Degenerate Representations and Fusion Rules in the $\widetilde{SL}_2(\mathbb{R})$ WZW Model

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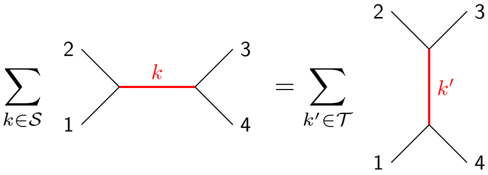
Introduction

String theory in AdS$_3$
Quantum Gravity

$\longrightarrow$

$\widetilde{SL}_2(\mathbb{R})$ WZW model
CFT with an $\widehat{\mathfrak{sl}}_2$ symmetry

The crossing symmetry equation:



Degenerate representations $\Longrightarrow$ Finite many terms

Symmetry algebra

- The $\mathfrak{sl}_2$ algebra:

$$\left[J^0, J^{\pm}\right] = \pm J^{\pm}, \quad \left[J^+, J^-\right] = 2J^0.$$

- The $\widehat{\mathfrak{sl}_2}$ algebra are defined by:

$$\left[J^a_m, J^b_n\right] = f^{ab}_c J^c_{m+n} + kmK^{ab}\delta_{m+n,0}, \quad a, b \in \{0, \pm\}, m, n \in \mathbb{Z}$$

where k is a constant, and $K^{ab} = \frac{1}{2} f^{ac}_d f^{bd}_c$ is the Killing tensor.

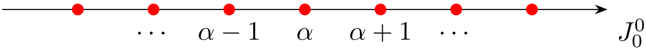
- The Sugawara construction of the dilation operator:

$$L_0 = \frac{K_{ab}}{2(k-2)} \left(J^a_0 J^b_0 + 2 \sum_{m>0} J^a_{-m} J^b_m \right).$$

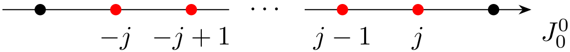
Irreducible representations of $\mathfrak{sl}_2$

We characterize different representations by the spin j and the eigenvalue m of J_0^0:

- $\mathcal{C}_\alpha^j$: $j \in -\frac{1}{2} + i\mathbb{R}_+$, $m \in \alpha + \mathbb{Z}$.



- $\mathcal{E}^j$: $j \in \mathbb{N}/2$, $m \in \{-j, -j+1, \cdots, j-1, j\}$.

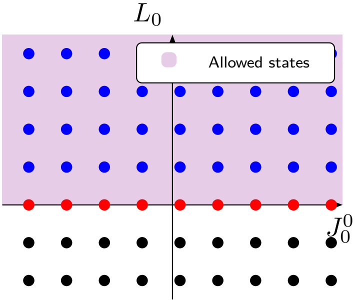


$\mathcal{E}^j$ is a level 0 degenerate representation, with the following null vector:

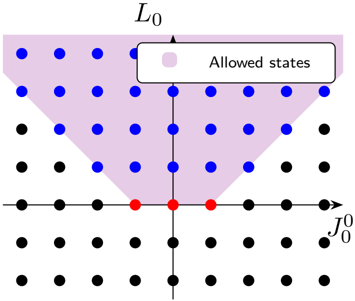
$$J_0^- |j, -j\rangle = 0.$$

Affine highest-weight representations of $\widehat{\mathfrak{sl}_2}$

$$\boxed{\mathcal{R}^j \text{ of } \mathfrak{sl}_2} \xrightarrow[J^a_{-n},\, n>0]{\text{Acting creation operators}} \boxed{\widehat{\mathcal{R}}^j \text{ of } \widehat{\mathfrak{sl}_2}}$$



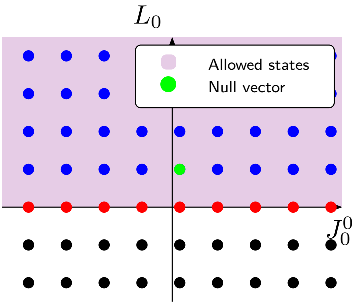
(a) $\widehat{\mathcal{C}}^j_\alpha$



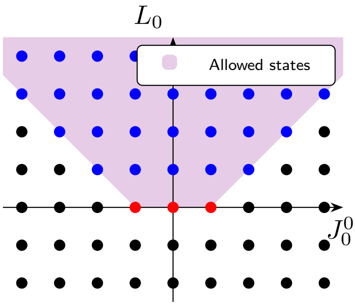
(b) $\widehat{\mathcal{E}}^j$

Affine highest-weight representations of $\widehat{\mathfrak{sl}_2}$

$$\boxed{\mathcal{R}^j \text{ of } \mathfrak{sl}_2} \xrightarrow[J^a_{-n},\, n>0]{\text{Acting creation operators}} \boxed{\widehat{\mathcal{R}}^j \text{ of } \widehat{\mathfrak{sl}_2}}$$



(a) $\widehat{\mathcal{C}}^j_\alpha$



(b) $\widehat{\mathcal{E}}^j$

Spectral flow

- The spectral flow is a family $(\rho_\omega)_{\omega\in\mathbb{Z}}$ of automorphisms of $\widehat{\mathfrak{sl}_2}$:

$$\rho_\omega(J_m^\pm) = J_{m\pm\omega}^\pm,$$
$$\rho_\omega(J_m^0) = J_m^0 + \frac{1}{2}k\omega\delta_{m,0}.$$

- The spectral flowed representation $\rho_\omega\left(\widehat{\mathcal{R}}\right)$ is related to $\widehat{\mathcal{R}}$ through

$$J_n^a \left|v,\omega\right\rangle = \rho_{-\omega}\left(J_n^a\right)\left|v\right\rangle.$$

My contributions

- The structure of spectral flowed degenerate representation
  - Spectral flowed vacuum representation
- Fusion rules with degenerate representations
  - level 0
  - level 1

Fusion rules

- The fusion rules are determined from the OPE

$$\phi^{|j_1,m_1,\omega_1\rangle}(z_1)\phi^{|j_2,m_2,\omega_2\rangle}(z_2) \sim \sum_{j_3,\omega_3} \left\langle \phi^{|j_1,m_1,\omega_1\rangle}(z_1)\phi^{|j_2,m_2,\omega_2\rangle}(z_2) \left(\phi^{|j_3,m_3,\omega_3\rangle}(z_3)\right)^{\dagger} \right\rangle \phi^{|j_3,m_3,\omega_3\rangle}(z_3).$$

$$\left\langle \phi^{|j_1,m_1,\omega_1\rangle}(z_1)\phi^{|j_2,m_2,\omega_2\rangle}(z_2) \left(\phi^{|j_3,m_3,\omega_3\rangle}(z_3)\right)^{\dagger} \right\rangle \neq 0$$

$$\Longleftrightarrow \widehat{\mathcal{R}}^{j_1,\omega_1} \times \widehat{\mathcal{R}}^{j_2,\omega_2} \supset \widehat{\mathcal{R}}^{j_3,\omega_3}$$

- The fusion rules involving degenerate representations is constrained by the null vector equations:

$$\hat{N}\,|j,m,\omega\rangle = 0 \Longrightarrow \left\langle \hat{N}\phi^{|j,m,\omega\rangle}(z) \prod_i \phi^{|j_i,m_i,\omega_i\rangle}(z_i) \right\rangle = 0.$$

Spectral flowed vacuum representation

- $\rho_{\pm 1}\left(\widehat{\mathcal{E}}^0\right)$ is a level 1 degenerate representation, with the corresponding null vector:

$$J_{-1}^{\mp 1}\left|k/2, \mp k/2\right\rangle = 0.$$

- The corresponding null vector equation:

$$\left\langle J_{-1}^{\pm}\phi^{|k/2,\pm k/2\rangle}(z_1)\phi^{|j_2,m_2\rangle}(z_2)\phi^{|j_3,m_3,\mp 1\rangle}(z_3)\right\rangle = 0,$$

$$\Longrightarrow j_2(j_2+1) = j_3(j_3+1)$$

- We have the following fusion rule:

$$\rho_{\pm 1}\left(\widehat{\mathcal{E}}^0\right) \times \widehat{\mathcal{R}}^j = \rho_{\pm 1}\left(\widehat{\mathcal{R}}^j\right).$$

Fusion rules with spectral flowed representations

- We conjecture that the spectral flow commutes with fusion,

$$\rho_{\omega_1}\left(\widehat{\mathcal{R}}^{j_1}\right)\times\rho_{\omega_2}\left(\widehat{\mathcal{R}}^{j_2}\right)=\rho_{\omega_1+\omega_2}\left(\widehat{\mathcal{R}}^{j_1}\times\widehat{\mathcal{R}}^{j_2}\right).$$

- The fusion rule with $\rho_{\pm1}\left(\widehat{\mathcal{E}}^{0}\right)$ proves the simplest case:

$$\begin{aligned}\rho_{\pm1}\left(\widehat{\mathcal{R}}^{j_1}\right)\times\widehat{\mathcal{R}}^{j_2}&=\left(\rho_{\pm1}\left(\widehat{\mathcal{E}}^{0}\right)\times\widehat{\mathcal{R}}^{j_1}\right)\times\widehat{\mathcal{R}}^{j_2}\\&=\rho_{\pm1}\left(\widehat{\mathcal{E}}^{0}\right)\times\left(\widehat{\mathcal{R}}^{j_1}\times\widehat{\mathcal{R}}^{j_2}\right)\\&=\rho_{\pm1}\left(\widehat{\mathcal{R}}^{j_1}\times\widehat{\mathcal{R}}^{j_2}\right).\end{aligned}$$

Level 1 degenerate representations

- Degenerate representations $\widehat{\mathcal{R}}^{\langle r,s\rangle}$ are labeled by two integers r and s for $r \geq 0, s \geq 1$, with the corresponding null vector at level $N = rs$.

$$j_{\langle r,s\rangle} = \frac{s-1}{2} - \frac{k+2}{2} r.$$

- The null vector for level 1 degenerate representation is given by the following null operator:

$$\hat{N}^c = K_{ab} J^a_{-1} J^b_0 J^c_0 + j_{\langle 1,1\rangle} f^c_{ab} J^a_{-1} J^b_0 - 2 j^2_{\langle 1,1\rangle} J^c_{-1}.$$

Fusion with level 1 degenerate representation

- Let's first consider the spectral flow preserving case. The null vector equation gives

$$\left\langle \hat{N}^c \phi^{|j_{\langle 1,1\rangle}, m_1\rangle}(z_1) \phi^{|j_2, m_2\rangle}(z_2) \phi^{|j_3, m_3\rangle}(z_3) \right\rangle = 0.$$

$$\Longrightarrow \left(j_{\langle 1,1\rangle}^2 - (j_2 - j_3)^2 \right) (1 + j_{\langle 1,1\rangle} + j_2 + j_3) = 0.$$

The solution to this equation is $j_3 = j_2 \pm j_{\langle 1,1\rangle}, -j_2 - j_{\langle 1,1\rangle} - 1$.

- We find the following fusion rule:

$$\widehat{\mathcal{R}}^{\langle 1,1\rangle} \times \widehat{\mathcal{R}}^{j} \supset \widehat{\mathcal{R}}^{j + j_{\langle 1,1\rangle}} \oplus \widehat{\mathcal{R}}^{j - j_{\langle 1,1\rangle}}.$$

Fusion with level 1 degenerate representation

- Next, let's consider the spectral flow violating case. The null vector equation becomes

$$\left\langle \hat{N}^c \phi^{|j_{\langle 1,1\rangle}, m_1\rangle}(z_1) \phi^{|j_2, m_2\rangle}(z_2) \phi^{|j_3, m_3, \pm 1\rangle}(z_3) \right\rangle = 0.$$

$$\Longrightarrow j_2(j_2+1) = j_3(j_3+1)$$

- We find:

$$\widehat{\mathcal{R}}^{\langle 1,1\rangle} \times \widehat{\mathcal{R}}^{j} \supset \rho_1\left(\widehat{\mathcal{R}}^{j}\right) \oplus \rho_{-1}\left(\widehat{\mathcal{R}}^{j}\right).$$

- In conclusion, we find the fusion rule with the level 1 degenerate representation to be:

$$\widehat{\mathcal{R}}^{\langle 1,1\rangle} \times \widehat{\mathcal{R}}^{j} = \widehat{\mathcal{R}}^{j+j_{\langle 1,1\rangle}} \oplus \widehat{\mathcal{R}}^{j-j_{\langle 1,1\rangle}} \oplus \rho_1\left(\widehat{\mathcal{R}}^{j}\right) \oplus \rho_{-1}\left(\widehat{\mathcal{R}}^{j}\right).$$

Conclusion and Outlook

- The fusion rules with degenerate representations are essential to solve the $\widetilde{SL}_2(\mathbb{R})$ WZW model.
- We determined the fusion rules with spectral flowed vacuum representation and level 1 degenerate representations.
- In the future, one possible generalizing of our results is to extend the fusion rules with spectral flowed vacuum representations to generic $\omega \in \mathbb{Z}$.

Thank You !