Degenerate Representations and Fusion Rules in the $\widetilde{SL}_2(\mathbb{R})$ WZW Model

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Contents

String theory on AdS_3

- The AdS/CFT correspondence is a powerful tool, which relats gravity in an Anti-de Sitter (AdS) spacetime with a conformal field theory (CFT) that lives on its boundary.

- We are interested in string theory on AdS_3, since:
  - Understanding string theory in AdS_3 enables us to study the AdS/CFT correspondence beyond gravity approximation.
  - It describes string theory on a curved spacetime, where the g_{00} component of metric is non-trivial.

- String theory on AdS_3 is described by the $\widetilde{SL}_2(\mathbb{R})$ Wess-Zumino-Witten (WZW) model.

$\widetilde{SL}_2(\mathbb{R})$ WZW model

- The $\widetilde{SL}_2(\mathbb{R})$ is a conformal field theory(CFT) with an $\widehat{\mathfrak{sl}}_2$ symmetry.
- The model has not been completely solved. The three-point functions are not fully known, and crossing symmetry has not been proved yet.
- In order to use conformal bootstrap to solve this model, we need the fusion rules for the representations of $\widehat{\mathfrak{sl}}_2$.

In this presentation, we are going to give some of the fusion rules between degenerate representations and affine highest representations.

Symmetry algebra

- The $\widehat{\mathfrak{sl}}_2$ algebra is generated by 3 holomorphic currents $J^a(z)$ through their operator product expansions (OPE),

$$J^a(z)J^b(w) = \frac{kK^{ab}}{(z-w)^2} + \frac{f^{ab}_c J^c(w)}{z-w} + \mathcal{O}(1), \tag{1}$$

  where the constant k is the level, K^{ab} is the Killing tensor $K^{ab} = \frac{1}{2g} f^{ac}_d f^{bd}_c$.

- The generators of $\widehat{\mathfrak{sl}}_2$ algebra are defined as $J^a_n = \oint \mathrm{d}z\, z^n J^a(z)$. We deduce the following commutation relations from the OPE

$$\left[J^a_m, J^b_n\right] = f^{ab}_c J^c_{m+n} + mkK^{ab}\delta_{m+n,0}. \tag{2}$$

- We introduce the Sugawara construction of the Virasoro generators:

$$L_n = \frac{K_{ab}}{2(k-2)} : \sum_{m\in\mathbb{Z}} J^a_{n-m} J^b_m : . \tag{3}$$

Spectral flow

- The spectral flow is a family $(\rho_\omega)_{\omega\in\mathbb{Z}}$ of automorphisms of $\widehat{\mathfrak{sl}}_2$ satisfying $\rho_{\omega_1}\circ\rho_{\omega_2}=\rho_{\omega_1+\omega_2}$, which are defined by

$$\begin{aligned}\rho_\omega(J^\pm_m) &= J^\pm_{m\pm\omega},\\ \rho_\omega(J^0_m) &= J^0_m+\frac{1}{2}k\omega\delta_{m,0}.\end{aligned} \tag{4}$$

- The spectral flow of Virasoro generators is

$$\rho_\omega(L_m)=L_m+\omega J^0_m+\frac{1}{4}kn^2\delta_{m,0}. \tag{5}$$

Affine primary fields

- An affine primary field $\phi^j(z)$ associated with representation $\mathcal{R}^j$ is defined by its OPE with current field $J^a(y)$:

$$J^a(y)\phi^j(z) \sim \frac{-(t^a)^T \phi^j(z)}{y-z} + \mathcal{O}(1), \tag{6}$$

where t^a is the generator of Lie algebra $\mathfrak{sl}_2$.

- The conformal dimension of $\phi^j(z)$ is propotional to the Casimir operator $C = K_{ab}t^a t^b = 2j(j+1)$:

$$\Delta_j = \frac{C(j)}{2(k-2)} = \frac{j(j+1)}{k-2}. \tag{7}$$

Isospin variables

- We introduce the isospin variables and represent the fields as functions of the isospin variables, where t^a acts on primary fields as differential operators $D^j(t^a)$.
- A field is represented as a function ϕ^j_x of x, and t^a acts as

$$\begin{cases} D^j_x(t^+) = x^2\partial_x - 2jx, \\ D^j_x(t^0) = x\partial_x - j, \\ D^j_x(t^-) = -\partial_x. \end{cases} \tag{8}$$

m-basis

- Another important basis is the m-basis, where J_0^0 is diagonalized. J_0^a act on the m-basis fields as

$$\begin{cases} J_0^+ \phi_m^j = (j-m)\phi_{m+1}^j, \\ J_0^0 \phi_m^j = m\phi_m^j, \\ J_0^- \phi_m^j = (j+m)\phi_{m-1}^j. \end{cases} \tag{9}$$

- The fields in the m-bases and x-basis are related by

$$\phi_m^j(z) \sim \int \mathrm{d}x\, x^{j+m} \phi_x^j(z). \tag{10}$$

Irreducible representations of $\mathfrak{sl}_2$

- A representation $\mathcal{R}$ of $\mathfrak{sl}_2$ can be extended to an affine highest-weight representation $\widehat{\mathcal{R}}$ by acting with creation operators $J^a_{n<0}$.
- We classify the irreducible representations of $\mathfrak{sl}_2$ into the following series:
  - Principle continuous series $\mathcal{C}^j_\alpha,\ j \in -\frac{1}{2} + i\mathbb{R}_+$.
  - Discrete series $\mathcal{D}^{j,\pm},\ j \in (-\infty, -\frac{1}{2})$.
  - Finite dimensional representations $\mathcal{E}^j,\ j \in \mathbb{N}/2$.

The representations can be characterized by the eigenvalues m of J^0_0.

Representations	Parameter values	Eigenvalues of J^0_0
$\mathcal{C}^j_\alpha$	$j \in -\frac{1}{2} + i\mathbb{R}_+$, $\alpha \in \mathbb{R}$ mod $\mathbb{Z}$	$\alpha + \mathbb{Z}$
$\mathcal{D}^{j,+}$	$j \in (-\infty, -\frac{1}{2})$	$-j + \mathbb{N}$
$\mathcal{D}^{j,-}$	$j \in (-\infty, -\frac{1}{2})$	$j - \mathbb{N}$
$\mathcal{E}^j$	$j \in \mathbb{N}/2$	$\{-j, -j + \frac{1}{2}, \cdots, j - \frac{1}{2}, j\}$

Irreducible representations of $\mathfrak{sl}_2$

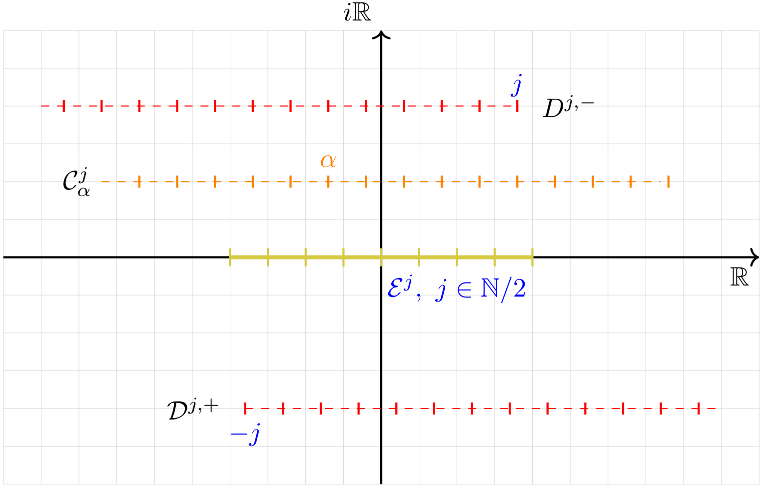


Figure: The spectra of irreducible representations of $\mathfrak{sl}_2$

Affine highest-weight representations of $\widehat{\mathfrak{sl}}_2$

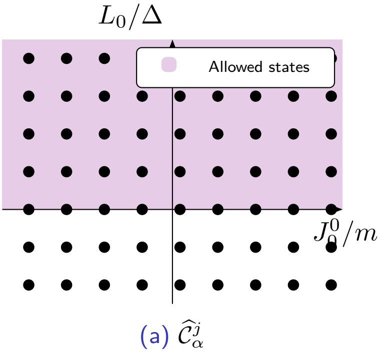
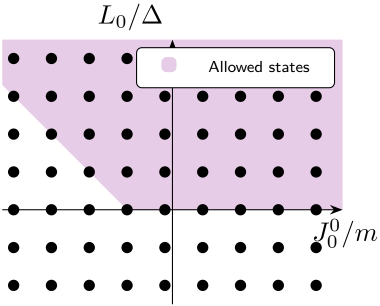
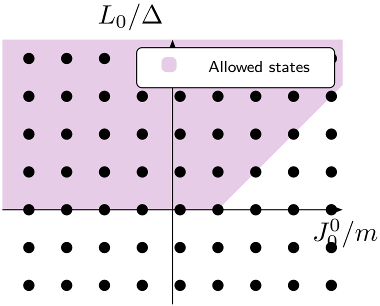


Figure: The spectra of irreducible representations of $\widehat{\mathfrak{sl}}_2(1)$

Affine highest-weight representations of $\widehat{\mathfrak{sl}}_2$



(a) $\widehat{\mathcal{D}}^{j,+}$



(b) $\widehat{\mathcal{D}}^{j,-}$

Figure: The spectra of irreducible representations of $\widehat{\mathfrak{sl}_2}(2)$

Spectral flowed representations

- A representation $\widehat{\mathcal{R}}$ provides the action of generators J_n^a on vector space V.
- The spectral flowed representation $\rho_\omega\left(\hat{\mathcal{R}}\right)$ acts on the vector space $V_\omega = \left\{|v,\omega\rangle \,\middle|\, |v\rangle \in V\right\}$, where the action of generators is defined to be

$$J_n^a\,|v,\omega\rangle = \rho_{-\omega}\left(J_n^a\right)|v\rangle \tag{11}$$

- The conformal dimension of states in $\rho_\omega\left(\hat{\mathcal{C}}^j\right), \omega \neq 0$ are not bounded from below. Hence it cannot be an affine highest-weight representation.
- On the other hand, we have

$$\begin{aligned}\rho_{-1}\left(\widehat{\mathcal{D}}^{j,+}\right) &= \widehat{\mathcal{D}}^{\frac{k}{2}-j,-}\\ \rho_{1}\left(\widehat{\mathcal{D}}^{j,-}\right) &= \widehat{\mathcal{D}}^{\frac{k}{2}-j,+}\end{aligned} \tag{12}$$

Degenerate representations

- A descendant state $|\chi\rangle = \hat{N}\left|j_{\hat{N}}\right\rangle$ is called a null vector if it is annihilated by all the positive modes:
$$J^a_{n>0}\,|\chi\rangle = 0, \tag{13}$$

- The null vector and its descendants form a non-trivial submodule $\widehat{\mathcal{V}}'$ of the representation $\widehat{\mathcal{V}}^j$. A degenerate representation is obtained by taking the quotient of the Verma module by this submodule:
$$\widehat{\mathcal{R}}^j = \widehat{\mathcal{V}}^j/\widehat{\mathcal{V}}' \tag{14}$$

- The degenerate representations $\widehat{\mathcal{R}}^{\langle r,s\rangle}$ are labeled by two integer r and s, whose null vector lives at level $N = rs$. The corresponding spin $j_{r,s}$ is given by:
$$j_{r,s} = \frac{s-1}{2} - \frac{k+2}{2}r \quad \text{for} \quad s \geq 1, r \geq 0. \tag{15}$$

Level 0 Degenerate representations

- The degenerate representation with a null vector at level 0 is of spin

$$j_{0,s} = \frac{s-1}{2} \tag{16}$$

.

- The level 0 degenerate representation is the affine highest-weight extension of the finite dimensional representations $\widehat{\mathcal{E}}^s$.
- The null vector is trivial

$$J_0^- \left| j_s, -j_s \right\rangle = 0. \tag{17}$$

Level 1 Degenerate representations

- The only possibility to have a level 1 null vector is $r = s = 1$, with spin

$$j_{1,1} = -\frac{k+2}{2}. \tag{18}$$

- The corresponding null vector is generated by the following null operator:

$$\hat{N}^c_{1,1} = K_{ab} J^a_{-1} J^b_0 J^c_0 + j_{1,1} f^c_{ab} J^a_{-1} J^b_0 - 2 j^2_{1,1} J^c_{-1}. \tag{19}$$

Spectral flowed vacuum representation

- The vacuum representation is nothing but the degenerate representation $\widehat{\mathcal{E}}^1$. One could view it as

$$\hat{\mathcal{E}}^{0,1} = \hat{\mathcal{D}}^{\langle 0,1\rangle,+} = \hat{\mathcal{D}}^{\langle 0,1\rangle,-} \tag{20}$$

- The identification implies the following relation:

$$\begin{aligned} \rho_1\left(\hat{\mathcal{E}}^{0,1}\right) &= \hat{\mathcal{D}}^{\langle 1,1\rangle,+} \\ \rho_{-1}\left(\hat{\mathcal{E}}^{0,1}\right) &= \hat{\mathcal{D}}^{\langle 1,1\rangle,-}. \end{aligned} \tag{21}$$

- The spectral flowed vacuum representation $\rho_\pm\left(\hat{\mathcal{E}}^{0,1}\right)$ is a degenerate representation, with the corresponding null vector:

$$J^{\mp}_{-1}\left|\frac{k}{2}, \mp\frac{k}{2}\right\rangle \tag{22}$$

Fusion rules

- In CFT, fusion rules form the algebra of primary fields, dictating how they combine. We could determine the fusion rules from the OPE

$$\phi^{j_1,\omega_1}(z_1)\phi^{j_2,\omega_2}(z_2) \sim \sum_{j_3,\omega_3} \left\langle \phi^{j_1,\omega_1}(z_1)\phi^{j_2,\omega_2}(z_2)\left(\phi^{j_3,\omega_3}(z_3)\right)^*\right\rangle \phi^{j_3,\omega_3}(z_3). \tag{23}$$

- The fusion $\widehat{\mathcal{R}}^{j_1,\omega_1}\widehat{\mathcal{R}}^{j_2,\omega_2} \ni \widehat{\mathcal{R}}^{j_3,\omega_3}$ is allowed only if the 3-point function $\left\langle \phi^{j_1,\omega_1}(z_1)\phi^{j_2,\omega_2}(z_2)\left(\phi^{j_3,\omega_3}(z_3)\right)^*\right\rangle$ is non-zero.
- The fusion rules involving degenerate representations is constrained by the null vector equations:

$$\left\langle \hat{N}\phi^{\langle r,s\rangle}(z_1)\phi^{j_2}(z_2)\phi^{j_3}(z_3)\right\rangle = 0. \tag{24}$$

Three-point functions

- In the x-basis, the three-point functions with three affine primary fields are

$$\left\langle \phi^{j_1}_{x_1,\bar{x}_1}(z_1,\bar{z}_1)\phi^{j_2}_{x_2,\bar{x}_2}(z_2,\bar{z}_2)\phi^{j_3}_{x_3,\bar{x}_3}(z_3,\bar{z}_3)\right\rangle = |z_{12}|^{-2\Delta^3_{12}}|z_{23}|^{-2\Delta^1_{23}}|z_{31}|^{-2\Delta^2_{31}} \times D\begin{bmatrix} j_1 & j_2 & j_3 \\ x_1 & x_2 & x_3 \end{bmatrix} C(j_1,j_2,j_3). \tag{25}$$

- The sturcture constant $C(j_1, j_2, j_3)$ is not fully determined yet. The x-dependence is included in the factor:

$$D\begin{bmatrix} j_1 & j_2 & j_3 \\ x_1 & x_2 & x_3 \end{bmatrix} = |x_{12}|^{2j^3_{12}}|x_{23}|^{2j^1_{23}}|x_{31}|^{2j^2_{31}}, \tag{26}$$

where $x_{ij} = x_i - x_j$ and $j^K_I = \sum_{i\in I} j_i - \sum_{k\in K} j_k$.

Three-point functions with spectral flow violation

- The three-point functions involving one spectral flowed fields are simplier in the m-basis:

$$\left\langle \phi^{j_1}_{m_1,\bar{m}_1}(z_1,\bar{z_1})\phi^{j_2}_{m_2,\bar{m}_2}(z_2,\bar{z_2})\phi^{j_3,-1}_{m_3,\bar{m}_3}(z_3,\bar{z_3})\right\rangle = \prod_i N^{j_i}_{m_i,\bar{m}_i}\delta(\sum_i m_i + \frac{k}{2})C(j_1,j_2,j_3) \\ \times \left| z_{12}^{\Delta^{j_3,\omega_3}_{m_3}-\Delta_{j_1}-\Delta_{j_2}} z_{23}^{\Delta_{j_1}-\Delta_{j_2}-\Delta^{j_3,\omega_3}_{m_3}} z_{31}^{\Delta_{j_2}-\Delta^{j_3,\omega_3}_{m_3}-\Delta_{j_1}} \right|^2 \tag{27}$$

  where $\Delta^{j,\omega}_m = \Delta_j - \omega m - \frac{1}{4}km^2$ is the conformal dimension of the spectral flowed fields $\phi^{j,\omega}_m(z)$.

- The normalization factor is

$$N^j_{m,\bar{m}} = \frac{\Gamma(j+1-m)}{\Gamma(\bar{m}-j)}. \tag{28}$$

Fusion with spectral flowed vacuum representation

- We conjecture that the spectral flow commutes with fusion, which means,

$$\rho_{\omega_1}\left(\hat{\mathcal{R}}\right) \times \rho_{\omega_2}\left(\mathcal{R}'\right) = \rho_{\omega_1+\omega_2}\left(\hat{\mathcal{R}} \times \mathcal{R}'\right). \tag{29}$$

- Since

$$\hat{\mathcal{E}}^1 \times \hat{\mathcal{R}}^j = \hat{\mathcal{R}}^j. \tag{30}$$

we have

$$\rho_\omega\left(\hat{\mathcal{E}}^1\right) \times \hat{\mathcal{R}}^j = \rho_\omega\left(\hat{\mathcal{R}}^j\right). \tag{31}$$

Fusion with spectral flowed vacuum representation

- We prove this fusion rule by calculating the null vector equation:

$$\left\langle J^-_{-1}\phi^{\frac{k}{2}}_{-\frac{k}{2},\bar{m_1}}(z_1,\bar{z_1})\phi^{j_2}_{m_2,\bar{z_2}}(z_2,\bar{z_2})\phi^{j_3,-1}_{m_3,\bar{m_3}}(z_3,\bar{z_3})\right\rangle = 0. \tag{32}$$

- The above equation gives the following constraint:

$$j_2(j_2+1) = j_3(j_3+1) \tag{33}$$

- Hence we proved the fusion rules with spectral flowed vacuum representation at $\omega = \pm 1$,

$$\rho_{\pm 1}\left(\hat{\mathcal{E}}^1\right) \times \hat{\mathcal{R}}^j = \rho_{\pm 1}\left(\hat{\mathcal{R}}^j\right). \tag{34}$$

Fusion with level 1 degenerate representation

- Let's first consider the spectral flow preserving case in the x-basis. The null vector equation gives

$$\left\langle \hat{N}^{c}_{1,1}\phi^{1,1}_{x_1,\bar{x_1}}(z_1,\bar{z_1})\phi^{j_2}_{x_2,\bar{x_2}}(z_2,\bar{z_2})\phi^{j_3}_{x_3,\bar{x_3}}(z_3,\bar{z_3})\right\rangle = 0. \tag{35}$$

- The above equation gives the following constraint on the spins

$$\left(j_{1,1}^2 - (j_2 - j_3)^2\right)(1 + j_{1,1} + j_2 + j_3) = 0. \tag{36}$$

The solution to this equation is $j_3 = j_2 \pm j_{1,1}, -j_2 - 1 + j_{1,1}$.

- Hence we find the following fusion rule:

$$\hat{\mathcal{R}}^{1,1} \times \hat{\mathcal{R}}^{j} \supset \hat{\mathcal{R}}^{j+j_{1,1}} \oplus \hat{\mathcal{R}}^{j-j_{1,1}}. \tag{37}$$

Fusion with level 1 degenerate representation

- Next, let's consider the spectral flow violating case in the m-basis. The null vector equation becomes

$$\left\langle \hat{N}^{c}_{1,1}\phi^{\langle 1,1\rangle}_{m_1,\bar{m}_1}(z_1,\bar{z_1})\phi^{j_2}_{m_2,\bar{z_2}}(z_2,\bar{z_2})\phi^{j_3,-1}_{m_3,\bar{m_3}}(z_3,\bar{z_3})\right\rangle = 0. \tag{38}$$

- It gives

$$j_2(j_2+1) = j_3(j_3+1) \tag{39}$$

- The equations for $\omega = \pm 1$ should be symmetric, which gives the same constraint. Hence we find:

$$\hat{\mathcal{R}}^{1,1} \times \hat{\mathcal{R}}^{j} \supset \rho_1\left(\hat{\mathcal{R}}^{j}\right) \oplus \rho_1\left(\hat{\mathcal{R}}^{j}\right). \tag{40}$$

Fusion with level 1 degenerate representation

Fusion rules with level 1 degenerate representation

In conclusion, we find the fusion rule with the level 1 degenerate representation to be

$$\widehat{\mathcal{R}}^{\langle 1,1\rangle} \times \widehat{\mathcal{R}}^{j} = \widehat{\mathcal{R}}^{j+j_{1,1}} \oplus \widehat{\mathcal{R}}^{j-j_{1,1}} \oplus \rho_1\left(\widehat{\mathcal{R}}^{j}\right) \oplus \rho_{-1}\left(\widehat{\mathcal{R}}^{j}\right). \tag{41}$$

Conclusion and Outlook

- We determined the fusion rules between spectral flowed vacuum representation with affine highest weight representations, which gives

$$\rho_{\pm 1}\left(\hat{\mathcal{E}}^{1}\right) \times \hat{\mathcal{R}}^{j} = \rho_{\pm 1}\left(\hat{\mathcal{R}}^{j}\right). \tag{42}$$

- We also give the fusion rule between level 1 degenerate representations with affine highest representations:

$$\widehat{\mathcal{R}}^{\langle 1,1\rangle} \times \widehat{\mathcal{R}}^{j} = \widehat{\mathcal{R}}^{j+j_{1,1}} \oplus \widehat{\mathcal{R}}^{j-j_{1,1}} \oplus \rho_{1}\left(\widehat{\mathcal{R}}^{j}\right) \oplus \rho_{-1}\left(\widehat{\mathcal{R}}^{j}\right). \tag{43}$$

- In the future, one possible generalizing of our results is to extend the fusion rules with spectral flowed vacuum represnetations to generic $\omega \in \mathbb{Z}$.

Thank You

Sugawara construction

- The symmetry algebra of 2D CFT is the Virasoro algebra, whose generators are $(L_n)_{n\in\mathbb{Z}}$. The commutation relations for Virasoro generators are

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(n-1)n(n+1)\delta_{n+m,0}. \tag{44}$$

The energy momentum $T(z)$ is a generating function of L_n:

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}. \tag{45}$$

- We introduce the Sugawara construction for the energy momentum tensor $T(z)$:

$$T(z) = \frac{K_{ab}}{2(k-2)} : J^a(z)J^b(z) :, \tag{46}$$

where $K_{ab} = K^{ab} = \frac{1}{2g} f^{ac}_{d} f^{bd}_{c}$