\widehat{\mathfrak{sl}_2}$ symmetry. The study of the $\widetilde{SL}_2(\mathbb{R})$ WZW model is motivated by its connection with string theory in AdS_3. The fusion rules of WZW models at fractional level have been well studied, where there exist doubly degenerate representations. One set of the fusion rules of these models have been proposed by Awata and Yamada [1]. However, this model at generic level has not been completely solved: the 3-point function is not fully known and crossing symmetry has not been proved yet.

One important feature of the $\widetilde{SL}_2(\mathbb{R})$ WZW model is the spectral flow. Maldacena and Ooguri have shown that the physical spectrum cannot be built only from affine primaries but should also include spectral flowed fields [2]. They proposed a well-tested and widely believed spectrum of this model as well as the fusion rules.

In order to solve this model, one needs to find the null vectors and the fusion rules between degenerate representations with other representations, as what is done in solving the Liouville theory [3] and the H_3^+-WZNW model [4]. One null vector of the $\widetilde{SL}_2(\mathbb{R})$ WZW model at level 1 and the fusion rule with the corresponding null vector was given in [5]. However, only the spectral flow preserving case is considered in this paper. Hence the deduced fusion rules might be incomplete.

In this paper, we determine the spectrum of spectral flowed degenerate representations. We find that the action of spectral flow on a might give another degenerate representation at higher level. We determine fusion rules between degenerate representation and generic representations by solving the null vector equations. We find the fusion with degenerate representations will also give spectral flowed representations.

This thesis is organized as follows. We give a brief review of $\widetilde{SL}_2(\mathbb{R})$ model in section 2. In section 3, we introduce the degenerate representations and give the spectrum the level 1 degenerate representations. In section 4, we apply the null vector to the three-point functions to calculate the fusion rules between degenerate representations and affine highest weight representations.

2 $\widetilde{SL}_2(\mathbb{R})$ WZW model

In this chapter, we introduce the theoretical foundation of the WZW model, especially concentrating on the $\widetilde{SL}_2(\mathbb{R})$ case. We begin by defining the non-Abelian affine symmetries and introducing the Sugawara construction. Then, we define fields and representations that appear in this model. At last, we give the correlation functions that are necessary to solve this model.

2.1 Non-Abelian affine symmetries

Reminders on Lie algebras

A Lie algebra is a vector space $\mathfrak{g}$ equipped with a binary operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, known as the Lie bracket, which satisfies the following axioms:

- Bilinearity: $[ax+by, z] = a\,[x,z] + b\,[y,z]\,,\quad [z, ax+by] = a\,[z,x] + b\,[z,y].$
- Antisymmetry: $[x,y] = -\,[y,x].$
- Jacobi identity: $[a,[b,c]] + [b,[c,a]] + [c,[a,b]] = 0.$

For a basis of generators t^a of $\mathfrak{g}$, the Lie bracket is defined by the commutation relations:

$$\left[t^a, t^b\right] = f_c^{ab} t^c. \tag{2.1}$$

The numbers f_c^{ab} are called structure constants of the Lie algebra. From these, one can define an invariant symmetric tensor called the Killing tensor:

$$K^{ab} = \frac{1}{2g} f_d^{ac} f_c^{bd}, \tag{2.2}$$

where g is the dual Coxeter number of $\mathfrak{g}$. For the case of $\mathfrak{sl}_2$, this number is $g = 2$. And the non-zero coefficients are:

$$K^{00} = \frac{1}{2}, \quad K^{+-} = K^{-+} = 1. \tag{2.3}$$

We define the quadratic Casimir operator C as

$$C = K_{ab} t^a t^b. \tag{2.4}$$

C is a central element of the universal enveloping algebra $U(\mathfrak{g})$, which means that C commutes with all Lie algebra generators: $[C, t^a] = 0$. This implies that in an irreducible representation, the Casimir operator is proportional to the identity.

For the algebra $\mathfrak{sl}_2$, the irreducible representations are labeled by a number j, called the spin. The eigenvalue of the Casimir operator is

$$C = 2t^0t^0 + t^+t^- + t^-t^+ = 2j(j+1). \tag{2.5}$$

Symmetry algebra

We now define the symmetry algebra of the WZW model. The G WZW model is a CFT with an infinite-dimensional affine Lie algebra, denoted as $\hat{\mathfrak{g}}_k$. This algebra is generated by $\dim\mathfrak{g}$ holomorphic currents $J^a(z)$ through their operator product expansions(OPE),

$$\boxed{J^a(z)J^b(w) = \frac{kK^{ab}}{(z-w)^2} + \frac{f^{ab}_c J^c(w)}{z-w} + \mathcal{O}(1),} \tag{2.6}$$

where the constant k is the level, which is the most important parameter of the model.

The modes of the current are defined by

$$J^a_n = \oint \mathrm{d}z\, z^n J^a(z). \tag{2.7}$$

From the OPEs (2.6), we deduce the following commutation relations, which formally define the affine Lie algebra $\widehat{\mathfrak{sl}}(2)_k$

$$\boxed{\left[J^a_m, J^b_n\right] = f^{ab}_c J^c_{m+n} + mkK^{ab}\delta_{m+n,0}.} \tag{2.8}$$

The Sugawara construction

The symmetry algebra of 2D CFT is the Virasoro algebra. The generators of Virasoro algebra are $(L_n)_{n\in\mathbb{Z}}$, and the commutation relations are

$$\boxed{[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(n-1)n(n+1)\delta_{n+m,0}.} \tag{2.9}$$

We define a holomorphic field $T(y)$ as a generating function for (L_n), known as the energy momentum tensor:

$$T(y) = \sum_{n\in\mathbb{Z}} \frac{L^{(z)}_n}{(y-z)^{n+2}}, \tag{2.10}$$

where $L_n^{(z)}$ is the Virasoro generator at z. The OPE between T and itself is

$$T(y)T(z) = \frac{c/2}{(y-z)^4} + \frac{2T(z)}{(y-z)^2} + \frac{\partial_z T(z)}{y-z} + \mathcal{O}(1). \tag{2.11}$$

A Virasoro primary field $V_\Delta(z)$ with conformal dimension Δ is defined by its OPE with $T(y)$:

$$T(y)V_\Delta(z) = \frac{\Delta V_\Delta(z)}{(y-z)^2} + \frac{\partial_z V_\Delta(z)}{y-z} + \mathcal{O}(1). \tag{2.12}$$

In the WZW model, we introduce the Sugawara construction for the energy momentum tensor $T(z)$:

$$\boxed{T(z) = \frac{K_{ab}}{2(k-g)} : J^a(z)J^b(z) : .} \tag{2.13}$$

The central charge of $T(z)$ is $c = \frac{k\dim\mathfrak{g}}{k-g}$. In our case of $\widetilde{SL}_2(\mathbb{R})$ WZW model, this gives

$$\boxed{c = \frac{3k}{k-2}} \tag{2.14}$$

The Virasoro generators are given by

$$L_n = \frac{K_{ab}}{2(k+g)} : \sum_{m\in\mathbb{Z}} J^a_{n-m} J^b_m : . \tag{2.15}$$

Using the commutation relations for the J_n^a (2.8), we can deduce commutation relation between L_m and J_n^a:

$$\boxed{[L_m, J_n^a] = -nJ^a_{m+n}.} \tag{2.16}$$

2.2 Affine primary fields

An affine primary field $\phi^j(z)$ associated with representation $\mathcal{R}^j$ is defined by its OPE with current field $J^a(y)$:

$$\boxed{J^a(y)\phi^j(z) \sim \frac{-(t^a)^T\phi^j(z)}{y-z} + \mathcal{O}(1),} \tag{2.17}$$

where t^a is the generator of Lie algebra $\mathfrak{sl}_2$. This OPE involves the transposition of Lie algebra generators, which ensures the associativity of the OPE $J^aJ^b\phi$. Since the transposition flips the sign of commutation relations, we need an additional minus sign.

The OPE between $T(y)$ and $\phi^j(z)$ can be obtained from the Sugawara construction (2.13):

$$T(y)\phi^j(z) = \frac{K_{ab}t^a}{k-g}\left(\frac{\frac{1}{2}t^b\phi^j(z)}{(y-z)^2} - \frac{\left(J^b\phi^j\right)(z)}{y-z}\right) + \mathcal{O}(1). \tag{2.18}$$

Hence an affine primary field is also a Virasoro primary field. The conformal dimension of $\phi^j(x)$ is propotional to the Casimir operator $C = K_{ab}t^at^b$:

$$\boxed{\Delta_j = \frac{C(j)}{2(k-g)} = \frac{j(j+1)}{k-g}.} \tag{2.19}$$

Based on the state-field correspondence, the field $\phi^j(z)$ corresponds to an affine primary state $\left|v^j\right\rangle$. The OPE

(2.17) implies the action of currents modes on this state:

$$\begin{cases} J^a_{n>0} \left| v^j \right\rangle = 0, \\ J^a_0 \left| v^j \right\rangle = -t^a \left| v^j \right\rangle. \end{cases} \tag{2.20}$$

Isospin variables

For practical calculations, especially for solving Ward identities, it is useful to realize the abstract action of the generators t^a as differential operators acting on functions. We introduce the isospin variables and represent the fields as functions of the isospin variables, where t^a acts on primary fields as differential operators $D^j(t^a)$. We introduce three important bases:

x-basis: In this basis, a field is represented as a function ϕ^j_x, and t^a acts as

$$\begin{cases} D^j_x(t^+) = x^2\partial_x - 2jx, \\ D^j_x(t^0) = x\partial_x - j, \\ D^j_x(t^-) = -\partial_x. \end{cases} \tag{2.21}$$

Because of the transposition in (2.17), in the isospin formalism the generator J^a_0 acts as

$$J^a_0 J^b_0 \phi^j_x(z) = \left(-D^j_x\left(t^b\right)\right)\left(-D^j_x\left(t^a\right)\right)\phi^j_x(z). \tag{2.22}$$

μ-basis: The t^a acts as

$$\begin{cases} D^j_x(t^+) = \mu\partial^2_\mu - \dfrac{j(j+1)}{\mu}, \\ D^j_x(t^0) = -\mu\partial_\mu, \\ D^j_x(t^-) = -\mu. \end{cases} \tag{2.23}$$

The x-basis field $\phi^j_x(z)$ and μ-basis field $\phi^j_\mu(z)$ are related by the Fourier transformation

$$\phi^j_x(z) \sim \int \mathrm{d}\mu\, \mu^{-j-1} \mathrm{e}^{\mu x} \phi^j_\mu(z). \tag{2.24}$$

m-basis: In this basis, the quadratic Casimir operator C and J^0_0 are diagonalized. We denote the state in m-basis to be $|j,m\rangle$. The action of $J^\pm_0$ on $|j,m\rangle$ is defined to be:

$$\begin{aligned} J^-_0 \left|j,m\right\rangle &= (j+m)\left|j,m-1\right\rangle, \\ J^0_0 \left|j,m\right\rangle &= m\left|j,m\right\rangle \\ J^+_0 \left|j,m\right\rangle &= (j-m)\left|j,m+1\right\rangle. \end{aligned} \tag{2.25}$$

The field $\phi^j_m(z)$ corresponding to the state $|j,m\rangle$ is related to the μ-basis field by

$$\phi^j_m(z) \sim \int \mathrm{d}\mu\, \mu^{-m} \phi^j_\mu(z). \tag{2.26}$$

2.3 Irreducible representations

Irreducible representations of $\mathfrak{sl}_2$

For a given affine Lie algebra $\widehat{\mathfrak{g}}$, we define the horizontal algebra to be the subalgebra generated by zero-mode generators J_0^a. The central extension in (2.8) vanishes for $m = 0$, hence the subalgebra is isomorphic to the underlying Lie algebra $\mathfrak{g}$.

We work in the m-basis. From (2.25), we find two null state $J_0^{\pm} |j, \pm\rangle$, which satisfy the following vanishing condition:

$$J_0^{\mp} \left(J_0^{\pm} |j, \pm j\rangle \right) = 0. \tag{2.27}$$

Hence we can classify different representations by the existence of these states.

The representations appear in the spectrum can be classified into several series:

- Principle continuous series $\mathcal{C}_\alpha^j$. They are labeled by spin $j \in -\frac{1}{2} + \mathrm{i}\mathbb{R}_+$ and a real number $\alpha \in (0, 1)$. The eigenvalue m of J_0^0 is unbounded both above and below, taking values $m \in \alpha + \mathbb{R}$.

- Discrete series $\mathcal{D}^{j,\pm}$. They are labeled by real spin $j \in (-\infty, -\frac{1}{2})$ and contain the state $|j, \mp j\rangle$ respectively. Hence the spectrum is either bounded from below and from above.

Another crucial class of representations are the finite dimensional representations $\mathcal{E}^j$, which contain both $|j, j\rangle$ and $|j, -j\rangle$. They exist only when j takes integer or half-integer values $j \in \mathbb{N}/2$, where m takes values in $\{-j, -j + \frac{1}{2}, \cdots, j - \frac{1}{2}, j\}$. We summarize the properties of these representations in the following table.

Representations	Parameter values	Eigenvalues of J_0^0	Conjugate representation
$\mathcal{C}_\alpha^j$	$j \in -\frac{1}{2} + i\mathbb{R}_+$, $\alpha \in \mathbb{R} \bmod \mathbb{Z}$	$\alpha + \mathbb{Z}$	$\mathcal{C}_{-\alpha}^j$
$\mathcal{D}^{j,+}$	$j \in (-\infty, -\frac{1}{2})$	$-j + \mathbb{N}$	$\mathcal{D}^{j,-}$
$\mathcal{D}^{j,-}$	$j \in (-\infty, -\frac{1}{2})$	$j - \mathbb{N}$	$\mathcal{D}^{j,+}$
$\mathcal{E}^j$	$j \in \mathbb{N}/2$	$\{-j, -j + \frac{1}{2}, \cdots, j - \frac{1}{2}, j\}$	$\mathcal{E}^j$

Affine highest representations

An irreducible representation $\mathcal{R}$ of $\mathfrak{g}$ can be naturally extended to an affine highest-weight representation $\widehat{\mathcal{R}}$, which is not necessarily irreducible though. From the above representations of $\mathfrak{sl}_2$, we can naturally induce the corresponding representations of $\widehat{\mathfrak{sl}_2}$. We plot the states by their eigenvalues of L_0 and J_0^0 in Figure.(1).

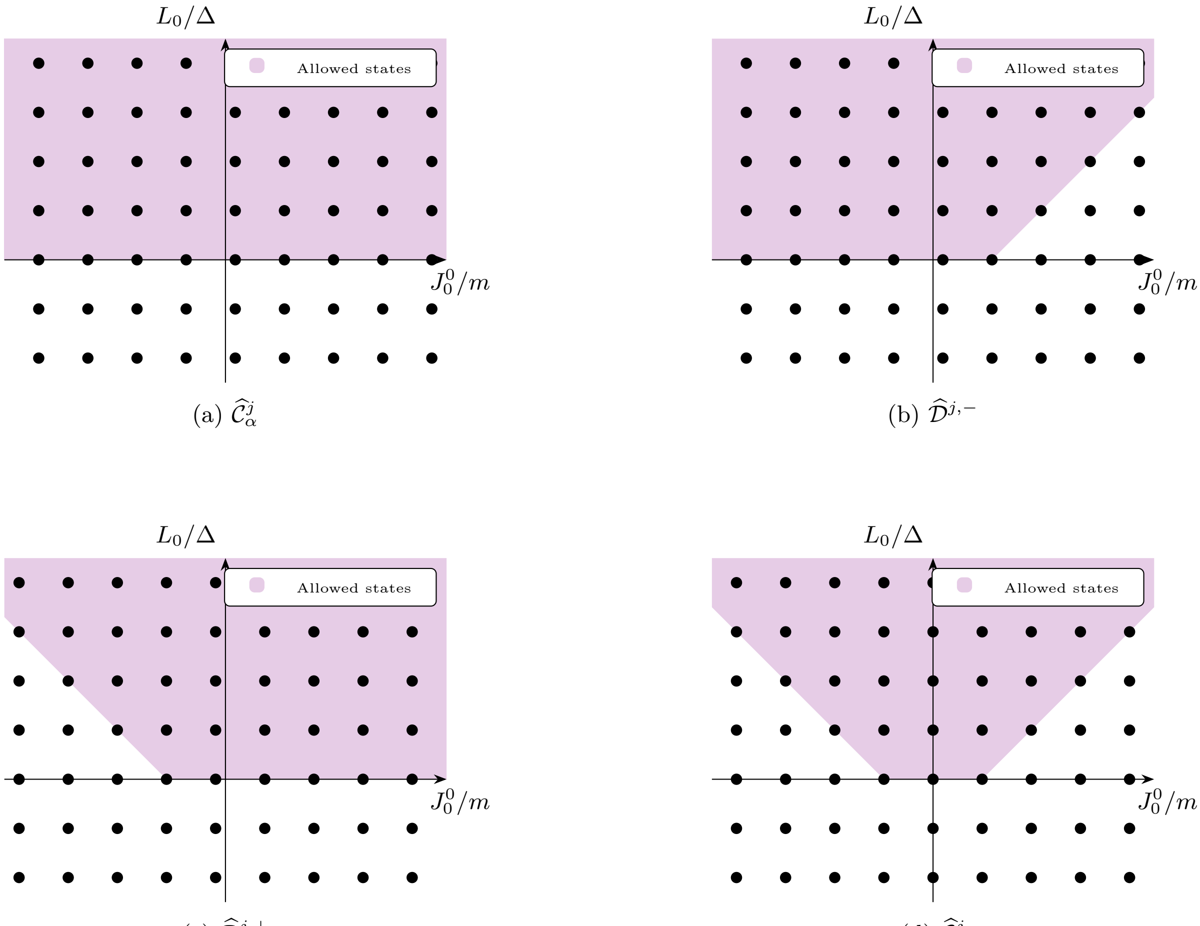
Spectral flow

The Spectral flow is a family $(\rho_\omega)_{\omega \in \mathbb{Z}}$ of automorphisms of $\widehat{\mathfrak{sl}}_2$ satisfying $\rho_{\omega_1} \circ \rho_{\omega_2} = \rho_{\omega_1 + \omega_2}$, which are defined by

$$\begin{aligned} \rho_\omega(J_m^{\pm}) &= J_{m \pm \omega}^{\pm}, \\ \rho_\omega(J_m^0) &= J_m^0 + \frac{1}{2} k\omega\delta_{m,0}. \end{aligned} \tag{2.28}$$

According to the Sugawara construction (2.15), the spectral flow of Virasoro generators is

$$\rho_\omega(L_m) = L_m + \omega J_m^0 + \frac{1}{4} kn^2 \delta_{m,0}. \tag{2.29}$$



(a) $\widehat{\mathcal{C}}^j_\alpha$ (b) $\widehat{\mathcal{D}}^{j,-}$ (c) $\widehat{\mathcal{D}}^{j,+}$ (d) $\widehat{\mathcal{E}}^j$

Figure 1: The spectra of irreducible representations of $\widehat{\mathfrak{sl}_2}$

Hence the conformal dimension of a spectral flowed primary field $\phi_m^{j,\omega}(z)$ is

$$\Delta_m^{j,\omega} = \Delta_j - \omega m - \frac{1}{4}km^2. \tag{2.30}$$

A representation $\widehat{\mathcal{R}}$ provides the action of generators J_n^a on vector space V. Based on this representation, we define the spectral flowed representation $\rho_\omega\left(\hat{\mathcal{R}}\right)$. The spectral flowed representation acts on the same vector space V, but the action of generators J_n^a is defined to be $\rho_{-\omega}\left(J_n^a\right)$.

For clarity in calculations, we label the states in $\rho_\omega\left(\hat{\mathcal{R}}\right)$ by $|j,m,\omega\rangle$. The definition of the action of generators J_n^a leads to

$$J_n^a\left|j,m,\omega\right\rangle \equiv \rho^{-\omega}\left(J_n^a\right)\left|j,m\right\rangle, \tag{2.31}$$

where we omit the label 0 in state $|j,m,0\rangle$ for simplicity. The conjugate representation of $\rho_n\left(\hat{\mathcal{R}}\right)$ is

$$\rho_\omega\left(\hat{\mathcal{R}}\right)^* = \rho_{-\omega}\left(\hat{\mathcal{R}}^*\right) \tag{2.32}$$

In addition, we conjecture that the spectral flow commutes with fusion, which has been proved in some specific cases [6],

$$\rho_{\omega_1}\left(\hat{\mathcal{R}}\right) \times \rho_{\omega_2}\left(\mathcal{R}'\right) = \rho_{\omega_1+\omega_2}\left(\hat{\mathcal{R}} \times \mathcal{R}'\right). \tag{2.33}$$

Let's consider the action of spectral flow on affine highest-weight representations. We introduce the following notation

$$\begin{aligned} \hat{\mathcal{C}}^{j,\omega} &= \rho_\omega\left(\hat{\mathcal{C}}^j\right), \\ \hat{\mathcal{D}}^{j,\frac{1}{2}+\omega} &= \rho_\omega\left(\hat{\mathcal{D}}^{j,+}\right). \end{aligned} \tag{2.34}$$

From (2.30), we find the conformal dimension of states in $\hat{\mathcal{C}}^{j,\omega}$ of non-zero ω are not bounded from below. Hence it cannot be an affine highest-weight representation.

On the other hand, the representations $\hat{\mathcal{D}}^{j,\pm}$ are characterized by the existence of state $|j,\mp j\rangle$, which satisfy the following conditions:

$$J_{n>0}^a\left|j,\pm j\right\rangle = J_0^{\pm}\left|j,\pm j\right\rangle = \left(J_0^0 \mp j\right)\left|j,\pm j\right\rangle = 0. \tag{2.35}$$

In particular, we notice that

$$\begin{aligned} &J_{n\geq 0}^{+}\left|j,-j,-1\right\rangle = J_{n+1}^{+}\left|j,-j\right\rangle = 0, \\ &J_{n>0}^{0}\left|j,-j,-1\right\rangle = J_n^0\left|j,-j\right\rangle = 0, \\ &\left(J_0^0 - \frac{k}{2} + j\right) = \left(J_0^0 + j\right)\left|j,-j\right\rangle = 0, J_{n>0}^{-}\left|j,-j,-1\right\rangle \quad = J_{n-1}^{-}\left|j,-j\right\rangle = 0. \end{aligned} \tag{2.36}$$

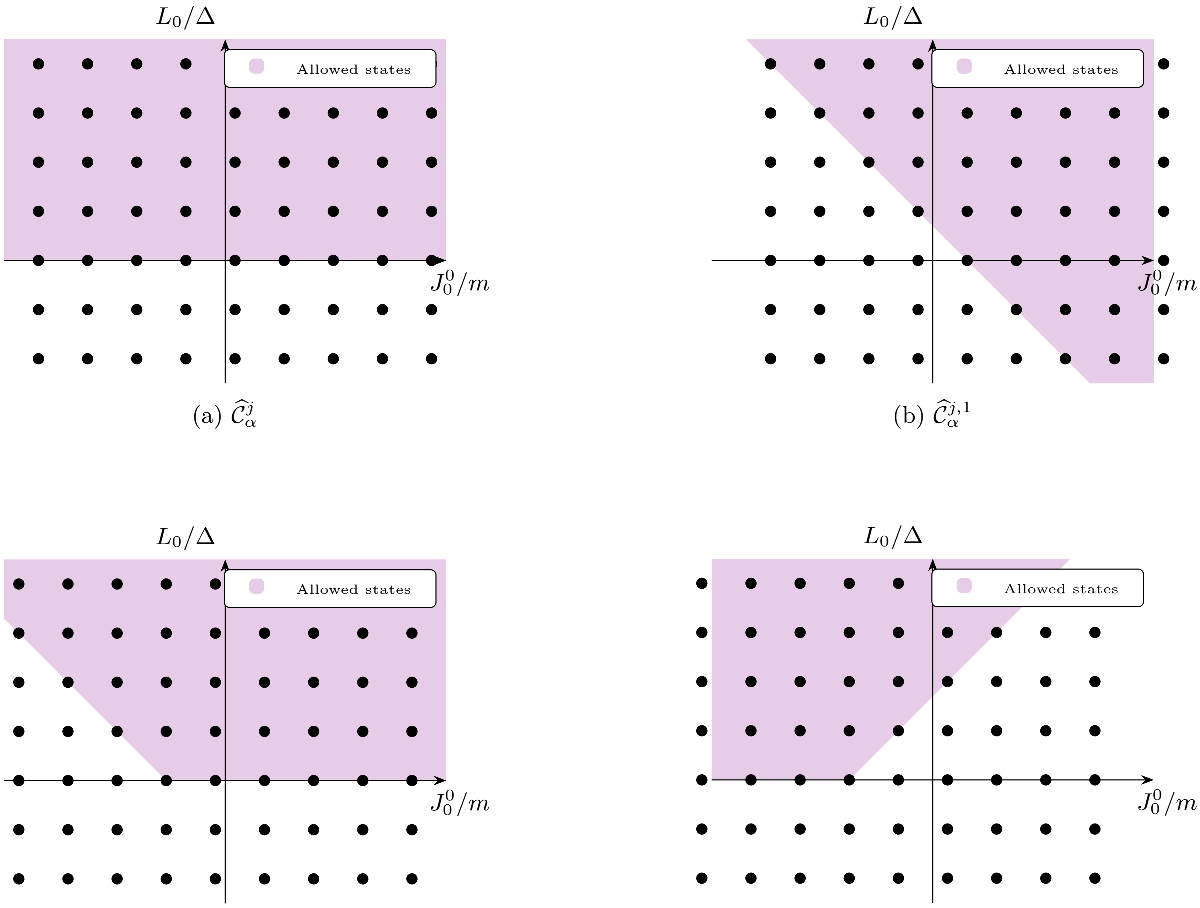
Hence we find $|j,-j,-1\rangle = \left|\frac{k}{2}-j,\frac{k}{2}-j\right\rangle$, and

$$\hat{\mathcal{D}}^{j,-\frac{1}{2}} = \rho_{-1}\left(\hat{\mathcal{D}}^{j,+}\right) = \hat{\mathcal{D}}^{\frac{k}{2}-j,-}. \tag{2.37}$$

Now we can use 2.34 to write the discrete series representations as

$$\widehat{\mathcal{D}}^{j,+} = \widehat{\mathcal{D}}^{j,\frac{1}{2}}, \quad \widehat{\mathcal{D}}^{j,-} = \widehat{\mathcal{D}}^{\frac{k}{2}-j,-\frac{1}{2}}. \tag{2.38}$$

Since the eigenvalue of L_0 are related to m, the spectrum of spectral flowed representations are twisted. We plot the spectrum of $\widehat{\mathcal{C}}_\alpha^{j,1}$ and $\widehat{\mathcal{D}}^{\frac{k}{2}-j,-\frac{1}{2}}$ in 2.



(a) $\widehat{\mathcal{C}}_\alpha^{j}$ (b) $\widehat{\mathcal{C}}_\alpha^{j,1}$

(c) $\widehat{\mathcal{D}}^{j,\frac{1}{2}}$ (d) $\widehat{\mathcal{D}}^{\frac{k}{2}-j,-\frac{1}{2}}$

Figure 2: Spectral flowed irreducible representations

2.4 Correlation functions

Ward identities

The OPE (2.6) implies the behaviour of $J^a(z)$ near $y = \infty$:

$$\boxed{J^a(y) \underset{y\to\infty}{=} \mathcal{O}\left(\frac{1}{y^2}\right)} \tag{2.39}$$

Consider a set of fields $\phi^{\sigma_i}(z_i)$, and a meromorphic function $\epsilon(y)$ such that $\epsilon(y) \underset{y\to\infty}{=} \mathcal{O}(1)$. Suppose $\epsilon(y)$ has no poles outside $\{z_1, \cdots, z_N\}$, we have

$$\oint_\infty \mathrm{d}y\, \epsilon(y) \left\langle J^a(y) \prod_i \phi^\sigma(z_i) \right\rangle = 0. \tag{2.40}$$

In case of $\epsilon(y) = 1$, we obtain the global Ward identities:

$$\left\langle \sum_i (J_0^a)^{(z_i)} \prod_i \phi^{\sigma_i}(z_i) \right\rangle = 0, \tag{2.41}$$

where $(J_0^a)^{(z_i)}$ means the operator acts only on the ith field. Especially, if all the fields are affine primary fields, the global Ward identities reduce to

$$\sum_i D_x^{j_i}(t^a)\left\langle \prod \phi_{x_i}^{j_i}(z_i)\right\rangle = 0. \tag{2.42}$$

On the other hand, if we take $\epsilon(y) = \frac{1}{(y-z_i)^n}$ and assume all fields but possibily the field at z_i to be affine primary fields, we find the following local Ward identities:

$$\left\langle J_{-n}^a \phi^{\sigma_i}(z_i) \prod_{k\neq i} \phi_{x_k}^{j_k}(z_k)\right\rangle = \sum_{k\neq i} \frac{D_x^{j_k}(t^a)}{(z_k - z_i)^n} \left\langle \phi^{\sigma_i}(z_i) \prod_{k\neq i} \phi_{x_k}^{j_k}(z_k)\right\rangle \tag{2.43}$$

Three-point functions

So far, we have only considered the chiral field $\phi^j(z)$ belongs to some representation $\widehat{\mathcal{R}^j}$ of $\widehat{\mathfrak{sl}_2}$. However, since the SL(2,R) WZW model is CFT with $\widehat{\mathfrak{sl}_2} \times \widehat{\mathfrak{sl}_2}$ symmetry, the field $\phi^j(z,\bar{z})$ transforms under both $\widehat{\mathfrak{sl}_2}$ algebras. One may assume that the fields are a product $\phi^j(z,\bar{z}) \sim \phi^j(z)\phi^j(\bar{z})$ of chiral fields. However, this chiral factorization fails at the level of zero modes. Hence we need to construct the correlation functions with both holomorphic and anti-holomorphic part.

When considreing both the left-moving and right-moving part, the transformation between x-basis and μ-basis is then given by

$$\phi_{x,\bar{x}}^j(z,\bar{z}) = \int_{\mathbb{C}} \mathrm{d}^2\mu\, |\mu|^{-2j-2} \mathrm{e}^{\mu x - \bar{\mu}\bar{x}} \phi_{\mu,\bar{\mu}}^j(z,\bar{z}). \tag{2.44}$$

The fields in m-basis are related to the μ-basis fields by

$$\phi_{m,\bar{m}}^j(z,\bar{z}) = N_{m,\bar{m}}^j \int_{\mathbb{C}} \frac{\mathrm{d}^2\mu}{|\mu|^2}\, \mu^{-m}\bar{\mu}^{-\bar{m}} \phi_{\mu,\bar{\mu}}^j(z,\bar{z}). \tag{2.45}$$

where the normalization factor is

$$N_{m,\bar{m}}^j = \frac{\Gamma(j+1-m)}{\Gamma(\bar{m}-j)}. \tag{2.46}$$

We use the following notation:

$$|F(z,m)|^2 = F(z,m) \times F(\bar{z},\bar{m}). \tag{2.47}$$

Please notice that although $\bar{z}$ is the complex conjugate of z, $\bar{m}$ is not necessarily the complex conjugate of m.

The 3-point functions with only affine primary fields can be determined from the global Ward identities (2.42). In x-basis, we find the solution to be:

$$\left\langle \phi_{x_1,\bar{x}_1}^{j_1}(z_1,\bar{z}_1)\phi_{x_2,\bar{x}_2}^{j_2}(z_2,\bar{z}_2)\phi_{x_3,\bar{x}_3}^{j_3}(z_3,\bar{z}_3)\right\rangle = |z_{12}|^{-2\Delta_{12}^3}|z_{23}|^{-2\Delta_{23}^1}|z_{31}|^{-2\Delta_{31}^2} \times D\begin{bmatrix} j_1 & j_2 & j_3 \\ x_1 & x_2 & x_3 \end{bmatrix} C(j_1,j_2,j_3), \tag{2.48}$$

where $z_{ij} = z_i - z_j$ and $\Delta_I^K = \sum_{i\in I}\Delta_{j_i} - \sum_{k\in K}\Delta_{j_k}$. The structure constant $C(j_1,j_2,j_3)$ is not fully determined yet. The x-dependence is included in the factor:

$$D\begin{bmatrix} j_1 & j_2 & j_3 \\ x_1 & x_2 & x_3 \end{bmatrix} = |x_{12}|^{2j_{12}^3}|x_{23}|^{2j_{23}^1}|x_{31}|^{2j_{31}^2}. \tag{2.49}$$

In the μ-basis, the corresponding factor is

$$D\begin{bmatrix} j_1 & j_2 & j_3 \\ \mu_1 & \mu_2 & \mu_3 \end{bmatrix} = \pi|\mu_2|^{-2j_1-2j_3-2}|\mu_1|^{2j_1+2}|\mu_3|^{2j_3+2} \times \left[\frac{\gamma(j_{23}^1+1)\gamma(j_{13}^2+1)}{\gamma(-j_{123}-1)\gamma(2j_3+2)}{}_2\mathcal{F}_1(j_{123}+2, j_{13}^2+1, 2j_3+2, -\frac{\mu_3}{\mu_2}) + \left|\frac{-\mu_3}{\mu_2}\right|^{-2(2j_3+1)}\frac{\gamma(j_{12}^3+1)}{\gamma(-2j_3)}{}_2\mathcal{F}_1(-j_{23}^1, j_{12}^3+1, -2j_3, -\frac{\mu_3}{\mu_2})\right] \tag{2.50}$$

where ${}_2\mathcal{F}_1$ is the product of two hypergeometric functions,

$${}_2\mathcal{F}_1(a,b,c,z) = F(a,b,c,z)\times F(a,b,c,\bar{z}). \tag{2.51}$$

The factor γ is defined by the gamma function

$$\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}. \tag{2.52}$$

Three-point functions with spectral flow violation

So far we have only discussed the 3-point function with affine primary fields. However, the spectral flow is not necessarily preserved under the fusion. Actually, in n-point function, the spectral flow can violate by at most $n-2$ units. To understand the full set of fusion rules, we must therefore consider these spectral flow-violating correlation functions.

The simplest non-trivial case is a three-point function where the total spectral flow is $\sum_i \omega_i = \pm 1$. Let us consider the case where one field is flowed, e.g., $\langle \phi^{j_1}(z_1)\phi^{j_2}(z_2)\phi^{j_3,-1}(z_3)\rangle$.

The J_0^0 global Ward identities now leads to the following selection rule:

$$m_1 + m_2 + m_3 + \frac{k}{2} = 0. \tag{2.53}$$

Since $J_0^+\phi^{j_3,-1}(z) = \rho_{-1}\left(J_1^+\phi^{j_3}(z)\right) = 0$, the J_0^+ global Ward identity now gives

$$\langle J_0^+\phi^{j_1}(z_1)\phi^{j_2}(z_2)\phi^{j_3,-1}(z_3)\rangle + \langle \phi^{j_1}(z_1)J_0^+\phi^{j_2}(z_2)\phi^{j_3,-1}(z_3)\rangle = 0. \tag{2.54}$$

The J_0^- global Ward identity now gives a more non-trivial constraint. By solving these relations, the final result of the spectral flow violating 3-point function in m-basis is :

$$\left\langle \prod_i \phi^{j_i,\omega_i}_{m_i,\bar{m}_i}(z_i)\right\rangle_{\sum\omega_i=-1} = \left| z_{12}^{\Delta^{j_3,\omega_3}_{m_3}-\Delta^{j_1,\omega_1}_{m_1}-\Delta^{j_2,\omega_2}_{m_2}} z_{23}^{\Delta^{j_1,\omega_1}_{m_1}-\Delta^{j_2,\omega_2}_{m_2}-\Delta^{j_3}_{m_3},\omega_3} z_{31}^{\Delta^{j_2,\omega_2}_{m_2}-\Delta^{j_3,\omega_3}_{m_3}-\Delta^{j_1,\omega_1}_{m_1}}\right| \times C(j_1,j_2,j_3). \times \delta(\sum_i m_i + \frac{k}{2})\frac{\Gamma(-j_1-m_1)}{\Gamma(\bar{m}_1+j_1+1)}\frac{\Gamma(-j_2-\bar{m}_2)}{\Gamma(m_2+j_2+1)}\frac{\Gamma(-j_3-\bar{m}_3)}{\Gamma(m_3+j_3+1)}. \tag{2.55}$$

The Gamma functions are the solution to the global Ward identities. They will be used to extract the fusion rules of the theory.

3 Degenerate representations

In addition the representations that appear in the spectrum, there exist a discrete set of representations known as the degenerate representations. Degenerate representations give constraints on the structure constant and hence play an essential role in making the model solvable.

The defining feature of a degenerate representation is the vanishing of null vectors (or null states) within its corresponding Verma module. A descendant state $|\chi\rangle = \hat{N} \left| j_{\hat{N}} \right\rangle$ is called a null vector if it is also an affine primary state, which means that it is annihilated by all the positive mode currents:

$$J^a_{n>0} |\chi\rangle = 0, \tag{3.1}$$

The existence of such a state implies that the representation $\widehat{\mathcal{V}}^j$ is reducible, as the null vector and its descendants form a non-trivial submodule $\widehat{\mathcal{V}}'$. The irreducible representation $\widehat{\mathcal{R}}^j$ is then obtained by taking the quotient of the Verma module by this submodule:

$$\widehat{\mathcal{R}}^j = \frac{\widehat{\mathcal{V}}^j}{\widehat{\mathcal{V}}'} \tag{3.2}$$

At the level of fields, the field $\phi^{j_{\hat{N}}}$ corresponding to $\left| j_{\hat{N}} \right\rangle$ is a degenerate field, which has a vanishing descendant

$$\hat{N} \phi^{j_{\hat{N}}} = 0. \tag{3.3}$$

This null vector equation leads to differential equations for the correlation functions involving the corresponding degenerate field, then giving constraints on the correlation functions and the structure constants.

The degenerate representations $\widehat{\mathcal{R}}^{\langle r,s \rangle}$ are labeled by two integer r and s. The spin $j_{r,s}$ of degenerate representation $\widehat{\mathcal{R}}^{\langle r,s \rangle}$ is given by the following formula:

$$j_{r,s} = \frac{s-1}{2} - \frac{k+2}{2} r \quad \text{for} \quad s \geq 1, r \geq 0. \tag{3.4}$$

The corresponding null vector is at level $N = rs$.

3.1 Level 0 null vector

The degenerate representation with a null vector at level 0 is of spin $j_s \equiv j_{0,s} = \frac{s-1}{2}$, which is nothing but the affine highest-weight extension of the finite dimensional representations. The $\widehat{\mathcal{R}}^{\langle 0,s \rangle}$ contains a trivial null vector

$$\left(J_0^-\right)^{2j_s+1} |j_s, j_s\rangle = 0, \tag{3.5}$$

which means nothing but

$$J_0^- |j_s, -j_s\rangle = 0. \tag{3.6}$$

3.2 Level 1 null vector

The only possibility to have a level 1 null vector is $r = s = 1$, with $j_{1,1} = -\frac{k+2}{2}$. The corresponding null vector is given in [5]:

$$\hat{N}^c_{1,1} = K_{ab} J^a_{-1} J^b_0 J^c_0 + j_{1,1} f^c_{ab} J^a_{-1} J^b_0 - 2 j^2_{1,1} J^c_{-1}. \tag{3.7}$$

We have no constraint on the m of the null vector, hence the level 1 degenerate representation could belong to either principle continuous series or discrete series. This null vector satisfies the null vector equation (3.1):

$$\begin{aligned} J_1^d, \hat{N}_{-1}^c \left|j_{1,1}, m\right\rangle &= \left(K_{ab}\left[J_1^d, J_{-1}^a\right] J_0^b J_0^c + j_{1,1} f_{ab}^c \left[J_1^d, J_{-1}^a\right] J_0^b - 2j_{1,1}^2 \left[J_1^d, J_{-1}^c\right]\right) \left|j_{1,1}, m\right\rangle \\ &= (2+k+2j_{1,1}) J_0^d J_0^c + (2j_{1,1} + kj_{1,1} + 2j_{1,1}^2) f_e^{ca} - K^{ac} j_{1,1}(K_{eb} J_0^e J_0^b - 2kj_{1,1}) \\ &= 0, \end{aligned} \tag{3.8}$$

where we use an identity $f_e^{ab} f_d^{ec} = 2(K_d^a K^{bc} - K^{ac} K_d^b)$ that holds for $\mathfrak{sl}_2$, and the definition of Casimir operator $C = K_{ab} J_0^a J_0^b = 2j_{1,1}(j_{1,1}+1)$. On the other hand, the commutation relation between J_0^a and $\hat{N}_{1,1}^b$ is

$$\left[J_0^a, \hat{N}_{1,1}^b\right] = f_c^{ab} \hat{N}_{1,1}^c. \tag{3.9}$$

It implies the null vector $\hat{N}_{1,1}^c \left|j_{1,1}\right\rangle$ generates a subrepresentation of $\widehat{\mathfrak{sl}_2}$.

Subrepresentations

The states $\hat{N}_{-1}^+ \left|j, m-1\right\rangle$, $\hat{N}_{-1}^0 \left|j, m\right\rangle$, and $\hat{N}_{-1}^- \left|j, m+1\right\rangle$ all have the same J_0^0 eigenvalue, namely m. Since the subrepresentation generated by the null vector is expected to be irreducible, these states should be linearly dependent. These states can be expanded in the basis of $J_{-1}^a \left|j_{1,1}, m\right\rangle$.

$$\begin{aligned} \hat{N}_{-1}^+ \left|j_{1,1}, m-1\right\rangle =& ((j_{1,1} - m + 1)(j_{1,1} + m) + 2j_{1,1}(m-1) - 2j_{1,1}^2)\, J_{-1}^+ \left|j_{1,1}, m-1\right\rangle \\ &+ (2(j_{1,1} - m + 1)m - 2j_{1,1}(j_{1,1} - m + 1))\, J_{-1}^0 \left|j_{1,1}, m\right\rangle \\ &+ (j_{1,1} - m + 1)(j_{1,1} - m)\, J_{-1}^- \left|j_{1,1}, m+1\right\rangle \\ =& (j_{1,1} + 1 + m)(-j_{1,1} - m)\Big(J_{-1}^+ \left|j_{1,1}, m-1\right\rangle + 2J_{-1}^0 \left|j_{1,1}, m\right\rangle - J_{-1}^{-1} \left|j_{1,1}, m+1\right\rangle\Big) \end{aligned} \tag{3.10}$$

We define

$$\left|N\right\rangle \equiv J_{-1}^+ \left|j, m-1\right\rangle + 2J_{-1}^0 \left|j, m\right\rangle - J_{-1}^{-1} \left|j, m+1\right\rangle \tag{3.11}$$

We find the other two states to be

$$\hat{N}_{-1}^0 \left|j_{1,1}, m\right\rangle = -(j_{1,1} - m)(m + j_{1,1}) N_{-1} \left|j_{1,1}, m\right\rangle \tag{3.12}$$

$$\hat{N}_{-1}^- \left|j_{1,1}, m+1\right\rangle = (j_{1,1} + m)(j_{1,1} + m + 1) N_{-1} \left|j_{1,1}, m\right\rangle \tag{3.13}$$

Hence all these three states are propotional to each other.

To find the spin of the subrepresentation, we first calculate the eigenstates of the Casimir operator. The commutation relation of Casimir operator $C = K_{ab} J_0^a J_0^b$ with J_{-1}^c is

$$\left[C, J_{-1}^c\right] = K_{ab}\left[J_0^a J_0^b, J_{-1}^c\right] = -f_{bd}^c J_{-1}^d J_0^b - f_{bd}^c J_0^b J_{-1}^d \tag{3.14}$$

The action of C on basis $J_{-1}^a \left|j, m\right\rangle$ can be writen as the following matrix:

$$C \begin{pmatrix} J_{-1}^+ \left|j, m-1\right\rangle \\ J_{-1}^0 \left|j, m\right\rangle \\ J_{-1}^- \left|j, m+1\right\rangle \end{pmatrix} = \begin{pmatrix} 4m + 2j(j+1) & -2(-j-1+m) & 0 \\ -4(-j-m) & 4 + 2j(j+1) & 4(-j+m) \\ 0 & 2(-j-1-m) & -4m + 2j(j+1) \end{pmatrix} \begin{pmatrix} J_{-1}^+ \left|j, m-1\right\rangle \\ J_{-1}^0 \left|j, m\right\rangle \\ J_{-1}^- \left|j, m+1\right\rangle \end{pmatrix} \tag{3.15}$$

After diagonalization, we find the eigenvectors and the corresponding eigenvalues are

$$\begin{cases} j+1: -\dfrac{j+1-m}{j+m} J_{-1}^{+} \left|j, m-1\right\rangle + 2J_{-1}^{0} \left|j, m\right\rangle + \dfrac{j+1+m}{j-m} J_{-1}^{-} \left|j, m+1\right\rangle \\ \quad\;\; j: \dfrac{-j-1+m}{m} J_{-1}^{+} \left|j, m-1\right\rangle + 2J_{-1}^{0} \left|j, m\right\rangle + \dfrac{-j-1-m}{m} J_{-1}^{-} \left|j, m+1\right\rangle \\ j-1: J_{-1}^{+} \left|j, m-1\right\rangle + 2J_{-1}^{0} \left|j, m\right\rangle - J_{-1}^{-} \left|j, m+1\right\rangle = \left|N\right\rangle \end{cases} \tag{3.16}$$

We find that $N_{-1}\left|j, m\right\rangle$ is exactly the eigenvector corresponding to $j-1$. Hence the subrepresentation has spin $j_{1,1}-1 = j_{1,-1}$.

3.3 Spectral flowed degenerate representations

For a given null vector $\left|\chi\right\rangle$, we could naturally define the spectral flowed null vector $\left|\chi, \omega\right\rangle$. It satisfies the flowed null vector equation automatically:

$$\rho_{\omega}\left(J_{n>0}^{a}\right)\left|\chi, \omega\right\rangle = 0. \tag{3.17}$$

We define a representation to be a spectral flowed degenerate representation if it contains a spectral flowed null vector, which can be obtained by applying spectral flow to the degenerate representations.

Since

$$\rho_{-1}\left(\widehat{\mathcal{D}}^{j,+}\right) = \widehat{\mathcal{D}}^{\frac{k}{2}-j,-}, \tag{3.18}$$

the spectral flowed degenerate representation $\rho_{-1}\left(\widehat{\mathcal{D}}^{\langle r,s\rangle,+}\right)$ is also a affine highest-weight representation. On the other hand, we have

$$\frac{k}{2} - j_{r,s} = \frac{k}{2} - \frac{s-1}{2} + \frac{k+2}{2} r = \frac{-s-1}{2} + \frac{k+2}{2}(r+1) = -j_{r+1,s} - 1. \tag{3.19}$$

Hence we find

$$\rho_{-1}\left(\widehat{\mathcal{D}}^{\langle r,s\rangle,+}\right) = \widehat{\mathcal{D}}^{\langle r+1,s\rangle,-}, \tag{3.20}$$

which means that the affine highest-weight degenerate representation $\widehat{\mathcal{D}}^{\langle r,s\rangle,+}$ is itself a spectral flowed degenerate representation.

spectral flowed vacuum representation

Now we consider the spectral flow of the vacuum representation, i.e. of the degenerate representation $\hat{\mathcal{E}}^{1}$. The vacuum representation is essential to the fusion rules since the fusion between any representation $\hat{\mathcal{R}}^{j}$ with the vacuum representation should give $\hat{\mathcal{R}}^{j}$ itself back:

$$\hat{\mathcal{E}}^{1} \times \hat{\mathcal{R}}^{j} = \hat{\mathcal{R}}^{j}. \tag{3.21}$$

Hence from our assumption (2.33), we find

$$\rho_{n}\left(\hat{\mathcal{E}}^{1}\right) \times \hat{\mathcal{R}}^{j} = \rho_{n}\left(\hat{\mathcal{R}}^{j}\right). \tag{3.22}$$

One could view $\hat{\mathcal{E}}^{0,1}$ as

$$\hat{\mathcal{E}}^{0,1} = \hat{\mathcal{D}}^{\langle 0,1\rangle,+} = \hat{\mathcal{D}}^{\langle 0,1\rangle,-} \tag{3.23}$$

which implies the following relation:

$$\begin{aligned}\rho_1\left(\hat{\mathcal{E}}^{0,1}\right) &= \hat{\mathcal{D}}^{\langle 1,1\rangle,+}\\ \rho_{-1}\left(\hat{\mathcal{E}}^{0,1}\right) &= \hat{\mathcal{D}}^{\langle 1,1\rangle,-}.\end{aligned} \tag{3.24}$$

We can verify that $|0,0,-1\rangle$ is indeed an affine highest-weight state:

$$\begin{aligned}J_1^+|0,0,-1\rangle &= J_2^+|0,0\rangle = 0,\\ J_1^0|0,0,-1\rangle &= J_1^0|0,0\rangle = 0,\\ J_1^-|0,0,-1\rangle &= J_0^-|0,0\rangle = 0,\end{aligned} \tag{3.25}$$

In addition, we have

$$\begin{aligned}J_0^+|0,0,-1\rangle &= J_1^+|0,0\rangle = 0,\\ J_0^0|0,0,-1\rangle &= \left(J_0^0+\frac{k}{2}\right)|0,0\rangle = \frac{k}{2}|0,0\rangle.\end{aligned} \tag{3.26}$$

Hence we find

$$|0,0,-1\rangle = \left|\frac{k}{2},\frac{k}{2}\right\rangle. \tag{3.27}$$

3.4 Spectra of degenerate representations

We plot the spectra of degenerate representations as 3. In these plots, each dot corresponds to eigenvalues of J_0^0 and L_0. The states appears in a representation are denoted by the shadowed area. The null vectors are labeled by the blue dashed line.

- Continuous series degenerate representation $\widehat{\mathcal{C}}_\alpha^{\langle 1,1\rangle}$: The null vector all live at level 1, hence there is a straight line in the 3a. After spectral flow, the spectrum 3b is twisted and the null states are still on one straight line, which is no-more horizontal.

- Discrete series degenerate representation $\widehat{\mathcal{D}}^{\langle 1,1\rangle,\frac{1}{2}}$: The same with $\widehat{\mathcal{C}}_\alpha^{\langle 1,1\rangle}$, we have null vectors at level 1. However, since $\widehat{\mathcal{D}}^{\langle 1,1\rangle,\frac{1}{2}}$ is itself a spectral flowed degenerate representation, we should also find spectral flowed null vectors, which is denoted by a second straight line in 3c. After spectral, we find another degenerate representation $\widehat{\mathcal{D}}^{-1-j_{2,1},-}$ with null vectors at level 2. Diagramatically, the spectrum and the null vectors are 'rotated' simutanuously. in 3d.

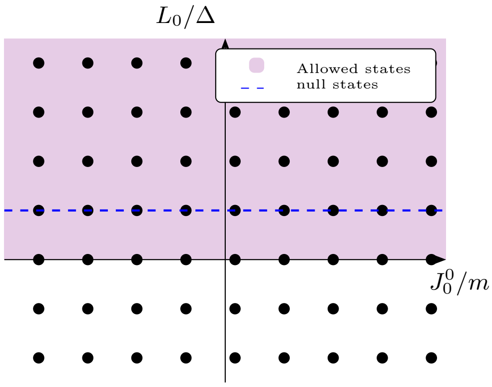
- Vacuum representation $\widehat{\mathcal{E}}^1$: Since the null vectors of the vacuum representation appears at level 0, the dashed line coincide with the axis $L_0=0$. After the spectral flow, the null vector at level one is $J_{-1}^+|-1-j_{1,1},-1-j_{1,1}\rangle$.

4 Fusion rules

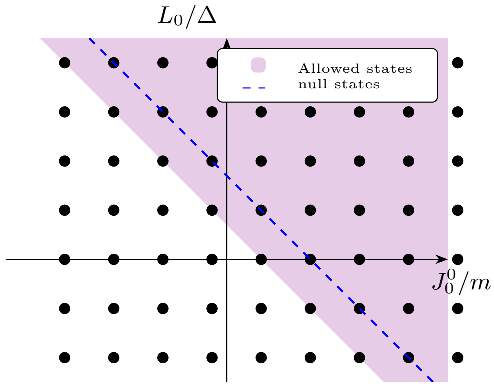
In CFT, fusion rules form the algebra of primary fields, dictating how they combine. We could determine the fusion rules from the OPE

$$\phi^{j_1,\omega_1}(z_1)\phi^{j_2,\omega_2}(z_2) \sim \sum_{j_3,\omega_3}\left\langle \phi^{j_1,\omega_1}(z_1)\phi^{j_2,\omega_2}(z_2)\left(\phi^{j_3,\omega_3}(z_3)\right)^*\right\rangle \phi^{j_3,\omega_3}(z_3). \tag{4.1}$$

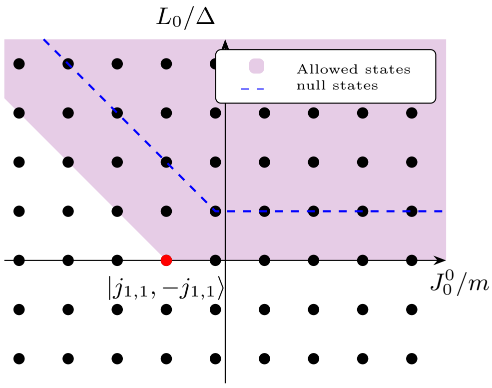
The fusion $\widehat{\mathcal{R}}^{j_1,\omega_1}\widehat{\mathcal{R}}^{j_2,\omega_2} \ni \widehat{\mathcal{R}}^{j_3,\omega_3}$ is allowed only if the 3-point function $\left\langle \phi^{j_1,\omega_1}(z_1)\phi^{j_2,\omega_2}(z_2)\left(\phi^{j_3,\omega_3}(z_3)\right)^*\right\rangle$ is non-zero.



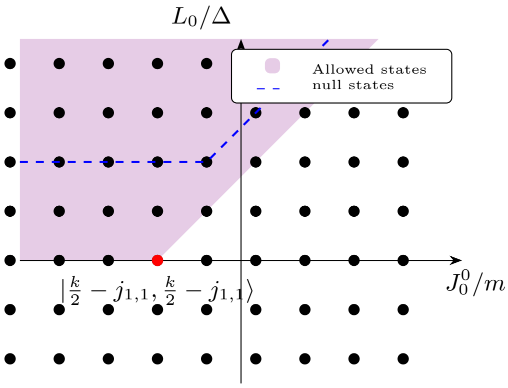
(a) $\widehat{\mathcal{C}}_\alpha^{\langle 1,1\rangle}$



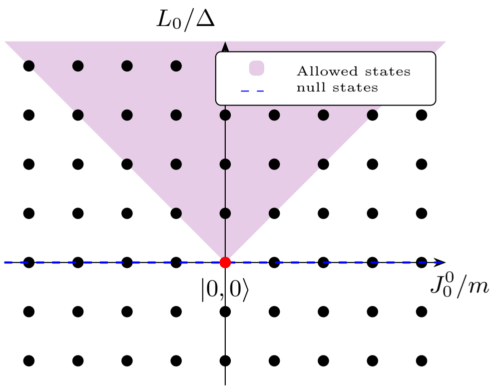
(b) $\widehat{\mathcal{C}}_\alpha^{\langle 1,1\rangle,1}$



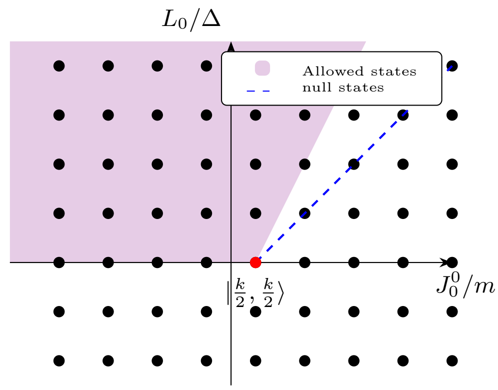
(c) $\widehat{\mathcal{D}}^{\langle 1,1\rangle,\frac{1}{2}}$



(d) $\widehat{\mathcal{D}}^{-1-j_{2,1},-}$



(e) $\widehat{\mathcal{E}}^1$



(f) $\widehat{\mathcal{D}}^{-1-j_{1,1},-}$

Figure 3: Spectra of degenerate representations

On the other hand, the null vector equations give constraint on the 3-point functions involving degenerate fields. In this chapter, we will use the null vector equations to determine the fusion rules of degenerate representations.

4.1 Fusion rules involving degenerate representations

Affine primary fields

The vanishing of null vector $\hat{N}^c_{1,1}\phi^{1,1}_x = 0$ gives the following equation:

$$\left\langle \hat{N}^c_{1,1}\phi^{1,1}_{x_1}(z_1)\phi^{j_2}_{x_2}(z_2)\phi^{j_3}_{x_3}(z_3)\right\rangle = 0. \tag{4.2}$$

This equation involves operators J^a_{-1}. We substitute the local Ward identity (2.43), and get the following differential equation:

$$\sum_{s=2,3}\frac{1}{z_{s1}}\left\{K_{ab}D_{x_s}\left(t^a\right)D_{x_1}\left(t^c\right)D_{x_1}\left(t^b\right)+j_{1,1}f^c_{ab}D_{x_s}\left(t^a\right)D_{x_1}\left(t^b\right)-2j^2_{1,1}D_{x_s}\left(t^c\right)\right\}\left\langle\phi^{1,1}_{x_1}(z_1)\phi^{j_2}_{x_2}(z_2)\phi^{j_3}_{x_3}(z_3)\right\rangle = 0. \tag{4.3}$$

We use the conformal symmetry to send $z_1, z_2, z_3 \to 0, 1, \infty$, where only the term propotional to $\frac{1}{z_{21}}$ survives. Then the null vector equation corresponding to $c = -$ is simplified to

$$\begin{aligned}&\left\{x^2_{12}\partial^2_1\partial_2 + 2j_2x_{12}\partial^2_1 + 2(1-2j_{1,1})x_{12}\partial_1\partial_2\right.\\ &\left.+2(1-2j_{1,1})j_2\partial_1 - 2j_{1,1}(1-2j_{1,1})\partial_2\right\} D^{\mathbf{1}}\begin{bmatrix} j_1 & j_2 & j_3\\ x_1 & x_2 & x_3\end{bmatrix} = 0.\end{aligned} \tag{4.4}$$

By substituting the 3-point function in x-basis (2.48), we get the following condition on the spins:

$$\left(j^2_{1,1} - (j_2 - j_3)^2\right)(1 + j_{1,1} + j_2 + j_3) = 0. \tag{4.5}$$

The solution to this equation is $j_3 = j_2 \pm j_{1,1}, -j_2 - 1 + j_{1,1}$. Since the above equation is symmetric for j_2 and j_3, the other term in (4.3) proportional to $\frac{1}{z_{31}}$ should give the same condition. Therefore we determine the following fusion rule:

$$\hat{\mathcal{R}}^{1,1} \times \hat{\mathcal{R}}^j \sup \hat{\mathcal{R}}^{j+j_{1,1}} + \hat{\mathcal{R}}^{j-j_{1,1}}. \tag{4.6}$$

In addition, the other two null vector equations of $c = 0, +$ can be deduced from the commutation relation $\left[J^a, \hat{N}^b_{1,1}\right] = f^{ab}_c\hat{N}^c_{1,1}$. Hence all three null vector equations give the same fusion rule.

Spectral flowed vacuum representation

The fusion between any representation $\hat{\mathcal{R}}^j$ with the vacuum representation $\widehat{\mathcal{E}}^1$ should give $\hat{\mathcal{R}}^j$ back:

$$\hat{\mathcal{E}}^1 \times \hat{\mathcal{R}}^j = \hat{\mathcal{R}}^j. \tag{4.7}$$

Hence from our conjecture (2.33), we find

$$\rho_{\pm1}\left(\hat{\mathcal{E}}^1\right) \times \hat{\mathcal{R}}^j = \rho_{\pm1}\left(\hat{\mathcal{R}}^j\right). \tag{4.8}$$

Let's prove this fusion rule by using the null vector appears in the spectral flowed vacuum representation, namely $J^-_{-1}\left|-j_{1,1}-1, j_{1,1}+1\right\rangle = 0$. The corresponding null vector equation in m-basis is

$$\left\langle J^-_{-1}\phi^{-j_{1,1}-1}_{j_{1,1}+1}(z_1)\phi^{j_2}_{m_2}(z_2)\phi^{j_3}_{m_3}(z_3)\right\rangle = 0. \tag{4.9}$$

By using the local Ward identity, we obtain

$$\begin{aligned}
0 =& \left\langle J^-_{-1}\phi^{j_1}_{m_1}(z_1)\phi^{j_2}_{m_2}(z_2)\phi^{j_3}_{m_3}(z_3)\right\rangle \\
=& -\frac{1}{z_{21}}\left\langle \phi^{j_1}_{m_1}(z_1)J^-_0\phi^{j_2}_{m_2}(z_2)\phi^{j_3}_{m_3}(z_3)\right\rangle \\
& -\frac{1}{z_{31}}\left\langle \phi^{j_1}_{m_1}(z_1)\phi^{j_2}_{m_2}(z_2)\rho_{-1}\left(J^-_{-1}\right)\phi^{j_3}_{m_3}(z_3)\right\rangle + \frac{1}{(z_{31})^2}\left\langle \phi^{j_1}_{m_1}(z_1)\phi^{j_2}_{m_2}(z_2)\rho_{-1}\left(J^-_{0}\right)\phi^{j_3}_{m_3}(z_3)\right\rangle .
\end{aligned} \tag{4.10}$$

Substituting the global Ward identity (2.42) to the above null vector equation, we find the null vector equation is simplified to

$$\left(\frac{1}{z_{21}}-\frac{1}{z_{31}}\right)\left\langle \phi^{j_1}_{m_1}(z_1)J^-_0\phi^{j_2}_{m_2}(z_2)\phi^{j_3}_{m_3}(z_3)\right\rangle - \frac{1}{(z_{31})^2}\left\langle \phi^{j_1}_{m_1}(z_1)\phi^{j_2}_{m_2}(z_2)\rho_{-1}\left(J^-_{0}\right)\phi^{j_3}_{m_3}(z_3)\right\rangle = 0. \tag{4.11}$$

Substitute the spectral flow violating 3-point function (2.55). Since $\Delta^{j_3,-1}_{m3-1} = \Delta^{j_3,-1}_{m_3} - 1$, we find the z-dependence of these two terms is exactly the same. The equation is simplified to

$$(-j_2-1+m_2)(m_2+j_2) - (-j_3-1+m_3)(m_3+j_3) = 0. \tag{4.12}$$

Since the m is conserved, we have $m_3 = -m_2+1$. We find the above equation has only two solutions $j_3 = j_2, -j_2-1$. Hence we show that

$$\rho_1\left(\widehat{\mathcal{E}}^1\right)\times\widehat{\mathcal{R}}^j = \rho_1\left(\widehat{\mathcal{R}}^j\right). \tag{4.13}$$

Generic spectral flowed representations

The above fusion rule involving vacuum representations can be generalized to generic level 1 degenerate representations. The null vector at level 1 gives the following equation:

$$\left\langle J^+_{-1}\phi^{\langle 1,1\rangle}_{m_1-1}\phi^{j_2}_{m_2}(z_2)\phi^{j_3,-1}_{m_3}(z_3)\right\rangle + 2\left\langle J^0_{-1}\phi^{\langle 1,1\rangle}_{m_1}\phi^{j_2}_{m_2}(z_2)\phi^{j_3,-1}_{m_3}(z_3)\right\rangle - \left\langle J^-_{-1}\phi^{\langle 1,1\rangle}_{m_1+1}\phi^{j_2}_{m_2}(z_2)\phi^{j_3,-1}_{m_3}(z_3)\right\rangle = 0. \tag{4.14}$$

Substituting the local Ward identity, the equation is equivalent to

$$\begin{aligned}
0 =& \frac{1}{z_{21}}\left\langle \phi^{\langle 1,1\rangle}_{m_1-1}J^+_0\phi^{j_2}_{m_2}(z_2)\phi^{j_3,-1}_{m_3}(z_3)\right\rangle + 2\frac{1}{z_{21}}\left\langle \phi^{\langle 1,1\rangle}_{m_1}J^0_0\phi^{j_2}_{m_2}(z_2)\phi^{j_3,-1}_{m_3}(z_3)\right\rangle \\
& + 2\frac{1}{z_{31}}\left\langle \phi^{\langle 1,1\rangle}_{m_1}\phi^{j_2}_{m_2}(z_2)J^0_0\phi^{j_3,-1}_{m_3}(z_3)\right\rangle - \frac{1}{z_{21}}\left\langle \phi^{\langle 1,1\rangle}_{m_1+1}J^-_0\phi^{j_2}_{m_2}(z_2)\phi^{j_3,-1}_{m_3}(z_3)\right\rangle \\
& - \frac{1}{z_{31}}\left\langle \phi^{\langle 1,1\rangle}_{m_1+1}\phi^{j_2}_{m_2}(z_2)\rho_{-1}\left(J^-_{-1}\right)\phi^{j_3,-1}_{m_3}(z_3)\right\rangle + \frac{1}{(z_{31})^2}\left\langle \phi^{\langle 1,1\rangle}_{m_1+1}J^-_0\phi^{j_2}_{m_2}(z_2)\rho_{-1}\left(J^-_{0}\right)\phi^{j_3,-1}_{m_3}(z_3)\right\rangle .
\end{aligned} \tag{4.15}$$

Substituting the 3-point function, again we find the z-dependence cancels and the equation reduces to

$$(-j_{1,1}-1+m_1+2m_2)(-j_{1,1}+m_1) - (m_3-m_2)((1-m_3-m_2)) = 0. \tag{4.16}$$

Since we have $j_{1,1} = -\frac{k+2}{2}$ and $m_1 + m_2 + m_3 + \frac{k}{2}$, we find the above equation is simplified to :

$$j_2(j_2+1) - j_3(j_3+1) = 0. \tag{4.17}$$

We find two solutions: $j_2 = j_3, -j_3 - 1$, which gives the following fusion rule:

$$\widehat{\mathcal{R}}^{\langle 1,1\rangle} \times \widehat{\mathcal{R}}^{j} \supset \rho_1\left(\widehat{\mathcal{R}}^{j}\right). \tag{4.18}$$

The fusion rule should be symmetric for $\rho_{\pm 1}$, hence we should also have

$$\widehat{\mathcal{R}}^{\langle 1,1\rangle} \times \widehat{\mathcal{R}}^{j} \supset \rho_{-1}\left(\widehat{\mathcal{R}}^{j}\right). \tag{4.19}$$

In conclusion, we conjecture the fusion rule between a degenerate representation and a generic affine highest weight representation to be:

$$\boxed{\widehat{\mathcal{R}}^{\langle 1,1\rangle} \times \widehat{\mathcal{R}}^{j} = \widehat{\mathcal{R}}^{j+j_{1,1}} \oplus \widehat{\mathcal{R}}^{j-j_{1,1}} \oplus \rho_1\left(\widehat{\mathcal{R}}^{j}\right) \oplus \rho_{-1}\left(\widehat{\mathcal{R}}^{j}\right).} \tag{4.20}$$

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