Degenerate Representations and Fusion Rules in the $\widetilde{SL}_2(\mathbb{R})$ WZW Model

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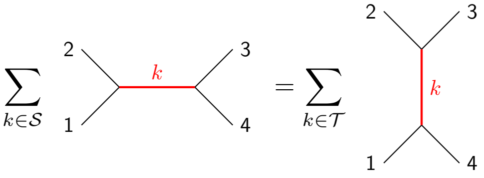
Introduction

String theory in AdS_3
Quantum Gravity

$\longrightarrow$

$\widetilde{SL}_2(\mathbb{R})$ WZW model
CFT with an $\widehat{\mathfrak{sl}}_2$ symmetry

The crossing symmetry equation:



Degenerate representations $\Longrightarrow$ Finite many terms

Symmetry algebra

- The $\mathfrak{sl}_2$ algebra:

$$\left[J^0, J^\pm\right] = \pm J^\pm, \quad \left[J^+, J^-\right] = 2J^0.$$

- The $\widehat{\mathfrak{sl}_2}$ algebra are defined by:

$$\left[J^a_m, J^b_n\right] = f^{ab}_c J^c_{m+n} + kmK^{ab}\delta_{m+n,0}, \quad m, n \in \mathbb{Z}$$

where k is the level, and $K^{ab} = \frac{1}{2g} f^{ac}_d f^{bd}_c$ is the Killing tensor.

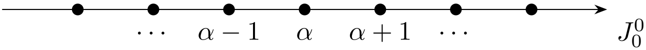
- The Sugawara construction of the conformal symmetry generator:

$$L_0 = \frac{K_{ab}}{2(k-2)} \left(J^a_0 J^b_0 + 2 \sum_{m>0} J^a_{-m} J^b_m \right).$$

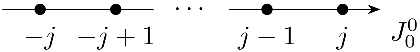
Irreducible representations of $\mathfrak{sl}_2$

We characterize different representations by the spin j and the eigenvalue m of J_0^0:

- $\mathcal{C}_\alpha^j$: $j \in \frac{1}{2} + i\mathbb{R}_+$, $m \in \alpha + \mathbb{Z}$.



- $\mathcal{E}^j$: $j \in \mathbb{N}/2$, $m \in \{-j, -j+1, \cdots, j-1, j\}$.

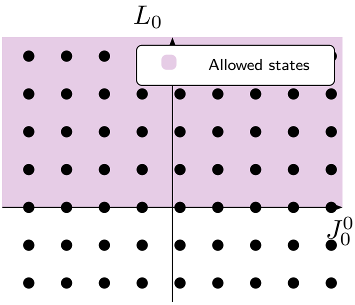


$\mathcal{E}^j$ is a level 0 degenerate representation, with the following null vector:

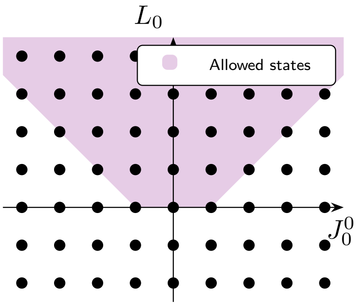
$$J_0^- \left| j, -j \right\rangle = 0.$$

Affine highest-weight representations of $\widehat{\mathfrak{sl}_2}$

$$\boxed{\mathcal{R}^j \text{ of } \mathfrak{sl}_2} \xrightarrow[J^a_{-n},\, n>0]{\text{Acting creation operators}} \boxed{\widehat{\mathcal{R}}^j \text{ of } \widehat{\mathfrak{sl}_2}}$$



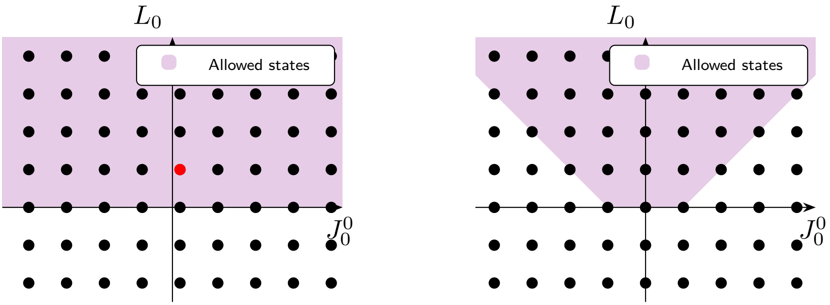
(a) $\widehat{\mathcal{C}}^j_\alpha$



(b) $\widehat{\mathcal{E}}^j$

Affine highest-weight representations of $\widehat{\mathfrak{sl}_2}$

$$\boxed{\mathcal{R}^j \text{ of } \mathfrak{sl}_2} \xrightarrow[J^a_{-n},\, n>0]{\text{Acting creation operators}} \boxed{\widehat{\mathcal{R}}^j \text{ of } \widehat{\mathfrak{sl}_2}}$$



(a) $\widehat{\mathcal{C}}^j_\alpha$

(b) $\widehat{\mathcal{E}}^j$

Spectral flow

- The spectral flow is a family $(\rho_\omega)_{\omega\in\mathbb{Z}}$ of automorphisms of $\widehat{\mathfrak{sl}_2}$:

$$\begin{aligned}\rho_\omega(J^\pm_m) &= J^\pm_{m\pm\omega},\\ \rho_\omega(J^0_m) &= J^0_m + \frac{1}{2}k\omega\delta_{m,0}.\end{aligned}$$

- The spectral flowed representation $\rho_\omega\left(\widehat{\mathcal{R}}\right)$ is related to $\widehat{\mathcal{R}}$ through

$$J^a_n \left|v,\omega\right\rangle = \rho_{-\omega}\left(J^a_n\right)\left|v\right\rangle .$$

My contributations

- Spectral flowed degenerate representation
    - Spectral flowed vacuum representation
- Fusion rules with degenerate representations
    - level 0
    - level 1

Fusion rules

- The fusion rules are determined from the OPE

$$\phi^{j_1,\omega_1}(z_1)\phi^{j_2,\omega_2}(z_2) \sim \sum_{j_3,\omega_3} \left\langle \phi^{j_1,\omega_1}(z_1)\phi^{j_2,\omega_2}(z_2) \left(\phi^{j_3,\omega_3}(z_3)\right)^* \right\rangle \phi^{j_3,\omega_3}(z_3).$$

$$\left\langle \phi^{j_1,\omega_1}(z_1)\phi^{j_2,\omega_2}(z_2) \left(\phi^{j_3,\omega_3}(z_3)\right)^* \right\rangle \neq 0 \Longleftrightarrow \widehat{\mathcal{R}}^{j_1,\omega_1} \times \widehat{\mathcal{R}}^{j_2,\omega_2} \supset \widehat{\mathcal{R}}^{j_3,\omega_3}$$

- The fusion rules involving degenerate representations is constrained by the null vector equations:

$$\hat{N}\left|j,m,\omega\right\rangle = 0 \Longrightarrow \left\langle \hat{N}\phi^{j,\omega}_m(z) \prod_i \phi^{j_i,\omega_i}_{m_i}(z_i) \right\rangle = 0.$$

Spectral flowed vacuum representation

- $\rho_\pm\left(\widehat{\mathcal{E}}^0\right)$ is a level 1 degenerate representation, with the corresponding null vector:

$$J^{\mp}_{-1}\left|k/2, \mp k/2\right\rangle = 0.$$

- The corresponding null vector equation:

$$\left\langle J^{\pm}_{-1}\phi\Big(\left|k/2, \pm k/2\right\rangle, z_1\Big)\phi\Big(\left|j_2, m_2\right\rangle, z_2\Big)\phi\Big(\left|j_3, m_3, \mp 1\right\rangle, z_3\Big)\right\rangle = 0,$$

$$\Rightarrow j_2(j_2+1) = j_3(j_3+1)$$

- We have the following fusion rule:

$$\rho_{\pm 1}\left(\hat{\mathcal{E}}^0\right) \times \widehat{\mathcal{R}}^j = \rho_{\pm 1}\left(\widehat{\mathcal{R}}^j\right).$$

Fusion with spectral flowed representations

- We conjecture that the spectral flow commutes with fusion, which means,

$$\rho_{\omega_1}\left(\hat{\mathcal{R}}\right) \times \rho_{\omega_2}\left(\mathcal{R}'\right) = \rho_{\omega_1+\omega_2}\left(\hat{\mathcal{R}} \times \mathcal{R}'\right).$$

- The fusion rule with $\rho_{\pm 1}\left(\widehat{\mathcal{E}}^0\right)$ proves the simplest case:

$$\begin{aligned}
\rho_{\pm 1}\left(\widehat{\mathcal{R}}^{j_1}\right) \times \widehat{\mathcal{R}}^{j_2} &= \left(\rho_{\pm 1}\left(\hat{\mathcal{E}}^0\right) \times \widehat{\mathcal{R}}^{j_1}\right) \times \widehat{\mathcal{R}}^{j_2} \\
&= \rho_{\pm 1}\left(\hat{\mathcal{E}}^0\right) \times \left(\widehat{\mathcal{R}}^{j_1} \times \widehat{\mathcal{R}}^{j_2}\right) \\
&= \rho_{\pm 1}\left(\widehat{\mathcal{R}}^{j_1} \times \widehat{\mathcal{R}}^{j_2}\right).
\end{aligned}$$

Level 1 degenerate representations

- The level 1 degenerate representation $\widehat{\mathcal{R}}^{\langle 1,1\rangle}$ is of spin

$$j_{1,1} = -\frac{k+2}{2}.$$

- The null vector is given by the following null operator:

$$\hat{N}^c = K_{ab} J^a_{-1} J^b_0 J^c_0 + j_{1,1} f^c_{ab} J^a_{-1} J^b_0 - 2 j^2_{1,1} J^c_{-1}.$$

Fusion with level 1 degenerate representation

- Let's first consider the spectral flow preserving case. The null vector equation gives

$$\left\langle \hat{N}^c \phi\Big(|j_{1,1}, m_1\rangle, z_1\Big)\phi\Big(|j_2, m_2\rangle, z_2\Big)\phi\Big(|j_3, m_3\rangle, z_3\Big)\right\rangle = 0.$$

$$\Longrightarrow \left(j_{1,1}^2 - (j_2 - j_3)^2\right)(1 + j_{1,1} + j_2 + j_3) = 0.$$

The solution to this equation is $j_3 = j_2 \pm j_{1,1}, -j_2 - 1 + j_{1,1}$.

- We find the following fusion rule:

$$\hat{\mathcal{R}}^{\langle 1,1\rangle} \times \hat{\mathcal{R}}^{j} \supset \hat{\mathcal{R}}^{j+j_{1,1}} \oplus \hat{\mathcal{R}}^{j-j_{1,1}}.$$

Fusion with level 1 degenerate representation

- Next, let's consider the spectral flow violating case. The null vector equation becomes

$$\left\langle \hat{N}^c \phi\Big(\left|j_{1,1}, m_1\right\rangle, z_1\Big) \phi\Big(\left|j_2, m_2\right\rangle, z_2\Big) \phi\Big(\left|j_3, m_3, \pm 1\right\rangle, z_3\Big) \right\rangle = 0.$$

$$\Longrightarrow j_2(j_2+1) = j_3(j_3+1)$$

- We find:

$$\hat{\mathcal{R}}^{\langle 1,1\rangle} \times \hat{\mathcal{R}}^j \supset \rho_1\left(\hat{\mathcal{R}}^j\right) \oplus \rho_1\left(\hat{\mathcal{R}}^j\right).$$

- In conclusion, we find the fusion rule with the level 1 degenerate representation to be:

$$\widehat{\mathcal{R}}^{\langle 1,1\rangle} \times \widehat{\mathcal{R}}^j = \widehat{\mathcal{R}}^{j+j_{1,1}} \oplus \widehat{\mathcal{R}}^{j-j_{1,1}} \oplus \rho_1\left(\widehat{\mathcal{R}}^j\right) \oplus \rho_{-1}\left(\widehat{\mathcal{R}}^j\right).$$

Conclusion and Outlook

- We determined the fusion rules between spectral flowed vacuum representation with affine highest weight representations, which gives

$$\rho_{\pm 1}\left(\hat{\mathcal{E}}^{1}\right) \times \hat{\mathcal{R}}^{j} = \rho_{\pm 1}\left(\hat{\mathcal{R}}^{j}\right).$$

- We also give the fusion rule between level 1 degenerate representations with affine highest representations:

$$\widehat{\mathcal{R}}^{\langle 1,1\rangle} \times \widehat{\mathcal{R}}^{j} = \widehat{\mathcal{R}}^{j+j_{1,1}} \oplus \widehat{\mathcal{R}}^{j-j_{1,1}} \oplus \rho_{1}\left(\widehat{\mathcal{R}}^{j}\right) \oplus \rho_{-1}\left(\widehat{\mathcal{R}}^{j}\right).$$

- In the future, one possible generalizing of our results is to extend the fusion rules with spectral flowed vacuum represnetations to generic $\omega \in \mathbb{Z}$.

Thank You

Sugawara construction

- The symmetry algebra of 2D CFT is the Virasoro algebra, whose generators are $(L_n)_{n\in\mathbb{Z}}$. The commutation relations for Virasoro generators are

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(n-1)n(n+1)\delta_{n+m,0}. \tag{1}$$

The energy momentum $T(z)$ is a generating function of L_n:

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}. \tag{2}$$

- We introduce the Sugawara construction for the energy momentum tensor $T(z)$:

$$T(z) = \frac{K_{ab}}{2(k-2)} : J^a(z) J^b(z) :, \tag{3}$$

where $K_{ab} = K^{ab} = \frac{1}{2g} f^{ac}_{d} f^{bd}_{c}$