3 point function with D series representation

March 13, 2025

Consider a discrete series representation $\hat{D}^{j_D,+}$. There is a $\mathfrak{sl}_2$ lowest weight state $|j_D, j_D\rangle$ such that $J_0^- |j_D, j_D\rangle = 0$. Hence the corresponding field satisfies

$$J_0^- \phi^{j_D}_{j_D,x}(z) = 0. \tag{0.1}$$

Insert this equation to 3 point functions, we find

$$\left\langle J_0^{-,(z_1)} \phi^{j_D}_{j_D,x_1}(z_1) \phi^{j_2}_{x_2}(z_2) \phi^{j_3}_{x_3}(z_3) \right\rangle = 0. \tag{0.2}$$

It means

$$\oint dz \left\langle J^-(z) \phi^{j_D}_{j_D,x_1}(z_1) \phi^{j_2}_{x_2}(z_2) \phi^{j_3}_{x_3}(z_3) \right\rangle = 0, \tag{0.3}$$

where the contour is chosen to be around z_1 but not including z_2 and z_3. Using the OPE between J and affine primary fields, we find

$$\oint dz \left\langle \frac{D^{j_D}_{x_1}(t^-)}{z - z_1} \phi^{j_D}_{j_D,x_1}(z_1) \phi^{j_2}_{x_2}(z_2) \phi^{j_3}_{x_3}(z_3) \right\rangle + \sum_{s=2,3} \oint dz \left\langle \phi^{j_D}_{j_D,x_1}(z_1) \frac{D^{j_s}_{x_s}(t^-)}{z - z_s} \phi^{j_2}_{x_2}(z_2) \phi^{j_3}_{x_3}(z_3) \right\rangle = 0 \tag{0.4}$$

The second term has no sigularity at z_1, hence has no contribution to the integral. The first term gives

$$\left\langle D^{j_D}_{x_1}(t^-) \phi^{j_D}_{j_D,x_1}(z_1) \phi^{j_2}_{x_2}(z_2) \phi^{j_3}_{x_3}(z_3) \right\rangle = 0. \tag{0.5}$$

In x-basis, it means

$$\partial_1 \left\langle \phi^{j_D}_{j_D,x_1}(z_1) \phi^{j_2}_{x_2}(z_2) \phi^{j_3}_{x_3}(z_3) \right\rangle = 0. \tag{0.6}$$

But since we know

$$\left\langle \phi^{j_D}_{j_D,x_1}(z_1) \phi^{j_2}_{x_2}(z_2) \phi^{j_3}_{x_3}(z_3) \right\rangle \sim x_{12}^{j_D + j_2 - j_3} x_{23}^{j_2 + j_3 - j_D} x_{31}^{j_3 + j_D - j_1}, \tag{0.7}$$

The derivative w.r.t. x_1 means

$$j_D + j_2 - j_3 = 0, \tag{0.8}$$

$$j_3 + j_D - j_2 = 0. \tag{0.9}$$

Which has no solution.

Problem solved. 0.1 should be writen in m basis since $|j, m\rangle$ is eigenstate of J_0^0. Hence we cannot define a field in x basis corresponding the highest weight state.