Fusion rule and spectrally flowed degenerate representations

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1 Fusion rules

We single out the contribution of a primary field ϕ^j_x and it's descendents in the OPE between two primary fields:

$$\phi^{j_1}_{x_1}(z_1)\phi^{j_2}_{x_2}(z_2) \supset \sum_{J\in\mathcal{J}} C^{j,J}_{j_1,j_2}(z_1,z_2) D\begin{bmatrix} -j_3-1 & j_2 & j_1 \\ x_3 & x_2 & x_1 \end{bmatrix} J\phi^j_x(z_2), \tag{1.1}$$

where $\mathcal{J}$ is the set of creation operators. In x-basis, the 3-point function is propotional to

$$D\begin{bmatrix} j_3 & j_2 & j_1 \\ x_3 & x_2 & x_1 \end{bmatrix} = x_{12}^{j_{12}^3} x_{23}^{j_{23}^1} x_{31}^{j_{31}^2}. \tag{1.2}$$

On both sides of the OPE, we insert $\oint_{z_1,z_2} dy\,(y-z_2)^m\, J^a(y)$, where the integration contour encloses both z_1 and z_2. Use the OPE between current operator $J^a(y)$ and primary field, we find

$$\left(J^{a,(z_2)}_m + z_{12}^m D^{j_1}_{x_1}\right)\phi^{j_1}_{x_1}(z_1)\phi^{j_2}_{x_2}(z_2) \supset C^{j,J}_{j_1,j_2}(z_1,z_2) D\begin{bmatrix} -j_3-1 & j_2 & j_1 \\ x_3 & x_2 & x_1 \end{bmatrix} J^a_m J\phi^j_x(z_2) \tag{1.3}$$

The vanishing of null vector $\hat{N}^0_{1,1}\phi^{1,1}_x = 0$ gives the following equation:

$$\begin{aligned}\phi^{j_2}_{x_{[2]}}\hat{N}^0_{1,1}\phi^{1,1}_{x_1} \supset \Bigg(&\frac{1}{z_{21}}K_{ab}D_{x_1}\left(t^0\right)D_{x_s}\left(t^a\right)D_{x_1}\left(t^b\right) + \frac{1}{z_{21}}j_{1,1}f^0_{ab}D_{x_s}\left(t^a\right)D_{x_1}\left(t^b\right) \\ &-\frac{2}{z_{12}}j^2_{1,1}D_{x_s}\left(t^0\right) + \hat{N}^0_{1,1}\Bigg) C^{j,J}_{j_1,j_2}(z_1,z_2) D\begin{bmatrix} -j_3-1 & j_2 & j_1 \\ x_3 & x_2 & x_1 \end{bmatrix} J\phi^j_x(z_2) = 0.\end{aligned} \tag{1.4}$$

The contribution of primary fields should vanish.This amounts to a differential equation of the 3-point function.

$$\left(K_{ab}D_{x_1}\left(t^0\right)D_{x_s}\left(t^a\right)D_{x_1}\left(t^b\right)+\frac{1}{z_{21}}j_{1,1}f^0_{ab}D_{x_s}\left(t^a\right)D_{x_1}\left(t^b\right)-\frac{2}{z_{12}}j^2_{1,1}D_{x_s}\left(t^0\right)\right)D\begin{bmatrix}-j_3-1 & j_2 & j_1\\ x_3 & x_2 & x_1\end{bmatrix}=0. \quad (1.5)$$

The solution gives us the fusion rule [1]:

$$\hat{\mathcal{R}}^{1,1}\times\hat{\mathcal{R}}^{j}=\hat{\mathcal{R}}^{j+j_{1,1}}+\hat{\mathcal{R}}^{j-j_{1,1}}. \quad (1.6)$$

Remark: We will not reduce the complexity by using the OPE Ward identities, since eventually we have to deal with the same differential function of 3-point functions.

2 Spectral Flow

2.1 Definition

The $\widehat{\mathfrak{sl}_2}$ algebra has a family of automorphisms $\rho_n, n\in\mathbb{Z}$, called spectral flow. They are defined by

$$\begin{aligned}\rho_n(J^{\pm}_m)&=J^{\pm}_{m\pm n},\\ \rho_n(J^0_m)&=J^0_m+kn\delta_{m,0}.\end{aligned} \quad (2.1)$$

According to the Sugawara constraction, their action on Virasoro generators are given by

$$\rho_n(L_m)=L_m+nJ^0_m+\frac{1}{4}kn^2\delta_{m,0}. \quad (2.2)$$

The spectral flows satisfy

$$\rho_{n_1}\circ\rho_{n_2}=\rho_{n_1+n_2}. \quad (2.3)$$

Given a representation $\hat{\mathcal{R}}$ of $\widehat{\mathfrak{sl}_2}$ on vector spece V, a spectrally flowed representation $\rho_n\left(\hat{\mathcal{R}}\right)$ is defined on vector space

$$V'=\{\rho_n(|v\rangle)|\ |v\rangle\in V\} \quad (2.4)$$

The action of $\widehat{\mathfrak{sl}_2}$ generators on spectrally flowed representations is

$$J^a_m\rho_n\left(|v\rangle\right)=\rho_n\left(\rho_{-n}\left(J^a_m\right)|v\rangle\right). \quad (2.5)$$

The conjugate representation of $\rho_n\left(\hat{\mathcal{R}}\right)$ is

$$\rho_n\left(\hat{\mathcal{R}}\right)^*=\rho_{-n}\left(\hat{\mathcal{R}}^*\right) \quad (2.6)$$

In addition, it's believed that the spectral flow commutes with fusion [2],

$$\rho_n\left(\hat{\mathcal{R}}\right)\times\rho_m\left(\mathcal{R}'\right)=\rho_{n+m}\left(\hat{\mathcal{R}}\times\mathcal{R}'\right). \quad (2.7)$$

2.2 Spectrally flowed representations

We introduce the following notation

$$
\begin{aligned}
\hat{\mathcal{C}}^{j,n} &= \rho_n\left(\hat{\mathcal{C}}^{j}\right), \\
\hat{\mathcal{D}}^{j,\frac{1}{2}+n} &= \rho_n\left(\hat{\mathcal{D}}^{j,+}\right).
\end{aligned}
\tag{2.8}
$$

From 2.2, we find the eigenvalues of L_0 in $\hat{\mathcal{C}}^{j,n}$ of non-zero n are not bounded from below. Hence it cannot be an affine highest-weight representation.

On the other hand, the representations $\hat{\mathcal{D}}^{j,\pm}$ are characterized by the existence of state $|j,\mp j\rangle$, which satisfy the following conditions:

$$
J^a_{n>0}\left|j,\pm j\right\rangle = J^{\pm}_0\left|j,\pm j\right\rangle = (J^0_0 \mp j)\left|j,\pm j\right\rangle = 0. \tag{2.9}
$$

In particular, we notice that

$$
\begin{aligned}
&J^{+}_{n\geq 0}\rho_{-1}(|j,-j\rangle) = \rho_{-1}\left(J^{+}_{n+1}\left|j,-j\right\rangle\right) = 0 \\
&J^{0}_{n>0}\rho_{-1}(|j,-j\rangle) = (J^0_0 - \frac{k}{2} + j)\rho_{-1}\left(|j,-j\rangle\right) = 0 \\
&J^{-}_{n>0}\rho_{-1}(|j,-j\rangle) = \rho_{-1}\left(J^{-}_{n-1}\left|j,-j\right\rangle\right) = 0.
\end{aligned}
\tag{2.10}
$$

Hence we find $\rho_{-1}(|j,-j\rangle) = \left|\frac{k}{2}-j,\frac{k}{2}-j\right\rangle$, and hence

$$
\hat{\mathcal{D}}^{j,-\frac{1}{2}} = \rho_{-1}\left(\hat{\mathcal{D}}^{j,+}\right) = \hat{\mathcal{D}}^{\frac{k}{2}-j,-}. \tag{2.11}
$$

If we take $j = j_{r,s} = \frac{s-1}{2} - \frac{k+2}{2}r$, we find

$$
\frac{k}{2} - j_{r,s} = \frac{-s-1}{2} - \frac{k+2}{2}(-r-1) = j_{-r-1,-s}. \tag{2.12}
$$

And since $j_{-r,-s} = -1 - j_{r,s}$, if we apply $\rho_{\mp 1}$ on degenerate representation $\hat{\mathcal{D}}^{\langle r,s\rangle,\pm}$, we should again obtain another degenerate representation $\hat{\mathcal{D}}^{\langle -r-1,-s\rangle,\mp}$.

2.3 Spectrally flowed vacuum representation

Now we consider the spectral flow of the vacuum representation, i.e. the degenerate representation $\hat{\mathcal{E}}^1$. On one hand, the fusion between any representation $\hat{\mathcal{R}}^j$ with the vacuum representation should give $\hat{\mathcal{R}}^j$ back:

$$
\hat{\mathcal{E}}^1 \times \hat{\mathcal{R}}^j = \hat{\mathcal{R}}^j. \tag{2.13}
$$

Hence from our assumption 2.7, we find

$$
\rho_n\left(\hat{\mathcal{E}}^1\right) \times \hat{\mathcal{R}}^j = \rho_n\left(\hat{\mathcal{R}}^j\right). \tag{2.14}
$$

On the other hand, one could view $\hat{\mathcal{E}}^{0,1}$ as

$$
\hat{\mathcal{E}}^{0,1} = \hat{\mathcal{D}}^{\langle 0,1\rangle,+} \cap \hat{\mathcal{D}}^{\langle 0,1\rangle,-} \tag{2.15}
$$

which implies the following relation:

$$\begin{aligned}\rho_1\left(\hat{\mathcal{E}}^{0,1}\right) &= \hat{\mathcal{D}}^{\langle -1,-1\rangle,+}\\ \rho_{-1}\left(\hat{\mathcal{E}}^{0,1}\right) &= \hat{\mathcal{D}}^{\langle -1,-1\rangle,-}.\end{aligned} \tag{2.16}$$

This can be verified by the following equations:

$$\begin{aligned}J_1^+\rho_{-1}\left(|0,0\rangle\right) &= \rho_{-1}\left(J_2^+\,|0,0\rangle\right) = 0\\ J_1^-\rho_{-1}\left(|0,0\rangle\right) &= \rho_{-1}\left(J_0^-\,|0,0\rangle\right) = 0\\ J_1^0\rho_{-1}\left(|0,0\rangle\right) &= \rho_{-1}\left(J_1^0\,|0,0\rangle\right) = 0\\ J_0^+\rho_{-1}(|0,0\rangle) &= \rho_{-1}\left(J_1^+\,|0,0\rangle\right) = 0\\ J_0^0\rho_{-1}\left(|0,0\rangle\right) &= \rho_{-1}\left(\left(J_0^0+\frac{k}{2}\right)|0,0\rangle\right) = \frac{k}{2}\rho_{-1}\left(|0,0\rangle\right)\end{aligned} \tag{2.17}$$

We find

$$\rho_{-1}\left(|0,0\rangle\right) = \left|\frac{k}{2},\frac{k}{2}\right\rangle. \tag{2.18}$$

Hence we have the following fusion rules:

$$\hat{\mathcal{D}}^{\langle -1,-1\rangle,\pm}\times\hat{\mathcal{R}}^j = \rho_\pm\left(\hat{\mathcal{R}}^j\right). \tag{2.19}$$

2.4 $j_{-1,-1}$ representations

Any affine highest representation with spin $j_{-1,-1}=\frac{k}{2}$ is a degenerate representation with level $-1\times -1 = 1$ null states. The corresponding null vector can be obtained by solving the following equations:

$$J_1^a\left(a_+J_{-1}^+J_0^- + a_0J_{-1}^0 + a_-J_{-1}^-J_0^+\right)|j_{-1,-1},m\rangle = 0. \tag{2.20}$$

We find null state to be

$$\left(-\frac{1}{j_{-1,-1}-m+1}J_{-1}^+J_0^- + 2J_{-1}^0 + \frac{1}{j_{-1,-1}+m+1}J_{-1}^-J_0^+\right)|j_{-1,-1},m\rangle \tag{2.21}$$

The null state is of spin $j_{-1,-1}+1$. We may compare the degenerate representations with $j_{1,1}$ and $j_{-1,-1}$:

$j_{1,1}=-\frac{k+2}{2}$	$j_{-1,-1}=\frac{k}{2}=-j_{1,1}-1$
$\hat{N}_{1,1}^a\,|j_{1,1},m\rangle\in\hat{\mathcal{R}}^{j_{1,1}-1}$	$\hat{N}_{-1,-1}^a\,|j_{-1,-1},m\rangle\in\hat{\mathcal{R}}^{j_{-1,-1}}$
-	$\hat{\mathcal{D}}^{\langle -1,-1\rangle,\pm}=\rho_{\pm 1}\left(\hat{\mathcal{E}}^1\right)$

It also implies there should be continuous series representation with $j_{-1,-1}$, whose fusion rules are still unknown. One conjecture is that the fusion gives a continuous transformation from $\rho_{-1}(\hat{\mathcal{R}}^j)$ to $\rho_1(\hat{\mathcal{R}}^j)$:

Representations	$\hat{\mathcal{D}}^{\langle -1,-1\rangle,+}$	$\hat{\mathcal{C}}_\alpha^{\langle -1,-1\rangle}$	$\hat{\mathcal{D}}^{\langle -1,-1\rangle,-}$
m	$-j_{-1,-1}+\mathbb{N}$	$\alpha+\mathbb{Z}$	$j_{-1,-1}-\mathbb{N}$
$\times\hat{\mathcal{R}}^j$	$\rho_1\left(\hat{\mathcal{R}}^j\right)$	?	$\rho_{-1}\left(\hat{\mathcal{R}}^j\right)$

2.5 Spectrally flowed degenerate representations

We try to list all degenerate representations, including both affine highest-weight and non-highest-weight:

Spectral flow	ρ_{-1}	ρ_1	$\rho_n, \|n\| > 1$
$\hat{\mathcal{C}}_\alpha^{\langle r,s\rangle}$	-	-	-
$\hat{\mathcal{C}}_\alpha^{\langle -r,-s\rangle}$	-	-	-
$\hat{\mathcal{D}}^{\langle r,s\rangle,+}$	$\hat{\mathcal{D}}^{\langle -r-1,-s\rangle,-}$	-	-
$\hat{\mathcal{D}}^{\langle r,s\rangle,-}$	-	$\hat{\mathcal{D}}^{\langle -r-1,-s\rangle,+}$	-
$\hat{\mathcal{E}}^s$	$\hat{\mathcal{D}}^{\langle -1,-s\rangle,-}$	$\hat{\mathcal{D}}^{\langle -1,-s\rangle,+}$	-

where '-' means the corresponding representation is non-highest-weight.

References

[1] Dario Stocco. "The torus one-point block of 2d CFT and null vectors in $\hat{\mathfrak{sl}}_2$". In: (Sept. 2022). arXiv: 2209.08653 [hep-th].

[2] Matthias R Gaberdiel. "Fusion rules and logarithmic representations of a WZW model at fractional level". In: *Nucl. Phys. B* 618 (2001), pp. 407–436. DOI: 10.1016/S0550-3213(01)00490-4. arXiv: hep-th/0105046.